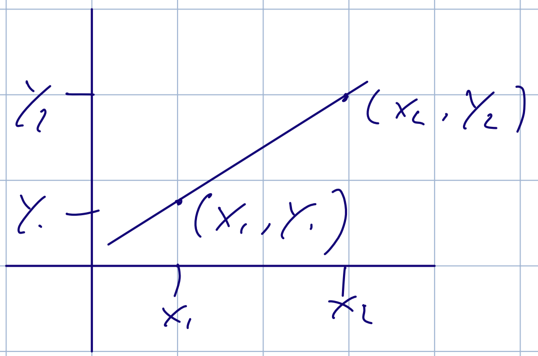
# Review

## 0.1 Lines

Lines have slopes




Any point on a 2d axis has x, y points

$$\text{Slope} \rightarrow \frac{\text{Rise}}{\text{Run}} \rightarrow \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \rightarrow \text{Slope formula}$$

Equations: by fixing one point we change the formula

$$m = \frac{y - y_1}{x - x_1} \quad \text{we know } x_1, y_1 \text{ are fixed}$$

how could we solve for $y - y_1$?

* by $x - x_1$

$$m(x - x_1) = \frac{y - y_1}{\cancel{x - x_1}}\left(\cancel{x - x_1}\right) \rightarrow m(x - x_1) = y - y_1$$

Point slope

$$m(x - x_1) = y - y_1 \rightarrow y - y_1 = m(x - x_1)$$

You need a point and a slope to solve PS

Equation of a line

Ex find the eq through

$(-2,-3), (8,2) \rightarrow y_2 - y_1 = m(x_2 - x_1)$

$x_1 \; y_1 \quad x_2 \; y_2$

$2 - -3 = m(8 - -2)$

$5 = m(10)$

$m = \frac{1}{2}$

We need a point and a slope

We have 2 points so the slope is trivial.

$y - -3 = \frac{1}{2}(x - -2)$

$y + 3 = \frac{1}{2}(x + 2)$

$2y + 6 = x + 2$

$2y - x = -4$

$y = \frac{1}{2}x - 2$

$y = mx + c$

$m =$ slope

$b = y$ - intercept

The y-intercept tells us to start on the origin up or down

We always graph to the right

Lines continued:

$y = c$, $c =$ constant

y intercept

We know this is a perfectly horizontal line

$x = c$,

This is a vertical line, has an x intercept

Ex: $4x + 2y - 3 = 0 \rightarrow y = mx + c$

$$2x + y - \frac{3}{2} = 0$$
$$y - \frac{3}{2} = -2x$$
$$y = -2x + \frac{3}{2}$$

Parallel lines: The same exact slope
Different intercepts

Perpendicular lines: Slopes are negative reciprocals
$2 \rightarrow -\frac{1}{2}$

Reciprocal is an inverted fraction.

Ex: Find eq. through $(6, 7)$ + Parallel to
$$2x + 3y = 12$$
$$3y = -2x + 12$$
$$y = -\frac{2}{3}x + 4 \qquad m = -\frac{2}{3}$$

$$y - y_1 = m(x - x_1)$$
$$y - 7 = -\frac{2}{3}(x - 6)$$ Multiply out the brackets
$$y - 7 = -\frac{2}{3}x + 4$$ Think about the signs
$$y = -\frac{2}{3}x + 11$$

Para slope $= -\frac{2}{3}$

Perp slope $= \frac{3}{2} \rightarrow y - y_1 = m(x - x_1)$
$$y - 7 = \frac{3}{2}(x - 6)$$
$$y = \frac{3}{2}x - 9$$

$$y = \frac{3}{2}x - 2$$

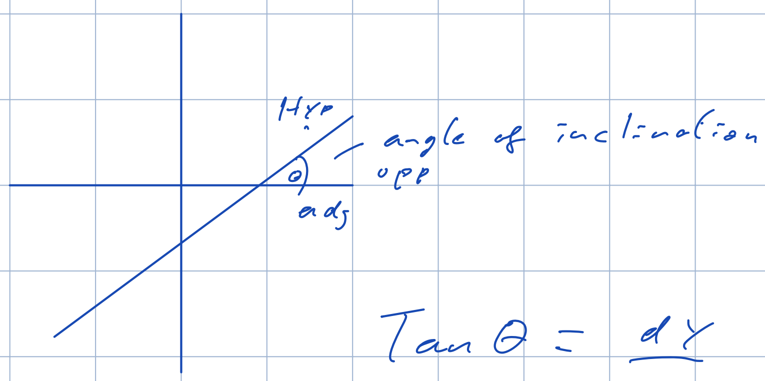
$$7 = 9 - 2 \quad \checkmark$$

Perpendicular intercept changed sign

Does it always

Angles of inclination:

The angle that any line makes with the x-axis




$$\tan\theta = \frac{dy}{dx} = \text{Slope} = m$$

$$m = \tan\theta$$

If you know the angle you can find the slope and vice versa.

Ex: $\theta = 30° = \frac{\pi}{6}$ radians

$$m = \tan\theta$$

$$= \tan(30) \text{ or } \tan\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = m$$

Ex 2.

$m = -1$

what is the angle of inclination?

$-1 = \tan\theta \qquad \tan^{-1}(-1) = \theta$

What is the angle so that a tangent of it gives me negative 1.

because $\tan\theta = -1$ we know $\dfrac{\sin\theta}{\cos\theta} = -1$

$\therefore \sin\theta = -\cos\theta$

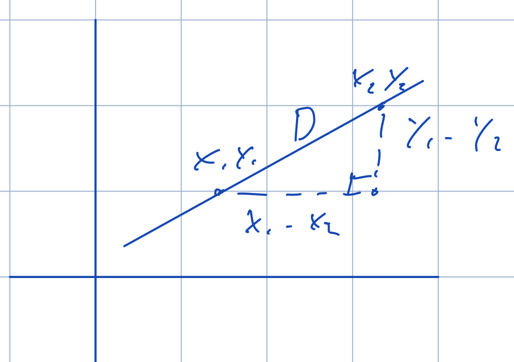
at $135°, 315°$ 2 angles because it cross two points

$\frac{3\pi}{4}, \frac{7\pi}{4}$

Slopes from angles and angles from slopes

Lots of trig and algebra in calc

Distance formula




Pythagorean theorem

$$D^2 = (Y_2 - Y_1)^2 + (X_2 - X_1)^2$$

$$hyp^2 = opp^2 + adj^2$$

$$\sqrt{D^2} = \sqrt{(X_2 - X_1)^2 + (Y_2 + Y_1)^2}$$

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}$$

## 0.2 Functions:

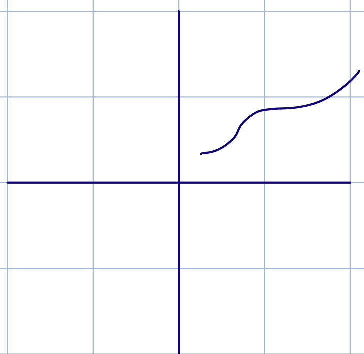
Some expression where each input determines exactly 1 output. $f(x)$

ex

| input<br>Fish | output<br>Weight |
|---|---|
| 1 | 3.2 |
| 2 | 1.4 |
| 3 | 2.8 |
| 4 | 2.3 |

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $y$ | 2 | 5 | -3 | 9 |

$$A = \pi r^2$$



Vertical line test for function

Ex. $x^2 + y^2 = 25 \rightarrow$ circle, not a function

for $y = 5, -5$

$$\sqrt{y^2} = \sqrt{25 - x^2}$$

$$y = \pm\sqrt{25 - x^2}$$

$f(x) = \sqrt{25 - x^2}$ top half of circle: function

$g(x) = -\sqrt{25 - x^2}$ bottom half of function

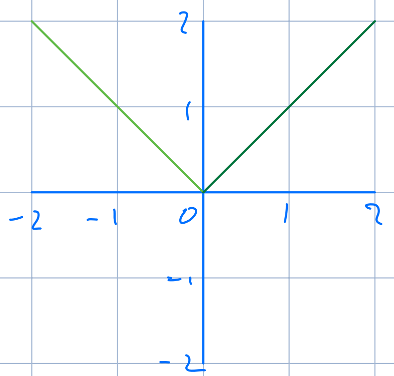
Piece-wise function: The function changes depending on the value of $x$

Absolute Value function:

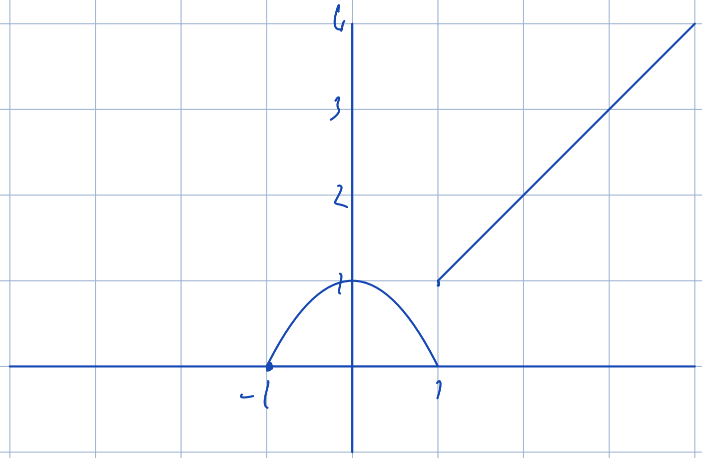
$|x|$ $\quad |5| = 5 \quad |-12| = 12$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

You can graph any piece-wise function by graphing any piece individually



$$g(x) = \begin{cases} 0 & , x \le -1 \\ \sqrt{1-x^2} & , -1 < x < 1 \\ x & x \ge 1 \end{cases}$$




Domain and range:

Domain : All Input values

Range : All Output Values

Limitations of functions

Area of a square

$A = s^2, \; s \ge 0$

$y = \frac{1}{x}, \; x \,!= 0$

$f(x) = \sqrt{x}, \; x \ge 0$

Natural domain: All values that work in the formula

$f(x) = x^3$ D: All real no.s

$x \in R$

To find out domain make sure to look for roots and denominators

$$g(x) = \frac{1}{(x-1)(x-3)}$$ $\rightarrow$ $x = 1, 3$ are not in our domain

D: All real no:s except $x \neq 1$, $x \neq 3$

Domains: Continued

$$h(x) = \tan x$$

tangent is not defined for certain values.

$$= \frac{\sin x}{\cos x} \rightarrow \cos x \neq 0$$

$$x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$f(x) = \sqrt{x^2 - 5x + 6}$$

$$x^2 - 5x + 6 \geq 0$$

Factor this

$$x^2 - 5x \geq -6$$

The radicand must be $\geq 0$

$$x^2 \geq 5x - 6$$

$$x \geq \sqrt{5x - 6}$$

$$(x-3)(x-2) \geq 0$$

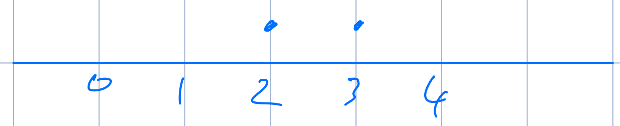
Partial

[Set each side equal to 0

$x - 3 = 0$ $\quad x = 3$ $\quad x \geq 3$

$x - 2 = 0 \quad x = 2 \quad \cancel{x \geq 2}$

Sign analysis test : Test the points : to the left, right



$x = 2 \quad x^2 - 5x + 6 = 4 - 10 + 6 = 0$

$x = 3 \quad x^2 - 5x + 6 = 9 - 15 + 6 = 0$

$x = 1 \quad 1 - 5 + 6 = 2$

$x = 4 \quad 16 - 20 + 6 = 1$

$x = 0 \quad 0 - 0 + 6 = 6$

every no. to the left of 2 is pos
because it's quadratic

We know that 2, 3 are solutions ∴ 0
It's a quadratic curve so we know the shape is U or ∩.

$D: (-\infty, 2] \cup [3, \infty)$

$$f(x) = \frac{x^2 - 4}{x - 2} \qquad x \neq 2$$

$D: \mathbb{R},\ x \neq 2$

$$f(x) = \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$f(x) = x + 2$

D: All real no.s? − No we have to keep the original domains

For simplification keep the original domain.

Can't eliminate anything from our domain.

If you can cancel out a domain problem its a hole

$g(x) = \dfrac{3x}{x-4}$ $x \neq 4$ $\dfrac{12}{0}$ Verticle asymptote a no. over 0. Can't Remove Domain issue

$f(x)$ $\dfrac{x^2-4}{x-2}$ $x \neq 2$ $\dfrac{0}{0}$ 'Hole': When you can "Remove" the discontinuity

So a 'hole' is simply a value in the domain that Could be represented by a simplified Version of the f(x) but because the f(x) originally had it in the Domain we have to keep it out of our solution.

Ex D and R of $f(x) = 2 + \sqrt{x-1}$

$2 + \sqrt{x-1}$

$x - 1 \geq 0$

$x \geq 1$

D: Real no.s if $x \geq 1 \rightarrow [1, \infty)$

R: $[2, \infty)$

Ex. $y = \dfrac{x+1}{x-1}$

D: $x \neq 1$

R: $y \neq 1$ horiz. asymptote

$x = 1 = \frac{2}{0}$ $\therefore$ Asymptote - Verticle

Find range by solving for $y$

$(x-1)y = x+1$

$yx - y = x+1$

$xy - x = y+1$

$x(y-1) = y+1$

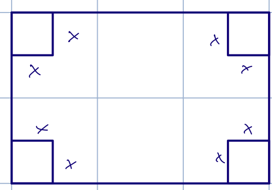
$x = \dfrac{y+1}{y-1}$ $\therefore$ $y \neq 1$

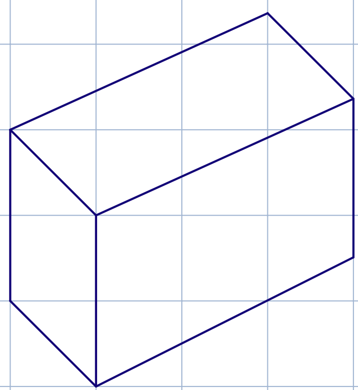
Word Problem

Making box from card board.

30cm

16cm





$V(x)$ find vol of box as function of $x$

vol = $h \times w \times d$

let $x = 1$

$h = x$

$w = 30 - 2x$

$d = 16 - 2x$

$$(x)(30-2x)(16-2x)$$

$$(x)(480 - 92x + 4x^2)$$

$$V(x) = 4x^3 - 92x^2 + 480x$$

$D: 0 \le x \le 8$

$R: 0 \le V(x) \le$

## Even + Odd

Even functions are symmetric about the y axis e.g. $x^2$. $f(-x) = f(x)$

Odd functions are symmetric about the origin

$f(-x) = -f(x)$

Symmetric about the origin.

ex $f(x) = x^4 - x^2 + 1$

$f(-x) = (-x)^4 - (-x)^2 + 1$

$= x^4 - x^2 + 1$

a negative to an even power = pos to the same power

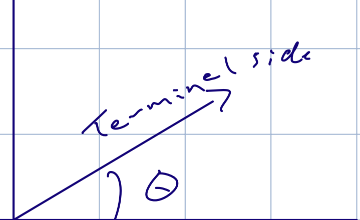
$g(x) = x^3 - x$

$g(-x) = (-x)^3 - (-x) + 1$

To check odd or even just plug in the negative x

Trig functions:

Angles: normally in reference to x axis




Positive angles

Initial side

negative angles

Radians vs Degrees:?

- Converting between:

$2\pi$ Radians $= 360°$

$1° = \frac{2\pi}{360} = \frac{\pi}{180}$

To convert D -> R x by $\frac{\pi}{180}$

Rad -> Deg x by $\frac{180}{\pi}$

Ex $200° \to$ Radians $= 200 \times \frac{\pi}{180} = 1\frac{20}{180}\pi$

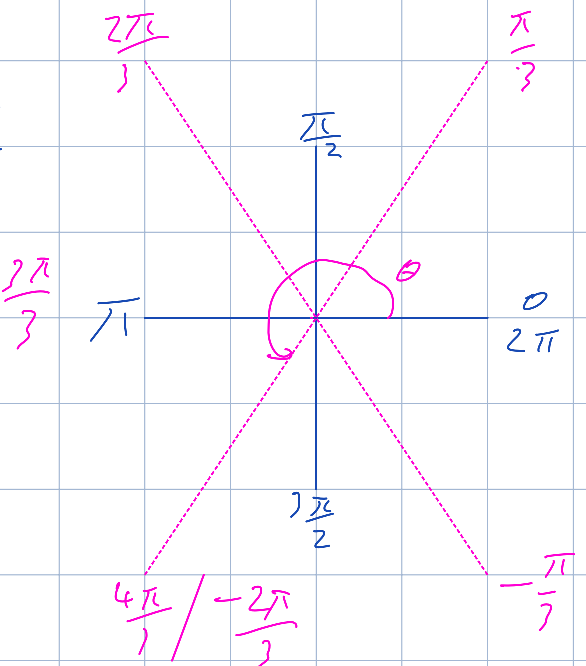
$1\frac{1}{9}\pi$ $\quad \frac{10\pi}{9}$

$-\frac{3\pi}{4} \to$ Radians $= -\frac{3\pi}{4} \times \frac{180}{\pi} = -\frac{540\pi}{4\pi} = -\frac{540}{4}$

$= -135°$

Graphing

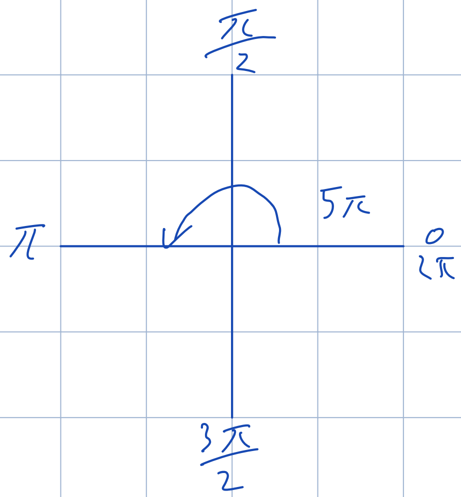
Ex: $\frac{4\pi}{3}$



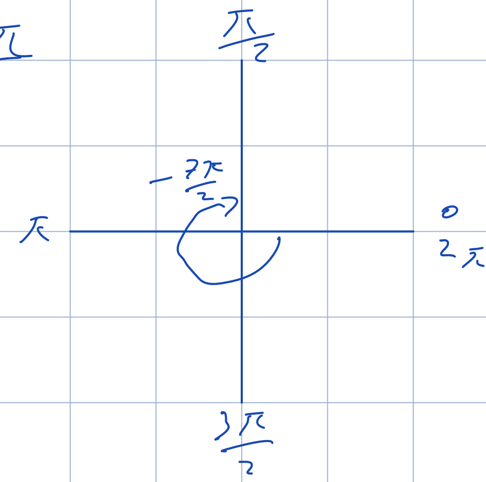

Co-terminal angles multiple ways to

measure

$5\pi$




$-\frac{7\pi}{2}$

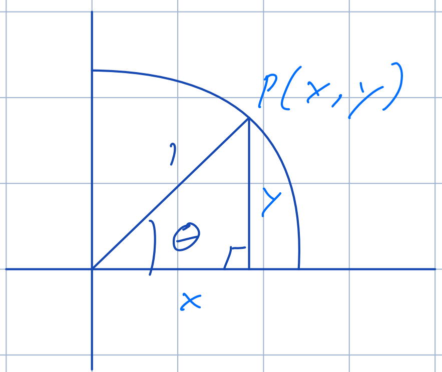


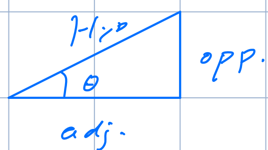

Sin Opp Hyp Cos Adj Hyp Tan Opp Adj

# Trig Functions

Unit circle has $r = 1$

$\sin = \frac{\text{opp}}{\text{Hyp}}$




$\sin\theta : \frac{\text{Opp}}{\text{Hyp}}$

$\cos\theta : \frac{\text{Adj}}{\text{Hyp}}$

$\tan\theta : \frac{\text{Opp}}{\text{Adj}}$

csc → cosecant
sec → secant
cot → cotangent

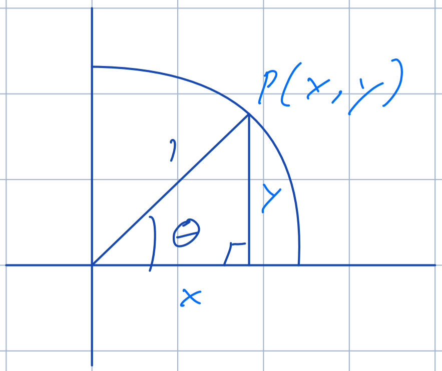
reciprocals

reciprocal = $\frac{1}{\text{no.}}$

$\csc^{-1} = \sin \therefore \csc\theta = \frac{\text{Hyp}}{\text{opp}}$

$\sec^{-1} = \cos \therefore \sec\theta = \frac{\text{Hyp}}{\text{adj}}$

$\cot^{-1} = \tan \therefore \cot\theta = \frac{\text{Adj}}{\text{opp}}$




$$\sin\theta = \frac{y}{1} = y$$
$$\cos\theta = \frac{x}{1} = x$$
$$\tan\theta = \frac{y}{x}$$

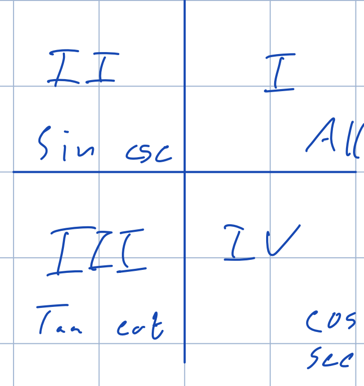
For unit circles where r = 1

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$ Tan identity

$$\sin\theta = \frac{y}{1} \qquad \csc\theta = \sin\theta^{-1} = \frac{1}{y}$$
$$\cos\theta = \frac{x}{1} \qquad \sec\theta = \cos\theta^{-1} = \frac{1}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \tan\theta^{-1} = \frac{x}{y}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$ or $\csc\theta(\cos\theta)$

4 Quadrants A S T C



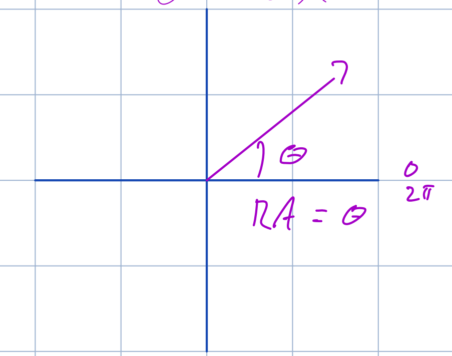

Using reference angles with ASTC we can find out the trig function for any angle θ

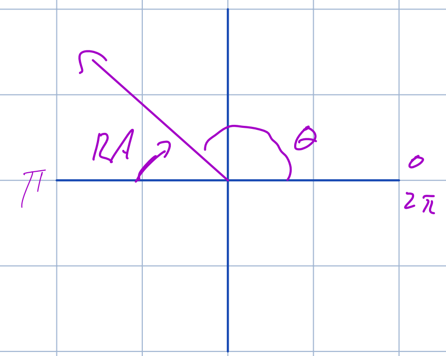
# Ex reference Angles:

Idea: make acute angle w/ x axis, then find trig func of that, use ASTC idea

The reference Angle is the angle that the terminal side makes with the x-axis that happens to be acute

$\theta - 0\pi$

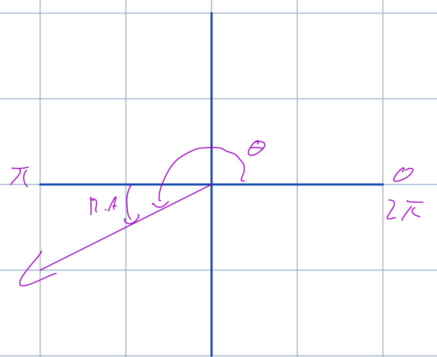


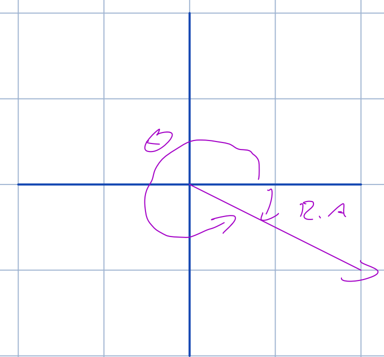




Terminal side is the vector.

