Homework 1

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For $f = u + iv$ to be analytic, u, v have to satisfy the Cauchy-Reimann Eqations.

$$\frac{\partial u}{\partial x} = 2x$$
$$\frac{\partial u}{\partial y} = -2y$$

The Cauchy-Reimann Equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So

$$\frac{\partial v}{\partial y} = 2x$$
$$\frac{\partial v}{\partial x} = 2y$$
$$\implies v = 2xy + c$$

Hence

$$f(x+iy) = (x^2 - y^2) + i(2xy + c)$$
$$f(z) = z^2 + ic \qquad \text{where } c \in \mathbb{R}$$

2.

One way

If f is differentiable as a complex function, then its derivative follow Cauchy Reimann Equation, then

$$Df(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}$$

which is of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

which computes with any complex number $(x + iy)$ of the from

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

as the product for the two matrices is

$$\begin{pmatrix} ax - by & ay + bx \\ -ay - ab & ax - by \end{pmatrix}$$

Converse

If the derivative of f commtues with multiplication with complex number $z = x + iy$

let the derivative of $f = Df$ be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$Df \times z = \begin{pmatrix} ax+cy & bx+dy \\ cx-ay & dx-by \end{pmatrix}$$
$$z \times Df = \begin{pmatrix} ax-by & ay+bx \\ cx-dy & cy+dx \end{pmatrix}$$

This implies $a = d$ and $b = -c$, hence the derivative is of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

which means it follows the Cauchy Reimann Inequality and hence complex function corresponding to f will be Analytic.

3.

To Prove

$$\left|\sum_{i=1}^{n} a_i \overline{b}_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{j=1}^{n} |b_j|^2$$

consider the term $\sum_{i=1}^{n} |a_i \overline{b}_j|$, then
by Triangle inequality

$$\left|\sum_{i=1}^{n} a_i b_j\right|^2 \leq \sum_{i=1}^{n} \left|a_i \overline{b}_j\right|^2$$

replacing the terms a_i and b_i with $|a_i|$ and $|b_i|$ in the vectors and using the cauchy inequality for real numbers gives

$$\sum_{i=1}^{n} \left|a_i \overline{b}_j\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b-i|$$

which gives the desired inequality.

4.

Let $f = u_1 + iv_1$ and $g = u_2 + iv_2$

$$\begin{aligned}
g \circ f &= u_2(u_1 + iv_1) + iv_2(u_1 + iv_2) \\
\frac{\partial(g \circ f)}{\partial x} &= \frac{\partial u_2(f)}{\partial x} \frac{(u_1' + iv_1')}{\partial x} + i \frac{\partial v_2(f)}{\partial x} \frac{(u_1' + iv_2')}{\partial x} \\
&= \left(\frac{\partial u_2(f)}{\partial x} \cdot \frac{\partial u_1}{\partial x} - \frac{\partial v_2(f)}{\partial x} \cdot \frac{\partial v_1}{\partial x}\right) + i\left(\frac{\partial u_2(f)}{\partial x} \cdot \frac{\partial v_1}{\partial x} + \frac{\partial v_2(f)}{\partial x} \cdot \frac{\partial u_1}{\partial x}\right) \\
\frac{\partial(g \circ f)}{\partial y} &= \frac{\partial u_2(f)}{\partial y} \frac{(u_1' + iv_1')}{\partial y} + i \frac{\partial v_2(f)}{\partial y} \frac{(u_1' + iv_2')}{\partial y} \\
&= \left(\frac{\partial u_2(f)}{\partial y} \cdot \frac{\partial u_1}{\partial y} - \frac{\partial v_2(f)}{\partial y} \cdot \frac{\partial v_1}{\partial y}\right) + i\left(\frac{\partial u_2(f)}{\partial y} \cdot \frac{\partial v_1}{\partial y} + \frac{\partial v_2(f)}{\partial y} \cdot \frac{\partial u_1}{\partial y}\right)
\end{aligned}$$

letting

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \left(\frac{\partial u_2(f)}{\partial x} \cdot \frac{\partial u_1}{\partial x} - \frac{\partial v_2(f)}{\partial x} \cdot \frac{\partial v_1}{\partial x}\right) \\
\frac{\partial v}{\partial x} &= \left(\frac{\partial u_2(f)}{\partial x} \cdot \frac{\partial v_1}{\partial x} + \frac{\partial v_2(f)}{\partial x} \cdot \frac{\partial u_1}{\partial x}\right) \\
\frac{\partial u}{\partial y} &= \left(\frac{\partial u_2(f)}{\partial y} \cdot \frac{\partial u_1}{\partial y} - \frac{\partial v_2(f)}{\partial y} \cdot \frac{\partial v_1}{\partial y}\right) \\
\frac{\partial v}{\partial y} &= \left(\frac{\partial u_2(f)}{\partial y} \cdot \frac{\partial v_1}{\partial y} + \frac{\partial v_2(f)}{\partial y} \cdot \frac{\partial u_1}{\partial y}\right)
\end{aligned}$$

Then by applying Cauchy Reimann Equation to the components, we can prove that $f \circ g$ also follows the Cauchy Reimann Equations, hence it is also analytic.

