Homework 2

Complex Analysis Homework 2

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1.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad \cos z = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

To show:

$$S_k(z) = \sin z - \sum_{i=0}^{k} (-1)^i \frac{z^{2i+1}}{(2i+1)!}, \quad C_k(z) = \cos z - \sum_{i=0}^{k} (-1)^i \frac{z^{2i}}{(2i)!}$$

have the same sign as $(-1)^{k+1}$
Inducting on n, which is the term in the considered series with the highest degree.
Base case: The condititon is clear for $n = 1, 0$
Induction Hypothesis: The condition if true for all $n \in \{1, 2, 3, \cdots n-1\}$
Induction Step:
If $n(2k+1)$ is odd: By the induction hypothesis, $C_k(z)$ satisfies the condition, by integrating it over $[0, z]$ we get that $S_k(z)$ also satisfies the condition.
If $n(2k)$ is even: By the induction hypothesis, $S_{k-1}(z)$ satisfies the condition, by integrating it over $[0, z]$ we get $-C_k(z)$ has the same sign as $(-1)^k$ hence $C_k(z)$ has the same sign as $(-1)^{k+1}$.
This completes the Induction.

2.

$$3 < \pi < 2\sqrt{3}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

Hence $\cos r$ is real if r is real.
also $\sin^2 z + \cos^2 z = 1$
Hence for all real z, $-1 \leq \sin z, \cos z \leq 1$
Derivative of $x - \sin x$ is $1 - \cos x$ which is always greater than or equal to 0, and since $\sin 0 = 0$, $\sin x < x \forall x > 0$
Hence

$$\sin \frac{\pi}{6} = \frac{1}{2} \implies \frac{1}{2} < \frac{\pi}{6} \implies 3 < \pi$$

From the previous question, we can say that

$$\sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

putting $2\sqrt{3}$ in the above polynomial returns a negative number. Hence $\sin 2\sqrt{3} < 0$ so there is a root $c_0 \in [0, 2\sqrt{3}]$, and c_0 is hence and integral multiple of π , thus $\pi \leq c_0 \leq 2\sqrt{3}$

4.

$$\begin{aligned}
e^z &= k \\
\log(k) &= \log(|k|) + i\arg(k) \\
\text{For } k &= 2 \\
z &= \log(2) + 2ni\pi \\
\text{For } k &= -1 \\
z &= i\pi + 2ni\pi \\
\text{For } k &= i \\
z &= \frac{i\pi}{2} + 2ni\pi \\
\text{For } k &= \frac{-i}{2} \\
z &= \log\left(\frac{1}{2}\right) + \frac{i3\pi}{2} + 2ni\pi \\
\text{For } k &= -1 - i \\
z &= \log\left(\sqrt{2}\right) + \frac{i5\pi}{8} + 2ni\pi \\
\text{For } k &= 1 + 2i \\
z &= \log\left(\sqrt{5}\right) + i\arctan\left(\frac{2}{\sqrt{5}}\right) + 2ni\pi
\end{aligned}$$

6.

$$\begin{aligned}
2^i &= (e^{\log 2})^i = e^{i\log(2)} = \cos(\log 2) + i\sin(\log 2) \\
i^i &= (e^{\frac{i\pi}{2} + 2in\pi})^i = e^{\frac{-\pi}{2} + 2n\pi} \\
(-1)^i &= (e^{\log -1})^i = e^{\pi + 2n\pi}
\end{aligned}$$

7.

$$\begin{aligned}
z^z &= e^{z \times \log z} \\
&= e^{z \times (\log(|z|) + i\arg(z))} \\
&= e^{Re(z)\cdot\log(|z|) - Im(z)\cdot(\arg(z) + 2n\pi)} \cdot e^{i(Re(z)\cdot(\arg(z) + 2m\pi) + Im(z)\cdot\log(z))} \\
&= e^{Re(z)\cdot\log(|z|) - Im(z)\cdot(\arg(z) + 2n\pi)} \cdot e^{i(Re(z)\cdot\arg(z) + Im(z)\cdot\log(z))}
\end{aligned}$$

9.

Let the triangle be formed by 3 complex numbers a, b, c. let $m = \frac{a+b+c}{3}$ let $f(z) = \frac{z-m}{a}$.
Since f is conformal, the angle between 2 lines does not change, hence the "angles of a trianlge" do not change
If $Im(b') > 0$:
let angles

$$\begin{aligned}
A &= \arg(c' - a') - \arg(b' - a') \\
B &= \arg(a' - b') - \arg(c' - b') \\
C &= \arg(b' - c') - \arg(a' - c')
\end{aligned}$$

where $a' = f(a), b' = f(b), c' = f(c)$
Then $A + B + C = \arg(b' - c') - \arg(c' - b') + \arg(c' - a') - \arg(a' - c') + \arg(a' - b') - \arg(b' - a')$
Thus $A + B + C = -\pi + \pi + \pi = \pi$

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1.

