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MSc Mathematics

Master Thesis

Local Flags: Bounding the Strong Chromatic Index

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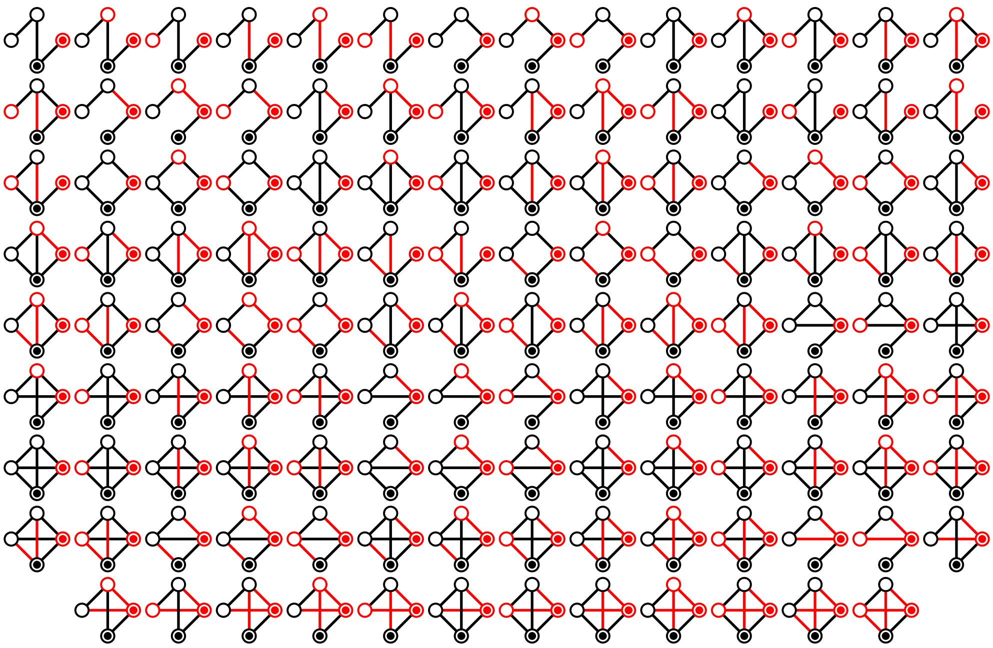

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Abstract

In this thesis we introduce a framework inspired by Razborov's flag algebras which we call *local flags*. This framework allows the use of the semidefinite method for obtaining new bounds on a wide family of "local" combinatorial parameters. We use this framework to make progress towards a conjecture of Erdős and Nešetřil, by an improved asymptotic upper bound on the strong chromatic index of a graph. We also make the first targeted progress on a bipartite version of this conjecture. Additionally, we introduce a bounded degree analogue of the Erdős pentagon conjecture. We state a conjecture and make significant progress toward proving it using our framework. This thesis serves as both an introduction and a handbook on applying the local flags method as we believe it has the potential to be used for many other applications.

Title: Local Flags: Bounding the Strong Chromatic Index

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Contents

Introduction

Consider a simple[1] graph G. How many colours do you need to colour the edges such that no two edges which touch have the same colour? What if no two edges which touch some common edge can have the same colour? Erdős and Nešetřil conjectured that you only need $1.25\Delta(G)^2$ colours where $\Delta(G)$ denotes the maximum degree of G, but the current best proof only shows an upper bound of $1.772\Delta(G)^2$ colours (for large enough $\Delta(G)$). In this thesis we show how we can lower this to $1.73\Delta(G)^2$ by introducing a novel framework which is a modification of Razborov's flag algebras. We also apply this new framework to some other problems including a bounded-degree variant of the Erdős's pentagon conjecture.

Strong Edge Colouring

An edge colouring of a simple graph G is an assignment $c\colon E(G) \to [k]$ for some $k \in \mathbb{N}$. Such a colouring is *proper* if no two incident[2] edges have the same colour. An edge colouring is *strong* if no two edges which share a common incident edge have the same colour. Put differently, proper edge colouring requires edges at distance 1 to have distinct colours and strong edge colouring extends this to distance 2. In figure 1 we see an example of a non-proper edge colouring, a proper (but not strong) edge colouring and a strong edge colouring of C_5, the cycle on 5 vertices.

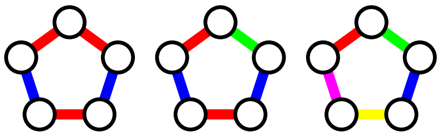


Figure 1.: Non-Proper, Proper & Strong Edge Colourings

The *chromatic index* of G, denoted $\chi'(G)$, is the minimum k such that a proper edge colouring of G with k colours exists. The *strong chromatic index* $\chi'_s(G)$ is the corresponding minimum number of colours required for a strong edge colouring.

Vizing's theorem is a well known result which tells us $\chi'(G)$ almost exactly in terms of the max degree of the graph $\Delta(G)$:

Theorem (Vizing, 1965 [23])**.** $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

[1] An undirected graph with no self loops and at most 1 edge between any two vertices.

[2] Have a vertex in common

Erdős and Nešetřil conjectured in 1985 that the strong chromatic index can also be bounded precisely by a function of the max degree:

Conjecture (Erdős and Nešetřil, see [11])**.** $\chi'_s(G) \leq \frac{5}{4}\Delta(G)^2$.

The simple construction of a C_5, with each vertex substituted with an independent set of size $\Delta(G)/2$, shows that this conjecture would be best possible if true.

A greedy argument shows a bound of $\chi'(G) \leq 2\Delta(G)^2 + o(\Delta(G)^2)$ but it wasn't until 1997 that Molloy and Reed broke the factor 2 barrier [19]. A series of papers have since made progress on lowering this bound closer to $\frac{5}{4}\Delta(G)^2$.

For $\Delta(G)$ large enough we have the following theorems:

1. *Molloy & Reed, 1997* [19]: $\chi'_s(G) \leq 1.998\Delta(G)^2$.
2. *Bruhn & Joos, 2015* [5]: $\chi'_s(G) \leq 1.93\Delta(G)^2$.
3. *Bonamy, Perrett & Postle, 2018* [2]: $\chi'_s(G) \leq 1.835\Delta(G)^2$.
4. *Hurley, de Joannis de Verclos & Kang, 2022* [17]: $\chi'_s(G) \leq 1.772\Delta(G)^2$.

In this thesis we will show how we brought this bound down even further to $1.73\Delta(G)^2$:

Theorem. *For $\Delta(G)$ large enough we have $\chi'_s(G) \leq 1.73\Delta(G)^2$.*

The 1997 paper by Molloy & Reed introduced a method for strong edge colouring we call the *2-step strategy*:

1. Find an upper bound for the *strong neighbourhood density* of G in terms of $\Delta(G)$.
2. Use a probabilistic colouring method which uses the previous bound to achieve a colouring with a low number of colours.

This method has been modified by the papers which followed the Molloy & Reed paper but this strategy has remained the core idea. We will look at this strategy in more detail (including defining strong neighbourhood density) in chapter 4. For this thesis we focus on the Step 1, using the Step 2 as a black box. We will find a new, lower, upper-bound on the strong neighbourhood density and hence achieve our new strong chromatic index bound.

Flag Algebras

Step 1 of the 2-step strategy asks us to find an upper bound on the strong neighbourhood density. The strong neighbourhood density belongs to a broad family of density functions which ask: "How many copies of some structure F do we find in some larger structure G, expressed as a real number $\in [0, 1]$"? These density functions usually count the number of copies of F in G, then normalise by the maximum possible number of such copies. For example, the density of edges in some graph G is $|E(G)|/\binom{|G|}{2}$.

Bounding densities is a common problem in combinatorics and in 2007 Razborov [21] introduced a framework called *flag algebras* which can be used to prove asymptotic results about densities in various combinatorial settings. These flag algebras are defined very generally in terms of finite model theory in [21] but we focus on their use with respect to simple graphs.

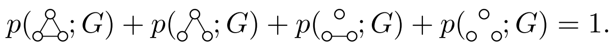
We give a brief flavour of flag algebras here but defer a full exposition until chapter 1.

A motivating example

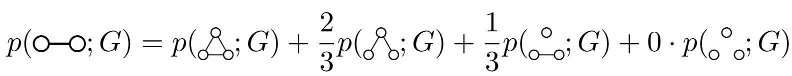
As a reminder, for a graph G and subset of vertices $U \subseteq V(G)$ the *induced subgraph* $G[U]$ is the subgraph of G consisting of the vertices in U and all edges between them. We can then define the *induced count* of F in G, denoted $c(F; G)$, as the number of subsets $U \subseteq V(G)$ such that $G[U] \cong F$. Then we define the *induced density* as $p(F; G) := c(F; G)/\binom{|G|}{|F|}$.

Note. *$p(F; G)$ is precisely the same as the probability that $G[U] \cong F$ if $U \subseteq V(G)$ is a uniformly random subset of size $|F|$. This is often the more useful way to interpret $p(F; G)$.*

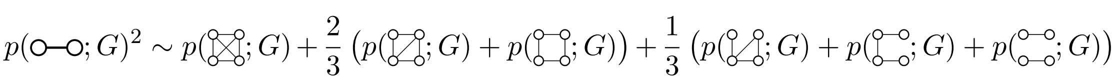
What are some simple algebraic relationships between small subgraphs? Consider picking 2 vertices at random: then either they form an edge or they don't. Hence $p(\circ\!-\!\circ; G) + p(\circ\ \ \circ; G) = 1$. In general, the sum of densities of all flags of some size k is always 1. e.g.



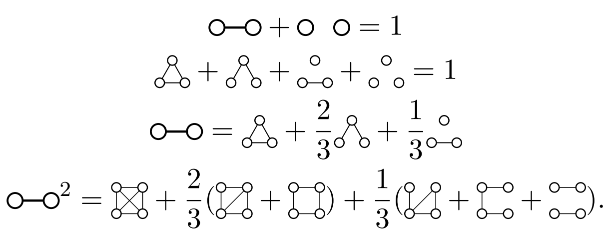
Now we note that we can sample 2 vertices uniformly at random by first sampling a triple of vertices at random, and then sampling 2 of those 3 uniformly randomly again. This lets us derive us the following relation:



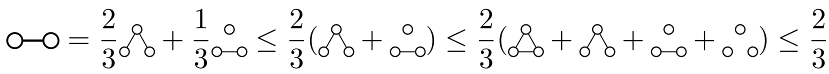
A similar thought experiment relating sampling two pairs uniformly at random to sampling 4 vertices and then splitting the 4 randomly into two halves tells us:



Simplifying our notation then we might arrive at a hypothetical symbolic algebra which has relations like:



We call these graph symbols *flags*. We can then prove results with simple symbolic manipulation: In a triangle-free graph we would have  $= 0$, hence



as flags are non-negative. This (given all the formal definitions and proofs we've deferred) is a formal proof that $p(\circ\!\!-\!\!\circ; G) \leq \frac{2}{3}$ for any triangle-free graph. The best possible result says that $p(\circ\!\!-\!\!\circ; G) \leq \frac{1}{2}$ and is known as *Mantel's theorem* [18]. This is also easily proved with flag algebras which is seen in chapter 1.

Computer Search

One of the most (if not *the most*) important aspects of flag algebras is that they lend themselves very well to computer search methods. The flag algebras allow us to prove results using only simple symbol manipulation in a very tractable way.

In practice we use the *semidefinite method* to optimise some objective function over the algebra, and due to duality this gives us a rigorous proof of an upper bound on our function. We will see in section 1.2 how we construct the semidefinite program, and how we can interpret the dual solutions in a more human understandable way.

Local Flags

One might be tempted to try to apply these flag algebras directly to our strong neighbourhood density problem, but in practice this problem doesn't fit well into the flag algebra model. In particular, Razborov's flag algebras are constructed to work well with density functions like the induced density function $p(F; G) = c(F; G)/\binom{|G|}{|F|}$ which have a denominator which is $\Theta(|G|^{|F|})$. This is convenient if we are trying to prove a bound on some function which is polynomial in $|G|$ (e.g. Mantel's theorem says the number of edges is $\leq \frac{1}{4}|G|^2$). But what if we want to prove a bound on a function which is polynomial in some other function of G? e.g. The Erdős-Nešetřil Conjecture wants to bound $\chi'_s(G)$ with a polynomial in $\Delta(G)$. This does not lend itself to the same methods.

Instead, we can define a new "density" function which instead normalises our induced count by a different denominator: one which captures the graph parameter we want to measure our count "relative to". In particular, in chapter 2 we introduce a new *local density function* and a concept we call a *local flag*. We show that, under certain conditions, these local flags also form a nice algebra with which we can apply the semidefinite method to prove bounds.

Contributions and Results

The work in this thesis was completed with valuable contributions from Rémi de Joannis de Verclos, Eoin Hurley, Jan Volec and my supervisor Ross Kang.

The concept for this new framework was originally explored[3] by Rémi de Joannis de Verclos in 2020, in collaboration with Eoin Hurley and Ross Kang. He conjectured the structure of the framework, then adapted flag algebra software[4] to test if this framework could in principle improve on existing results. This experiment showed that if the framework could be realised formally then it could improve the best known bound on the strong chromatic index.

The conjecture which motivates chapter 3, the proof of lemma 3.7 and the definition of $Q(G, v)$ in section 3.7 were conceived in discussion with Eoin Hurley.

Using our new framework which is introduced in chapter 2 we have made progress on several open problems:

- In chapter 3 we show a "warmup" application of the new framework: We state a new conjecture which is a bounded-degree version of the famous Erdős pentagon conjecture [10]. We then show how a relatively straightforward application of the local flags method makes non-trivial progress towards proving the conjecture, and show that a slightly more complex application then gets even closer to the full result. This problem came about naturally as the original pentagon conjecture was originally resolved using the classic flag algebras ([16], [15]).
- In chapter 4 we apply this new framework to make progress on the Erdős-Nešetřil Conjecture, achieving the best-yet bound of $\chi'_s(G) \lesssim 1.73\Delta(G)^2$. The approach used here is a modification of the approach conjectured by de Joannis de Verclos.
- At the end of chapter 4 we alter the method to make the first targeted progress on the special bipartite version of this conjecture, showing that if G is bipartite then we have the bound $\chi'_s(G) \lesssim 1.6254\Delta(G)^2$. We also investigate the asymmetric version of this bipartite case and make an interesting discovery where the chromatic bound is constant across all degrees of asymmetry.

At the end of the document you will find 3 appendices which might be of interest.

- Appendix A lists some notation as a reference.
- Appendix B gives some practical details on how the SDP software was written, and where to find it.
- Appendix C gives some light details on how the solutions to SDP problems were, or could be, verified.

[3] In unpublished notes.
[4] https://crates.io/crates/flag-algebra

1. Background: Classic Flag Algebras

In section 1.1 we will briefly introduce Razborov's flag algebras as they apply to graphs. If the reader is already familiar with flag algebras this section can be skipped. For more detailed introductions to flag algebras we recommend reading *Flag Algebras: A First Glance* by Silva, Filho and Sato [22] and *A Brief Introduction to Flag Algebra* by Qi [20]. We include this section for completeness.

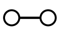
In section 1.2 we will discuss how the semidefinite method is applied to flag algebras. The method we use is slightly different to the method used in other works (e.g. [22]) but is more easily adapted to our new framework. This section may be of interest even to a reader familiar with flag algebras.

Finally in section 1.3 we will introduce *coloured graphs* as combinatorial objects with their own structure. These will help us unlock the full potential of our local flags in our applications (e.g. chapters 3, 4).

1.1. Flag Algebras

All the following definitions, theorems etc. are concepts from [21], rephrased to focus only on the simple graph case. We saw in the motivating example above that we want to construct a symbolic algebra out of small graphs with some nice properties:

- We would like the algebraic operations to be easily computable[1].
- We want the algebra to describe true relationships about densities in large graphs. i.e. If $A + B \geq C^2 \cdot D$ is a true statement in our algebra then that must imply that $p(A;G) + p(B;G) \geq p(C;G)^2 \cdot p(D;G)$ for any G large enough.

When we defined the induced count $c(F;G)$ we counted all possible instances of F in G; Sometimes we want to only consider a subset of those instances, those where we force some fixed vertices in F to be mapped to some fixed vertices in G. (e.g. if we count the copies of  in G where we force the first vertex to be mapped to some fixed $v \in V(G)$ then $c(F;G) = \deg v$). This motivates the last desirable properties of our algebra.

- We would like our symbolic algebra to be able to capture densities where we have fixed some vertices.
- We want to be able to relate these "fixed" densities to the un-fixed densities.

[1] At least with some computer software assistance.

Razborov's flags algebras give us all these nice properties. This section covers the basic formalisms; Introducing a density function which accounts for pre-fixing some vertices, defining what it means for something to hold "for large graphs", defining the actual flag algebras with their properties, introducing an ordering $\geq$ and finally covering the averaging operator which relates "fixed vertex" densities to their un-fixed counterparts.

1.1.1. Flags

The fundamental object of our algebra is the *flag* which is a partially labelled graph, meaning some of the vertices of the graph have integer labels assigned to them[2]. There may be no labelled vertices.

The *type* of the flag is the subgraph induced by the labelled vertices. In figure 1.1 we see some example flags and their types.

Note 1.1.1. *Visually we represent labelled vertices with a partially filled vertex. Additionally we almost always drop the specific integer labels from the diagram as the specific values usually do not matter.*

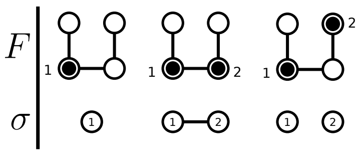


Figure 1.1.: Example flags and their types

We give the formal definitions:

Definition 1.1 (Type)**.** *A* ***type*** σ *of size* k *is a graph with vertex set* $[k]$*. We write* $|\sigma|$ *to denote the size of the underlying graph. We write* $\emptyset$ *to denote the type consisting of the empty graph.*

Definition 1.2 (σ-Embedding)**.** *Given a type* σ *and a graph* F*, a* ***σ-embedding*** *is an injective function* $\theta\colon [|\sigma|] \to V(F)$ *which is a graph isomorphism between* σ *and* $F[\operatorname{im}\theta]$*.*

Definition 1.3 (σ-Flag)**.** *A* ***σ-flag*** *is a tuple* (F, θ) *where* θ *is an* σ*-embedding into* F*. If the embedding is implicit (e.g. if* $\sigma = \emptyset$*) we often drop the* θ *from the notation.*

Note 1.1.2. *A type* σ *is implicitly itself a* σ*-flag when taken with the identity embedding* $\mathrm{id}\colon [|\sigma|] \to [|\sigma|]$*. We often use this fact implicitly.*

We then say that two flags are isomorphic if there is a graph isomorphism between the underlying graphs which preserves the labels.

Definition 1.4 (Flag Isomorphism)**.** $f\colon V(F) \to V(F')$ *is a* σ*-flag isomorphism from* $(F, \theta) \to (F', \theta')$ *if it is graph isomorphism* $F \to F'$ *such that* $f(\theta(i)) = \theta'(i)\ \forall\ i \in [|\sigma|]$*. We can write* $(F, \theta) \cong (F', \theta')$ *if such an* f *exists.*

[2] The term flag apparently refers to the fact that some of the graph is fixed and the rest hangs loose, like a flag in the wind.

1.1.2. Induced Counts and Density

In the introduction we defined the induced count $c(F;G)$ to be the number of $U \subseteq V(F)$ such that $G[U] \cong F$. If both F and G are partially labelled and have the same type (so are "compatible") then we can ask how many copies of F are there in G where the fixed part of F maps to the fixed part in G.

Definition 1.5 (Induced Count)**.** *Fix two σ-flags $(F,\theta),(G,\eta)$. We define the induced count of (F,θ) in (G,η) written $c((F,\theta);(G,\eta))$ as the number of subsets $\mathrm{im}(\eta) \subseteq U \subseteq V(G)$ such that $(F,\theta) \cong (G[U],\eta)$.*[3]

We can extend this notion of counting how many copies of F there are in G to multiple flags, asking how many tuples of disjoint copies of $F_1,\dots,F_t$ are there in G (where disjoint means vertex-disjoint apart from the fixed labelled vertices).

Precisely we define $c(F_1,\dots,F_t;G)$ to be the number of $U_1,\dots,U_t \subseteq V(G)$ such that $U_i \cap U_j = \mathrm{im}\,\eta\ \forall\ i,j \in [t]$ where $i \neq j$ and $(F_i,\theta_i) \cong (G[U_i],\eta)$ for all $i \in [t]$.

Note that if $c(F_1,\dots,F_t;G) > 0$ we know that G must be large enough to fit disjoint copies of $F_1,\dots,F_t$ leading us to the following definition.

Definition 1.6 (Fit)**.** *We say σ-flags $F_1,\dots,F_t$ **fit** in σ-flag G if*

$$|G| - |\sigma| \geq \sum_{i=1}^{t} |F_i| - |\sigma|.$$

Example. *Consider the type $\sigma = \circ$, a single vertex. Any σ-flag (G,η) is just a graph with a specific distinguished vertex (the unique labelled vertex). Consider then the σ-flag $F = \odot\!\!-\!\!\circ$, an edge with a single labelled vertex. For any G with distinguished vertex v we have $c(\odot\!\!-\!\!\circ;G^v) = \deg_G(v)$.*[4] *Similarly, $c(\odot\!\!-\!\!\circ,\odot\!\!-\!\!\circ;G^v) = \deg(v)(\deg(v)-1)$. This is as we are counting how many distinct ordered pairs of vertices (a,b) there are in G such that $G[\{a,v\}] \cong \odot\!\!-\!\!\circ$ and $G[\{b,v\}] \cong \odot\!\!-\!\!\circ$. In other words, how many distinct pairs of vertices are there connected to v, which is clearly $\deg(v)(\deg(v)-1)$.*

Now we can define the induced density by normalising the induced count by the max possible number of such non-overlapping t-tuples.

Definition 1.7 (Induced Density)**.** *Given σ-flags $F_1,\dots,F_t$ and G define the **induced density** of $F_1,\dots,F_t$ in G as:*

$$p(F_1,\dots,F_t;G) := \frac{c(F_1,\dots,F_t;G)}{\binom{|G|-|\sigma|}{|F_1|-|\sigma|,\dots,|F_t|-|\sigma|,R}}$$

where we use multinomial coefficient notation with $R = (|G|-|\sigma|) - \sum_{i=1}^{t} |F_i| - |\sigma|$.

Note 1.1.3. *Again, as in the motivating example in the introduction, we can interpret $p(F_1,\dots,F_t;G)$ in a precise probabilistic way. $p(F_1,\dots,F_t;G)$ is exactly the probability that $(F_i,\theta_i) \cong (G[U_i],\eta)\ \forall\ i \in [t]$ where $U_1,\dots,U_t \subseteq V(G)$ is a uniformly random t-tuple of subsets such that $U_i \cap U_j = \mathrm{im}\,\eta\ \forall i,j \in [t], i \neq j$.*

[3] Where we restrict the codomain of η appropriately.

[4] G^v is the $\circ$-flag (G,θ) where $\theta(1) = v$.

1.1.3. Graph Classes

Often, we want to limit our view to a subset of all possible graphs. For example, we will want to consider only triangle-free graphs when proving Mantel's theorem.

We pick some class of graphs $\mathcal{G}$. For Razborov's flag algebras we assume that $\mathcal{G}$ is *hereditary*, meaning it is closed under taking induced subgraphs. This will change when we introduce our new local framework. Then for any type σ we write $\mathcal{G}^\sigma$ for the set of all σ-flags up to isomorphism. We write $\mathcal{G}^\sigma_n$ for the set of all σ-flags of size n. We generally assume $\mathcal{G}^\sigma$ is infinite for any type $\sigma \in \mathcal{G}$. If $\sigma = \emptyset$ we often skip the superscript and just refer to $\mathcal{G}$ or $\mathcal{G}_n$.

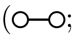
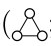
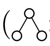
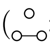
From this point all definitions and results are relative to some fixed graph class $\mathcal{G}$. Given some graph class $\mathcal{G}^\sigma$ we get the very powerful *chain rule*.

Lemma 1.1 (The Chain Rule, Lemma 2.2 [21])**.** *If $F_1, \dots, F_t \in \mathcal{G}^\sigma$ are σ-flags which fit in $G \in \mathcal{G}^\sigma$ then for all $1 \le s \le t$ and every n such that $F_1, \dots, F_s$ fit into a σ flag of size n and a σ-flag and $F_{s+1}, \dots, F_t$ fit in G we have:*

$$p(F_1, \dots, F_t; G) = \sum_{F \in \mathcal{G}^\sigma_n} p(F_1, \dots, F_s; F) p(F, F_{s+1}, \dots, F_t; G).$$

This chain rule is one of the crucial properties that we will lose when we define our local flags framework.

Example. *The chain rule tells us if $\mathcal{G}$ is all graphs then for any G where $|G| \ge 3$ we have*

$p($$; G) = p($$; G) + \frac{2}{3} p($$; G) + \frac{1}{3} p($$; G).$

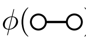
1.1.4. Limit Functionals

Consider a sequence of σ-flags $(G_k)_{k \in \mathbb{N}}$ which is *increasing*, meaning the sequence $(|G_k|)_{k \in \mathbb{N}}$ is strictly increasing. Then for $F \in \mathcal{G}^\sigma$ consider the function $\lim_{k \to \infty} p(F; G_k)$. We call a sequence of σ-flags $(G_k)_{k \in \mathbb{N}}$ **convergent** if this limit exists for all $F \in \mathcal{G}^\sigma$. It is a consequence of Tychonoff's theorem that any increasing sequence of graphs contains a convergent subsequence (as $[0,1]^{\mathcal{G}^\sigma}$ is compact). For this reason it suffices to only consider convergent sequences. Now we can introduce the limit functional which precisely links our symbolic algebra to it's underlying interpretation in terms of density functions.

Definition 1.8 (Limit Functional)**.** *Given some convergent σ-flag sequence we construct the corresponding* ***limit functional*** *$\phi \colon \mathcal{G}^\sigma \to [0,1]$ as*

$$\phi(F) := \lim_{k \to \infty} p(F; G_k)$$

We write Φ^σ for the collection of all limit functionals for type σ.

Example. *If we prove that $\phi($$) \le \frac{1}{2}$ for all $\phi \in \Phi^\emptyset$ where $\mathcal{G}$ is the class of triangle-free graphs, then we will have proved Mantel's theorem.*[5]

[5] We will have proved the bound asymptotically but this actually suffices.

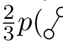
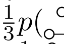
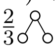
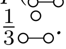
1.1.5. The Algebra

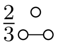
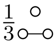
We would like to formally describe a symbolic structure on these flags such that relations in the structure describe true relations about their associated density functions. As a start we want to be able to describe linear combinations of flags. Take then the formal real vector space $\mathbb{R}\mathcal{G}^\sigma$ for some fixed type σ, this gives us a proper notion of these linear combinations of flags. We can then linearly extend our density function p to this space in the first argument giving us a function $p\colon \mathbb{R}\mathcal{G}^\sigma \times \mathcal{G}^\sigma \to \mathbb{R}$ capturing that $p(\circ\!\!-\!\!\circ + \circ\ \ \circ; G) = p(\circ\!\!-\!\!\circ; G) + p(\circ\ \ \circ; G) = 1$ and similar relations.

The chain rule (lemma 1.1) tells us then that for any $F, G \in \mathcal{G}^\sigma$ and $n \geq |F|$ we have $p(F; G) = \sum_{H \in \mathcal{G}_n^\sigma} p(F; H)p(H; G)$. In In particular $p(F; H)$ is just some real, computable number for each H meaning $p(F; G) - \sum_{H \in \mathcal{G}_n^\sigma} p(F; H)p(H; G) = 0$ is a linear relation on density functions which holds for all F, G. This tells us that in our algebra any vector of the form $v = F - \sum_{H \in \mathcal{G}_n^\sigma} p(F; H)H$ has the property that $p(v; G) = 0$ for all $G \in \mathcal{G}^\sigma$.

We can define the space $\mathcal{K}^\sigma$ as the span of vectors of the form $F - \sum_{H \in \mathcal{G}_n^\sigma} p(F; H)H$ for $F \in \mathcal{G}^\sigma$, $n \geq |F|$ and quotient out this relation from $\mathbb{R}\mathcal{G}^\sigma$. Because $p(v; G) = 0$ for all $v \in \mathcal{K}^\sigma, G \in \mathcal{G}$ our linear extension of p is still well defined on this space.

Not only is p well defined on this space, we can now also linearly extend any limit functional ϕ to be a function $\phi\colon \mathbb{R}\mathcal{G}^\sigma/\mathcal{K}^\sigma \to \mathbb{R}$.