5.

The solution to the first question verifies the Cauchy Reimann Equations for $f(z) = z^2$

For $f(z) = z^3$ we can write f as

$$f(x+iy) = x^3 + 3ix^2y - 3xy^2 - iy^3$$
$$f(x+iy) = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

and let

$$u = x^3 - 3xy^2$$
$$v = 3x^2y - y^3$$

Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

And

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy$$

Hence $f(z) = z^3$ follows the Cauchy Reimann Equations.

6.

Since $ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic

$$6ax + 2by = 0$$
$$2cx + 6dy = 0$$

Hence $3a + c = 0$ and $b + 3d = 0$
Hence the cubic is of the form $ax^3 + 3dx^2y - 3axy^2 - dy^3$
Integration Method
For the conjugate harmonic function

$$\frac{\partial u}{\partial x} = 3ax^2 + 6dxy - 3ay^2 = \frac{\partial v}{\partial y}$$

Hence v is of the form

$$v = 3ax^2y + 3dxy^2 - ay^3 + f(x)$$

$$\frac{\partial u}{\partial y} = 3dx^2 - 6axy - 3dy^2 = -\frac{\partial v}{\partial x}$$

Hence v is of the form

$$v = -x^3 + 3a^2xy + 3dxy^2 + f(y)$$

Hence

$$v = -dx^3 + 3ax^2 + 3dxy^2 - ay^3$$

Formal Method

$$
\begin{aligned}
f(z) &= 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ic \\
u(0,0) &= 0 \\
2u\left(\frac{z}{2}, \frac{z}{2i}\right) &= a\frac{z^3}{8} + 3b\frac{z^3}{8i} + 3a\frac{z^3}{8} + b\frac{z^3}{8i} \\
\therefore f(z) &= z^3\left(a + \frac{b}{3i}\right) \\
\therefore f(z) &= ax^3 + 3dx^2y - 3axy^2 - dy^3 + i(-dx^3 + 3ax^2 + 3dxy^2 - ay^3)
\end{aligned}
$$

The conjugate harmonic of the equation of the following will be of the form $-dx^3 + 3ax^2y + 3dxy^2 - ay^3 + K$, because they follow the Cauchy Reimann Equations.

7.

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Let $|f(z)| = c \neq 0$ be a function which has constant absolute value but is not constant value

$$
\begin{aligned}
|f(z)|^2 &= c^2 \\
f(z)\overline{f(z)} &= c^2 \\
\overline{f(z)} &= \frac{c^2}{f(z)}
\end{aligned}
$$

Hence, $\overline{f(z)}$ is also analytic, so it follows the Cauchy Reimann Equations, comparing Cauchy Reimann Equations for $f(z)$ and $\overline{f(x)}$ is $u(x,y) = v(x,y) = 0$ where $f = u + iv$ and $\overline{f} = u - iv$, which means that the functions is constant, which is a contradiction, hence, if an analytic function has constant absolute value then it reduces to a constant number.

8.

let $f(x+iy) = u(x,y) + iv(x,y)$
then

$$
\begin{aligned}
\overline{f(\overline{z})} &= u(x,-y) - iv(x,-y) \\
\frac{\partial \overline{f(\overline{z})}}{\partial x} &= \frac{\partial u(x,-y)}{\partial x} - i\frac{\partial v(x,-y)}{\partial x} \\
\frac{\partial \overline{f}}{\partial y} &= -\frac{\partial u(x,-y)}{\partial y} + i\frac{\partial v(x,-y)}{\partial y}
\end{aligned}
$$

$f(z)$ is analytic $\iff$ $\overline{f(\overline{z})}$ is analytic because they satisfy the Cauchy Reimann Equations together.

9.

$$
\begin{aligned}
u(z) &= a(x,y) + ib(x,y) \\
u \text{ is harmonic} \implies v(x,y) &= \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \\
\frac{\partial^2 u(\overline{z})}{\partial x^2} + \frac{\partial^2 u(\overline{z})}{\partial y^2} &= v(x,-y) = 0 \\
\therefore u(\overline{z}) &\text{ is also harmonic}
\end{aligned}
$$