The domain for a single valued branch of $\sqrt{z}$ is the complement of $\{x : x \in \mathbb{R}, x \leq 0\}$, hence for $\sqrt{1+z}$ the domain would be $\{x : x \in \mathbb{R}, x \leq -1\}$ and for $\sqrt{1-z}$ the domain would be $\{x : x \in \mathbb{R}, x \geq 1\}$ Hence, for the function $f(z) = \sqrt{1+z} + \sqrt{1-z}$ the domain would be the intersection of two which is $\{x + iy : x, y \in \mathbb{R}, y \neq 0\} \cup \{x : x \in \mathbb{R}, -1 \leq x \leq 1\}$

3.

if $Re\ f(z) = 0$ then $f(z) = iy$ and $|f(z)^2 - 1| < 1$ implies $|-y^2 - 1| < 1$ which is false for all $y \in \mathbb{R}$ hence $Re\ f(z)$ is never zero in the domain. Since the domain is connected $Re\ f(z)$ is either greater than 0 or less than 0 in the entire domain.

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1.

FTSOC: If there exists a transformation, let that be

$$\overline{z} = \frac{az + b}{cz + d}$$

$\overline{0} = 0$ hence $b = 0$
if $r \in \mathbb{R}, \overline{r} = r$

$$\begin{aligned} r &= \frac{ar}{cr + d} \\ cr^2 &= (a - d)r \\ cr &= a - d \end{aligned}$$

which is a contradiction, hence such a linear transformation does not exist

2.

If a linear transform is represented in the following way

$$Tz = \frac{az + b}{cz + d} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then composition of maps will correspond to multiplication of matrices

$$T_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$T_1^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

Then

$$T_1 T_2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

$$T_1 T_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$T_1^{-1} T_2 = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Hence

$$T_1 T_2 z = \tfrac{3z+2}{4z+3} \quad T_2 T_1 z = \tfrac{z+2}{2z+5} \quad T_1^{-1} T_2 z = z - 2$$

3.

$|az - aw| = |a||z - w| = |z = w|$ if $|a| = 1$ Hence rotation preserves distances.
Since distance is preserved $|z| = |Tz|$. Hence all points of a circle of radius r around 0 will remain on the cirlcle.
Let $a = T1$
Let $b = Tz$ for some $z \in \mathbb{C}$
$|\frac{b}{a}| = |z|$
and $|z - 1| = \left|\frac{b}{a} - \frac{a}{a}\right| = \left|\frac{b}{a} - 1\right|$
Since the distance from 0 and 1 is the same for the points z and $\frac{b}{a}$, $\frac{b}{a} = z$ or $\frac{b}{a} = \overline{z}$.
Hence the transformation followed by a rotation that sends 1's image to 1 sends all points to themselves or thier relfection in the real number line.
Let that transformation be $T'z = \frac{Tz}{T1}$
For non real numbers $a = x + iy, b = z + iw$ and $T'a = \overline{a}$
Since T' is distance preserving

$$|a - b| = |Ta - Tb|$$
$$|(x - z) + i(y - w)| = |(x - z) + i(-y - w')|$$

is true if and only if $y - w = -y - w'$ where $w' = w$ or $-w$ and for $w, y \neq 0$ the statement is true iff $w' = -w$ or $T'b = \overline{b}$
Hence if one of the points goes to its mirror image, all points go to their mirror image. so T' is either identity or reflection about the real number line.
any such T can be obtained by following T' with a rotation that maps 1 to $T1$.
Hence the most general distance preserving transformations are identitiy or relfection about the real numberline followed by a rotation.
If the transfomation is relfection followed by rotation, it maps $r \cdot e^{i\theta}$ to $r \cdot e^{-i\theta + i\alpha}$ which can be rewritten as $r \cdot e^{-(i\theta - i\alpha)}$ which is rotation by $-\alpha$ followed by reflection.
Hence any transformation that preserves distances is of the form, a rotation followed by identity or reflection along the real number line.

4.

Let the transformation be

$$Tz' = \frac{a'z + b'}{c'z + d'}$$

If $a' \neq 0$
divide the numerator and denominator by $|a'|^2$

$$Tz = \frac{z + b}{cz + d}$$

putting $z = 0$ we get b/d is real and taking the sequence of natural numbers Tz converges to $\frac{1}{c}$ Hence c is also real.

Taking $b = x + iy$ and $z = -x$ we get

$$T(-x) = \frac{iy}{(-cx + kx) + kiy}$$

Which is real iff $c = k$ so

$$Tz = \frac{z + b}{cz + cb} = 1/c$$

otherwise,
if $c = rd$ where r is real we get

$$Tz = \frac{b}{d(rz + 1)} = \frac{b/d}{rz + 1}$$

which would make $\frac{b}{d}, r, 1$ real
otherwise,
We can find real numbers a and b such that they would make the real and imaginary parts of the denominator 0 respectively. Hence To make the fraction always real, we b would have to be both real and imaginary so $T(z) = 0$.