Example. *If $\mathcal{G}$ is the class of triangle-free graphs then we know from our motivating example that $p(\circ\!\!-\!\!\circ; G) = \frac{2}{3}p($$; G) + \frac{1}{3}p($$; G)$ for all $G \in \mathcal{G}$. We would like this to translate to a relation like $\circ\!\!-\!\!\circ = \frac{2}{3}$ $+ \frac{1}{3}$.*

In the space $\mathbb{R}\mathcal{G}^\emptyset/\mathcal{K}^\emptyset$ both the vectors $\circ\!\!-\!\!\circ$ and $\frac{2}{3}$ $+ \frac{1}{3}$ belong to the same coset, so they are indeed equal. Hence this space does capture this relationship.

However, linear combinations are not powerful enough. We also want to be able to make statements about products of densities. Ideally we would be able to define a product on vectors $f, g \in \mathbb{R}\mathcal{G}^\sigma/\mathcal{K}^\sigma$ such that for any $G \in \mathcal{G}^\sigma$ we have $p(f \cdot g; G) = p(f; G) \cdot p(g; G)$. Unfortunately we don't achieve the ideal relation, but we do get the result asymptotically: $\phi(f \cdot g) = \phi(f) \cdot \phi(g)\ \forall\ \phi \in \Phi^\sigma$.

Definition 1.9 (σ Flag Algebra)**.** *For fixed type σ define the following product $\mathcal{G}^\sigma \times \mathcal{G}^\sigma \to \mathbb{R}\mathcal{G}^\sigma/\mathcal{K}^\sigma$ on σ-flags $F, G \in \mathcal{G}^\sigma$.*

$$F \cdot G := \sum_{H \in \mathcal{G}_\ell^\sigma} p(F, G; H) \cdot H$$

for any $\ell \geq |F| + |G| - |\sigma|$. Extend this product then bilinearly to the space $\mathbb{R}\mathcal{G}^\sigma \times \mathbb{R}\mathcal{G}^\sigma$. This then induces a bilinear map $\mathbb{R}\mathcal{G}^\sigma/\mathcal{K}^\sigma \times \mathbb{R}\mathcal{G}^\sigma/\mathcal{K}^\sigma \to \mathbb{R}\mathcal{G}^\sigma/\mathcal{K}^\sigma$.

*This is all well defined due to lemma 2.4 in [21]. This turns the space $\mathbb{R}\mathcal{G}^\sigma/\mathcal{K}^\sigma$ into an algebra. We call this the σ **flag algebra** $\mathcal{A}^\sigma$.*

This algebra is then associative, commutative and unital (also lemma 2.4 in [21]). Again this product can be computed with a small number of $p(F, G; H)$ calculations where F, G, H are fixed meaning this is computable.

Example. *Let $\sigma =$. Then $(\ ) \cdot (\ )$ is computed by choosing ℓ large enough so that and fit. We can pick $\ell = 3$. Then $\mathcal{G}_3^\sigma = \{\ \}$ and we can compute $p(\ ,\ ;G)$ for each $G \in \mathcal{G}_3^\sigma$ and find*

$$(\ ) \cdot (\ ) = 0 \cdot \ + 0 \cdot \ + \frac{1}{2} \cdot \ + 0 \cdot \ + \frac{1}{2} \cdot \ + 0 \cdot $$
$$= \frac{1}{2} \cdot \ + \frac{1}{2} \cdot \ .$$

(This isn't technically precise as we should be taking the coset).

Now we see the theorem which tells us why this product is useful:

Theorem 1.2 (Theorem 2 [22])**.** *For fixed type σ, vectors $f, g \in \mathcal{A}^\sigma$ we have*

$$p(f;G) \cdot p(g;G) = p(f \cdot g;G) + O\left(\frac{1}{|G|}\right)$$

where $G \in \mathcal{G}^\sigma$.

What this theorem tells us is that our symbolic product corresponds to a valid product of the underlying density functions in the limit, which we can use to prove asymptotic results. We make this precise in the next section.

Example. *Returning to our example with $\sigma =$ and $F =$ we can compute that (choosing representative of cosets) we have $\ ^2 = \ + \ $. Let G be a -flag with labelled vertex v. Then we know $c(\ ;G^v) = \deg v$ from before. Then we can see that $c(\ ;G^v) + c(\ ;G^v)$ counts all possible ways of choosing pairs from the neighbourhood of v hence $c\left(\ ;G^v\right) + c\left(\ ;G^v\right) = \binom{\deg v}{2}$. Therefore:*

$$\begin{aligned} p(\ ;G^v)^2 - p(\ ^2;G^v) &= \left(\frac{c(\ ;G^v)}{|G|-1}\right)^2 - p(\ ;G^v) - p(\ ;G^v) \\ &= \left(\frac{\deg v}{|G|-1}\right)^2 - \frac{\binom{\deg(v)}{2}}{\binom{|G|-1}{2}} \\ &= \frac{\deg v}{(|G|-1)^2(|G|-2)} + \frac{\deg v}{(|G|-1)(|G|-2)} \\ &= O\left(\frac{1}{|G|}\right). \end{aligned}$$

as $\deg(v) \in O(|G|)$.

Now theorem 1.2 proves that our algebra does exactly capture the algebraic behaviour of densities in the limit.

Lemma 1.3. *Any limit functional $\phi\colon \mathcal{A}^\sigma \to \mathbb{R}$ is an algebra homomorphism. Meaning for any $f, g \in \mathcal{A}^\sigma$ we have $\phi(f+g) = \phi(f) + \phi(g)$ and $\phi(f \cdot g) = \phi(f) \cdot \phi(g)$.*

1.1.6. Positivity

We want to introduce a (pre)order on our algebra so that statements like $\leq$ $+$ $+$ $+$ have meaning.

Definition 1.10 (Positive)**.** *We say that $f \in \mathcal{A}^\sigma$ is* ***positive*** *if $\phi(f) \geq 0\ \forall\ \phi \in \Phi^\sigma$.*
We write this as $f \geq_{\mathcal{A}^\sigma} 0$ or most often just $f \geq 0$.

Example. *All flags are positive elements of the algebra as densities are non-negative. Additionally any squared vector f^2 is positive as $\phi(f^2) = \phi(f)^2 \geq 0$. The zero vector is also positive.*

Lemma 1.4. *The set of positive elements of $\mathcal{A}^\sigma$ forms a convex cone, meaning it is closed under non-negative linear combination. We call this the* ***semantic cone*** *$\mathcal{C}^\sigma_{\text{sem}}$.*

We then extend this notation writing $f \geq g$ iff $f - g \geq 0$ meaning $f \geq g$ iff $\phi(f) \geq \phi(g)$ for all $\phi \in \Phi^\sigma$. [6]

Note 1.1.4. *Sometimes we will write something like $\leq \frac{1}{2}$ which is technically not a valid statement. You can make this precise by replacing $\frac{1}{2}$ by $\frac{1}{2}\emptyset$ as the empty graph always has $\phi(\emptyset) = 1$. This gives $\leq \frac{1}{2}$ the intended meaning that $\phi(\ ) \leq \frac{1}{2}\ \forall\ \phi \in \Phi^\sigma$.*

Finally we show how the *averaging operator* connects $\mathcal{A}^\sigma$ and $\mathcal{A}^\emptyset$.

1.1.7. Averaging Operator

We introduce some notation for the unlabelled version of a partially labelled flag.

Definition 1.11 (Downward operator)**.** *For a σ-flag (F, θ) define $\downarrow F$ as the $\emptyset$-flag obtained by forgetting the partial labelling.*

We then define a computable normalising factor $q_\sigma(F)$:

Definition 1.12. *For a σ-flag F define $q_\sigma(F)$ to be the probability that a uniformly random injective $\theta\colon [|\sigma|] \to V(F)$ is such that $(\downarrow F, \theta) \cong F$.*

See figure 1.2 for example flags F and their $q_\sigma(F)$ normalising factors.

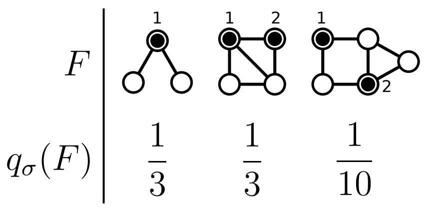


Figure 1.2.: Flags and their normalising factors

Now we can define the averaging operator:

[6]This is a preorder as $f \geq g$ and $g \geq f$ does not imply that $f = g$.

Definition 1.13 (Averaging Operator)**.** *We define the map* $[\![\cdot]\!]\colon \mathcal{A}^\sigma \to \mathcal{A}^\emptyset$ *as follows. For a fixed* σ*-flag* F *define* $[\![F]\!] := q_\sigma(F){\downarrow}F$*. We then extend this function linearly to the space* $\mathcal{A}^\sigma = \mathbb{R}\mathcal{G}^\sigma/\mathcal{K}^\sigma$*. This is well defined by theorem 2.5 in [21].*

Note 1.1.5. *This map* $[\![\cdot]\!]$ *is a linear map, but not an algebra homomorphism. i.e. We do not have* $[\![f \cdot g]\!] = [\![f]\!][\![g]\!]$ *in general.*

The following result is the key to linking $\mathcal{A}^\sigma \to \mathcal{A}^\emptyset$:

Lemma 1.5 (Lemma 4 [22])**.** *Let* F *be a* σ*-flag and* G *a graph with* $p(\sigma; G) > 0$*. Then let* θ *be a uniformly random* σ*-embedding into* G*, then:*

$$\mathbb{E}_\theta[p(F;(G,\theta)] = \frac{p([\![F]\!];G)}{p([\![\sigma]\!];G)} = \frac{q_\sigma(F)p({\downarrow}F;G)}{q_\sigma(\sigma)p(\sigma;G)}$$

In particular by linearity of expectation this can be extended $\mathcal{A}^\sigma$*.*

Note 1.1.6. *In Razborov's paper he describes (definition 10) a measure theoretic way of constructing a random distribution of limit functionals* ψ *from some fixed limit functional* ϕ *in such a way that* $\mathbb{E}_\psi[\psi(F)] = \frac{\phi([\![f]\!])}{\phi([\![\sigma]\!])}$*. We do not use this construction in this thesis.*

An important corollary of lemma 1.5 is the following:

Lemma 1.6. *The averaging operator* $[\![\cdot]\!]$ *maps* $\mathcal{C}^\sigma_{\text{sem}}$ *to* $\mathcal{C}^\emptyset_{\text{sem}}$*, meaning it preserves positivity. In particular,* $[\![f^2]\!] \geq 0 \ \forall \ f \in \mathcal{A}^\sigma$*.*

This gives us yet another important tool to prove results using this algebra. In particular we are now able to prove Mantel's theorem.

Example (Proof of Mantel's theorem)**.** *Let* $\mathcal{G}$ *be the class of triangle-free graphs. Mantel's theorem says that* $\phi(\circ\!\!-\!\!\circ) \leq \frac{1}{2} \ \forall \ \phi \in \Phi^\emptyset$*. We prove this by showing* $\circ\!\!-\!\!\circ \leq \frac{1}{2}$ *meaning* $\frac{1}{2}\emptyset - \circ\!\!-\!\!\circ \geq 0$*. Choosing representatives of the cosets of* $\emptyset$ *and* $\circ\!\!-\!\!\circ$ *this is the same as showing* $\frac{1}{2}(\overset{\circ}{_{\circ}\wedge_{\circ}} + \overset{\circ}{\circ\!-\!\circ} + \overset{\circ}{\circ\;\circ}) - (\frac{2}{3}\overset{\circ}{_{\circ}\wedge_{\circ}} + \frac{1}{3}\overset{\circ}{\circ\!-\!\circ}) \geq 0$*. Simplifying coefficients this means we need to show* $-\frac{1}{6}\overset{\circ}{_{\circ}\wedge_{\circ}} + \frac{1}{6}\overset{\circ}{\circ\!-\!\circ} + \frac{1}{2}\overset{\circ}{\circ\;\circ} \geq 0$*.*

Consider then the type $\sigma = \circ$*. We must have* $(\odot\!\!-\!\!\circ - \odot\ \circ)^2 \in \mathcal{C}^\sigma_{\text{sem}}$ *so we have* $[\![(\odot\!\!-\!\!\circ - \odot\ \circ)^2]\!] \in \mathcal{C}^\emptyset_{\text{sem}}$*. We can compute that* $[\![(\odot\!\!-\!\!\circ - \odot\ \circ)^2]\!] = -\frac{1}{3}\overset{\circ}{_{\circ}\wedge_{\circ}} - \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} + \overset{\circ}{\circ\;\circ}$*.*

Therefore as $\mathcal{C}^\emptyset_{\text{sem}}$ *is a convex cone we have* $\frac{1}{2}(-\frac{1}{3}\overset{\circ}{_{\circ}\wedge_{\circ}} - \frac{1}{3}\overset{\circ}{\circ\!-\!\circ} + \overset{\circ}{\circ\;\circ}) + \frac{1}{3}\overset{\circ}{_{\circ}\wedge_{\circ}} \geq 0$ *which does simplify to exactly* $-\frac{1}{6}\overset{\circ}{_{\circ}\wedge_{\circ}} + \frac{1}{6}\overset{\circ}{\circ\!-\!\circ} + \frac{1}{2}\overset{\circ}{\circ\;\circ}$*. In one line this is:*

$$\phi(\circ\!\!-\!\!\circ) \leq \phi\left(\circ\!\!-\!\!\circ + [\![(\odot\!\!-\!\!\circ - \odot\ \circ)^2]\!] + \frac{1}{3}\overset{\circ}{\circ\!-\!\circ}\right) = \phi\left(\frac{1}{2}(\overset{\circ}{_{\circ}\wedge_{\circ}} + \overset{\circ}{\circ\!-\!\circ} + \overset{\circ}{\circ\;\circ})\right) = \phi\left(\frac{1}{2}\emptyset\right) = \frac{1}{2}.$$

1.2. The Semidefinite Method

In our proof of Mantel's theorem we took some fixed known elements of the semantic cone and took a conic combination of those elements to show $\frac{1}{2}\emptyset - \circ\!\!-\!\!\circ \geq 0$. This is a powerful method to prove bounds on densities, taking some fixed "axiomatic" positive vectors and combining them to get new positive vectors. In general to prove $\phi(f) \leq \lambda \ \forall \ \phi \in \Phi^\emptyset$ we need to show $\lambda\emptyset - f \geq_{\mathcal{A}^\emptyset} 0$. It also suffices to find $\lambda\emptyset - (f + r) \geq 0$ for some $r \geq 0$.

Given some fixed $f \in \mathcal{A}^\sigma$ then we want to solve $\min\{\lambda\colon \lambda\emptyset - f \in \mathcal{C}^\emptyset_{\text{sem}}\}$.

1.2.1. Linear Programming

One direct way to approach this is to take some fixed set of known positive elements $v_1, \ldots, v_k \in \mathcal{A}^{\emptyset}$ and try to minimise our objective over non-negative linear combinations of these vectors. This corresponds to a method called *linear programming (LP).*

Fix our *objective vector* $f \in \mathcal{A}^{\sigma}$. Take some finite list of flags $F_1, \ldots, F_\ell$ such that $v_1, \ldots, v_k$ and f can be expressed in the basis $\mathcal{B} = (F_1, \ldots, F_l, \emptyset)$. This gives us a notion of a coefficient function coef_i for each $i \in [\ell]$ and $\mathrm{coef}_\emptyset$. WLOG f has no $\emptyset$ coefficient.

If $c = [c_1, \ldots, c_k]^T \in \mathbb{R}^k$ is elementwise non-negative then $\sum_{i=1}^k c_i v_i$ is $\in \mathcal{C}^{\emptyset}_{\text{sem}}$. We want this to equal some $\lambda\emptyset - (f + r)$ where $r \in \mathcal{C}^{\emptyset}_{\text{sem}}$. Each element of the basis is in the cone so it suffices to find $c_1, \ldots, c_k$ such that $\mathrm{coef}_j(\sum_{i=1}^k c_i v_i) \leq -\mathrm{coef}_j(f)\ \forall\ j \in [\ell]$. Then our λ is given by $\mathrm{coef}_\emptyset(\sum_{i=1}^k c_i v_i)$. We can express this as an optimisation problem over real matrices. Construct a matrix A as

$$A := \begin{bmatrix} \mathrm{coef}_1(v_1) & \mathrm{coef}_2(v_1) & \ldots & \mathrm{coef}_\ell(v_1) \\ \mathrm{coef}_1(v_2) & \mathrm{coef}_2(v_2) & \ldots & \mathrm{coef}_\ell(v_2) \\ \vdots & \vdots & & \vdots \\ \mathrm{coef}_1(v_k) & \mathrm{coef}_2(v_k) & \ldots & \mathrm{coef}_\ell(v_k) \end{bmatrix}$$

and let $\beta = [\mathrm{coef}_\emptyset(v_1), \ldots, \mathrm{coef}_\emptyset(v_k)]^T \in \mathbb{R}^k$. WLOG f has no $\emptyset$ component so we write $f = [\mathrm{coef}_1(f), \ldots, \mathrm{coef}_\ell(f)]^T \in \mathbb{R}^\ell$. Then our problem is expressed as:

$$\begin{aligned} \min_{c \in \mathbb{R}^k} \quad & c^T\beta \\ \text{such that} \quad & c^T(-A) \geq f \\ & c \geq 0. \end{aligned}$$

This is the standard form of a linear programming problem. Many fast solvers exist to find good solutions.

Example (Proving Mantel's theorem via LP)**.** *Our objective vector is* $f = \frac{2}{3} \ + \frac{1}{3}$. *We can use two known elements of the semantic cone:* $v_1 = \llbracket(\ -\ )^2\rrbracket = \ - \frac{1}{3} \ - \frac{1}{3}$ *and* $v_2 =$. *We choose* $\mathcal{B} = (\ ,\ , \emptyset)$ *and use the fact that* $\emptyset = \ + \ +$ *to write* $v_1 = \emptyset - \frac{4}{3} \ - \frac{4}{3}$. *Then in vector notation* $f = [\frac{1}{3}, \frac{2}{3}, 0]^T$, $v_1 = [-\frac{4}{3}, -\frac{4}{3}, 1]^T$ *and* $v_2 = [1, 0, 0]^T$. *Then we find*

$$A = \begin{bmatrix} -\frac{4}{3} & -\frac{4}{3} \\ 1 & 0 \end{bmatrix}.$$

Hence we want to find $c = [c_1, c_2] \geq 0$ *such that* $c(-A) \geq [\frac{1}{3}, \frac{2}{3}]^T$ *minimising* $c^T[1, 0]^T$. *It can be found that an optimal solution is* $c = [\frac{1}{2}, \frac{1}{3}]^T$ *giving value* $\frac{1}{2}$. *Hence* $\leq \frac{1}{2}$ *as expected.*

This method works when we have a fixed list of known elements of $\mathcal{C}^{\emptyset}_{\text{sem}}$ but how can we incorporate an entire space of these elements?

1.2.2. Semidefinite Programming

We know that $[\![\nu^2]\!] \in \mathcal{C}^{\emptyset}_{\text{sem}} \ \forall\ \nu \in \mathcal{A}^{\sigma}$ for any type σ. How do we encode this into our optimisation problem? One way to do this would be to add a search variable for $\nu \in \mathcal{A}^{\sigma}$ and consider $[\![\nu^2]\!]$ as one of our known elements of the semantic cone. i.e. take our list of known positive elements $v_1, \ldots, v_k$ then try to minimise $\sum_{i=1}^{k} c_i \operatorname{coef}_{\emptyset}(v_i) + \operatorname{coef}_{\emptyset}([\![\nu^2]\!])$ over $c \in \mathbb{R}^k, \nu \in \mathcal{A}^{\sigma}$ such that $c \geq_{\mathbb{R}^k} 0$ and $\sum_{i=1}^{k} c_i \operatorname{coef}_j(v_i) + \operatorname{coef}_j([\![\nu^2]\!]) \leq -\operatorname{coef}_j(f) \ \forall\ j \in [\ell]$. We can express this in the language of matrices and column vectors.

Assume as before we have a basis $\mathcal{B} = (F_1, \ldots, F_\ell, \emptyset)$ of a subspace of $\mathcal{A}^{\emptyset}$ as above. Then for a type σ consider some basis $\mathcal{B}^{\sigma} = (F^{\sigma}_1, \ldots, F^{\sigma}_m)$ of $\mathcal{A}^{\sigma}$ such that $[\![\nu^2]\!] \in \operatorname{span} \mathcal{B} \ \forall\ \nu \in \operatorname{span} \mathcal{B}^{\sigma}$. It can be shown then that, as $[\![\cdot]\!]$ is a linear function and writing $\nu = \sum_{i=1}^{m} a_i F^{\sigma}_i$ we have the following for all $k \in [\ell]$ and similarly for $\operatorname{coef}_{\emptyset}$.

$$\operatorname{coef}_k([\![\nu^2]\!]) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \operatorname{coef}_k([\![F^{\sigma}_i \cdot F^{\sigma}_j]\!]).$$

meaning we can define matrices C^{σ}_i for each $i \in [\ell]$ and $C^{\sigma}_{\emptyset}$ as as

$$C^{\sigma}_i := \begin{bmatrix} \operatorname{coef}_i([\![(F^{\sigma}_1)^2]\!]) & \operatorname{coef}_i([\![F^{\sigma}_1 F^{\sigma}_2]\!])) & \ldots & \operatorname{coef}_i([\![F^{\sigma}_1 F^{\sigma}_m]\!]) \\ \operatorname{coef}_i([\![F^{\sigma}_2 F^{\sigma}_1]\!]) & \operatorname{coef}_i([\![(F^{\sigma}_2)^2]\!])) & \ldots & \operatorname{coef}_i([\![F^{\sigma}_2 F^{\sigma}_m]\!]) \\ \vdots & \vdots & & \vdots \\ \operatorname{coef}_i([\![F^{\sigma}_m F^{\sigma}_1]\!]) & \operatorname{coef}_i([\![F^{\sigma}_m F^{\sigma}_2]\!])) & \ldots & \operatorname{coef}_i([\![(F^{\sigma}_m)^2]\!]) \end{bmatrix}$$

and find that $\operatorname{coef}_i([\![\nu^2]\!]) = \nu^T C^{\sigma}_i \nu$ where ν is considered a column vector over $\mathcal{B}^{\sigma}$. By cyclicity of the trace we have $\nu^T C^{\sigma}_i \nu = \operatorname{tr}(\nu\nu^T C^{\sigma}_i) = \langle \nu\nu^T, C^{\sigma}_i \rangle$ where $\langle\cdot\rangle$ is the standard inner product on real symmetric matrices[7].

The collection of matrices C^{σ}_i are fixed by choice of $\mathcal{B}^{\sigma}$ so we now know how to compute the coefficients of $[\![\nu^2]\!]$ over $\mathcal{B}$ as the inner product of $\nu\nu^T$ with some C^{σ}_i. The power of this is now we can do better than just searching over a single vector ν.

Note 1.2.1. *As a reminder, a symmetric matrix $P \in \mathbb{R}^{m \times m}$ is positive semidefinite (PSD) iff $v^T P v \geq 0$ for all $v \in \mathbb{R}^m$. In particular $\nu\nu^T$ is PSD and the collection of PSD matrices is a convex cone. We use the notation $P \succ 0$.*

If we search instead over the space of positive semidefinite (PSD) matrices then it is a well known result (the spectral theorem) that any such P has spectral decomposition $P = \sum_{i=1}^{m} \nu_i \nu_i^T$ where $\nu_1, \ldots, \nu_m$ are orthogonal eigenvectors[8] of P. Then

$$\langle P, C^{\sigma}_i \rangle = \sum_{j=1}^{m} \langle \nu_j \nu_j^T, C^{\sigma}_i \rangle = \sum_{j=1}^{m} \operatorname{coef}_i([\![\nu_j^2]\!]).$$

Hence searching over PSD matrices allows us to search over several orthogonal vectors ν_i at the same time. This gives us a new optimisation problem: Given our objective vector

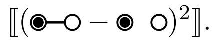
[7] $\langle A, B \rangle = \operatorname{tr}(A^T, B)$. This is also the elementwise dot-product.

[8] Not necessarily normalised.

$f \in \mathcal{A}^{\emptyset}$, known elements of the semidefinite cone $v_1, \dots, v_k$, basis $\mathcal{B} = (F_1, \dots, F_\ell, \emptyset)$ such that $v_1, \dots, v_k, f \in \operatorname{span}\mathcal{B}$, type σ and basis $\mathcal{B}^\sigma = (F_1^\sigma, \dots, F_m^\sigma)$ such that $[\![\nu^2]\!] \in \operatorname{span}\mathcal{B}\ \forall\ \nu \in \operatorname{span}\mathcal{B}$ we precompute the matrices C_i^σ for $i \in [\ell]$ and $C_\emptyset$ and then try to solve:

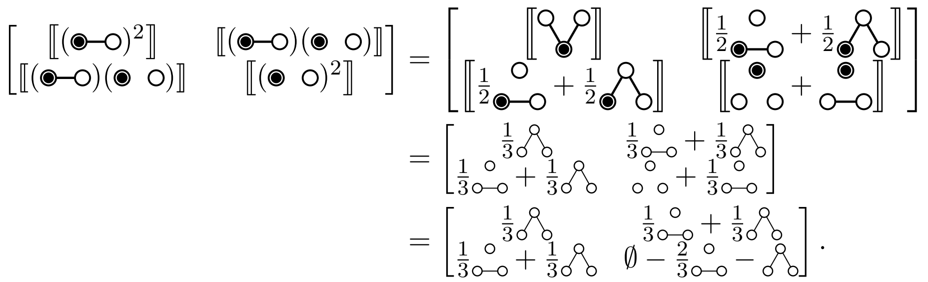
$$\begin{aligned} \min_{c \in \mathbb{R}^k, P \in \mathbb{R}^{m\times m}} \quad & \sum_{i=1}^k c_i \operatorname{coef}_\emptyset(v_i) + \langle P, C_\emptyset^\sigma \rangle \\ \text{such that} \quad & \sum_{i=1}^k c_i \operatorname{coef}_j(v_i) + \langle P, C_j^\sigma \rangle = -f_j \ \forall\ j \in [\ell] \\ & c_i \geq 0 \ \forall\ i \in [k] \\ & P \succ 0. \end{aligned}$$

This is a semidefinite program. It can easily be converted to a more standard form. It can also easily be extended to more than a single type σ.

Now we can see how we could have used this approach to prove Mantel's theorem, showing how optimising over P allows us to discover the vector .

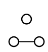
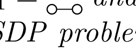
Example (Proving Mantel's theorem via SDP)**.** *Let $\sigma =$*  *and take the bases*

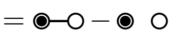
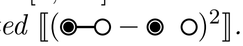
Then we can compute the matrix of all $[\![F \cdot G]\!]$ for $F, G \in \mathcal{B}^\sigma$:



giving us the coefficient matrices

$$C_1^\sigma = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \ C_1^\sigma = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 \end{bmatrix} \ C_\emptyset^\sigma = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We bring in then a single known element of the semantic cone $v_1 =$  *and remembering that our objective vector is $f =$*  *we compose the SDP problem as outlined above and can use a solver to output $c_1 = \frac{1}{3}$ and $P = \frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right) \succ 0$. Then we can verify that $\lambda = c_1 \operatorname{coef}_\emptyset(v_1) + \langle P, C_\emptyset^\sigma \rangle = \frac{1}{2}$ as expected and we do satisfy the constraints that $c_1 \operatorname{coef}_1(v_1) + \langle P, C_1^\sigma \rangle = \frac{1}{3} = -\operatorname{coef}_1(f)$ and $c_2 \operatorname{coef}_1(v_1) + \langle P, C_2^\sigma \rangle = \frac{2}{3} = -\operatorname{coef}_2(f)$ as required. Therefore $\frac{1}{2}\emptyset - f \in \mathcal{C}_{\text{sem}}^\emptyset \implies \phi(f) \leq \frac{1}{2}\ \forall\ \phi \in \Phi^\emptyset$ as expected.*

This value of P shouldn't be surprising as $P = \frac{1}{2}\nu\nu^T$ where $\nu = [1, -1]^T$  *so the "discovered" element of the semantic cone was the expected* *.*

1.2.3. Duality

Consider again our linear program: "Minimise $\text{coef}_\emptyset(\sum_i c_i v_i)$ over $c \in \mathbb{R}^k_{\geq 0}$ such that $\text{coef}_j(\sum_i c_i v_i) \leq -\text{coef}_j(f) \ \forall\ j \in [\ell]$", which we expressed as a matrix problem: "Minimise $c^T\beta$ over $c \in \mathbb{R}^k$ such that $c^T(-A) \geq f$ and $c \geq 0$".

The powerful thing about linear programming is *duality*; We can convert our minimisation problem above into a dual maximisation problem which asks us to maximise $f^T x$ over $x \in \mathbb{R}^\ell$ such that $x \geq 0$ and $(-A)x \leq \beta$. By weak duality of linear programming we know that any solution to this problem is a lower bound to any possible solution to our original (primal) problem.