$RA = \pi - \theta$

Acute angle between x and terminal side




$R.A = \theta - \pi$

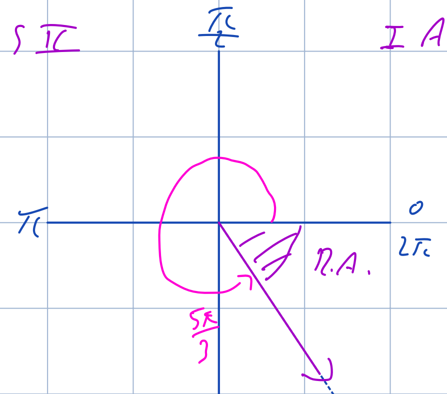
$RA = 2\pi - \theta$

Ref angles are the acute angles between x axis and terminal side

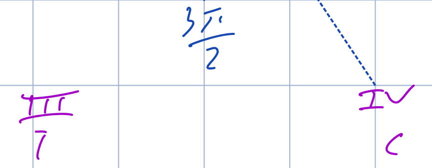
Ex. find sin, cos, TAN of $\frac{5\pi}{3}$

Steps: Find Quadrant? ✓

1. Find quadrant
2. Find reference angle




$$R.A = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$




3. Find all trig functions of the reference angle

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$
$$\tan\left(\frac{\pi}{3}\right) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

We know that Sin is neg in Q4 $\therefore \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

In Q4 only cos is +ive

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$$

Tan is neg on RA.

$$\tan\left(\frac{5\pi}{3}\right) = -\sqrt{3}$$

The ref angle corresponds to the angle in the quadrant that we have the terminal side is in. We just need to assign the correct sign to the angle

Reference Angle is all based in Q1

Graphing: $Y = A\sin(Bx)$ $\quad Y = A\cos(Bx)$

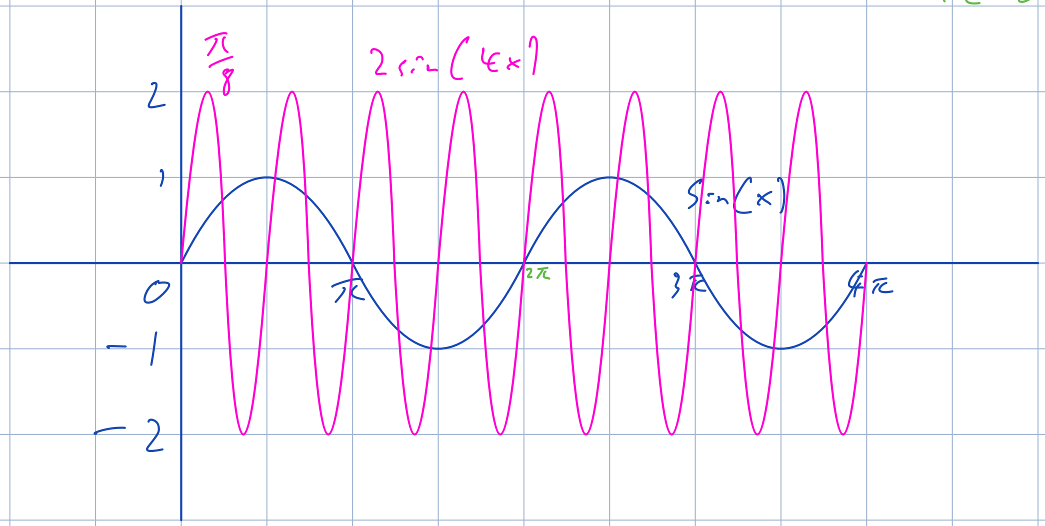
Ex $Y = \sin(x)$ , $Y = 2\sin(4x)$

Y starts at 0, oscillates every $2\pi$, moves between -1, 1

• Amplitude

$|A|$ = Amplitude

• Period $= \frac{2\pi}{B}$




$Y = A\sin(Bx)$

$|A|$ = amplitude

$P = \frac{2\pi}{B}$

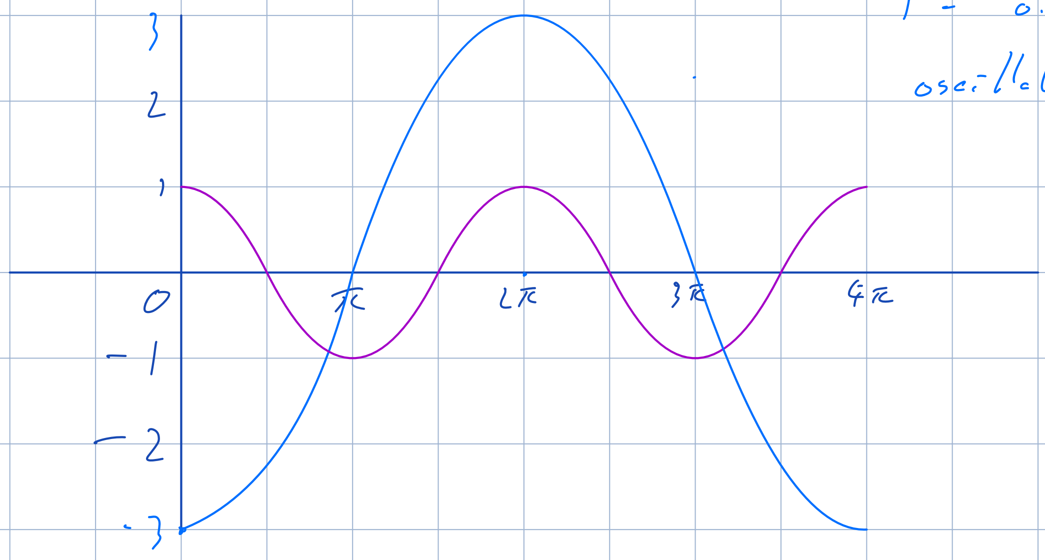
P = frequency

Ex $Y = -3\cos(.5x)$

Amp $= |A| = 3$

$P = \frac{2\pi}{0.5} = 4\pi$

oscillates every $4\pi$




$\cos(0) = 1$

$-3\cos(\frac{0}{2}) = -3$

$\cos(\frac{\pi}{2}) = 0$

$-3\cos(0.5(\frac{\pi}{2})) = -3\cos(\frac{\pi}{4}) = -3(0.866...)$

Need a unit circle

# Shifting trig functions

$|A| = \text{Amp} \quad B = \text{Period, freq}$

$$Y = A\sin(Bx - C)$$
$$Y = A\cos(Bx - C)$$

lets factor out $B$: write as $Y = A\sin\left[B\left(x - \frac{C}{B}\right)\right]$

$$Y = A\cos\left[B\left(x - \frac{C}{B}\right)\right]$$

The $\frac{C}{B}$ = our translation. (shift along x-axis)

If $-\frac{C}{B}$ shift right

$+\frac{C}{B}$ shift left

pos B

$$\left(x - \frac{C}{B}\right)$$

neg B

$$\left(x + \frac{C}{B}\right) = \left(x - \left(-\frac{C}{B}\right)\right)$$

Adding a no. brings the curve forward in time $(x)$ — which makes it earlier, pushing it left.

Subtracting pushes the occ later

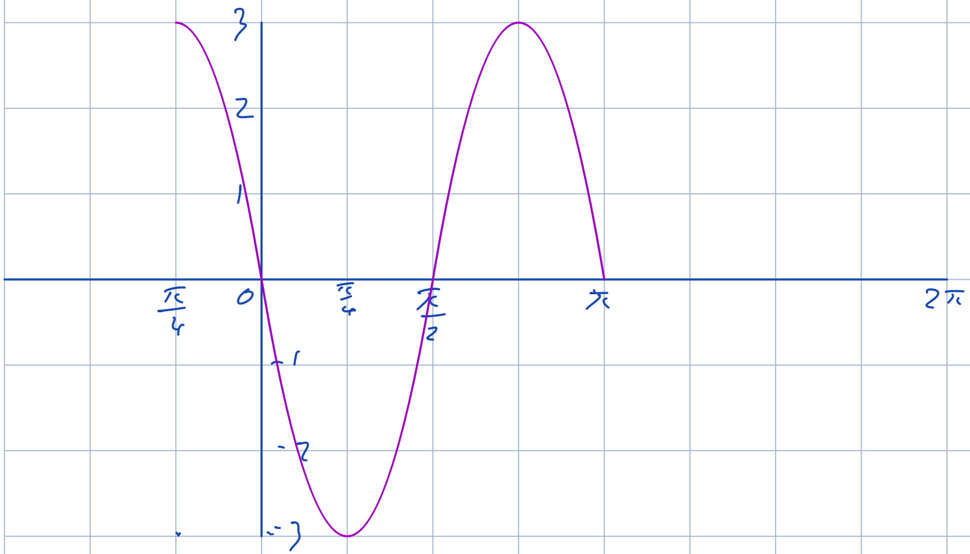
Ex. $Y = 3\cos\left(2x + \frac{\pi}{2}\right)$ fac. $B$

$$= 3\cos\left(2\left(x + \frac{\pi}{4}\right)\right)$$

$|A| = 3$

$B = \frac{2\pi}{2} = \pi$

$\text{Shf} = +\frac{\pi}{4}$




## Combining and Composition of functions:

Functions: $+$, $-$, $\div$, $\times$ and compose functions

$$f(x) = 1 + \sqrt{x-2}, \quad g(x) = x - 3$$

$$(f+g)(x) = (1 + \sqrt{x-2}) + (x-3)$$
$$= -2 + x + \sqrt{x-2}$$

$$(f-g)(x) = 4 + \sqrt{x-2} - x$$

$$(f\cdot g)(x) = (1+\sqrt{x-2})(x-3)$$
$$x - 3 + x\sqrt{x-2} - 3\sqrt{x-2}$$

$$(f \div g)(x) = \frac{(1+\sqrt{x-2})}{(x-3)}$$

Domains of basic operations of 2 functions is the intersection of the original func:

Ex. $f(x) = \sqrt{x}$ , $g(x) = \sqrt{x}$

$$(f \cdot g)(x) = x$$

but the domain is still restricted by $\sqrt{x}$

$D: \cancel{\mathbb{R}}$

$D: = D(f(x)) \cap D(g(x))$

Compositions:

$f(x) = x^3 - 4$

$g(x) = \sqrt{x}$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x})$$

$$= (\sqrt{x})^3 - 4$$

$$(g \circ f)(x) = g(f(x))$$

$$= g(x^3 - 4)$$

$$= \sqrt{(x^3 - 4)}$$

$f(x) = \sqrt{x}$, $g(x) = \frac{1}{x}$, $h(x) = x^3$

$(f \circ g \circ h)$ $\quad f(g(x^3))$

$f\left(\frac{1}{(x^3)}\right)$

$\sqrt{\frac{1}{x^3}} = \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$

$(f \circ g \circ h)(8) = \frac{1}{8^{1.5}}$

$h(x) = (x-7)^3 \rightarrow (f \circ g)(x)$

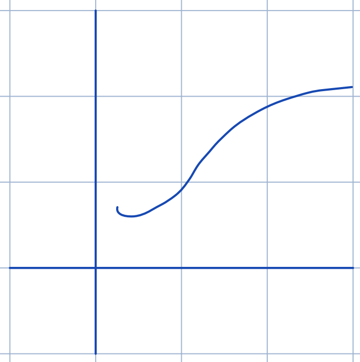
$= f(x) = x^3$

$g(x) = x - 7$

Calculus: Limits are basis of calculus.

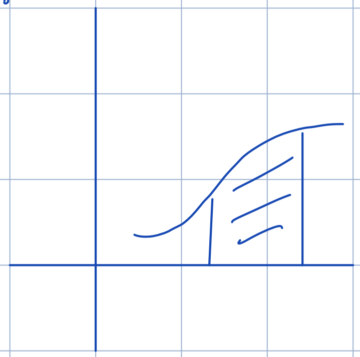
2 basic goals:

1. Given a curve: find the slope at a point on the curve



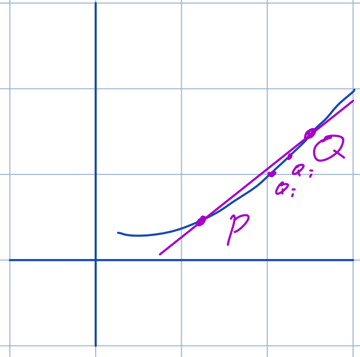
Find the tangent to a curve at a point

2. Area under a curve between 2 points



Tangent Problem:

- Finding limits




What is the slope of the curve at point P

We need the curve

We can use the secant PQ

Is PQ a good Appoximation?

Can we make a better appr?

We can move Q to improve the appr.

Moving Q down the curve we create a more accurate approximation.

We can move Q almost infinitely close to P we will have 2 points that are so close the approximation will essentially be correct.

As Q gets closer to P the secant line resembles the tangent.

Note: $Q \neq P$ because we need 2 points to make a line.

Limits: how close can we move point Q to P without them being the same?

Points can be infinitessimally close

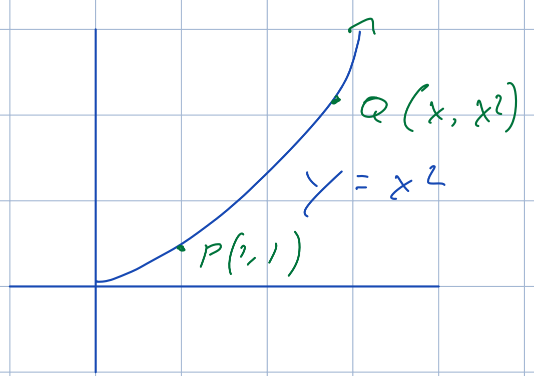
they can be so close the secant will = tangent.

The secant will be identical to the tangent

Moving Q to P but not P is a limit.

we use limits because we need 2 points

Ex. Understanding limits




Find eq. of Tangent line to $y=x^2$ at $(1,1)$

Q is $(x, y)$ but we know $y=x^2$

$\therefore Q = (x, x^2)$

PQ = secant line, line eq $y - y_1 = m(x - x_1)$

For tangent line:

$y - y_1 = m_{tan}(x - x_1)$

$m_{tan}$ is the slope of the Tangent line

Try to find slope of tangent
look a secant line
Use the idea of a secant and make it a tangent line by moving Q really close to P.

slope of Secant $\quad m_{sec} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_Q - y_P}{x_Q - x_P}$

$(x_1, y_1) = P \quad = 1, 1$

$(x_2, y_2) = Q \quad = x, x^2$

$m_{sec} = \dfrac{x^2 - 1}{x - 1} \to \dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$

$\to x + 1 = m_{sec}$

Step 1:

As $Q \to P$    As Q gets close to P the secant gets close to the tangent

$m_{sec} \to m_{tan}$

Step 2: We want the slope at 1,1

If we take $m_{sec}$ at $(1,1)$ we get slope $= 0$

for $\dfrac{(x^2 - 1)}{x - 1} \quad \therefore x \neq 1 \quad \therefore Q \neq P$

So $m_{sec} = \frac{x^2-1}{x-1} \to \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x+1$

There is no domain issue because we never let $x=1$, just very very close.

$m_{sec} = x+1$ As $x \to 1$ what happens?

$Q = 4$, $m_{sec} = 5$

$Q = 2$, $= 3$

$Q = 1.5$ $= 2.5$

$= 1.1$ $= 2.1$

As $x \to 1$ $x$ gets close to 1

$m_{sec} \to 2$ $m_{sec}$ gets close to 2

We know the limit of the secant line $= 2$

$\therefore$ We know the actual slope of the tangent line $= 2$

$m_{tan} = 2$

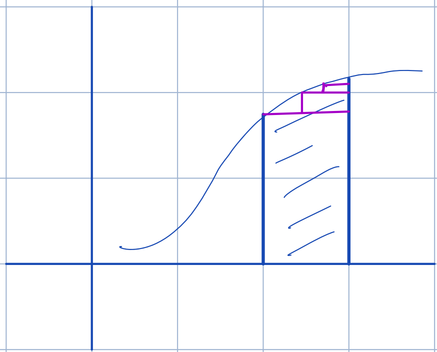
Tangent line: $y - y_1 = m_{tan}(x - x_1)$

$\to y - 1 = 2(x-1)$

$y = 2x - 1$

Using limits to find tan of curve is basic idea of Calc

Area under curve intro:



We estimate the area using increasingly smaller circles infitessimally smaller, we do this using limits.

Define the limit:

What does the function do as the variable approaches a given value?

The limit is about getting close, not what happens at the value.

Ex: what happens to $x^2$ as $x \to 2$. $\lim f(x^2)$

| $x$ | 1 | 1.75 | 2 | 2.1 | 3 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 3.06 | X | 4.41 | 9 |

$f(x) \to L$ as $x \to a$

$\lim_{x \to 2} f(x) = 4$ or $f(x) \to 4$ as $x \to 2$

$\lim_{x \to a} f(x) = L$

$x \neq a$ is part of the definition of a limit. We only care about $f(x)$ near $a$.

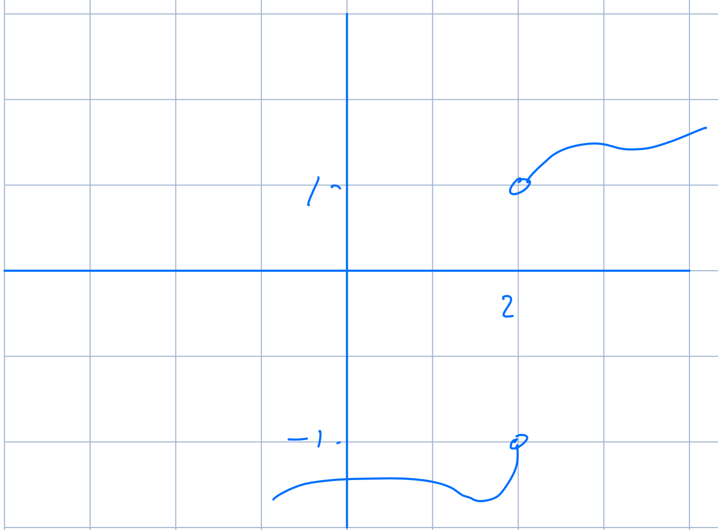
What does $f(x)$ do near $a$.

Ex. $\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$ What does the function val do

| $x$ | .5 | .99 | .999 | 1 | 1.001 | 1.01 | 1.5 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | .67 | .503 | .5002 | | .499 | .497 | .5 |

One sided limits

$\lim_{x \to 2} f(x) = ?$




Right sided limit:

$\lim_{x \to 2^+} f(x)$

Left sided limit:

$\lim_{x \to 2^-} f(x)$

Finding limits for multi sided eqs.

What is the val of $f(x)$ as $x$ approaches 2 from the right from the right side and the left from the left side?

From the right $\lim_{x \to 2^+} f(x) = 1$

From the left: $\lim_{x \to 2^-} f(x) = -1$

Note: For the limit to exist at a point 'a'

$$\lim_{x \to a} f(x) = L$$

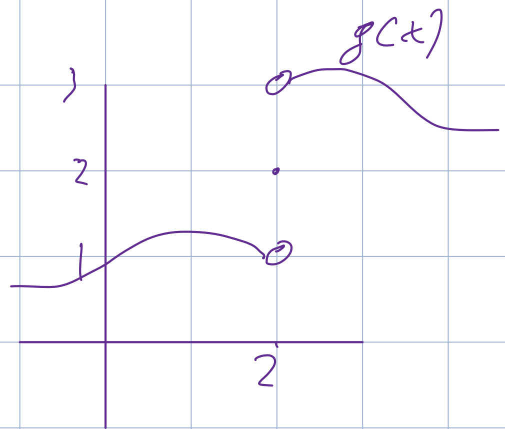
You must have:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

The function from the left and the func from the right have to have the same value

$\therefore$ lim above does not exist

Ex:



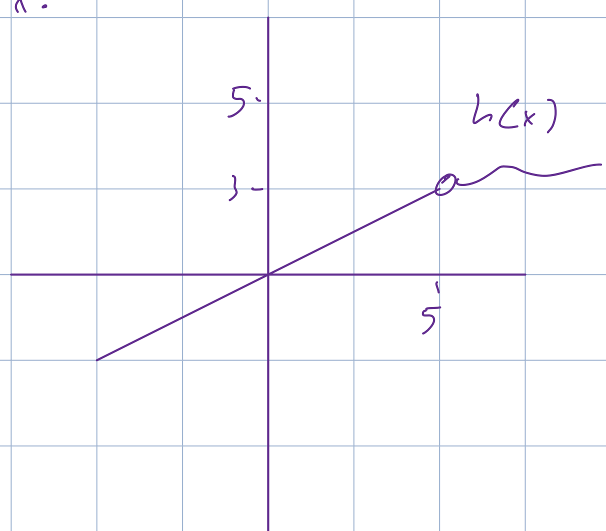

Can we show!

$$\lim_{x \to 2^+} g(x) = 3$$

$$\lim_{x \to 2^-} g(x) = 1$$

$$\lim_{x \to 2} g(x) = n/a - \text{D.N.E}$$

Ex.




$$\lim_{x \to 5^+} h(x) = 3$$

$$\lim_{x \to 5^-} h(x) = 3$$

$$\lim_{x \to 5} h(x) = 3$$

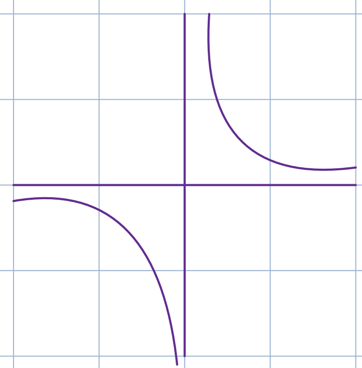
Ex Find Limit of $f(x) = \frac{1}{x}$ as $x \to 0$

| $x$ | $-.5$ | $-.01$ | $-.001$ | $0$ | $.001$ | $.01$ | $.5$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-2$ | $-100$ | $-1000$ | // | $1000$ | $100$ | $2$ |

$\lim\limits_{x \to 0^+} f(x) = \infty$

$\lim\limits_{x \to 0^-} f(x) = -\infty$

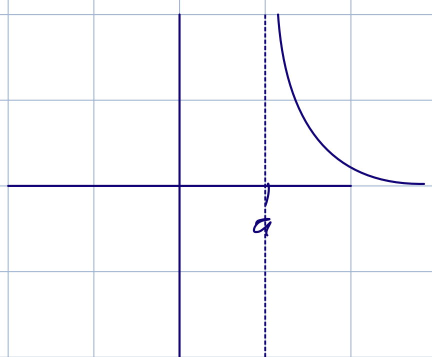
0 is an asymptote
limit D.N.E



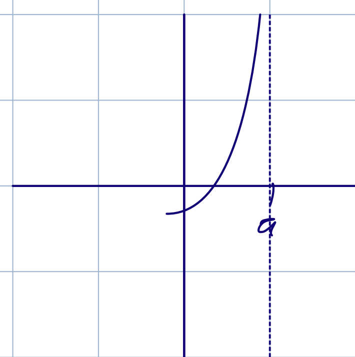
If $\lim\limits_{x \to a^{\pm}} f(x) = \pm\infty$

If there is a limit as $x$ approaches $a$ from either the right or left of some func is $\pm\infty$ its an asymptote

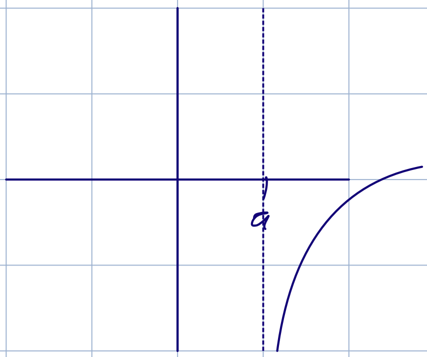
There 4 cases of Asymptotes:



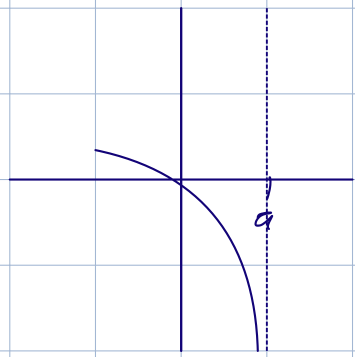

$\lim\limits_{x \to a^+} f(x) = +\infty$




$\lim\limits_{x \to a^-} f(x) = +\infty$



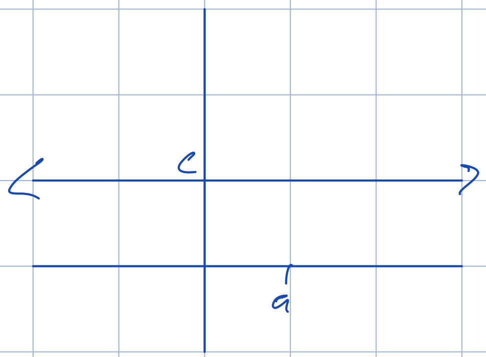
$\lim_{x \to a^+} f(x) = -\infty$



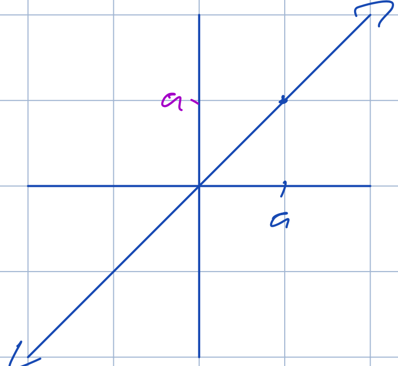
$\lim_{x \to a^-} f(x) = -\infty$

Computing limits – Properties of limits:

Basics: 1. $\lim_{x \to a} C = a$ limit of contsants = constant itself



2. $\lim_{x \to a} x = a$ limit for some function x at point a = x



3. $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

Properties: $\lim_{x \to a} f(x) = L_1$ $\quad \lim_{x \to a} g(x) = L_2$

Note the approach is a for both limits

5 properties:

1. $\lim_{x \to a} \left[ f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

Adding and subtracting limits works, limits are additive

2. $\lim_{x \to a} \left[ f(x) \cdot g(x) \right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

3. $\lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ $\quad \lim_{x \to a} g(x) \neq 0$

4. $\lim_{x \to a} \left[ f(x) \right]^n = \left[ \lim_{x \to a} f(x) \right]^n$

$\rightarrow \lim_{x \to a} \sqrt[n]{f(x)} \rightarrow \sqrt[n]{\lim_{x \to a} f(x)}$

Ex

$$\lim_{x\to 2}\left(x^3-2x+7\right) \to \lim_{x\to 2} x^3 - \lim_{x\to 2} 2x + \lim_{x\to 2} 7$$

$$\to \left[\lim_{x\to 2} x\right]^3 - \left(\lim_{x\to 2} 2\right)\cdot\left(\lim_{x\to 2} x\right) + \lim_{x\to 2} 7$$

broke up complex polynomial into limits of constants and x-terms

$$\to 2^3 - 2\cdot 2 + 7$$

$$\to 8 - 4 + 7$$

$$\to 11 \qquad \lim_{x\to 2}\left(x^3-2x+7\right)$$

This works for every polynomial.

