Complex Analysis Assignment 3

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1. Map the common part of the disks $|z| < 1$ and $|z-1| < 1$ on the inside of the unit circle. Choose the mapping so that the two symmetries are preserved.

The two circles intersect at points $a = \frac{1}{2} + \frac{\sqrt{3}i}{2}$ and $b = \frac{1}{2} - \frac{\sqrt{3}i}{2}$, hence we can map these two points at 0 and ∞ respectively by

$$f(z) = -\frac{z-a}{z-b}$$

This makes a sector of angle $\frac{\pi}{3}$ passing through $f(0) = b$, $f(1) = a$ and $f\left(\frac{1}{2}\right) = 1$
applying

$$g(z) = z^{\frac{3}{2}}$$

considering a branch cut which makes it the half plane $Re(z) \geq 0$
and the map

$$h(z) = \frac{z-1}{z+1}$$

sends it to the unit circle
The map $h \circ g \circ f$ takes the intersection of the two discs to the unit circle

2. Map the reigon between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ on a half plane.

The map

$$f(z) = \frac{-i\pi z}{z-1}$$

to the strip $0 \leq Im(z) \leq \pi$
The map

$$g(z) = e^z$$

sends the strip to the half plane $Im(z) > 0$
the map $g \circ f$ takes the reigon between the two circles to a half plane

3. map the complement of the arc $|z| = 1, y \geq 0$ on the outside of the unit circle so that the points at infinity correspont to each other.

The map

$$a(z) = -i\frac{z+1}{z-1}$$

this maps the the arc to the line $x \leq 0, y = 0$, sending ∞ to $-i$

$$b(z) = \sqrt{z}$$

with the branch cut being the negative number real line, maps the points complement to the line on the half plane $Re(z) > 0$, sending $-i$ to $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$
The function

$$c(z) = i\frac{z-1}{z+1}$$

maps the half plane onto the unit circle. sending $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$ to $\sqrt{2} - 1$
The function

$$d(z) = \left(\sqrt{2} - 1\right) \cdot \frac{z+1}{1-z}$$

sends the circle to the right plane and send $\sqrt{2} - 1$ to 1
The function

$$e(x) = \frac{z+1}{z-1}$$

Send the half plane to the complement of the circle and sends 1 to ∞
Hence the function

$$e \circ d \circ c \circ b \circ a$$

sends the complement of the arc to outside the unit circle and sends ∞ to itself

4. Compute

$$\int_\gamma xdz$$

where γ is the directed line segment from 0 to $i+1$

Let $z(t) = t + it, t \in [0, 1]$
Then

$$\int_\gamma xdz = \int_0^1 x(z(t)) \cdot z'(t)dt = \int_0^1 t(1+i)dt = \frac{1+i}{2}$$

5. Compute

$$\int_{z=|r|} xdz$$

for the positive sense in two ways, first by the use of parameters and second, by observing that $x = \frac{z+\bar{z}}{2} = \frac{1}{2}\left(z + r^2 z\right)$ on the circle

1. **Parameters**:
   $z(t) = re^{it}, 0 \leq t \leq 2\pi$
   $z'(t) = ire^{it}$
   Hence

$$\begin{aligned}\int_{|z|=r} x dz &= \int_0^{2\pi} x(z(t)) \cdot z'(t) dt \\ &= \int_0^{2\pi} (r\cos t)(ir)(\cos t + i \sin t) dt \\ &= r^2 i \left(\int_0^{2\pi} cos^2 t \, dt + \int_0^{2\pi} \sin t \cos t \, dt \right) \\ &= \pi r^2 i\end{aligned}$$

2. **Observation:**
   $f(z) = z$ is an analytic function, Hence the integral in a closed loop is 0 thus

$$\begin{aligned}\int_{|z|=r} x dz &= \int_{|z|=r} \frac{r^2 dz}{2z} \\ &= \frac{r^2}{2} \cdot 2\pi i = \pi i r^2\end{aligned}$$

6. Compute

$$\int_{|z|=2} \frac{dz}{z^2 - 1}$$

in the positive sense of the circle

This is equal to

$$\frac{1}{2} \left(\int_{|z|=2} \frac{dz}{z-1} - \int_{|z|=2} \frac{dz}{z+1} \right)$$

for both the integral, the roots of the follwoing the linear polynimals, lie inside the loop, hence both of them evaluate to $2\pi i$ Hence the total integral evaluates to 0.

7.

a)

$n(\sigma_1, x_1) = 0$, hence $n(\sigma_1, z) = 0$ for $z \in \gamma_2$
We know that σ_1 intersects the X-axis. Let σ_1 intersect the X-axis at x' where $x' \neq 0$. Such that x' is the rightmost intersection of σ_1 and X-axis.