We can interpret this maximisation problem in a useful way: We can think of this dual problem as being a relaxation of the search over possible limit functionals ϕ. Take the search vector $x \in \mathbb{R}^\ell$ and construct a linear functional ϕ_x on the space $\operatorname{span}\mathcal{B}$ by defining $\phi_x(F_i) := x_i$ and $\phi_x(\emptyset) = 1$, then linearly extending. We can then see that $x \geq 0$ means $\phi(F_i) \geq 0 \ \forall\ i \in [\ell]$ which is a requirement of true limit functionals. Similarly the requirement that $(-A)x \leq \beta$ corresponds to requiring that $\phi(v_i) \geq 0 \ \forall\ i \in [k]$, meaning our known elements of the semantic cone must have positive value as expected. Finally then $\max_{x\in\mathbb{R}^\ell} f^T x$ corresponds to $\max \phi_x(f)$.

Hence the dual problem attempts to maximise some hypothetical $\phi_x(f)$ on the constraints that ϕ_x must behave like a true limit functional. This interpretation makes it clear that our fixed known elements of the semantic cone correspond to constraints on the possible space of linear functionals; Adding new positive elements to the list may exclude previously possible invalid limit functionals and get us closer to a true answer.

We expanded our linear program into a semidefinite program. Luckily for us semidefinite programs also have duality. The dual version of the SDP above asks us to maximise $\sum_{x\in\mathbb{R}^\ell} f^T x$ such that $\sum_{j=1}^\ell x_j \,\text{coef}_j(v_i) + \text{coef}_\emptyset(v_i) \geq 0 \ \forall\ i \in [k]$ and $\sum_{j=1}^\ell x_j C^\sigma_j + C^\sigma_\emptyset \succ 0$.

We can again interpret this dual program in the same way, where we take $x \in \mathbb{R}^\ell$, interpret it as a linear functional ϕ_x on $\mathcal{B}$ with $\phi_x(F_i) := x_i$ and $\phi_x(\emptyset) = 1$. Then the dual problem is to maximise $\phi_x(f)$ such that $\phi(v_i) \geq 0 \ \forall\ i \in [k]$ but now we have the additional constraint that the matrix $Q = \sum_{i=1}^\ell \phi(F_j)C^\sigma_j + C^\sigma_\emptyset$ is positive semidefinite. But this exactly means that $\nu^T Q \nu \geq 0$ for all ν which means $\phi_x([\![\nu^2]\!]) \geq 0 \ \forall \nu$. Hence our extension from linear to semidefinite programming has had the effect on our dual problem of adding the requirement that $\phi_x([\![\nu^2]\!]) \geq 0$ for all $\nu \in \operatorname{span}\mathcal{B}^\sigma$.

This gives us our overall strategy then for using flag algebras and semidefinite programming. If you want to prove an upper bound on some density function, express it as $\phi(f)$ for some vector f, then generate as many linear constraints on the space of linear functionals ϕ_x as possible and finally add the requirement that $\phi_x([\![\nu^2]\!]) \geq 0$ for all ν in some subspace. Then the problem of maximising $\phi_x(f)$ can be converted rigorously to a dual problem of finding convex combinations of elements of the semantic cone, proving true upper bounds on $\phi(f) \forall \phi \in \Phi^\emptyset$.

1.3. Coloured Graphs

We have already discussed graphs with colourings in the introduction. In that context we were considering some underlying graph G and asking questions about what possible colourings G can admit under certain constraints. In this section we are going to take a very different view of graphs and colourings, instead considering graphs with colourings as combinatorial objects in their own right.

Definition 1.14 ((c_v, c_e)-Graph)**.** *Given $c_v, c_e \in \mathbb{N}$ where we define a (c_v, c_e)-**graph** to be a simple undirected graph where each edge and each vertex are assigned colours from $[c_v]$ and $[c_e]$ respectively.*

We often call $(k, 1)$-graphs or $(1, k)$-graphs k-vertex-colour graphs or k-edge-colour graphs respectively.

Note 1.3.1. *It's very important to highlight here that we do **not** require that the colourings are proper: Adjacent vertices and incident edges can have the same colour.*

Definition 1.15. *Two (c_v, c_e)-graphs G, G' are isomorphic if there is a graph isomorphism $f : G \to G'$ which preserves colours.*

Example. *A $(2, 1)$-graph is a graph where each vertex can have 1 of 2 colours, and each edge is fixed to same colour. We usually visually show colour 1 as black and colour 2 as red. See figure 1.3 for all 2-vertex-coloured graphs of size 3.*

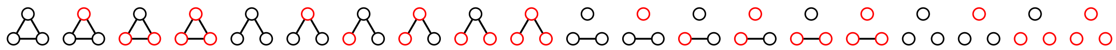


Figure 1.3.: All $(2, 1)$-graphs of size 3 up to isomorphism.

Example. *The category of $(1, 1)$-graphs is equivalent to the category of simple graphs.*

All of what we have seen in the previous sections applies exactly the same to coloured graphs as to graphs. We have types and flags and flag algebras in the exact same way. The induced count function $c(F; G)$ now counts those $U \subseteq V(G)$ such that $G[U] \cong F$ as coloured graphs. See figure 1.4 for an example expression in σ-flags derived from $(2, 2)$-graphs where $\sigma = $ ○, a single red vertex (colour 2).

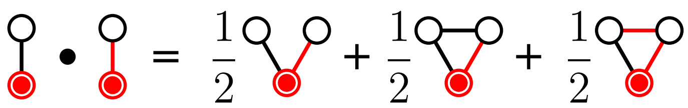
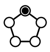


Figure 1.4.: An example expression in an $(2, 2)$-graph flag algebra.

2. Local Flags

Consider the following TODO question from chapter 3: "Given a triangle-free graph G with max degree $\Delta(G)$ and some fixed vertex $v \in V(G)$, how many pentagons (5-cycles) are there containing v"? The upper bound of $\Delta(G)^4$ is easily seen, but we can do better. Phrased differently: If $P(G, v)$ is the number of pentagons in G containing v then we want to find an asymptotic upper bound on $P(G, v)/\Delta(G)^4$. If we pick a flag $F =$  which is a pentagon with a single labelled vertex then this could be written as $c(F; G^v)/\Delta(G)^4$. This looks a lot like a density function, except that our denominator is of the wrong form: It should be $\in \Theta(|G|^4)$, not $\Delta(G)^4$. It is for this reason that applying classic flag algebras directly is not promising: Our function is of the wrong form.

In this chapter we will introduce the *local density* function $\rho(F; G) := c(F; G)/\binom{\Delta(G)}{|F|-|\sigma|}$ and define a product on vectors of flags such that

$$\rho(f; H)\rho(g; H) = \rho(f \cdot g; H) + O\left(\frac{1}{\Delta(G)}\right).$$

We also show that we have an averaging operator which works in the limit:

$$\mathbb{E}_\theta[\rho(F; (G, \theta))] \sim \frac{\rho(\llbracket F \rrbracket; G)}{\rho(\llbracket \sigma \rrbracket; G)}$$

We show then that these constructions suffice to allow us to apply the semidefinite method in order to prove rigorous bounds on $\rho(f; G)$ in the limit as $\Delta(G) \to \infty$.

This method of using local flags is the novel framework at the core of this thesis. In chapter 3 and 4 we will use this framework to prove several interesting results, including making progress on the Erdős and Nešetřil conjecture described in the introduction.

2.1. Local Flags

Let $\mathcal{G}$ be some graph class as before. Let Δ be some graph parameter $\Delta\colon \mathcal{G} \to \mathbb{N}_0$.

Note 2.1.1. *We almost exclusively use the maximum degree function Δ so you can effectively always interpret $\Delta(G)$ in this way. In particular, all of our examples and applications use the max degree function for $\Delta(G)$.*

We start by defining the *local density* in the same way as the induced density (definition 1.7) but with a different denominator. We use the definition of σ-flags (definition 1.3) from the classic flag algebras directly.

Definition 2.1 (Local Density)**.** *Let $(F,\theta),(G,\eta)$ be σ-flags as before. Then $c((F,\theta);(G,\eta))$ is the induced count (definition 1.5). Define the* ***local density*** *$\rho((F,\theta);(G,\eta))$ to be*

$$\rho(F;G) := \frac{c(F;G)}{\binom{\Delta(G)}{|F|-|\sigma|}}.$$

Note the $\rho \neq p$ notation.

Because of our choice of normalisation we are not guaranteed the $[0,1]$ codomain as in the classic case. In particular, this destroys the nice probabilistic sampling interpretation that we had with the induced density function. In general, this function may be bounded or unbounded:

Example. *If $\mathcal{G}$ is all graphs then $c(\circ;G) = |V(G)|$ hence $\rho(\circ;G) = \frac{|G|}{\Delta(G)}$ which is an unbounded function.*

Example. *If $\mathcal{G}$ is all graphs then consider the $\circ$-flag $\bullet\!\!-\!\!\circ$. Then for any G with distinguished vertex v we have $c(\bullet\!\!-\!\!\circ;G^v) = \deg(v)$ so $\rho(\bullet\!\!-\!\!\circ;G^v) = \frac{\deg v}{\Delta(G)} \leq 1$.*

This bounded/unbounded distinction is the key distinction we want to make. Those flags F with bounded behaviour are those with the exploitable algebraic structure. Indeed we will eventually define our *local flags* to be those with bounded local density but we first need to address some technical definitions to avoid pathological cases.

We need a way to technically describe the action of taking a σ-flag with some unlabelled vertices and labelling one of those vertices:

Definition 2.2 (Label Extension)**.** *Given a σ-flag (F,θ) and some $v \in V(F)$ which is unlabelled ($\notin \operatorname{im}\theta$) we can construct a* ***label extension*** *$\theta' : [|\sigma|+1] \to V(F)$ as*

$$\theta'(i) = \begin{cases} \theta(i) & \text{if } i \in [|\sigma|] \\ v & \text{otherwise} \end{cases}$$

This is an embedding of the type σ' obtained by adding new vertex $|\sigma|+1$ to σ such that $\sigma' \cong F[\operatorname{im}\theta \cup \{v\}]$. See figure 2.1 for an example of a flag and all its possible label extensions.

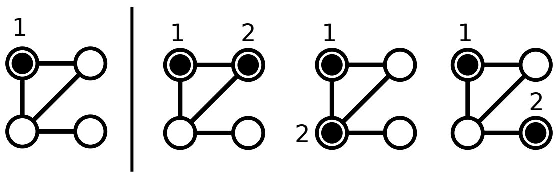


Figure 2.1.: All possible label extensions of a flag

The other technicality we need to address is that of the hereditary nature of graph classes. In the classic flag algebra we always assumed that $\mathcal{G}$ was hereditary. We want to be able to talk about some non-hereditary graph classes $\mathcal{G}$. This is not a major obstacle but means we need to adjust our notation.

Definition 2.3 (Hereditary Closure)**.** *Given a graph class $\mathcal{G}$ we define the* ***hereditary closure*** *of $\mathcal{G}$ to be the smallest graph class which contains $\mathcal{G}$ and is closed under taking induced subgraphs. i.e. it's the graphs $G \in \mathcal{G}$ along with their induced subgraphs. We denote this as $\overline{\mathcal{G}}$.*

Now we are ready to define our local flags:

Definition 2.4 (Local σ-Flag)**.** *Fix some graph class $\mathcal{G}$ and takes its hereditary closure $\overline{\mathcal{G}}$. Let σ be a type. Then a σ-flag $(F, \theta) \in \overline{\mathcal{G}}^\sigma$ is a* ***local σ-flag*** *if we have the following properties:*

1. *$(G, \eta) \to \rho((F, \theta); (G, \eta))$ is a bounded function as a function $\mathcal{G}^\sigma \to \mathbb{R}_{\geq 0}$. (We are very intentionally using $\mathcal{G}$ and not its closure here).*

2. *If we label any of F's unlabelled vertices we get another local flag.*

To state this 2nd property more precisely: We require that for any label extension θ' of θ, the induced extended flag (F, θ') is also a local flag. This is not a circular definition as any label extension of (F, θ) reduces the number of unlabelled vertices by 1; We could define inductively starting with those flags with no unlabelled vertices.

What we're trying to capture here is that any "subflag" of F is also a local flag, meaning we can pin down F's vertices and continue to get bounded behaviour.

Example. *As in the example above $F =$ ◉–○ gives rise to a bounded local density function ρ. The only label extension of F is the edge with both vertices labelled $F' =$ ◉–◉. This (as with all flags with no unlabelled vertices) has $c(F'; G) = 1$ so $\rho(F; G) = 1/\binom{\Delta(G)}{0} = 1$ which is bounded so F' is a local flag. Hence $F =$ ◉–○ is a local flag.*

We write $\mathcal{G}^\sigma_{\text{loc},n} \subseteq \overline{\mathcal{G}}^\sigma_n$ for the set of local σ-flags of size n up to isomorphism, and $\mathcal{G}^\sigma_{\text{loc}}$ for all local σ-flags. As usual we can drop the σ superscript if $\sigma = \emptyset$.

Note 2.1.2. *For σ-flag F the requirement that $\rho(F; G)$ is a bounded function is equivalent to requiring that $c(F; G) \in O(\Delta(G)^{|F|-|\sigma|})$.*

Comparing this section to the classic flag algebra case (section 1.1) you might expect us to introduce something akin to the chain rule (lemma 1.1). Unfortunately, no such relation exists in general for local flags. This is the first big loss when moving to the new framework: Prima facie, it's unclear how one can "project" small flags into a space of larger flags.

Note 2.1.3. *This second property we require of local flags is not necessarily implied by the first, which we show now in lemma 2.1.*

Lemma 2.1. *There is a class of graphs $\mathcal{G}$ and σ-flag F such that $G \mapsto \rho(F; G)$ is a bounded function $\mathcal{G}^\sigma \to \mathbb{R}$ but F is not a local σ-flag. i.e. F has a labelled extension with unbounded density function.*

Proof. Take $\mathcal{G}$ to be the class of 3-vertex-coloured graphs (black, red and blue) G which have a single red vertex and $\Delta(G)^2$ blue vertices[1] such that there are no edges between red and blue vertices.

Then take $\emptyset$-flag $F = \textcolor{red}{\circ}\ \textcolor{blue}{\circ}$. For any $G \in \mathcal{G}$ we have $c(F;G) = \Delta(G)^2$ as there is 1 choice for the red vertex and $\Delta(G)^2$ choices for the blue. Hence $\rho(F;G) = c(F;G)/\binom{\Delta(G)}{2} \leq 2$. Consider then the labelled extension $F' = \textcolor{red}{\odot}\ \textcolor{blue}{\circ}$ of type $\textcolor{red}{\circ}$. We have $c(F';G) = \Delta(G)^2$ again for any $G \in \mathcal{G}^{\textcolor{red}{\circ}}$ but now $\rho(F';G) = \Delta(G)^2/\Delta(G) = \Delta(G)$ which is an unbounded function. Hence F' is not a local $\textcolor{red}{\circ}$-flag meaning F is not a local $\emptyset$-flag. $\square$

Intuitively, F is a local σ-flag (relative to our choice of $\mathcal{G}$ and Δ) if $\Delta(G)$ bounds the "degree of freedom" for the possible mapping of F's unlabelled vertices.

Lemma 2.2. *If $\Delta(G)$ is the max degree function then any σ-flag F is a local σ-flag if all connected components of F contains a labelled vertex.*

Proof. We give a sketch of the proof: The full details of the proof can be found in the proof of lemma 3.10.

Consider bounding the number of isomorphisms $F \to G[U]$. The labelled vertices are fixed. The vertices directly connected to labelled vertices have $\Delta(G)$ choices. After mapping those we again have $\Delta(G)$ choices for their unmapped neighbours etc. leading to a total bound of $O(\Delta(G)^{|F|-|\sigma|})$ choices. $\square$

2.2. Algebraic Structure

Now that we have described local σ-flags $\mathcal{G}^{\sigma}_{\text{loc}}$ we can describe their algebraic structure. Ideally, we would like to construct a product structure on $\mathcal{G}^{\sigma}_{\text{loc}}$ such that we get a result like theorem 1.2 but for the local density function ρ. In fact, this is exactly what we get: An algebraic structure such that $\rho(f;G)\rho(g;G) = \rho(f \cdot g;G) + o(1)$ (theorem 2.3).

First, we take the concept of limit functionals (section 1.1.4) from the classic flag algebras with some minor modifications: We require now that a sequence of graphs $(G_k)_{k\in\mathbb{N}}$ is Δ-increasing:

Definition 2.5. *A sequence $(G_k)_{k\in\mathbb{N}}$ is Δ-increasing if the sequence $(\Delta(G))_{k\in\mathbb{N}}$ is strictly increasing*[2].

Then given some Δ-increasing sequence of σ-flags $(G_k)_{k\in\mathbb{N}}$ and some local σ-flag F we can look at $\lim_{k\to\infty} \rho(F;G_k)$. As F is a local σ-flag $\rho(F;\cdot)$ is bounded so the image is compact. For this reason (again via Tychonoff's theorem) any sequence of σ-flags $(G_k)_{k\in\mathbb{N}}$ contains a convergent subsequence meaning $\lim_{k\to\infty}\rho(F;G_k)$ exists for all $F \in \mathcal{G}^{\sigma}_{\text{loc}}$. Hence we can define a limit functional $\phi\colon \mathcal{G}^{\sigma}_{\text{loc}} \to \mathbb{R}$ from such a convergent sequence $(G_k)_{k\in\mathbb{N}}$ as $\phi(F) := \lim_{k\to\infty}\rho(F;G_k)$.

[1] Implicitly the rest of the vertices are black

[2] We required that the codomain of Δ was $\mathbb{N}$ so strictly increasing implies unbounded. Generalising Δ to a function with codomain $\mathbb{R}$ is likely possible so we would need to add the unbounded requirement here.

As with the classic case we can take the space of formal linear combinations of local flags $\mathbb{R}\mathcal{G}^\sigma_{\text{loc}}$ and linearly extend ρ and ϕ to these spaces. As before we denote the set of all limit functionals on type σ by Φ^σ.

Note 2.2.1. *Unlike in the classic case we will not be quotienting the space $\mathbb{R}\mathcal{G}^\sigma_{\text{loc}}$ by a subspace. This is as we do not have an equivalent relation to the chain rule. The side effect of this is that there may be vectors $f \in \mathbb{R}\mathcal{G}^\sigma_{\text{loc}}$ which are formally non-zero but have $\phi(f) = 0 \ \forall \ \phi \in \Phi^\sigma$. This doesn't affect the correctness of our arguments, but does mean the semidefinite programs will be larger and therefore may take longer to solve.*

We now define the product on $\mathbb{R}\mathcal{G}^\sigma_{\text{loc}}$ which will turn it into an algebra.

Definition 2.6 (Local Flag Product)**.** *Let $F, F' \in \mathcal{G}^\sigma_{\text{loc}}$ be given. Let $n = |F| + |F'| - |\sigma|$, the minimum size of a flag which can fit F and F'. Then we define:*

$$F \cdot F' := \sum_{H \in \mathcal{G}^\sigma_{\text{loc},n}} p(F, F'; H) \cdot H$$

Note: This is the induced density function p, not the local density function ρ. Extend this product bilinearly to the space $\mathbb{R}\mathcal{G}^\sigma_{\text{loc}}$ to create an algebra $\mathcal{L}^\sigma$.

Note 2.2.2. *It will be useful in later chapters to refer to the subspace of $\mathcal{L}^\sigma$ spanned by flags of a fixed size n. Call this subspace $\mathcal{L}^\sigma_n$.*

This product has the exact limiting behaviour that we need and some nice algebraic properties:

Theorem 2.3. *For $f, g \in \mathcal{L}^\sigma$ and σ-flag $G \in \mathcal{G}^\sigma$ we have*

$$\rho(f; G)\rho(g; G) = \rho(f \cdot g; G) + O\left(\frac{1}{\Delta(G)}\right)$$

and in particular any limit functional $\phi \in \Phi^\sigma$ is an algebra homomorphism $\mathcal{L}^\sigma \to \mathbb{R}$.

Lemma 2.4. *The algebra $\mathcal{L}^\sigma$ is commutative, associative and unital.*

We will prove both of these results after first proving a key technical result which makes this product work:

Theorem 2.5. *Let $F, F' \in \mathcal{G}^\sigma_{\text{loc}}$ be local σ-flags and H a σ-flag $\in \overline{\mathcal{G}}^\sigma$ of size $n = |F| + |F'| - |\sigma|$ such that $p(F, F'; H) > 0$, then H is a local σ-flag.*

Proof. Let θ, θ' be the σ embeddings for F, F' and η the σ-embedding for H. To prove H is a local flag we first need to show that $\rho(H; \cdot)$ is a bounded function. i.e. $(G \mapsto c(H; G)) \in O(\Delta(G)^{|H|-|\sigma|})$.

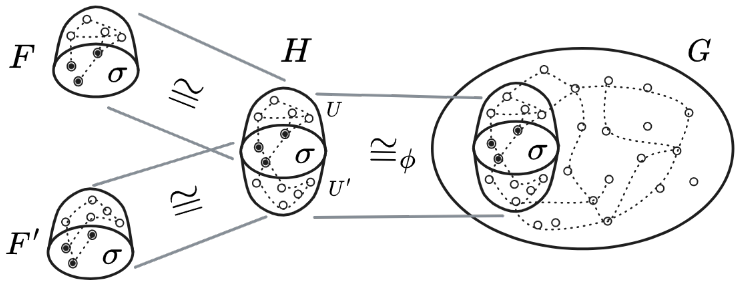
As $p(F, F'; H) > 0$ there is some $U, U' \subseteq V(H)$ such that $U \cap U' = \operatorname{im} \eta$ and $F \cong H[U]$ and $F' \cong H[U']$ as σ flags. As $|H| = |F| + |F'| - |\sigma$ we have $U \cup U' = V(H)$.

Let (G, ζ) be another σ-flag. If $c(H; G) = 0$ we're done so assume otherwise and let $\operatorname{im} \zeta \subseteq V \subseteq V(G)$ be such that $H \cong G[V]$ as σ-flags. Let ϕ be this isomorphism. Then

In particular ϕ induces an embedding of U, U' into $V(G)$ such that $\operatorname{im}\zeta = \phi(U) \cap \phi(U')$ and $G[\phi(U)] \cong F$ and $G[\phi(U')] \cong F'$ as σ-flags. Hence any choice of an instance of H in G gives rise to a pair of instances of F and F' in G.

Note then also that any choice of a pair of instances of F, F' in G can be derived from at most 1 instance of H, as the size of H was chosen to be the minimum possible such that both F, F' fit. If the two instances overlap (outside of the required intersection at $\operatorname{im}\zeta$) then they don't correspond to an instance of H. If they don't overlap then their union corresponds to a single *possible* instance of H.

In summary, each instance of H gives rise to at least one pair of instances of F, F' in G, and each pair of instances of F, F' is induced by at most 1 instance of H. Therefore $c(H; G) \leq c(F; G) \cdot c(F'; G)$. We use then the fact that F and F' are local σ-flags so $c(F; \cdot)$ and $c(F'; \cdot)$ are $\in O(\Delta(G)^{|F|-|\sigma|})$ and $\in O(\Delta(G)^{|F'|-|\sigma|})$ respectively. Hence their product is $\in O(\Delta(G)^{|F|+|F'|-2|\sigma|}) = O(\Delta(G)^{|H|-|\sigma|})$ so $c(H; \cdot) \in O(\Delta(G)^{|H|-|\sigma|})$ showing $\rho(H; \cdot)$ is a bounded function as required.



It remains to prove that H still has bounded density after fixing any unlabelled vertices. This argument goes almost identically so we just give the high level details. Once again pick $U, U' \subseteq V(H)$ where $U \cap U' = \operatorname{im}\eta$ and $F \cong H[U]$ and $F' \cong H[U']$. Pick some unlabelled vertex $v \in V(H) \setminus \operatorname{im}\eta$. Let H^v be the labelled extension fixing v. Then any copy of H^v in G induces via U, U' copies of F, F' in G. WLOG $v \in U \setminus \operatorname{im}\eta$ so each copy of F induced by U is also a copy of F^v, the labelled extension of f fixing the preimage of v. Then as F is local F^v has bounded corresponding density function. Once again we get $c(H^v; G) \leq c(F^v; G) \cdot c(F'; G) \in O(\Delta(G)^{|F|-(|\sigma|+1)+|F'|-|\sigma|}) = O(\Delta(G)^{|H|-(|\sigma|+1)})$ as required. $\therefore H$ is a local flag. $\square$

The value of this theorem is that it tells us that our local product definition is equivalent to summing over all σ-flags, not just the local flags:

Corollary 2.5.1. *For local flags $F, F' \in \mathcal{G}^\sigma_{\text{loc}}$ we have*

$$F \cdot_{\mathcal{L}^\sigma} F' = \sum_{H \in \mathcal{G}^\sigma_{\text{loc},n}} p(F, F'; H) \cdot H = \sum_{H \in \overline{\mathcal{G}}^\sigma_n} p(F, F'; H) \cdot H$$

where $n = |F| + |F'| - |\sigma|$.

This allows us to prove our product limiting result (Theorem 2.3).

Proof of theorem 2.3. Let $(F,\theta),(F',\theta')\in\mathcal{G}^\sigma_{\mathrm{loc}}$ be local σ-flags and $(G,\eta)\in\mathcal{G}^\sigma$ a σ-flag. We compute

$$\begin{aligned}\rho(F;G)\cdot\rho(F';G)-\rho(F\cdot F';G)&=\rho(F;G)\cdot\rho(F';G)-\left(\sum_{H\in\mathcal{G}^\sigma_{\mathrm{loc},n}}p(F,F';H)\rho(H;G)\right)\\&=\rho(F;G)\cdot\rho(F';G)-\left(\sum_{H\in\overline{\mathcal{G}}^\sigma_n}p(F,F';H)\rho(H;G)\right)\\&=\frac{c(F;G)\cdot c(F';G)}{\binom{\Delta(G)}{|F|-k}\binom{\Delta(G)}{|F'|-k}}-\frac{\sum_{H\in\overline{\mathcal{G}}^\sigma_n}c(F,F';H)c(H;G)}{\binom{n-k}{|F|-k}\binom{\Delta(G)}{n-k}}\end{aligned}$$

where $k=|\sigma|$ and $n=|F|+|F'|-k$. First we note the denominators are asymptotically equivalent. It is a standard binomial identity that $\binom{\Delta(G)}{n-k}\binom{n-k}{|F|-k}=\binom{\Delta(G)}{|F|-k}\binom{\Delta(G)-(|F|-k)}{|F'|-k}$ hence

$$\lim_{\Delta(G)\to\infty}\frac{\binom{n-k}{|F|-k}\binom{\Delta(G)}{n-k}}{\binom{\Delta(G)}{|F|-k}\binom{\Delta(G)}{|F'|-k}}=\lim_{\Delta(G)\to\infty}\frac{\binom{\Delta(G)}{|F'|-k}}{\binom{\Delta(G)-(|F|-k)}{|F'|-k}}=1.$$

Both denominators are asymptotically equivalent and $\in\Omega(\Delta(G)^{n-k})$ so we can focus on the numerators:

$$c(F;G)c(F';G)-\sum_{H\in\overline{\mathcal{G}}^\sigma_n}c(F,F';H)c(H;G).\qquad(\dagger)$$

It suffices to show $(\dagger)\in O(\Delta(G)^{n-k-1})$.

The term $c(F;G)c(F';G)$ counts the number of pairs of subsets $\operatorname{im}\eta\subseteq U,U'\subseteq V(G)$ such that $F\cong G[U]$ and $F'\cong G[U']$. Comparatively the sum $\sum_{H\in\overline{\mathcal{G}}^\sigma_n}c(F,F';H)c(H;G)$ counts the number of subsets $\operatorname{im}\eta\subseteq U,U'\subseteq V(G)$ such that $F\cong G[U]$, $F'\cong G[U']$ *and* $U\cap U'=\operatorname{im}\eta$. This relies on corollary 2.5.1 which let us sum over all possible flags of size n.

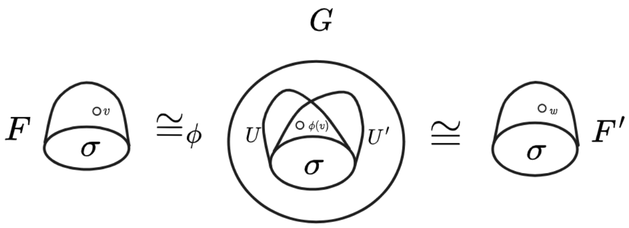
Clearly then $(\dagger)$ counts the number of pairs of subsets $\operatorname{im}\eta\subseteq U,U'\subseteq V(G)$ such that $F\cong G[U],F'\cong G[U']$ and $U\cap U'\neq\operatorname{im}\eta$ meaning there is an overlap of the image of the unlabelled vertices of F,F'.