The limit of a polynomial as $x \to a = P(a)$

To find the limit of a polynomial plug in the no.

$$\lim_{x\to a} P(x) = P(a)$$

Ex

$$\lim_{x\to 2}\left(x^5-3x+4\right)^3 = \left(2^5-6+4\right)^3$$

$$=(30)^3$$

$$=27{,}000$$

Ex:

$$\lim_{x \to 2} \frac{4x^2+1}{x-3}$$

$$\rightarrow \frac{\lim_{x \to 2} 4x^2+1}{\lim_{x \to 2} x-3} = -17$$

Ex $$\lim_{x \to 1} \sqrt[3]{\frac{5x+7}{x^2+1}} = \sqrt[3]{\frac{5+7}{2}} = \sqrt[3]{6}$$

with limits we don't care factor out

Ex $$\lim_{x \to 2} \frac{x^2-4}{x-2} \rightarrow \lim_{x \to 2} \frac{(\cancel{x-2})(x+2)}{\cancel{x-2}}$$ Domain range

$$\rightarrow \lim_{x \to 2} x+2 = 4$$

Ex $$\lim_{x \to -4} \frac{2x+8}{x^2+x-12} \rightarrow \frac{2\,(\cancel{x+4})}{(\cancel{x+4})(x-3)}$$

$$\rightarrow \lim_{x \to -4} \frac{2}{x-3} = -3.5$$

Ex:

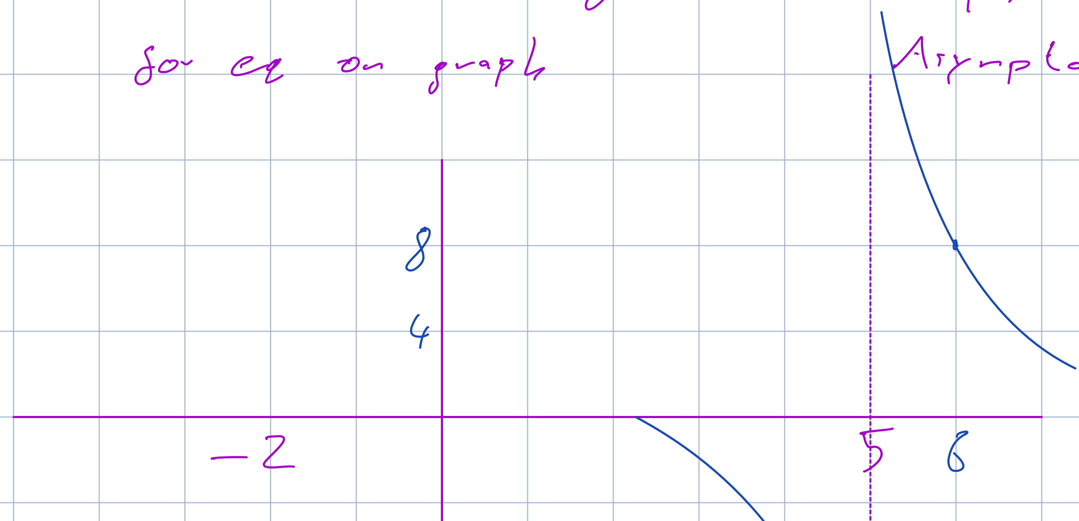
$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} \to \lim_{x \to 5} \frac{(x-5)(x+2)}{(x-5)(x-5)}$$

$$\to \lim_{x \to 5} \frac{x+2}{x-5} \to \lim_{x \to 5} \frac{x+2}{x-5}$$

→ Do a sign analysis test

Put the numbers gon 0 on top, bottom

for eq on graph




Plug in nos to either side of Asym

$6 \to \frac{x+2}{x-5} = 8$

$5.1 = \frac{7.1}{0.1} = 71$

$4 = \frac{6}{-1} = -6$

$4.9 = \quad = -69$

$$\lim_{x \to 5} \frac{x+2}{x-5} \quad \text{D.N.E}$$

Note:

① if $\frac{0}{0}$ -> common factor at that point.
We can simplify

② If you can't cancel the problem then there's an asymptote.
— limit might not exist.

ex.

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} \quad \text{for } x=1 \to \frac{0}{0}$$

$$\cancel{\lim_{x\to 1} \frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1} = \lim_{x\to 1} \sqrt{x}-1}$$

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} \cdot \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)} \to \frac{\cancel{(x-1)}(\sqrt{x}+1)}{\cancel{x-1}}$$

$$\lim_{x\to 1} \sqrt{x}+1 \qquad L = 2$$

Ex. $$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

at $x=0 \to \frac{0}{0}$

$\therefore$ factor

$\rightarrow \lim_{x \to 0} \dfrac{x}{x(\sqrt{x+1}+1)} \rightarrow \lim_{x \to 0} \dfrac{1}{\sqrt{x+1}+1}$

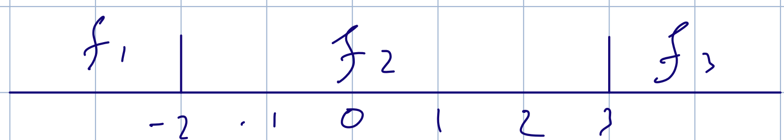
at $x = 0 = \frac{1}{2}$

Piece·Wise limit

Take one sided limits and see if they're equal:

Ex: $f(x) = \begin{cases} f_1 \quad \dfrac{1}{x+2}, & x < -2 \\ f_2 \quad x^2 - 5, & -2 < x \leq 3 \\ f_3 \quad \sqrt{x+13}, & x > 3 \end{cases}$

look at $x \to -2$, $x \to 3$




$\lim_{x \to -2^+} x = f_1 \quad \dfrac{1}{x+2} = -\infty$

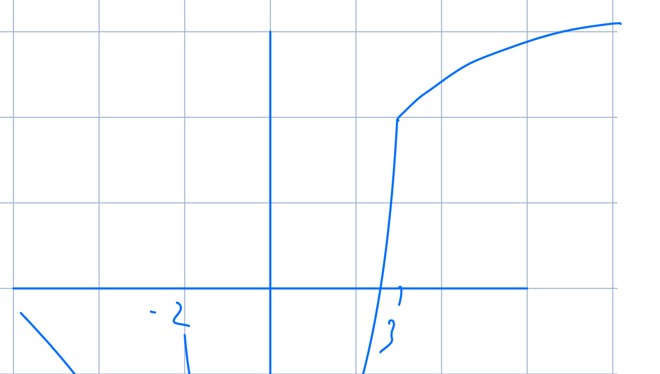
$= \frac{1}{0}$ at $x = -2$, at $x = -1.8 = 5$, at $-1.9 = 10$

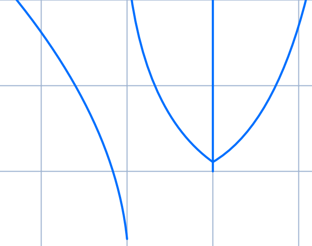
$\lim_{x \to -2^-} x = f_2 \quad x^2 - 5$

$= -1$, $L = -1$

$\lim_{x \to 3^+} x = f_2 \quad x^2 - 5$

at $x : 3 = 4$





$$\lim_{x \to 3^-} x = f, \sqrt{x+13}$$

at $x: 3 = \sqrt{16} = 4$

$$\lim_{x \to 3} f(x) = 4 \qquad \lim_{x \to -2} \quad \text{D.N.E}$$

Trig: limits of trig functions

$\sin(x)$ $\cos(x)$ are continuous every where.

$$\lim_{x \to a} \sin(x) = \sin(a) \qquad \text{ditto for } \cos$$

for tan: $$\lim_{x \to a} \tan(x) = \lim_{x \to a} \frac{\sin(x)}{\cos(x)}$$

$$\rightarrow \frac{\lim_{x \to a} \sin(x)}{\lim_{x \to a} \cos(x)} \rightarrow \frac{\sin(a)}{\cos(a)} = \tan(a)$$

$$\lim_{x \to a} \tan(x) = \tan(a) \qquad \cos(a) \neq 0$$

$$x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$$

Ex

$$\lim_{x\to1} \cos\left(\frac{x^2-1}{x-1}\right) \to$$ Cont. functions can be treated as compositions

$$\cos\left[\lim_{x\to1} \frac{x^2-1}{x-1}\right] \to \cos\left[\lim_{x\to1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}\right]$$

$$\to \cos\left(\lim_{x\to1} x+1\right) \to \cos(2)$$

Ex

$$\lim_{x\to\frac{\pi}{2}} \left[3x^2 + \cos x\right]$$

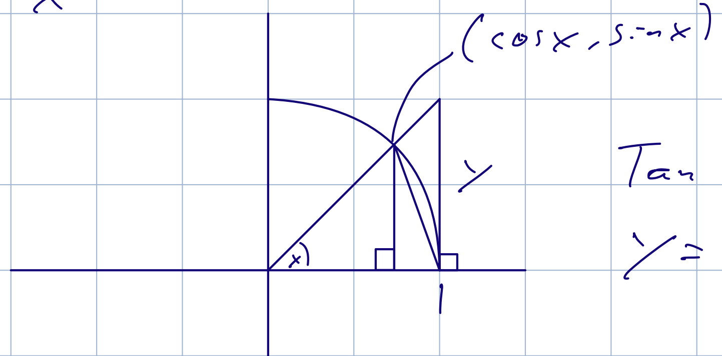
at $\frac{\pi}{2}$

$$3\left(\frac{\pi}{2}\right)^2 + \cos\frac{\pi}{2}$$

$$\frac{3\pi^2}{4} + 0$$

$$= \frac{3\pi^2}{4}$$

Ex. $\lim_{x\to0} \frac{\sin(x)}{x}$ at $0 = \frac{0}{0}$




$\tan x = \frac{y}{1}$

$y = \tan x$

Area

Big : $\frac{1 \cdot \tan x}{2}$

Small : $\frac{1 \cdot \sin x}{2}$

Sector : $r \cdot \text{angle} = \frac{1 \cdot x}{2}$

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}$$

$$\sin x < x < \tan x \quad \div \sin x$$

$$1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \qquad \text{take}^{-1}$$

$$= 1 > \frac{\sin(x)}{x} > \cos(x)$$

$$\lim_{x \to 0} 1 = 1, \quad \lim_{x \to 0} \cos(x) = 1$$

Squeeze theorem means the bounds dictate the limit, both are 1 $\therefore \lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\lim_{x \to 0} \frac{1-\cos(x)}{x} \quad \text{at } 0 = \frac{0}{0}$$

Find identity: Multiply by conjugate

$$\lim \frac{1-\cos(x)}{x} \times \frac{1+\cos(x)}{1+\cos(x)}$$

$$\to \lim_{x \to 0} \frac{1-\cos^2(x)}{x(1+\cos x)} \to \frac{\sin^2(x)}{x(1+\cos x)}$$

$$\sin^2 x + \cos^2 x = 1 \quad \therefore \quad 1-\cos^2 x = \sin^2 x$$

$$\to \lim \frac{\sin(x) \cdot \sin(x)}{x \cdot (1+\cos(x))}$$

$$\to \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin(x)}{1+\cos(x)}$$

$$\to 1 \cdot \frac{0}{2}$$

$$L = 0 \quad \therefore \quad \lim \frac{1-\cos x}{x} = 0$$

Ex $\lim_{x \to 0} \frac{\tan x}{x}$ at $0 = \frac{0}{0}$

$$\to \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} \to \lim_{x \to 0} \frac{\sin x}{\cos x} \cdot \frac{1}{x}$$

$$\to \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$$\to 1 \cdot \lim \frac{1}{\cos x}$$

$$\to 1 \cdot 1 = 0 \qquad \lim_{x \to 0} \frac{\tan x}{x} = 1$$

$\lim_{x \to 0} \frac{\sin(2x)}{x}$ at $0 = \frac{0}{0}$ keep the ratio somehow

$$\lim \frac{\sin(2x)}{x} \cdot \frac{2}{2} \to \frac{2\sin(2x)}{2x}$$

$$\to 2 \cdot \lim_{x \to 0} \frac{\sin(2x)}{2x} = 2 \cdot 1$$

$$\rightarrow \lim_{x \to 0} \frac{\sin(2x)}{x} = 2$$

ex. $$\lim_{x \to 0} \frac{\sin(5x)}{\sin(6x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$\rightarrow \lim_{x \to 0} \frac{\frac{\sin(5x)}{x}}{\frac{\sin(6x)}{x}}$$

Sin x over x is an ID we can use

$$\lim_{x \to 0} \frac{\sin(5x)}{x} \cdot \frac{5}{5} = \lim_{x \to 0} \frac{5\sin(5x)}{5x}$$

$$\rightarrow 5 \lim_{x \to 0} \frac{\sin(5x)}{5x} \rightarrow 5 \lim_{x \to 0} 1$$

$$\therefore \frac{5}{6}$$

$$\lim_{x \to 0} \frac{\sin(x^2)}{x} \cdot \frac{x}{x} \rightarrow \lim_{x \to 0} \frac{x \sin(x^2)}{x^2}$$

$$\rightarrow \lim_{x \to 0} x \cdot \lim_{x \to 0} \frac{\sin(x^2)}{x^2}$$

$L = 0 \cdot 1 = 0$

ex:

$$\lim_{x\to 0} \frac{\sin^2(x)}{x} = \lim_{x\to 0} \sin(x)\cdot \sin(x)$$

$$\to \lim_{x\to 0} \sin(x) \cdot \lim_{x\to 0} \frac{\sin(x)}{x}$$

$$\to L = 0 \cdot 1$$

ex:

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ : has no limit.

$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$ Also has no limit?

Sin is bound between 1, -1

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|$$

$$\underbrace{\lim_{x\to 0} -|x| = 0 \qquad \lim_{x\to 0} |x| = 0}_{\text{Squeeze theorem}} \to \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

ex:
$$\lim_{x\to 0} \frac{2-\cos(3x)-\cos(4x)}{x}$$

$$\lim_{x\to 0} \frac{1-\cos(3x)+1-\cos(4x)}{x}$$

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x} + \lim_{x\to 0} \frac{1-\cos(4x)}{x}$$

$$\lim_{x\to 0} \frac{3}{3} \cdot \frac{1-\cos(3x)}{x} \to 3 \lim_{x\to 0} \frac{1-\cos(3x)}{3}$$

$$\to 3 \cdot 0$$

$$\lim_{x\to 0} \frac{4}{4} \cdot \frac{1-\cos(4x)}{x} \to 4 \lim_{x\to 0} \frac{1-\cos(4x)}{4}$$

$$\to 4 \cdot 0 \qquad L = 0$$

ex:
$$\lim_{x\to 0} \frac{x^2-3\sin(x)}{x} \to \lim \frac{x^2}{x} - \frac{3\sin(x)}{x}$$

$$\to x - 3 \cdot \frac{\sin(x)}{x} \to x - 3 \cdot 1$$

$$\to 0 - 3 \cdot 1$$

$$L = -3$$

ex $\lim_{T \to 0} \dfrac{T^2}{1-\cos^2(T)} \to \dfrac{T^2}{\sin^2 T}$

$\to \lim_{T \to 0} \dfrac{T}{\sin(T)} \cdot \dfrac{T}{\sin(T)}$ $\quad L = 1 \cdot 1$

ex. $\lim_{x \to 0} \dfrac{x}{\cos(\frac{1}{2}\pi - x)}$ $\quad \cos(\frac{1}{2}\pi) = 0 \therefore \frac{0}{0}$

$\lim_{x \to 0} \dfrac{x}{\sin(x)}$ $\quad L = 1$

ex

$\lim_{\theta \to 0} \dfrac{\theta^2}{1-\cos\theta} \cdot \dfrac{1+\cos\theta}{1+\cos\theta}$

$\lim_{\theta \to 0} \dfrac{\theta^2(1+\cos\theta)}{1-\cos^2\theta} \Rightarrow \lim_{x \to 0} \dfrac{\theta^2(1+\cos\theta)}{\sin^2\theta}$

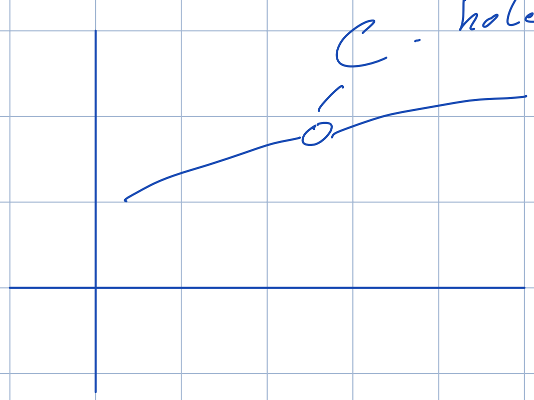
$\lim_{\theta \to 0} \dfrac{\theta^2}{\sin^2\theta} \cdot (1+\cos\theta) \to 1 \cdot 2 \quad L = 2$

Continuity: a function is continuous if it has no holes, breaks or asymptotes

No holes, breaks, asymptotes

$f(x)$ is cont. at point $c$ if

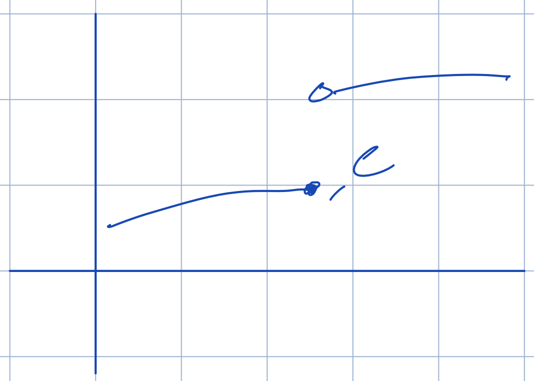
1. $f(c)$ is defined.
2. The limit of $f(x)$ $\lim_{x \to c} f(x)$ must exist
3. $\lim_{x \to c} f(x) = f(c)$




hole: not cont. at c

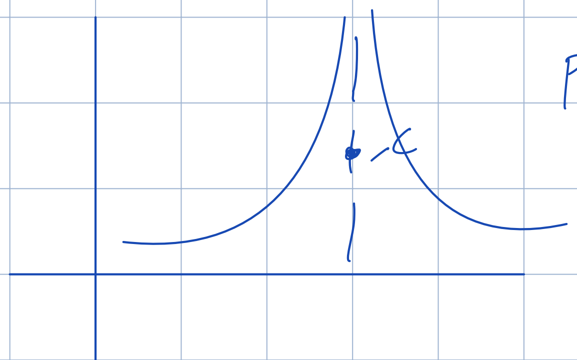
a removable discontinuity

: can fill with single point.




Point at c, but no limit.

Jump discont.




Point at c, limit but $\lim \neq f(c)$

Infinite discontinuity.

ex: are these cont at $x=2$

$f(x) = \frac{x^2-4}{x-2}$ ; no because theres a hole at $x=2$

$\frac{0}{0}$ = hole

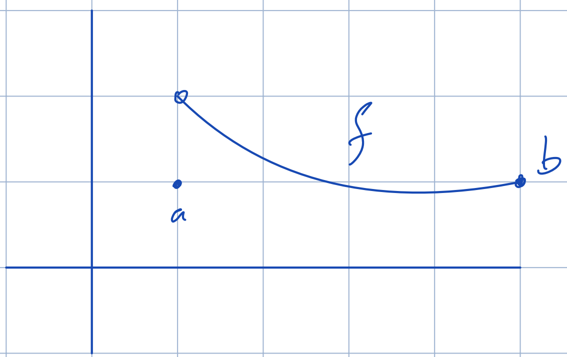
$g(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$ no because $g(2) \neq \lim_{x \to 2} g(x)$

$h(x) \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$ yes $\lim_{x \to 2} h(2) = 4$, $h(2) = 4$

$\lim_{x \to 2} \frac{x^2-4}{x-2} \to \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$ $\lim_{x \to 2} x+2$

$L = 4$

If $f$ is cont at every point between $a, b$
then $f$ is cont. on $(a, b)$




Cont. on $f(x)$ $a < x \le b$

In general:

cont. from left at $c$:

$$\lim_{x \to c^-} f(x) = f(c)$$

Cont from right at $c$:

$$\lim_{x \to c^+} f(x) = f(c)$$

Proving Continuity

Prove $f(x) = \sqrt{16 - x^2}$ is cont $(-4 : 4)$

1. check $(-4, 4)$ $\sqrt{16 - (-4^2)}$
$= \sqrt{0} = 0$
$\sqrt{16 - (4^2)}$
$= \sqrt{0} = 0$ No issue bet. $-4 : 4$

$$\lim_{x \to c} f(c) = f(c)$$

2. Check $\lim_{x \to 4^-} f(x)$ $\sqrt{16 - x^2}$

at $x = 4$ $L = 0 = f(4)$

3. $\lim_{x \to -4^+} f(x)$

at $x = -4$ $L = 0 = f(-4)$

Properties: $f, g$ are cont at $c$

$\therefore$ True

1. $f + g$
2. $f - g$
3. $f \cdot g$

All cont. at $c$

4. $f \div g$ is cont. at 'c' unless $g(c) = 0$

If $g(c) = 0$ there is a discont. @ 'c' for $\frac{f}{g}$

If a func is cont @ $c$ the $\lim_{x \to c} f(x) = f(c)$

$P(x)$ = polynomial

$\lim_{x \to c} P(x) = P(c)$ $\therefore$ every polynomial is cont. everywhere.

This means:

1. Every Polynomial is cont. everywhere

2. Every rational func is cont everywhere as all rational funcs are $\frac{\text{polynom}}{\text{polynom}}$ except where denominator = 0. at that point there will be a discontinuity.

   1: hole $\frac{0}{0}$ 2. Asymptote $\frac{n}{0}$

ex. $f(x) = \dfrac{x^2-4}{x^2+x+6}$ find disconts at denom = 0

$$x^2 + x - 6 = 0$$

$$(x-2)(x+3) = 0$$

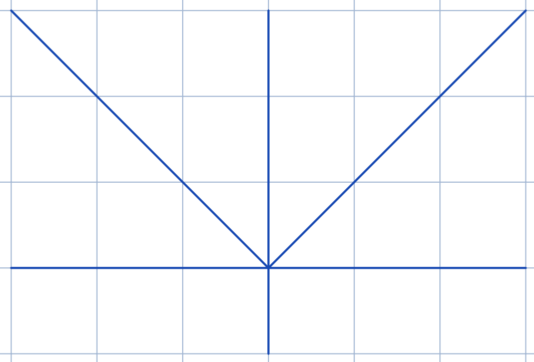
$\left.\begin{array}{l} x = -3 \\ x = 2 \end{array}\right\}$ disconts at $x = -3, 2$

$\dfrac{(x+2)(x-2)}{(x+3)(x-2)}$ we can cross out $(x-2)$ $\therefore$ its a hole because $\dfrac{a(0)}{c(0)} = \dfrac{0}{0}$

Can't factor out $(x+3)$

$\therefore$ its $\dfrac{ab}{0(d)}$

Prove $f(x) = |x|$ is cont. every where



$f(x) = |x|$

prove for $x = (0, +\infty)$ $f(x) = f(-x)$

$$f(x) = |x| = \begin{cases} x : & x > 0 = \text{Polynom} \therefore \text{cont.} \\ 0 : & x = 0 = \text{Is the limit.} \\ -x : & x < 0 = \text{Poly} \therefore \text{cont.} \end{cases}$$

$$\lim_{x \to 0^+} x = 0 \qquad \lim_{x \to 0^-} -x = 0$$

Compositions: If $\lim_{x \to c} g(x) = L$ and $f$ is cont at $L$

Then $\lim_{x \to c} f(g(x)) = \lim_{x \to c} f(g(c))$

$= f(L) = f\left(\lim_{x \to c} g(x)\right)$

We can seperate limits by composition

ex:

$$\lim_{x \to 4} |10 - 3x^2| \to \left|\lim_{x \to 4} 10 - 3x^2\right|$$ is a composition

$$|-38| = 38$$

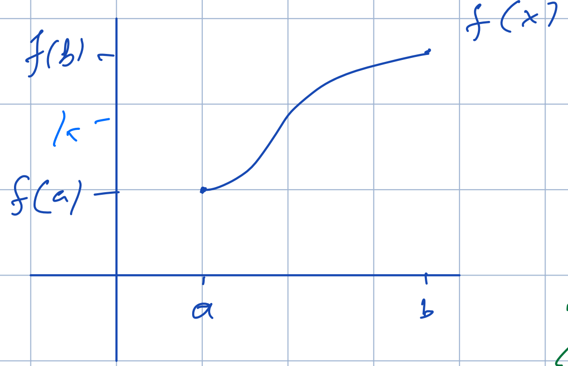
If 2 funcs are cont. every where then their comp is cont. every where

If $f$ is cont in its domain the $f^{-1}$ is cont is also cont but in its respective domain, the range of

$f(x) = x^3$ Poly cont on $(-\infty$ to $\infty)$

$y = x^3$ $\quad$ $x = y^3$, $\sqrt[3]{x} = y$ $\to$ $f^{-1}(x) = \sqrt[3]{x}$

Intermediate value theorem




$f(x)$ is cont on $[a, b]$

Say $f(a) < k < f(b)$

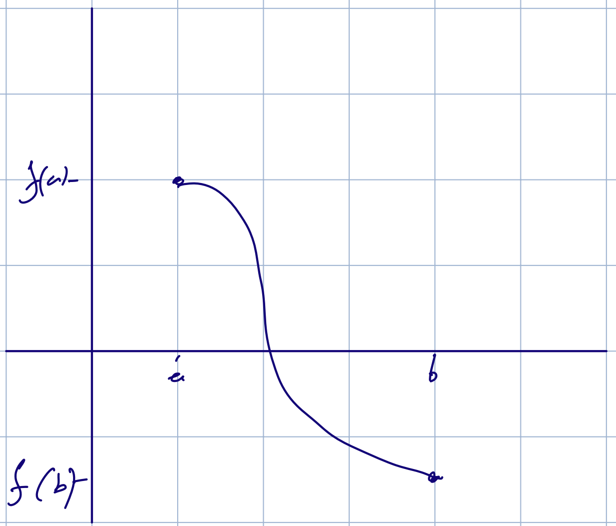
is $f^{-1}(k)$ between $a : b$

Yes - there is at least one no. $c$ such that $f(c) = k$, where

$a < c < b$

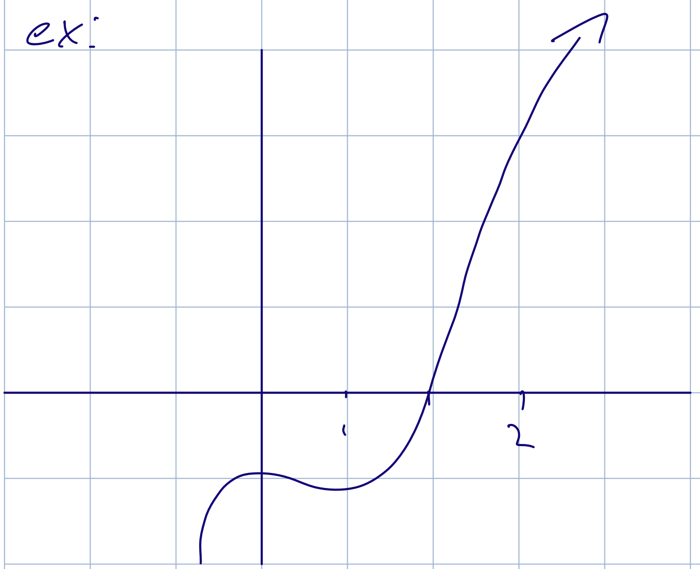
# Intermediate value theory

Can approx roots w/ IVT




The point of 0 is an interval
find the input that gives 0
we know that there is a 0 val
because our 2 points have
different signs
There must be at least one
Root on [a, b]

ex:




$y = x^3 - x - 1$

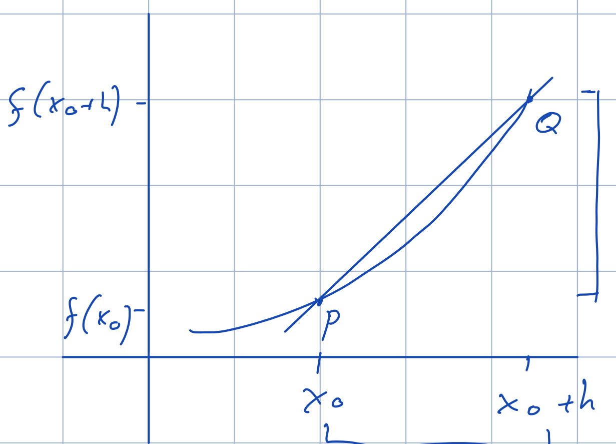
Limits and slope of a curve

1.5. Limits to rates of change

Finding tangent to curve.

