1. The geometry of linear equations:

In the 1st lecture:

- The row pic
- The col. pic
- Matrix form:

example 2 equations with 2 unknowns

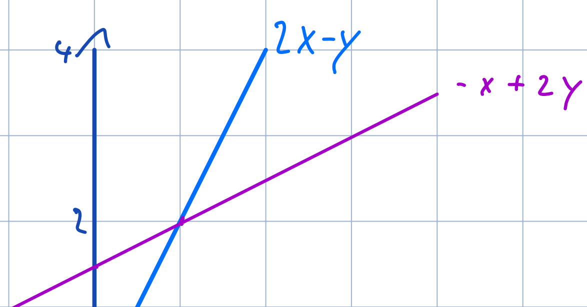
$$2x - y = 0$$
$$-x + 2y = 3$$

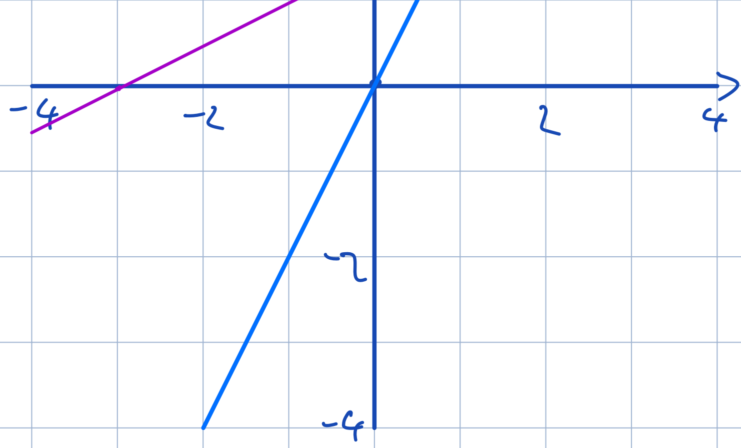
what is the co-efficient matrix?

A matrix is just a rectangular array of num

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$
$$Ax = b$$

lets step back to see the bigger picture here,
a row picture:
We take one row at a time and we plot all the points that satisfy the 1st equation: $2x - y = 0$





In linear equations all lines are straight and all points on the line solve one of the row eq.s
both eqs are solved at the point where they meet at $x = 1$, $y = 2$

The row picture plots the co-efficients of our system of equations.
It's a simple start.

Now we look at the col. pic., the cols of a matrix
We can think of our system as

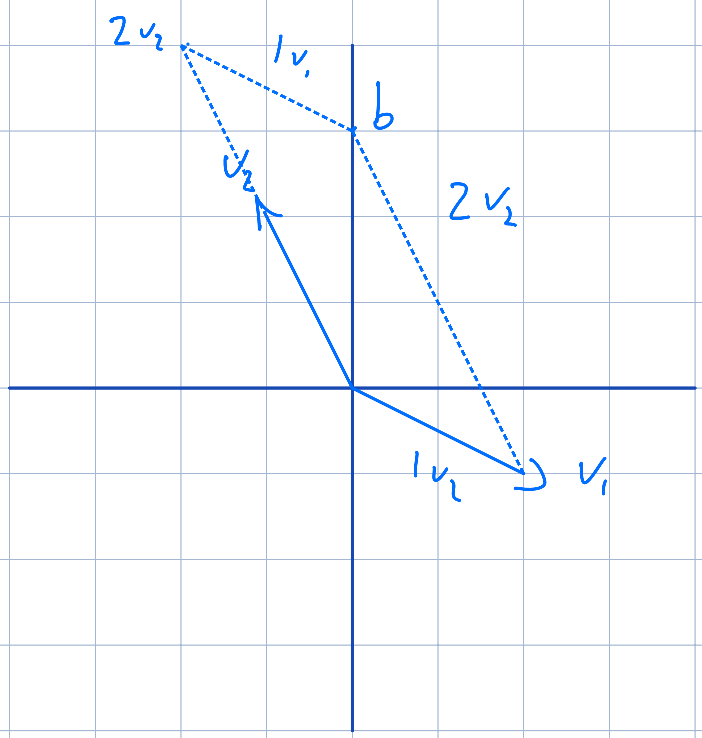
$$x \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$x \cdot v_1 + y \cdot v_2 = b$$

We think of this as a combination of the right amount of v_1 + the right amount of v_2 to get b.

This is called a linear combination of the cols of A. Take a multiple of col 1 and a multiple of col 2 and add.

Now for the geometry of the col pic



In the col pic we draw our col vectors then complete the parallelogram to find the values for x, y so that we finish on the answer b.

We know that we can solve $Ax = b$ with A but what if we took all the combinations of v_1, v_2 what if we had all the x's and all the y's?

We could get any right hand side we wanted because v_1, v_2 are independent and their linear combos span all of $\mathbb{R}^2$, it fills the whole plane.

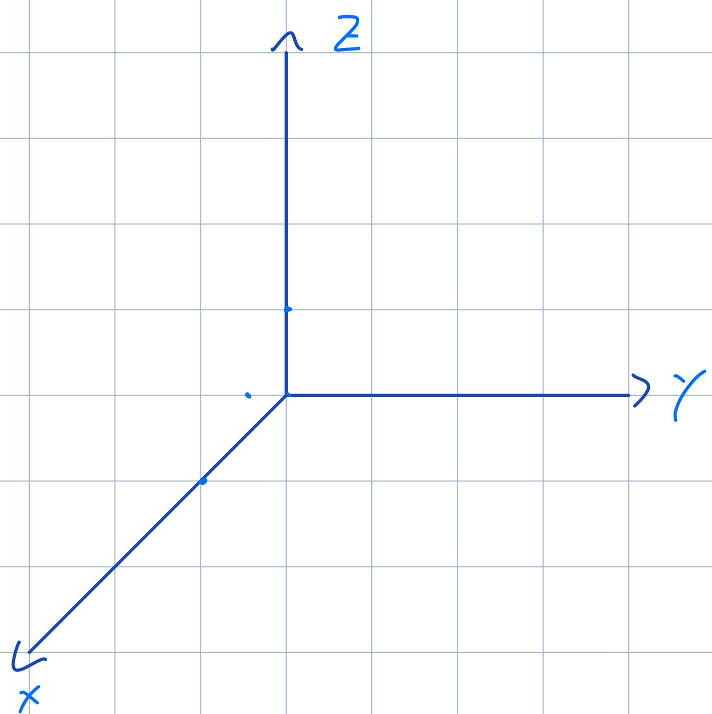
example 2: 3 equations with 3 unknowns.

$$
\begin{aligned}
2x - y \quad &= 0 \\
-x + 2y - z &= -1 \\
-3y + 4z \quad &= 4
\end{aligned}
$$

$$
\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}
$$

$A \qquad x = b$

Row Pic: We're in 3 dimensions X, Y, Z



We're no longer dealing with lines, we now draw 2D planes for our eqs.

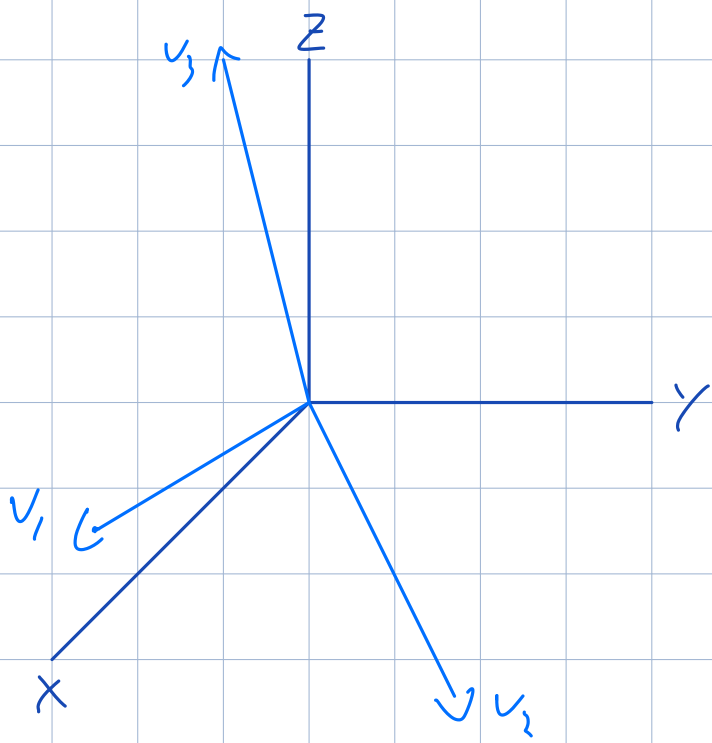
Very hard to draw, I'll plot it instead.

The point is: The planes can still cross each other at certain points, 2 independent planes will cross over a line and 3 planes will meet at a point, where A is solved.

The planes would not meet if they are parallel.
Anyway the row pic was simple for 2 dimensions
complex for 3 and for 4 and up even more
convoluted.

The 3 dim col pic:

$$x \cdot \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{v_1} + y \cdot \underbrace{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}}_{v_2} + z \cdot \underbrace{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}_{v_3} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$



In this problem finding
the right row combo is
easy because we
have the rhs in
the lhs
it's simply
[0, 0, 1]

For right hand sides the ans. is unlikely to
be as simple. Which makes the row pic would
still be very difficult in 3d or higher.

looking again at the big picture of L.A.

$Ax = b$ for all b's

If we change our b to be $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

Then our solution x would be $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

In the row picture we had 3 planes meeting at a point, in the col pic we had a combo of col vectors used to get to the point.

What about all b's? can we solve $Ax = b$ for all right hand sides?

- This is the fundamental algebra question for LA.
- Elimination is a way to find out

Do the linear combinations of the cols fill 3 dimensional space?

In the col picture what is really happening is $A \cdot \vec{x}$ Where A is a matrix, $\vec{x}$ is a vector and the result b is a combination of the columns of A.

linear combinations to find all possible b's

The matrix A does fill 3-D space,
It's non-singular and invertible

When would a 3x3 A not produce some 3x1 b
when 1 or more cols of A are dependent
When they share a plane e.g if col 3 = col 1 + Col 2. Col 3 wouldn't give us anything new our solutions would exist only on a plane in R^3. It would be singular, not invertable.

Imagine a v with 9 components and we had 9 equations, 9, unknowns
each one would be a $\vec{v}$ in 9D space and we would be looking at their linear combos.
We would be looking at the rhs b and asking if we can find a solution and, can we always do it?
It depends on our matrix, we could have cols that are not indy. If the $8^{th} = 9^{th}$ col.

then no.

The matrix form:

$$A \cdot x = b$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ how to multiply a matrix times a vector.

2 ways to do it.

1. the col way

$$1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$$x_1 \cdot [v_1] + x_2 \cdot [v_2] = b$$

2. 1 row at a time

$$1 \cdot 2 + 2 \cdot 5 = 12$$
$$1 \cdot 1 + 2 \cdot 3 = 7$$

Ax is a combination of the cols of A.

7. elimination w/ matrices

Using elimination to solve
a system of eqs

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\begin{aligned} x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \\ 4y + z &= 2 \\ Ax &= b \end{aligned}$$

1^{st} step of elim.

Multiply the 1^{st} eq by the right no. to subtract it from the second eq to remove the x-term

$eq1 \cdot 3 = 3x + 6y + 3z = 6$

$eq2 - eq1 = 0x$

because we are using the 1st term in the 1st eq to elim the x from the 2nd eq we req. to the 1st x val as the pivot.

$$A = \begin{bmatrix} \underline{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

We want our pivot to remove x from the second line so we multiply it by 3.

$$\rightarrow A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & \underline{2} & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

The 1st col is now cleaned up

The 2, 2 pos is now our next pivot

We can now elim 3, 2

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Now we have 3 equations 1 of which only has 1 unknown

We call this U

U is an upper triangular matrix with pivots in the diagonal

The whole purpose of elimination is to get from $A \rightarrow U$

The most common comp in scientific comp.

This is a good matrix with 3 pivots

How could this matrix have failed?
If we have a 0 in the entire 1st col.

If we had a 0 in a pivot we can swap rows to move the 0 out of a pivot pos

We could have had 0 in the 2nd pivot.
if we had a 6 in the 2-2 pos.
also a -4 in the 3-3 pos would have given a fully dep final row.
The matrix would not have been invertible.

Back substitution:
lets bring b in as an additional col.
We see this as an augmented matrix:
Our A is now

$$\overset{A}{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]} \rightarrow \overset{A'}{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]}$$

$$\rightarrow \begin{array}{c} \begin{array}{cccc} & A^2 & & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right] \end{array} \therefore$$

We can see now that
$5z = -10$
$z = -2$

now $2y + 4 = 6$
$y = 1$

$x + 2 - 2 = 2$
$x = 2$

c is the resulting vector from b after elimination.

The Matrix form
how do we denote the operations in Matrix form?
$A = LU$ where L is a matrix of row operations (elimination steps)

The big pic of elimination.
When we multiply A by a $\vec{v}$?

$$\begin{bmatrix} — \\ — \\ — \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

We get a combination of the cols of A. $3 \cdot col_1$, $4 \cdot col_2$ ...

There is a parallel in row operations:

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} - \\ = \\ - \end{bmatrix}$$

1 of row a + 0 of row b + 1 of row c

$1 \times 3 \cdot 3 \times 3$

$\therefore$ we can express our row ops used in elimination as 3 row multiplications

A matrix · a col is a col. a row times a matrix is a row

$$\underset{e_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\underset{e_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \underset{U}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}$$

We now have the pieces of the elim. matrix in 2 elementary matrices. E_{21}, E_{32}

$$E_{32} \cdot (E_{21} \cdot A) = U$$

L is the multiplication of the elementary matrices

$L = (E_{32} \cdot E_{21})$ - by moving the braces from one place to another we can express the equation in a simple format.