To see intuitively that $(\dagger)\in O(\Delta(G)^{n-k-1})$ remember that F,F' being local flags means $\Delta(G)$ bounds the degree of freedom of choosing the image of each of their unlabelled vertices. Then $\Delta(G)^{n-k}$ represents the degree of freedom of choosing pairs of embeddings of F,F' with no constraints; Adding the constraint that the embeddings must overlap means reducing the degree of freedom by at least one, giving $\Delta(G)^{n-k-1}$. We show the argument in detail here:

We calculate an upper bound on $(\dagger)$ by summing for each U embedding F into G and over each unlabelled $v\in V(F)$ the maximum number of U' embeddings of F' which overlap on the image of v.

Fix $\operatorname{im}\eta\subseteq U\subseteq V(G)$ such that $F\cong G[U]$ and $v\in V(F)\setminus\operatorname{im}\eta$. Call the isomorphism ϕ. We ask then how many $\operatorname{im}\eta\subseteq U'\subseteq V(G)$ are there such that $\phi(v)\in U'$ and

$F' \cong G[U']$? We can upper bound this by summing over each unlabelled $w \in V(F')$ and asking how many U' embeddings are there where the isomorphism maps w to $\phi(v)$. This is exactly what is answered by taking a labelled extension of F' labelling w and extending the label of G to map w to $\phi(v)$. We are guaranteed that F' is a local flag so this labelled extension also has bounded density. In particular $c((F')^w; G^{\phi(v)}) \in O(\Delta(G)^{|F'|-(k+1)})$.



We have a constant number of choices of w and v and $c(F;G) \in O(\Delta(G)^{|F|-k})$ choices for U giving us a total upper bound of $O(\Delta(G)^{|F|-k+|F'|-(k+1)}) = O(\Delta(G)^{n-k-1})$ as required.

$$\therefore \rho(F;G)\rho(F';G) - \rho(F \cdot F';G) = \frac{O(\Delta(G)^{n-k-1})}{\Omega(\Delta(G)^{n-k})} = O\left(\frac{1}{\Delta(G)}\right).$$

This then extends linearly to $\mathcal{L}^\sigma = \mathbb{R}\mathcal{G}^\sigma_{\text{loc}}$ as the vector space only contains finite combinations. This has the immediate corollary of showing that any limit functional $\phi \in \Phi^\sigma$ is an algebra homomorphism $\mathcal{L}^\sigma \to \mathbb{R}$. □

Finally we can prove lemma 2.4, showing $\mathcal{L}^\sigma$ to be associative, commutative and unital.

Proof of Lemma 2.4. It is clear by definition that the product is commutative.

To show associativity let F_1, F_2, F_3 be local σ-flags. Then let $k = |\sigma|$, $\ell = |F_1|+|F_2|-k$ and $n = |F_1| + |F_2| + |F_3| - 2k$. Then we can calculate:

$$\begin{aligned}
(F_1 \cdot F_2) \cdot F_3 &= \left(\sum_{H \in \mathcal{G}^\sigma_{\text{loc},\ell}} p(F_1, F_2; H)H\right) \cdot F_3 \\
&= \left(\sum_{H \in \overline{\mathcal{G}}\ell^\sigma} p(F_1, F_2; H)H\right) \cdot F_3 \\
&= \sum_{H \in \overline{\mathcal{G}}^\sigma_\ell} p(F_1, F_2; H)(H \cdot F_3) \\
&= \sum_{H \in \overline{\mathcal{G}}^\sigma_\ell} p(F_1, F_2; H) \left(\sum_{G \in \mathcal{G}^\sigma_{\text{loc},n}} p(H, F_3; G)G\right) \\
&= \sum_{G \in \mathcal{G}^\sigma_{\text{loc},n}} \left(\sum_{H \in \overline{\mathcal{G}}^\sigma_\ell} p(F_1, F_2; H)p(H, F_3; G)\right) G
\end{aligned}$$

Then as we used corollary 2.5.1 to get a sum over all flags, not just local ones, we can apply the chain rule for induced densities (lemma 1.1) to get:

$$(F_1 \cdot F_2) \cdot F_3 = \sum_{G \in \mathcal{G}^{\sigma}_{\text{loc},n}} p(F_1, F_2, F_3; G) \cdot G$$

This is symmetric in all 3 terms so clearly $(F_1 \cdot F_2) \cdot F_3 = F_1 \cdot (F_2 \cdot F_3)$ so $\mathcal{L}^{\sigma}$ is associative.

Finally take the type σ and view it as a σ-flag implicitly. Then $c(\sigma; G) = 1$ for any σ-flag G so $\rho(\sigma; G) = 1/\binom{\Delta(G)}{0} = 1$ showing σ is a local σ-flag $\implies \sigma \in \mathcal{L}^{\sigma}$. Then it is easy to see that σ is a unit in this algebra as for any $F \in \mathcal{L}^{\sigma}$ we have $F \cdot \sigma = \sum_{H \in \mathcal{G}^{\sigma}_{\text{loc},|F|}} p(F, \sigma; H) H = F$ as the only flag of size $|F|$ which contains a copy of F is F itself. □

2.3. Positivity

We previously adopted the idea of limit functionals almost directly from the classic flag algebra, we now do the same with positivity.

Definition 2.7 (Positive Element)**.** *We say $f \in \mathcal{L}^{\sigma}$ is a positive element, denoted $f \geq_{\mathcal{L}^{\sigma}} 0$ or just $f \geq 0$, if $\phi(f) \geq 0 \ \forall \phi \in \Phi^{\sigma}$.*

We then denote the convex cone of all positive elements of $\mathcal{L}^{\sigma}$ by $\mathcal{C}^{\sigma}_{\text{sem}}$. We again call this the (local) semantic cone.

2.4. Averaging Local Flags

So far we have made restrictions no restrictions on what types σ we consider, but it will prove useful to look at only a select set of types we call *local types*.

Definition 2.8 (Local Type)**.** *We say that a type (definition 1.1) is a* ***local type*** *if $\downarrow F$ is a local $\emptyset$-flag for all $F \in \mathcal{G}^{\sigma}_{\text{loc}}$.*

We also give a useful equivalent condition:

Lemma 2.6. *A type σ is a local type iff σ is itself a local $\emptyset$-flag.*

Proof. One direction is easy to see. If σ is a local type then as $\sigma \in \mathcal{G}^{\sigma}_{\text{loc}}$ we have $\downarrow\sigma \in \mathcal{G}^{\emptyset}_{\text{loc}}$ but $\downarrow\sigma = \sigma$ so σ is a local $\emptyset$-flag.

To show the other direction assume σ is itself a local $\emptyset$-flag meaning $c(\sigma; G) \in O(\Delta(G)^{|\sigma|})$. Let $(F, \theta) \in \mathcal{G}^{\sigma}_{\text{loc}}$ be given, then we also have $c((F, \theta); (G, \eta)) \in O(\Delta(G)^{|F|-|\sigma|})$. Let C_{σ} be the constant such that $c(\sigma; G) \leq C_{\sigma} \Delta(G)^{|\sigma|} \forall G \in \mathcal{G}$ and C^{σ}_{F} the constant such that $c((F, \theta); (G, \eta)) \leq C^{\sigma}_{F} \Delta(G)^{|F|-|\sigma|} \forall (G, \eta) \in \mathcal{G}^{\sigma}$. For fixed σ and $U \subseteq V(G)$ there is a finite number of possible embeddings η such that (G, η) is a σ-embedding and $\operatorname{im} \eta = U$, namely the size of the automorphism group of σ $|\operatorname{Aut}(\sigma)|$.

We want to show that $\downarrow(F,\theta) = F$ is a local $\emptyset$-flag so first need to show $c(F;G) \in O(\Delta(G)^{|F|})$. Given any embedding (G,η) and $\operatorname{im}\eta \subseteq U \subseteq V(G)$ such that $(F,\theta) \cong (G[U],\eta)$ we must have $F \cong G[U]$ as $\emptyset$-graphs. This gives us a map into copies of F in G which is just forgetting the labels in both (G,η) and (F,θ). Conversely consider some copy of F in G $U \subseteq V(G)$ ($F \cong G[U]$ as $\emptyset$-flags). Then as (F,θ) is a σ-flag there is some $V \subseteq U$ such that $\sigma \cong G[V]$. We can then construct a σ-flag (G,η) where $\operatorname{im}\eta = V$ such that $(F,\theta) \cong (G[U],\eta)$ as σ-flags. This shows us that the previous map must be surjective. Hence

$$
\begin{aligned}
c(F;G) &\leq |\{((G,\eta),U)\colon \operatorname{im}\eta \subseteq U \subseteq V(G) \text{ such that } (F,\theta) \cong (G[U],\eta)\}| \\
&\leq \sum_{(G,\eta)\in\mathcal{G}^\sigma} |\{U\colon \operatorname{im}\eta \subseteq U \subseteq V(G) \text{ such that } (F,\theta) \cong (G[U],\eta)\}| \\
&\leq \sum_{(G,\eta)\in\mathcal{G}^\sigma} c((F,\theta),(G,\eta)) \\
&\leq \sum_{(G,\eta)\in\mathcal{G}^\sigma} C_F^\sigma \Delta(G)^{|F|-|\sigma|} \\
&\leq |\mathrm{Aut}(\sigma)| c(\sigma;G) C_F^\sigma \Delta(G)^{|F|-|\sigma|} \\
&\leq |\mathrm{Aut}(\sigma)| C_\sigma \Delta(G)^{|\sigma|} C_F^\sigma \Delta(G)^{|F|-|\sigma|} \\
&\in O(\Delta(G)^{|F|})
\end{aligned}
$$

as required. It remains to show that any labelled extension of $\downarrow F$ also has bounded density function. $\downarrow F$ is an $\emptyset$-flag so a labelled extension F' of $\downarrow F$ is just F with a single labelled vertex. There are 2 cases but they both proceed similarly. If this labelled vertex is in $\operatorname{im}\theta \subseteq V(F)$ then as σ is a local $\emptyset$-flag by an almost identical argument to the previous we get a bound of $c(F';G^v) \in O(\Delta(G)^{|F|-1})$. The other case is that the newly labelled vertex of $\downarrow F$ is one of the unlabelled vertices in F, in which case we use that F is a local σ-flag to again bound $c(F';G^v) \in O(\Delta(G)^{|F|-1})$ as required. Therefore $\downarrow F$ is a local $\emptyset$-flag. $\square$

We adopt this definition of local type because we would like to introduce an averaging operator $[\![\cdot]\!]\colon \mathcal{L}^\sigma \to \mathcal{L}^\emptyset$ akin to definition 1.13 but we cannot just apply this map directly as we don't generally have a guarantee that $[\![f]\!] \in \mathcal{L}^\emptyset$ even if $f \in \mathcal{L}^\sigma$. By our definition above this map is well defined if σ is local type.

Definition 2.9 (Averaging operator)**.** *As a reminder we have a normalisation function $q_\sigma\colon \mathcal{G}^\sigma \to [0,1]$ given in definition 1.12. Define the **averaging operator** $[\![\cdot]\!]\colon \mathbb{R}\mathcal{G}^\sigma \to \mathbb{R}\mathcal{G}^\emptyset$ as before by defining $[\![F]\!] = q_\sigma(F)\downarrow F$ for $F \in \mathcal{G}^\sigma$ and extend linearly.*

If σ is a local type and we restrict the domain to $\mathbb{R}\mathcal{G}^\sigma_{\mathrm{loc}} = \mathcal{L}^\sigma$ we get a map $[\![\cdot]\!]\colon \mathcal{L}^\sigma \to \mathcal{L}^\emptyset$.

We have a very nice result (lemma 1.5) in the classic case where this operator represents an average in a very concrete way: $\mathbb{E}_\theta[p(F;(G,\theta))] = \frac{p([\![F]\!];G)}{p([\![\sigma]\!];G)}$ We do not get quite such a neat relation for local densities, but we do get the relation in the limit:

Lemma 2.7. *For σ-flag F and graph ($\emptyset$-flag) $G \in \mathcal{G}$ we have*

$$\mathbb{E}_\theta[\rho(F;(G,\theta))] \sim \frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} \text{ as } \Delta(G) \to \infty$$

where θ is a uniformly random σ-embedding into G.

Proof. This proof uses the fact that this relation holds for induced densities (lemma 1.5) and the following relation between induced and local densities: For any σ-flags H and G we have

$$\rho(H;G) = \frac{c(H;G)}{\binom{\Delta(G)}{|H|-|\sigma|}} = \frac{\binom{|G|-|\sigma|}{|H|-|\sigma|}c(H;G)}{\binom{|G|-|\sigma|}{|H|-|\sigma|}\binom{\Delta(G)}{|H|-|\sigma|}} = \frac{\binom{|G|-|\sigma|}{|H|-|\sigma|}}{\binom{\Delta(G)}{|H|-|\sigma|}}\frac{c(H;G)}{\binom{|G|-|\sigma|}{|H|-|\sigma|}} = \frac{\binom{|G|-|\sigma|}{|H|-|\sigma|}}{\binom{\Delta(G)}{|H|-|\sigma|}}p(H;G)$$

Therefore

$$\frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} = \frac{p([\![F]\!];G)}{p([\![\sigma]\!];G)}\frac{\binom{|G|}{|F|}}{\binom{\Delta(G)}{|F|}}\frac{\binom{\Delta(G)}{|\sigma|}}{\binom{|G|}{|\sigma|}}.$$

Now we use lemma 1.5 to show

$$\begin{aligned}\frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} &= \mathbb{E}_\theta[p(F;(G,\theta))]\frac{\binom{|G|}{|F|}}{\binom{\Delta(G)}{|F|}}\frac{\binom{\Delta(G)}{|\sigma|}}{\binom{|G|}{|\sigma|}}\\ &= \mathbb{E}_\theta[\rho(F;(G,\theta))]\frac{\binom{\Delta(G)}{|F|-|\sigma|}}{\binom{|G|-|\sigma|}{|F|-|\sigma|}}\frac{\binom{|G|}{|F|}}{\binom{\Delta(G)}{|F|}}\frac{\binom{\Delta(G)}{|\sigma|}}{\binom{|G|}{|\sigma|}}\\ &= \mathbb{E}_\theta[\rho(F;(G,\theta))]\frac{\binom{\Delta(G)}{|F|-|\sigma|}\binom{\Delta(G)}{|\sigma|}}{\binom{\Delta(G)}{|F|}}\frac{\binom{|G|}{|F|}}{\binom{|G|-|\sigma|}{|F|-|\sigma|}\binom{|G|}{|\sigma|}}\end{aligned}$$

We can use standard binomial relations to reduce this to

$$\frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} = \mathbb{E}_\theta[\rho(F;(G,\theta))]\frac{\binom{\Delta(G)}{|F|-|\sigma|}\binom{\Delta(G)}{|\sigma|}}{\binom{\Delta(G)}{|F|}\binom{|F|}{|\sigma|}} = \mathbb{E}_\theta[\rho(F;(G,\theta))]\frac{\binom{\Delta(G)}{|F|-|\sigma|}}{\binom{\Delta(G)-|\sigma|}{|F|-|\sigma|}}$$

Then $\lim_{\Delta(G)\to\infty}\binom{\Delta(G)}{|F|-|\sigma|}/\binom{\Delta(G)-|\sigma|}{|F|-|\sigma|} = 1$ so we do indeed get

$$\frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} = (1+o(1))\mathbb{E}_\theta[\rho(F;(G,\theta))] \implies \frac{\rho([\![F]\!];G)}{\rho([\![\sigma]\!];G)} \sim \mathbb{E}_\theta[\rho(F;(G,\theta))].$$

□

Now we can show that the averaging operator preserves positivity as in the classic case (lemma 1.6). We only get this behaviour if σ is a local type.

Lemma 2.8. *Let σ be a local type. Then the averaging operator preserves positive elements of $\mathcal{L}^\sigma$: $[\![\mathcal{C}^\sigma_{\text{sem}}]\!] \subseteq \mathcal{C}^\emptyset_{\text{sem}}$. In particular as with the classic case we have $[\![f^2]\!] \geq 0$ for all $f \in \mathcal{L}^\sigma$.*

Proof. As σ is a local type we have that $[\![\mathcal{C}^{\sigma}_{\text{sem}}]\!] \subseteq \mathcal{L}^{\emptyset}$.

Let $f \in \mathcal{C}^{\sigma}_{\text{sem}}$ be given and assume for sake of contradiction that $[\![f]\!] \notin \mathcal{C}^{\emptyset}_{\text{sem}}$. Then there must exist some limit functional $\phi \in \Phi^{\emptyset}$ such that $\phi(f) < 0$. Equivalently there is some Δ-increasing convergent sequence of graphs $(G_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \rho(f; G_k) < 0$.

By lemma 2.7 we know that

$$\rho([\![f]\!]; G_k) = (1 + o(1))\rho([\![\sigma]\!]; G_k)\mathbb{E}_{\theta}[\rho(f; (G_k, \theta))]$$

where $o(1)$ is as $k \to \infty$. In particular this means we must have

$$\lim_{k\to\infty} \rho([\![\sigma]\!]; G_k)\mathbb{E}_{\theta}[\rho(f; (G_k, \theta))] < 0.$$

Then as σ is a local type σ is a local $\emptyset$-flag so has bounded density: $\exists$ a constant C such that $C \geq \rho([\![\sigma]\!]; G_k) \geq 0 \forall k \in \mathbb{N}$. In particular then we must have

$$\lim_{k\to\infty} \mathbb{E}_{\theta}[\rho(f; (G_k, \theta))] < 0.$$

Therefore there is some k_0 large enough such that $\mathbb{E}_{\theta}[\rho(f; (G_k, \theta))] < 0 \ \forall\ k \geq k_0$. In particular then for each $k \geq k_0$ by the properties of the expected value there must exist some σ-embedding θ_k such that $\rho(f; (G_k, \theta_k)) \leq \mathbb{E}_{\theta}[\rho(f; (G_k, \theta))] < 0$. This gives us a Δ-increasing sequence of σ-flags which must contain a convergent subsequence $(G'_k, \theta'_k)_{k\in\mathbb{N}}$ which in turn gives us a limit functional $\phi \in \Phi^{\sigma}$. This limit functional then must have the property that

$$\phi(f) = \lim_{k\to\infty} \rho(f; (G'_k, \theta'_k)) \leq \lim_{k\to\infty} \mathbb{E}_{\theta}[\rho(f; (G'_k, \theta)] < 0$$

but by assumption we have $\phi(f) \geq 0 \ \forall \phi \in \Phi^{\sigma}$ so this is a contradiction. Therefore $[\![f]\!] \in \mathcal{C}^{\emptyset}_{\text{sem}}$. $\square$

2.5. The Semidefinite Method

We can adopt the semidefinite method as outlined in section 1.2 almost identically. Once again we have a concept of a semantic cone and can prove asymptotic bounds of the form $\phi(f) \leq \lambda$ by finding elements of the cone of the form $\lambda\emptyset - f$.

One key difference is that we that we can only use constraints of the form $[\![f^2]\!] \geq 0 \ \forall f \in \mathcal{L}^{\sigma}$ if σ is a local type, so our set of types under consideration is slightly limited.

In practical terms the lack of the chain rule and its corresponding quotient sets means that expressing our combinatorial questions in the form of $\sup_{\phi} \phi(f)$ for $f \in \mathcal{L}^{\sigma}$ is more difficult; We will see in the next chapter a "warmup" application which shows how we can exploit regularity to give us a wide set of constraints and allow us to project small flags into a basis of larger flags.

3. Application: Pentagons in Triangle-Free Graphs

Given a graph G let $P(G)$ denote the number of pentagons (5-cycles)[1] in G. In 1983 Erdős made the following conjecture:

Conjecture (Erdős 1983 [9]). *If G is triangle-free then $P(G) \leq \left(\frac{|G|}{5}\right)^5$.*

This conjecture remained open until 2012 when it was proved by both Grzesik [15] and Hatami, Hladký, Kráľ, Norine and Razborov [16]. Importantly for us, both of these papers used Razborov's flag algebras to prove the result!

Inspired by Erdős's conjecture we ask the following question: Can we bound the number of pentagons in a triangle-free graph as a function of the maximum degree $\Delta(G)$? This leads us to the following *bounded-degree pentagon conjecture*:

Conjecture 3.1. *If G is triangle-free then $P(G) \leq \frac{|G|}{5}\left(\frac{\Delta(G)}{2}\right)^4 = \frac{|G|\Delta(G)^4}{5\cdot 16}$.*

We claim then the following theorem:

Theorem 3.1. *If G is triangle-free then $P(G) \leq 0.02073 \cdot |G|\Delta(G)^4 \approx 1.658\frac{|G|}{5}\frac{\Delta(G)^4}{16}$.*

We proved this result using local flags, but the approach is slightly too complex to serve as the "warmup" application, so we first prove this slightly weaker result:

Theorem 3.2. *If G is triangle-free then $P(G) \leq \frac{|G|}{5}\frac{\Delta(G)^4}{8} = 2 \cdot \frac{|G|}{5}\frac{\Delta(G)^4}{16}$.*

We will focus initially on proving this weaker result, then in section 3.7 we will adapt the method to prove the stronger result.

3.1. Tightness

Lemma 3.3. *If the bounded-degree pentagon conjecture is true then it is tight.*

Proof. Let $k \in \mathbb{N}$ even be given and take G to be the $k/2$-blowup of C_5 (figure 3.1) meaning take 5 independent sets of size $k/2$ as "supernodes" then densely connect the supernodes into a 5-cycle. This graph is triangle-free (clear by case analysis). Then we see that choosing a vertex from each of the 5 supernodes gives a distinct 5-cycle so there are at least $\left(\frac{k}{2}\right)^5$ pentagons in the graph (this is actually exact). But $|G| = \frac{5k}{2}$ and $\Delta(G) = k$ so we can rewrite this as $\left(\frac{k}{2}\right)^5 = \frac{|G|}{5}\left(\frac{\Delta(G)}{2}\right)^4$ meeting the bound. We can do this for any even $k = \Delta(G)$ so this holds asymptotically. $\square$

[1] Counted as subsets: the order does not matter

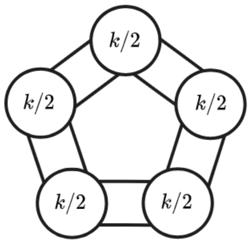


Figure 3.1.: Balanced 5-partite cycle graph on $5k/2$ vertices

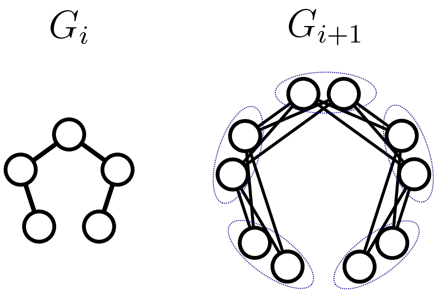
3.2. Reductions

First we argue that an asymptotic bound suffices. Essentially, this is true as any small graph with a high pentagon density can be expanded to arbitrary high degree.

Lemma 3.4. *If $P(G) \lesssim \lambda|G|\Delta(G)^4$ as $\Delta(G) \to \infty$ for G triangle-free then $P(G) \leq \lambda|G|\Delta(G)^4$ for all triangle-free G.*

Proof. More precisely our condition states that for any Δ-increasing sequence of triangle-free graphs $(G_k)_{k\in\mathbb{N}}$ we have $\limsup_{k\to\infty} \frac{P(G_k)}{|G|\Delta(G)^4} \leq \lambda$.

Assume then for sake of contradiction that there exists G_0 triangle-free such that $\frac{P(G_0)}{|G_0|\Delta(G_0)^4} > \lambda$. Let $\delta := \frac{P(G_0)}{|G_0|\Delta(G_0)^4}$ and consider the following sequence of graphs $(G_k)_{k\in\mathbb{N}}$: Start with G_0 and construct G_{i+1} from G_i by doubling every vertex of G_i, then for every $u \sim v$ in G_i connect both copies of u to both copies of v.



Then we have $|G_{i+1}| = 2|G_i|$ and $\Delta(G_i) = 2\Delta(G_{i+1})$ for all $i \in \mathbb{N}$. Note also that G_i triangle-free implies G_{i+1} triangle-free. To see this consider assume G_i triangle-free and G_{i+1} has a triangle $\{u, v, w\}$. As each of u, v, w are connected they must be copies of three different nodes from G_i. However such copies are connected iff their originals are connected so there must be a corresponding triangle in G_i which is a contradiction. Hence each G_i is triangle-free.

Take any pentagon in G_i. This correspond to 2^5 pentagons in G_{i+1}. Each such pentagon in G_{i+1} we get by expanding a C_5 in G_i comes from some unique pentagon in G_i hence $P(G_{i+1}) \geq 2^5 P(G_i)$ for all $i \geq 1$. Hence for every $i \in \mathbb{N}$ we have

$$\frac{P(G_i)}{|G_i|\Delta(G_i^4)} \geq \frac{(2^5)^i P(G_0)}{2^i|G_0|(2^i\Delta(G_0))^4} = \frac{2^{5i}P(G_0)}{2^{5i}|G_0|\Delta(G_0)^4} = \delta$$

and in particular

$$\limsup_{k\to\infty} \frac{P(G_k)}{|G_k|\Delta(G_k)^4} \geq \liminf_{k\to\infty} \frac{P(G_k)}{|G_k|\Delta(G_k)^4} \geq \delta > \lambda$$

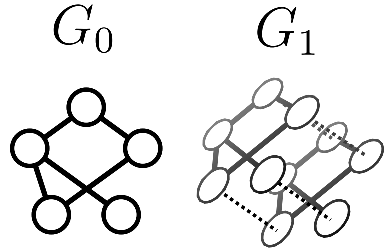
but $(G_k)_{k\in\mathbb{N}}$ is a Δ-increasing sequence of triangle-free graphs so this contradictions our assumption. Hence no such G_0 exists. $\square$

Now we show not only is it enough to show this bound asymptotically, it also suffices to show the bound only for regular graphs.

Lemma 3.5. *For any triangle-free G there exists a regular triangle-free G' such that $\Delta(G') = \Delta(G)$ and*

$$\frac{P(G')}{|G'|\Delta(G')^4} \geq \frac{P(G)}{|G|\Delta(G)^4}$$

Proof. Let G_0 be such a triangle-free graph. Construct a sequence $(G_k)_{k\in\mathbb{N}}$ as follows: To construct G_{i+1} take two copies of G_i then for each vertex $v \in V(G_i)$ with $\deg v < \Delta(G_i)$ add an edge to G_{i+1} between the two copies of v.



We show by induction that each G_i is triangle-free, $\Delta(G_i) = \Delta(G_0)$ $\forall i \in \mathbb{N}$ and $\frac{P(G_i)}{|G_i|\Delta(G_i)^4} \geq \frac{P(G_0)}{|G_0|\Delta(G_0)^4}$. G_0 satisfies these conditions by assumption; Assume they hold for G_i. First we argue that G_{i+1} has no triangles. Clearly each copy of G_i has no triangles so we need to show the addition of edges between two 2 copies of G_i doesn't induce any triangles. Each edge we add is between two copies of a vertex $v \in V(G_i)$. This means each vertex in G_{i+1} has at most 1 neighbour outside of its own copy of G_i. Therefore adding an edge between two copies of some v cannot induce a 3-cycle as the neighbourhoods of each copy of v will be disjoint so G_{i+1} is triangle-free. Next we show $\Delta(G_{i+1}) = \Delta(G_i) = \Delta(G_0)$; This is clear as we add an edge only to vertices v which have $\deg v < \Delta(G_i) = \Delta(G_0)$ meaning we do not increase the maximum degree. Finally we show that $\frac{P(G_{i+1})}{|G_{i+1}|\Delta(G_{i+1})^4} \geq \frac{P(G_0)}{|G_0|\Delta(G_0)^4}$. This is also easy as $|G_{i+1}| = 2|G_i|$ but $P(G_{i+1}) \geq 2P(G_i)$ as we take two copies of G_i.