Tangent lines and rates of change.

Slope of curve at a point.




Find slope of secant line

$x_0$ is a point, $h$ is a tiny distance.

$(x_0+h) - x_0 = h$

Take $h$ and make it approach 0

$\Delta x = h$

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x}$$

$\Delta y = f(x_0+h) - f(x_0)$

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{h}$$ diff. quotient

This is the slope of a secant but if we move Q infinitely close to P we will get a tangent to P. We do this by using a limit of $h \to 0$

letting $h$ approach 0 allows us to get the slope of the tangent line.

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

ex. Find eq of tangent line to $y = x^2$ @ $1,1$

$f(x) = x^2 \qquad x_0 = 1$

$$f(x_0 + h) = f(1+h) = (1+h)^2$$
$$= h^2 + 2h + 1$$

$$m_{tan} = \lim_{h \to 0} \frac{f(h+1) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2h + 1 - 1}{h} = \lim_{h \to 0} h + 2$$

@ $h = 1 \quad l = 2$ $\qquad (1+h)^2 \to 2(1+h)^1$

$$= h + 2$$

$m_{tan} = 2 \qquad$ @ $h = 0 = 2$

$m_{tan} = 2$ @ $1,1 \qquad y - y_1 = m(x - x_1)$

$$y - 1 = 2(x - 1)$$

$$y = 2x - 1$$

$$y = mx + c$$
$$1 = 2x + c$$
$$1 = 2 + c$$
$$c = -1$$
$$y = 2x - 1$$

This is only the slope at 1 point on the curve.

ex. eq of $m_{tan}$ $y = \frac{3}{x}$ @ $3, 1$

$$y = 3x^{-1}$$

$$m_{tan} = -3x^{-2} ?$$ $x_0 = 3$

$$f(x) = \frac{3}{3}$$ $f(x+h)$

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\to \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$\to \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h}$$

$$\frac{\frac{1}{2}}{2} = \frac{1}{2(2)} = \frac{1}{4}$$

$$\rightarrow \lim_{h\to 0} \frac{\frac{3}{3+h} - \frac{3+h}{3+h}}{h}$$

$$\rightarrow \lim_{h\to 0} \frac{\frac{3-3-h}{3+h}}{h} \rightarrow \lim_{h\to 0} \frac{\frac{-h}{3+h}}{\frac{h}{1}}$$

Mult:

$$\frac{\frac{-h}{3+h}}{\frac{h}{1}} \rightarrow \frac{-h}{3+h} \cdot \frac{1}{h} \rightarrow \frac{-h}{h(3+h)}$$

Reciprocate

$$\lim_{h\to 0} \frac{-1}{3+h} \quad @\ h=0 \quad m_{tan} = -\frac{1}{3}$$

$$m_{tan} = -3x^{-2}$$

$$= -3\left(x^{-2}\right)$$

$$= -3\left(\frac{1}{x^2}\right)$$

$$= -3\left(\frac{1}{9}\right)$$

$$= -\frac{3}{9}$$

$$= -\frac{1}{3}$$

$y = mx + c$

$y = -\frac{1}{3}x + c$

$1 = -\frac{1}{3}(3) + c$

$1 = -1 + c$

$c = 2 \quad \therefore$

$y = -\frac{1}{3}x + c$

How do we get the eq for entire slope?

Slope of tangent line to $y\sqrt{x}$ @ any point

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\to \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

Change sign a mult. by the conjugate

$$\to \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \to \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$\to \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \quad @\ h = 0 \to \frac{1}{2\sqrt{x}}$$

$m = \dfrac{1}{2\sqrt{x}}$ this is the slope of our curve

The slope isn't the same as the equation of a line "$Y = mx + c$", it is simply the slope $m_{tan}$ at any point in the curve

Applications: diff. between avg. velocity and instantaneous velocity

$$V_{avg} = \frac{\text{Displacement}}{\text{Time}} = \frac{f(T_0 + h) - f(T_0)}{h}$$

ex. find avg Vel of $S(T) = 1 + 3T - 2T^2$ on $[1, 3]$

$T_0 = 1$, $T_1 = 3$ $\quad h = T_1 - T_0 = 2$

$$\lim_{h \to 2} \frac{S(1+2) - S(1)}{2} \to \frac{S(3) - S(1)}{2}$$

$$\to \lim_{h \to 2} \frac{1 + 9 - 18 - (1 + 3 - 2)}{2}$$

$$= \frac{-8 - 2}{2} = -5$$

# Instantaneous Vel

$$V_{ave} = \frac{f(T_0+h) - f(T_0)}{h} \rightarrow h = \text{Time} \rightarrow 0$$

Almost 0 time lapses in an instant. $\therefore h \rightarrow 0$

$$\lim_{h\to 0} \frac{f(T+h) - f(T)}{h}$$

$$\rightarrow \frac{1+3(T+h) - 2(T+h)^2 - 1+3T - 2T^2}{h}$$

ex: $s(T) = 500 - 16T^2$ $\quad V_{int}$ @ 5s

$$\lim_{h\to 0} \frac{s(T_0+h) - s(T_0)}{h}$$

$$\rightarrow \lim_{h\to 0} \frac{s(5+h) - s(5)}{h}$$

$$\rightarrow \lim_{h\to 0} \frac{(500 - 16(5+h)^2) - 500 + 16(5)^2}{h}$$

$$\rightarrow \lim_{h\to 0} \frac{400 - 16(5+h)^2}{h}$$

$$\to \lim_{h\to 0} \frac{400 - 16(25 + 10h + h^2)}{h}$$

$$\to \lim_{h\to 0} \frac{-16h^2 - 160h}{h} = -16h - 160$$

$$m_{tan} = -160 \quad @ \; t = 5$$
$$h \to 0$$

generic $\quad s(T+h) = 500 - 16(T+h)^2$

$$s(T) = 500 - 16T^2$$

$$= \lim_{h\to 0} \frac{500 - 16(T+h)^2 - (500 - 16T^2)}{h}$$

$$\lim_{h\to 0} \frac{500 - 16T^2 - 32Th + 16h^2 - 500 + 16T^2}{h}$$

$$\to \lim_{h\to 0} \frac{-16h^2 - 32Th}{h} \to \lim_{h\to 0} -16h - 32T$$

as $h \to 0$ $\quad -32T =$ General formula for slope

$V_{init} = -32T$ $\quad$ slope @ $5_s$ $= -32(5) = -160$

# Rates of change:

Slope is a rate of change

$Y = 3x - 1 \qquad roc = \frac{3}{1}$

$Y = -x - 1 \qquad roc = -1$

Istantaneous R.O.C is the slope of a curve at a point.

Rate of change

$$r_{ave} : \frac{f(x_0+h) - f(x_0)}{h}$$

$$r_{inst} : \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

ex: Avg, Ins ROC of a curve

$f(x) = 3x^2 - 4$

Avg. $[2, 5]$ $\quad x_0 = 2, \; h = 3$

Inst : $-2$ $\quad x_0 = -2, \; h = 0$

# Introduction to the derivative of a function

## 2.1 : The derivative

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ * Slope of a curve at a point

Taking the derivative is finding a curve at a point.

The deriv of x w/ respect to x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)\ f(x)}{h}$$ also f prime of x

Ex: Find deriv of

$f(x) = 2x^2 - 3$ then find eq. of tan line to $f(x)$ at $2:5$

$= 4x$

$f(x+h) = 2(x+h)^2 - 3$

$f(x)\ \ 2x^2 - 3$

$$\lim_{h \to 0} \frac{(2x^2 + 4xh + 2h^2) - 3 - 2x^2 + 3}{h}$$

$$\to \lim \frac{4xh + 2h^2}{h} \to \lim_{h \to 0} 4x + 2h$$

@ $h \to 0$ $\quad 4x$

$\therefore\ f'(x) = 4x = \text{slope of } 2x^3 - 3$

$y - y_1 = m(x - x_1)$ @ 2, 5

$y - 5 = 8(x - 2)$

$y = 8x - 16 + 5$

$y = 8x - 11$

ex. find $f'(x)$ for $f(x) = 2x^3 - x$

$= 6x^2 - 1$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^3 - (x+h) - (2x^3 - x)}{h}$$

$$\to \lim_{h \to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - x - h - 2x^3 + x}{h}$$

$$\to \lim_{h \to 0} \frac{2x^3 - 2x^3 + 3x^2h + 3xh^2 + h^3 - x + x - h}{h}$$

$$\to \lim_{h \to 0} 6x^2 + 6xh + 2h^2 - 1$$

$$= 6x^2 + 0 + 0 - 1 = 6x^2 - 1$$

Slope of $2x^3 - x = 6x^2 - 1$

@ $x = 3$

$m = f'(3) = 53$

$y = f(3) = 51$

Find $f'(x)$ for $f(x) = 3x + 2$

$3x + 2$ is a line, not a curve

$f'(x) = 3$

$\therefore$ slope $= m$.

ex. $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ $f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

eq. of tangent @ $x = 4$

$y = \sqrt{4} = 2$

$x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} * \frac{1}{2}$

$= \frac{1}{2\sqrt{x}}$

$y - y_1 = m(x - x_1)$

$y = \frac{1}{2}\left(4^{-\frac{1}{2}}\right)(x - 4) = 0.25(x - 4) + 2$

$\frac{1}{2\sqrt{4}} = \frac{1}{4}$

$4^{-\frac{1}{2}} = \frac{1}{4}^{\frac{1}{2}} = \frac{1}{\sqrt{4}} = 0.5$

$y = \frac{x}{4} + 1$

ex.

$V_{inst}$ = derivative of vel curve

$$= \lim_{h \to 0} \frac{f(T+h) - f(T)}{h}$$

$$= f'(T)$$

$V_{inst}$ is the $f'$ of a position curve.

ex. find $S(T) = 1250 - 16T^2$

$$V(T) = -32T = S'(T)$$

When hit ground?

$$\lim_{h \to 0} \frac{1250 - 16(T+h)^2 - (1250 - 16T^2)}{h}$$

$$= \frac{1250 - 16T^2 - 32Th - 16h^2 - 1250 + 16T^2}{h}$$

$$\lim_{h \to 0} 32T - 16h \quad @\ h = 0 \to -32T$$

hits ground at $16T^2 = 1250$

$$T^2 = \frac{1250}{16} = 78.125$$

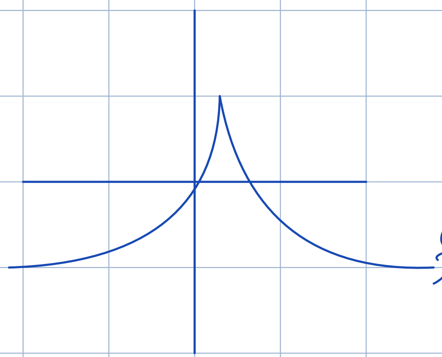
$$T = 8.83\ldots$$

how fast? $v = -32(8.83...)$
$= -282.843$

When is something differentiable
Ability to take a derivative at a point.

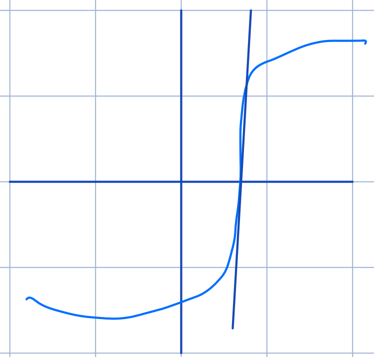
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ This must exist

2. implications:



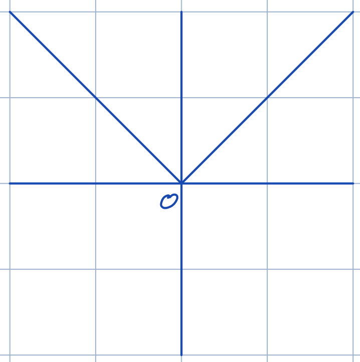
$f(x)$ At any sharp point our limit does not exist

1. can't derive a sharp point.



can't take a derivative where the slope is undefined.

ex: Is $|x|$ diff everywhere?

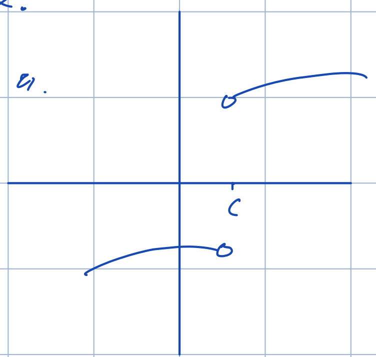



no, not at 0.
but everywhere else.

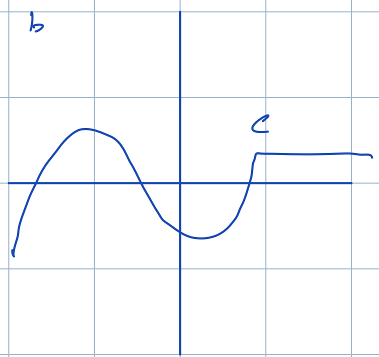
Some functions will not be diff. at specific points.

ex.

a.




b




c not diff @ c
because its sharp

a. not diff. @ c

All diff functions are cont. but not all cont funcs are diff. At any point!

Derivatives representations

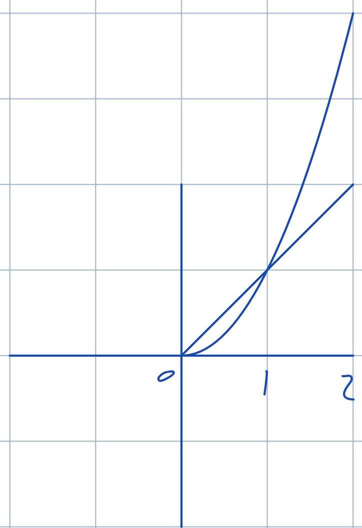
$$f'(x) = \frac{d}{dx}\left[f(x)\right]$$

$$y' = \frac{dy}{dx}$$

Slope of deriv at a point:

$$f'(a) = \frac{d}{dx}\left[f(x)\right]\Big|_{x=a}$$

$$y'(a) = \frac{dy}{dx}\Big|_{x=a}$$




$y = x$ intersection points are

$0,0$ : $1,1$

Cross sectional are is found by calculating the area between curves, using integration:

$$A(x) = \pi x^2 - \pi (x^2)^2 = \pi (x^2 - x^4)$$

$$V = \int_0^1 \pi (x^2 - x^4) = \left[\pi \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\right]_0^1$$

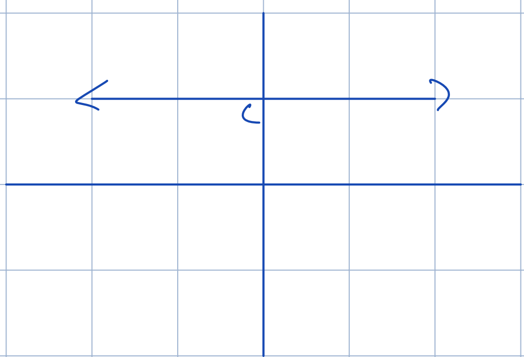
$$= \frac{2\pi}{15}$$

Area under a curve using midpoint riemann sum from $x=0$ to $x=4$ with $n=2$

$$f(x) = \frac{1}{2}x^3 + 4$$

2.2 techiques of derivations.

Slope of a constant $= 0 = m$




if $y = c$, $m = 0$

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(3) = 0, \quad \frac{d}{dx}(-1) = 0 \quad \frac{d}{dx}(\pi) = 0$$

$$f(x) = x^3 \qquad \frac{d}{dx} = 3x^2 = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^3$$
$$f(x) = x^3$$

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$\lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$\lim_{h \to 0} 3x^2 + 3xh + h^2 \qquad @\ h = 0$$
$$= 3x^2$$

$$\frac{d}{dx} = 3x^2$$

In general: $\frac{d}{dx}(x^n) = nx^{n-1}$

ex: $\frac{d}{dx}(x^2) = 2x$

ex $\frac{d}{dx}(x^5) = 5x^4$, $\frac{d}{ds}(s^{15}) = 15s^{14}$

ex. $\frac{d}{dx}(x^{-3}) = -3x^{-4}$

$$\frac{d}{dp}\left(p^{-2}\right) = -2p^{-3} \qquad \frac{d}{dx}\left(\frac{1}{x}\right) = -x^{-2}$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}\left(c \cdot f(x)\right) = c \cdot \frac{d}{dx}\left(f(x)\right)$$

$$\frac{d}{dx}\left(5x^{4}\right) = 5 \cdot \frac{d}{dx}\left(x^{4}\right) = 5\left(4x^{3}\right) = 20x^{3}$$

$$\frac{d}{dx}\left(-x^{7}\right) = -1 \cdot \frac{d}{dx}\left(x^{7}\right) = -1\left(7x^{6}\right)$$
$$= -7x^{6}$$

$$\frac{d}{dx}\left(\frac{\pi}{x^{2}}\right) = \pi \cdot \frac{d}{dx}\left(\frac{1}{x^{2}}\right) = \pi\left(-2x^{-3}\right)$$
$$= -2\pi x^{-3}$$

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}\left(f(x)\right) \pm \frac{d}{dx}\left(g(x)\right)$$

ex.

$$\frac{d}{dx}\left(3x^{9} - x^{-3}\right) = 27x^{8} + 3x^{-4}$$

$$\frac{d}{dx}\left(3x^{9}\right) - \frac{d}{dx}\left(x^{-3}\right)$$

ex: $\frac{d}{dx}\left(4 - 3\sqrt{x}\right) = \frac{d}{dx}\left(4 - 3x^{\frac{1}{2}}\right)$

$= -\frac{3}{2} x^{-\frac{1}{2}} = -\frac{3}{2} \cdot \frac{1}{\sqrt{x}} = -\frac{3}{2\sqrt{x}}$ $\quad \frac{-3}{2\sqrt{x}}$

$y = 5x^7 - 3x^4 + 2x^3 + x - 1$

$\frac{dy}{dx} = 35x^6 - 12x^3 + 6x^2 + 1$

ex. at what points does $y = x^3 - 3x + 4$ have horizontal tangent lines

$\frac{dy}{dx} = 3x^2 - 3$

horizontal $m = 0$, $3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = 1, -1$

$y = 1^3 - 3 + 4$

$y = 2$

$y = -1^3 + 3 + 4$

$y = 6$

horizontal tangents

Points $(1, 2)$, $(-1, 6)$

Slopes of 0 are often max and mins of graphs

higher derivatives

$2^{nd}$, $3^{rd}$ derivatives

$1^{st}$ : $\frac{d}{dx}$, $f'(x)$, $y'$, $\frac{dy}{dx}$

$2^{nd}$ : $f''(x)$, $y''$, $\frac{d^2y}{dx^2}$

$3^{rd}$ : $f'''(x)$, $\frac{d^3y}{dy^3}$

$$f(x) = 7x^4 - 3x^3 - 5x^2 + 9x - 347$$

$$\frac{d}{dx} = 28x^3 - 9x^2 - 10x + 9$$

$$\frac{d^2}{dx^2} = 84x^2 - 18x - 10$$

$$\frac{d^3}{dx^3} = 168x - 18$$

$$\frac{d^4}{dx^4} = 168$$

$$\frac{d^5}{dx^5} = 0$$

ex $\quad y = \dfrac{x^5 - 2x - 3}{3\sqrt{x}} \rightarrow$

$$\rightarrow \frac{x^5}{3x^{\frac{1}{2}}} - \frac{2x}{3x^{\frac{1}{2}}} - \frac{3}{3x^{\frac{1}{2}}}$$

$$\rightarrow \frac{x^{4\frac{1}{2}}}{3} - \frac{2x^{\frac{1}{2}}}{3} - \frac{1}{x^{\frac{1}{2}}}$$

$$\rightarrow \frac{1}{3}x^{\frac{9}{2}} - \frac{2}{3}x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

$$\frac{d}{dx} = \frac{3}{2}x^{\frac{7}{2}} - \frac{1}{3}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

Product and quotient rules

ex: $\quad f(x) = x^2 \quad , \quad g(x) = x^3$

$$\frac{d}{dx}\left(x^2 \cdot x^3\right) = \frac{d}{dx}\Big(f(x) \cdot g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big) \cdot \frac{d}{dx}\Big(g(x)\Big)$$

True? No

$$\frac{d}{dx}\left(x^5\right) = 5x^4$$

$$\frac{d}{dx}\left(x^2\right) = 2x \cdot \frac{d}{dx}\left(x^3\right) = 3x^2 \quad = 6x^2$$

We can't seperate derivs by multiplication, we need to use the product rule

The product rule:

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{d}{dx}\left(f(x)\right)\cdot g(x) + \frac{d}{dx}\left(g(x)\right)\cdot f(x)$$

$$\frac{d}{dx}(x^2)\cdot x^3 = 2x^4 \quad + \quad \frac{d}{dx}(x^3)\cdot x^2 = 3x^4$$

$$= 5x^4$$

ex $y = (x^2-1)(3x^4+2x)$

$$\frac{dy}{dx} = \frac{dy}{dx}(x^2-1)\cdot 3x^4+2x + \frac{dy}{dx}(3x^4+2x)\cdot(x^2-1)$$

$$= \left(2x\cdot(3x^4+2x)\right) + \left((12x^3+2)\cdot(x^2-1)\right)$$

$$= \left(6x^5+4x^2\right) + \left(12x^5+2x^2-12x^3-2\right)$$

$$= 18x^5-12x^3+6x^2-2$$

$$f(x) = (1+x^2)\sqrt{x}$$

$$= \frac{d}{dx}(1+x^2)\cdot\sqrt{x} + \frac{d}{dx}(\sqrt{x})\cdot(1+x^2)$$

$$= 2x\cdot x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\cdot(1+x^2)$$

$$= 2x^{\frac{3}{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{x}}\cdot(1+x^2)$$

$$= 2x^{\frac{3}{2}} + \frac{1}{2x^{\frac{1}{2}}} + \frac{x^2}{x^{\frac{1}{2}}}$$

$$= 2x^{\frac{3}{2}} + \frac{1}{2x^{\frac{1}{2}}} + \frac{1}{2}x^{\frac{3}{2}}$$

$$\frac{5}{2}x^{\frac{3}{2}} + \frac{1}{2\sqrt{x}}$$

ex: $g(x) = (x^2+1)\,f(x)$ $\quad f(2) = 3$, $f'(2) = -1$

find $g'(2)$

$$g(x) = \frac{d}{dx}(x^2+1)\cdot 3 + -1\cdot(x^2+1)$$

$$= 3(2x) - (x^2+1)$$

$$= 6x - (x^2+1)$$

@ $x=2$: $= 12-(5) = 7. = g'(2)$

Quotient rule: for division of derivatives

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$$

To take the $\frac{d}{dx}$ of a quo

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot\frac{d}{dx}(f(x)) - f(x)\cdot\frac{d}{dx}(g(x))}{(g(x))^2}$$

low d high - high d low

□ the bottom and you're good to go

ex:

$$y = \frac{x^3 - 3x^2 - 5}{2x+5}$$

$$\frac{dy}{dx} = \frac{\left((2x+5)\cdot\frac{d}{dx}(x^3-3x^2-5)\right) - x^3-3x^2-5\cdot\left(\frac{d}{dx}\,2x+5\right)}{(2x+5)^2}$$

$$= \frac{\left((2x+5)(3x^2-6x)\right) - \left((x^3-3x^2-5)\cdot(2)\right)}{(2x+5)^2}$$

$$= \cancel{(6x^3 - 12x^3 + 15x^2 - 30x)} - 2x^3 + 6x^2 + 10$$

$$\rightarrow \frac{4x^3 + 9x^2 \cancel{- 30x} + 10}{\cancel{4x^2} + 20x + 25}$$ don't factor btm

$$\rightarrow \quad \frac{4x^3+9x^2-30x+10}{(2x+5)^2}$$

$$f(x) = \frac{(3x-1)(x^2+4)}{x^2+2}$$

1st use quotient rule

$$\frac{\text{low d hi} - \text{hi d low}}{\text{bottom}^2} =$$

$$\frac{d}{dx} = \frac{(x^2+2)\cdot\frac{d}{dx}\Big((3x-1)(x^2+4)\Big) - (3x-1)(x^2+4)\frac{d}{dx}(x^2+2)}{(x^2+2)^2}$$

$$= \frac{(x^2+2)\Big(3(x^2+4)+2x(3x-1)\Big) - (3x-1)(x^2+4)\cdot 2x}{(x^2+2)^2}$$

Applications of derivative:

DVD : $S(T) = \frac{7T}{T^2+1}$, $T$ = years, $T \geq 0$

find: 1. rate of change for sales what is m

$$\frac{d}{dT} = \frac{(T^2+1)\cdot\frac{d}{dT}7T - 7T\cdot\frac{d}{dT}(T^2+1)}{(T^2+1)^2}$$

$$= \frac{7T^2+7-14T^2}{(T^2+1)^2} = \frac{-7T^2+7}{(T^2+1)^2} = ROC$$

2. Peak of sales $m=0$

T can't be neg.

$$\frac{-7T^2+7}{(T^2+1)^2} = 0 \quad @\, T=1 \quad = \frac{0}{4}$$

Peak is $T=1$

3. how fast do sales increase when movie is released.

release is $t=0$ $\rightarrow$ $\frac{-7t^2+7}{(T^2+1)^2} = 7$

The rate is 7.

