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Tags : Logic

Isomorphism Between First Order Interpretations

Motivation

Let $R = \{<\}, F = \emptyset$
Consider two structures, one where $S = \mathbb{Q} \cap (0, 1)$ and one where $S = \mathbb{Q} \cap (0, \infty)$
Then we cannot have any **First Order Formula** which is valid for one of the model and not for the other one, hence that is not strong enough to differentiate between the two models.

Here we show that the two models are isomorphic.

Let $\mathcal{I}_1 = \{\mathcal{M}_1, \sigma_1\}$ and $\mathcal{I}_2 = \{\mathcal{M}_2, \sigma_2\}$
Then we say that the above two interpretations **Isomorphic** if there exists a bijection $\sigma : S_1 \to S_2$ such that:

- For every $r \in R$ we have $(e_1, e_2 \dots e_n) \in r_1^{\mathcal{M}_1}$ iff $(\sigma(e_1), \sigma(e_2) \dots \sigma(e_n)) \in r_2^{\mathcal{M}_2}$
- For every $f \in F$ we have $\sigma(f^{\mathcal{M}_1}(e_1, e_2 \dots e_n)) = f^{\mathcal{M}_2}(\sigma(e_1), \sigma(e_2) \dots \sigma(e_n)))$
- For every $c \in C$ we have $\sigma(c^{\mathcal{M}_1}) = c^{\mathcal{M}_2}$
- For every $x \in Var$ we have $\sigma(\sigma_1(x)) = \sigma_2(x)$

lemma

If $\mathcal{I}_1, \mathcal{I}_2$ are isomorphic, then for every formula φ, $\mathcal{I}_1 \models \varphi$ iff $I_2 \models \varphi$.

Proof: Induction on structure of φ

References

Syntax of First Order Logic
Semantics of First Order Logic