$L = (E_n \cdot E_{n-1} \cdot \ldots E_1)$

So $L = E_{32} \cdot E_{21}$ There is a better way

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

Notes on permutation matrices:

P is a matrix that exchanges rows

$$P \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

To exchange rows we use $P \cdot A$
to exchange cols we use $A \cdot P$
Multiplying on the left gives row operations
Multiplying on the right gives col ops.

The order of matrices is immutable.
We can't change the order of matrices

To go back to finding L we want to go back from U to A
$A = LU$

Inverses:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Multiplying $e_{32} \cdot e_{31}$ will give us the elimination matrix L
But there a better way to do this

We should think, not how do we get from A to U but how do we get from U back to A.

We use inverse matrices to reverse the steps taken to produce U

What then is the inverse of a Matrix?

All matrices so far have been invertible.

The inverse of a matrix is the matrix A^{-1} for a given A such that $A^{-1}A = I$

example: the matrix that undoes the step E_{21}

E_{21}^{-1} is below; it takes 3 of the 1^{st} row added to the second. giving us I

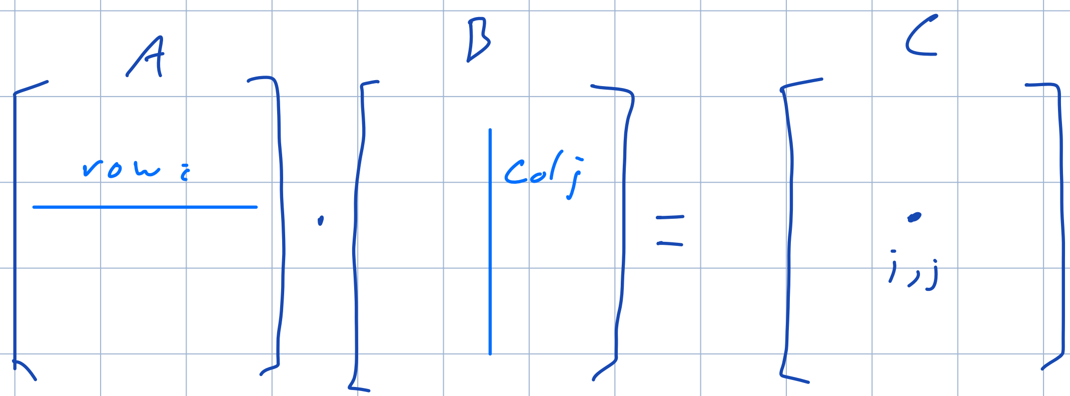
$$\overset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix undoes elimination.

Linear Algebra 3:

- 4 ways of matrix multiplication.
- Inverse of A, AB, A^T
- Gauss-Jordan elimination / A^{-1}

How to multiply 2 matrices



How do we calculate the val of C_{ij}?
We take the dot prod. of A row i with B col j

ex: $C_{34} = (\text{row } 3 \text{ of } A) \cdot (\text{col } 4 \text{ of } B)$

$$= (a_{31} \cdot b_{14}) + (a_{32} \cdot b_{24}) + \cdots (a_{3i} \cdot b_{4j})$$

$$= \sum_{k=1}^{n} a_{3k}\, b_{k4}$$

The sum goes along the row a and down col b

When are we allowed to multiply these matrices?
What are their shapes?

Can we multiply 2×3 by a 3×2?

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} (a_1x_1 + b_1x_2 + c_1x_3), & (a_1y_1 + b_1y_2 + c_1y_3) \\ (a_2x_1 + b_2x_2 + c_2x_3), & (a_2y_1 + b_2y_2 + c_2y_3) \end{bmatrix}$$

Yes, we get a $2 \cdot 2$ matrix because:
We do a dot product for each row of A
and each col. in B
The no. of cols in A must be equal to the no.
of rows in B for multiplication to work.

We can multiply matrices by thinking of A
as an $m \times n$ matrix and B as $n \times p$

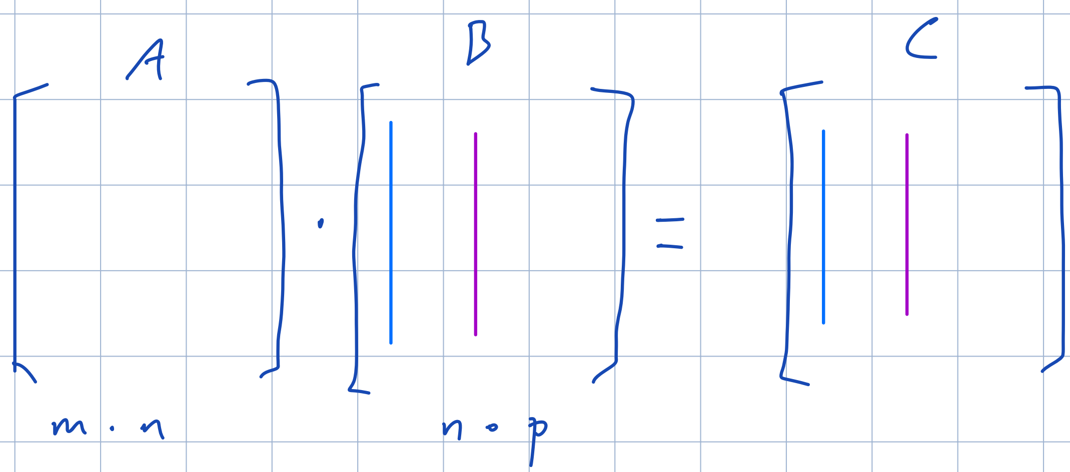
A has m rows and n cols.
B must have n rows. to be combatible

C, the output will have $m \times p$ output shape.
each $m \cdot p$ entry will be a dot prod. of
a single row with a single col.

$[m \times n] \cdot [n \times p] = [m \times p]$

Other ways to look at the same calculation.
looking at whole cols

looking at $A \cdot B = C$ again



looking at whole cols of C
we know how to multiply a matrix · a col
It's the combination of the cols of the matrix

Col 1 of C: $A \cdot B_1 = C_1$
$A \cdot B_2 = C_2$

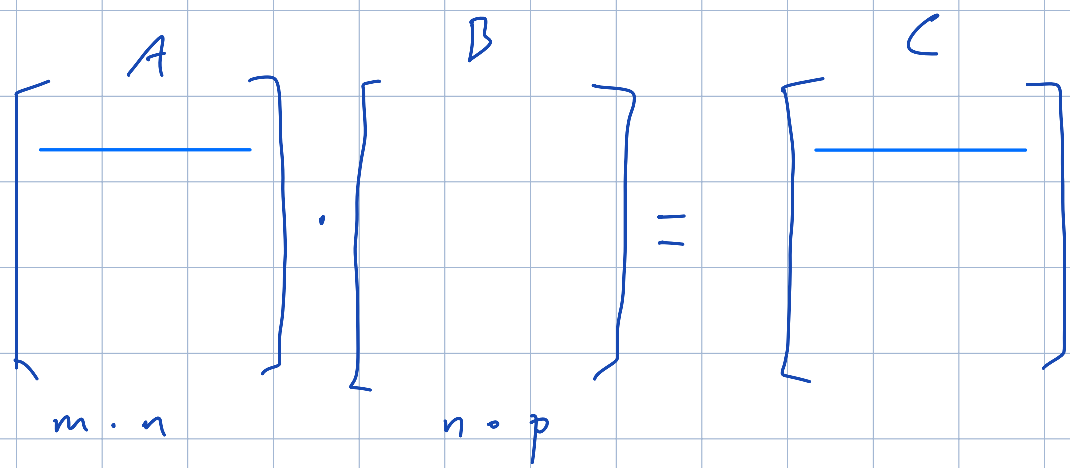
We can think of multiplying a matrix by a $\vec{v}$ in matrix multiplication as having p cols side by side giving us C with m rows for each p vectors.

C is simply A multiplied by the cols of B

The cols of C are combinations of the cols of A

The cols of A, C ∴ have len(m)

3rd way: Rows



Same as the col view except that a row of A takes the combination of the rows of B to give a row of C.

Rows of C are combinations of the rows of B

The 4th way: Cols · rows

A Row · col gave us a single no. at C_{ij} essentially a dot product.

Cols · Rows will give an $m \times p$ size matrix

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{array}{c} C_1 \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \end{array}$$

We do this for Col and row combo and ΣC_K

AB is a sum of (cols of A) · (Rows of B)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \end{bmatrix}$$

This is a special matrix as all the rows of the resulting answer lie on the line $\begin{bmatrix} 1 & 6 \end{bmatrix}$

The row space for this matrix is just a line.

The cols space is also a line

Matrix multiplication by blocks:

$$\underbrace{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]}_{A} \underbrace{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]}_{B} = \underbrace{\left[\begin{array}{c|c} A_1B_1+A_2B_3 & \\ \hline & \end{array}\right]}_{C}$$

Inverses: (Square Matrices)

Not all □ matrices are invertible
If it is invertible then there exists some other matrix A^{-1} such that:

$$A^{-1}A = I$$

- When does it exist
- How do we find it?

Normally Inverses are left inverses. For non-□ matrices they're left inverse but for □ matrices we have $AA^{-1} = I = A^{-1}A$

For square matrices a left inverse is also a right inverse

If $A^{-1}A$ exists for square matrices so does AA^{-1}

Cases w/ no inverse
- We call invertible matrices non-singular
Non-invertible matrices are singular
The singular case

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Why does A have no A^{-1}?
What I know:
- Rows are on the same line
- Cols. are on the same line

$\therefore$ Any combinations of the rows or cols of A will either be:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ - Zero

Or simply multiples of the 1st row (or 1st col for right inverse

Also determinant of $A = 0$

$$\det(A) = ad - bc = (1 \cdot 6) - (2 \cdot 3) = 0$$

Because both cols lie on the same line every combination will be on that line and we can't get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A square matrix won't have an inverse if we can solve $Ax = 0$ for a non 0 x

$$\overset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}} \cdot \overset{x}{\begin{bmatrix} 3 \\ -1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Key equation for inverse

We can't have an inverse if some combination of the cols gives 0 because

$$A^{-1}A = I, \quad Ix = x$$

yet if $Ax = 0$, $A^{-1}Ax = A^{-1}0 = 0 \therefore A^{-1}A \neq I$

Conclusion: Non-invertible singular matrice produces a 0 for some non-zero X

Invertible example:

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}$$

A has a non 0 $\det(A)$

$\det(A) = 7 - 6 = 1$

$$A^{-1}A = I = AA^{-1}$$

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} & \\ & \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -2 \end{array}\right]$$

$x_2 = -2$

$x_1 - 3(-2) = 1$

$x_1 = 7$

$$X = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\begin{array}{ccc} A & X & b \\ \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \cdot & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 1 \end{array}\right] \qquad x_2 = 1$$

$$x_1 + 3 = 0$$
$$x_1 = -3$$

$$\begin{array}{c} X \\ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} A & B & I \\ \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \cdot & \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

finding the inverse is like solving n-systems for square matrices

Gauss-Jordan idea:

Solving 2 equations at once

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

to solve them together we extend the augmented matrix idea

$$\begin{array}{cc} A & I \\ \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \end{array}$$

We start elimination

And once we create an upper triangular we continue to get the identity on the left hand side

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$
$$I \qquad A^{-1}$$

We now have A^{-1} on the rhs.

Why does this work?

$$E_n \dots E_2 \cdot E_1 [A \mid I] \to E[A \mid I]$$

We're applying the elimination steps to A to get I and applied to I we get $A^{-1} = E$

$$EA = I \therefore E = A^{-1}$$

$$E[A \;\; I] = [I \;\; A^{-1}]$$ Gauss jordan elimination.

Lecture 4. $A = LU$

Overview:

- The inv. of of a product AB
- Inv. of A^T
- Product of elimination matrices
- $A = LU$ (no row exchanges)

Inv of a product:

If we multiply $A \cdot B$ and we know A^{-1}, B^{-1}
what is AB^{-1}

let $B = A^{-1}$
$AB = I \qquad I^{-1} = I$
let $B = I$
$AI = A, \quad A^{-1} = A^{-1}I$
let $A = I$
$IB = B, \quad B^{-1} = IB^{-1}$

$$AB = A^{-1}B^{-1}$$

We know A and B are invertible
what's the inverse of AB?

To get the inv. of AB
we multiply $B^{-1}A^{-1}$
B^{-1} goes first here
We have to move the paren
around.

$$AB(B^{-1}A^{-1}) = I$$
$$ABB^{-1}A^{-1} = I$$
$$AIA^{-1} = I$$
$$AA^{-1} = I$$
$$I = I$$

And reversed:

$$B^{-1}A^{-1}(AB) = I$$
$$B^{-1}A^{-1}AB = I$$
$$B^{-1}IB = I$$
$$B^{-1}B = I$$

Inverse of A^T:

A is $\square$, invertible
If we transpose A, what is the inverse?
What is A^TA?

$$\underset{A}{\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}} \longrightarrow \underset{A^T}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}$$

$$A \quad \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ \hline 1 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \\ \hline 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right] \quad I \qquad I \quad A^{-1}$$

$$A^T \quad \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 2 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 \end{array}\right] \quad I \qquad I \quad (A^{-1})^T$$

The inv. of A^T is $(A^{-1})^T$

how to prove?

$I^T = I$

$(A^{-1}A)^T = I$

$A^T(A^{-1})^T = I$ $\therefore$ the inv of A^T is the transpose of A^{-1} : $(A^{-1})^T$

When transposing a product, we have to rev. the order $\therefore$ $(AB)^T = B^T A^T$

About elimination:

Elim is the right way to understand what the matrix has got. $A = LU$ is the most basic factorization.

Matrix form of elimination.

A look at how L connects A to U.