Finally we note that the minimum degree increases by 1 every iteration if the graph is non-regular: $\delta(G_i) < \Delta(G_i) \implies \delta(G_{i+1}) = \delta(G_i) + 1$. Then as $\Delta(G_i) = \Delta(G_0) \forall i \in \mathbb{N}$ this means there can be at most $\Delta(G_0) - \delta(G_0) \leq \Delta(G_0)$ iterations until we arrive at a regular G_k. We found then a regular graph G_k which satisfies our conditions. $\square$

Corollary 3.5.1. *It suffices to show that $\frac{P(G)}{|G|\Delta(G)^4} \lesssim \lambda$ only for regular triangle-free G.*

Proof. Assume the bound holds for regular graphs. Assume then for contradiction that there is some triangle-free Δ-increasing sequence $(G_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \frac{P(G_k)}{|G_k|\Delta(G_k)^4} > \frac{1}{5\cdot 8}$. Then by the previous lemma we can construct a sequence $(G'_k)_{k\in\mathbb{N}}$ where each G'_k is triangle-free and regular such that $\Delta(G'_k) = \Delta(G_k)$ and $\frac{P(G'_k)}{|G'_k|\Delta(G'_k)^4} > \frac{P(G_k)}{|G_k|\Delta(G_k)^4}$ for all $k \in \mathbb{N}$. Hence this is a Δ-increasing triangle-free regular sequence such that

$$\limsup_{k\to\infty} \frac{P(G'_k)}{|G'_k|\Delta(G'_k)} \geq \lim_{k\to\infty} \frac{P(G_k)}{|G_k|\Delta(G_k)} > \lambda$$

which is a contradiction. $\square$

Now we know we only need to prove the bound asymptotically for regular graphs. The final reduction step is the following simple lemma

Lemma 3.6. *Let $P(G, v)$ be the number of pentagons in G containing v. If we have some $\lambda \in \mathbb{R}$ such that $\frac{P(G,v)}{\Delta(G)^4} \lesssim \lambda$ as $\Delta(G) \to \infty$ then $\frac{P(G)}{|G|\Delta(G)^4} \lesssim \frac{1}{5}\lambda$ as $\Delta(G) \to \infty$.*

Proof. Note $\sum_{v\in V(G)} P(G, v) = 5P(G)$. Hence

$$\begin{aligned}\frac{P(G)}{|G|\Delta(G)^4} = \frac{\sum_{v\in V(G)} P(G,v)}{5|G|\Delta(G)^4} &= \frac{\sum_{v\in V(G)} \lambda\Delta(G)^4 + o(\Delta(G)^4)}{5|G|\Delta(G)^4} \\ &= \frac{\lambda|G|\Delta(G)^4 + o(|G|\Delta(G)^4)}{5|G|\Delta(G)^4} = \frac{\lambda}{5} + o(1)\end{aligned}$$

$\square$

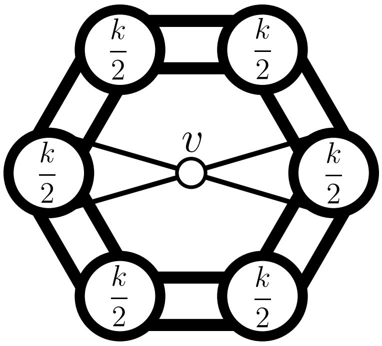
Now we see how our proof of theorem 3.2 will go. We need only show that for a regular triangle-free graphs G and $v \in V(G)$ we have an asymptotic bound of $P(G, v) \lesssim \frac{\Delta(G)^4}{8}$. This is something that local flags can prove.

Unfortunately we know that this approach cannot be directly improved to show the full $\frac{|G|}{5}\frac{\Delta(G)^4}{16}$ bound as this result is tight:

Lemma 3.7. *There exists regular graphs G with arbitrarily large $\Delta(G)$ such that some $v \in V(G)$ sits on $\frac{|G|}{5}\frac{\Delta(G)^4}{8}$ 5-cycles.*

Proof. For any $k \in \mathbb{N}$ even we construct the following graph: Construct a blowup of the 6-cycle of size $k/2$. This consists of 6 independent sets of size $k/2$ which are densely connected to their neighbours in a 6-cycle. We then add a single extra vertex which will be our distinguished vertex $v \in V(G)$ and connect it to densely to 2 of the supernodes which are on opposite ends of the cycle.

This graph is almost regular in that every vertex has degree k, except those in the 2 supernodes connected to v which have degree $k + 1$. This is a detail that doesn't matter and could be addressed but would distract from the core of the construction. Asymptotically speaking this construction is regular.



Note then that we can construct a pentagon going through v by first choosing a vertex from each of the connected supernodes which is $\left(\frac{k}{2}\right)^2$ choices. Then we can choose the final 2 vertices of the pentagon by either going "up" or "down" and picking any vertex from each of the supernodes. This gives us $2\left(\frac{k}{2}\right)^2$ choices leading to an overall number of $\frac{k^4}{8}$ 5-cycles as required. □

3.3. Local Flags for Regular Graphs

We show now why focusing on only regular graphs is so powerful in the context of local flags. As we saw in section 1.2 we want to find interesting elements of the semantic cone $\mathcal{C}^{\emptyset}_{\text{sem}}$. This is dual to finding general linear relations of density limits of local flags.

Let $\mathcal{G}$ be a class of regular graphs. We start by defining the extension of a flag:

Definition 3.1 (Extension)**.** *Let σ be a type of size k. Then we define the **extension** ext_i^σ as the sum of all σ flags $\in \overline{\mathcal{G}}^\sigma$ of size $k+1$ which have an edge between the unlabelled vertex the and vertex labelled i.*

Note 3.3.1. *By lemma 2.2 such flags are all local flags so $\text{ext}_i^\sigma \in \mathcal{L}^\sigma$.*

Example. *Let $\mathcal{G}$ be the class of red-black vertex coloured regular graphs then see figure 3.2 for an example σ and two possible extensions of σ corresponding to extending on vertex 1 or 2.*

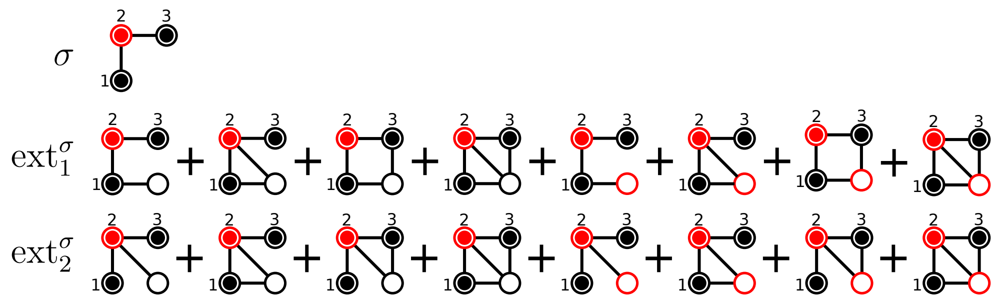


Figure 3.2.: Example type σ with two possible extensions

These extensions are important for the following reason:

Lemma 3.8. *If $\mathcal{G}$ consists of only regular graphs then for any σ and $\phi \in \Phi^\sigma$ we have $\phi(\mathrm{ext}_i^\sigma) = 1\ \forall\ i \in [|\sigma|]$.*

Proof. Let $\mathrm{ext}_i^\sigma = \sum_{\alpha \in I} F_\alpha$ for some index set I. Let (G, η) be any σ-flag with $G \in \mathcal{G}$. Then any instance of some F_α is a subset $\mathrm{im}\,\eta \subseteq U \subseteq V(G)$ where $G[U] \cong F_\alpha$. In particular $|U| = |F_\alpha| = |\sigma| + 1 = |\mathrm{im}\,\eta| + 1$ so $U = \mathrm{im}\,\eta \cup \{v\}$ for some $v \in V(G)$. In particular by definition of ext_i^σ this v must be connected in G to $\eta(i)$. Hence $v \in N(\eta(i)) \setminus \mathrm{im}\,\eta$. This map from a copy of F_α in G to a specific vertex is injective as we can take the σ-flag $(G[\mathrm{im}\,\eta \cup \{v\}], \eta)$ which must be isomorphic to F_α. This map is also surjective for the same reason, given any $v \in N(\eta(i)) \setminus \mathrm{im}\,\eta$ we take $G[\mathrm{im}\,\eta \cup \{v\}$ which must be isomorphic to some flag F_α where the unlabelled vertex is connected to the vertex labelled i, so appears in the ext_i^σ expression.

Therefore $\sum_{\alpha \in I} c(F_\alpha; (G, \eta)) = |N(\eta(i)) \setminus \mathrm{im}\,\eta|$. In particular then

$$\rho(\mathrm{ext}_i^\sigma; (G, \eta)) = \sum_{\alpha \in I} \frac{c(F_\alpha; (G, \eta)}{\Delta(G)} = \frac{|N(\eta(i)) \setminus \mathrm{im}\,\eta|}{\Delta(G)}$$

The size of $\mathrm{im}\,\eta$ is constant and $|N(\eta(i))| = \Delta(G)$ so this is in the range $[1 - \frac{|\mathrm{im}\,\eta|}{\Delta(G)}, 1]$. Hence $\rho(\mathrm{ext}_i^\sigma; (G, \eta)) = 1 - o(1)$ so we do get that $\phi(\mathrm{ext}_i^\sigma) = 1\ \forall\ \phi \in \Phi^\sigma$. $\square$

Note 3.3.2. *This proof only really required that sequences of graphs in $\mathcal{G}$ are asymptotically regular so the conditions for this lemma could be relaxed slightly.*

Corollary 3.8.1. *For any type σ and $\phi \in \Phi^\sigma$ we have $\phi(\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma) = 0$ for all $i, j \in [|\sigma|]$ and $\phi(f \cdot \mathrm{ext}_i^\sigma) = \phi(f)$ for all $f \in \mathcal{L}^\sigma, i \in [|\sigma|]$.*

In particular $\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma$, $f \cdot \mathrm{ext}_i^\sigma - f$ and $f - f \cdot \mathrm{ext}_i^\sigma$ are all elements of the semantic cone $\mathcal{C}_{\mathrm{sem}}^\sigma$.

Corollary 3.8.2. *If σ is a local type then for any $\phi \in \Phi^\emptyset$ we have $\phi([\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]) = 0$ for all $i, j \in [|\sigma|]$ and $\phi([\![f \cdot \mathrm{ext}_i^\sigma]\!]) = \phi([\![f]\!])$ for all $f \in \mathcal{L}^\sigma$ and $i \in [|\sigma|]$.*

Proof of Corollary 3.8.2. By the previous corollary $\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma$ are in the semantic cone $\mathcal{C}_{\mathrm{sem}}^\sigma$ so by lemma 2.8 we must have $[\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!] \in \mathcal{C}_{\mathrm{sem}}^\sigma$ so $\phi([\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]) \geq 0$. The same goes for swapping i and j so $\phi([\![\mathrm{ext}_j^\sigma - \mathrm{ext}_i^\sigma]\!]) \geq 0$ implying $\phi([\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]) = 0$.

The same argument works for $[\![f \cdot \mathrm{ext}_i^\sigma]\!]$ and $[\![f]\!]$. $\square$

Note 3.3.3. *These relations suggest that if we focused only on regular graph classes from the beginning we could have quotiented out the set of relations $[\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]$ from our vector space $\mathbb{R}\overline{\mathcal{G}}^\emptyset$ to get a "cleaner" algebra. We have not verified that this is possible. If this is true it could theoretically enable us to generate smaller semidefinite programs but we would need an algorithmic way of reducing vectors over this subspace which prima facie is not a straightforward task.*

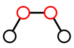
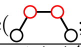
The value of these results for us is twofold: Firstly, this gives us a wide set of easy to generate elements of the semantic cone which we can use in our semidefinite program. Secondly, this property where $\phi([\![f]\!]) = \phi([\![f \cdot \mathrm{ext}_i^\sigma]\!])$ gives us a way of expressing a vector in a subspace of larger flags. In particular multiplying by ext_i^σ gives a vector over flags which are 1 larger than those in the original vector. This will be very useful to us.

3.4. Counting Pentagons with Local Flags

We want to use local flags to show an asymptotic bound that for any regular, triangle-free graph G we have $P(G, v) \lesssim \frac{\Delta(G)^4}{8}$ for any $v \in V(G)$. We first reduce this to a problem on coloured graphs.

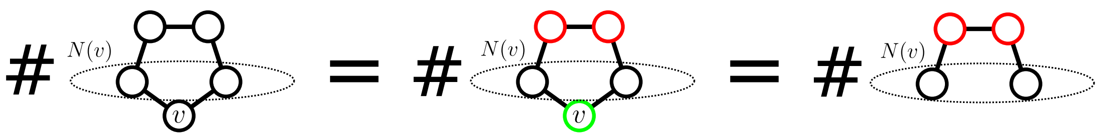
Let $\mathcal{G}$ be the class of red-black vertex coloured graphs which are triangle-free, regular and such that for each $G \in \mathcal{G}$ the number of black vertices in G is exactly $\Delta(G)$ and the set of black vertices in G is independent. Note then that $\overline{\mathcal{G}}$ is the same class without the regularity as all other properties are hereditary.

Then we have the following reduction:

Lemma 3.9. *Let*  *be the black-red-red-black path. Then for any* $\lambda \in \mathbb{R}$ $c($$;G)/\Delta(G)^4 \lesssim \lambda$ *as* $\Delta(G) \to \infty$ *over* $\mathcal{G}$ *implies* $\frac{P(H,v)}{\Delta(H)^4} \lesssim \lambda$ *as* $\Delta(H) \to \infty$ *over the class of regular triangle-free graphs.*

Proof. Let G be a simple regular triangle-free graph and fix some $v \in V(G)$. Then $|N(v)| = \Delta(G)$ and the set $N(v)$ is independent.

Any pentagon containing v then must contain 2 vertices of $N(v)$ and 2 vertices outside of $N(v)$. In particular if we construct a 3-vertex-coloured graph H which is a copy of G where v is green, $N(v)$ is black and all others are red then we want to count how many pentagons contain 1 green vertex, 2 black vertices and 2 red vertices. Then as all black vertices are connected to the green vertex it suffices to count how many black-red-red-black paths there are in H. In fact removing v from H has no effect on this count so WLOG we can remove it.



This H graph has (up to asymptotically 0 terms) the same max degree as G so any bound on H asymptotically applies to G. □

3.4.1. Local Flags Setup

We have decided on a class of graphs $\mathcal{G}$ so we need to now find a description of the local σ-flags.

Lemma 3.10. *A σ-flag F is a local σ-flag (relative to our choice of $\mathcal{G}$) iff each connected component of F contains a labelled vertex or a black vertex.*

Proof. First we show if each connected component of F contains a labelled or black vertex then F has bounded density function. We do this by induction on $|F| - |\sigma|$. The base case is that $|F| = |\sigma|$ which implies F consists only of labelled vertices meaning $c(F; G) = 1 \in O(\Delta(G)^0)$ for any $G \in \mathcal{G}^\sigma$ as required.

Otherwise assume $|F| > |\sigma|$. Note then that as each connected component of F contains at least one labelled or black vertex each vertex u of F has a finite minimum distance $d(u)$ to some labelled or black vertex (where $d(u) = 0$ for all black and labelled u and $d(u) > 0$ for all unlabelled red vertices). Let v be some unlabelled vertex in F with maximum $d(v)$. Consider then removing v from F: $F' := F[V(F) \setminus \{v\}]$. This F' must not have any connected components with no black or labelled vertices because we chose $d(v)$ large as possible. Hence $c(F'; G) \in O\left(\Delta(G)^{|F|-|\sigma|-1}\right)$ by our induction hypothesis.

Consider then some subset $\operatorname{im} \eta \subseteq U' \subseteq V(G)$ such that $G[U'] \cong F'$. We want to bound how many copies U of F can we obtain by adding one more vertex $u \in V(G)$ onto which our removed v can be mapped. There are 2 cases: If the $v \in V(F)$ we removed was a black vertex then u must be a black vertex, there are only $\Delta(G)$ of those meaning there are $\leq \Delta(G)$ possible such U's. Otherwise v was a red vertex meaning it was connected to $V(F) \setminus \{v\}$. This means u must be in the neighbourhood of one of the vertices of U of which there are $O(\Delta(G))$ such options. In either case there are only $O(\Delta(G))$ such extensions of U'. In particular though every copy U of F can be reached in this way. This is easily seen by taking any such copy and removing the vertex u that v is mapped to, this must give you a copy U' of F' with which we can re-add u to get back to U. Therefore this process counts all copies U of F in G giving us a bound of $c(F'; G) \cdot O(\Delta(G)) = O\left(\Delta(G)^{|F|-|\sigma|}\right)$ as required.

Now that we know any such F has bounded density function we simply note that any label extension of F gives you a new flag which also has a labelled or black vertex in each connected component. Therefore label extension preserves bounded density functions so F is a local σ-flag.

We also show that this encapsulates all local σ-flags. This isn't strictly required for our application so I give a brief idea of the proof. Taking any flag with a connected component consisting of all unlabelled red vertices we can easily construct a sequence of graphs $(G_k)_{k \in \mathbb{N}}$ in $\mathcal{G}$ where we keep increasing the number of copies of the unlabelled red component. We can do this maintaining the fact that each G_k is regular and has only $\Delta(G)$ black vertices. Then the number of copies of F increases unbounded along this sequence proving F is not local. $\square$

Now that we know our set of local flags, which tells us our local types by lemma 2.6 we can now formulate our problem as a semidefinite problem.

3.5. The Semidefinite Program

We fixed our class of graphs $\mathcal{G}$ to be the class of red-black vertex coloured regular triangle-free graphs with $\Delta(G)$ black vertices such that the set of black vertices is independent. We want to asymptotically bound the number of black-red-red-black paths in this class. This means we have the following objective function: . This is a local flag by lemma 3.10. We will now use the semidefinite method (section 1.2) to find a bound $\lambda \in \mathbb{R}$ such that $\phi(\;) \leq \lambda \;\forall\; \phi \in \Phi^{\sigma}$.

Note 3.5.1. *It proves to be more intuitive to describe our problem in terms of maximising* $\phi(\;)$. *We then use duality (section 1.2.3) to convert this to a problem of finding some* $\lambda\emptyset - \; \in \mathcal{C}^{\emptyset}_{\text{sem}}$ *which proves the* λ *upper bound rigorously.*

We need to pick a subspace of $\mathcal{L}^{\emptyset}$ to optimise our objective function over. A standard method (e.g. [14], [6]) is picking the subspace spanned by flags of some fixed size. In this case it suffices to consider the subspace of flags of size 5.[2].

By our choice of subspace we need to find a vector f over flags of size 5 such that bounding $\phi(f)$ gives a bound on $\phi(\;)$. To achieve this we view as a -flag in itself: . Letting $\sigma =$ we note that σ is a local type and we can compute $[\![\;]\!] = \frac{2}{4!} = \frac{1}{12}$. We can then compute ext^{σ}_1 and then use corollary 3.8.2 which tells us that

$$\frac{1}{12}\phi(\;) = \phi([\![\;]\!]) = \phi([\![\; \cdot \text{ext}^{\sigma}_1]\!]). = \phi([\![\text{ext}^{\sigma}_1]\!]).$$

where we used that σ is the unit of the algebra $\mathcal{L}^{\sigma}$. Then ext^{σ}_1 is a vector of flags of size 5: See figure 3.3. It suffices then to try to maximise $[\![\text{ext}^{\sigma}_1]\!]$: Call this vector O.

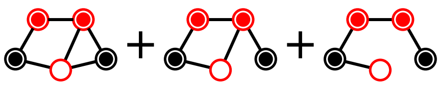


Figure 3.3.: ext^{σ}_1 where $\sigma =$

To apply our semidefinite method we first need a list of known elements of the semantic cone $\mathcal{C}^{\emptyset}_{\text{sem}}$ or dually a list of linear constraints on limit functionals. For any type σ we immediately get the elements $[\![\text{ext}^{\sigma}_i - \text{ext}^{\sigma}_j]\!]$ from corollary 3.8.2. We want only those that are in the subspace of flags of size 5 so we take all such vectors where $|\sigma| = 4$.

We know that in any $G \in \mathcal{G}$ the set of black vertices is independent and has size $\Delta(G)$. Let B_k be the $\emptyset$-flag consisting of k independent black vertices. e.g. $B_1 = \circ, B_2 = \circ\;\;\circ, B_3 = {}^{\circ}_{\circ\;\circ}, \ldots$. Then we know $c(B_k; G) = \binom{\Delta(G)}{k}$ for any $G \in \mathcal{G}$ which implies $\phi(B_k) = 1 \;\forall\; \phi \in \Phi^{\emptyset} \;\forall\; k \in \mathbb{N}$. We would like to add these to the list of constraints. For $k = 5$ this is already in the subspace of flags of size 5 so we can add it

[2]Picking larger flags intuitively allows the search to find tighter bounds but comes at the cost of computation time

directly. For $k < 5$ we need to do the same extension trick as with our objective function where we use that $[\![B_k]\!] = B_k$ to get

$$
\begin{aligned}
1 = \phi(B_k) = \phi([\![B_k]\!]) = \phi([\![B_k \cdot \mathrm{ext}_1^{B_k}]\!]) = \phi\left(\left[\!\!\left[B_k \cdot \left(\mathrm{ext}_1^{B_k}\right)^{5-k}\right]\!\!\right]\right) \\
= \phi\left(\left[\!\!\left[\left(\mathrm{ext}_1^{B_k}\right)^{5-k}\right]\!\!\right]\right).
\end{aligned}
$$

$\left(\mathrm{ext}_1^{B_k}\right)^{5-k}$ is a vector over flags of size 5 so this is a constraint we can use.

Finally then we want to encode constraints of the form $\phi([\![f^2]\!]) \geq 0$ for all $f \in \mathcal{L}^\sigma$. To do this we need to find local types σ and sizes n such that f^2 is an expression of flags of size 5 for all $f \in \mathcal{G}^\sigma_{\mathrm{loc},n}$. Those pairs (σ, n) are (, 3), (, 4), (, 4), (, 4), (, 4), and (, 4).

Let $\mathcal{B} = (F_1, \ldots, F_\ell)$ be the basis of all flags of size 5. In summary then our problem is:

$$
\begin{aligned}
\max_{x \in \mathbb{R}^\ell} \quad & \phi_x(O) \\
\text{such that} \quad & \phi_x(F_i) \geq 0 \;\forall\; i \in [\ell] \\
& \phi_x([\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]) = 0 \;\forall\; |\sigma| = 4, i, j \in [4] \\
& \phi_x(B_5) = 1 \\
& \phi_x\left(\left[\!\!\left[\left(\mathrm{ext}_1^{B_k}\right)^{5-k}\right]\!\!\right]\right) = 1 \;\forall\; k \in [4] \\
& \phi_x([\![f^2]\!]) \geq 0 \forall f \in \mathcal{L}_n^\sigma \;\forall (\sigma, n) \text{ from above list.}
\end{aligned}
$$

We know from section 1.2 how to construct the corresponding SDP. A solution to this SDP finds a convex combination of elements of the semantic cone which prove an upper bound on $\phi(O) \;\forall\; \phi \;\in \Phi^\emptyset$.

We use a SDP solver (see appendix B) to show the following result:

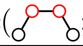
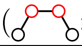
Lemma 3.11.

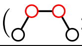
$$\frac{1}{4}\emptyset - O \in \mathcal{C}^\emptyset_{\mathrm{sem}}.$$

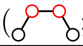
In the next section (3.6) we will show how we make this SDP solution rigorous. For now we show that this bound suffices to prove theorem 3.2.

Proof of theorem 3.2. By lemma 3.11 we have that $\phi(O) \leq \frac{1}{4} \;\forall\; \phi \in \Phi^\sigma$. We know that $\frac{1}{12}\phi(\;) = \phi(O)$ so $\phi(\;) \leq 3$. Therefore for any Δ-increasing sequence of graphs $(G_k)_{k \in \mathbb{N}}$ we have $\lim_{k \to \infty} \frac{c(\;; G_k)}{\binom{\Delta(G_k)}{4}} \leq 3$. Intuitively then $\binom{\Delta(G_k)}{4}$ is asymptotically $\frac{\Delta(G_k)^4}{4!}$ so we get $\lim_{k \to \infty} c(\;; G_k) \leq \frac{3}{4!} = \frac{1}{8}$. We show this argument more precisely now:

We use the fact that $\frac{\Delta(G_k)!}{(\Delta(G_k)-4)!} = \Delta(G_k)^4 + o(\Delta(G_k)^4)$ to show that

$\lim_{k\to\infty} c($$; G_k) / \Delta(G_k)^4 = \lim_{k\to\infty} c($$; G_k) / \binom{\Delta(G_k)}{4} \cdot \frac{\Delta(G_k)!}{4!(\Delta(G_k)-4)!\Delta(G_k)^4}$

$= \lim_{k\to\infty} c($$; G_k) / \binom{\Delta(G_k)}{4} \cdot \frac{\Delta(G_k)^4 + o(\Delta(G_k)^4}{4!\Delta(G_k)^4}$

$= \lim_{k\to\infty} (1+o(1)) \frac{1}{4!} c($$; G_k) / \binom{\Delta(G_k)}{4}$

$$\leq \frac{1}{8}.$$

Then by lemma 3.9 $\frac{P(G,v)}{\Delta(G)^4} \lesssim \frac{1}{8}$ for any regular triangle-free G and $v \in V(G)$. Lemma 3.6 tells us then $\frac{P(G)}{|G|\Delta(G)^4} \lesssim \frac{1}{5\cdot 8}$ for any regular triangle free G. Corollary 3.5.1 shows that we can drop the regular requirement. Finally, lemma 3.4 shows that this asymptotic bound implies the non-asymptotic bound.

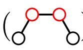
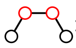
$$\therefore P(G) \leq \frac{|G|}{5} \frac{\Delta(G)^4}{8} \; \forall \; G \text{ triangle-free.}$$

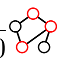
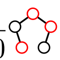
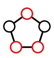
□

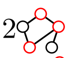
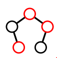
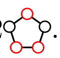
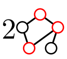
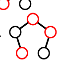
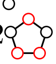
3.6. SDP Proof Validation

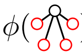
We've shown in the previous section how to express our problem in the form of a semidefinite program and we claimed that SDP software found a solution proving a bound of 1/8. Clearly though appealing to a software implementation, no matter how well tested, is not rigorous: We should verify the outcome ourselves. We saw in section 1.2 how such an SDP solution corresponds to constructing a specific element of the semantic cone by taking convex combinations of known elements of the cone. In this section we will show the combination of elements that was found by the SDP software, verifying that it is indeed a valid proof.

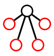
We will defer the specific details of how we converted the SDP solution into the following proof to appendix C.

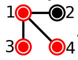
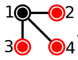
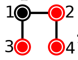
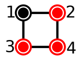
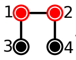
First, as a reminder we were trying to bound $\phi(O) = \frac{1}{12}\phi($$)$ where we defined O as an extension of $\sigma_O =$ : $O = [\![\mathrm{ext}_1^{\sigma_O}]\!]$. We expand this and find that

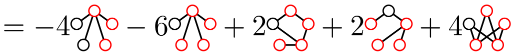
$O = \frac{1}{60}$ $+ \frac{1}{120}$ $+ \frac{1}{60}$

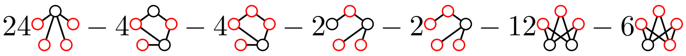
For convenience we scale this by 120 to get 2 $+$  $+ 2$. Then we prove that $120\phi(O) \leq \lambda \; \forall \; \phi \in \Phi^\emptyset$ by showing $\lambda\emptyset - 2$ $+$  $+ 2$ $\in \mathcal{C}^\emptyset_{\text{sem}}$.

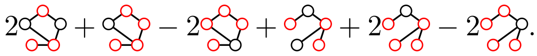
First we take $B_1 = \circ$ as defined above. We know $1 = \phi(\circ) = \phi([\![(\mathrm{ext}_1^{\circ})^4]\!]) = \frac{1}{5}\phi($$)$ for all $\phi \in \Phi^\emptyset$ hence

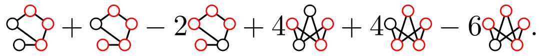
$$5\emptyset -$$  $$\in \mathcal{C}^\emptyset_{\mathrm{sem}}. \quad (3.1)$$

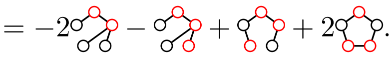
Next, by corollary 3.8.2 for any local type σ and $i, j \in [|\sigma|]$ and we have $[\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!] \in \mathcal{C}^\emptyset_{\mathrm{sem}}$. It also proves easier to scale these by 120 as with O. We compute some such vectors for the following fixed types: $\sigma_1 :=$ , $\sigma_2 :=$ , $\sigma_3 :=$ , $\sigma_4 :=$  and $\sigma_5 :=$ .

$$120[\![\mathrm{ext}_2^{\sigma_1} - \mathrm{ext}_1^{\sigma_1}]\!] =$$  $$\in \mathcal{C}^\emptyset_{\mathrm{sem}}. \quad (3.2)$$

$$120[\![\mathrm{ext}_1^{\sigma_2} - \mathrm{ext}_2^{\sigma_2}]\!] =$$  $$\in \mathcal{C}^\emptyset_{\mathrm{sem}}. \quad (3.3)$$

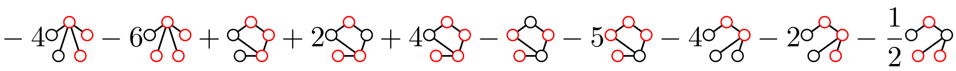
$$120[\![\mathrm{ext}_2^{\sigma_3} - \mathrm{ext}_1^{\sigma_3}]\!] =$$  $$(3.4)$$

$$120[\![\mathrm{ext}_2^{\sigma_4} - \mathrm{ext}_1^{\sigma_4}]\!] =$$  $$(3.5)$$

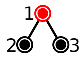
$$120[\![\mathrm{ext}_3^{\sigma_5} - \mathrm{ext}_1^{\sigma_5}]\!] =$$  $$(3.6)$$

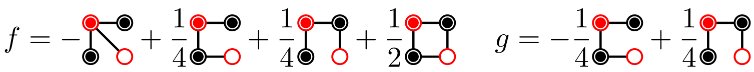
Now we can take a convex combination of these elements of the semantic cone to get another element of the semantic cone.

$$6 \cdot (3.1) + 1 \cdot (3.2) + \frac{1}{4} \cdot (3.3) + 1 \cdot (3.4) + 1 \cdot (3.5) + 2 \cdot (3.6)$$

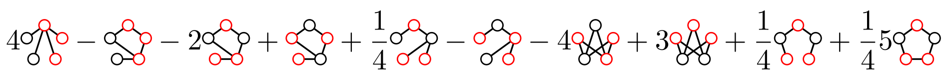
$$= 30\emptyset$$ 

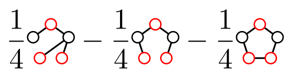
 $$(3.7)$$

Consider then the type $\sigma_6 =$  and the following vectors $f, g \in \mathcal{L}^{\sigma_6}$.