With word problems look for an axis.

Applications cont.

$s = f(t)$ a position function

$v(t) = \frac{ds}{dt}$ = Velocity

What happens if we take another derivative

what is $v'(t)$? its acceleration

$a(t) = v'(t) = s''(t)$ = acceleration

$j(\tau) = a'(t) = s'''(t)$ = change in acceleration

ex:

$$s(t) = 2t^3 - 15t^2 + 24t$$ find $a(t)$, $j(t)$

$v(t) = 6t^2 - 30t + 24$ vel

$a(t) = 12t - 30$ acc

$j(t) = 12$ $j(t)$

What is a @ $t = 3$ $= 12(3) - 30$

$= 6$

When is $a = 0$? $12t = 30$

$t = \frac{30}{12} = 2.5$

ex. fireworks

$s(t) = -16t^2 + 256t$ find max height

$v = 0, t \neq 0$

$v(t) = \frac{d6}{ds} = -32t + 256$

@ $v = 0$, $32t = 256$

$t = \frac{256}{32} = 8s$

$s(8) = -16(64) + 256(8) = 1024$

Cost function

$$C(x) = -0.2x^2 + 200x + 9000$$

Marginal cost is the rate of change of your cost. i.e how much the next item costs, cost of prod 1 additional cost.

$$M(x) = \frac{dc}{dx} = -0.4x + 200$$

@ $x = 100$ $\quad M(x) = 160$

$x = 101$ $\quad M(x) = 159.6$

$150$ $\quad M(x) = 140$

Derivatives of trig functions

2 points to know

$$\lim_{h \to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{h \to 0} \frac{1 - \cos(x)}{x} = 0$$

if $f(x) = \sin(x)$ $\quad f'(x) = ?$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \sin(x)$$
$$f(x+h) = \sin(x+h)$$

$$f' = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Use addition rule for sin

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$\to \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x) + \cos(x)\sin(h)}{h}$$

$$\lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h}$$

$$\lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\sin(h)}{h}$$

$$\to \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \cos(x) \cdot 1$$

$$\lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \cos(x)\cdot 1$$

$$\to \lim_{h\to 0} \frac{-\sin(x)(1-\cos(h))}{h} + \cos x$$

$$\to \lim_{h\to 0} -\sin(x)\cdot \frac{(1-\cos(h))}{h} + \cos(x)$$

$$\to \lim_{h\to 0} -\sin(x)\cdot 0 + \cos(x)$$

$$\to \lim_{h\to 0} \cos(x)$$

$$L = \cos(x)$$

$$\frac{dy}{dx} \sin(x) = \cos(x)$$

Identities

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

ex:

$\frac{dy}{dx}$ $\quad y = x\sin(x)$

Use product rule $= f'(x)\cdot g(x) + g'(x)\cdot f(x)$

$f(x) = x \qquad g(x) = \sin(x)$

$f'(x) = 1 \qquad g'(x) = \cos(x)$

$$\frac{dy}{dx} \quad 1\cdot \sin x + x\cdot \cos x = x\cos(x) + \sin(x)$$

Find tangent line at $x = \frac{\pi}{2}$

$$\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2} \cdot \cos\left(\frac{\pi}{2}\right)$$
$$= 1 + \frac{\pi}{2} \cdot 0$$
$$= 1$$

$$y = \frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}\right)\right.$$
$$= \frac{\pi}{2}$$

$m = 1$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore y = x$

ex: $y = \dfrac{\sin(x)}{1 + \cos(x)}$

quotient rule $\dfrac{l\, dhi - hi\, dlo}{b(n^2}$

$\frac{d}{dx}\sin(x) = \cos(x)$ $\quad$ $\frac{d}{dx}(1 + \cos(x)) = -\sin(x)$

$$\frac{dy}{dx} = \frac{((1 + \cos(x)) \cdot \cos(x)) - (\sin(x) \cdot -\sin(x))}{(1 + \cos(x))^2}$$

$$= \frac{\cos(x) + \cos^2(x) + \sin^2(x)}{(1 + \cos(x))^2}$$

$\sin^2(x) = 1 - \cos^2(x)$ ?, $\quad \cos^2(x) + \sin^2(x) = 1$

$\rightarrow$

$$\frac{1 + \cos(x)}{(1 + \cos(x))^2} = \frac{1}{1 + \cos(x)}$$

$$\frac{dy}{dx} = \frac{1}{1 + \cos(x)}$$

$y = \sin x$

$$\frac{dy}{dx} = \cos(x)$$

$$\frac{d^2y}{dx^2} = -\sin(x)$$

$$\frac{d^3y}{dx^3} = -\cos(x)$$

$$\frac{d^4y}{dx^4} = \sin(x)$$

Word problem:

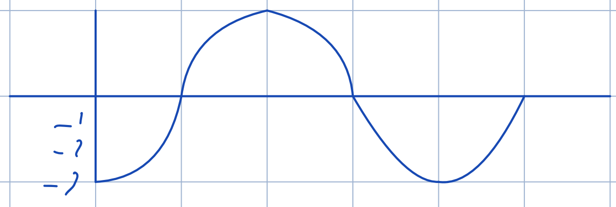
Spring on ceiling with weight

the weight is oscillating

A spring is stretched 3 cm beyond resting pos and

released @ $t = 0$

$S(t) = -3\cos(t)$




$V(t) = S'(t) = \frac{d}{dx} -3\cos(t)$

$= 3\sin(t)$

ex: $\frac{d}{dx}(3x^2-4)^3$ how to distribute?

Using chain rule

Chain Rule:

Take derivatives of compositions.

for e.g. $\frac{d}{dx}(3x^2-4)^{100}$

ex: Can it be expressed as a composition?

$\frac{d}{dx}(3x^2-4)^{100}$

let a function $y = u^{100}$, $u = 3x^2-4$

take deriv w/ respect to $u$

$\frac{dy}{du} = 100u^{99}$

take $\frac{du}{dx} = 6x$

what is $\frac{dy}{dx}$? = $\cancel{100(6x)^{99}}$? no

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

$\frac{dy}{dx}$ is the chain of $\frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{d}{du}(u^{100}) \cdot \frac{d}{dx}(3x^2-4)$$

$$= 100u^{99} \cdot 6x$$

$$= 600x(3x^2-4)^{99}$$

Quick chain rule

do power rule, multiply by $\frac{d}{dx}$ of internal eq

This is the general power rule

says: if you need $\frac{d}{dx}$ of $\left(f(x)^n\right)$

$y = u^n \qquad u = f(x)$

$\frac{dy}{du} = nu^{n-1} \qquad \frac{du}{dx} = \frac{d}{dx}\left(f(x)\right)$

$= nu^{n-1} \cdot \frac{d}{dx}\left(f(x)\right)$

$= n\left(f(x)\right)^{n-1} \cdot \frac{d}{dx}\left(f(x)\right)$

ex: $y = (x^3 - 2x + 38)^4$

$\frac{dy}{dx} = nu^{n-1} \cdot \frac{d}{dx}(u)$

$= 4(u)^3 \cdot (3x^2 - 2)$

$\frac{dy}{dx} = (12x^2 - 8)(x^3 - 2x + 38)^3$

ex. $\frac{d}{dx}(x^6)$

$= \frac{d}{dx} = nu^{n-1} \cdot \frac{d}{dx} u$

$= 6u^5 \cdot \frac{d}{dx}(x)$

$= 1 \cdot 6x^5$

ex: $y = (2x-3)(x^2-5)^3$

product rule, chain rule

$\frac{dy}{dx} = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$f'(x) = 2$

$g'(x) = 2x \cdot 3(x^2-5)^2$

$= 6x(x^2-5)^2$

$\frac{dy}{dx} = 2 \cdot (x^2-5)^3 + (2x-3)(6x(x^2-5)^2)$

$= 2(x^2-5)^3 + 6x(2x-3)(x^2-5)^2$

$= 2(x^2-5)^3 + (12x^2-18x)(x^2-5)^2$

$= (x^2-5)^2(2(x^2-5) + (12x^2-18x))$

$$= (x^2-5)^2 \cdot (2x^2 - 10 + 12x^2 - 18x)$$

$$= (x^2-5)^2 (14x^2 - 18x - 10)$$

Chain rule w/ trig function

$$y = \sqrt{5x^2 - 1} \qquad y = (5x^2-1)^{\frac{1}{2}}$$

$$y = u^{\frac{1}{2}}$$

$$\frac{dy}{dx} = nu^{n-1} \cdot \frac{d}{dx}(5x^2-1)$$

$$\rightarrow \quad \frac{u^{-\frac{1}{2}}}{2} \cdot 10x$$

$$= \frac{5x}{\sqrt{5x^2-1}}$$

Chain rule w/ trig functions:

$$y = \cos(x^4) \qquad \text{let } y = \cos(u)$$
$$u = x^4$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{du} = -\sin(u)$$
$$\frac{du}{dx} = 4x^3$$

$$\frac{dz}{dx} = -4x^3 \sin(x^4)$$

Applications of rules: Chain Rule, Prod rule, Quo rule

$$\frac{d}{dx}\Big( f(g(x)) \Big) = f'(g(x)) \cdot g'(x)$$

ex. $y = \sin(4x^5)$ this is a composition

we use chain rule

$$\frac{dy}{dx} = \cos(4x^5) \cdot 20x^4$$
$$= 20x^4 \cos(4x^5)$$

ex: $\frac{d}{dx}\left( \cos^2(x^4) \right) \rightarrow \frac{d}{dy}\left( (\cos(x^4))^2 \right)$

$$= 2\cos(x^4) \cdot \frac{d}{dx}\left( \cos(x^4) \right)$$
$$= 2\cos(x^4) \cdot (-\sin(x^4) \cdot 4x^3)$$
$$= 2\cos(x^4) \cdot (-4x^3 \sin(x^4))$$
$$= 2 \cdot \cos(x^4) \cdot -4x^3 \cdot \sin(x^4)$$
$$= -8x^3 \cdot \cos(x^4) \cdot \sin(x^4)$$

ex: $\frac{d}{dx}\left(\tan(3x^2-2x)\right)$ use chain rule

$$\frac{d}{dx}\left(\tan(u)\right) \cdot \frac{d}{dx}(3x^2-2x)$$

$$= \sec^2(3x^2-2x) \cdot (6x-2)$$

$$= (6x-2)\left(\sec^2(3x^2-2x)\right)$$

ex: $\frac{d}{dx}\left(\sqrt{x^3+\csc(x^3)}\right) \to \frac{d}{dx}\left[x^3+\csc(x^3)\right]^{\frac{1}{2}}$

$$\to \frac{1}{2}\left(x^3+\csc(x^3)\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(x^3+\csc(x^3)\right)$$

$$\to \frac{1}{2}\left(x^3+\csc(x^3)\right) \cdot \left(3x^2 + \frac{d}{dx}\left[\csc(x^3)\right]\right)$$

$$\to -\csc(x^3)\cot(x^3) \cdot 3x^2$$

$$= \frac{1}{2}\left(x^3+\csc(x^3)\right)^{-\frac{1}{2}} \cdot \left(3x^2 - \left(3x^2\cot(x^3)\csc(x^3)\right)\right)$$

$$\frac{d}{dx}(x^2+x)^{-1} = -(x^2+x)^{-2} \cdot \frac{d}{dx}\, x^2+x$$

$$= -(2x+x)(x^2+x)^{-2}$$

ex: $$\frac{d}{dx}\left[\left(3+x^2\cot(x^2)\right)^{-3}\right]$$

$$\rightarrow -3\left(3+x^2\cot(x^2)\right)^{-4}\cdot\underbrace{\frac{d}{dx}\left(3+x^2\cot(x^2)\right)}$$

$$\rightarrow 2x\cdot\cot(x^2)+\underbrace{x^2\cdot\frac{d}{dx}\cot x^2}$$

$$\rightarrow x^2\left(-\csc^2(x^2)\cdot x^2\cdot\frac{d}{dx}x^2\right)$$

$$\rightarrow x^2\left(-x^2\csc^2(x^2)\cdot 2x\right)$$

$$\rightarrow -3\left(3+x^2\cot(x^2)\right)^{-4}\cdot\left(2x\cot(x^2)+x^2\left(-\csc^2(x^2)\cdot 2x\right)\right)$$

$$\frac{d}{dx}\left(\frac{1+\cos(x^2)}{1-\sin(x^2)}\right)$$

Quotient rule

$$\frac{\text{lodhi} - \text{hidlo}}{\text{btm}^2}$$

ldh

$$\rightarrow \left(1-\sin(x^2)\right)\cdot\frac{d}{dx}\left(1+\cos(x^2)\right)$$

$$\frac{d}{dx} = -\sin(x^2)\cdot\frac{d}{dx}x^2$$

$$= -2x\sin x^2$$

$$=\left(1-\sin(x^2)\right)\left(-2x\sin(x^2)\right)$$

hdl

$$\rightarrow \left(1+\cos(x^2)\right)\cdot\frac{d}{dx}\left(1-\sin(x^2)\right)$$

$\frac{d}{dx} \to -\cos(x^2) \cdot 2x$

$hd l = -\left(1+\cos(x^2)\right) \cdot \left(2x\cos(x^2)\right)$

$$\frac{\left(1-\sin(x^2)\right) \cdot \left(-2x\sin(x^2)\right) - \left(1+\cos(x^2)\right) \cdot \left(2x\cos(x^2)\right)}{(1-\sin(x^2))^2}$$

$y = x^2 \sin(3x)$

$\frac{dy}{dx} = 2x\sin(3x) + x^2 \frac{d}{dx} \sin(3x)$

$x^2 \cdot \left(\cos(3x) \cdot \frac{d}{dx} 3x\right)$

$= 3x^2 \cos(3x)$

# Implicit differentiation

explicit is when $y$ is given

Implicit

$$y + xy = x \quad \text{Implicit}$$

Can convert to exp $\rightarrow$ $y(1+x) = x$

$$y = \frac{1}{1+x}$$

ex $x^2 + y^2 = 4$

$$y^2 = 4 - x^2$$

$$y = \pm(4-x^2)^{\frac{1}{2}}$$

implicit eqs can def. more than 1 func of $x$.

ex $2x^3 + 3y^3 = 9xy$

can't be written explicitly

Can be implicitly differentiated.

ex: $x^3 - y^3 = 5$

Treat $y$ as a function of $x$

derive both sides w/ respect to $x$

Every time you take a deriv of $y$ must get $\frac{dy}{dx}$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(5)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}((y)^3) = 0$$

$$3x^2 + 3y^2 \cdot \frac{d}{dx}(y) = 0$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 0$$

$$3y^2 \cdot \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

function is implicit ∴ deriv is usually implicit.

Solve for $\frac{dy}{dx}$

ex: $3y^2 + \sin(y) = 4x^5$

Y's are functions of x ∴ we always get a $\frac{dy}{dx}$

$$\frac{d}{dx} 3y^2 + \frac{d}{dx} \sin(y) = \frac{d}{dx} 4x^5$$

$$= 20x^4$$

$$6y \cdot \frac{dy}{dx} + \cos(y) \cdot \frac{dy}{dx} = 20x^4$$

$$\frac{dy}{dx}(6y + \cos(y)) = 20x^4$$

$$\frac{dy}{dx} = \frac{20x^4}{6y + \cos(y)}$$

ex: $xy = 1$

$y = \frac{1}{x}$

$\frac{dy}{dx} = -x^{-2}$

exp

$\frac{d}{dx}(x \cdot y)$

$\frac{d}{dx}(x) \cdot y + x \cdot \frac{d}{dx} y = 0$

$1 \cdot y + x \frac{dy}{dx} = 0$

$y = -x \frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{y}{x}$

imp

ex: $3x^2 - y^2 = 16$

find $\frac{d^2y}{dx^2}$

$\frac{d}{dx} 3x^2 - \frac{d}{dx} y^2 = 0$

$6x - 2y \frac{dy}{dx} = 0$

$2y \frac{dy}{dx} = 6x$

$\frac{dy}{dx} = \frac{3x}{y}$

$\frac{d}{dx} \frac{dy}{dx} = \frac{d}{dx} \frac{3x}{y}$

$\frac{d^2y}{dx^2} = \frac{l\,dh - h\,dl}{l^2}$

$\frac{3x}{y}$

$$\frac{y\cdot 3 - \frac{dy}{dx}3x}{y^2}$$

$$= \frac{3y - \frac{dy}{dx}3x}{y^2}$$

$$\frac{dy}{dx}(3x) = \frac{3(3x)}{y}$$

$$= \frac{9x}{y}$$

$$\frac{3y - \frac{9x}{y}}{y^2} \qquad \frac{3y}{y^2} - \frac{\frac{9x}{y}}{y^2} = \frac{3}{y} - \frac{9x}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{3}{y} - \frac{9x^2}{y^3}$$

ex $y^2 - x + 1 = 0$ find slopes @ $(2,-1),(2,1)$

$$\frac{d}{dx}y^2 - \frac{d}{dx}x = 0$$

$$2y\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2y} \quad @ \ (2,-1),(2,1)$$

$$= \frac{1}{2}, -\frac{1}{2}$$

2 different slopes

a b c d

ex: $4x^4 + 8x^2y^2 - 25x^2y + 4y^4 = 0$

a: $16x^3$

b: prod rule: $f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$$16x \cdot y^2 + 8x^2 \cdot \frac{dy}{dx} \cdot 2y$$

$$= 16xy^2 + \frac{dy}{dx} 16x^2y$$

C. p-rule $-25x^2y$

$$= -50xy + \frac{dy}{dx} \cdot -25x^2$$

d. $\frac{dy}{dx} 16y^3$

$$16x^3 + 16xy^2 + \frac{dy}{dx} 16x^2y - 50xy + \frac{dy}{dx} \cdot -25x^2 + \frac{dy}{dx} 16y^3$$

$$\frac{dy}{dx}\left(16x^2y - 25x^2 + 16y^3\right) = -16x^3 - 16xy^2 + 50xy$$

$$\frac{dy}{dx} = \frac{-16x^3 - 16xy^2 + 50xy}{16y^3 + 16x^2y - 25x^2}$$

eq @ 2, 1

$$m = \frac{-16(2^3) - 16(2)(1^2) + 50(2)(1)}{16(1^3) + 16(2^2)(1) - 25(2^2)}$$

$$= \frac{-128 - 32 + 100}{16 + 64 - 100} = \frac{-60}{-20} = 3$$

$$y = 3x - 5$$

Related rates:

How to relate a formula to its change according to time

e.g how is cost changing according to time.

Relating a formula w/ time.

e.g. cone: Cone is being filled but also leaking

What is the rate of change of the vol of water w/ respect to time

For vol of cone we need h, r

Get a formula that relates vol to time

first get vol of cone

$V = \pi r^2 h$ : cylinder

$V = \frac{1}{3}\pi r^2 h$ = cone

We can use implicit diff to take a derivative of an equation w/ respect to any variable we want. take w/ r to t

v, r, h are all functions of time

$$v = \frac{\pi}{3} r^2 h$$

$\frac{d}{dt}$ : rate of change acc. to time

$$\frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dv}{dt} = \frac{d}{dt}\frac{\pi}{3} r^2 \cdot h + \frac{d}{dt} h \cdot \frac{\pi}{3} r^2$$

$$\frac{dv}{dt} = \frac{\pi}{3}\left(2r \cdot \frac{dr}{dt} \cdot h + \frac{dh}{dt} \cdot r^2\right)$$

$$\frac{dv}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

The roc of Vol — roc of rad — roc of height ex: $y = x^3$ find $\frac{dy}{dt}$

$$\frac{d}{dt}(y) = \frac{d}{dt}x^3$$

$$\frac{dy}{dt} = \frac{dx}{dt}3x^2$$

@ $t = 1$?

We need $x, y, \frac{dx}{dt}$

$\rightarrow$ If $x = 2$, $\frac{dx}{dt} = 4$

@ $t = 1$

$$\frac{dy}{dt} = 4\left(3(2^2)\right)$$

$$\frac{dy}{dt} = 48 \quad @\ t = 1$$

ex: Oil spill: Radius is spread @ 3 ft/s

how fast is the area increasing when $r = 30$

let $t$ = time, $a$ = area, $r$ = rad. $\frac{dr}{dt} = 3$

$a = \pi r^2$, take d. w/ respect to time

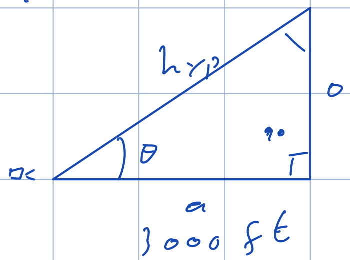
$$\frac{da}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

@ $r = 30$

$$\frac{da}{dt} = 2\pi \cdot 30 \cdot 3$$

$$= 180\pi \text{ ft}^2/\text{s}$$

ex:




soh
cah
toa

Rocket is climbing @ 600 ft/s when h is @ 4000 f how fast will angle of elevation have to change @ 4k ft to keep up w/ rocket

$\theta = \sin \frac{o}{hyp}$, $o = h$, $a = 3000$, $x$: cons.

$hyp^2 = a^2 + h^2$

$\frac{dh}{dt} = 600 ft/s$

$h = 600(t)$

$t = \frac{4000}{600} = 6\frac{2}{3}s$

$\theta$ = angle we want roc of $\theta$ acc to $t$

$\tan(\theta) = \frac{o}{a} = \frac{h}{3000}$

$\frac{d}{dt}\tan(\theta) = \frac{d}{dt}\frac{1}{3000}h$

$\frac{d\theta}{dt}\cdot \sec^2\theta = \frac{1}{3000}\frac{dh}{dt}$

$\left.\frac{d\theta}{dt}\right|_{4000ft} = \frac{1}{3000}\cdot 600 ft/s \div \sec^2\theta$

@ $h = 4000$, $\sec\theta = \frac{5000}{3000}$

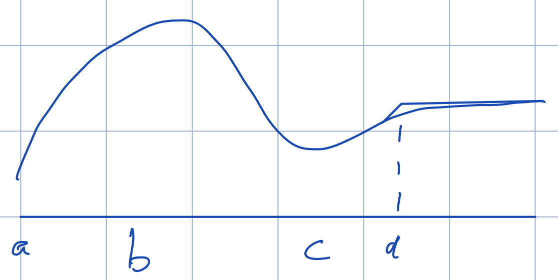
$$\frac{d\theta}{dt} \cdot \left(\frac{5}{3}\right)^2 = \frac{1}{3000} \cdot 600 \text{ ft/s}$$

$$\frac{d\theta}{dt} = \frac{1}{5} \cdot \left(\frac{5}{3}\right)^2$$

$$\left.\frac{d\theta}{dt}\right|_{h=4000} = \frac{9}{125} \text{ rad/s}$$

The rate of change of the angle w/ respect to time.

Chapter 3: Increasing, decreasing and concavity:




Where is this increasing?

decreasing?

$(a, b), (c, d)$ increasing - positive slope

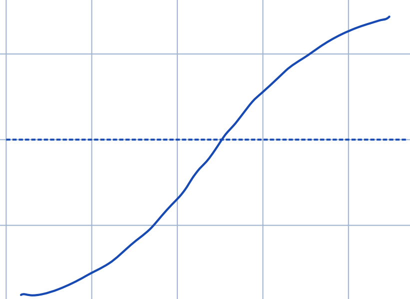
$(b, c)$ decreasing - negative slope

Implies: 1. If our deriv $f'(x) > 0$ our slope is increasing

2. If $f'(x) < 0$ then $f(x)$ is decreasing on our interval

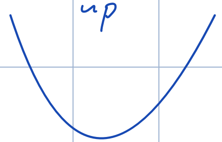
3. If $f'(x) = 0$ the $f(x)$ is constant at a point or @ the interval.

Concavity: the direction of the curvature of our line. It is the second derivative, the way that it is curved.
How a curve is increasing or decreasing
It lets us know the change in the intensity of the curve



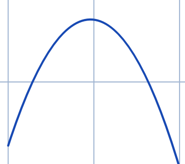
- Increasing everywhere, but not at a constant rate.
- Changes from concave up to concave down at an inflection point.

