Cana Assignment 4

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1. PG 130, Problem 5

If f has an essential singularity, it would come arbitrarily close to every point, then f is would not be bounded, hence f does not have an essential singularity.

Let z_0 be a pole of f, then by the open mapping theorem, an open set around z_0 would be mapped to an open neighbourhood of ∞, which would mean, either the Re or the Im parts are unbounded, Hence f does not have a pole either.

Hence an isolated singularity of $f(z)$ is removable.

2. PG 130, Problem 6

If $e^{f(z)}$ has a pole at z_0 then the real part of $f(z)$ goes to ∞ as z goes to z_0. Hence $f(z)$ would also have a pole at z_0.

Having a pole at z_0 implies, given any $M > 0$, $\exists \epsilon > 0$ Such that $z \in B_\epsilon(z_0) \implies |f(z_0)| > M$.
Pick $z \in B_\epsilon(z_0) \neq z_0$, its image is finite and the sequence $f(z) + 2n\pi i$ converges to ∞., take the inverse images of each of the points of the sequence. which form a sequence $z_n \to z_0$. $e^{f(z_n)}$ is a constant sequence which is finite, hence z_0 is not a pole of $e^{f(z)}$ which is a contradiction.

3. PG 133, Problem 3

Let $f(z) = \cos z$ and let $z_0 = 0$, $f(z) - 1$ has a root at z_0
The derivative of $f'(0) = \sin 0 = 0$,
second derivative is $-\cos 0 = -1$. Hence the root has degree 2.
Hence $\cos z - 1 = z^2 g(z) = \zeta^2(z)$, hence $\zeta(z) = z\sqrt{g(z)}$ and we get

$$\begin{aligned} f(z) - 1 &= \cos z - 1 \\ &= -2\sin^2\left(\frac{z}{2}\right) \\ &= \zeta^2(z) \\ \therefore \zeta(z) &= \sqrt{2} i \sin\left(\frac{z}{2}\right) \end{aligned}$$

4. PG 133, Problem 4

$f(z^n) - f(0)$ has all i order derivatives for $i \in \{1 \dots n-1\}$ and 0 is a root
hence $f(z^n) - f(0) = z^n h(z)$ where $h(0) \neq 0$

Hence we can choose a neighborhood of 0 such that $|h(z)-h(0)| \leq |h(0)|$ for all z in the neighborhood, so we can define a single valued branch of $\sqrt[n]{h(z)}$ in the neighbourhood of 0

$$f(z^n) - f(0) = \left(z\sqrt[n]{h(z)}\right)^n$$

and have $g(z) = z\sqrt[n]{h(z)}$

5. PG 136, Problem 1

The equation 36 in ahlfors states that, given a function f analytic on $|z| < R$, such taht $|f(z)| < M, f(z_0) = w_0$, then we have

$$\left|\frac{M(f(z)-w_0)}{M^2 - \bar{w}_0 f(z)}\right| \leq \left|\frac{R(z-z_0)}{R^2 - \bar{z}_0 z}\right|$$

putting $R = M = 1$ we get

$$\left|\frac{f(z)-w_0}{1-\bar{w}_0 f(z)}\right| \leq \left|\frac{z-z_0}{1-\bar{z}_0 z}\right|$$

as $z \to z_0$ we have $f(z) \to w_0$

$$\left|\frac{f'(z_0)}{1-|f^2(z_0)}\right| \leq \left|\frac{1}{1-|z_0|^2}\right|$$

but z_0 can be any point

$$\left|\frac{f'(z)}{1-|f^2(z)}\right| \leq \left|\frac{1}{1-|z|^2}\right|$$

6. PG 136, Problem 2

Consider the map $g : \mathcal{H} \to \mathcal{D}$, where $\mathcal{D}$ is an open unit disk and $\mathcal{H}$ is the upper half plane, given by $g(z) = \frac{z-z_0}{z-\bar{z}_0}$, where $z_0 \in \mathcal{H}$ is a fixed point.
Consider the map $h(z) = \frac{z-f(z_0)}{z-\overline{f(z_0)}}$

And consider the map $h \circ f \circ g^{-1} : \mathcal{D} \to \bar{\mathcal{D}}$
This satisfies the conditions that the function is analytic on $\mathcal{D}$, and $|h(f(g^{-1}(z)))| \leq 1$ for all $z \in \mathcal{D}$ also $h(f(g^{-1}(0))) = 0$.

This gives $|h(f(g^{-1}(z)))| \leq |z| \implies |h(f(z))| \leq |g(z)| \implies \left|\frac{f(z)-f(z_0)}{f(z)-\overline{f(z_0)}}\right| \leq \frac{z-z_0}{z-\bar{z}_0}$
taking $z \to z_0$ we get

$$\frac{|f'(z_0)|}{Im(f(z_0))} \leq \frac{1}{Im(z_0)}$$

7. PG 136, Problem 3

In problem 5, equality holds iff

$$\left|\frac{f(z)-w_0}{1-\overline{w_0} f(z)}\right| = \left|\frac{z-z_0}{1-\overline{z}_0 z}\right|$$

which holds iff equality holds in (36) with $M = R = 1$ hence $|Sf(T^{-1}\zeta)| = |\zeta|$ where S adn T are linear fractional transformations.

This gives $SfT^{-1}(\zeta) = c\zeta$ for some c. This gives $f = S^{-1}(cT(\zeta))$ which is a compostition of 3 linear fractional transformationsm, and hence is a Linear fractional transformation.

In Problem 6, equality holds iff $|h(f(g^{-1}(z)))| = |z|$ which gives $h(f(g^{-1}(z))) = cz$ for some constant c. That gives f as $f(z) = h^{-1}(cg(z))$ which is also a linear fractional transformation.

8. PG 148, Problem 2

Let S be a simply connected space.
Let $P = S \setminus M$ where M is a set with m points
The $P^c = S^C \cup M$ points which are $m+1$ connected components, hence p has connectivity $m+1$

The homology basis consists of m loops each centered around the m points. So for each of the m points, we can find a circle γ_i around the points $a_i \in M$ and we will have $n(\gamma_i, a_j) = \delta_{i,j}$ which forms a basis

9. PG 148, Problem 5

Any closed curve γ in the domain Ω that winds around 1 also winds around -1. consider some point $z_0 \in \gamma$ and choose some value of $\theta_1 = arg(1-z)$ such that $(1-z_0) = r_1 e^{i\theta_1}$. and take a branch of $\sqrt{1-z_0}$.
Now as we travel around γ, as we come back to z_0 we add 2π to the argument of $(1-z)$. The same works for the argument of $(1+z)$ and so the argument of the product changes by 4π. Now when we take the square root, we divide the argument by 2 which means that as we move around γ and come back to z_0, the argument of $\sqrt{1-z^2}$ changes by 2π.

Now we compute

$$\int_\gamma \frac{1}{\sqrt{1-z^2}} dz.$$

And by Cauchy's Theorem, we can assume γ is a very large cirle and so acts as a small disc about infinity. we apply the change of variables $z = \frac{1}{w}$ and we get the integral

$$-\int_{|w|=\delta} \frac{dw}{w\sqrt{w^2-1}}$$

$\sqrt{w^2-1}$ is analytic in a neighbourhood of 0 so we can compute the integral by cauchy's formula to be 2π