Question 1

- If $|x-y|<\delta$ such that $x, y \in [1,\infty)$
- $$\begin{aligned} |\log(x)-\log(y)| &= \left|\log\left(\frac{x}{y}\right)\right| \\ &\leq \left|\log\left(\frac{y+\delta}{y}\right)\right| \\ &\leq \left|\log\left(1+\frac{\delta}{y}\right)\right| \\ &\leq \left|\frac{\delta}{y}\right| \\ &\leq \delta \end{aligned}$$
- Therefore, $\forall\epsilon>0, \exists\delta=\epsilon$ such that $|x-y|<\delta \implies |\log(x)-\log(y)|<\epsilon$
- Hence $\log$ is uniformly continuous in $[1,\infty)$

Question 2

- If ϕ is a continuous function and $f$is Riemann integrable, then $\phi\circ f$ is Riemann integrable
- let $g(x)=x^2,\ g$ is continuous
- then $g\circ f_r + g\circ f_i$ is riemann integrable
- let $h(x)=\sqrt{x}$ which is continuous
- hence, $\sqrt{f_r^2+f_i^2}$ is Riemann integrable
- let f be a Riemann integrable complex function, $c=\alpha+\iota\beta$
- $$\begin{aligned} c\int_a^b f(x)dx &= (\alpha+\iota\beta)\left(\int_a^b f_r(x)dx+\iota\int_a^b f_i(x)dx\right) \\ &= \left(\alpha\int_a^b f_r(x)dx-\beta\int_a^b f_i(x)dx\right)+\iota\left(\alpha\int_a^b f_r(x)dx-\beta\int_a^b f_i(x)dx\right) \\ &= \int_a^b cg(x)dx \end{aligned}$$
- Let $z=\int_0^1 f(x)dx$. Define $O=\frac{z}{|z|}$

- $$\begin{aligned}\left|\int\limits_0^1 f(x)dx\right| = |z| &= O\int\limits_0^1 f(x)dx \\ &= \int\limits_0^1 Of(x)dx \\ &= \int\limits_0^1 Re(Of(x))dx + \iota\int\limits_0^1 Im(Of(x))dx\end{aligned}$$
- As LHS is a real number,
- $\left|\int\limits_0^1 f(x)dx\right| = \left|\int\limits_0^1 Re(Of(x))dx\right| \leq \int\limits_0^1 |Of(x)|dx = \int\limits_0^1 |O||f(x)|dx$
- Therefore $|\int_0^1 f(x)dx| \leq \int_0^1 |f(x)|dx$

- # Question 3
  - ## a)
    - $\frac{1}{p} + \frac{1}{q} = 1$
    - let $f(u) = \frac{u^p}{p} + \frac{v^q}{q} - uv$
    - $f'(u) = u^{p-1} - v$
    - $u^{p-1} = v$ for critical points
    - $f''(u) = (p-1)u^{p-2} = (p-1)\frac{v}{u}$
    - $\frac{1}{p} < 1 \implies p > 1 \implies f''(u) > 0$
    - hence $u_0^{p-1} = v, u_0$ is a minimum
    - $$\begin{aligned}\frac{u^p}{p} + \frac{v^q}{q} - uv &\geq \frac{u_0^p}{p} + \frac{u_0^{(p-1)q}}{q} - u_0^p \\ &\geq \frac{u_0^p}{p} + \frac{u_0^p}{q} - u_0^p \\ &\geq 0\end{aligned}$$
  - ## b)
    - $u = \frac{f(x)}{(\int_0^1 f^p(x))^{\frac{1}{p}}}, v = \frac{g(x)}{(\int_0^1 g^q(x))^{\frac{1}{q}}}$
    - applying part a
    - $$\begin{aligned}\frac{fg}{(\int_0^1 f^p dx)^{\frac{1}{q}}(\int_0^1 g^q dx)^{\frac{1}{q}}} &\leq \frac{1}{p}\left(\frac{f}{\left(\int_0^1 f^p dx\right)^{\frac{1}{p}}}\right)^p + \frac{1}{q}\left(\frac{g}{\left(\int_0^1 g^q dx\right)^{\frac{1}{q}}}\right)^q \\ &\leq \frac{1}{p}\frac{f^p}{\int_0^1 f^p dc} + \frac{1}{q}\frac{g^q}{\int_0^1 g^q dc}\end{aligned}$$