Concave up



Slope is increasing - function may not be
- Opens upward

Concave down

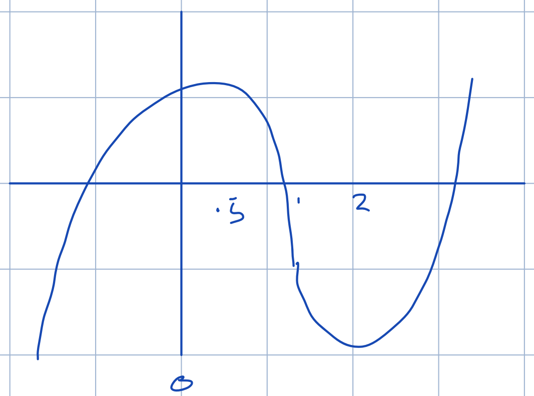


Slope is decreasing

Second derivatives on concavity:

1. $f''(x) > 0 \rightarrow$ Slope is increasing : Concave up
2. $f''(x) < 0 \rightarrow$ Slope decreasing : Concave down
3. $f''(x) = 0 \rightarrow$ Slope is constant

Inflection points are the point @ which the slope changes from inc. to dec.




Inc: $(-\text{inf}, 0)$, $(2, \text{inf})$
Dec: $(.5, 2)$
C up: $(1, \text{inf})$
C down: $(-\text{inf}, 1)$
inflection $x = 1$

A: $f'(x) < 0$
$f''(x) > 0$

C: $f'(x) < 0$
$f''(x) < 0$

B $f'(x) > 0$
$f''(x) > 0$ not past point of inflexion

Relative extrema:

Relative max or min

high point or low point on an interval

Max: Where a function changes from increasing to decreasing

Low: Where $f(x)$ changes from decreasing to increasing

At inflexion slope = 0 @ critical no.s

Critical point

Slope = 0 is not always a crit point.

To find crit no.s take 1st derivatives and set equal to 0

$f'(x) = 0$ * if denominator, set to 0 also

ex: find critical no.s

$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3$$

$3x^2 - 3 \rightarrow (3x + 3)(x - 1)$

$x = 1 \quad x = -1$

How to prove if they're max or mins?

$2^{nd}$ order?

Absolute max/min:

highest/lowest point on interval

Some graphs don't have abs. max/min

e.g graphs to infinity

All continuous have max/mins on any closed interval [a, b]

These can occur @ an endpoint or a critical point

If interval isn't closed then our absolute maxs and mins can occur only at critical points but they don't always occur

IF they occur they do so @ crit points

1. Find crit and eval
2. find end points

ex:

$$f(x) = 2x^3 - 15x^2 + 36x \qquad [1, 5]$$

$$f'(x) = 6x^2 - 30x + 36$$

$$= (3x - 6)(2x - 6)$$

$$x = 2, \quad x = 3$$

| | | |
|---|---|---|
| @ $x = 1$ | $f(x)$ | $= 2 - 15 + 36 = 23$ |
| $x = 2$ | " | $= 16 - 60 + 72 = 28$ |
| $3$ | " | $= 54 - 135 + 108 = 27$ |
| $5$ | | $= 250$ |

ex: abs Max/min $y = 6x^{\frac{4}{3}} - 3x^{1/3}$ on $1 : -1$

$$\frac{dy}{dx} = 8x^{\frac{1}{3}} - x^{-2/3}$$

@ = 0 factor out $x^{-\frac{2}{3}}$

$$\frac{8x^{\frac{1}{3}}}{x^{-\frac{2}{3}}} = 8x^{\frac{3}{3}} = 8x$$

$$\frac{x^{-2/3}}{x^{-\frac{2}{3}}} = 1$$

$$\rightarrow x^{-\frac{2}{3}}(8x - 1) = 0$$

$x^{-\frac{2}{3}} = 0$, $x = 0$ *

$8x - 1 = 0$,

$x = \frac{1}{8}$ * Crit points

$y = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$ @ end points, Crit points

| $x$ | $y$ | |
|---|---|---|
| $-1$ | $9$ | max |
| $0$ | $0$ | |
| $\frac{1}{8}$ | $-\frac{9}{8}$ | min |

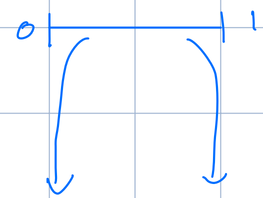
1 | 3

ex:

$$f(x) = \frac{1}{x^2 - x} = (x^2 - x)^{-1}$$

Can I check [0, 1]: no

We have to find crit nos as usual then find one sided limits.



$$\lim_{x \to 0^+} \frac{1}{x^2 - x} = -\infty$$

We don't have an abs min

$$\lim_{x \to 1^-} \frac{1}{x^2 - x} = -\infty$$

$$\frac{d}{dx}(x^2 - x)^{-1} = -(x^2 - x)^{-2} \cdot \frac{d}{dx} x^2 - x$$

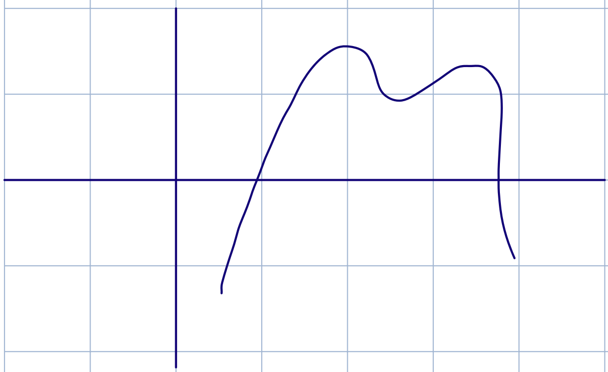
$$= -1(2x - 1)(x^2 - x)^{-2}$$

$$= \frac{-(2x-1)}{(x^2 - x)^2} \qquad x = 0.5$$

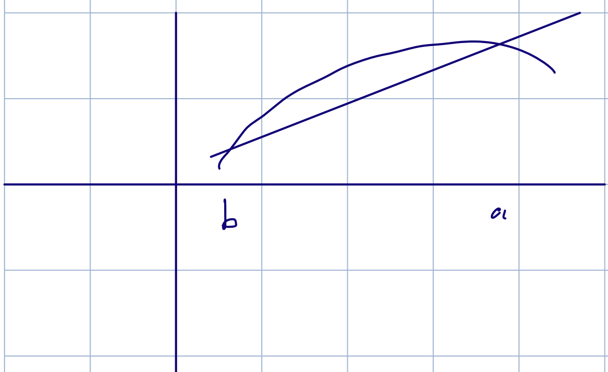
$$f(0.5) = \frac{1}{(.5^2 - .5)} = -4 = \text{abs max}$$

Rolles theorem and Mean-value theorem:

If your func crosses the x-axis and crosses the x-axis again and continuous between then slope = 0 at at LEAST one point.



Mean Value theorem says take 2 points, draw a secant at some point on the curve, the slope will be parallel to the slope of a, b




Mean Value theorem is just a change of direction applied to Rolles theorem

## 1st derivative test for increasing / Decreasing

Recall:

$f'(x) > 0$

1st derivative is +, slope is positive, line is increasing

$f'(x) < 0$:

Slope is negative, decreasing

$f'(x) = 0$:

Critical points, slope is constant

How to find Relative extrema:

1st derivative test says:

1. Take 1st derivative
2. Set = 0 for critical #'s
3. Create table for graph of critical no.s

| | $x_1$ | $x_2$ . . . | $x_i$ |
|---|---|---|---|
| $f(x)'$ | | | |

4. Note that either side of the CN $x_i$ is going to be inc or dec and we can use this to find our max/mins or constants.

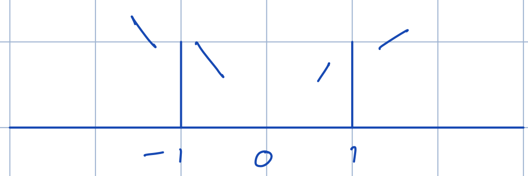
Plug in $f(x)$ nos approaching $x_i$

Also inc. undefined vals as they are asymptotes ex: $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3 = 0$ $\qquad 3x^2 = 3$

$x^2 = 1$

$x = \pm 1$




@ $x = -0.9,\ 0.9$

$3(-0.9)^2 - 3 = 2.43 - 3 = -0.57$

@ $x = -1.1,\ 1.1$

$3(-1.1^2) - 3 = 3.63 - 3 = 0.63$

| -1.1 | -1 | -.9 | .9 | 1 | 1.1 |
|---|---|---|---|---|---|
| + | 0 | − | − | 0 | + |
| ↗ | | ↘ | ↘ | | ↗ |

$-1$ is a relative max, $1$ is a relative min

To find R. max: plug into $f(x)$ $x = -1$

$f(-1) = x^3 - 3x + 1$
$= -1 + 3 + 1 = 3$

R max $= (-1, 3)$

Find R. min @ $f(1)$

$f(1) = 1 - 3 + 1$
$= -1$

R. min $= (1, -1)$

ex: $f(x) = 3x^{5/3} - 15x^{2/3}$

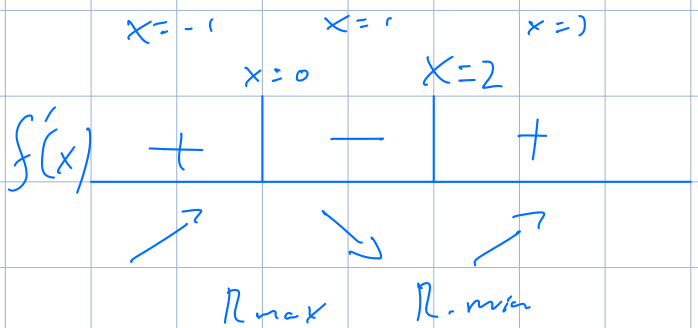
$f'(x) = 5x^{2/3} - 10x^{-1/3} = 0$

$x = 0,$

factor out 5x. smallest power

$5x^{-1/3}(x-2) = 0$ , $x = 0$ , $x = 2$

$= \dfrac{5.(x-2)}{\sqrt[3]{x}}$ $\quad \sqrt[3]{x} = 0 \quad x = 0$




for $f'(x)$

$f'(-1) = \dfrac{5(-3)}{\sqrt[3]{-1}} = \dfrac{-15}{-1} = 15$

$f'(1) = \dfrac{5(1-2)}{\sqrt[3]{1}} = \dfrac{-5}{1} = -5$

$f'(3) = \dfrac{5(3-2)}{\sqrt[3]{3}} = \dfrac{5}{\sqrt[3]{3}}$

$x = 0$ = R. max $\quad f(0) = 3(0)^{5/3} - 15(0)^{2/3} = 0$

$R.\ max = (1, 0)$

$R\ min \quad x = 2 \quad f(2) = 3(\sqrt[3]{32}) - 15(\sqrt[3]{4})$

$$3 \cdot (\sqrt[3]{8} \cdot \sqrt[3]{4})$$
$$= 3 \cdot (2\sqrt[3]{4}) = 6\sqrt[3]{4} - 15\sqrt[3]{4}$$
$$= -9\sqrt[3]{4}$$

$R\ min = (2, -9\sqrt[3]{4})$

$2^{nd}$ derivative test:

If

$f''(x) > 0$ then slope is increasing concave up

* $f''(x) = 0$ slope is unchanging or changinging concavity

$f''(x) < 0$ slope is decreasing concave down

* Possible inflexion point at $f''(x) = 0$

How to find inflexion points

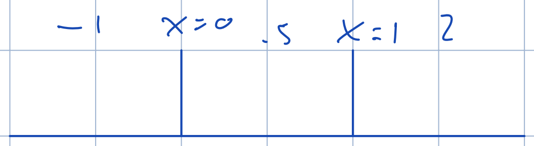
do $2^{nd}$ derivative test:

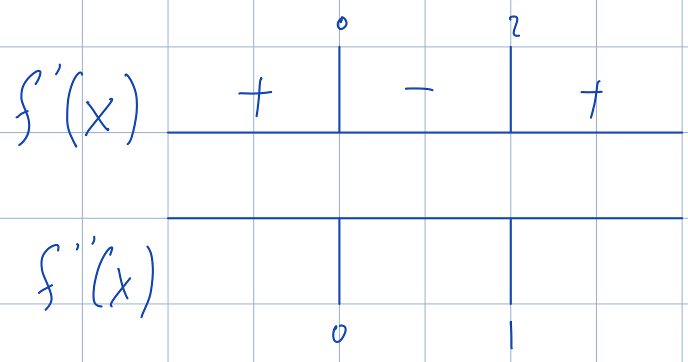
Take $2^{nd}$ deriv, find $f''(x) = 0$

$f''(x)$ of $5x^{\frac{-1}{3}}(x-2) \to 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}}$

$$= 3\tfrac{1}{3}x^{-\frac{1}{3}} - 3\tfrac{1}{3}x^{-\frac{4}{3}} = \frac{10}{3}x^{-\frac{4}{3}}(x-1)$$

$f''(x) = 0$ @ $x = 1$, $x = 0$




1st $\frac{d}{dx}$ gives relative max or min, 2nd gives concavity

If we have $f'(X) = 0$ it's a critical point

at a critical point we either have a max, min or inflexion

The concavity tells us?

ex: do 2nd $\frac{d}{dx}$ test for: $f(x) = x^4 - 4x^3 + 12$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$ @ inflexion $f''(x) = 0$

$12x^2 - 24x = 0$

$12x(x-2)$ $x = 0, x = 2$

| | C. up ↗ | | C. down ↘ | | C. up ↗ |
|---|---|---|---|---|---|
| $f''(x)$ | + | \| | − | \| | + |
| | −1 | 0 | 1 | 2 | 3 |

$f''(-1) = 12 + 24$
$= 36$

$f(1) = -12$

$f(3) = 108 - 72$
$= 36$

When $x = 0$ there is no concavity ∴ we can get all possible inflexion pts.

0, 2 are possible inflexion points

0 and 2 are x vals of inflexion points to get our inflex points ∴ plug into $f(x)$
Use the o.g. function $f(x)$

$f(0) = 12$

$f(2) = 2^4 - 4(2^3) + 12$
$= 16 - 32 + 12$
$= -4$

inf. pts @ $(0, 12)$, $(2, -4)$

ex: $g(x) = (x-1)^{\frac{1}{3}}$     Use chain rule

$$g'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}} \cdot \frac{d}{dx} x-1$$

$$= \frac{1}{3}(x-1)^{-\frac{2}{3}}$$

$$g''(x) = -\frac{2}{9}(x-1)^{-\frac{5}{3}} \cdot \frac{d}{dx}(x-1)$$

$$= -\frac{2}{9}(x-1)^{-\frac{5}{3}}$$

$$\frac{-2(x-1)^{-5/3}}{9} = \frac{-2}{9\sqrt[3]{(x-1)^5}} = 0$$

@ $x-1 = 0 \therefore x = 1$

we get 1 possible inflx point

| | c. up ↗ | c. down ↘ |
|---|---|---|
| $g''(x)$ | $+$ | $-$ |
| | 0   $x=1$ | 2 |

$$g''(0) = \frac{-2}{9(\sqrt[3]{-1^5})}$$

$$= \frac{-2}{-9} = \frac{2}{9}$$

$$g''(2) = \frac{-2}{9(\sqrt[3]{1^5})}$$

$$= -\frac{2}{9}$$

inflexion $x = 1$

$$g(1) = (x-1)^{\frac{1}{3}} = 0$$

$\therefore$ inflexion @ $(1, 0)$

Using 1st, 2nd derivative test

ex $f(x) = x^3 - 3x^2 - 24x + 32$

$$f'(x) = 3x^2 - 6x - 24 \qquad @\ f'(x) = 0$$
$$3(x^2 - 2x - 8)$$
$$3(x+2)(x-4) \qquad x = -2,\ x = 4$$

$$f''(x) = 6x - 6 \qquad @\ f''(x) = 0$$
$$x = 1$$

| | -3 | -2 | 1 | 4 | 5 |
|---|---|---|---|---|---|
| $f'(x)$ | Inc + | | dec − | | Inc + |

| | 0 | 1 | 2 |
|---|---|---|---|
| $f''(x)$ | c. down − | | c. up + |

$$f'(1) = 3 - 6 - 24$$
$$= -27$$
$$f'(-3) = 27 + 18 - 24$$
$$= 21$$
$$f'(5) = 75 - 30 - 24$$
$$= 21$$

$$f''(0) = -6$$
$$f''(2) = 6$$

$$f(1) = 1 - 3 - 24 + 32$$
$$= 6$$

$4 \times 24 =$

Inflexion at $(1, 6)$

$16 \times 4 = 40 + 24 = 64$

$16 \times 3 = 30 + 18 = 48$

R. min @ $x = 4$ $\quad f(4) = 64 - 48 - 96 + 32$

$= -48$

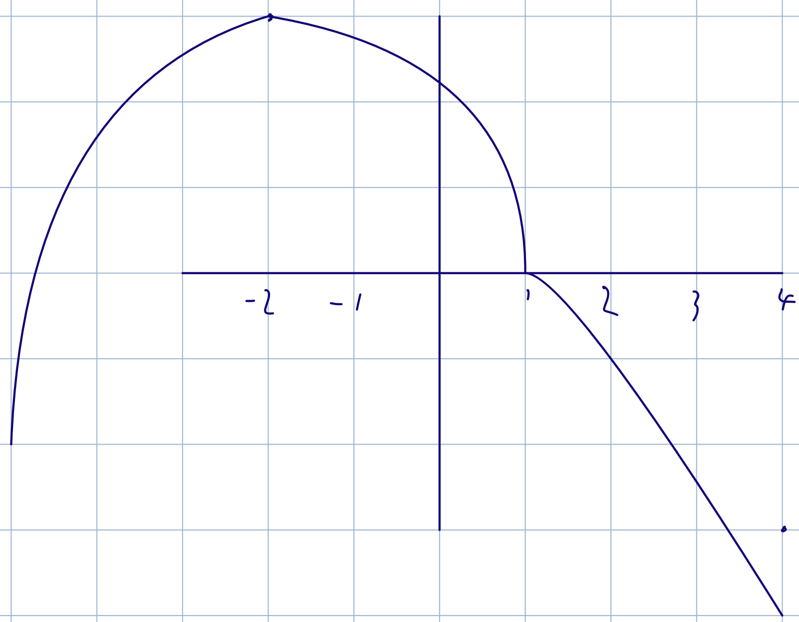
R. min @ $(4, -96)$

R. max @ $-2$ $\quad f(-2) = (-2)^3 - 3(-2^2) - 24(-2) + 32$

$= -8 - 12 + 48 + 32$

$= 60$

R. max @ $(-2, 60)$



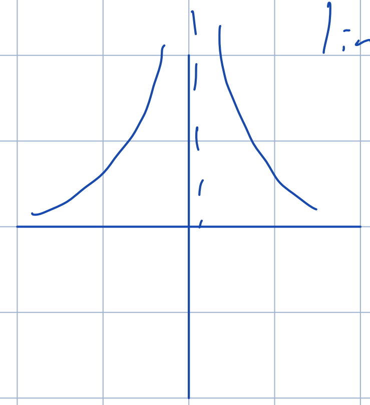

limits @ ins

Recall:

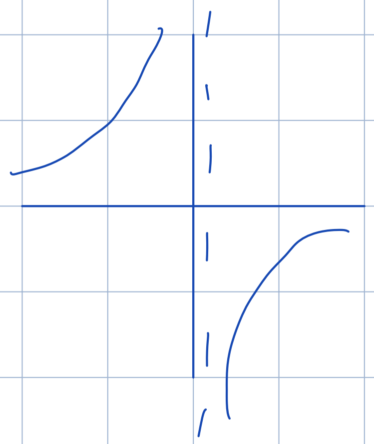
If $\lim_{x \to a} f(x) = \pm \infty$

We have an asymptote at the val of a.

We had



lim @ inf



not a limit.

$$f(x) = \frac{x}{(x-3)(x-1)}$$ discontinuities @ $x=3$, $x=1$

To find discontinuities set denom = 0

Case 1: holes - removable discontinuity

Only a single point is missing

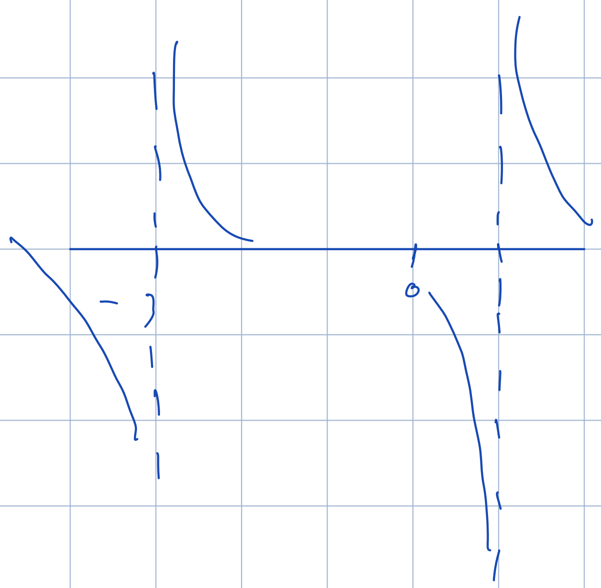
i.e $\frac{0}{0}$

Case 2: Asymptote, non-removable
- Can't cancel out the discontinuity

ex: $f(x) = \dfrac{x}{(x+3)(x-1)}$

Discont: @ $x = -3$, $x = 1$
They are both asymptotes

Can do sign analysis to check the intervals




The limits do not exist as $x \to -3, 1$

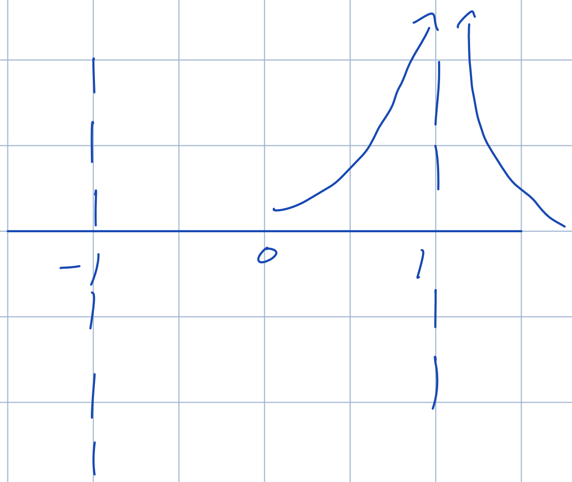
@ $x = -4$

$f(x) = -\frac{4}{5}$

ex:

$$\lim_{t \to 1} \frac{t^3}{(t^2-1)^2} \qquad \text{at } t = 1, -1$$

Discontinuities



$$f(0) = \frac{0}{1}$$

= hole

as $t \to 1$ the lim goes to $+\infty$

What if our function goes to $\infty$

$$\lim_{x \to \infty^{\pm}} f(x)$$

ex: $\lim_{x \to +\infty} \frac{1}{x}$

| $x$ | 1 | 10 | 100 | 1000000 |
|---|---|---|---|---|
| $f(x)$ | 1 | 0.1 | 0.01 | 0.000001 |

$x$ is approaching 0

$$\lim_{x \to -\infty} \frac{1}{x}$$ for $-\infty$ it also approaches 0

If $f(x)$ approaches some no. as $x \to \pm\infty$
we can say that the limit exists

What does this mean?

If never reaches the limit, but it does give us
a horizontal asymptote.

$$\lim_{x \to +\infty} \frac{1}{x^n} \to \left( \lim_{x \to +\infty} \frac{1}{x} \right)^n = 0$$

Any $\frac{1}{x^n}$ will approach 0 because $0^n = 0$
for +inf or −inf.

Note: The limit of a polynomial as $x \to \pm\infty$

As polynomials approach $\pm$ inf the graph also appr. $\pm$ inf on the $y$ axis

Depends on the largest power of $f(x)$
and its sign

Ex: $\lim_{x \to +\infty} x^3 = +\infty$ $\lim_{x \to -\infty} x^3 = -\infty$

$\lim_{x \to +\infty} x^2 = +\infty$ $\lim_{x \to -\infty} x^2 = +\infty$

Also: limits of polynom follow behavior of its highest power term.

i.e $x^4 + x^2 + x$ follows behavior of $x^4$

$$\lim_{x \to -\infty} -3x^3 - 2x^2 - x + 9 = \lim_{x \to -\infty} -3x^3 = +\infty$$

$$\lim_{x \to +\infty} \frac{5x-2}{3x+9} = \frac{+\infty}{+\infty} = 1 ? \text{ no}$$

$$\rightarrow \frac{\frac{5x}{x} - \frac{2}{x}}{\frac{3x}{x} + \frac{9}{x}} \rightarrow \frac{5 - \frac{2}{x}}{3 + \frac{9}{x}}$$

now take the limit at inf

$$\frac{5 - \frac{2}{\infty}}{3 + \frac{9}{\infty}} = \frac{5-0}{3+0} = 1\frac{2}{3}$$

Divide every term by the largest power of $x$ in the denominator

So that you are not undefined.

