Measure Theory and Integration

MATH 320, Yale University, Fall 2022

These are lecture notes for MATH 320, "Measure Theory and Integration" taught by Charlie Smart at Yale University during the fall of 2022. These notes are not official, and have not been proofread by the instructor for the course. These notes live in my lecture notes respository at

https://github.com/Eph97/Eph97/Math320.

If you find any errors, please open a bug report describing the error, and label it with the course identifier, or open a pull request so I can correct it.

Contents

Syllabus

Instructor	Prof. Charlie Smart, `charlie.smart@yale.edu`
Lecture	MW 11:35 AM–12:50 PM, Hum 207
Recitation	TBA
Textbook	Walter Rudin. *Real & Complex Analyis*. 3rd. McGraw-Hill, 1987
Midterms	Mon Oct 3, 2022 (In-Class)
	Mon Nov 7, 2022 (In-Class)
Final	TBD

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Following the universal rule for first class, we covered the syllabus and logistics. We then dove into review on the Riemann integral to motivate measure theory. Prof. Smart seemed much more comfortable talking about math than logistics.

1.1 *Course Plan*

- Review Riemann Int
- Abstract Measure Theory (Majority of class)
- Applications in Probability
- Apps in Dynamics
- Some brief apps in geometry

Problem 1.1. The Riemann Integral

Works for most functions of interest but notably

1. Is not closed under important limits
2. It is hard to eneralize to new geometries.

Definition (Riemann Integral)**.** We can define the riemann integral as $\int_{\mathbf{Q}} f$ of a bounded $f : \mathbf{Q} \to \mathbf{R}$ defined in closed rectangles $\mathbf{Q} = [a_1, b_1] \times [a_2, b_2] \ldots [a_d, b_d] \subseteq \mathbf{R}$. We use Darboux sums

Riemann Integral

Definition (partition)**.** A partition of $\mathbf{Q}$ is a finite set of *closed* rectanlges whose |underlineinteriors are disjoint and whose union is $\mathbf{Q}$.

partition

The upper and lower Darboux sums are $U(p, f) = \sum_{R \in P} \sup f \cdot |R|$ and $L(p, f) = \inf f \cdot |R|$

Lemma 1.1. *For any 2 partitions p_1 and p_2 of $\mathbf{Q}$, $L(p_1, f) \leq U(p_2, f)$*

Solution. We have

$$
\begin{aligned}
L(p_1, f) &= \sum_{R} inf f \cdot |R| = \sum_{R_1 \in P_1} inf f \sum_{R_2 \in P_2} |R_1 \cap R_2| \\
&= \sum_{\substack{R_1 \in P_1 \\ R_2 \in P_2 \\ |R_1 \cap R_2| > 0}} \inf_{R_1} f \cdot |R_1 \cap R_2| \leq \sum_{\substack{R_1 \in P_1 \\ R_2 \in P_2 \\ |R_1 \cap R_2| > 0}} \inf_{R_1 \cap R_2} f \cdot |R_1 \cap R_2| \\
&\leq \sum_{\substack{R_1 \in P_1 \\ R_2 \in P_2 \\ |R_1 \cap R_2| > 0}} \sup_{R_1 \cap R_2} f \cdot |R_1 \cap R_2| \leq \sum_{\substack{R_1 \in P_1 \\ R_2 \in P_2 \\ |R_1 \cap R_2| > 0}} \sup_{R_2} f \cdot |R_1 \cap R_2| \\
&= \sum_{R_2 \in P_2} \sup_{R_2} f \sum_{\substack{R_1 \in P_1 \\ |R_1 \cap R_2| > 0}} |R_1 \cap R_2| = \sum_{R_2 \in P_2} \sup_{R_2} f \cdot |R_2| = U(p_2, f)
\end{aligned}
$$

■

Note we say a bounded $f : \mathbf{Q} \to \mathbf{R}$ is Riemann integrable when $\sup_p L(p, f) = \inf_p U(p, f)$ in which case we let $\int_{\mathbf{Q}} f$ denote the common value.