If $n(\sigma_1, x_2) \neq 0, x_2$ lies in the interior of σ_1, hence $x_2 < x'$ But this contradicts the fact that x_2 is the rightmost intersction of γ_1 with the X-axis.
$\therefore n(\sigma_1, x_2) = 0$
as $x_2 \in \gamma_2$, for $z \in \gamma_2, n(\sigma_1, z) = 0$ as γ_2 is a connected curve and n is a continuous function, connected component get mapped to connected components. Image of $\mathbb{N}$ is singleton

b)

$n(\sigma_1, x) = n(\sigma_2, x) = 1$ for small $x > 0$.
Let x' be the point where γ intersects X-axis for the first time. As $Im\ z_1 < 0$ and $Im\ z_2 > 0$ and subarc $z_1 \to z_2$ passing through the origin. If σ_1 and σ_2 does not meet the X-axis at any point greater than $\frac{x'}{2}$ and the subarc $z_1 \to z_2$ passing through x' of γ does not meet the X-axis at any point less than $\frac{x'}{2}, n(\sigma_1, \frac{x'}{2}) = n(\sigma_2, \frac{x'}{2}) = 1$. As $x' > 0, \frac{x'}{2} > 0$

c)

The first intersection of x_1 of the positive real axis with γ lies on γ_1.
If not, the first intersection x_1 of the positive real axis with γ lies on γ_2.
$Im\ z_1 < 0$ and $Im\ z_2 > 0$, the subarc $z_1 \to z_2$ whcih γ_1 does not meet the X-axis at any point less than x_1 and the subserc $z_1 \to z_2$ of $\sigma_1$passing through the origin does not meet the X-axis at any point greater than $x_1 \therefore n(\sigma_1, x_1) = 1$
But if $x_1 \in \gamma_2, n(\sigma_1, x_1) = 0$ which is a contradiction
$\therefore$ the first intersection x_1 of the positive real axis with γ lies on γ_2

d)

$n(\sigma_2, x_2) = 1$, hence $n(\sigma_2, z) = 1$ for $z \in \gamma_1$
$Im\ z_1 < 0$ and $Im\ z_2 > 0$, the subarc $z_1 \to z_2$ of σ_2 which γ_2 does not meet the X-axis at any point less than x_1 and the subarc $z_1 \to z_2$ of σ_2 which passes through the origin does not meet the x axis at any point greater than x_1,
$\therefore n(\sigma_2, x_1) = 1$
$\therefore n(\sigma_2, z) = 1$, for $z \in \gamma_1$ as it is connected and n is a continuous function, $n(\sigma_2, z)$ is connected $\forall z \in \gamma_1$ and hence a singleton image of $\mathbb{N}$

8. Compute

$$\int_{|z|=1} \frac{e^z}{z} dz$$

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e^z is an analytic function, and by cauchy's integral formula

$$e^0 = 1 = \frac{1}{2i\pi} \int_{|z|=1} \frac{e^z}{z-0} dz$$

Thus the integral computes to $2i\pi$

9. Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

by decomposing the integrand in partial fraction

$$\int_{|z|=2} \frac{dz}{z^2+1} = \frac{1}{2i}\left(\int_{|z|=2} \frac{dz}{z-i} - \int_{|z|=2} \frac{dz}{z+i}\right)$$

as i and $-i$ both lie in the interior of $|z|=2$

$$\int_{|z|=2} \frac{dz}{z+i} = \int_{|z|=2} \frac{dz}{z-i}$$

Hence

$$\int_{|z|=2} \frac{dz}{z^2+1} = 0$$

10 Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition that $|a| \neq \rho$

$$\begin{aligned}
\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= -i\rho \int_{|z|=\rho} \frac{dz}{z(z-a)(\bar{z}-\bar{a})} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2-\bar{a}z)} \\
&= \frac{i\rho}{\bar{a}} \int_{|z|=\rho} \frac{dz}{(z-a)\left(z-\frac{\rho^2}{\bar{a}}\right)} \\
&= \frac{i\rho}{|a|^2-\rho^2}\left(\int_{|z|=\rho} \frac{dz}{z-a} - \int_{|z|=\rho} \frac{dz}{z-\frac{\rho^2}{\bar{a}}}\right) \\
a<\rho \Longrightarrow & \int_{|z|=\rho} \frac{dz}{z-a} = 2\pi i, \int_{|z|=\rho} \frac{dz}{z-\frac{\rho^2}{\bar{a}}} = 0 \\
a>\rho \Longrightarrow & \int_{|z|=\rho} \frac{dz}{z-a} = 0, \int_{|z|=\rho} \frac{dz}{z-\frac{\rho^2}{\bar{a}}} = 2\pi i \\
\text{Thus } \int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= \frac{2\pi\rho}{\rho^2-|a|^2} \\
&= \frac{2\pi\rho}{|a|^2-\rho^2}
\end{aligned}$$

11. Compute

$$\int_{|z|=1} e^z z^{-n} dz, \qquad \int_{|z|=2} z^n(1-z)^m dz, \qquad \int_{|z|=\rho} |z-a|^{-4}|dz| \; (|a| \neq \rho)$$

Part 1:

$$\int_{|z|=1} e^z z^n dz = \frac{2i\pi}{(n-1)!} e^z \bigg|_{z=0} = \frac{2i\pi}{(n-1)!}$$