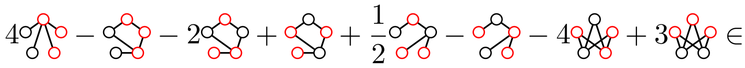


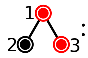
Then by 2.8 both of the following are elements of the semantic cone:

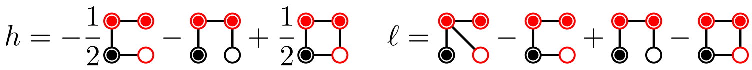
$$120[\![f^2]\!] =$$ 

$$120[\![g^2]\!] =$$ 

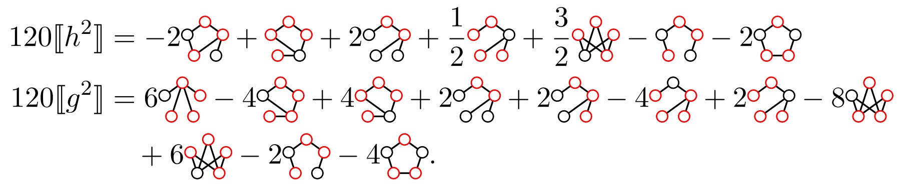
The sum of these two elements gives a new element of the semantic cone

 $$\in \mathcal{C}^\emptyset_{\mathrm{sem}} \quad (3.8)$$

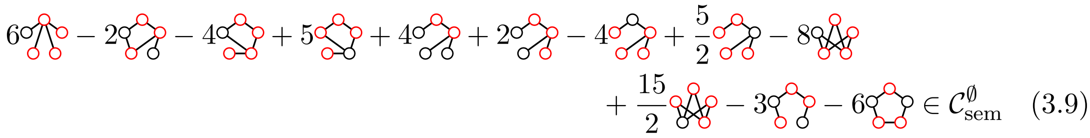
We do the same thing with the type $\sigma_7 :=$ : Take the following vectors $h, \ell \in \mathcal{L}^{\sigma_7}$.



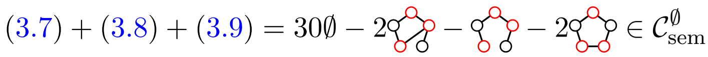
Then we can compute

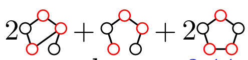


The sum of these two elements of the semantic cone is:



Finally, we take the combination of equations 3.7, 3.8 and 3.9 to get our final element of the cone.



We know $120O =$  so $30\emptyset - 120O \in \mathcal{C}^{\emptyset}_{\text{sem}}$. This implies $\frac{1}{4}\emptyset - O \in \mathcal{C}^{\emptyset}_{\text{sem}}$ as required. This proves lemma 3.11 completing our proof of theorem 3.2.

3.7. Proving the Stronger Result

We can use local flags in a more complex way to get the stronger result of theorem 3.1.

To prove the bound in theorem 3.2 we defined $P(G, v)$ to be the number of pentagons containing a $v \in G$ and used local flags find an upper bound. To prove the strong result in theorem 3.1 we will bound the following function. For fixed $v \in G$ define

$$Q(G, v) := \Delta(G)P(G, v) + \sum_{u \in N(v)} P(G, u).$$

Then we claim

Lemma 3.12. *For G triangle-free and $v \in V(G)$*

$$Q(G, v) \lesssim 0.2073\Delta(G)^5 \text{ as } \Delta(G) \to \infty$$

We will prove this lemma now using local flags. We use the same class of graphs $\mathcal{G}$ as before, where the black vertices represent those in the neighbourhood of v and the others are red. We've seen already then how to convert a triangle-free regular graph G into the corresponding $G' \in \mathcal{G}$ such that $\rho(\ , G') = P(G,v)/\binom{\Delta(G)}{4}$.

Note that any pentagon passing through the neighbourhood of v contains at most 2 such vertices. Therefore

$$\rho(\ + 2 \ ; G') = \frac{\sum_{u \in N(V)} P(G,v)}{\binom{\Delta(G)}{5}}.$$

Now we use the same extensions trick from before to embed our desired vector into a space of flags of size n and get the following objective vector

$$O := [\![\ \cdot (\mathrm{ext}_1^{\sigma_1})^{n-4}]\!] + [\![\ \cdot (\mathrm{ext}_1^{\sigma_2})^{n-5}]\!] + 2[\![\ \cdot (\mathrm{ext}_1^{\sigma_3})^{n-5}]\!]$$

where $\sigma_1 = \ , \sigma_2 = \ , \sigma_3 = \ .$ Then using the exact same set of constraints as from section 3.5 we can use an SDP solver (see appendix B) using flags of size $n = 8$ to find:

Lemma 3.13.

$$0.4146\emptyset - O \in \mathcal{C}_{\mathrm{sem}}^{\emptyset}$$

hence $\phi(O) \leq 0.4146 \ \forall \ \phi \in \Phi^{\emptyset}$.

Calculating the normalising factors (definition 1.12) we find this implies

$$\frac{2}{4!}\phi(\ ) + \frac{2}{5!}\phi(\ + 2 \ ) \leq 0.4146$$

Hence for any convergent Δ-increasing sequence of graphs $(G_k)_{k \in \mathbb{N}}$ we have

$$\begin{aligned}
&\lim_{k\to\infty} \frac{2}{4!}\rho(\ ; G_k) + \frac{2}{5!}\rho(\ ; G_k) + \frac{4}{5!}\rho(\ ; G_k) \leq 0.4146 \\
\implies &\lim_{k\to\infty} \frac{2}{4!}\frac{c(\ ; G_k)}{\binom{\Delta(G_k)}{4}} + \frac{2}{5!}\frac{c(\ ; G_k)}{\binom{\Delta(G_k)}{5}} + \frac{4}{5!}\frac{c(\ ; G_k)}{\binom{\Delta(G_k)}{5}} \leq 0.4146 \\
\implies &\lim_{k\to\infty} 2\frac{c(\ ; G_k)}{\Delta(G_k)^4} + 2\frac{c(\ ; G_k)}{\Delta(G_k)^5} + 4\frac{c(\ ; G_k)}{\Delta(G_k)^5} \leq 0.4146 \\
\implies &\lim_{k\to\infty} \frac{c(\ ; G_k)}{\Delta(G_k)^4} + \frac{c(\ ; G_k)}{\Delta(G_k)^5} + 2\frac{c(\ ; G_k)}{\Delta(G_k)^5} \leq 0.2073
\end{aligned}$$

Therefore

$$\Delta(G)c(\ ; G) + c(\ ; G) + 2c(\ ; G) \lesssim 0.2073\Delta(G)^5$$

By construction though this proves the same bound holds for the $Q(G,v)$ function

$$\Delta(G)P(G,v) + \sum_{u \in N(V)} P(G,u) \lesssim 0.2073\Delta(G)^5 \implies Q(G,v) \lesssim 0.2073\Delta(G)^5$$

proving lemma 3.12.

Now we can prove the full result of theorem 3.1.

Proof of 3.1. For a triangle-free regular graph G we sum $Q(G,v)$ over all $v \in V(G)$:

$$\begin{aligned}\sum_{v\in V(G)} Q(G,v) &= \sum_{v\in V(G)} \Delta(G)P(G,v) + \sum_{u\in N(V)} P(G,u)\\ &= 5\Delta(G)P(G) + \sum_{v\in V(G)}\sum_{u\in N(V)} P(G,u)\end{aligned}$$

Then as G is regular we this comes out to

$$\begin{aligned}\sum_{v\in V(G)} Q(G,v) &= 5\Delta(G)P(G) + \Delta(G)\sum_{u\in V(G)} P(G,u)\\ &= 5\Delta(G)P(G) + 5\Delta(G)P(G)\\ &= 10\Delta(G)P(G)\end{aligned}$$

Therefore

$$10\Delta(G)P(G) \lesssim 0.2073|G|\Delta(G)^5 \implies P(G) \lesssim 0.02073|G|\Delta(G)^4$$

$\square$

4. Application: Strong Edge Colouring

As defined in the introduction, a strong edge colouring of a graph is an edge colouring where two edges which are incident to some common edge must have different colours. The minimum number of colours required for a strong edge colouring of G is called the strong chromatic index $\chi'_s(G)$. In 1985 Erdős and Nešetřil conjectured that $\chi'_s(G) \leq 1.25\Delta(G)^2$, and in 1989 Faudree et al. conjectured that $\chi'_s(G) \leq \Delta(G)^2$ if G is bipartite [11].

In this chapter we will use the semidefinite method on local flags to prove the following theorems:

Theorem. *For $\Delta(G)$ large enough we have*

$$\chi'_s(G) \leq 1.73\Delta(G)^2.$$

Theorem. *If G is bipartite then for $\Delta(G)$ large enough we have*

$$\chi'_s(G) \leq 1.6254\Delta(G)^2.$$

Both of these theorems are the new improved bounds in their respective cases. We start by showing the first result, and leave the bipartite case until section 4.5. Additionally, in section 4.6 we will investigate the asymmetric version of the bipartite case, leading to an interesting discovery which suggests the strong neighbourhood density bound coefficient is constant across all asymmetry ratios.

4.1. Line Graph Equivalence

A strong edge colouring of a graph G is equivalent to a proper vertex colouring of $L(G)^2$, the square of the line graph of G (see [19]). The line graph $L(G)$ is a graph with a vertex v_e for each $e \in E(G)$ such that $\{v_e, v_f\} \in E(L(G))$ iff e is incident to f in G. The square of a graph G^2 then is a copy of the graph where vertices at distance ≤ 2 are connected. See figure 4.1 for an example construction.

Definition 4.1 (Strong Neighbourhood)**.** *Given a graph G and edge $E \in E(G)$ define the strong neighbourhood $N_s(e)$ of e as those edges $f \in E(G)$ which are adjacent to e in $L(G)^2$ (i.e. those which cannot share a colour with e in a strong edge colouring).*

4.2. The Conjecture of Erdős and Nešetřil

In 1985 Erdős and Nešetřil [11] conjectured an upper bound on the strong chromatic index $\chi'_s(G)$, the minimum colours needed for a strong edge colouring.

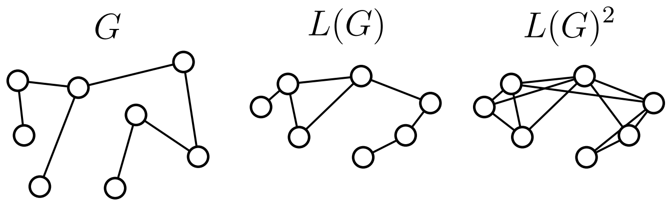


Figure 4.1.: Example G, $L(G)$ and $L(G)^2$

Conjecture 4.1 (Erdős, Nešetřil 1985 [11]). *$\chi'_s(G) \leq 1.25\Delta(G)^2$ for all graphs G.*

The bound of $2\Delta(G)^2$ was the best known bound until 1997 when Molloy and Reed showed the following.

Theorem (Molloy, Reed 1997 [19]). *$\chi'_s(G) \leq 1.998\Delta(G)^2$ for $\Delta(G)$ sufficiently large.*

Their proof consisted of colouring $L(G)^2$ with 2 discrete steps: The first is a bound on the edge density of any neighbourhood of $L(G)^2$. We call this the *strong neighbourhood density*.

Lemma (Lemma 1 from [19]). *If G has maximum degree Δ then for each $e \in E(G)$ $|N_s(e)| \leq (1 - \frac{1}{36})\binom{2\Delta^2}{2}$.*

After showing the strong neighbourhoods in $L(G)^2$ are sparse they use a colouring lemma to colour $L(G)^2$:

Lemma (Lemma 2 from [19]). *Let $\delta, \gamma > 0$ satisfying some condition. Then if $\Delta(H) \leq X$ such that $N(v)$ has at most $(1-\delta)\binom{X}{2}$ edges then $\chi(H) \leq (1-\gamma)X$.*

This strategy of bounding the edge density of strong neighbourhoods of G and using a probabilistic colouring was iterated on through successive papers: Bruhn and Joos found an asymptotically tight bound on the strong neighbourhood density and improved the colouring lemma.

Theorem (Bruhn & Joos, 2015 [5]). *$\chi'_s(G) \leq 1.93\Delta(G)^2$ for $\Delta(G)$ sufficiently large.*

Bonamy, Perett and Postle introduced a modification of the method where rather than bounding the strong edge neighbourhood density for the entire graph we instead focus on a subgraph of $L(G)^2$ of high degree vertices. They show that the neighbourhood density in this subgraph can go below the tight bound of Bruhn and Joos and so can be coloured with fewer colours. The rest of the graph has low degree so can be coloured greedily.

Theorem (Bonamy, Perrett & Postle, 2018 [2]). *If $\Delta(G)$ is sufficiently large then $\chi'_s(G) \leq 1.835\Delta(G)^2$.*

Most recently then Hurley, de Joannis de Verclos and Kang improved on the colouring lemma from the Bonamy et al. paper to achieve the current lowest known bound.

Theorem (Hurley, de Joannis de Verclos & Kang, 2022 [17]). *If $\Delta(G)$ is sufficiently large then $\chi'_s(G) \leq 1.772\Delta(G)^2$.*

In this chapter we use the semidefinite method on local flags to improve the strong neighbourhood density bound, then applying the colouring lemma from Hurley et al. as a black box we claim the following result.

Theorem 4.1. *If $\Delta(G)$ is sufficiently large then $\chi'_s(G) \leq 1.73\Delta(G)^2$.*

The rest of this chapter is the proof of this theorem. We will follow the same structure as used in chapter 3, first reducing the problem to a problem on a certain class of coloured graphs, finding an objective vector O of local flags for which an asymptotic bound on the density of O will allow us to derive an asymptotic bound on the size of $|E(H[N_{H[F]}(f)])|$. We then enumerate some elements of the semantic cone and apply the semidefinite method to get a bound on O.

4.3. Reduction

The theorem that we want to improve on is the strong edge neighbourhood theorem from the Bonamy et al. paper which appears in Hurley et al. as follows:

Theorem (Theorem 3.1 [17]). *Fix $\eta \in [0, 0.3]$. For any graph G let $H = L(G)^2$. Let F be a maximal subset of $V(H)$ such that $H[F]$ has minimum degree $\geq (2-\eta)\Delta(G)^2$. Then for any $f \in F$ the number of edges in the subgraph $H[N_{H[F]}(f)]$ induced by the neighbourhood of f (in $H[F]$) is at most*

$$\left(\frac{31}{6} - \frac{128}{10-3\eta} - \eta^2\right)\Delta(G)^4.$$

In other words we are given a graph G and a maximal subset of the edges $F \subseteq E(G)$ such that the subgraph induced by F in $H = L(G)^2$ has minimum degree $\geq (2-\eta)\Delta(G)^2$ for some η. This means each $f \in F$ has at least $(2-\eta)\Delta(G)^2$ other edges $f' \in F$ adjacent in $L(G)^2$ ($|N_s(f) \cap F| \geq (2-\eta)\Delta(G)^2 \ \forall \ f \in F$). We can then just consider the graph $H[F]$ induced by this high degree F subset. Then for some fixed $f \in F$ we want to find an upper bound on the number of edges in its neighbourhood in $H[F]$. That is, the number of pairs $e, e' \in N_s(f) \cap F$ such that e, e' adjacent in $L(G)^2$.

Note 4.3.1. *The first reduction we can note is that WLOG we can assume G is regular, as outlined in each of the above papers.*

We note now that asymptotically it suffices to count only the number of edges in $H[N_{H[F]}(f)]$ which correspond to pairs $e, e' \in E(G)$ which are not incident. i.e. those e, e' which are adjacent in $L(G)^2$ but not $L(G)$.

Lemma 4.2. *Let G, F, H and $f \in F$ be as above. Let A be those pairs $e, e' \in F$ such that e, e' adjacent in $L(G)^2$ but not $L(G)$. Then*

$$|E(H[N_{H[F]}(f)])| = |E(H[N_{H[F]}(f)]) \cap A| + o(\Delta(G)^4).$$

Proof. We count how many pairs $e, e' \in N_s(f) \cap F$ are adjacent in $L(G)$. Clearly $|N_s(f)| \in O(\Delta(G)^2)$ and each $e \in N_s(f)$ has $O(\Delta)$ incident edges leading to a bound of $O(\Delta(G)^3) \subseteq o(\Delta(G)^4)$ as required. □

Similarly, the condition that $\deg_{H[F]}(f) \geq (2-\eta)\Delta(G)^2$ for all $f \in F$ is dominated only by non-incident edges.

Lemma 4.3. *Let G, F, H and $f \in E(G)$ be as above. Let B be those edges $e \in F$ such that e is adjacent to f in $L(G)^2$ but not in $L(G)$. Then*

$$\deg_{H[F]}(f) = |B| + o(\Delta(G)^2).$$

Proof. The number of edges incident to a fixed f in $L(G)$ is $O(\Delta(G))$. □

As we did in section 3.4 we want to reduce our problem to a problem on coloured graphs. Given G, $F \subseteq E(G)$, $\eta > 0$ and $f \in F$ as above we can construct a new corresponding $(2,2)$-graph (red/black vertices, red/black edges) G' as follows: Let $f = \{u, v\}$. Take a copy of G and colour all neighbours of u and v black and all other vertices red. Then colour every edge $e \in F$ black and all other edges red. See figure 4.2 for example.

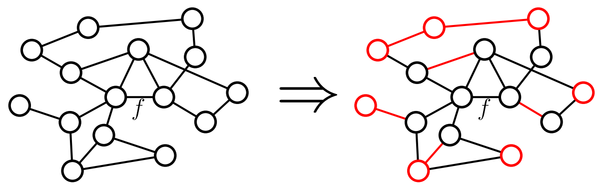


Figure 4.2.: Example coloured graph transformation

Lemma 4.4. *Let $G, H = L(G)^2, F, \eta, f$ and G' be as above. Define $E_O(G') \subseteq \binom{E(G')}{2}$ as the set of pairs of non-incident edges e, e' in G' where e, e' are black edges with a common incident edge and at least one of the vertices of each edge is black. Then the number of edges in $H[N_{H[F]}(f)]$ where $H = L(G)^2$ is equal to $|E_O(G')| + o(\Delta(G)^4)$.*

Proof. Let G, $H = L(G)^2$, F, η, f and G' be as above. By lemma 4.2 it suffices to show the result for number of pairs e, e' in $H[N_{H[F]}(f)]$ where e, e' are not incident.

Consider some edge $\{e, e'\}$ in $H[N_{H[F]}(f)]$ where e, e' are not incident. Both of these e, e' are in F so are coloured black in G'. Similarly both are in the strong neighbourhood of f meaning each has a common incident edge with f. Edges incident to f have both vertices coloured black in G' so each of e, e' has at least one black vertex. Hence this edge $\{e, e'\}$ corresponds to a pair of non-incident black edges with at least one black vertex each in G' at distance ≤ 2 from each other.

Consider then some pair $e, e' \in E_O(G')$ As each e, e' has a black vertex they must be either incident to f *or* equal to f. If both $e, e' \neq f$ then e, e' satisfy exactly the conditions to be an edge in $H[N_{H[F]}(f)]$. Hence we overcount the number of edges in

$H[N_{H[F]}(f)]$ by exactly the number of these pairs $e, e' \in E_0$ where one of $e, e' = f$. Assume $e = f$, this leaves only $\leq 2\Delta(G)^2$ choices for e' so there are at most $2\Delta(G)^2$ such pairs which is $o(\Delta(G)^4)$ as required. $\square$

Corollary 4.4.1. *It suffices to bound the size of E_O over the class of all regular $(2,2)$-graphs where there are $\leq 2\Delta(G)^2$ black vertices and the subgraph of $L(G)^2$ induced by black edges has minimum degree $\geq (2-\eta)\Delta(G)^2$.*

This is something we can bound with local flags.

4.4. Local Flag Setup

Let $\mathcal{G}$ be the class of graphs described in corollary 4.4.1. As before the hereditary closure $\overline{\mathcal{G}}$ only drops the regularity requirement.

Lemma 4.5. *Given the class $\mathcal{G}$ and Δ as the max-degree function then a σ-flag $(F, \theta) \in \overline{\mathcal{G}}^\sigma$ is a local-σ flag (definition 2.4) iff each connected component of F contains at least one black vertex or labelled vertex.*

Proof. See proof of lemma 3.10. $\square$

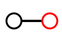
4.4.1. Objective Vector

First we want to express the size of $E_O(G)$ as a local flag vector. Note if $\mathcal{B} \subseteq G$ is the set of black edges with at least one black vertex then

$$2|E_O(G)| = \sum_{e \in \mathcal{B}} |\{e' \in \mathcal{B} \colon e, e' \text{ have common edge}\}|$$

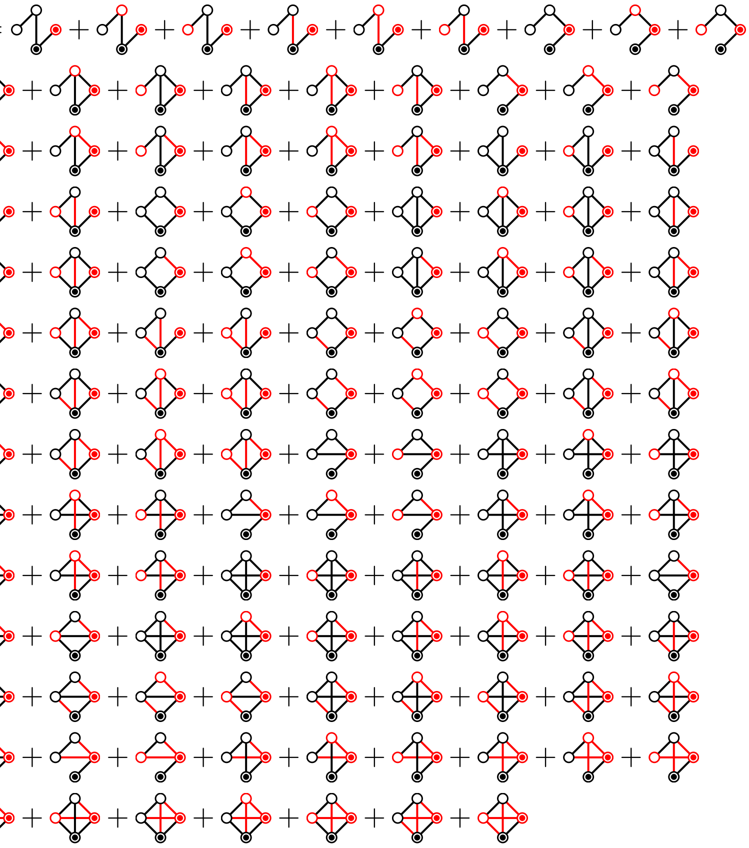
We can split $\mathcal{B}$ further into $\mathcal{B}_1$ and $\mathcal{B}_2$, the set of edges with one black and one red vertex and the set of edges with two black vertices respectively.

$$2|E_O(G)| = \sum_{e \in \mathcal{B}_1} |\{e' \in \mathcal{B} \colon e, e' \text{ have common edge}\}| \\ + \sum_{e \in \mathcal{B}_2} |\{e' \in \mathcal{B} \colon e, e' \text{ have common edge}\}|$$

The $\sigma_1 =$  is a local type by lemma 2.6. Clearly for any $e \in \mathcal{B}_1$ we can view G as a σ_1-flag where σ_1 is mapped to e. Call this G^e. Similarly any σ_1-embedding corresponds to precisely one $e \in \mathcal{B}_1$. Fix some $e \in \mathcal{B}_1$, then take some $e' \in \mathcal{B}$ such that e, e' have a common incident edge. Then the vertices of e, e' in G^e induce some subgraph of size 4 where the two unlabelled vertices (those corresponding to e') have a black edge and there is a single connected component. Conversely any such induced subgraph corresponds to exactly one $e' \in \mathcal{B}$. Hence we can count such e''s there are by counting how many such induced subgraphs there are. These subgraphs are easily enumerated.

Define $D(\circ\!\!-\!\!\circ) \in \mathcal{L}^{\sigma_1}$ to be the vector consisting of the sum of all local σ_1-flags of size 4 with a black edge between the unlabelled vertices and a single connected component:

$D(\circ\!\!-\!\!\circ) :=$



Then by the above $\rho(D(\circ\!\!-\!\!\circ); G^e) = \frac{|\{e' \in \mathcal{B} \colon e, e' \text{ have common edge}\}|}{\binom{\Delta(G)}{2}}$.

We can then use lemma 2.7 to note that

$$\begin{aligned}\rho([\![D(\circ\!\!-\!\!\circ)]\!]; G) &\sim \rho([\![\circ\!\!-\!\!\circ]\!]; G)\mathbb{E}_\theta[\rho(D(\circ\!\!-\!\!\circ), (G, \theta))] \\ &= \frac{1}{2}\frac{\sum_{e \in \mathcal{B}_1} |\{e' \in \mathcal{B} \colon e, e' \text{ have common edge}\}|}{\binom{\Delta(G)}{2}\binom{\Delta(G)}{2}}\end{aligned}$$

We can do the exact same construction with $\sigma_2 = \circ\!\!-\!\!\circ$. The only thing we have to note is that for any $e \in \mathcal{B}_2$ there are two ways to embed σ_2 so defining $D(\circ\!\!-\!\!\circ)$ in the same way we get a double count of $\rho(D(\circ\!\!-\!\!\circ); G^e) = 2\frac{|\{e' \in \mathcal{B} \colon e, e' \text{ have common edge}\}|}{\binom{\Delta(G)}{2}}$ hence applying lemma 2.7 again we get

$$\begin{aligned}\rho([\![D(\circ\!\!-\!\!\circ)]\!]; G) &\sim \rho([\![\circ\!\!-\!\!\circ]\!]; G)\mathbb{E}_\theta[\rho(D(\circ\!\!-\!\!\circ), (G, \theta))] \\ &= \frac{\sum_{e \in \mathcal{B}_2} |\{e' \in \mathcal{B} \colon e, e' \text{ have common edge}\}|}{\binom{\Delta(G)}{2}\binom{\Delta(G)}{2}}\end{aligned}$$

Therefore

$$\rho(2[\![D(\circ\!-\!\textcolor{red}{\circ})]\!] + [\![D(\circ\!-\!\circ)]\!]; G) \sim \frac{2|E_O(G)|}{\binom{\Delta(G)}{2}\binom{\Delta(G)}{2}}$$

so we pick $2[\![D(\circ\!-\!\textcolor{red}{\circ})]\!] + [\![D(\circ\!-\!\circ)]\!]$ as the vector we want to find a density bound for.

In general when applying the semidefinite method we will want a vector over local flags of some fixed size n. Luckily our class $\mathcal{G}$ consists of regular graphs so we can use the extension vectors from section 3.3 and corollary 3.8.2 to get a vector equivalent to $2[\![D(\circ\!-\!\textcolor{red}{\circ})]\!] + [\![D(\circ\!-\!\circ)]\!]$. We define then our final objective vector O to be

$$O := 2\left[\!\!\left[D(\circ\!-\!\textcolor{red}{\circ}) \cdot \left(\mathrm{ext}_1^{\circ\!-\!\textcolor{red}{\circ}}\right)^{n-4}\right]\!\!\right] + \left[\!\!\left[D(\circ\!-\!\circ) \cdot \left(\mathrm{ext}_1^{\circ\!-\!\circ}\right)^{n-4}\right]\!\!\right]$$

for some $n \geq 4$ yet to be chosen.

