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Tags : Logic

Semantics of First Order Logic

To give meaning to our formulas, we first fix a set S, which we call the **Universe**. And we then interpret our formulas in terms what we have in the universe. To do that we have an interpretation function ι. Together they make a **First Order Structure**

First Order Structures

Given a First Order Language L, A **First Order Structure** L is a pair $\mathcal{M} = (S, \iota)$ where S is a *non-empty* set and ι a function defined over $R \sqcup F \sqcup C$ such that

- For each relation symbol $r \in R$ with arity n, we have $\iota(r) \subseteq S^n$
- For each function symbol $f \in F$ with arity n, we have $\iota(f) : S^n \to S$
- For each constant $c \in C$ we have $\iota(c) \in S$.

For more readability we denote $\iota(x)$ as $x^{\mathcal{M}}$. That we have enough structure for our **First Order Formulas** we can build an interpretation for it. This corresponds to *Evaluations* for propositional logic

Interpretation

An **Interpretation** of L is the tuple $\mathcal{I} = (\mathcal{M}, \sigma)$ where $\mathcal{M}$ is a **First Order Structure** and $\sigma : Vars \to S$ is an assignment of elements of S to variables.

Given any σ, we denote by $\sigma[x_1 \to s_1, \dots x_n \to s_n]$ the modified assignment σ' where

$$\sigma'(x) = \begin{cases} \text{corresponding } s_i & x \in \{x_1, x_2 \dots x_n\} \\ \sigma(x) & \text{otherwise} \end{cases}$$

Similarly $\mathcal{I}[x \to s] = (\mathcal{M}, \sigma[x \to s]) = I'$

Now, Given an interpretation $\mathcal{I}$, each term t over L maps to a unique element in S.

- If t is a constant $c \in C$, $t^{\mathcal{I}} = c^{\mathcal{M}}$.
- If t is a variable $x \in Var$, $t^{\mathcal{I}} = \sigma(x)$
- If t is of the form $f(t_1, t_2 \dots t_n)$ where $f \in F$, then $t^{\mathcal{I}} = f^{\mathcal{M}}(t_1^{\mathcal{I}}, t_2^{\mathcal{I}} \dots t_n^{\mathcal{I}})$

Satisfactory Interpretation

A **Satisfactory Interpretation** corresponds to *Satisfying valuation* for *propositional logic*.
We say $\mathcal{I} \models \varphi$ ($\mathcal{I}$ satisfies φ) if

- $\mathcal{I} \models t_1 \equiv t_2$ if $t_1^{\mathcal{I}} = t_2^{\mathcal{I}}$
- $\mathcal{I} \models r(t_1 \dots t_n)$ if $(t_1^{\mathcal{I}} \dots t_n^{\mathcal{I}}) \in r^{\mathcal{M}}$
- $\mathcal{I} \models \neg\varphi$ if $\mathcal{I} \not\models \varphi$
- $\mathcal{I} \models \varphi \vee \psi$ if $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$
- $\mathcal{I} \models \exists x \; \varphi$ if there is an element $s \in S$ such that $\mathcal{I}[x \to s] \models \varphi$

And as usual, we say a formula is *satisfiable* if there exists a satisfactory interpretation for it. And a formula is *valid* if all Interpretations satisfy it.

Free Variables

The *Quantifiers* change the behavior of variables they interact with. For both $\exists$ and $\forall$ the variables they "bind" become independent of the **Interpretation**. Hence for any Interpretation $(\mathcal{M}, \sigma)$, σ needs to only assign to the free variables. We build a function FV that gives the set of free variables in a formula, it can be defined as

- If φ is the atomic formula $r(t_1 \dots t_n)$ $FV(\varphi)$ is the set of variables in $\{t_1 \dots t_n)\}$
- If φ is $t_1 \equiv t_2$ then $FV(\varphi)$ is the set of variables in $\{t_1, t_2\}$
- $FV(\varphi) = FV(\neg\varphi)$
- $FV(\varphi \vee \psi) = FV(\varphi) \cup FV(\psi)$
- $FV(\exists x\ \varphi) = FV(\varphi) \setminus \{x\}$

And Considering them we have the following:
If σ and σ' agree on $FV(\varphi)$ then $(\mathcal{M}, \sigma) \models \varphi$ iff $(\mathcal{M}, \sigma') \models \varphi$

Logical Consequence

Logical Consequence has the same meaning as that in *Propositional logic*. Given a set X of first order equations. We say $X \models \varphi$ if for every $\mathcal{I}$ such that $\mathcal{I} \models X$ we have $\mathcal{I} \models \varphi$.

References

First Order Logic
Syntax of First Order Logic