- Integrating on both sides from 0 to 1
- $$\frac{\int_0^1 fgdx}{(\int_0^1 f^p dx)^{\frac{1}{q}}(\int_0^1 g^q dx)^{\frac{1}{q}}} \leq \frac{1}{p} + \frac{1}{q} = 1$$
- therfore

$$\int_0^1 fgdx \leq \left(\int_0^1 f^p dx\right)^{\frac{1}{p}} + \left(\int_0^1 g^q dx\right)^{\frac{1}{q}}$$

- **c)**
  - as proved in problem 2
  - $|\int_0^1 fgdx| \leq \int_0^1 |fg|dx = \int_0^1 |f||g|dx$
  - taking $|f|$ and $|g|$ be function for path (b)
  - $|\int_0^1 fgdx| \leq (\int_0^1 |f|^p)^{\frac{1}{q}} + (\int_0^1 |g|^q)^{\frac{1}{q}}$

- # Question 4
- ## a)
  - S is not Jordan Measurable
  - Consider a partition $P_1 \times P_2 = P$
  - Since S is dense in I^2, $L(P, \chi_S) = 0$, $U(P, \chi_S) = 1$ for any partition P
  - $\inf L(P, \chi_S) = 0$
  - $\sup U(P, \chi_S) = 1$
  - hence X_S is not Integrable
- ## b)
  - Since no open balls can be made around any point in S which are a subset of S, $S^o = \emptyset$
  - $\overline{S} = I^2$
  - $\delta S = \overline{S} \setminus S^o = I^2$
  - I^2 is a closed and bounded subset of $\mathbb{R}^2$, hence I^2 is compact
  - Since I^2 is compact, there exits an open cover with finite subcover ${R_n}
  - $I^2 \textcolor{red}{\text{\\sub}} \bigcup_{1 \leq i \leq n} R_i$
  - $Area(I^2) \leq Area\left(\bigcup_{1 \leq i \leq n} R_i\right)$
  - Therefore area of any union of sets that make a cover of I^2, hence The jordan measure of δS is non zero

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- # Question 5
- ## a)
  - $\delta S = \left\{\left(\frac{1}{n}, y\right) | n \in \mathbb{N}, y \in I\right\}$
  - The set $\left(\frac{1}{n}, y\right)$ for a particulat n can be covered by the rectangle $I \times \left[\frac{1}{n} - \frac{\epsilon}{3}, \frac{1}{n} + \frac{\epsilon}{3}\right]$
  - The area of the cover of the set is $\frac{2\epsilon}{3} < \epsilon$, hence the has measure 0
  - countable union of zero measure sets has measure 0, hence δS has measure 0, hence S is Integrable
- ## b)
  - $Area(S) = Area(I^2) - \sum_{i=1}^{\infty} Area\left(\left[\frac{1}{n} + \frac{\epsilon}{2 \cdot 2^i}, \frac{1}{n} - \frac{\epsilon}{2 \cdot 2^i}\right]\right)$
  - $Area(S) = 1 - \epsilon$
  - $Area(S) = 1$

- # Question 6
- Since f is continuous, it is integrable in a closed interval, let η be the closed figure which is the union of Γ_f, and the area enclosed in the curve.
- since f is integrable, we know that χ_η is integrable $\implies \Gamma_f$ has content 0, as it is a subset of the boundary of η
- ## If f is only integrable, the above argument still holds as the only Information used in the above proof is that f is integrable.
- # Question 7
- $\int_{\mathbb{R}} \tilde{D} = \int_I D$
- $$\int_0^1 (1-x)dx = x - \frac{x^2}{2}|_{[0,1]}$$
  $$= \frac{1}{2}$$
- Using Fubini's theorem

- $$\begin{aligned}\int_{I\times I} D(x)D(xy)dxdy &= \int_{I\times I}(1-x)(1-xy)dxdy \\ &= \int_0^1 (\int_0^1 (1-x)(1-y)dy)dx \\ &= \int_0^1 (1-x)*(1-\frac{x}{2})dx \\ &= \frac{5}{2}\end{aligned}$$