We have A an want to operate on it with our elementary matrix E_{ij}

The 1^{st} E will be E_{21} because we elim the 8 at pos 2,1

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

here we have $EA = U$

but we want $A = LU$

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$EA = U$$
$$E^{-1}EA = E^{-1}U$$
$$E^{-1}E = I$$
$$IA = E^{-1}U$$
$$A = E^{-1}U$$
$$E^{-1} = L$$

L is the inverse of the product of elimination steps

L is a lower triangular matrix

With 1's on the diagonal.

We can break this up further into LDU, where D has the pivots

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}$$

Examples w/ larger matrices

$E_{32} E_{31} E_{21} A = U$ (no row exchanges)

$A = \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_{L} U$

We multiply in rev. order because thats the order of removal from the LHS

L is the prod of inverses.

example:

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{31}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\therefore \overset{E_{32}}{\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}}\overset{E_{21}}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}=\overset{E}{\begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}}$$

(left of A)
$EA = U$

$$\overset{E_{21}^{-1}}{\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}}\overset{E_{32}^{-1}}{\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}}=\overset{L}{\begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix}}$$

(left of U)

$A = LU$

L works out to be a nicer factorisation as it introduces no unnecessary vals as exist in E

$A = LU$:

If no row exchanges, the multipliers go directly into L.

L comes out to be the ideal factorisation

How expensive is Matrix factorisation?

How many operations do we need to do?

Ops per $n \times n$ Matrix?

Worst case? We have to elim the entire lower triangle which is

$$n-1 + n-2 + \dots + n-n+2 + n-n+1$$

for $n = 2$ $n = 3$ $n = 4$

$O = 1$ $O = 1 + 2 = 3$ $O_{ps} = 3 + 3 = 6$

$n = 5$ $n = 6$ $n = 7$

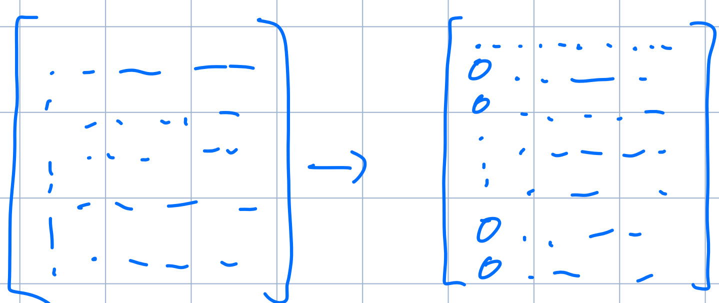
$O = 10$ $O_{ps} = 15$ $O_{ps} = 21$

$$= \sum_{k=1}^{n-1} k = n-1\left(\frac{n}{2}\right) = \frac{n^2 - n}{2}$$

let $n = 100$ $O_{ps} = \dfrac{10{,}000 - 100}{2}$

$$O_{ps} = \frac{9{,}900}{2}$$

$$O_{ps} = 4950$$



elimination is $O(n^2)$ bounded

Actual max ops is $\dfrac{n^2 - n}{2}$

looking at the idea that ops happen to each i,j

we have 100 ops to eliminate a single row

to begin $\therefore$ its 100^2 ops for the 1st col

So we have $n^2 + (n-1)^2 + (n-2)^2 + \dots (n-n+2)^2 + (n-n+1)^2$

$$= \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{(n^2+n)(2n+1)}{6}$$

$$= \frac{2n^3+2n^2+n^2+n}{6}$$

$$= \frac{1}{n^3} + \frac{1}{2}n^2 + \frac{1}{6}n$$

let $n = 2$		$= 5$
3	"	$= 14$
4,	Ops	$= 30$
5	"	$= 55$
6	"	$= 91$
7	"	$= 140$

$$n = 4 = \frac{1}{3}(4^3) + \frac{1}{2}(16) + \frac{4}{6}$$

$$= \frac{1}{3}(64) + 8 + \frac{2}{3}$$

$$= 21 + 8 + 1 = 30$$

When doing elim we also operate on the RHS

$$\left[\begin{array}{c|c} & b \\ & - \\ & - \\ & - \\ & \vdots \\ & - \end{array}\right]$$

and the cost of ops on b is

$(n-1) + (n-2) + \dots + 2 + 1$

$= \frac{n^2}{2}$

Including row exchanges:
Row exchanges are needed if we have a 0 in a pivot position.

Transposes and permutations
Permutations are used to do row exchanges

example a 3 by 3 P for 2 row exchanges

$$P_{231} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Others:

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{213} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{312} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

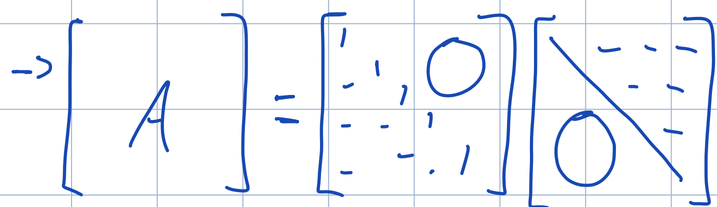
Any $P \cdot P = P$, The set of P matrices is closed under multiplication. $(n \times n)$

The inv. of any P is its transpose.

5. Transposes, Permutations and n-dimensional spaces

Permutation matrices P exec. row exchanges

Reminder $A = LU$



This assumes no row exchanges, so how do we account for row exchanges?

If we think of the good case w/ no row exchanges as $IA = LU$

here we can think of I as the non-altering permutation Matrix

How Matlab handles elim is to do row exchanges on not only 0-pivots but to identify numerically small pivots and exchange them for larger pivots

Matlab doesn't like small no.s.

Elimination w/ row exchanges becomes $PA = LU$

For any invertible A : $PA = LU$

* P = I w/ reordered rows

No of possible P's for $n \times m$ cols?

$3 \cdot m = 6$, $4 \cdot m = 24$

$4 \cdot 4$ matrices have 24 P's   1, 2, 6, 24

$n \times n$ will have $\prod_{k=1}^{n} k = n!$

Counts possible reorderings

P is always invertible and $P^{-1} = P^T$

$P^T P = I = P^{-1} P = P P^{-1}$

$n!$ Permutations

Transposes:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

$3 \cdot 2$ $2 \cdot 3$

Transpose general formula = $(A^T)_{ij} = A_{ji}$

Symmetric matrices

In S the transpose $S^T = S$

There is a reflection along the diagonal

$$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 3 & 9 \\ 7 & 9 & 4 \end{bmatrix}$$

When do we get symmetric matrices?

When we multiply any $A^T \cdot A$

e.g.

$$\underset{R}{\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

how do we know that $R^T R = S$?

Take the transpose of S

$$S^T = (R^T R)^T = R^T R^{TT}, \quad R^{TT} = R$$

$$\therefore R^T R = (R^T R)^T$$

Vector Spaces:

What are vector spaces?

What are subspaces?

Think about the vector ops we can do?

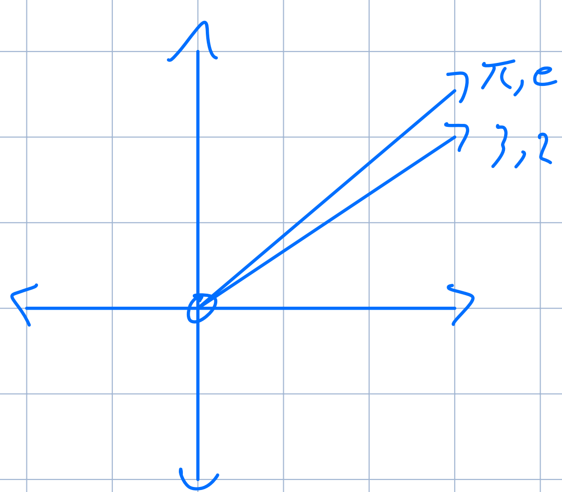
Adding and multiplication within $\vec{v}$ spaces

Vector Space

A space is a collection of vectors where performing vector addition and scalar multiplication stays within the space

examples: R^2 = all 2D real $\vec{v}$'s = X, Y plane

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}$$



What isn't a $\vec{v}$ space:

If we have a set S with vectors $s_1, \dots s_n$ where S_k is in R_2 except for $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If we multiply any S_i by $0 \rightarrow 0 \cdot S_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is not $\in S \therefore S \neq V$

$\mathbb{R}^3$ = All vectors with 3 components

e.g. $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

$\mathbb{R}^n$ = All vectors with n-real components

Definition of a v-space.

key
1. Closed under addition $x_1 + x_2 \in V$
2. Closed under scalars $ax_1 \in V$
3. Closed under $ax_1 + bx_2$

A. Commutative: $v + w = w + v$

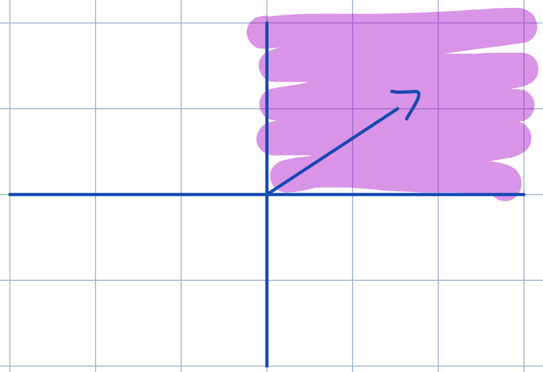
B. Asociative: $(v + w) + u = (u + w) + v$

C. Includes zero vector: There exist $Z \in V: v + Z = v$

D. Additive inverse: for each $v \in V: -v \in V: -V + V = Z$

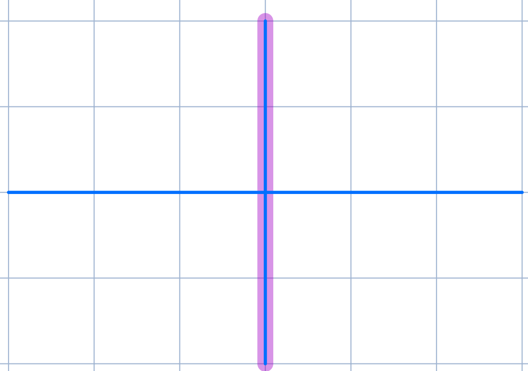
E. Distribution of scalar multiplication: $a(u + w) = au + aw$

Example of a non-Vect space



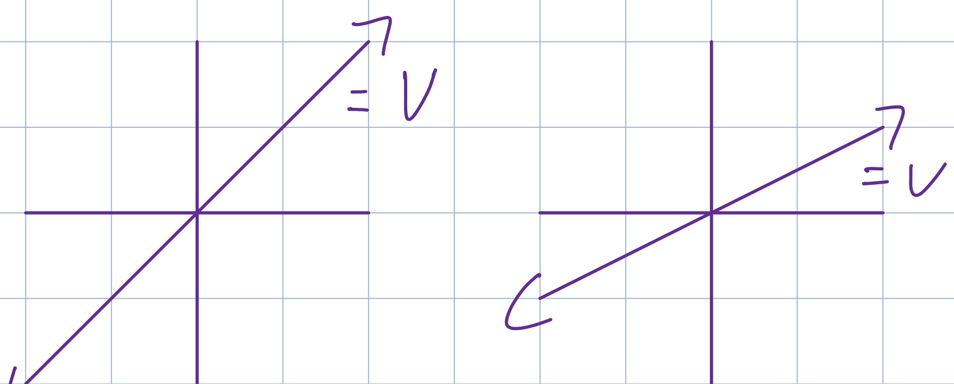
This would be closed under addition but not negative scalars

We can have vector spaces within R^n however example if we have R^2 and we cover the line $\begin{bmatrix} 0 \\ y \end{bmatrix}$ we can say that we span a line in R^2



Any linear combo of $\begin{bmatrix} 0 \\ y \end{bmatrix}$ for any y will stay in the line

It can be any line in R^2, that crosses the origin



, etc.

We can say that the vectors span a line in R^2

All possible subspaces in R^2:

1. All of R^2
2. Any line through the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
3. The zero vector Z

All 3 of these possibilities satisfy the definition of a subspace

All possible subspaces in R^2:

1. All of R^3
2. Any plane through the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
3. Any line through the origin
4. The zero vector Z

How do we get a subspace from a matrix?

- One way is from the cols - The Col Space

  Cols are in R^3

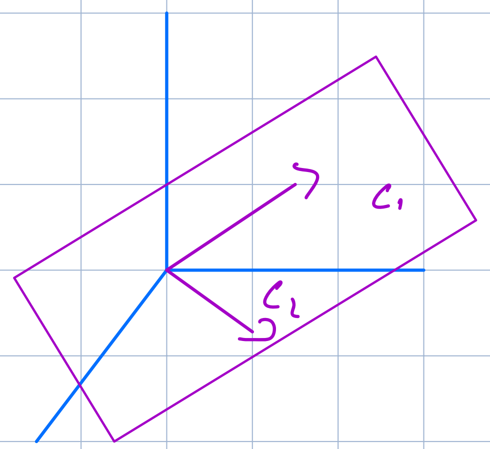
  To make a subspace from the cols of A we to be able to take linear combos of the cols of A

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$c\begin{bmatrix}1\\2\\4\end{bmatrix} + d\begin{bmatrix}3\\3\\1\end{bmatrix} = V$$

How do we know what space the col space of A spans: $C(A)$

Can't manually eval all linear combos



Geometrically we can fill out a plane in R^3

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6. Vector space and subspaces
   - Cols space of A: solving $Ax=b$
   - Null space of A

What is a vector space:

A bunch of vectors where any linear combo. stays in the v. space.

$v+w \in V$, $cv \in V$, $cv+dw \in V$

Examples:
- All of R^3
- A plane through the origin in R^3 – Subspace
- A line through the origin – Subspace

If we have 2 subspaces P, L in R^3
P is a plane, L is a line

$P \cup L$: all $v \in P$ or L
Is $P \cup L$ a subspace?
No, most vectors aren't in $P \cup L$
$v + w$ would likely not be in $P \cup L$

$P \cap L$: all $v \in P$ and L
Yes because it's either:
The 0 vector or it's L

The intersection of any 2 subspaces is also a subspace.
$S \cap T$ is a subspace

The col space of A

$$\begin{array}{c} A \\ \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \end{array}$$

The $C(A)$ is a subspace of R^4
$C(A)$ has the cols of A and all linear combos.