Theorem 1.2. *If $f : \mathbf{Q} \to \mathbf{R}$ is continuous, then f is Rieman integrable.*

Solution. $\mathbf{Q}$ is compact, so f is uniformly cont. Thus, given a $\varepsilon > 0$, choose a $\delta > 0$ so $|x - y| < \delta \implies |f(x) - f(y)| > \varepsilon$. Let p be any partition of $\mathbf{Q}$ whose rectangles have diameter $< \delta$
Next Computer

$$
\begin{aligned}
0 &\leq \inf_{p_1} U(p_1, f) - \sup_{p_2} L(p_2, f) \leq U(p_1, f) - L(p_1, f) \\
&= \sum_{R \in P} \left(\sup_R f - \inf_R f \right) \cdot |R| \\
&\leq \sum_{R \in P} \varepsilon \cdot |Q|
\end{aligned}
$$

but since $\varepsilon > 0$ is arbitary, f is riemann integrable. ■

Example 1.1. If $f : [0, 1] \to \mathbf{R}$ and $f(x) = \begin{cases} 1 & x \notin \mathbf{Q} \\ 0 & x \in \mathbf{Q} \end{cases}$ Then f is not Riemann integrable.

Solution. for any P, we can see $U(p, f) = 1$ and $L(p, f) = 0$ ■

1.2 *Measures*

We have two notions of volume

1. The outer jordan measure of a set $x \subseteq \mathbf{R}^d$ is

$$J^*(x) = \inf\{\sum_{k=1}^{n} |Q_k| : n \geq 1, Q_1, \ldots, Q_n \subseteq R^n \text{ closed rectangles and } x \subseteq \cup_{k=1}^{b} Q_k\}$$

2. Outer Lebesgue measure of a set $X \subseteq \mathbf{R}^n$ is

$$L^*(x) = \inf\{\sum_{k=1}^{\infty} |Q_k| : Q_1, Q_2, \ldots \subseteq R^n \text{ closed rectangles and } x \subseteq \cup_{k=1}^{\infty} Q_k\}$$

We have some important properties for these measures, namely

(a) $L^*(x) \leq J^*(x)$

(b) If X is compact, then $L^*(x) = J^*(x)$

Solution. Suppose $x \subseteq \cup_{k=1}^{\infty} Q_k$, pick $\varepsilon > 0$ and let $(1+\varepsilon)Q_k$ be he $(1+\varepsilon)$-dialation of Q_k. Note $(1+\varepsilon)Q_k$ are open cover of x so $x \subseteq \sum_{k=1}^{n}(1+\varepsilon)Q_k$ and

$$\sum_{k=1}^{n} |(1+\varepsilon)Q_k| = (1+\varepsilon)^d \sum_{k=1}^{n} |Q_k| \leq (1+\varepsilon)^d \sum_{k=1}^{\infty} |Q_k|$$

■

3. $L^*(\cup^{\infty} x_k) \leq \sum^{\infty} L^*(X_k)$

Solution. sketch by "Dovetail" (Come back to) ■

Example 1.2. The set $x = \mathbf{Q} \cap [0,1]$ has $L^*(x) = 0$ and $J^*(x) = 1$ write $x = \{q_1, q_2, \ldots\}$ then let $Q_k = [q_k = \frac{\varepsilon}{2^k}, q_k + \frac{\varepsilon}{2^k}]$ and observe that $x \subseteq \cup_1^{\infty} Q_k$ and $\sum_{k=1}^{\infty} |Q_k| = 2\varepsilon$

Example 1.3. Let $X \subseteq [0,1]$ be generalized cantor set. Let $1_{\mathbf{C}}$ be the indicator fct of $\mathbf{C}$.

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Observe $1_{\mathbf{C}}$ is not Riemann int on $[0,1]$. Indeed, C is the set of discontinuities of $1_{\mathbf{C}}$ and $L^*(\mathbf{C}) = \frac{1}{2}$. Moreover $1_{\mathbf{C}}(x) = \lim_{n\to\infty} 1_{\mathbf{C}_n}(x)$ and $0 \le 1_{\mathbf{C}} \le 1_{C_{n+1}} \le 1_{\mathbf{C}_n} \le 1$ and $\lim_{n\to\infty} \int_{[0,1]} 1_{C_n} = \frac{1}{2}$