Part 2:

If $n, m > 0$

since z^n and $(1-z)^m$ are analytic, the integral evalutes to 0

If $n < 0$ and $m > 0$

$$\int_{|z|=2} \frac{(1-z)^m}{z^{|n|}} dz = \frac{2i\pi}{(n-1)!} \frac{d^{|n|-1}}{dz^{|n|-1}} (1-z)^m \bigg|_{z=0} = 2i\pi(-1)^{|n|-1} \binom{m}{|n|-1}$$

If $n > 0$ and $m < 0$, similarly we get

$$\int_{|z|=2} \frac{z^n}{(1-z)^{|m|}} dz = 2i\pi(-1)^m \binom{n}{|m|-1}$$

Part 3:

$$I = \int_{|z|=\rho} \frac{|dz|}{|z-a|^4} = \frac{-i\rho}{a^2} \int_{|z|=\rho} \frac{zdz}{(z-a)^2(z-\rho^2 a^2)^2}$$

if $|a| < \rho$ then $1/(z - \frac{\rho^2}{a^2})$ is analytic and

$$I = \frac{-i\rho}{a^2} \frac{2i\pi}{1!} \frac{d}{dz} \left(\frac{z}{z - \frac{\rho^2}{a^2}} \right) \bigg|_{z=a}$$

The derivative evalutates to

$$\frac{d}{dz} \left(\frac{z}{(z - \rho^2/a^2)} \right) = a^2 \frac{\rho^4 - a^4}{(az - \rho^2)^4}$$

hence the integral evalueates to

$$I = 2\pi\rho \frac{a^2 + \rho^2}{(\rho^2 - a^2)^3}$$

Similarly if $|a| > \rho$

$$I = 2\pi\rho \frac{a^2 + \rho^2}{(a^2 - \rho^2)^3}$$

12. Prove that a function which is analytic in the whole plane and satisfies the inequality $|f(z)| < |z|$ for some n and all sufficiently large $|z|$ reduces to a polynomial

$$\begin{aligned} |f^{(n+1)}(a)| &\leq \left| \frac{(n+1)!}{2i\pi} \int_{|z|=r} \frac{f(z)dz}{(z-a)^{n+2}} \right| &&\leq \frac{(n+1)!}{2\pi} \int_{|z|=r} \frac{|f(z)|dz}{|z-a|^{n+2}} \\ &< \frac{(n+1)!}{2\pi} \int_{|z|=r} \frac{|z|^n dz}{|z-a|^{n+2}} &&< \frac{(n+1)!}{2\pi} \int_{|z|=r} \frac{r^n dz}{|r-a|^{n+2}} \\ &\leq \frac{(n+1)! r^n}{2\pi(r-a)^{n+2}} \int_{|z|=r} |dz| &&= \frac{(n+1)! r^n}{(r-a)^{n+2}} \\ \text{As } r \to \infty, & \frac{(n+1)! r^n}{(r-a)^{n+2}} \to 0 \end{aligned}$$

Hence $f^{(n+1)} = 0$, thus f is a polynomial of degree upto n.

13. If $f(z)$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find and upper bound for $f^{(n)}(z)$ in $|z| \leq \rho < R$

$$
\begin{aligned}
|f^{(n)}(a)| &= \left| \frac{n!}{2i\pi} \int_{|z|=R} \frac{f(z)dz}{(z-a)^{n+1}} \right| \\
&\leq \frac{n!}{2\pi} \int_{|z|=R} \frac{|f(z)||dz|}{|z-a|^{n+1}} \\
&\leq \frac{n!M}{2\pi} \int_{|z|=R} \frac{|dz|}{|z-a|^{n+1}} \\
&\leq \frac{n!MR}{(R-a)^n}
\end{aligned}
$$

14. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leq \frac{1}{1-|z|}$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy Inequality will yield.

$$
\begin{aligned}
|f^{(n)}(0)| &= \left| \frac{n!}{2i\pi} \int_{|z|=R} \frac{f(z)dz}{z^{n+1}} \right| \\
&\leq \frac{n!}{2\pi} \int_{|z|=R} \frac{|f(z)||dz|}{|z|^{n+1}} \\
&= \frac{n!}{R^n(1-R)} \\
\text{On minimizing by setting } R &= \frac{n}{n+1} \\
|f^{(n)}(0)| &\leq (n+1)!\left(1+\frac{1}{n}\right)^n
\end{aligned}
$$

15. Show that successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. Fromulate a sharper theorem of the same kind

$$
|f^{(n)}(z)| = \frac{n!}{2\pi} \int_{|z|=R} \frac{|f(z)||dz|}{|z-a|^{n+1}} \qquad \leq \frac{n!M}{R} \text{ where } M = \sup_{|z|=R} |f(z)|
$$

M exists as f is continuous adn $|z| = R$ is compact
if $f^{(n)}(z) > n!n^n$ then $\frac{n!M}{R} > n!n^n$ hence $\frac{M}{R} > n^n$ which is a contradictoin.