How big then is $C(A)$?
Does it cover R^4?
No, but how do we know and how big is it?

The question we are trying to answer is:
for the equation $Ax = b$
When can we solve x for b?
for which b can we get a sol x?

We have 4 equations but only 3 unknowns
There are some points in a 4D plane we can't reach.

$$\begin{array}{c} A \\ \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \end{array} \cdot \begin{array}{c} X \\ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \end{array} = \begin{array}{c} b \\ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \end{array}$$

We can't solve <u>all</u> b's but we can solve a subspace which is the col space of A.

We can solve any b that is a combination of the cols of A.

b's we can solve

$$A \cdot \begin{bmatrix}0\\0\\0\end{bmatrix} = \overset{b}{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}, \quad A\begin{bmatrix}1\\0\\0\end{bmatrix} = \overset{b}{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}}, \quad \text{etc.}$$

Any combo of cols.

We can solve $Ax = b$, when b is in the col space of A.

Dependence and Independence:

$$\overset{A}{\begin{bmatrix}1 & 1 & 2\\2 & 1 & 3\\3 & 1 & 4\\4 & 1 & 5\end{bmatrix}}$$

The cols of A are not independent:

$x = \begin{bmatrix}1\\1\\0\end{bmatrix} \equiv x = \begin{bmatrix}0\\0\\1\end{bmatrix}$

The 3^{rd} col. is a combo of the 1^{st} 2

1, 2 are pivot cols.

A is a 2D subspace of R4.

Null space:

If the col space contains all b's for $Ax=b$ then the null space contains all sols. to $Ax=0$

Also a subspace / vector space.

The $N(A)$ contains x's, not b's

$N(A) \in R^3$

Examples: $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} n \\ n \\ -n \end{bmatrix}, \begin{bmatrix} -n \\ -n \\ n \end{bmatrix}$

Written as $c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ | $N(A)$ is a 1D Line $\in R^3$ going through the origin.

How do we know that it is a subspace?

- Check that our sols. to $Ax=0$ always give a subspace:

If $A(v)=0$, $A(w)=0$ then:

$$A(v+w) = A(v) + A(w)$$

$$Av = 0$$

$$Aw = 0$$

$$A(v+w) = 0$$

This is the distributive law of vector spaces

So what's the point of a Vector space?

$$\overset{A}{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}} \cdot \overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}$$

What sols do we have for $Ax = b$?

Do they form a subspace?

No, they break the rules of spaces.

The $\vec{OV}$ isn't a sol, if $Av = b$ then $2Av = 2b$

The sols for b are a plane/line but it does not cross the origin

7. Solving $Ax = 0$

Pivot vars and special sols:

How to compute the $C(A)$, $N(A)$

$$\overset{A}{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}}$$

4x3

Col 2 is not indy

indy: Simple to see w/ row elimination

The 3rd row is not indy

Elim for the rectangular case

$$\overset{A}{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}} \to \begin{array}{l} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \end{array} \begin{array}{l} \\ -2\text{ row }1 \\ -3\text{ row }1 \end{array}$$

A_{11} was our first pivot but now we have 0 in E_{22} and E_{32} $\therefore$ we look at the next col to get a pivot @ E_{23}

Elimination does not change the null space because the sols to the system don't change.

The $C(A)$ however does change.

$$\overset{A}{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \overset{U \text{ upper echelon}}{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \begin{matrix} \\ \\ -1 \cdot \text{row}2 \end{matrix}$$

U has 2 pivots u_{11}, u_{23}

A then is said to have a rank of 2 as elim prods 2 pivots

U is in upper echelon form (upside down staircase)

We only say upper triangular when the Matrix is square.

The rank $r(A)$ = no. of pivots used in elim.

We now have U, and we cal solve $Ux = 0$ to find our null space because elim does not affect the solutions x

$$\overset{U}{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \cdot \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2 pivot cols

+2 free cols

The non-pivot cols are free. Can be freely assigned any val for x_2, x_4

We then solve for x_1, x_3

X-particular: A special sol using the pivot vals to solve $Ux = 0$

$$X = \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix}$$

Our equations w/ $Ux = 0$ now become

$$x_1 + 2(1) + 2x_3 + 2(0) = 0$$
$$2x_3 + 4(0) = 0$$

$\rightarrow \quad 2x_3 = 0$

$\hookrightarrow \quad x_1 + 2 = 0$

$\quad x_1 = -2$

$$X_{p_1} = c \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X_{p_2} = \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

$$x_1 + 2x_3 + 2 = 0$$
$$2x_3 + 4 = 0$$
$$2x_3 = -4$$
$$x_3 = -2$$
$$x_1 - 4 + 2 = 0$$
$$x_1 = 2$$

$$X_{p_2} = d \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

We now have a basis for our $N(A)$

$$N(A) = c \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

This is the sol for $Ux = 0$, $Ax = 0$

If the rank of $A = 2$ we will have $n-2$ special solutions of $Ax = 0$

We have $n-r$ free variables

To find all sols for $Ax = 0$

- Do elim
- find pivots, free
- Solve for special sols.

Reduced row echelon form: U with all pivots reduced to 1, all possible 0's above and below diagonal

Rref has 0's above pivots

Pivots = 1

$$\underset{A}{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \underset{U}{\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}$$

R tells us the pivot rows and cols with I in the diag.

R also has the special sols in the rows

R takes the form $\begin{bmatrix} I & F \\ 00 & 00 \end{bmatrix}$

I here is the pivot cols, F is the free cols

$$N(A) = \overset{N}{\begin{bmatrix} -F \\ I \end{bmatrix}} = \text{Nullspace Matrix}$$

$$\overset{R}{\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}} \cdot \overset{N}{\begin{bmatrix} -F \\ I \end{bmatrix}} = \begin{bmatrix} -F \\ 0 \end{bmatrix} + \begin{bmatrix} F \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Rx = 0 = RN$$

Second example:

$$\overset{A^T}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \overset{U}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}} \rightarrow \overset{rref}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Pivots $= 2$, $r(A) = r(A^T)$

The rank of A is equal to the rank of A^T

Find the nullspace:

$$\begin{array}{c} R \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} X \\ \begin{bmatrix} x_1 \\ x_2 \\ F \end{bmatrix} \end{array} = \begin{array}{c} 0 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

2 pivots, 1 free var
= 1 special sol: set f to be 1
$x_1 = -1$
$x_2 = -1$

If f was 0 we would get all 0's

$$x_p = c \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = N(A)$$

8. $Ax=b$ row reduced form R

- Completing the sol. of $Ax=b$
  ↳ If it has a sol.

Identify:

- Do we have a sol?
- How many?

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\begin{array}{rl|l} & x_1 + 2x_2 + 2x_3 + 2x_4 & b_1 \\ + & 2x_1 + 4x_2 + 6x_3 + 8x_4 & b_2 \\ \hline = & 3x_1 + 6x_2 + 8x_3 + 10x_4 & b_3 \end{array}$$

In this example we have solutions only when $b_1 + b_2 = b_3$ e.g $\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$, if b_3 is anything other than $b_1 + b_2$ we do not have a sol. $\vec{x}$

Elim w/ augmented matrix

$$\overset{b}{\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}} \quad \overset{A}{\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]} \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right] = \overset{b}{\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}}$$

$$\left[\begin{array}{cccc|l} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 - b_2 + 2b_1 = b_3 - b_1 - b_2 \end{array}\right] \quad = \begin{array}{c} b \\ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \end{array}$$

$$0 = b_3 - b_1 - b_2$$
$$b_3 = b_1 + b_2$$

Here we have a solution

The condition of solving $Ax=b$ is when the b is in the $C(A)$ - Solvability

We see that if a combination of rows of A gives the 0 row then the same combination of the entries of b must give 0 for it to be solvable with $Ax=b$

Finding the complete sol. for $Ax = b$:
Start by finding 1 sol. called x-particular

Steps: 1. To find x_p set all free vars to 0
1.a. Solve $Ax = b$ for the pivots
We can use our calculated $b = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

$$\underset{U}{\begin{bmatrix} \underline{1} & 2 & 2 & 2 \\ 0 & 0 & \underline{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \cdot \underset{x_p}{\begin{bmatrix} ? \\ 0 \\ ? \\ 0 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}}$$

Pivot 1 Pivot 2

This b is taken from elim. process.

$$x_1 + 2x_3 = 1$$
$$2x_3 = 3$$
$$x_3 = 3/2$$
$$x_1 = 1 - 3 = -2$$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

Step 2: Add on the nullspace in the col forms

$c \cdot x_{01}$ $d \cdot x_{02}$

$$c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}2\\0\\2\\1\end{bmatrix}$$

Step 3: Complete the sol

$$X = X_p + X_{01} + \dots X_{0n}$$

This works because $A \cdot x_p$ will give the right hand side b that we are finding the solution(s) for. We include the nullspace of A as we can take any linear combo of Ax_{0i} will give the 0 vector thus staying in the family of $Ax = b$

$$Ax_p = b, \quad Ax_0 = 0, \quad A(x_p + c \cdot x_{01} + d x_{02}) = b$$

$$X_{\text{complete}} = \begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix} + c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

Plotting all sols x in R^4 would give us a 2d plane that goes through x_p

Higher rank A's

We have an $m \times n$ with rank r

how is m, n related to r?

$r \leq m$, $r \leq n$

Full col rank, $r = n$ implies 0 free vars in sol, nullspace is 0 vector only.

$Ax = b$ has one sol. x_p for each b, if it exists

$Ax = b$ has 0 or 1 solution

Example:

$$\begin{array}{c}A\\ \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}\end{array} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 0 & -17 \\ 0 & -14 \end{bmatrix} \rightarrow \begin{array}{c}U\\ \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\end{array} \rightarrow \begin{array}{c}\text{rref}\\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\end{array} = \begin{bmatrix} I \\ F \end{bmatrix}$$

Full row rank, $r = m$

Infinitely many sols for $Ax = b$ for any b

$n-r$ dimension null space / free vars

example:

$$\overset{A}{\begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}} \to \begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & -5 & -17 & -14 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & f_1 & f_2 \\ 0 & 1 & f_3 & f_4 \end{bmatrix}$$

$$\begin{bmatrix} I & F \end{bmatrix}$$

Square Matrix $r = m = n$

True full rank, 1 sol for each b, $N(A) = \vec{0}$

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}} \to \overset{U}{\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}} \to \overset{rref}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$r = m = n$ are invertible

Summary

$r = m = n$	$r = n < m$	$r = m < n$
$rref = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \end{bmatrix}$-ish
1 sol	0 or 1 sol	∞ sols, always a sol

$r < m,\ r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

0 or ∞ sols

9. Independence, Basis and Dimension

let A be $m \times n$, $m < n$

$\rightarrow$ there are non-0 sols to $Ax = 0$

More unknowns than equations

There will be some free cols with free vars at least 1.

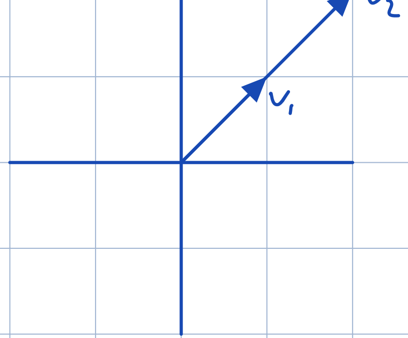
What does a vectors independence mean?

Vectors $X_1, \dots X_n$ are indy if no non-0 $\vec{v}$ gives the $\vec{0}$

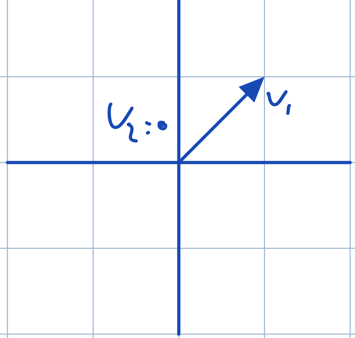
$$c_1 X_1 + c_2 X_2 + \dots c_n X_n \neq 0 : \sum_{i=0}^{n} c_i \neq 0$$

e.g. dependent systems.

$V \in \mathbb{R}^2$



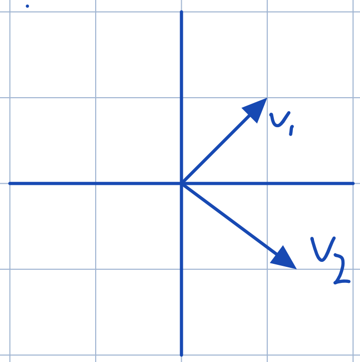
$2v_1 - v_2 = 0$



$5v_2 + 0v_1 = 0$

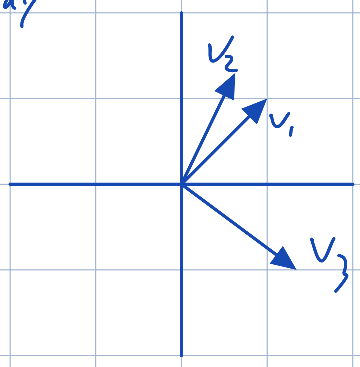
If one of our $\vec{v}$'s is $\vec{0}$ then we don't have ind.