Lemma 4.6.

$$\rho(O; G) \sim \frac{2|E_O(G)|}{\binom{\Delta(G)}{2}\binom{\Delta(G)}{2}} \text{ as } \Delta(G) \to \infty \ \forall\ G \in \mathcal{G}$$

Proof. As outlined above □

4.4.2. Search Constraints

As in chapter 3, we want to collect some known elements of the semantic cone where each vector consists of local flags of some fixed size n. (Dually this corresponds with finding linear constraints on limit functionals). As $\mathcal{G}$ is a class of regular graphs we get the constraints of the form $[\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]$ from corollary 3.8.2 as we did in section 3.5.

First we find some constraints which encode the property that the minimum degree of the subgraph of $L(G)^2$ induced by black edges is $\geq (2-\eta)\Delta(G)^2$. In other words for each black edge e the number of other black edges in G with a common incident edge is $\geq (2-\eta)\Delta(G)^2$. Following how we defined $D(\circ\!-\!\textcolor{red}{\circ})$ we define $D'(\sigma)$ for $\sigma \in \{\circ\!-\!\circ, \circ\!-\!\textcolor{red}{\circ}\}$ as the sum of local σ-flags of size 4 where the unlabelled vertices form a black edge. Let $e \in E(G)$ be an edge of the form $\circ\!-\!\textcolor{red}{\circ}$. Then by lemma 4.3 we have:

$$\rho(D'(\circ\!-\!\textcolor{red}{\circ}); G^e) = \frac{\deg_{H[F]}(e)}{\binom{\Delta(G)}{2}} - o(1) \geq \frac{2\deg_{H[F]}(e)}{\Delta(G)^2} - o(1) \geq 2(2-\eta) - o(1)$$

Hence $\phi(D'(\circ\!-\!\textcolor{red}{\circ})) \geq 2(2-\eta)\ \forall\ \phi \in \Phi^{\circ\!-\!\textcolor{red}{\circ}}$. In particular then

$$\phi((D'(\circ\!-\!\textcolor{red}{\circ}) - 2(2-\eta)(\mathrm{ext}_1^{\circ\!-\!\textcolor{red}{\circ}})^2) \cdot \left(\mathrm{ext}_1^{\circ\!-\!\textcolor{red}{\circ}}\right)^{n-4}) \geq 0\ \forall\ \phi \in \Phi^{\circ\!-\!\textcolor{red}{\circ}}$$
$$\implies \phi\left(\left[\!\!\left[(D'(\circ\!-\!\textcolor{red}{\circ}) - 2(2-\eta)(\mathrm{ext}_1^{\circ\!-\!\textcolor{red}{\circ}})^2) \cdot \left(\mathrm{ext}_1^{\circ\!-\!\textcolor{red}{\circ}}\right)^{n-4}\right]\!\!\right]\right) \geq 0\ \forall\ \phi \in \Phi^{\emptyset}$$

by corollary 3.8.2 and lemma 2.8 giving us another constraint. We could add the same constraint for $\circ\!-\!\circ$ but in practice it suffices to only add this one.

Next we want to encode the fact that there are at most $2\Delta(G)$ black vertices. For any type σ we can define the *black vertex extension vector of size* k $\mathrm{ext}_{B,k}^\sigma$ as the sum of all local flags of size $|\sigma| + k$ where the unlabelled vertices are black vertices.

Lemma 4.7. *For a graph G let $B(G)$ denote the set of black vertices and $i \in \mathbb{N}$. Consider a σ-flag (G, η) then*

$$\rho(\mathrm{ext}^{\sigma}_{B,i}; (G,\eta)) = \frac{\binom{|B(G)\setminus \operatorname{im}\eta|}{i}}{\binom{\Delta(G)}{i}} \sim \left(\frac{|B(G)\setminus \operatorname{im}\eta|}{\Delta(G)}\right)^{i}$$

Proof. Consider a σ-flag (G,η). Given any set of black vertices $V \subseteq B(G) \operatorname{im}\eta$ of size i we can induce a subgraph $\operatorname{im}\eta \cup V$ of size $|\sigma| + i$ which is counted by some local σ-flag of size $|\sigma| + i$ where the unlabelled vertices are black. Similarly every $U = \operatorname{im}\eta \cup V$ which is isomorphic to some local σ-flag where the unlabelled vertices are black implies $V \subseteq B(G) \setminus \operatorname{im}\eta$. □

Corollary 4.7.1. *For any type σ $\phi(\mathrm{ext}^{\sigma}_{B,i}) \leq 2^{i}$ for all $\phi \in \Phi^{\sigma}$, $i \in \mathbb{N}$.*

Corollary 4.7.2. *For any type σ, $i \in \mathbb{N}$ we have $\phi(\mathrm{ext}^{\sigma}_{B,i}) = \phi(\mathrm{ext}^{\sigma}_{B,1})^{i}$*

Corollary 4.7.3. *For any type σ, $i \in [|\sigma|]$ $\phi(2\mathrm{ext}^{\sigma}_{i} - \mathrm{ext}^{\sigma}_{B,1}) \geq 0$ $\forall \phi \in \Phi^{\sigma}$. If σ is a local type then $\phi([\![2\mathrm{ext}^{\sigma}_{i} - \mathrm{ext}^{\sigma}_{B,1}]\!]) \geq 0$ $\forall$ $\phi \in \Phi^{\emptyset}$.*

Then we can add the constraint that

$$[\![2\mathrm{ext}^{\sigma}_{i} - \mathrm{ext}^{\sigma}_{B,1}]\!] \geq 0$$

for all local types σ of size $n-1$. Similarly for local type $\sigma = \circ$ we can add the constraints

$$\phi([\![(\mathrm{ext}^{k}_{B,1} - \mathrm{ext}_{B,k}) \cdot (\mathrm{ext}^{\sigma}_{1})^{n-k-1}]\!]) = 0$$

for all $2 \leq k \leq n-1$ which are all constraints over flags of size n.

We then we add the constraint using $\sigma = \circ$ that $\phi([\![\circ \cdot (\mathrm{ext}^{\circ}_{1})^{n-1}]\!]) \leq 2$ which holds because $[\![\circ]\!] = \circ$.

Finally then we add the constraints of the form $\phi([\![f^2]\!]) \geq 0$ for all $f \in \mathcal{L}^{\sigma}$ for all local types σ. As we did in chapter 3 we pick a subspace of $\mathcal{L}^{\sigma}$ of flags of some size m such that f^2 is a vector over flags of size n for each f in the subspace. In particular this means $2m - |\sigma| = n$.

Letting $\mathcal{B} = (F_1, \ldots, F_\ell)$ be a list of local σ-flags of size n we have the following optimisation problem:

$$\begin{array}{rl}
\max\limits_{x \in \mathbb{R}^{\ell}} & \phi_x(O) \\
\text{such that} & \phi_x(F_i) \geq 0 \ \forall\ i \in [\ell] \\
& \phi_x([\![\mathrm{ext}^{\sigma}_{i} - \mathrm{ext}^{\sigma}_{j}]\!]) = 0 \ \forall \text{ local types } |\sigma| = n-1, i, j \in [n-1] \\
& \phi_x\left(\left[\!\!\left[(D'(\circ\!-\!\circ) - 2(2-\eta)(\mathrm{ext}^{\circ - \circ}_{1})^2) \cdot \left(\mathrm{ext}^{\circ - \circ}_{1}\right)^{n-4}\right]\!\!\right]\right) \geq 0 \\
& \phi_x([\![2\mathrm{ext}^{\sigma}_{1} - \mathrm{ext}^{\sigma}_{B,1}]\!]) \geq 0 \ \forall \text{ local types } |\sigma| = n-1 \\
& \phi_x([\![(\mathrm{ext}^{k}_{B,1} - \mathrm{ext}_{B,k}) \cdot (\mathrm{ext}^{\sigma}_{1})^{n-k-1}]\!]) = 0 \text{ for } \sigma = \circ, \ \forall\ 2 \leq k \leq n-1 \\
& \phi_x([\![f^2]\!]) \geq 0 \forall f \in \mathcal{L}^{\sigma}_{m} \ \forall\ \sigma \text{ local type } \forall\ 2m - |\sigma| = n
\end{array}$$

As with chapter 3 we know from section 1.2.3 how to convert this to a rigorous SDP which will find an element $\lambda\emptyset - O \in \mathcal{C}^{\emptyset}_{\text{sem}}$. This program however has a dependency on the parameter $\eta > 0$. For any fixed value of η we can find an upper bound on $\phi(O)$ but we cannot use this approach to find a general closed form for such a bound in terms of η. In practice though to prove theorem 4.1 we need only to find some "good" value of η. We found such an η using a search method described in section 4.4.3.

We use our SDP software (appendix B) and choose $n = 5$ to find the following bound:

Lemma 4.8. *For $\eta = 0.2703$ we have*

$$10.644\emptyset - O \in \mathcal{C}^{\emptyset}_{\text{sem}}$$

We will not translate the SDP solution back to a separate proof here as we did in section 3.6, instead we leave details on verifying the construction to appendix C.

Now we can prove theorem 4.1.

Proof of theorem 4.1. Let $\eta = 0.2703$ and $\lambda \in \mathbb{R}$ be such that we have a bound of $\phi(O) \leq \lambda \ \forall \ \phi \in \Phi^{\emptyset}$. Then by lemma 4.6 we have

$$\lim_{\Delta(G)\to\infty} \frac{2|E_O(G)|}{\binom{\Delta(G)}{2}\binom{\Delta(G)}{2}} \leq \lambda.$$

This then implies that

$$\lim_{\Delta(G)\to\infty} \frac{|E_O(G)|}{\Delta(G)^4} \leq \frac{\lambda}{8}.$$

We use corollary 4.4.1 to translate from coloured graphs back to simple graphs and get that for any graph G, $H = L(G)^2$, maximal subset $F \subseteq E(G)$ such that $H[F]$ has minimum degree $\geq (2-\eta)\Delta(G)^2$ and some $f \in F$ we have

$$\lim_{\Delta(G)\to\infty} \frac{|E(H[N_{H[F]}(f)])| + o(\Delta(G)^4)}{\Delta(G)^4} \leq \frac{\lambda}{8} \implies \lim_{\Delta(G)\to\infty} \frac{|E(H[N_{H[F]}(f)])|}{\Delta(G)^4} \leq \frac{\lambda}{8}.$$

In order to apply the colouring lemma from Hurley, de Joannis de Verclos and Kang [17] we need to find $\sigma \in \mathbb{R}$ such that $|E(H[N_{H[F]}(f)])| \leq (1-\sigma)\binom{2\Delta(G)^2}{2}$ for $\Delta(G)$ large enough. We can use the fact that $\binom{2\Delta(G)^2}{2} = 2\Delta(G)^4 - o(\Delta(G)^4)$ to see that $\sigma = (1 - \frac{\lambda}{16})$ suffices.

Now we can apply Theorem 1.2 from Hurley et al [17] which states that for any $\iota > 0$ there is some Δ_0 large enough such that $\chi(H[F]) \leq (1 - \varepsilon(\sigma) + \iota)2\Delta(G)^2$ where $\varepsilon(\sigma) = \sigma/2 - \sigma^{3/2}/6$.

Evaluating $\varepsilon(\sigma)$ for $\lambda = 10.644$ from lemma 4.8 gives a bound of $\chi(H[F]) \leq (1.72981 + \iota)\Delta(G)^2$ Note then $2 - \eta = 1.7297 < 1.73$ so by the proof of Theorem 1.6 in [17] this proves $\chi(H) \leq 1.73\Delta(G)^2$ proving $\chi'_s(G) \leq 1.73\Delta(G)^2$ for $\Delta(G)$ large enough. □

Note 4.4.1. *A more careful validation of the dual solution from the SDP solver is required, but for the purposes of this thesis we considered it unnecessary as it is is not* *mathematically interesting. I give a brief discussion of how you can do this in appendix C. After such a validation of the floating point error you could in theory get a strictly below 1.73. We don't do this as any such result is negligibly better than 1.73 so not very interesting.*

4.4.3. Implementation Notes

- Our choice of η here is provided above without explanation. In practice we found an optimal η by a *ternary search process*. Ternary search allows you to numerically minimise a function $f(x)$ over x in some range assuming that f is convex over that range. In particular, our choice of η dictates our bound on $\chi'_s(G)$. If η is small then $2-\eta$ is large and our bound is always $\geq 2-\eta$. On the other hand if η is large then we expect our strong neighbourhood density bound to approach the $\frac{3}{2}$ bound of Bruhn and Joos [5]. Hence we hypothesise that the bound obtained by choosing $\eta \in [0,2]$ is a convex function. We apply the ternary search process to find that $\eta = 0.2703$ is a good value.
- The SDP program associated which finds the bound in lemma 4.8 is very slow. We know by experimentation that limiting our class $\mathcal{G}$ to only have red edges (those not in F) between black and red vertices gives the same bounds but is much faster. If further work is done on this problem with this SDP we recommend either proving we can assume this WLOG or at the very least ensure your derived bound is still true for the case where we don't have this limitation.

4.5. The Bipartite Case

In 1989 Faudree, Gyárfas, Schelp and Tuza made a conjecture that for the special case where G is bipartite we get a tighter upper bound on the strong chromatic index.

Conjecture (Faudree, Gyárfas, Schelp, Tuza [11])**.** *If G is bipartite then $\chi'_s(G) \leq \Delta(G)^2$.*

From what we can find, there are no papers showing a better bound than the general bounds for the special case of bipartite graphs. Luckily for us the method of local flags is easily specialised to the case of bipartite graphs meaning we can make the first targeted progress toward this conjecture. This is a big advantage of the method of local flags, the methods used in previous papers to bound the strong neighbourhood density do not seem to be so easily adaptable to a special case like this.

We claim the following theorem:

Theorem 4.9. *If G is bipartite then for $\Delta(G)$ large enough we have*

$$\chi'_s(G) \leq 1.6254\Delta(G)^2.$$

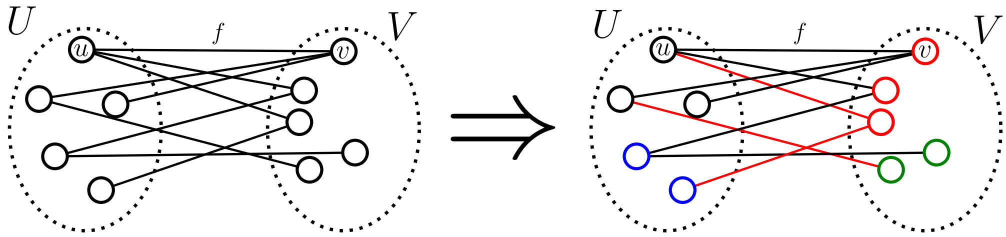


Figure 4.3.: Example coloured graph transformation

The first thing we do is adapt our reduction from section 4.3. Once again we can assume WLOG that G is regular. Assume we are given G, $F \subseteq E(G)$, $\eta > 0$ and $f \in F$ as before except now G is bipartite.

Once again we show how to convert (G, F, f) to an equivalent coloured graph. In this case we will use a $(4, 2)$-graph. First we pick some fixed bipartition of the vertices of G, $V(G) = U \sqcup V$. This is possible as G is bipartite and we can take any such partition. Let $f = \{u, v\}$ such that $u \in U$ and $v \in V$.

Colour the neighbours of u red, and the neighbours of v black. Note $N(u) \subseteq V$ and $N(v) \subseteq U$. Then colour the remaining vertices of U and V blue and green respectively. Finally colour edges black if they are $\in F$, otherwise red. See figure 4.3 for an example.

Now, we can adopt lemmas 4.2 and 4.3 as in the previous section. Then we can also use the following lemma which is identical to lemma 4.4:

Lemma 4.10. *Let $G, H = L(G)^2, F, \eta, f$ and G' be as above. Define $E_O(G') \subseteq \binom{E(G')}{2}$ as the set of pairs of non-incident edges e, e' in G' where e, e' are black edges with a common incident edge and at least one of the vertices of each edge is black **or red**. Then the number of edges in $H[N_{H[F]}(f)]$ where $H = L(G)^2$ is equal to $|E_O(G')| + o(\Delta(G)^4)$.*

Proof. Follow the proof of lemma 4.4. □

Corollary 4.10.1. *It suffices to bound the size of $E_O(G)$ over the class $\mathcal{G}$ of all regular $(4, 2)$-graphs where there are $\Delta(G)^2$ black vertices, $\Delta(G)$ red vertices, the subgraph of $L(G)^2$ induced by black edges has minimum degree $\geq (2 - \eta)\Delta(G)^2$ and the graph can be bi-partitioned into the components of blue/black vertices and red/green vertices.*

We therefore choose our graph class $\mathcal{G}$ as the one from corollary 4.10.1 and as with the previous section we get $\overline{\mathcal{G}}$ as the same class without the regularity requirement and the following description of local flags.

Lemma 4.11. *Given the class $\mathcal{G}$ and Δ as the maximum degree function a σ-flag $(F, \theta) \in \overline{\mathcal{G}}^\sigma$ iff each connected component of F contains at least one black, red or labelled vertex.*

Proof. See proof of lemma 3.10. □

4.5.1. Objective Vector

As with section 4.4.1 we'll define $D(\sigma)$ where $\sigma \in \{\circ\!-\!{\color{red}\circ}, {\color{blue}\circ}\!-\!{\color{red}\circ}, \circ\!-\!{\color{green}\circ}\}$ as the sum of all local σ-flags of size 4 where the unlabelled vertices form a black edge which is connected to the labelled vertices and at least one unlabelled vertex is red or black. Then we have:

$$\rho\left([\![D(\circ\!-\!{\color{red}\circ})]\!] + [\![D({\color{blue}\circ}\!-\!{\color{red}\circ})]\!] + [\![D(\circ\!-\!{\color{green}\circ})]\!]; G'\right) \sim \frac{|E_O(G)|}{\binom{\Delta(G)}{2}\binom{\Delta(G)}{2}}$$

by the same derivation as in section 4.4.1. Hence we define the objective vector O as

$$O := \left[\!\!\left[D(\circ\!-\!{\color{red}\circ}) \cdot \left(\mathrm{ext}_1^{\circ-{\color{red}\circ}}\right)^{n-4} \right]\!\!\right] + \left[\!\!\left[D({\color{blue}\circ}\!-\!{\color{red}\circ}) \cdot \left(\mathrm{ext}_1^{{\color{blue}\circ}-{\color{red}\circ}}\right)^{n-4} \right]\!\!\right] + \left[\!\!\left[D(\circ\!-\!{\color{green}\circ}) \cdot \left(\mathrm{ext}_1^{\circ-{\color{green}\circ}}\right)^{n-4} \right]\!\!\right]$$

for $n \in \mathbb{N}$ yet to be chosen and get the following result:

Lemma 4.12.
$$\rho(O; G) \sim \frac{|E_O(G)|}{\binom{\Delta(G)}{2}^2}.$$
as $\Delta(G) \to \infty$ for all $G \in \mathcal{G}$.

Proof. As above. □

4.5.2. Constraints

Following the pattern of this thesis we now list some constraints on the space of limit functionals and find ways to express them in the basis of flags of fixed size n. As $\mathcal{G}$ is a regular class we get the constraints of the form $\phi([\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]) = 0$ for all local types σ, $i, j \in [|\sigma|]$. We can add these for all local σ of size $n-1$.

We then define $D'(\sigma)$ as we did in section 4.4.2 for $\sigma \in \{\circ\!-\!{\color{red}\circ}, {\color{blue}\circ}\!-\!{\color{red}\circ}, \circ\!-\!{\color{green}\circ}\}$ to be the sum of all σ-flags of size 4 where the unlabelled vertex is a black edge connected to the labelled vertices, then again in the same construction find that (using lemma 4.3)

$$\rho(D'(\sigma); G^e) \geq 2(2-\eta) - o(1)$$

for any $G \in \mathcal{G}$, $e \in E(G)$ of type σ. Then as we did in section 4.4.2 we can add the following constraints to our list:

$$\phi\left(\left[\!\!\left[\left(D'(\sigma) - 2(2-\eta)(\mathrm{ext}_1^\sigma)^2\right) \cdot (\mathrm{ext}_1^\sigma)^{n-4}\right]\!\!\right]\right) \geq 0 \;\forall\; \phi \in \Phi^\emptyset$$

for all σ above.

Finally then we add some constraints to add the fact that there are $\Delta(G)$ black and red vertices. As we did in section 4.4.2 we define the *black and red vertex extension vectors of size k* denoted $\mathrm{ext}_{B,k}^\sigma$ and $\mathrm{ext}_{R,k}^\sigma$ respectively, as the sum of all local flags of size $|\sigma| + k$ where the unlabelled vertices are black and red respectively. Then as with lemma 4.7 we get the following result:

Lemma 4.13. *For a graph $G \in \mathcal{G}$ let $B(G)$ and $R(G)$ denote the set of black vertices and $k \in \mathbb{N}$. Then for a σ-flag $(G, \eta) \in \mathcal{G}^\sigma$ we have*

$$\rho\left(\mathrm{ext}^\sigma_{B,i}; (G,\eta)\right) = \frac{\binom{|B(G) \setminus \mathrm{im}\,\eta|}{i}}{\binom{\Delta(G)}{i}} \sim \left(\frac{|B(G) \setminus \mathrm{im}\,\eta|}{\Delta(G)}\right)^i$$
$$\rho\left(\mathrm{ext}^\sigma_{R,i}; (G,\eta)\right) = \frac{\binom{|R(G) \setminus \mathrm{im}\,\eta|}{i}}{\binom{\Delta(G)}{i}} \sim \left(\frac{|R(G) \setminus \mathrm{im}\,\eta|}{\Delta(G)}\right)^i.$$

Proof. Same as 4.7. □

Then as we always have $\Delta(G)$ black and red vertices we get the following:

Corollary 4.13.1. *For any type σ, $k \in \mathbb{N}$ we have $\phi(\mathrm{ext}_{B,k}) = \phi(\mathrm{ext}_{R,k}) = 1$ for all $\phi \in \Phi^\sigma$.*

Corollary 4.13.2. *For any type σ, $i \in [|\sigma|]$, $k \in \mathbb{N}$ we have $\phi(\mathrm{ext}^\sigma_1 - \mathrm{ext}_{B,k}) = \phi(\mathrm{ext}^\sigma_1 - \mathrm{ext}_{R,k}) = 0$. Therefore for any local type σ $\phi([\![\mathrm{ext}^\sigma_1 - \mathrm{ext}_{B,k}]\!]) = \phi([\![\mathrm{ext}^\sigma_1 - \mathrm{ext}_{R,k}]\!]) = 0$.*

We can add these constraints for all local types of size $n-1$.

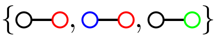
Define B_k, R_k for $k \in \mathbb{N}$ to be the graph of k black vertices and k red vertices respectively. Then

$$c(B_k; G) = c(R_k; G) = \binom{\Delta(G)}{k}$$

as the set of black vertices and set of red vertices are both independent. Hence $\phi(B_k) = 1$ for all $k \in \mathbb{N}, \phi \in \Phi^\emptyset$. We can then do the usual trick where we use extension vectors to project this equality into the space of flags of size n.

This leaves us finally with the following optimisation problem where $(F_1, \ldots, F_\ell)$ is the ordered collection of local flags of size n.

$$\begin{aligned}
\max_{x \in \mathbb{R}^\ell} \quad & \phi_x(O) \\
\text{such that} \quad & \phi_x(F_i) \geq 0 \ \forall\ i \in [\ell] \\
& \phi_x([\![\mathrm{ext}^\sigma_i - \mathrm{ext}^\sigma_j]\!]) = 0 \ \forall \text{ local types } |\sigma| = n-1, i, j \in [n-1] \\
& \phi_x\left(\left[\!\!\left[(D'(\sigma) - 2(2-\eta)(\mathrm{ext}^\sigma_1)^2) \cdot (\mathrm{ext}^\sigma_1)^{n-4}\right]\!\!\right]\right) \geq 0 \ \forall\ \sigma \in \{\cdots\} \\
& \phi_x([\![\mathrm{ext}^\sigma_1 - \mathrm{ext}^\sigma_{B,1}]\!]) = 0 \ \forall \text{ local types } |\sigma| = n-1 \\
& \phi_x([\![\mathrm{ext}^\sigma_1 - \mathrm{ext}^\sigma_{R,1}]\!]) = 0 \ \forall \text{ local types } |\sigma| = n-1 \\
& \phi_x\left(\left[\!\!\left[B_k \cdot \left(\mathrm{ext}^{B_k}_1\right)^{n-k}\right]\!\!\right]\right) = 1 \ \forall\ 1 \leq k \leq n \\
& \phi_x\left(\left[\!\!\left[R_k \cdot \left(\mathrm{ext}^{R_k}_1\right)^{n-k}\right]\!\!\right]\right) = 1 \ \forall\ 1 \leq k \leq n \\
& \phi_x([\![f^2]\!]) \geq 0 \forall f \in \mathcal{L}^\sigma_m \ \forall\ \sigma \text{ local type } \forall\ 2m - |\sigma| = n
\end{aligned}$$



We then use the SDP method (appendix B) again to find the following result:

Lemma 4.14. *For $\eta = 0.3746$ we have $4.0928\emptyset - O \in \mathcal{C}^{\emptyset}_{\text{sem}}$*

Then we can prove theorem 4.9 exactly as we proved theorem 4.1 in the previous section.

Proof. Let $\eta = 0.3746$ and $\lambda \in \mathbb{R}$ be such that we have a bound of $\phi(O) \leq \lambda \; \forall \; \phi \in \Phi^{\emptyset}$. Then by lemma 4.12 we have

$$\lim_{\Delta(G)\to\infty} \frac{|E_O(G)|}{\binom{\Delta(G)}{2}^2} \leq \lambda.$$

This then implies that

$$\lim_{\Delta(G)\to\infty} \frac{|E_O(G)|}{\Delta(G)^4} \leq \frac{\lambda}{4}.$$

We use corollary 4.10.1 to translate from coloured graphs back to simple graphs and get that for any bipartite graph G, $H = L(G)^2$, maximal subset $F \subseteq E(G)$ such that $H[F]$ has minimum degree $\geq (2-\eta)\Delta(G)^2$ and some $f \in F$ we have

$$\lim_{\Delta(G)\to\infty} \frac{|E(H[N_{H[F]}(f)])| + o(\Delta(G)^4)}{\Delta(G)^4} \leq \frac{\lambda}{4} \implies \lim_{\Delta(G)\to\infty} \frac{|E(H[N_{H[F]}(f)])|}{\Delta(G)^4} \leq \frac{\lambda}{4}.$$

In order to apply the colouring lemma from Hurley, de Joannis de Verclos and Kang [17] we need to find $\sigma \in \mathbb{R}$ such that $|E(H[N_{H[F]}(f)])| \leq (1-\sigma)\binom{2\Delta(G)^2}{2}$ for $\Delta(G)$ large enough. We can use the fact that $\binom{2\Delta(G)^2}{2} = 2\Delta(G)^4 - o(\Delta(G)^4)$ to see that $\sigma = (1 - \frac{\lambda}{8})$ suffices.

Now we can apply Theorem 1.2 from Hurley et al [17] which states that for any $\iota > 0$ there is some Δ_0 large enough such that $\chi(H[F]) \leq (1 - \varepsilon(\sigma) + \iota)2\Delta(G)^2$ where $\varepsilon(\sigma) = \sigma/2 - \sigma^{3/2}/6$.

Evaluating $\varepsilon(\sigma)$ for $\lambda = 4.0928$ from lemma 4.14 gives a bound of $\chi(H[F]) \leq (1.62538 + \iota)\Delta(G)^2$ Note then $2 - \eta = 1.6254$ so by the proof of Theorem 1.6 in [17] this proves $\chi(H) \leq 1.6254\Delta(G)^2$ proving $\chi'_s(G) \leq 1.6254\Delta(G)^2$ for $\Delta(G)$ large enough. □

4.6. Asymmetric Bipartite Strong Edge Colouring

In the previous section we found a bound on the strong edge colouring for bipartite $\Delta(G)$-regular graphs. A natural question to consider is what if the graph is bipartite, but each of the components have different maximum degrees. Brualdi and Quinn Massey made the following conjecture in 1993:

Conjecture (Brualdi, Quinn Massey [4])**.** *If $G = A \sqcup B$ is a bipartition then $\chi'_s(G) \leq \Delta(A)\Delta(B)$.*

In particular we will focus on the following problem: Given some $p \in (0,1]$ consider a bipartite graph $G = A \sqcup B$ where A is $\Delta(A)$-regular, B is $\Delta(B)$-regular and $\Delta(B) =$

$p\Delta(A)$, what asymptotic bound can be derived on the strong chromatic index of G? Clearly $p = 1$ is the regular bipartite case from the previous section.