If the powers are equal the asymptote is the co-efficient over the co-efficient

ex: $\lim_{x \to -\infty} \dfrac{5x^2 - 4x}{15x^3 - 3} \to 0$

$\to \lim_{x \to -\infty} \dfrac{\frac{5x^2}{x^3} - \frac{4x}{x^3}}{\frac{15x^3}{x^3} - \frac{3}{x^3}} = \lim_{x \to 0} \dfrac{\frac{5}{x} - \frac{4}{x^2}}{15 - \frac{3}{x^3}}$

@ $x = -\infty$ $\dfrac{0 - 0}{15 - 0} = 0$

ex: $\lim_{x \to -\infty} \dfrac{7x^3 - 2x^2 + 1}{-2x + 9}$

$\to \lim_{x \to -\infty} \dfrac{\frac{7x^3}{x} - \frac{2x^2}{x} + \frac{1}{x}}{\frac{-2x}{x} + \frac{9}{x}} \to \lim_{x \to -\infty} \dfrac{7x^2 - 2x + \frac{1}{x}}{-2 + \frac{9}{x}}$

@ $x = -\infty = -\infty$

ex:

$$\lim_{x \to +\infty} \sqrt[3]{\frac{2x^2-3}{3x^2-5}}$$

$$= \left( \lim_{x \to +\infty} \frac{2x^2-3}{3x^2-5} \right)^{\frac{1}{3}}$$

$$= \frac{2}{3}^{\frac{1}{3}}$$

ex:

$$\lim_{x \to \infty} \frac{(x^2+2)^{\frac{1}{2}}}{3x-6} \quad x \quad \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$\to \frac{\frac{\sqrt{x^2+2}}{|\sqrt{x^2}|}}{\frac{3x}{|x|} - \frac{6}{|x|}} =$$

$$\lim_{x \to \infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{2}{x^2}}}{3 - \frac{6}{x}} = \lim_{x \to \infty} \frac{\sqrt{1 + 2x^{-2}}}{3 - \frac{6}{x}}$$

$$= \frac{1}{3} \quad @ \ x \to \infty$$

ex:

$$\lim_{x \to \infty} \sqrt{x^4 + 2} - x^2$$

$$\to \lim_{x \to \infty} \frac{\sqrt{x^4+2} - x^2}{1} \cdot \frac{\sqrt{x^4+2} + x^2}{\sqrt{x^4+2} + x^2}$$

$$\to \lim_{x \to \overset{+}{\infty}} \frac{x^4 + 2 - x^4}{\sqrt{x^4+2} + x^2} \to \lim_{x \to 0} \frac{2}{\sqrt{x^4+2} + x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\to \frac{\frac{2}{x^2}}{\frac{\sqrt{x^4+2}}{\sqrt{x^4}} + \frac{x^2}{x^2}} = \frac{2x^{-2}}{\sqrt{\frac{x^4}{x^4} + \frac{2}{x^4}} + 1}$$

$$= \frac{2x^{-2}}{\sqrt{1 + \frac{2}{x^4}} + 1} \quad @ -\infty$$

$$= \frac{-0}{1+1} = \frac{-0}{2} = 0$$

ex

$$\lim_{x \to +\infty} \sqrt{7-x} \to \lim_{x \to +\infty} \frac{(7-x)^{\frac{1}{2}}}{1}.$$

$\to$ DNE

Curve Sketching:

Use x, y intercepts, rel max, rel min + concavity to sketch graph

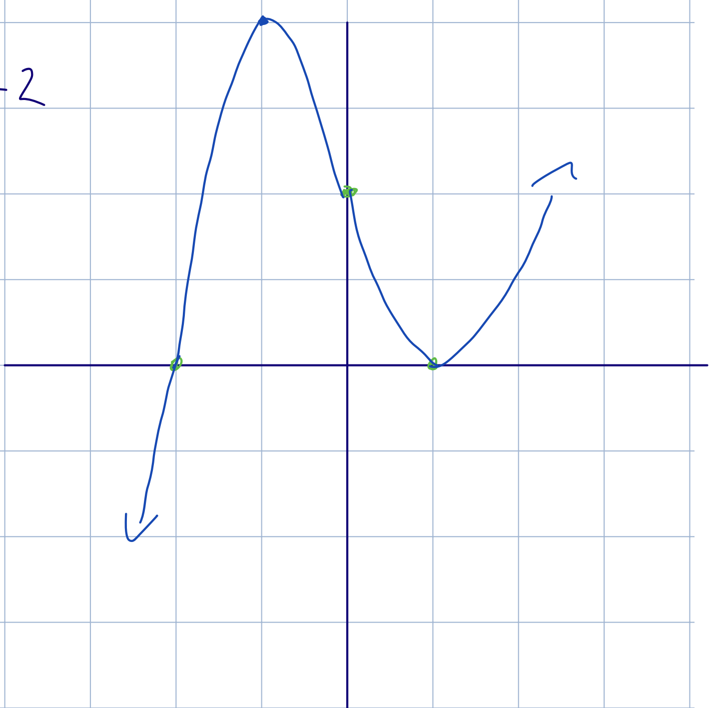
ex: $y = (x+2)(x-1)^2$

Step 1: find intercepts

for $y = 0$, $x = 0$

$y = 0$ @ $x = 1$, $x = -2$

$x = 0$ @ $y = 2$



$y = x^2 - 2x + 1(x+2)$

$= x^3 - 3x + 2$

Rel max: if $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 3x^2 - 3$ R. max/min? @ $x = 1, -1$ min at $x = 1$

$x = -1$ $y = 2$ $y = 0$

2. for rational funcs find all asymptotes

3. first deriv test for crit nos / inc, dec

$y' = 3x^2 - 3$ @ $y = 0$

$3x(x-1)$ c.n

$x = 1, -1$

at $x = 2$

$y' = 27 - 3 = 24$ ↑

$x = .5$ $y' = -2.25$

$y'' = 6x$ @ $y'' = 0$ $6x = 0$

$x = 0$ inflex point

let $x = 1$ $y'' = 6$

$x = -1$ $y'' = -6$

| | ↗ $-1$ | ↘ $1$ | ↗ |
|---|---|---|---|
| $y'$ | pos | neg | pos |
| $y''$ | c. down | c. up | |
| | ∩ | $x = 0$ | ∪ |

ex:

$$y = \frac{x^2-1}{x^3}$$

find x, y intercepts
find asymptote
do 1st, 2nd deriv tests

$y = 0$ @ $x^2 - 1 = 0$

$x^2 = 1$, $x = \pm 1$ $= \frac{0}{1} =$ holes

$x = 0$, $y = \frac{1}{0}$ = Asymptote DNE

1st derive

$y = (x^2-1)(x^{-3})$

$y = x^{-1} - x^{-3}$

$y' = -x^{-2} + 3x^{-4}$

$\frac{1}{3x^4 - x^2}$

$(3x^4 - x^2)^{-1} = 0$

$(3x^2 - 1)^{-1} = 0$

$(3x^2)^{-1} = 1$

$(x^2)^{-1} = \left(\frac{1}{3}\right)^{-1}$

$x^2 = 3$

$x = \pm\sqrt{3}$

There's a horizontal asymptote at $y = 0$ for $\pm\infty$

Crit nos at $x = \sqrt{3}$, $x = -\sqrt{3}$

4. $y'' = 2x^{-3} - 12x^{-5}$

@ $y'' = 0$ $\frac{1}{2x^3} - \frac{1}{12x^5} = 0$

$(2x^3 - 12x^5)^{-1} = 0$ at $x = 0$

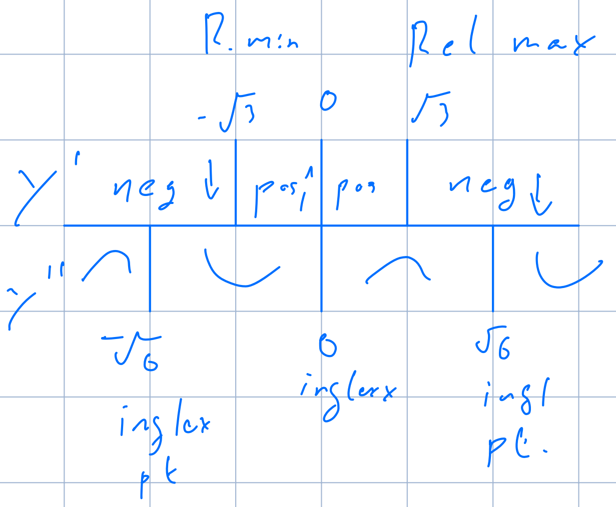
$2x^{-3}(1 - 6x^2)^{-1}$

$2x^{-3}(-6x^2 + 1)^{-1}$ $\qquad \frac{1}{-6x^2+1}$ $\qquad (-6x^2+1)^{-1} = 0$

$(6x^2)^{-1} = 1^{-1}$

$x^2 = \frac{1}{6}^{-1}$

$x = \pm\sqrt{6}$



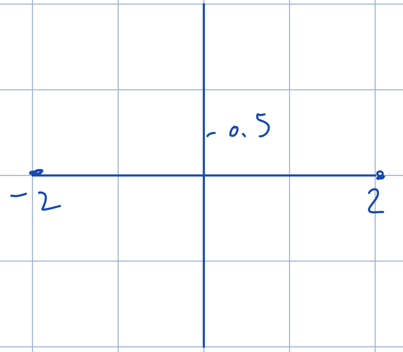

$f(x) = (2x^2 - 8)(x^2 - 16)^{-1}$ $\qquad$ Y int $= 2x^2 - 8 = 0$

$= 2 - \frac{2x^2}{16} - 8x^{-2} + \frac{1}{2}$ $\qquad$ Y int @ $x^2 = 4$

$= \frac{1}{8}x^2 - 8x^{-2} + \frac{5}{2}$ $\qquad x = \pm 2$

X int at $\frac{1}{2} = y$

$f'(x) = \frac{1}{4}x + 16x^{-3}$



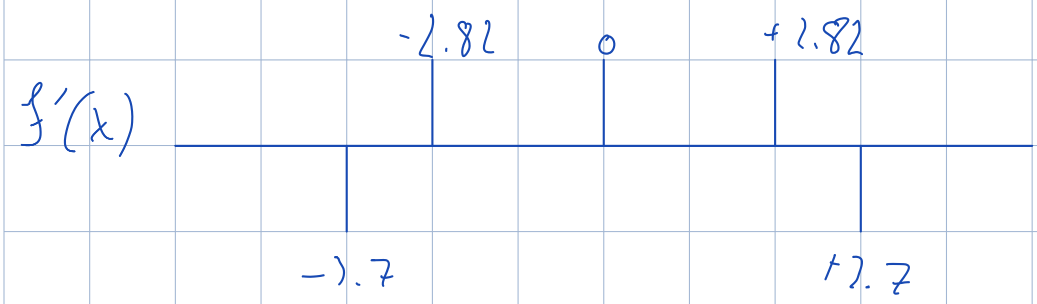

$f'(x) = 0$ @ $\frac{x}{4} = -\frac{16}{x^3}$

$x = -\frac{64}{x^3}$

$x^4 = -64$

$$x = \pm\sqrt[4]{64}$$
$$= \pm 2\sqrt{2} = \pm 2.82$$

$$f''(x) = \frac{1}{4} - 48x^{-4}$$
$$= \frac{1}{4} - \frac{48}{x^4} = 0$$
$$1 - 4\cdot\frac{48}{x^4} = 0$$
$$\frac{192}{x^4} = 1$$
$$x^4 = 192$$
$$x = \pm 2\sqrt[4]{12} = \pm 3.72\ldots$$



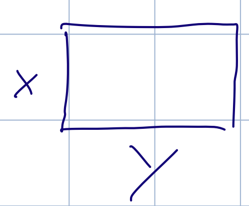

Max / Min Application problems

1. Cont. functions. Always
2. Open intervals: not always.

ex:

fencing a rectangular region. Maximise area.
We have 100 ft of fence

$f(x)$ = Area



$A = x \cdot y$ what is max $(A)$

$L = 2x + 2y$ when $2x + 2y = 100$

$L(x, y) = 100$

Use our constraint, solve for one var, sub into

$A$ / $f(x)$

$2y = 100 - 2x$ $y = 0$ @ $x = 50$, not possible

$y = 50 - x$ $x$ can not be $<0, \geq 50$

$y' = -1$ $y'(x) = -1$

$y'' = 0$

$y(x) = 50 - x$ $0 < x < 50$

$A = x \cdot y$ $y = 50 - x$

$A = x(50 - x)$ $A$ int. @ $x = 0$, $x = 50$

$A = 50x - x^2$

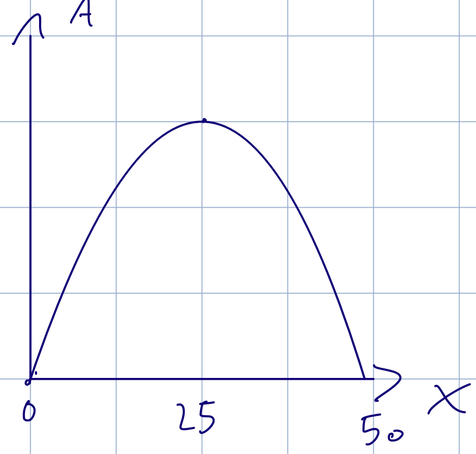
$A' = 50 - 2x$ Crit. no. @ $2x = 50$

$A'' = -2$ $x = 25$

25 25 is local max

$A'$      pos | neg

$A''$   $\cap$   neg



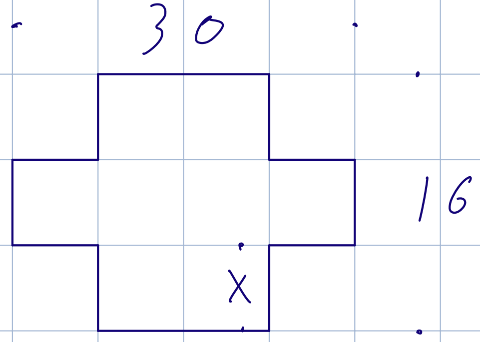

$A = \max$ @ $x = 25$

$y = 50 - x, \quad y = 25$

$\max(A) = 25^2$

Max area of box

$h \cdot w \cdot d$    let




$A = x(30-2x)\cdot(16-2x)$     $2x \leq 16$, $x \geq 0$

$A = x(480 - 60x - 32x + 4x^2)$

$= 4x^3 - 92x^2 + 480x$

$A' = 12x^2 - 184x + 480$    at $A' = 0$

$$\frac{-b \pm \sqrt{4(ac)}}{2a}$$

$x = 12, \quad x = \frac{10}{3}$

| | | $\frac{10}{3}$ | | $12$ | |
|---|---|---|---|---|---|
| $A'$ | $+$ | \| | $-$ | \| | $+$ |
| | | $R_{max}$ | | $R_{min}$ | |

let $x = 1$, $A' = 12 - 184 + 480$

$= $ pos

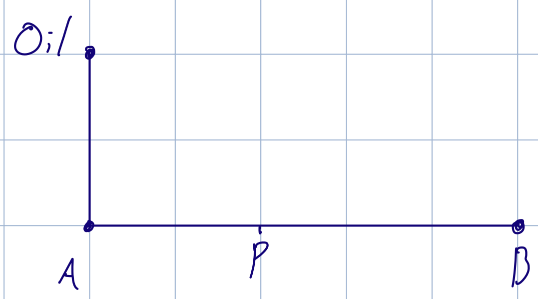
$x = 4 : 192 -$

LeEx = 13

$\frac{10}{3}$ = R. max

$$A = \frac{10}{3}\left(30 - \frac{20}{3}\right)\left(16 - \frac{20}{3}\right) = 1125.925\_$$

Ex:




Pipe = 1.0 per km in ocean
$\frac{1}{2}$ per km

Minimise cost

$AB = 8$

$OA = 5$

$O \to A$ = ocean, $A \to B$ = land

$$\text{Ocean} = OA^2 + AP^2$$

$$\text{land} = \frac{1}{2}(AB - AP)$$

$$C(P) = \underbrace{OA^2 + AP^2}_{\sqrt{5^2 + P}} + \underbrace{\frac{1}{2}(AB - AP)}_{+\frac{1}{2}(5 - P)}$$

$$= (p^2 + 25)^{\frac{1}{2}} + \frac{1}{2}(8 - p)$$

$$= (p^2 + 25)^{\frac{1}{2}} - \frac{p}{2} + 4$$

$$C'(p) = \frac{1}{2}(p^2 + 25)^{-\frac{1}{2}} \cdot 2p - \frac{1}{2}$$

$C' = p(p^2+25)^{-\frac{1}{2}} - \frac{1}{2}$ @ $C' = 0$

$\frac{p}{(p^2+25)^{-2}} = \frac{1}{2}$

$2p = \sqrt{(p^2+25)}$

$4p^2 = p^2+25$

$3p^2 = 25$

$p^2 = \frac{25}{3}$

$p = \sqrt{\frac{25}{3}} = + \frac{5}{\sqrt{3}}$

$= \frac{5\sqrt{3}}{3}$

Plug $P_1$ into $C(P)$

$C\left(\frac{5\sqrt{3}}{3}\right) = 8.33$

Can min vol $vol = \pi r^2 h$ for $1000 cm^3$

$Vol = \pi r^2 h$ , material $= 2\pi r^2 + 2\pi r h$

S. Area $= S$ let $x = r$, $y = h$

$Vol = V$

$Vol = 1000 = \pi x^2 y$

$y = \frac{\pi x^2}{1000}$

$y = \frac{1000}{\pi x^2}$

$y \geq 0$

$x \geq 0$

$S = 2\pi x^2 + 2\pi x y = 2\pi x^2 + 2\pi x \cdot \frac{1000}{\pi x^2}$

$S = 2\pi x^2 + \frac{2000}{x}$

$S' = 4\pi x - \frac{2000}{x^2}$ let $S' = 0$

$$4\pi x = \frac{2000}{x^2}$$

$$\pi x = \frac{2000}{4x^2}$$

$$\pi x^3 = \frac{500}{x^2}$$

$$x^3 = \frac{500}{\pi}$$

$$x = \sqrt[3]{\frac{500}{\pi}}$$

$$x = 5.42$$

$$y = \frac{1000}{\pi x^2} = 10.84\text{cm}$$

$$S'' = 4\pi + \frac{4000}{r^3}$$

$f(t) =$ temp in fa° at time $t$,

The diff. $dx = \Delta x$, it's another way of writing a small change in $x$

$$df = f'(x)\,dx = f'(x)\,\Delta x$$

The change in $f$ which is written

$$\Delta f = f(x + \Delta x) - f(x)$$

$$\Delta f \approx df$$

for $f(x) = x\ln x$ find

$$df = f'(x)\,dx$$

$$f'(x) = x \cdot \frac{1}{x} + \ln x$$

$$df = (1 + \ln x) \cdot dx$$

2. $df$ when $x = 2$, $dx = -0.3$

$$df = (1 + \ln 2) \cdot -0.3$$

$$df = -0.5079$$

3. $\Delta f$ when $x = 2$, $\Delta x = -0.3$

$$\Delta f = f(x + \Delta x) - f(x)$$

$$= (x + \Delta x)\ln(x + \Delta x) - x\ln x$$

$$= 1.7\ln 1.7 - 2\ln 2$$

$$= -0.484$$ = Change in tan line

R of sphere = 8, possible e of .5

linear app..

It's useful to app. a func near a particular val with its tangent line

$f(t) = f^\circ$ at time $t$, $t$ = time in hours

estimate temp at 7, 8.

$f(6) = 60^\circ$
$f'(6) = 3^\circ$

$\int_0^T 3 = f(t) = c + 3t + df$

Approximating is easy when using the tangent line but we should try include the possible err.

$f(6) = c + 18$
$c = 42$
$f(7) = 63 + df$
$f(8) = 66 + df$

let $y = f(x)$
we know $f(a)$
want to know $f(a + \Delta x)$ $\Delta x$ = small no.

Can app. with tangent line
Moving $\Delta x$ on our graph to the left we also move up the y-axis by $f'(a) \cdot \Delta x$
$\therefore$ the height of the tangent line at $f(a + \Delta x)$

$$= f(a) + f'(a) \cdot \Delta x$$

$$f(a + \Delta x) = f(a) + f'(a) \cdot \Delta x$$

Can write using diff symbols:

let $x = a + \Delta x$

$\Delta x = x - a$

$$f(x) = f(a) + f'(a) \cdot \Delta x$$

$$L(x)$$

eq. of tangent line at $x = a$

$$y - y_1 = m(x - x_1)$$

$y = f(a + \Delta x)$

$y_1 = f(a)$

$x_1 = a$

$x = a + \Delta x$

$m = f'(a)$

$$y = f(a) + f'(a) \cdot (x - a)$$

Can appr. with tangent line

$$f(a + \Delta x) \approx f(a) + f'(a) \cdot \Delta x$$

$\sqrt{59}$

$f(a) = a^{\frac{1}{2}}$

1, 4, 9, 16, 25, 36

$f(64) = 8$

$\sqrt{54} = \sqrt{6} \cdot \sqrt{9} = 3\sqrt{6}$

$f(54) = 3\sqrt{6}$

$f(a+5) = f(a) + f'(a) \cdot \Delta x$

$f'(a) = \frac{1}{2} a^{-\frac{1}{2}}$

$f'(54) = \frac{1}{2} \cdot \frac{1}{\sqrt{54}} = \frac{1}{2} \frac{1}{3\sqrt{6}}$

$= \frac{1}{6\sqrt{6}}$

$f(59) = 3\sqrt{6} + 5 \cdot \frac{1}{6\sqrt{6}}$

$= 3\sqrt{6} + \frac{5}{6\sqrt{6}}$ ✓

---

$L(x) = f(a) + f'(a) \cdot \Delta x$

$y = \sin x$

estimates $\sin(33)$ $\quad$ $\sin(30) = 1$ $\quad$ let $a = 30°$ in rad

$= \frac{\pi}{6}$

$L(33) = f(\frac{\pi}{6}) + f'(\frac{\pi}{6}) \cdot (x - \frac{11\pi}{60}) dx = \frac{33\pi}{180} = \frac{11\pi}{60}$

$$f(x) = \sin x$$
$$f'(x) = \cos x$$

$$L\left(\frac{11\pi}{60}\right) = \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) \cdot \left(\frac{11\pi}{60} - \frac{\pi}{6}\right)$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{60}$$
$$= \frac{1}{2} + \frac{\pi\sqrt{3}}{120}$$

What is the diff.? the $df$ is

$$f(a + \Delta x) = f(a) + f'(a) \cdot \Delta x$$

$$f(a + \Delta x) - f(a) = f'(a) \cdot \Delta x$$

The differential $dx$ is another way of writing $\Delta x$

$$dx = \Delta x$$

The differential $df = f'(x) \cdot dx = f'(x) \cdot \Delta x$

The change in $f$ $\Delta f = f(x + \Delta x) - f(x)$
/ change in $y$

$\Delta f \approx df$ The change in the function / Y is $\approx$ to the differential

$df = f'(x) \cdot dx$

$f'(x) = \frac{1}{x} \cdot x + 1 \cdot \ln x$

$= 1 + \ln x$

$df = ((1 + \ln x) \cdot dx)$

Rad of sphere $= 8$, error max $= 0.5cm$

Vol of sphere $= \frac{4\pi r^3}{3}$

$dx = \pm .25$ ?

let $x = r$

$f(x) = \frac{4}{3}\pi x^3$

$f'(x) = 4\pi x^2$, $dx = 0.5$

$df =$ error

$df = f'(x) \cdot dx$

$= 4\pi 8^2 \cdot 0.5 = 128\pi \ cm^3$

The relative error $= \frac{\text{err.}}{f(x)} = 18.75\ \%$

L'hopital's rule

$\frac{0}{0}$ is indeterminate form

A limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ is called a $\frac{0}{0}$ indet. form if the $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

Area under curve by making infinitely smalle rectangles then adding them togethe

Antiderivate

$A(x) =$ area under curve for $f(x)$ on $[a, b]$

then $A'(x) = f(x)$

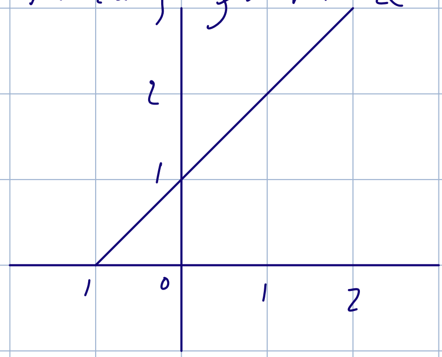
To find area under $f(x)$ we must undo $A'(x)$

Antiderivative = Integral

$f(x) = x + 1$, find area on $[1, x]$

$A(x) = \frac{1}{2}x^2 + x + d$

Area, from $x = -1$ to $1$ $= \frac{2 \cdot 2}{2} = 2$ cm




from $-1$ to $2$

$\frac{3 \cdot 3}{2} = 4.5$

$A(1) = \frac{1}{2}x^2 + x + d.$

$A(1) = \frac{1}{2}1^2 + 1 + 0.5$

$A(x) = \frac{1}{2}x^2 + x + \frac{1}{2}$ $\qquad A(2) = \frac{1}{2} \cdot 2^2 + 2 + \frac{1}{2}$

$= 4.5$

$f(x) = x^2$

$A(x) = \frac{1}{3}x^3 + c$ $\qquad$ @ $x = 0$, $A(x) = 0$

$\therefore C = 0$

$A(1) = \frac{1}{3}$

Indefinite Integral

Given $f$ on some interval $I$

$F$ is antiderivative $(A(x))$

if $F'(x) = f(x)$

$$F(x) = ax^i + bx^h \ldots + c$$

Integration = anti derivation

$$\int f(x)\,dx = F(x) + c$$

Indefinite Integration is w/ out boundaries

Integration table:

$$\frac{d}{dx}(x) = 1 \qquad \int 1\,dx = x + c$$

$$\frac{d}{dx}(x^r) = rx^{-1} \qquad \int x^r\,dx = \frac{1}{r+1}x^{r+1} + c$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \int \cos x\,dx = \sin x + c$$

$$\frac{d}{dx}(-\cos x) = \sin x \qquad \int \sin x\,dx = -\cos x + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \int \sec^2 x\,dx = \tan x + c$$

$$\frac{d}{dx}(-\cot x) = \csc^2 x \qquad \int \csc^2 x\,dx = -\cot x + c$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \int \sec x \tan x\,dx = \sec x + c$$

$\frac{d}{dx}(-\csc x) = \csc x \cot x \quad \int \csc x \cot x = -\csc + c$

ex: $\int x^4 \, dx = \frac{1}{5}x^5 + c$

$\int \frac{1}{x^3} \, dx = \qquad + c$

$\int x^{-3} \, dx = -\frac{1}{2} x^{-2} + c$

$\int \sqrt{x} \, dx \to \int x^{\frac{1}{2}} \, dx = \frac{2}{3} x^{\frac{3}{2}} + c$

$\int \frac{1}{x} \, dx \to \int x^{-1} \, dx = \ln x + c$

Properties:

1. $\int c \cdot f(x) \, dx = c \cdot \int f(x) \, dx$

2/3 $\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

$\int f(x) \pm g(x) \, dx \neq \int f(x) \, dx \cdot \int g(x) \, dx$

ex $\int 2x \, dx = x^2 + c$

ex: $\int 1 + x \, dx = x + \frac{1}{2}x^2 + c$

$\int 4\cos x \, dx \to 4 \cdot \int \cos x \, dx$

$= 4\sin x + c$

$\int x + x^2 \, dx = \frac{1}{2}x^2 + \frac{1}{3}x^3 + c$

ex $\int 2x^5 - 3x^2 + 4x + 8 \, dx$

$= \frac{1}{3}x^6 - x^3 + 2x^2 + 8x + c$

ex $\int (x^2 - 2)(x^2 - 3) \, dx$

$\to \int x^4 - 5x^2 + 6 \, dx$

$= \frac{1}{5}x^5 - \frac{5}{3}x^3 + 6x + c$

ex $\int \frac{\cos x}{\sin^2 x} \, dx$

$\to \int \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \, dx \to \int \csc x \cdot \cot x \, dx$

$= -\csc x + C$

ex $\int \frac{T^2 - 2T^4}{T^4} \, dT$

$\to \int (T^2 - 2T^4) \cdot T^{-4} \, dT$

$\to \int T^{-2} - 2 \, dT$

$= -T^{-1} - 2T + C$

ex find eq of a curve such that
the slope @ each point is $x^2$

$f'(x) = x^2$

$f(x) = \int x^2$

$\cancel{\therefore} f(x) = \frac{1}{3}x^3 + C$

given it passes $(2, 1)$

$f(2) = \frac{1}{3} 2^3 + C$

$= \frac{1}{3} 8 + C$

$1 = \frac{8}{3} + C$

$C = 1 - \frac{8}{3}$

$C = -\frac{5}{3}$

$y = \frac{1}{3}x^3 - \frac{5}{3}$

ex: Diff e.q.s: 

$\frac{dy}{dx} = f(x)$

Idea: can you find $F(x)$ such that $y = F(x)$ satisfies $\frac{dy}{dx} = f(x)$

ex: $\frac{dy}{dx} = x^4$ $\qquad f(x) = x^4$

$F(x) = \int x^4 \, dx$

$= \frac{1}{5}x^5 + c$

$\frac{dy}{dx} = x^4 \quad \cdot dx$

$\to dy = x^4 \, dx$

$\int dy = y$

$\therefore y = \int x^4 \, dx$

$y = \frac{1}{5}x^5 + c$

ex: Initial val problems: We want 1 sol. from our family of sols. Given init sol.

$$\frac{dy}{dx} = \frac{1}{(2x)^3} \quad , \quad y(1) = 0$$

$$dy = (2x)^{-3} \, dx$$

$$y = \int 2^{-3} x^{-3} \, dx \rightarrow \int \frac{1}{8} \cdot x^{-3} \, dx$$

$$= -\frac{1}{16} x^{-2} + c$$

$$y = -\frac{1}{16x^2} + c \qquad y(1) = 0$$

$$\therefore -\frac{1}{16} + c = 0$$

$$c = \frac{1}{16}$$

$$y = -\frac{1}{16x^2} + \frac{1}{16}$$

ex $$\frac{dy}{dx} = \cos x \quad , \quad y(0) = 1$$

$$dy = \cos x \, dx$$

$y = \int \cos x \, dx$

$y = \sin x + c$ $\quad y(0) = 1$

$\sin(0) = 0 \therefore c = 1$

$y = \sin x + 1$

ex: Catapult: acc. shoots straight up.
w/ init vel of 128 ft/s, $s(0) = 16$

- find position func for the height
- find max height
- When will it hit ground $\quad$ let $g = -32$ ft/s

$v(t) = -32(t) + 128$

$s(t) = \int -32(t) + 128 + c$

$s(t) = -16t^2 + 128t + c$ $\quad s(0) = 16$

$s(0) = 0 + 0 + c$
$c = 16$
$s(t) = 128t - 16t^2 + 16$ $\quad$ max height at $v(t) = 0$

$128 - 32t = 0$
$32t = 128$

$$t = \frac{128}{32}$$

$$t = 4$$

$S(4)$ = max height

$$S(4) = -16(4^2) + 128(4) + 16$$
$$= 512 - 256 + 16$$
$$S_{max} = 256 + 16 = 272 \text{ ft}$$

time at ground when $S(t) = 0 \quad t \neq 0$

$$S(t) = -16t^2 + 128t + 16 = -16$$
$$128t = 16t^2 - 16$$
$$8t = t^2 - 1$$
$$8t = t^2 - 1$$

$$t^2 + 8t - 1 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \frac{-8 \pm \sqrt{64+4}}{2}$$

$$= -4 + \frac{1}{2}\sqrt{68} \qquad \sqrt{68} > 8$$

$s' = 6t$
$h' = -t$

$vol = \frac{1}{3} s^2 h$

$V(t) = \frac{1}{3} s^2 h$ @ $s = 3$, $h = 9$

$v(t) = \frac{1}{3} 9 \cdot 9 = 27 cm^3$

$h' = -t$
$h = -\frac{1}{2} t^2 + C_2$

$s' = 6t$
$s = 3t^2 + C_2$

$v(t) =$

$v(t) = \frac{1}{2} t^2 + C_1 + C_2$

$v(t) = 27$

$v'(t) = t$

$$s'(t) = 6$$
$$h'(t) = -1$$

$$V(t) = \frac{1}{3} s^2 \cdot h$$

$$s(t_0) = 9$$
$$h(t_0) = 3$$

$$V(t) = \frac{1}{3} s(t)^2 \cdot h(t)$$

$$V'(t) = \frac{1}{3} \cdot 2S(t)\, s'(t)\, h(t) + \frac{1}{3}[s(t)]^2 h'(t)$$

$$\frac{d}{dx}\left(\frac{2x+3}{3x^2-4}\right) \rightarrow \frac{\text{lo d hi} - \text{hi d lo}}{\text{btm}^2}$$

$$\frac{3x^2-4(2) - (2x+3(6x))}{(3x^2-4)(3x^2-4)}$$

$$= \frac{6x^2 - 12x^2 - 18x - 8}{(3x^2-4)(3x^2-4)}$$

$$= -\frac{6x^2 - 18x + 4}{(3x^2-4)^2}$$

@ $x = -1$

$$= \frac{-6 + 18 - 8}{(3-4)^2}$$

$= \quad 4$

$$\lim_{x \to -\frac{\pi}{2}} \tan x = \tan\left(\frac{\pi}{2}\right) = 0$$

limit dne

$$x^2 + 4y^2 = 7 + 3xy$$

$$\frac{d}{dx} = 2x + 8y\frac{dy}{dx} = 3y + 3x \cdot \frac{dy}{dx}$$

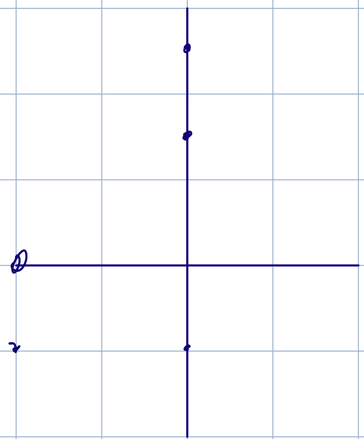
$$2x + 8y \cdot \frac{dy}{dx} = 3y + 3x \cdot \frac{dy}{dx}$$

$$8y\frac{dy}{dx} - 3x \cdot \frac{dy}{dx} = 3y - 2x$$

$$\frac{dy}{dx}(8y - 3x) = 3y - 2x$$

$$\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$$

| $x$ | $g(x)$ | $h(x)$ | $h'(x)$ |
|---|---|---|---|
| $-2$ | $7$ | $0$ | $-1$ |
| $0$ | $-2$ | $5$ | $3$ |



$h(-2) = 0$ $\quad h(0) = 5$

$h'(-2) = -1$

$ax^2 + bx + c$

Applications of integration:

Area between 2 curves, Vol of a solid of revolution

vol using multiple methods

Rules: • Integration by parts:

• Trig integrals

Trig substitution:

$$\sec\theta = \frac{hyp}{adj}$$

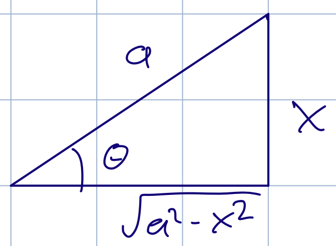
When to use trig substitution

for $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$

Then we use $1+\tan^2\theta = \sec^2\theta$

or $1-\sin^2\theta = \cos^2\theta$

1. $\sqrt{a^2-x^2}$ has to be a side, it's got a minus in it.




$hyp = \sqrt{\lambda^2 + \phi^2}$, $\lambda = \sqrt{hyp^2 - \phi^2}$

let $a = hyp$

hyp has to be the pos no.

We want $\frac{x}{a}$ for $\tan\theta$, $\sin\theta$ or $\sec\theta$

$\tan\theta = \frac{opp}{adj} =$ won't work

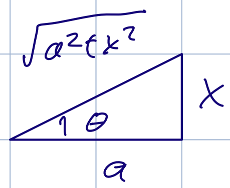
$\sin\theta = \frac{opp}{hyp} = \frac{opp}{a}$ could work

$\sec\theta = \frac{a}{adj} =$ won't work

$\sin\theta = \frac{x}{a} =$

$x = a\sin\theta$

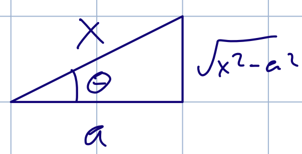
ex: $\sqrt{a^2+x^2}$




$\tan\theta = \frac{opp}{adj} = \frac{x}{a}$ $\quad x = a\tan\theta$

soh cah toh
sha

ex: $\sqrt{x^2-a^2}$

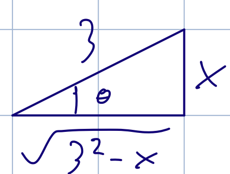



$\frac{x}{a} = \frac{hyp}{adj} = \sec\theta$

$x = a\sec\theta$

ex:

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx \qquad \sqrt{9-x^2} \to \sqrt{3^2-x^2}$$




$\frac{x}{3} = \frac{opp}{hyp}$

$= \sin\theta$

$x = 3\sin\theta \qquad dx = 3\cos\theta\,d\theta$

$$\sqrt{3^2-(3\sin\theta)^2} \to \sqrt{9-9\sin^2\theta}$$

$\rightarrow \sqrt{9\cos^2\theta}$

$\rightarrow 3\cos\theta = \sqrt{3^2 - x^2}$

$$\rightarrow \int \frac{(3\sin\theta)^2}{3\cos\theta} \cdot 3\cos\theta \, d\theta$$

$$\rightarrow \int \frac{3^2\sin^2\theta \cdot 3\cos\theta \, d\theta}{3\cos\theta}$$

$$\rightarrow \int 9\sin^2\theta \, d\theta$$

$$\rightarrow 9 \cdot \int \sin^2\theta \, d\theta$$

$$\rightarrow 9 \int \frac{1}{2}\left(1 - \cos 2\theta\right) d\theta$$

$$\rightarrow \frac{9}{2} \int 1 - \cos 2\theta \quad d\theta$$

$$\rightarrow \frac{9}{2} \int 1 \, d\theta - \frac{9}{2} \int \cos 2\theta \, d\theta$$

$$\rightarrow \frac{9}{2} \theta - \frac{9}{2} 2 \cdot \sin\theta + C$$

$$\rightarrow \frac{9\theta}{2} - \frac{9}{2}\sin\theta + C \qquad x = 3\sin\theta$$

1. $\ln 1 = 0$ ln has e as its base

$\log_2 x = 4$ base to the power gives argument of our log

$2^4 = x$

$e^0 = 1$

$\ln_e 1 = 0$

2. $\ln xy = \ln x + \ln y$

$\ln x + \ln y = \ln xy$

3. $\ln \frac{x}{y} = \ln x - \ln y$

$\ln x - \ln y = \ln \frac{x}{y}$

4. $\ln x^r = r \cdot \ln x$

$r \cdot \ln x = \ln x^r$

ex: $\ln \frac{3x^4}{2y^2} \rightarrow \ln 3x^4 - \ln 2y^2$

$\rightarrow \ln 3 + \ln x^4 - (\ln 2 + \ln y^2)$

$\rightarrow \ln 3 + 4\ln x - (\ln 2 + 2\ln y)$

$\rightarrow \ln 3 - \ln 2 + 4\ln x - 2\ln y$

ex: $\ln \dfrac{x^2+1}{\sqrt{x}} \to \ln(x^2+1) - \ln x^{\frac{1}{2}}$

$\to \ln(x^2+1) - \frac{1}{2} \ln x$

ex: $\ln\left(\dfrac{x^5 \sin^2(\pi x)}{\sqrt[3]{x^4+2}}\right) \to \ln x^5 + \ln \sin^2(\pi x) - \ln (x^4+2)^{\frac{1}{3}}$

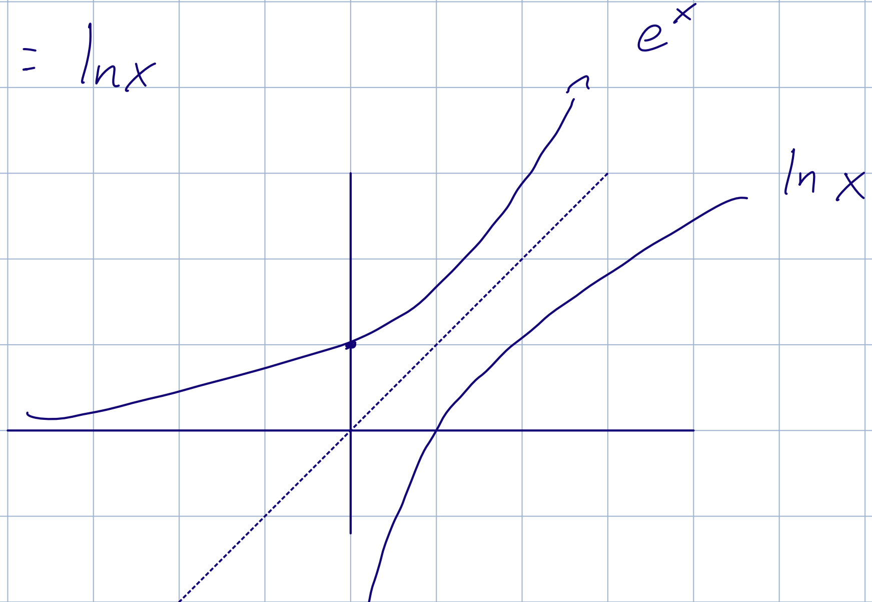
$\to 5\ln(x) + 2\ln(\sin(\pi x)) - \frac{1}{3}\ln(x^4+2)$

ex: $\frac{2}{3}\left[5\ln x + \frac{1}{2}\ln(x^2+1) - 4\ln(x-2)\right]$

$\to \frac{2}{3} \cdot \dfrac{\ln(x^5 \cdot (x^2+1)^{\frac{1}{2}})}{\ln(x-2)^4} \to \ln\left(\dfrac{(x^5 \cdot (x^2+1)^{\frac{1}{2}})}{(x-2)^4}\right)^{\frac{2}{3}}$

Graph of $\ln(x)$

$f(x) = \ln x$



ex: $\int x^3\,dx$

$= \frac{1}{4}x^4 + C$

$$\int_1^2 \frac{x^4-1}{x^2}\,dx \to \int_1^2 \frac{x^4}{x^2} - \frac{1}{x^2}\,dx$$

$$\to \int_1^2 x^2 - x^{-2}\,dx$$

$$= \frac{x^3}{3} + x^{-1}\Big|_1^2 = \frac{11}{6}$$

ex: $\int \frac{1}{x}\,dx$

$= \ln|x| + C$

because $\frac{1}{T}$ is cont. on $[a,b]$ by the fundamental theory of calc there is an integral of $\frac{1}{T}$

$$F(x) = \int_a^x \frac{1}{T}\,dx$$

$$\int_a^b \frac{1}{x}\,dx = \ln x \Big|_a^b$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$$

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}\Big((x^2+1)^3\Big) = 3(x^2+1)^2 \cdot 2x$$

$$= 6x(x^2+1)^2$$

$$\frac{d}{dx}\Big(\ln(g(x))\Big) = \frac{1}{g(x)} \cdot \frac{d}{dx} g(x)$$

ex: $$\frac{d}{dx}\Big(\ln(3x^2-1)\Big) = \frac{1}{3x^2-1} \cdot 6x$$

$$= \frac{6x}{3x^2-1}$$

ex: $$\frac{d}{dx}\left(\sqrt[3]{\ln x}\right) \to \frac{d}{dx}(\ln x)^{\frac{1}{3}} \to \frac{1}{3}(\ln x)^{-\frac{2}{3}} \cdot \frac{d}{dx}\ln x$$

$$= \frac{1}{3}(\ln x)^{-\frac{2}{3}} \cdot \frac{1}{x} = 3x^{-1} \cdot (\ln x)^{-\frac{2}{3}}$$

ex $\frac{d}{dx} \ln\left(2x + \sqrt{x^3 - 1}\right)$ $u = 2x + (x^3-1)^{\frac{1}{2}}$

$\rightarrow \frac{d}{dx} \ln\left(2x + (x^3-1)^{\frac{1}{2}}\right) \rightarrow \frac{1}{2x + (x^3-1)^{\frac{1}{2}}} \cdot \frac{d}{dx} u$

$\frac{du}{dx} = 2 + \frac{1}{2} \cdot (x^3-1)^{-\frac{2}{2}} \cdot 3x^2$

$= 2 + \frac{3x^2}{2}(x^3-1)^{-\frac{1}{2}}$

$= \frac{2 + \frac{3x^2}{2}(x^3-1)^{-\frac{1}{2}}}{2x + (x^3-1)^{\frac{1}{2}}}$

ex: $\frac{d}{dx}\left(x^3 \ln 5x\right) \rightarrow$

$\frac{d}{dx} f(x) \cdot g(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$

$= \frac{d}{dx} x^3 \cdot \ln 5x + \frac{d}{dx} \ln 5x \cdot x^3$

$= 3x^2 \cdot \ln 5x + \frac{1}{5x} \cdot \frac{d}{dx} 5x \cdot x^3$

$= 3x^2 \ln 5x + \frac{5}{5x} \cdot x^3$

$= 3x^2 \ln 5x + x^2$

$$\frac{d}{dx}\left[\ln|\cos x|\right] \rightarrow \frac{d}{dx} f(g(x)) = f'(gx)\cdot g'(x)$$

$$\rightarrow \frac{1}{\cos x}\cdot -\sin x$$

$$= -\frac{\sin x}{\cos x} = -\tan x$$

ex: $\frac{d}{dx}\left[\ln\left(\frac{x^2(2x^2+1)^3}{\sqrt{5-x^2}}\right)\right]$

$$\frac{d}{dx}\begin{cases} \rightarrow \ln\left(x^2(2x^2+1)^3\right) - \ln\left((5-x^2)^{\frac{1}{2}}\right) \\ = \ln(x^2) + \ln\left((2x^2+1)^3\right) - \frac{1}{2}\ln(5-x^2) \\ = \underbrace{2\ln(x)}_{a} + \underbrace{3\ln(2x^2+1)}_{b} - \underbrace{\frac{1}{2}\ln(5-x^2)}_{c} \end{cases}$$

$$= \frac{d}{dx}a + \frac{d}{dx}b - \frac{d}{dx}c$$

$$\frac{d}{dx}a = 2\frac{d}{dx}\ln x = \frac{2}{x}$$

$$\frac{d}{dx}b = 3\frac{d}{dx}\frac{1}{2x^2+1}\cdot 4x = 3\cdot\frac{4x}{2x^2+1} = \frac{12x}{2x^2+1}$$

$$\frac{d}{dx} c = \frac{1}{2}\frac{d}{dx}\ln(5-x^2) = \frac{}{5-x^2}\cdot -2x$$

$$= -\frac{1}{2}\cdot\frac{-2x}{5-x^2} = +\frac{x}{5-x^2}$$

$$= \frac{2}{x} + \frac{12x}{2x^2+1} + \frac{x}{5-x^2}$$

ex: logarithmic differentiation

$$y = \frac{(x-1)^4}{\sqrt[3]{2x-1}}$$

$$\ln y = \ln\left(\frac{(x-1)^4}{\sqrt[3]{2x-1}}\right)$$

$$\ln y = \ln(x-1)^4 - \ln(2x-1)^{\frac{1}{3}}$$

$$= 4\ln(x-1) - \frac{1}{3}\ln(2x-1)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(4\ln(x-1) - \frac{1}{3}\ln(2x-1)\right)$$

$$\frac{1}{y}\cdot\frac{d}{dx}y = 4\cdot\frac{d}{dx}\ln(x-1) - \frac{1}{3}\frac{d}{dx}\ln(2x-1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 4\cdot\frac{1}{x-1} - \frac{1}{3}\cdot\frac{2}{2x-1}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{x-1} - \frac{2}{3(2x-1)}$$

$$\frac{dy}{dx} = \frac{4}{x-1} - \frac{2}{3(2x-1)} \cdot y$$

$$\frac{dy}{dx} = \left(\frac{4}{x-1} - \frac{2}{3(2x-1)}\right) \cdot \frac{(x-1)^4}{\sqrt[3]{2x-1}}$$

ex: $\ln \frac{2x}{y} - \sin y + x^2 = 0$

$$\ln 2x - \ln y - \sin y + x^2 = 0$$

$$\ln y + \sin y = \ln 2x + x^2$$

$$= \ln 2 + \ln x + x^2$$

$$\frac{d}{dx}\left[\ln y + \sin y\right] = \frac{d}{dx}\left[\ln 2 + \ln x + x^2\right]$$

$$= \frac{1}{y} \cdot \frac{dy}{dx} + \cos y \frac{dy}{dx} = \frac{1}{2} + \frac{1}{x} + 2x$$

$$\frac{dy}{dx}\left(\frac{1}{y} + \cos y\right) = \frac{1}{2} + \frac{1}{x} + 2x$$

$$\frac{dy}{dx} = \frac{\frac{1}{2} + \frac{1}{x} + 2x}{\frac{1}{y} + \cos y}$$

Integrals $\int \frac{1}{u}\, du = \ln|u| + C$

ex $\int \frac{1}{5x-2}\, dx$ let $u = 5x-2$

$du = 5\,dx$ $dx = \frac{1}{5}\, du$

$$\int \frac{1}{u} \cdot \frac{du}{5} \to \frac{1}{5} \int \frac{1}{u} \cdot du$$

$$\to \frac{1}{5} \ln|u| + C \to \frac{1}{5} \ln|5x-2| + c$$

ex: $\displaystyle\int \frac{(\ln x)^{\frac{1}{2}}}{3x} \, dx$ let $u = \ln x$

$du = \frac{1}{x} dx$

$dx = x\, du$

$$\to \int \frac{u^{\frac{1}{2}}}{3x} \cdot x \, du$$

$$\to \int \sqrt{u} \cdot \frac{1}{3x} \cdot x \cdot du$$

$$\to \int u^{\frac{1}{2}} \cdot \frac{\cancel{x}}{3\cancel{x}} \cdot du$$

$$\to \int u^{\frac{1}{2}} \cdot \frac{1}{3} \cdot du \to \frac{1}{3} \int u^{\frac{1}{2}} \, du$$

$$\to \frac{1}{3} \frac{2}{3} u^{\frac{3}{2}} \, du$$

$$\to \frac{2}{9} u^{\frac{3}{2}} + c$$

$$\to \frac{2}{9} \ln x^{\frac{3}{2}} + c$$

Integrals:

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \sec^2 x = \tan x + c$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\int \csc x \cot x \, dx = -\csc x + c$$

$$\int \csc^2 dx = -\cot x + c$$

ex: $\int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} dx \qquad \text{let } u = \cos x$$

$$du = -\sin x \, dx$$

$$= \int \frac{\sin x}{u} \cdot \frac{du}{-\sin x} \qquad dx = \frac{du}{-\sin x}$$

$$= -\int \frac{\sin x}{u} \cdot \frac{du}{\sin x}$$

$$= -\int \frac{1}{u} du$$

$$= -\ln|u| + c$$

$$= -\ln|\cos x| + C = \ln|\cos^{-1} x| + C$$
$$= \ln|\sec x| + C$$

$$\int \tan x \, dx = \ln|\sec x| + C$$

ex: $\int \sec x \, dx$

$\rightarrow \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$

$\rightarrow \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$

let $u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x \, dx$

$dx = \frac{du}{\sec^2 x + \sec x \tan x}$

$\rightarrow \int \frac{\sec^2 x + \sec x \tan x}{u} \cdot \frac{du}{\sec^2 x + \sec x \tan x}$

$\rightarrow \int \frac{1}{u} \, du = \ln|u| + C$

$= \ln|\sec x + \tan x| + C$

$$\int \tan u \, du = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = \ln|\csc u - \cot u$$

ex $\int x \sec(x^2)\, dx$ let $u = x^2$

$du = 2x\,dx$ $dx = \frac{du}{2x}$

$$\int x \cdot \sec(u) \cdot \frac{du}{2x} \to \int \frac{1}{2} \cdot \sec(u)\, du$$

$$\to \frac{1}{2} \int \sec(u) \cdot du$$

$$= \frac{1}{2}\left(\ln|\sec x^2 + \tan x^2|\right) + C$$

ex: $\int \frac{\sin 2x}{1 + \sin^2 x}\, dx \to$ let $u = 1 + \sin^2 x$

$du = 2\sin x \cdot \cos x\, dx$

$dx = \frac{du}{2\sin x \cdot \cos x}$

$$\to \int \frac{\sin 2x}{u} \cdot \frac{du}{2\sin x \cos x}$$

$$\sin 2x = 2\sin x\cos x$$

$$= \int \frac{2\sin x\cos x}{u} \cdot \frac{du}{2\sin x\cos x}$$

$$= \frac{1}{u}\quad du$$

$$= \ln|u| + C$$

$$= \ln|1+\sin^2 x| + C = \ln(1+\sin^2 x) + C$$