Indy:



$c \cdot v_1 + d \cdot v_2 = b : b \ \forall \ b \in R^2$

Not indy



$\rightarrow$

$$\begin{array}{c} A \\ \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \end{array}$$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0} \ \exists \ c_1, c_2, c_3$

When $v_1, \ldots v_n$ are cols of A, they are indy
if $N(A)$ is the $\vec{0}$ only
The rank of indy cols $= r = n \leq m$

They are dep. if $A\vec{c} = 0$ for some non-0 $\vec{c}$
$r < n$

Spanning a space:

$v_1, v_2, \dots v_n \in V$ is said to span a space S if

S contains $c_1 v_1 + c_2 v_2, \dots + c_n v_n$ for all $c_i \in R$

The $C(A)$ is an example where:

$$C(A) = c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots c_n \vec{a}_n = S$$

S is the smallest possible space for those linear combos.

Basis: is a sequence of vectors for a v.space V

$v_1, v_2, \dots v_z$ w/ 2 properties

1. They are independent
2. They span the space.

Example: Spaces in R^3

- The identity I: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$N(I): \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Any permutation P

Other :

$$\underbrace{\overbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}}^{\text{Not a basis for } \mathbb{R}^3}, \begin{bmatrix}3\\3\\5\end{bmatrix}}_{\text{likely a basis}}$$

For $\mathbb{R}^n$: n vectors give a basis if the $n \times n$ Matrix is invertible.

$$\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$$ Span a plane in $\mathbb{R}^3$ and are a basis for this plane.

Bases for spaces are not unique, there are many different possibilities
But they all have at least n-vectors for the $\mathbb{R}^n$ space they cover.
Given a space, every basis has the same no. of vectors. which = the dimension of the space.

Example: Space is $C(A)$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Ask is A a basis for some space?

No, some cols are dep.

$\therefore N(A) \neq \{\vec{0}\}$

$N(A)$

$$c \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad d \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$B(A)$:

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

The pivot cols of A

The rank of A is the no. of pivot cols, also the dim. of $C(A)$

Its the dim of $C(A)$ not of A itself

Matrices have rank, spaces have dimensionality

The $\dim(N(A))$ is the no. of free vars

If $\dim(C(A))$ is r, then

$$\dim(N(A)) = n - r$$

10. The 4 fundamental subspaces:

1. The $C(A)$
2. The $N(A)$
3. The Row space $C(A^T)$
4. The $N(A^T)$ null space of A^T / left null space

When A is $m \times n$

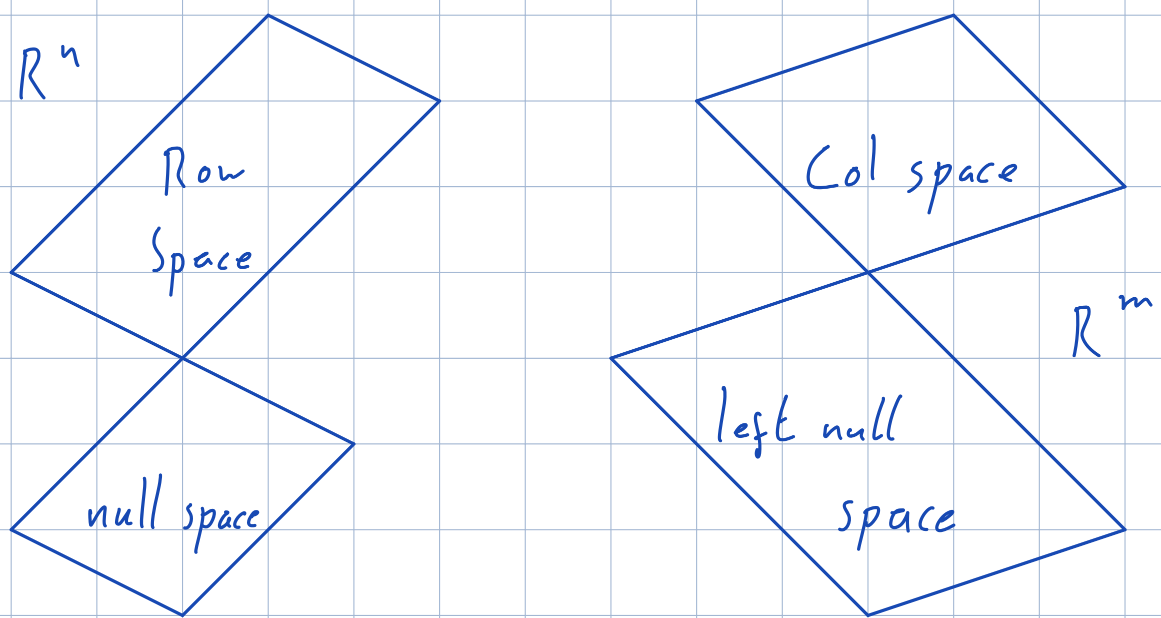
$C(A) \in R^m \quad = r$

$N(A) \in R^n \quad = n-r$

$C(A^T) \in R^n$

$N(A^T) \in R^m \quad =$

4. Subspaces



Finding a basis and the dim of our spaces

$\dim(C(A)) = r = \text{rank}$

And, the dim of the row space $C(A^T)$ is also r

$\dim(C(A^T)) = r$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$ $r = 2$, 2 dependent rows

The dim of $N(A) = n-r$, the $\dim(N(A^T)) = m-r$

The pivot cols give a basis for $C(A)$, how can we get a basis for $C(A^T)$ w/ doing rref twice

the special sols for $A = C(A^T)$

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}} \to \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \overset{\text{rref}}{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix},$$

$$C(A^T) = \begin{bmatrix} I \\ F \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

A basis for the row space of A can be found by finding R and using the pivot rows

Finding the left nullspace $N(A^T)$

If the $N(A)$ is $Ax = 0$ for our special sols
$c_1 \cdot x_1 + \dots c_n x_n$

Then $N(A^T)$ is $A^T y = 0$ for special sols
$c_1 y_1 + c_n y_n$

$$\begin{bmatrix} & & \\ & A^T & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \\ y \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

lets take the transpos of $A^T y = 0$

$$(A^T y)^T = y^T A^{TT} = y^T A = 0^T$$

This is a row $\cdot$ a matrix

$$\begin{bmatrix} & y^T & \end{bmatrix} \cdot \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \overset{0^T}{\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}}$$

The y^T here is on the left, ergo the left nullspace of A

How do we get a basis for $y^T A = 0^T$?
We already have a row of 0's in rref

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}} \rightarrow \overset{\text{rref}}{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

If we find the equation for the 3rd row we have a basis
We know that the process of $A \rightarrow R$ will have a row e.q giving 0^T

how does $A \to R$? through the elimination matrix E, $EA = R$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ -a_1 \\ -a_1 \end{array}$$

$$\downarrow$$

$$\begin{array}{cc} \quad R \qquad\qquad\qquad E & \\ \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] & \begin{array}{l} -2a_2 \\ \cdot -1 \\ \end{array} \end{array}$$

$$E = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$N(A^T)$ has $\dim(m-r) = 1$

This one row is in E and it is the one that gives 0^T

The 3rd row of $R = 0^T$ $\therefore$ $E_3 \in N(A^T)$

but not only E_3 but any $c \cdot E_3$

$N(A^T) = c \cdot [-1 \ 0 \ 1]$

For $\square$ invertible A's the rref was I

We know that $EA = R$

$\therefore EA = I$, $E = A^{-1}$

Review of Subspaces

$N(A)$, $C(A^T) \in \mathbb{R}^n$ and they add to n

$N(A^T)$, $C(A) \in \mathbb{R}^m$, add to m

New vector space type

Matrix spaces, All 3×3 matrices

$\therefore$ Matrices are our $v_i \in V$

They follow our rules:

$A + B \in V$

$cA \in V$

$cA + dB \in V$

What subspaces exist?

- Upper triangular U's
- Symmetric Matrices S
- $S \cap U$ = diagonal matrices D

11. Matrix Spaces:

- Vector Spaces without traditional vectors
  V-spaces can be made up of more than just 1D vectors.

New space M of all 3·3 spaces.

Subspaces:

- S - All symmetric
- U - Upper triangular
- D - Diagonal.

What are the basis' for the subspaces, what are the dimensions?

The dim (M) is 9, we need 9 no.s to reach all A's ∈ M.

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These 9 A's are the simplest basis possible for a 3·3 space

For Symmetric matrices we hav

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

When choosing a basis for S we need 1 for each el. above/below the diagonal as well as one for each diag.

$$\dim(M : 4 \cdot 4) = 16 \ , \ \dim(S : 4 \times 4) = 4 + 6 = 10 = \sum_{i=1}^{n} i$$

The dimension of U, upper triangular is also 6 as it's simply a non reflective S.

The $\dim(S) = \dim(S \cap U) =$ all A's in both S, U which are diagonal $= 3$

Spaces are closed under intersection $S \cap U \subset M$

Not under union though $S \cup U \not\subset M$.

The union of $S \cup U$ can't be a subspace of M because they each only fill a plane in M and the union of 2 planes does not cover a space

If we take combinations of S, U we can fill M

$S + U$

$S + U$ = Any $A \in S$ + any $B \in U$

which equals all $C \in M$

We know then that the dimension of 2 subspaces

$$\dim(V) + \dim(W) \equiv \dim(V \cap W) + \dim(V + W)$$

The Vector space of differential eqs

$$\frac{d^2 y}{dx^2} + y = 0$$

Whats in the sols?

$y = \cos x, \sin x, e^{ix}$

$\therefore$ The nullspace of this e.q

The complete sol. is

$$c_1 \cdot \cos x + c_2 \sin x$$

This is a vector space with dim = 2

basis = sinx, cosx

Not traditional vectors but still can treat as vectors

Rank:

lets look at rank 1 matrices

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

$n = 2$, $m = 3$

$r =$ one because row2 = 2 · row1

r is the no. of pivot cols / no. of indy rows.

The $\dim(C(A)) = r = \dim(C(A^T)) = 1$

This fact that the dim of the col. space = the dim of the row space = the rank is easily seen by deconstructing A into a col · row product.

$$\overset{A}{\begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} = A$$

Every rank 1 matrix has the form

$A = u \cdot v^T$ (some col · some row)

Rank 1 matrices can be seen as building blocks for all other matrices

A matrix with rank n can be deconstructed into n rank 1 matrices.

In a 5x17 matrix with $r = 4$ we can create 4 rank 1 matrices that construct the 5x17 A.

Can we use the subset of $r:4$ matrices to construct a subspace of M for 5x17 ?

If we add 2 rank 4 matrices will it still be rank 4?

No

Subspace:

$V \in R^4$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

S all $v \in R^4 : v_1 + v_2 + v_3 + v_4 = 0$

Is S a subspace?

Yes because any combo of $V_1 + \dots V_4$ will $= 0$

Find the dim(S)

$Ax = 0$, the components of v sum to give 0

So if we take v and multiply

it from the left by the row $[1\ 1\ 1\ 1]$

our dot product is equivalent to $v_1 + v_2 + v_3 + v_4$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{v} = v_1 + v_2 + v_3 + v_4 = 0$$

$Av = 0$, we now have our matrix A and our $x = v$. We want the space for $Av = 0$ which is the $N(A)$

The rank of $A = 1$ $\therefore$ the $\dim(N(A)) = n - r = 3$

So the dimension of $S = 3$

The dimension of $S = 3$, what is a basis?

When $\dim(N(A)) = 3$, we have $r = 1$, then we have 3 free variables

$$N(A) = \begin{bmatrix} v_1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering $v_1 + v_2 + v_3 + v_4 = 0$ once we fix one var it's simple to see that $v_1 = -1$ for all special sols.

What are the other subspaces of A?

$$\overset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}}$$

$\dim(C(A)) = 1$, all cols $= [1]$

$\therefore C(A) = [1]$

$\dim(C(A^T)) = r = 1$, only one row

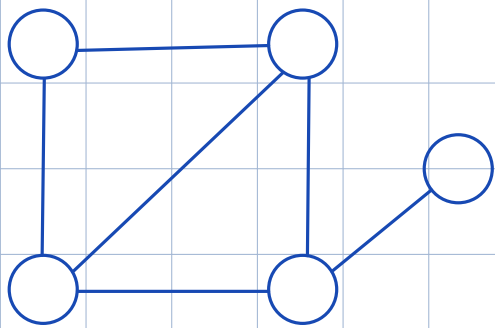
$\therefore$

$$C(A^T) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\dim(N(A^T)) = m - r = 0 \therefore$ only $\vec{0}$

$N(A^T) = [0]$

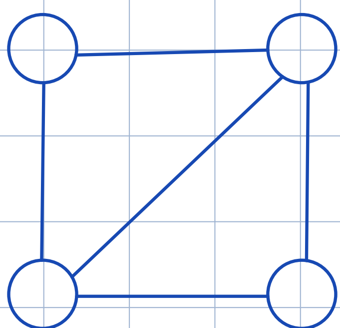
Graphs: A collection of nodes and edges



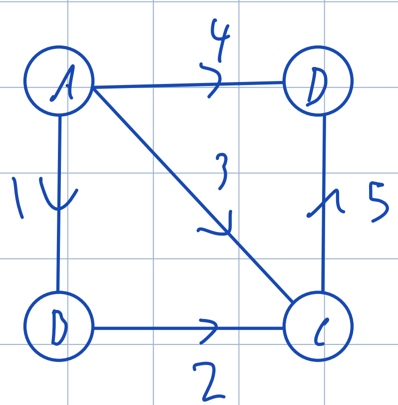
A graph w/ 6 edges, 5 nodes
Describable by a 5x6 A

We want to know if some amount of nodes w/ some amount of edges makes a graph

Graphs, Networks, Incidence matrices:



Graph w/ 4 nodes, 5 edges
Non-directed.
A row for each edge
A col for each node

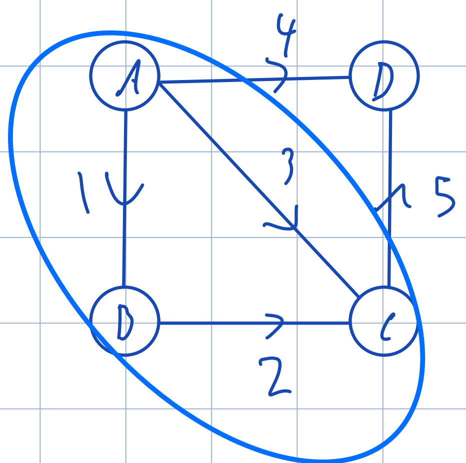


A directed graph w/ the direction indicating current/flow

$$
\begin{array}{cc|cccc|}
 & & \multicolumn{4}{c}{\text{Nodes}} \\
e & 1 & -1 & 1 & 0 & 0 \\
d & 2 & 0 & -1 & 1 & 0 \\
g & 3 & -1 & 0 & 1 & 0 \\
e & 4 & -1 & 0 & 0 & 1 \\
 & 5 & 0 & 0 & -1 & 1 \\
 & & A & B & C & D
\end{array}
$$

Arrows leaving nodes have -1 charge, entering have +1

We create an incidence matrix



The graph has a loop in edges 1, 2, 3. (subgraph)

The rows of edges 1, 2, 3 are not linearly indy. Loops correspond to dep. rows.