Previously, we have been using the method introduced by Bonamy, Perrett and Postle [2] where we have a subset $F \subseteq E(G)$ with minimum degree $\geq (2-\eta)\Delta(G)^2$ in $L(G)^2$ and want to bound the strong edge density only within this subgraph. For this section we will revert back to the simpler method originally introduced by Molloy and Reed [19] where we find a general bound on the strong neighbourhood density, then use the colouring lemma of Hurley, de Joannis de Verclos and Kang [17] to get a bound on the chromatic number. The reason for this is that this method was more readily adapted. There is no reason to believe that the stronger method using the high degeneracy subgraph can't be adapted but we will see that this simpler method already reveals something very interesting about this problem: Namely that we get the same scalar coefficient in the bound for $\chi'_s(G)$ for every value of p we tested.

Theorem 4.15. *For $p \in [0.1, 0.2, 0.3, \ldots, 1]$ we get the following result: For $\Delta(G)$ large enough where $G = A \sqcup B$ is a bipartition with $\Delta(B) = p\Delta(A)$ we have the bound*

$$\chi'_s(G) \leq 1.6632\Delta(A)\Delta(B)$$

This is not a stronger result than theorem 4.9[1], but it is interesting that the different values of p do not give different bounds on $\chi'_s(G)$.

We will now give a brief description of how this was shown. We will skip the proofs as they follow exactly as in the previous sections with minor modifications.

4.6.1. Reduction

We will follow a similar approach to section 4.5. Let $G = A \sqcup B$ be as described above and let $e = \{a, b\} \in E(G)$ be given where $a \in A, b \in B$. We construct G' a 4-coloured graph where $N(a)$ is coloured red, $N(b)$ coloured black and the rest of A and B are coloured blue and green respectively in the same fashion as figure 4.3. The primary thing to note is that we are not looking at some subgraph induced by $F \subseteq E(G)$ as before so we do not need to introduce edge colours.

Note then $|N(a)| = \Delta(A) = \Delta(G)$ and $|N(b)| = \Delta(B) = p\Delta(G)$. Hence we take our class $\mathcal{G}$ to be 4-coloured bipartite graphs G' which have $\Delta(G)$ red vertices, $p\Delta(G)$ black vertices and the components of black/blue and red/green vertices form a bipartition $G = A \sqcup B$ where A, B are $\Delta(G)$-regular and $p\Delta(G)$-regular respectively.

Lemma 4.16. *For any bipartite $G = A \sqcup B$ where A is $\Delta(G)$ regular and B is $p\Delta(G)$, $f \in E(G)$ we can asymptotically bound the strong neighbourhood density of G at f by any asymptotic bound on $|E_O(G')|$ where G' is derived from G by the above process and $E_O(G') \subseteq \binom{E(G')}{2}$ is the set of pairs of non-incident edges e, e' with a common incident edge and at least one vertex of both e, e' is red or black.*

[1]This could likely be remedied by using the method of Bonamy et. al.

Now, these graphs $G \in \mathcal{G}$ are no longer regular so we cannot rely on the extension vectors from section 3.3 in the same way as we have been. However, we can modify our approach slightly to take care of this issue.

For any type σ consider vertex i. If i is black or blue (corresponding to component A) then we will still have $\phi(\mathrm{ext}_i^\sigma) = 1$ as A is $\Delta(G)$-regular. Otherwise i corresponds to a vertex in component B which is $p\Delta(G)$ regular. Hence $\phi(\mathrm{ext}_i^\sigma) = p$. However this implies that $\phi(\frac{1}{p}\mathrm{ext}_i^\sigma) = 1$. For this reason we define the *unit extension* at i to be

$$\mathrm{uext}_i^\sigma = \begin{cases} \mathrm{ext}_i^\sigma & \text{if } i \text{ black or blue} \\ \frac{1}{p}\mathrm{ext}_i^\sigma & \text{otherwise} \end{cases}$$

Lemma 4.17. *For any type σ, $i, j \in [|\sigma|]$ we have $\phi(\mathrm{uext}_i^\sigma - \mathrm{uext}_j^\sigma) = 0$ for all $\phi \in \Phi^\sigma$. In particular $[\![\mathrm{uext}_i^\sigma - \mathrm{uext}_j^\sigma]\!] \in \mathcal{C}_{\mathrm{sem}}^{\emptyset}$ if σ is a local type.*

This gives us an analogue of the $[\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]$ constraints we have been using in our other applications.

We then follow the approach of section 4.5.1 to define $D(\sigma)$ as the sum of local σ-flags of size 4 which are connected and at least one unlabelled vertex is black or red. This gives us an objective vector O:

$$O := \left[\!\!\left[D(\circ\!\!-\!\!\circ) \cdot \left(\mathrm{uext}_1^{\circ\!-\!\circ}\right)^{n-4} \right]\!\!\right] + \left[\!\!\left[D(\circ\!\!-\!\!\circ) \cdot \left(\mathrm{uext}_1^{\circ\!-\!\circ}\right)^{n-4} \right]\!\!\right] + \left[\!\!\left[D(\circ\!\!-\!\!\circ) \cdot \left(\mathrm{uext}_1^{\circ\!-\!\circ}\right)^{n-4} \right]\!\!\right]$$

which is defined in terms of $n \in \mathbb{N}$ as usual.

Lemma 4.18. *For $G \in \mathcal{G}$*

$$\rho(O; G) \sim \frac{|E_O(G)|}{\binom{\Delta(G)}{2}^2}$$

4.6.2. Constraints

First, we take the set of constraints $\phi([\![\mathrm{uext}_i - \mathrm{uext}_j]\!]) = 0$ from lemma 4.17.

Next we define an *black vertex extension* ext_B^σ and *red vertex extension* vector as in section 4.4.2. We then note that as there are $\Delta(G)$ red vertices and $p\Delta(G)$ black vertices we have $\phi(\mathrm{ext}_B^\sigma) = p$ and $\phi(\mathrm{ext}_R^\sigma) = 1$ for any type σ. We add the constraints then that $[\![\mathrm{ext}_1^\sigma - \frac{1}{p}\mathrm{ext}_B^\sigma]\!] = 0$ and $[\![\mathrm{ext}_1^\sigma - \mathrm{ext}_R^\sigma]\!] = 0$ for all local types of size $n-1$. Finally then if B_1, R_1 are the flags consisting of a single black and red vertex we know $\phi(B_1) = p$ and $\phi(R_1) = 1$ for all $\phi \in \Phi^\emptyset$. Hence $\phi([\![B_1 \cdot \mathrm{uext}_1^{B_1}]\!]) = p$ and $\phi([\![R_1 \cdot \mathrm{uext}_1^{R_1}]\!]) = 1$ are valid constraints we can add.

4.6.3. Results

Using the constraints from the previous section we can construct a valid SDP as in the other applications for some fixed $p \in (0,1]$. Solving this gives us a bound on the number of edges in the strong neighbourhood of the form $|E(H[N_s(e)])| \leq \lambda_p \Delta(G)^4$.

In order to apply theorem 1.2 from Hurley et. al. [17] we need an asymptotic bound of the form $|E(H[N_s(e)])| \leq (1-\sigma)\binom{\Delta(H)}{2}$ where $H = L(G)^2$. We can calculate that WLOG $\Delta(H) = 2p\Delta(G)^2$ so $\sigma = 1 - \lambda_p/2p^2$ suffices. Theorem 1.2 from [17] then gives an asymptotic bound

$$\chi'_s(G) \leq (1-\varepsilon(\sigma)+\iota)\Delta(H) = 2(1-\varepsilon(\sigma)+\iota)p\Delta(G)^2 = 2(1-\varepsilon(\sigma)+\iota)\Delta(A)\Delta(B)$$

Performing this process for varying values of p proves theorem 4.15.

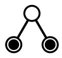
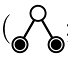
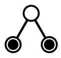
We found then that for every value of p that we tried we would get the same value for σ, meaning the bound on $\chi'_s(G)$ was also constant across p. This suggests that the bound λ_p is of the form Cp^2 for some constant $C \approx 1.13723$.

There is a lot of interesting ways to look further into this including:

- If we implement the high-degeneracy approach of Bonamy et. al. do we still get this constant bound across p?
- Can we generalise the SDP solution to prove that we do get the same sparsity bound for all $p \in (0,1]$, rather than just experimentally trying for fixed p?
- The results of the SDP programs suggest that the supremum on the strong neighbourhood density is some function $\lambda_p = Cp^2$. This is a topic worth investigating in it's own right, to find out if this is just a limitation of the method or if there is some underlying truth to this assertion.

5. Future Directions

We list here a few directions in which this work could be continued:

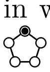
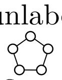
- We identified in note 2.1.1 that we always assumed Δ referred to the maximum degree function. There is no reason to believe that this is the only graph parameter which could be used. For example, consider the maximum codegree function Δ_2 (the max size of $N(u) \cap N(v)$ for any $u, v \in V(G)$). Then the flag  has the property that $c($$; (G, \eta)) \leq \Delta_2(G)$ for any embedding η. Hence choosing this function as our normalisation function makes  a local flag.

- In this thesis we focused entirely on flags as applied to simple graphs. However, flags have been applied more broadly, including both closely related concepts such as directed graphs [12], permutations [1] and discrete geometry [13]. An interesting line of investigation would be to see if this method can be adapted to those areas.

- In chapter 3 we investigated bounding pentagons in a triangle-free graph. This method could be adapted to other problems in this area, referred to as *generalised Turán numbers*. e.g. Bounding how many C_7s you can have in a C_5 free graph.

- In chapter 3 we showed that $P(G, v) \lesssim 1/8$ for G triangle-free, and showed that this bound is tight (lemma 3.7). A natural question to ask is whether this extremal graph is unique.

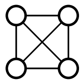
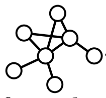
- In section 4.6.3 we list several ways the asymmetric bipartite strong edge colouring work could be continued.

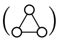
- We defined a local type (definition 2.8) so that we had a well defined, positivity preserving, averaging operator (lemma 2.8). However we can note that the averaging lemma (lemma 2.7) does not depend on σ being a local type. There is a sense in which we can still "unlabel" these flags. For example, if $\mathcal{G}$ is all graphs then  is a local -flag but  is not a local $\emptyset$-flag. However, if we introduce a new "density function" on $\emptyset$-flags as

$$\zeta(F; G) := \frac{c(F; G)}{|G|\binom{\Delta(G)}{|F|-1}}$$

Then we believe we can linearly extend this over $\mathbb{R}\mathcal{G}$ and define limit functionals. Then applying lemma 2.7 similar to the proof of lemma 2.8 we can show that $f \geq 0$ does imply $\zeta([\![f]\!]; G) \gtrsim 0$. We believe this approach could improve even further on the bound on our pentagon conjecture.

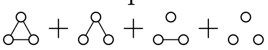
Popular summary

A graph is a set of points (vertices) and lines (edges). e.g.  or .

The study of graphs (Graph Theory) has been a central pillar of mathematics since Euler studied the bridges of Königsberg in 1736. One of the first things we can study is the fact that graphs can contain other graphs: Take some subset of the vertices and look at only those vertices and the edges between them and you end up with another graph. One can then ask, "How many copies of a graph F are there in a graph G"? A famous Dutch result is *Mantel's theorem* which says a graph of size n with no triangles () has $\leq \frac{n^2}{4}$ edges.

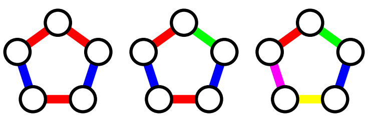
We can then ask, "What is the *density* of a graph F in another graph G"? i.e. How many copies of F are there in G, divided by the max possible number of copies of F. We call this $p(F; G)$. Consider a graph with n vertices and m edges: There are at most $\binom{n}{2}$ edges in a graph of size n, so we say the *edge density* is $p(\circ\!\!-\!\!\circ; G) = m/\binom{n}{2}$. We can re-write Mantel's theorem to say the density of edges in a triangle-free graph is at most $m/\binom{n}{2} \leq \frac{n^2}{4}/\binom{n}{2} \approx \frac{1}{2}$. It turns out these densities have some nice properties. For example, in any graph the density of edges ($\circ\!\!-\!\!\circ$) and non-edges ($\circ\ \ \circ$) always add up to 1. This is as the number of pairs of vertices which don't have an edge is $\binom{n}{2} - m$ hence:

$$p(\circ\!\!-\!\!\circ; G) + p(\circ\ \ \circ; G) = \frac{m}{\binom{n}{2}} + \frac{\binom{n}{2} - m}{\binom{n}{2}} = 1$$

In general, the sum of the densities of all graphs of some fixed size is always 1! If we wanted to come up with some nice notation to work with we might like to write this as $\circ\!\!-\!\!\circ + \circ\ \ \circ = 1$,  $= 1$, etc. It turns out then that we can rigorously create this notation (formally an algebra) out of graphs where statements like $\circ\!\!-\!\!\circ + \circ\ \ \circ = 1$ are true. Then, any statement we prove with this notation turns out to be a true statement about densities of graphs. These algebras are called *flag algebras*. These algebras are very easily manipulated by computers so we can use them in combination with computer search to prove new results about densities.

In this thesis we create a new type of algebra very similar to flag algebras which we use to make progress on some classic graph theory problems, as well as some new problems.

One classic problem we made progress on is the *strong edge colouring conjecture*: Given a graph we can assign a colour to each edge. A *proper* colouring is one where no edges which touch (share a vertex) have the same colour. A *strong* colouring then also requires edges which touch a common edge have different colours. We can then ask "What is the minimum colours needed for a strong edge colouring"? Erdős and Nešetřil conjectured in 1985 that you only need $1.25\Delta(G)^2$ colours where $\Delta(G)$ is the max degree (the max number of edges at any vertex). We prove, using our local flags, that you need at most $1.73\Delta(G)^2$ colours, which is the best known bound.



Non-proper, proper and strong edge colourings.

A. Notation

- $[k]$: The set $\{1, \ldots, k\} \subseteq \mathbb{N}$. $[0] = \emptyset$.
- $G[U]$: Induced subgraph.
- $|\sigma|, |F|$: The type size and flag size (number of vertices) respectively.
- $\cong$: Isomorphism.
- $N_G(v)$: The neighbourhood of v in G (subscript often omitted).
- $a := b$: Define a to be equal to b.
- $\operatorname{im} f$: The image of a function f.
- $X \succ 0$: X is positive semidefinite.
- coef_j: Coefficient function. Only well defined relative to some ordered basis.
- G^v for $v \in V(G)$ is the $\circ$-flag where v is labelled.
- $\emptyset$: Can denote the empty set or the empty graph.
- $\binom{X}{k}$: for X a set is the set of k-subsets of X.

We use the following asymptotic notation:[1]

- $f(n) \in O(g(n))$ means $\exists k > 0, \exists N$ such that $f(n) \leq kg(n) \; \forall \; n \geq N$. Alternatively $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$.
- $f(n) \in o(g(n))$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
- $f(n) \in \Omega(g(n))$ means $\exists k > 0, \exists N$ such that $f(n) \geq kg(n) \; \forall \; n \geq N$. Alternatively $\liminf_{n\to\infty} \frac{f(n)}{g(n)} > 0$.
- $f(n) \in \omega(g(n))$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.
- $f(n) \in \Theta(g(n))$ means $f \in O(g(n))$ and $f \in \Omega(n)$.
- $f(n) \sim g(n)$ means $f = (1 + o(1))g(n)$.

[1] We tend to write $f \in O(g(n))$ where it is more customary to write $f = O(g(n))$. This is as we prefer to avoid "one way equalities", this is just notational convention.

B. Software Implementation

Our implementations of the SDP programs are built on the flag software written by Rémi de Joannis de Verclos. In particular there are the following two projects:

- An implementation of the classic flag algebras based on work done during his PhD ([7]) https://github.com/avangogo/rust-flag-algebra.
- A library using the previous implementation written to test if the local flags idea would work in principle: https://github.com/avangogo/local-flags.

The modifications made to these projects for this thesis can be found in the following forks: https://github.com/EoinDavey/rust-flag-algebra and https://github.com/EoinDavey/local-flags. These implementations use Rust[1] to generate SDP programs which are then solved using CSDP[2], a standard open source SDP solver [3].

Note. *Rust is a good choice of language for this application. It is a strongly, statically typed language with high level constructions meaning we can define concepts such as flags, types etc directly in the code. It is however also extremely fast, meaning the bulk computation of matrices of coefficients over large bases is still extremely fast.*

The implementations for the applications in this thesis can be found at these locations:

- The “direct” pentagon bounding result, lemma 3.11: `local-flags/examples/bounded_pentagon.rs`.
- The stronger pentagon bounding result, lemma 3.13: `local-flags/examples/bounded_pentagon_alt_approach.rs`.
- The strong edge colouring conjecture bound, lemma 4.8: `local-flags/examples/strong_density.rs`.
- The bipartite strong edge colouring conjecture bound, lemma 4.14: `local-flags/examples/bipartite_strong_density.rs`.

At an extremely high level, the local flag algebra programs do the following steps:

1. Define the class of local $\emptyset$-flags by defining a sub-class of coloured graphs.
2. Compute an ordered basis of flags of fixed size n.

[1] https://www.rust-lang.org/
[2] https://github.com/coin-or/Csdp/wiki

3. Construct the objective vector over this basis.
4. Enumerate linear constraint vectors over this basis.
5. Calculate all local types σ and sizes k such that f^2 is a vector of size n if $f \in \mathcal{L}_m^\sigma$. Generate the coefficient matrices C_i^σ as described in section 1.2.
6. Write all these vectors and matrices in standard format (SDPA Sparse) such that it can be used with CSDP.
7. Call CSDP as a subroutine to solve the generated SDP problem.
8. (Optionally) read the generated SDP solution and generate a HTML report.

After these programs have completed and found an optimal bound for the SDP they will create a *certificate* file containing the optimal (dual and primal) solution in SDPA-Sparse format. This certificate contains the linear coefficients and PSD matrices needed to extract a rigorous proof that $\lambda\emptyset - O \in \mathcal{C}_{\text{sem}}^{\emptyset}$ where λ is our optimal bound and O is the objective vector. We give more information on recovering the proof in appendix C.

Example

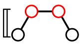
We show now some of the details of how the proof of lemma 3.11 is implemented. I'll skip all boilerplate and show only the code which is of high importance:

First we have some code describing the local $\emptyset$-flags which are those where each connected component contains a black vertex and there are no triangles or edges between black vertices.

```
impl SubFlag<G> for TriangleFreeConnected {
    const HEREDITARY: bool = false;

    fn is_in_subclass(flag: &G) -> bool {
        // Each connected component contains a vertex colored 0
        if !flag.is_connected_to(|i| flag.color[i] == 0) {
            return false;
        }
        // No black-black edges.
        if flag.content.edges()
            .any(|(u, v)| flag.color[u] == 0 && flag.color[v] == 0) {
            return false;
        }
        // No triangles.
        let n = flag.content.size();
        for (u, v, w) in iproduct!(0..n, 0..n, 0..n) {
            if u == v || u == w || v == w {
                continue;
```

```
            }
            if !flag.content.edge(u, v)
                || !flag.content.edge(u, w)
                || !flag.content.edge(v, w) {
                continue;
            }
            return false;
        }
        true
    }
}
```

We then create the basis of flags of size 5 and construct our objective vector $O = [\![$$\cdot \mathrm{ext}_1^\sigma]\!]$. We use a helper function `Degree::project` to construct the extension vectors.

```
pub fn main() {
    let n = 5;
    let basis = Basis::new(n);

    let c5_path: F = Colored::new(
            Graph::new(4, &[(0, 1), (1, 2), (2, 3)]), vec![0, 1, 1, 0])
            .into();
    let obj = Degree::project(&c5_path, n).untype();
    ...
}
```

Then we create a vector of inequalities, which we fill with the inequalities meaning $\phi(F) \geq 0$ for all flags, $\phi([\![B_k{\cdot}\mathrm{ext}_1^{B_k}]\!]) = 1$ and the regularity constraints $\phi([\![\mathrm{ext}_i^\sigma - \mathrm{ext}_j^\sigma]\!]) = 0$.

```
let mut ineqs = vec![flags_are_nonnegative(basis)];
for i in 1..=n {
    ineqs.push(ones(n, i).untype().equal(1.));
}

ineqs.append(&mut Degree::regularity(basis));
```

Finally we construct a `Problem` object with these linear constraints and all $\phi([\![f^2]\!]) \geq 0$ constraints[3]. We then use the convenience object `FlagSolver` to execute this problem and to print a HTML report.

```
let pb = Problem::<N, _> {
```

[3]These are referred to as Cauchy-Schwarz inequalities in the codebase

```
    ineqs,
    cs: basis.all_cs(),
    obj: -obj,
}
.no_scale(); // Means we don't have to rescale the output.

let mut f = FlagSolver::new(pb, "bounded_pentagon");
f.init();
f.print_report(); // Write some informations in report.html
```

Executing this code with `cargo run --release --example bounded_pentagon` prints output `Optimal value: 0.25000001` which is 1/4 up to floating point error as expected. The program writes the dual and primal solutions to disk as a file `certificate`, and the HTML report as `report.html`.

The report should allow a quick visual check that the flag vectors look as expected. This is most useful for small n, as for large n the flags vectors are too large.

Note. *The programs generate a lot of precomputed data which is stored locally in a directory* `data/`*. This can be quite a large amount of data, so users should take care to not run flag programs for* n *too large unchecked, as they could in theory use up all disk space.*

Notes on Correctness

There are a few features and properties of the flag-algebra library which are valid to use for classic flag algebras, but either are not valid for local flags or we did not prove that they were valid.

- Some operations in the flag-algebra library assume that the sum of flags of a fixed size is always equal to 1, which is not true for local flags. Having `HEREDITARY = false` in the graph class definition should disable these operations.
- The `rust-flag-algebra` library as written by de Joannis de Verclos has some behaviour where it modifies some of the Cauchy-Schwarz matrices. I could not identify why it does this but I think it was intended to reduce the size of the generated SDP programs by taking advantage of some invariants across flags. As I could not validate what this behaviour was doing I disabled it in my fork of this library. Hence it is important (unless you can identify what these modifications are doing) to use the modified version of the `rust-flag-algebra` library.
- There is a function `multiply_and_unlabel` in the `rust-flag-algebra` library which I did not prove was correct for local flags, but I suspect it is.

C. SDP Verification

When we apply an SDP solver to our problems as in appendix B we have claimed that the SDP solver will give us a matrix which proves the claimed bound. However, these SDP solvers are numerical algorithms meaning they can only give us a very close approximation of what they claim. This is in particular due to the mechanics of floating point arithmetic. To be rigorous we need to take these *approximate* solutions and find a true solution which achieves the same results. For some results we have a very tight bound (e.g. lemma 3.11 is best possible by lemma 3.7). In these cases we cannot afford to introduce any error when finding a true solution close to the approximate one. In other cases there is some room for “rounding” of the approximate solution.

In section 3.6 we showed the results of converting an SDP solution back into a standalone proof. This was done for elucidatory purposes, confirming that the SDP solutions do indeed correspond to a convex combination of elements of the semantic cone. In this first section we will discuss how this was done. However, this full conversion to a standalone proof is not necessary: In the second section we will discuss how you might verify that SDP solution is sound without as much manual effort.

The certificate files generated by the search programs can be found in the `certificates/` directory of the `https://github.com/EoinDavey/local-flags` repository.

Conversion to Standalone Proof

The SDP software outlined in appendix B contains some functionality to find a minimised set of constraints. It does this by running the search with all constraints, then removing them one at a time so long as the optimum solution is maintained. The function which does this is `FlagSolver::minimize`. There are several such methods implemented but I only verified the correctness of `minimize` as implemented in the `github.com/EoinDavey/rust-flag-algebra` fork of the `rust-flag-algebra` library.

Note. *CSDP will fail to run if there are any variables which have no associated constraints. For that reason I recommend using the* `.protect` *method to prevent the minimising process from removing the* $F \geq 0 \ \forall F$ *inequalities.*

This minimisation process reduces the size of the dual matrices that we need to deal with. However, if the search process itself is slow, then this process will be extremely slow. Hence it has limited applications for larger problems.

Given then a dual certificate (minimised or not) from our SDP program it will be in SDPA-Sparse format. This is a sparse block-diagonal format of describing a matrix.

These blocks then correspond to the coefficients associated with our inequalities. i.e. they are blocks of size 1 for linear inequalities or PSD matrices for the Cauchy-Schwarz $[\![f^2]\!] \geq 0$ inequalities. In the github.com/EoinDavey/rust-flag-algebra repository you will find a Python[1] script `sdpa_parse.py` which can interpret the SDPA problem input file and certificate file.

At this point the main goal is to remove floating point inaccuracy. In practice you will find that most coefficients are clearly floating point approximations of identifiable rationals, 1/4, 7/5 etc. To do this process of replacing floating point approximations with their associated rational I copied the `sdpa_parse.py` file and modified it to overwrite the associated coefficients, while asserting that we still get an objective function bound close to what we expected. This was a tedious manual process but after some time I had converted the coefficients to nice rationals.

Modifying the entries of PSD matrices may render them no longer PSD so I needed to check that each such matrix was still PSD. After obtaining these rational PSD matrices I wanted to extract the associated vectors ν_i which give us the elements $[\![\nu_i^2]\!]$ as described in section 1.2.3. The most direct way to do this is to compute the spectral decomposition, I used https://wolframalpha.com/ to do this symbolically. However, one of the PSD matrices gave me some "ugly" coefficients such as $\sqrt{11+\sqrt{89}}$. For that matrix I instead used the Z3 Solver[2] to find a nicer set of vectors ν_i such that $\sum_i \nu_i \nu_i^T$ gave the same matrix. This was just an aesthetic choice.

Note. *When researching this project I found the paper Exact Semidefinite Programming Bounds for Packing Problems by Dostert, de Laat and Moustrou [8] which describes a more mathematical approach to making a floating point SDP solution exact. I didn't use this approach but I think it's valuable to mention it here.*

Verification Without Conversion

Most of the results in this thesis are not "tight", in the sense that we are rounding the SDP solutions up. e.g. In lemma 4.8 we give the result that $10.644\emptyset - O \in \mathcal{C}_{\text{sem}}^{\emptyset}$, but in fact the SDP solver claims a better result of 10.643189. Rounding up gives us some room to maneuver with respect to the floating point, while still achieving the nice 1.73 bound of theorem 4.1.

For the results in this thesis (other than section 3.6) we did not go through a full verification as we felt it fell outside the scope of this research project, being more a manual process required for full publication of the results. However, I will briefly describe here how this process would work.

First, we should ensure that the SDP problems our programs generate are full precision. In practice we do this by expressing every problem in terms of rationals and then scaling expressions by common denominators to get SPD problems expressed entirely as integers. Then, the SDP solver will give us a solution as a matrix of floating point

[1] https://python.org/

[2] https://www.microsoft.com/en-us/research/project/z3-3/

numbers. We can interpret this matrix as a list of coefficients $c_i > 0$ and PSD matrices P_k (section 1.2.3) as implemented in `sdpa_parse.py`. Each of the coefficients can be interpreted from its floating point representation to some rational $\overline{c_i} > 0$. We could try to cast the entries of the matrices P_k in the same fashion but we need the resulting $\overline{P_k}$ to definitely be PSD. We know P_k is PSD up to floating point precision so we apply a numerical algorithm to compute it's spectral decomposition $P_k = \sum_j \nu_{k,j}\nu_{k,j}^T$. We then cast each of these vectors $\nu_{j,k}$ as rationals $\overline{\nu_{j,k}}$ and compute $\overline{P_k} = \sum_j \overline{\nu_{j,k}\nu_{j,k}}^T$ which by construction is definitely PSD.

Now we expect to get

$$\sum_i \overline{c_i} v_i + \sum_k \sum_j [\![\overline{\nu_{j,k}}^2]\!] = \lambda\emptyset - O + \varepsilon \in \mathcal{C}_{\text{sem}}$$

where O is our objective vector and v_i are the fixed elements of $\mathcal{C}^{\emptyset}_{\text{sem}}$. We expect that λ should be close enough to the bound we want to prove and ε is some small error vector which is left over. This proves that $\phi(O) \leq \lambda - \phi(\varepsilon)$ for all $\phi \in \Phi^{\emptyset}$. If we can bound the size of $\phi(\varepsilon)$ then we can get a proper bound on $\phi(O)$.

An alternative strategy is to try and minimise the size of ε. To do this we could take our list of known elements $v_1, \ldots, v_k$ and the vectors $[\![\overline{\nu_{j,k}}^2]\!]$ and treat this as a linear programming problem. There exist exact LP solvers over there which use rational arithmetic so don't suffer from floating point error. eg. https://www.math.uwaterloo.ca/~bico/qsopt/ex/.

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