Graphs will have 2m 0's because theres only 2 connections

What can we ask of matrices to learn about our graph?

- The nullspace of M

  looking at the cols of M which combinations of nodes, edges give a 'balanced' graph sum.

Mx tells us the output of the graph, when $Mx = 0$, the graph has 0 total difference on positive and negative charges.

Are the 4 cols of M indep or dep.

Solve $Mx = 0$

$$\overset{M}{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}} \overset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \overset{b}{\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix}}$$

Think of X as containing the potential @ the nodes

The potential $\cdot M$ gives us the diff. in potentials across the edges.

When will $Mx = 0$? When the sum of pot. across the edges $= 0$.

Guaranteed : $\vec{0}$ vector

We know > 1 sol because we have at least one loop

$\therefore$ at least 1 dependent row so the $\dim(N(A)) > 0$

$$X \in N(A) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Const pot.

Also a basis for the nullspace of A

What does this mean for our application?

- It means if all node pot. are the same, the graph is grid locked, all potentials are the same.

Typically we would ground a node in the graph. Setting its pot. to 0

The x_4 col / 4^{th} node is set to 0, we then have 3 indy cols

Our rank of A is $n - \dim(N(A)) = 3$

any 3 cols, pots are a basis.

What is $N(A^T)$, $A^T y = 0$

$$\overset{M^T}{\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}} \cdot \overset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}} = \overset{0}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\begin{aligned} -y_1 - y_2 - y_4 &= 0 \\ y_1 - y_2 &= 0 \\ y_2 + y_3 - y_5 &= 0 \\ y_4 + y_5 &= 0 \end{aligned}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & F_1 & P_3 & F_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\dim(N(A^T)) = m - r = 5 - 3 = 2$, $\therefore$ 2 special sols.

Kirkhoffs Current law : KCL

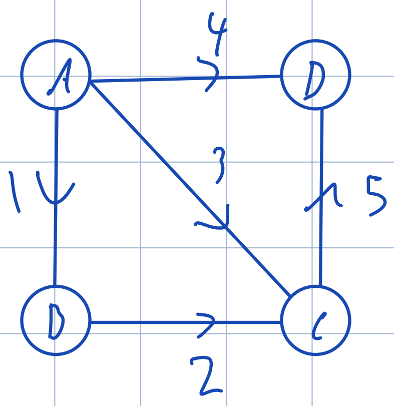
There is some matrix C connecting the potential difference in the nodes to the currents on the edges.

The currents on the edge are the y_i vals in $A^T y$

$$\overset{M^T}{\begin{bmatrix} -1 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}} \cdot \overset{y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}} = \overset{0}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\begin{aligned} -y_1 - y_2 - y_4 &= 0 \\ y_1 - y_2 &= 0 \\ y_2 + y_3 - y_5 &= 0 \\ y_4 + y_5 &= 0 \end{aligned}$$

$-y_1 - y_2 - y_3 = 0$ | Is our 1^{st} eq of $A^T y = 0$
It says that the current is flowing out of node 1 with the net flow = 0



Finding a basis for $N(A^T)$:

$$\begin{array}{c} \text{rref} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

free vars $= y_3, y_5$

1. $\begin{bmatrix} y_1 \\ y_2 \\ 1 \\ y_4 \\ 0 \end{bmatrix}$

$y_1 + 1 = 0, \quad y_1 = -1$

$y_2 + 1 = 0 \quad y_2 = -1$

$y_4 + 0 = 0, \quad y_4 = 0$

$y_a = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

2. $\begin{bmatrix} y_1 \\ y_2 \\ 0 \\ y_4 \\ 1 \end{bmatrix}$

$y_1 + 0 = 0, \quad y_1 = 0$

$y_2 + 0 = 0 \quad y_2 = 0$

$y_4 + 1 = 0, \quad y_4 = -1$

$y_a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$N(A^T) =$

$$c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The rowspace of A, it's the col space of A^T

$\dim(C(A^T)) = r = 3$

any 3 non-looping, indy cols.

Graphs w/out loops are trees.

Quiz one review:

1. u, v, w are $v \in \mathbb{R}^7$, non-0

What possible dimensions could they span?

$1, 2, 3$

2. U is $5 \cdot 3$, in echelon form w/ 3 pivots

a. What is the $N(u)$?

$$\dim(N(U)) = 3 - r = 0$$

$N(u)$ = 0-vector only

$$N(u) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b. There is a $10 \cdot 3$ matrix:

$$B = \begin{bmatrix} U \\ 2U \end{bmatrix}$$

B has rank 3, echelon form $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U \\ 0 \end{bmatrix}$

c. $C = \begin{bmatrix} U & U \\ U & 0 \end{bmatrix}$

echelon form of $C = \overset{\text{rref}}{\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}}$, $r = 6$

d. what is the dim(C^T)

$\dim(N(C^T) = m - r = 10 - 6 = 4$

3. $Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Whats the dim$(C(A^T))$

$\dim(C(A)^T) = r$

A has 3 cols $\therefore$ x has 3 parts

$\dim(N(A)) = 2 = n - r \therefore r = 1$

$\dim(C(A^T)) = 1$

A also has 3 rows $\therefore$ b has 3 parts

$\therefore A = 3 \times 3$ rank 1

The $N(A)$ tells us that the third col is the $\vec{0}$
and col2 = - col1

X_p says that $2 \cdot col_1 = b$

$$\therefore A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

c. When can $Ax=b$ be solved?

$Ax=b$ if b is on the line $c \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = C(A)$

4. If A is square, $N(A) = \vec{0}$
what do we know abt $N(A^T)$
it's also the $\vec{0}$
$\dim(N(A)) \equiv \dim(N(A^T))$

5. Matrix space of 5x5 As
do the invertible matrices form a subspace?
No, no gurantee that $A+B$ is invertible

6. If $B^2 = 0$ then $B = 0$?

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \overset{B^2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

No

7. A is $n \times n$ is solvable for any rhs b if the cols are independent.

Yes if $A, b \in \mathbb{R}^n$

Maybe not for $A, b \in \mathbb{C}^n$

8. $$\overset{L}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \cdot \overset{U}{\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} = \overset{B}{\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}}$$

Solve $Bx = 0$ with a basis for null space

$B = LU \quad U = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

U is rank 2, $\therefore \dim(C(B)) = 2$, $\dim(N(B)) = n - 2$
$= 2$

$$Ux = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = Bx \, , \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

L is invertible

for any $A = BC$, if B is invertible then

$B^{-1}A = C$, $N(A) = N(C)$

multiplying by invertible B's doesn't change the null space

Basis for $N(B) = N(U)$

$$\begin{array}{c} U \\ \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} x_1 \\ x_2 \\ f_1 \\ f_2 \end{bmatrix} =$$

$x_1 - f_1 + 2f_2 = 0$

$x_2 + f_1 - f_2 = 0$

let $f_1 = 1$, $f_2 = 0$

$x_1 = 1$, $x_2 = -1$

$f_1 = 0$, $f_2 = 1$

$x_1 = -2$, $x_2 = 1$

$$N(B) = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Now solve:

$$Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \overset{U_1:}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \overset{B_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$Bx = 0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

9. If $m=n$ then $C(A) = C(A^T)$

False $C(A) = C(A^T)$ only if A is symmetric.

10. Do A, $-A$ share the same 4 subspaces?

Yes

11. If A, B share the same 4 subspaces is A a multiple of B?

No e.g I and the permutation matrices

12. Exchanging 2 rows changes the $C(A)$ but not the $C(A^T)$ or the $N(A)$?

True: The location of free vars in the $N(A)$ might change?

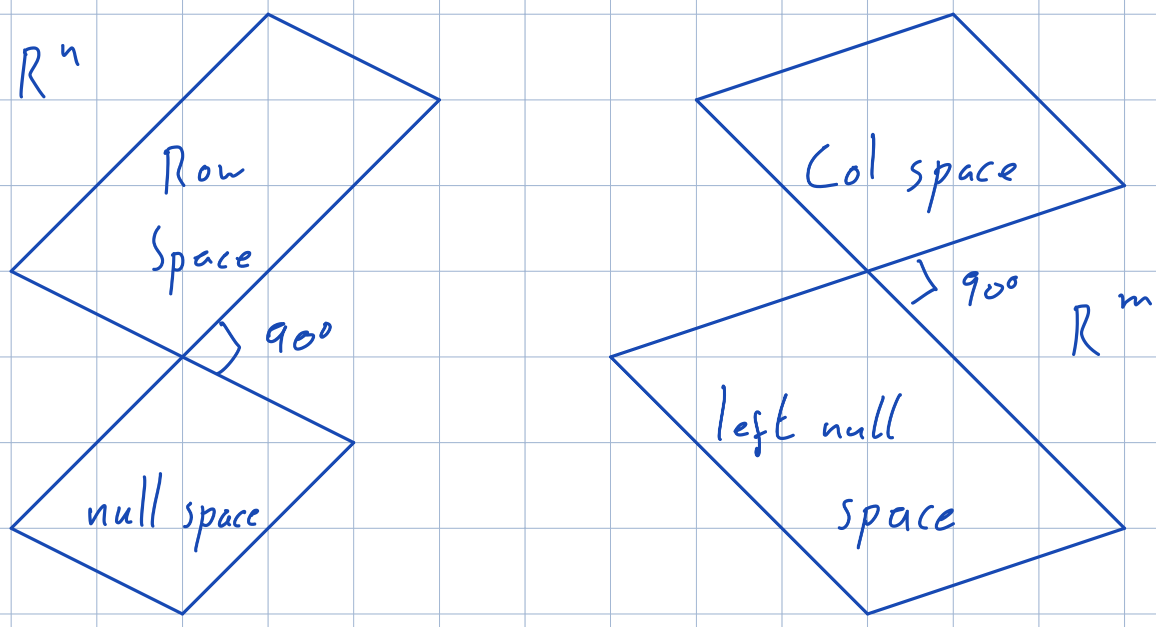
13. $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can't be both in the $N(A)$ and in $C(A^T)$

because Av would give $v^T \cdot v$ as part of the ans which is the $\|v\|^2$ of v

$$v^T v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 14 \text{ for } b_i$$

14. Orthogonal Vectors and subspaces

What does it mean for A's, $\vec{v}$'s to be orthogonal?



Orthogonal spaces are perpendicular to one another
the angle between the 2 is 90°.

2 ortho vectors form a right angle triangle

$\therefore \|x\|^2 + \|y\|^2 = \|x+y\|^2$

It's pythagorous theorem for $\vec{v}$'s

$\|x\|^2 = (\text{length of } x)^2$

$(\text{length of } x)^2 = x^T x$

$= (x_1 \cdot x_1) + (x_2 \cdot x_2) \ldots + (x_n \cdot x_n)$

let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\|x\|^2 = x^T x = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 14$

$$\|X\| = \sqrt{\|X\|^2} = \sqrt{x^T x} = \sqrt{14}$$

We have $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, let $y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\|y\|$ $\|X\|$ $\|y\|^2$ $\|X\|^2$

Y is orthogonal to X we can form a right angle triangle in $\mathbb{R}^3$ using

$\|y\|^2 + \|X\|^2 = \|Y + X\|^2$ Only for orthogonality

$$\|y\|^2 = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 4 + 1 = 5$$

$$\|Y + X\|^2 = 19$$

lets add $x + y = \begin{array}{c} x \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{array} + \begin{array}{c} y \\ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \end{array} = \begin{array}{c} x+y \\ \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \end{array}$

$$\|Y + X\|^2 = \begin{bmatrix} 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = 9 + 1 + 9 = 19$$

Orthogonal vectors also give a dot product of 0 when multiplying one by the transpose of another orthogonal vector:

$$x^T y = 0 : \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 - 2 + 0 = 0$$

Proof:

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$
$$x^T x + y^T y = (x+y)^T (x+y)$$
$$x^T x + y^T y = x^T(x+y) + y^T(x+y)$$
$$\cancel{x^T x} + \cancel{y^T y} = \cancel{x^T x} + x^T y + y^T x + \cancel{y^T y}$$
$$0 = x^T y + y^T x$$
$$y = x(y^T x)$$
$$0 = y^T x (y^T x)$$
$$0 = (y^T x)^2$$
$$0 = y^T x = x^T y$$

let x = the 0 vector, when is y orthogonal?

when $x^T y = 0$, true for any y with the 0-vector

$\therefore$ all vectors are orthogonal to the $\vec{0}$

Subspaces:

let a space S be orthogonal to T

A space is orthogonal to another when $v^T w = 0$ is true for all $v \in S$, $w \in T$

If 2 spaces meet at some point other than the origin they are not orthogonal

One space is orthogonal to another only when it is the same dimension or higher than the other space

- A line is never orthogonal to the whole plane
- A line is always ortho to the $\vec{0}$
- A line can be ortho to another line.

The row space is ortho to the null space?

why?

$$N(A) = Ax = 0$$

$$\begin{array}{c} \text{row}_1 \\ \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ & \text{—} & \\ & \text{—} & \end{array}\right] \end{array} \cdot \begin{array}{c} X \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (a_{11} \cdot x_1) + (a_{12} \cdot x_2) + (a_{13} \cdot x_3)$$

$Ax = 0$ says that each row dot prod. w/ $x = 0$

$\therefore$ x is orthogonal to all the rows of A

We have to show that it's ortho to all $v \in C(A^T)$

Well any $r \cdot x = 0$ $\therefore$ any $c_1 r + c_2 x \equiv 0$

The 4 subspaces carve n into 2 subspaces, m into 2 supspaces so what happens when working with matrices with less than full rank.

In R^3 we can't have a col space be a line and the nullspace be a line, because they have to equal $n = 3$

$$\overset{A}{\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}} \cdot \overset{X}{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}} = \overset{\vec{0}}{\begin{bmatrix} \\ \end{bmatrix}}$$

$N(A)$ is the plane ortho to $[1\ 2\ 5]$

The nullspace and the row space are orthogonal compliments of each other

For all $v \in C(A^T)$, $w \in N(A)$ $v^T w = 0$

Solving $Ax = b$ when theres no sols.

How to get a 'nearest' sol for when we have more unknowns than equations?

We want a 'best' sol

We can use A^TA to find a best sol

it gives us a projection of A and b onto n dim. space allowing us to use equations

We take an $m \times n$ A multiply from the left by A^T to get a new $n \times n$

$$\underset{n \times m}{A^T} \cdot \underset{m \times n}{A} = \underset{n \times n}{(A^TA)}$$

A^TA is:

- Square
- Symmetric $(A^TA)^T = A^TA^{TT} = A^TA$

Invertible?

Central equation:

If we can't solve $A^Tx = b$

we use $A^TA\hat{x} = A^Tb$

$\hat{x}$ is our best sol

Invertibility of A^TA

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = x_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Only solvable if b is in $C(A)$

$$\overset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}} = \overset{A^TA}{\begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}} \circ \overset{\hat{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}} \circ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

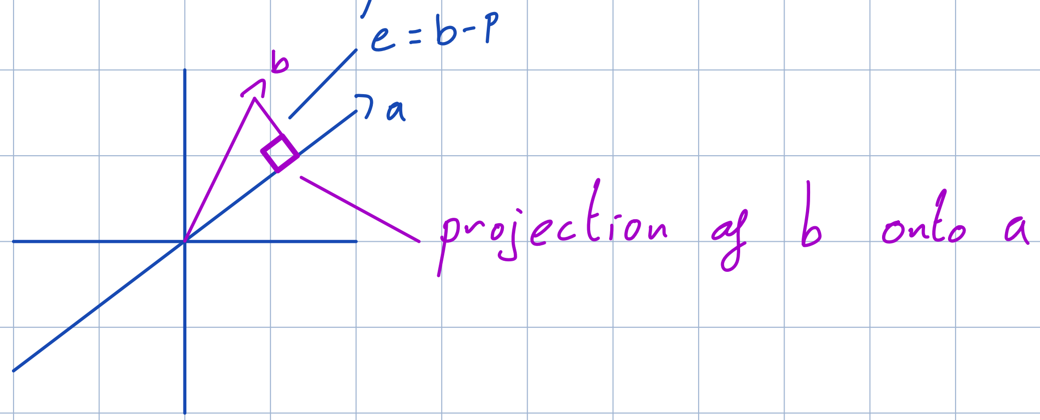
$$\overset{A^TA}{\begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}} \circ \overset{\hat{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{A^Tb}{\begin{bmatrix} (b_1 + b_2 + b_3) \\ (b_1 + 2b_2 + 5b_3) \end{bmatrix}}$$

We have a sol $\hat{x}$ for all A^Tb only if the cols of A are indep

The nullspace of A^TA is invertible if the cols of A are independent.

The rank of A^TA is always the same as A

15. Projections into subspaces



To find the point on b, closest to 'a' (the best sol)
we want to find a transformation of b onto 'a'
Solve for the transformed b, then project back to the original state.

The point on 'a' closest to b is the projection p
and it's orthogonal to a

The length of p : $\|p\|$ will be the distance of b
from our line a called the error.

p will be some multiple x of a
such that

$$e = b - p, \quad a \perp e, \quad p = xa$$

$$a^T e = 0$$

$$a^T(b - xa) = 0$$

$$x a^T a = a^T b$$

$$x = \frac{a^T b}{a^T a}$$

And if $p = xa$, $x = \dfrac{a^T b}{a^T a}$

$$p = \frac{a^T b}{a^T a} \cdot a = a\left(\frac{a^T b}{a^T a}\right)$$

If we double b what happens?

p is also doubled

Doubling a does nothing, it's a line, not a vector.

Matrices: The projection is carried out by some Matrix P

P acts on b to prod. p

$$Pb = p = \left(\frac{a a^T}{a^T a}\right) b$$

$$P = \frac{a a^T}{a^T a}$$

The col space of P is the line through 'a'

The rank is 1

P is Symmetric

$P^T P = P$

$P^2 = P$

Applying P twice does not move the matrix

Matrix Projections: Projecting n-dimensional subspaces

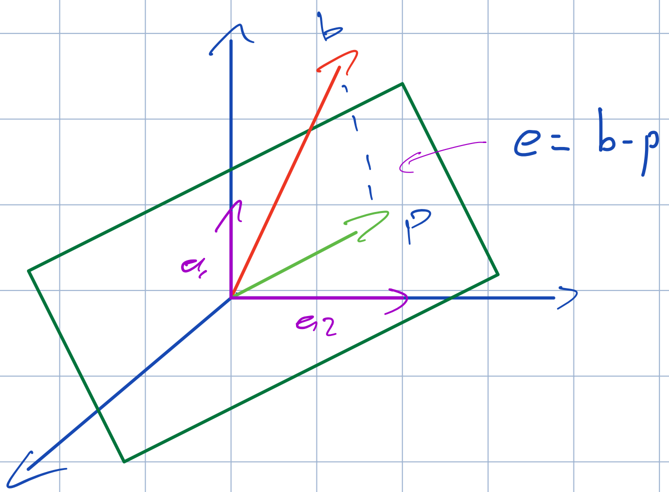
Why project?

To find a 'best' sol where none exists

Solve the closest problem we can.

$A\hat{x} = p$ is the nearest sol to $Ax = b$ we can have

x doesn't exist so we use $\hat{x}$



how to project b into the plane?

The plane is defined by a basis: a_1, a_2

lets make our basis the col space of A

$$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

we have a b not in the $c(A)$

how to project b into p using $Pb = p$

There is an error e between b and A, $e = b - p$

and $e \perp A$

p will be some multiple of our cols of A

$p = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$\therefore\ p = A\hat{x}$

find the right combination of the cols of $A\hat{x}$

so that $A\hat{x} = p$, $e = b - p = b - A\hat{x}$

$b - A\hat{x}$ is perpendicular to the plane $C(A)$

The plane has cols a_1, a_2

$$a_1^T(b - A\hat{x}) = 0,\quad a_2^T(b - A\hat{x}) = 0$$

$$A^T(b - A\hat{x}) = 0$$

$$A^TA\hat{x} = A^Tb$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$\overset{A^T}{\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}} \cdot (b - Ax) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$A^Te = 0$, e is in the left nullspace $N(A^T)$

$N(A^T)$ is orthogonal to $C(A)$

$$A^T(b - A\hat{x}) = 0$$

$$A^TA\hat{x} = A^Tb$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

A^TA is $n \times n$, invertible we can take inverse to find $\hat{x}$

$$p = A\hat{x}$$
$$p = A(A^TA)^{-1}A^Tb$$

$$Pb = p = A(A^TA)^{-1}A^Tb$$
$$P = p/b = A(A^TA)^{-1}A^T$$
$$= A(A^{-1})(A^T)^{-1}A^T$$
$$= I\,I$$
$$= I$$

Wrong because A is not square so we can't seperate the inverse.

If A was already in vertibe then it would cover R^n and the projection would be I because b would be in the $C(A)$

The properties of P hold for non square matrices:

$$P^T = P$$
$$(A(A^TA)^{-1}A^T)^T = A(A^TA)^{-1}A^T$$
$$A^T((A^TA)^{-1}A^T)^T = \text{''}$$

A^TA is square, symmetric

The inv. of is also symm

$\therefore ((A^TA)^{-1})^T = (A^TA)^{-1}$

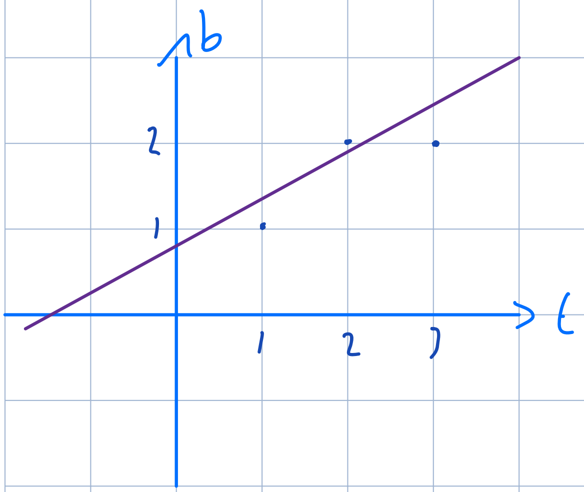
$$A(A^TA)A^T = P$$

$$P^2$$

$$A(A^TA)^{-1}\cancel{A^TA}\cancel{(A^TA)^{-1}}A^T = A(A^TA)^{-1}A^T = P$$

Application:

Fitting a line using least squares



$b = C + Dt$

Points: $(1, 1)$ 3 eqs 2 unknowns
$(2, 2)$
$(3, 2)$

$b = C + Dt$: $1 = C + D$
$2 = C + 2D$
$2 = C + 2D$

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \overset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

Unsolvable, need to project b to p

Solve $A^TA\hat{x} = A^Tb$

$$\underset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$\underset{A^T}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}} \cdot \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\underset{A^T A}{\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}} \quad \underset{\hat{x}}{\begin{bmatrix} c \\ D \end{bmatrix}} = \underset{A^T b}{\begin{bmatrix} 5 \\ 11 \end{bmatrix}}$$

$$\underset{E}{\left[\begin{array}{cc|c} 3 & 6 & 5 \\ 0 & 2 & 1 \end{array}\right]} \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/2 \end{array}\right] \qquad D = 1/2$$

best sol is $b = \frac{2}{3} + \frac{1}{2} t$

16. Projection Matrices and least squares

Pb projects b onto nearest point in $C(A)$

If b is already in the col space then
$Pb = b$

If b is orthogonal to $C(A)$ then $Pb = \vec{0}$

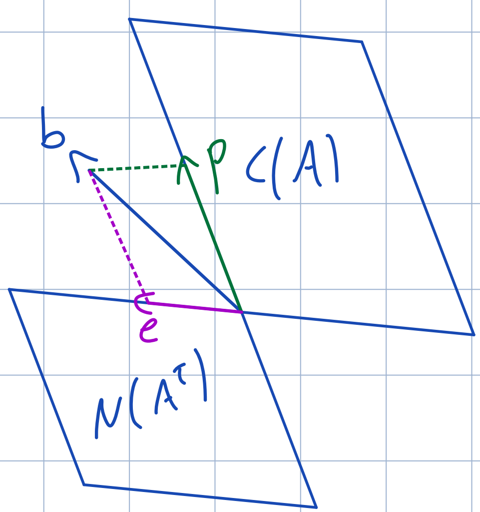
Which vectors are orthogonal to the $C(A)$?

All $x \in N(A^T)$: the left nullspace

For the vector in the $N(A^T)$ the error e has length
$2\,\|b\|^2$

If b is in the col space then it has the form
$A \cdot x$, a combo of some cols of A

Geometric Pic:

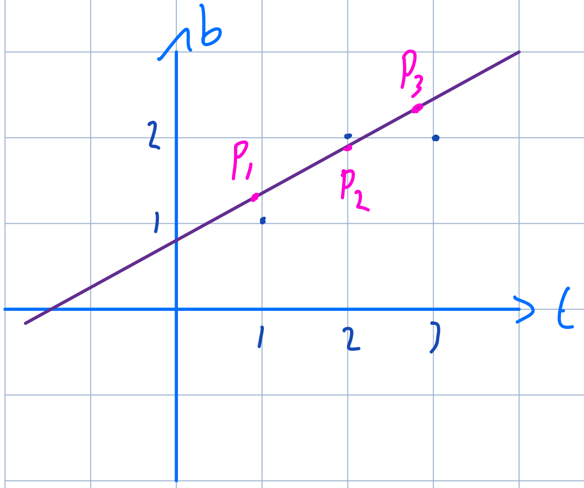


$b = p + e$
$p = Pb$
$e = ?b$

$p + e = b$
$Pb + e = b$
$e = b - Pb$
$e = (I - P)b$

$Pb = A(A^TA)^{-1}A^T$

There are other regression tools but least squares is an evergreen way of initial analysis



Looking again at the system but using the 3 vals that are closest to the 3 initial points.

The distance between p_i and b_i is e_i, how to find e_i?

We are minimising $\|Ax-b\|^2 = \|e\|^2$

Minimising:

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \overset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} \quad \overset{\|Ax-b\|^2}{\rightarrow (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2}$$

$$= \|e\|^2$$

How to minimise?

find C, D to find p

$$p_i = C + Dt$$

$$p_1 = \tfrac{2}{3} + \tfrac{1}{2} = \tfrac{7}{6}$$

$$p_2 = \tfrac{2}{3} + 1 = \tfrac{10}{6}, \quad p = \begin{bmatrix} 7/6 \\ 10/6 \\ 13/6 \end{bmatrix}$$

$$p_3 = \tfrac{2}{3} + 1\tfrac{1}{2} \quad \tfrac{13}{6}$$

$$\|Ax - b\|^2 = \|e\|^2$$

$$Ax = p$$

$$\|p - b\|^2 = \|e\|^2$$

$$\begin{array}{ccccccc} p & & b & & p-b & & \\ \begin{bmatrix} 7/6 \\ 10/6 \\ 13/6 \end{bmatrix} & - & \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} & = & \begin{bmatrix} 1/6 \\ -2/6 \\ 1/6 \end{bmatrix} & = & e \end{array}$$

, take negative: $e = \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$

$$e^T \cdot e \qquad p + e = b$$

$$\begin{bmatrix} 1/6 & 2/6 & -1/6 \end{bmatrix} \begin{bmatrix} 1/6 \\ 2/6 \\ -1/6 \end{bmatrix} = 1/36 + 4/36 + 1/36 = \|e\|^2 = 1/6$$

p is orthogonal to e

e is also orthogonal to the $\vec{0}$ and the col space $C(A)$

The values of $e : \sum e_i = 0$

$p : \sum p_i = 5$

$\left.\begin{matrix} \sum e_i = 0 \\ \sum p_i = 5 \end{matrix}\right\} + = \sum b_i = 5$

Invertibility of A^TA

We need A^TA to solve $Pb = A(A^TA)^{-1}A^T$

If A has indep cols, prove A^TA is invertible

Suppose $A^TAx = 0$

show that $x = 0$

Matrices are invertible iff. $N(A) = \vec{0}$ only

Take the dot.p of both sides with x

$x^TA^TAx = x^T0$

$(Ax)^TAx = 0$

If $y^Ty = 0$

then $\|y\| = 0$

$\therefore Ax = 0$

If the cols of A are indep, $Ax = 0$ then $x = 0$

All other x's would be in the $C(A)$

Orthonormal

Cols are definitely indep if they are perpendicular unit vectors

e.g.: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Orthogonal basis, Matrices and Grahm-Schmidt

In an orthonormed basis all $q_i \in Q$ is orthog to $q_k \in Q$
and $q_i^T q_i = 1$, $\|q\|^2 = 1$

$$q_i^T q_j = \begin{cases} 0 : i \neq j \\ 1 : i = j \end{cases}$$

Ortho regers to the relationship
Normal rgers to the length.

Matrix of q_i

$$\begin{array}{c} Q \\ \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \end{array}$$

$q_i^T q_j$ in $Q^T Q$

$$\begin{array}{c} Q^T \\ \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \end{array} \begin{array}{c} Q \\ \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \end{array} = \begin{bmatrix} q_1^T q_1 & q_i^T q_j & \cdots \\ & q_i^T q_i & \\ & & \ddots \ q_n^T q_n \end{bmatrix} = \begin{array}{c} I \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & & \vdots \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \end{array}$$

$Q^T Q = I$

Q is an orthonormal matrix

Orthogonal is used when Q is square

If Q is orthog $\rightarrow Q^{-1} = Q^T$

example: any permutation Matrix

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{Q^T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

example

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Counter example:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \begin{matrix} \|q_1\|^2 = 2 \; \therefore \; \|q\| = \sqrt{2}, \text{ not o-normal.} \\ \|q_2\|^2 = 2 \end{matrix}$$

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The above Q is an Adhemar Matrix which has a pattern

$$Q_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Why do we want orthogonal A's?
lets look at rectangular.

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$ This is an orthonormal basis for a 2D plane in R^3

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

If we want to complete a basis for R^3 we can look look for patterns or use G-S

If we have Q with orthonormal cols and we want to project onto its $C(Q)$

$$P = A(A^TA)^{-1}A^T = Q(Q^TQ)^{-1}Q^T$$
$$= Q(I)^{-1}Q^T$$

$$= QQ^T$$

If Q is square then it fills R^n already, $QQ^T = I$

else $P = QQ^T$

P is symmetric, so is QQ^T

$P^2 = P$, $QQ^TQQ^T = QQ^T$

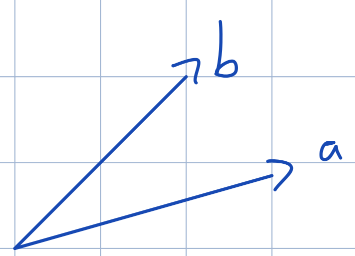
Orthonormality simplifies:

$$A^TA\hat{x} = A^Tb$$
$$Q^TQ\hat{x} = Q^Tb$$
$$I\hat{x} = Q^Tb$$
$$\hat{x} = Q^Tb$$
$$\hat{x}_i = q_i^Tb$$

- The i'th component of $\hat{x}$ is the i'th row of Q : $q_i^T \cdot b$

Grahm Schmidt:

From any indy vectors to orthonormal matrix



From a, b we want q_1, q_2

lets create A using a, q_1 to be an orthonormal $\frac{A}{\|A\|}$

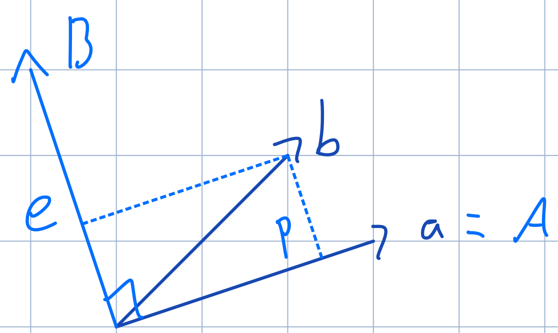
We want B to be $\perp$ to A

use bandlets project B into the null space of A.

Creating A, B.

$$q_1 = \frac{A}{\|A\|^2} \qquad q_2 = p\left(\frac{B}{\|B\|^2}\right)$$

We want a vector that takes b and projects makes it orthogonal to A a



$$e = b - p$$
$$b = e + p$$

Want to project b away from A, into the $N(A^T)$ onto e.

$B = b - p$ | The original vector - minus the projection

$$p = \frac{A^T b}{A^T A} \cdot A \rightarrow B = b - \frac{A^T b}{A^T A} \cdot A$$

If B is orthog to A, then what would that mean?

$$A^T B = 0 \ , \ B^T A = 0$$

So $A^TB = A^T\left(b - \frac{A^Tb}{A^TA} \cdot A\right) = A^Tb - \frac{A^Tb}{A^TA} \cdot A^TA$

$$= A^Tb - \frac{A^Tb}{1} \cdot 1 = A^Tb - A^Tb = 0$$

If we had 3 vectors a, b, c and wanted ortho

orthogonal A, B, C

Where $q_1 = \frac{A}{||A||^2}$ $q_2 = \frac{B}{||B||^2}$ $q_3 = \frac{C}{||C||^2}$

C needs to be $\perp$ to A and B

C needs to be in the $N([AB]^T)$

lets start with C $\perp$ to A

To find C $\perp$ to A we subtract the components in the Direction of A

$$C \perp A = c - \frac{A^Tc}{A^TA} A$$

For C $\perp$ A and B we want to remove the components in the direction of A, B

$$\frac{C}{||C||} \perp (A, B) = c - \frac{A^Tc}{A^TA} A - \frac{B^Tc}{B^TB} B$$

Example: a, b

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$$

$\|b\|^2 = \sqrt{5}$

$\|a\|^2 = \sqrt{3}$

let $a = A$, $B = b - \frac{A^Tb}{A^TA}A$

$$= b - \frac{A^Tb}{3}A$$

$$= b - \tfrac{1}{3}(3)\cdot A$$

$$\begin{array}{ccccc} b & - & A & = & B \end{array}$$

$$= \begin{matrix}1\\0\\2\end{matrix} \quad \begin{matrix}1\\1\\1\end{matrix} \quad \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

Prove A is $\perp$ B

$$\overset{A^T}{\begin{bmatrix}1 & 1 & 1\end{bmatrix}}\begin{bmatrix}0\\1\\-1\end{bmatrix} = 0 + 1 - 1 = 0$$

To make an orthogonal Q normalise A, B

$$q_1 = \frac{A}{\|A\|} = \frac{A}{\sqrt{3}} = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix} \qquad q_2 = \frac{B}{\|B\|} = \frac{B}{\sqrt{5}} = \begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

How is the $C(Q)$ related to $C(A)$?

They're the same space but one is an ortho basis

For elimination we had $A = LU$

In G-S we use $A = QR$

R is the G-S relation / steps / expression

Say $A =$

$$\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \cdot \overset{R}{\begin{bmatrix} 1/\sqrt{} & \\ 0 & 1/\sqrt{} \end{bmatrix}}$$

$$= \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ a_1^T q_2 & a_2^T q_2 \end{bmatrix}$$

$a_1^T q_2 = 0$ because they're $\perp$

18. Properties of Determinants:

Every □ has a det.
Dets needed for λ's
When $|A| != 0$ then A is invertible.
If $|A| = 0$ then A is singular.

3 key properties

1. $\det(I) = 1$

2. Exchange rows, reverse the ± sign of the det
∴ All permutation matrices are ±1

Property 1,2 for 2·2

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3. linear combos of row 1

Multiply any 1 row by t can be factored out of the det

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix} = t(ad-bc)$$

3. b Also applies to other linear combos in a <u>single</u> row <u>only</u>

$$\begin{vmatrix} a & b \\ c+c' & d+d' \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix}$$

4. If 2 rows are equal the det = 0

for $n \times n$

$$\begin{vmatrix} - & a & - \\ - & b & - \\ - & a & - \end{vmatrix} = 0$$

because acc. to P2 exchanging 2 rows must change the sign but if they're the same it must be 0

5. Subtracting some $l \cdot$ row i from row k does not change the det.

$\therefore$ Elimination does not affect the det.

$$|A| = |U|$$

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ la & lb \end{vmatrix} \right)$$

$$= \left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix} \right)$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - 0$$

6. A row of 0's gives a det $= 0$ because of prop. 3

$$\begin{vmatrix} 5\cdot 0 & 5\cdot 0 \\ c & d \end{vmatrix} = 5 \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$$

7. We can eliminate to U: upper-triangular.

$$U = \begin{vmatrix} d_1 & t_i & t_i \\ & d_2 \ldots & t_i \\ 0 & & d_n \end{vmatrix}$$

And having U we can take a product of the diagonals to get $\det(U)$

$$|U| = d_1 \cdot d_2 \cdot \ldots d_n = \prod_{i=1}^{n} U_{ii}$$

We can simplify the det of $n \cdot n$ elimination by taking the product of the pivots

* If we have no row exchanges.

8. The determinant is an invertability test, $\det A = 0$ only if A is singular

If A is singular we would have a row of 0's after elim

So if $\det A \neq 0$, A is invertible. A^{-1} exists

Elimination essentially gives us

$$\overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \to \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} \cdot b \end{bmatrix}$$

Take product of pivots

$$a \cdot (d - c/a \cdot b) = ad - bc$$

9. The $\det(A \cdot B) = (\det A) \cdot (\det B)$

True for diag. matrices

$$|A^{-1}| = \frac{1}{|A|}$$

$$A^{-1}A = I \qquad |A^2| = (|A|)^2$$

$$|I| = 1$$

$$|A^{-1}||A| = 1$$

$$|2A| = 2^n \cdot |A|$$

This essentially is a factor of 2 on all rows

10. a $\det A^T = \det A$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$(ad - bc) = (ad - cb)$$

The transpose shows that all rules above for cols also apply to rows

b. $|A^T| = |A|$

$$|U^T \cdot L^T| = |L \cdot U|$$

$$|U^T| \cdot |L^T| \quad |U^T| \cdot |L^T|$$

We know that $|A| = |U|$

$\therefore |L| = 1 = |L^T|$

Determinant shifts:

The formula for det A

Key props:

1. $\det I = 1$
2. Row exchange = sign reverse $|PA| = -|A|$

   If P is a single exchange

3. Linear combos of a single row can be factored into a seperate det.

$$\overset{A}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

Apply prop 3 to row 1 split into 2

$$(ad-bc) = ad + (-bc)$$

$$= \overset{1}{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \overset{2}{\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}} + \overset{3}{\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}} \quad \overset{4}{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}$$

Apply 3 to row 2 of our 2 new A's to get 4 easy dets

$$0 + (ad) + (-bc) + 0$$

We can see that the composites 1, 4 = 0
looking at 2, 3 we have a diagonal matrix: 2
and a matrix that is Anti-diagonal: 3
If we row exchange 3 we have a diagonal matrix w/ a negative det.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} - \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix}$$

$$= (ab - 00) - (cd - 00)$$
$$= ab - cd$$

Applying the method of splitting one row at a time to $n \times n$ matrices

3·3 example:

- Split row 1 into 3 pieces
- We could the split into 9 pieces or realise that we already know that we need at least 1 val in each col and :

$$\begin{vmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ 0 & c_1 & 0 \end{vmatrix}$$

Won't be a factor, neither will any with the $\vec{0}$ in the col space

So $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} \begin{vmatrix} 0 & a_2 & 0 \\ b_1 & 0 & 0 \\ 0 & 0 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & 0 & 0 \\ 0 & 0 & b_3 \\ 0 & c_2 & 0 \end{vmatrix}$$

$$(a_1 b_2 c_3) \quad -(a_2 b_1 c_3) \quad -(a_1 b_3 c_2)$$

$$+\begin{vmatrix} 0 & 0 & a_3 \\ b_1 & 0 & 0 \\ 0 & c_2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_3 \\ 0 & b_2 & 0 \\ c_1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & 0 \\ 0 & 0 & b_3 \\ c_1 & 0 & 0 \end{vmatrix}$$

$$--(a_3 b_1 c_2)$$