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optimal Representation
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Contents

Setup

1. Suppose a physician can only see see ATE and some measure of representativeness. They have prior $\bar{\beta}$ and $\beta_{ATE} = (1/N)\sum \beta_i$.

2. need model for betas related to each other based on x's. WLOG, suppose

$$\beta(x_i) = x_i\gamma$$

   Where x_i is a vector of characteristics and γ is a vector of coefficients.
   If you know γ, then you know β for any given patient.

3. However, you don't observe γ, you instead observe: $\beta_{ATE} = \bar{x}\gamma$ where $\bar{x} = (\frac{1}{N})\sum x_i$

4. We know β_i for patients with characteristics $\bar{x}$ (it is β_{ATE}).

5. For other patients, need to solve

$$\beta_{i,post} = \mathrm{E}(x_i\gamma|\bar{x}\gamma = \beta_{ATE})$$

6. to solve

   (a)

$$\beta_{i,post} = \mathrm{E}(x_i\gamma|\bar{x}\gamma = \beta_{ATE})$$

$$= \mathrm{E}((x_i - c_i\bar{x})\gamma|\bar{x}\gamma = \beta_{ATE}) + c_i\mathrm{E}(\bar{x}\gamma|\bar{x}\gamma = \beta_{ATE})$$

$$= \mathrm{E}((x_i - c_i\bar{x})\gamma|\bar{x}\gamma = \beta_{ATE}) + c_i\beta_{ATE}$$

   For any constant c_i.
   Choose c_i so that

$$\mathrm{Cov}((x_i - c_i\bar{x})\gamma, \bar{x}\gamma) = 0$$

   maybe assume normality so that this guarantees independence. Then,

$$\mathrm{E}((x_i - c_i\bar{x})\gamma|\bar{x}\gamma = \beta_{ATE}) = (x_i - c_i\bar{x})\mathrm{E}(\gamma)$$

So then

$$(x_i - c_i\bar{x})\mathrm{E}(\gamma) + c_i\beta_{ATE} = x_i\mathrm{E}(\gamma) + c_i(\beta_{ATE} - \bar{x}\mathrm{E}(\gamma))$$

(c_i depends on x_i)
In other words, your belief is your prior, adjusted based on the difference between the observed ATE and your prior about the ATE. The key question is how much adjustment you do which depends on "c_i". We choose c_i to solve:

$$\mathrm{Cov}((x_i - c_i\bar{x})\gamma, \bar{x}\gamma) = 0$$

$$\iff \mathrm{Cov}(x_i\gamma, \bar{x}\gamma) - c_i\mathrm{Cov}(\bar{x}\gamma, \bar{x}\gamma) = 0$$

$$\iff \mathrm{Cov}(x_i\gamma, \bar{x}\gamma) = c_i\mathrm{Var}(\bar{x}\gamma)$$

$$\iff c_i = \frac{\mathrm{Cov}(\beta_i, \beta_{ATE})}{\mathrm{Var}(\beta_{ATE})}$$

The random variable in this context is γ (the coefficients on the x's) in this case $\mathrm{Var}(\beta_{ATE})$ is a measure of how uncertain one was about what β_{ATE} would be before doing the trial.
c_i is the equation for a regression of β_i on β_{ATE}. In other words, we take a bunch of patients with characteristics x_i and we keep redrawing the gammas from our prior distribution the we ask how correlated β_i and β_{ATE} are. If they are more correlated (as they would be for patients where the x_i are closer to $\bar{x}$ we update more.
To compute c_i, we just need to know x_i $\bar{x}$, and the distribution of γ.
Suppose we want to design the trial to minimize:

$$\min \mathrm{E}[(\beta_i - \beta_{i,post})^2]$$

Simple Cases

1. There is just one x and it is binary (old v young). Can it be solved analytically?

2. Can you solve a 2-dimensional case?

First observe that in our current setup, c does not depend on the γ's
We have that

$$c = \frac{\mathrm{Cov}(x\gamma, \bar{x}\gamma)}{\mathrm{Var}(\bar{x}\gamma)}$$

so for individual i, c reduces to

$$c = \frac{(1^2 + x\bar{x})\mathrm{Var}(\gamma)}{(1^2 + \bar{x}^2)\mathrm{Var}(\gamma)} = \frac{1 + x\bar{x}}{1 + \bar{x}^2} = \frac{x'\bar{x}}{\bar{x}'\bar{x}}$$

1 Brute Force Solution

recall we want to minimize

$$\min \mathrm{E}_x \left(\mathrm{E}_{\gamma_{0,1}}[(\beta_i - \beta_{i,post})^2]\right)$$

let

$$J = \beta_i - \beta_{i,post} = \underbrace{[(1-x)\gamma_0 + x\gamma_1]}_{\beta_i} - \underbrace{[((x-c\bar{x})\mathrm{E}(\gamma) + c\beta_{ATE})]}_{\beta_{i,post}}$$

$$J = ([(1-x)\gamma_0 + x\gamma_1 - ((x-c\bar{x})\mathrm{E}(\gamma) + c\beta_{ATE})]^2)$$

$$= ([(1-x)\gamma_0 + x\gamma_1 - (\mathrm{E}(\gamma_0)(1-x-c(1-\bar{x})) - \mathrm{E}(\gamma_1)(x-c\bar{x}) - c((1-\bar{x})\gamma_0 + \bar{x}\gamma_1)]^2)$$

$$= ([\gamma_0 - x\gamma_0 + x\gamma_1 - \mathrm{E}(\gamma_0) + \mathrm{E}(\gamma_0)x + \mathrm{E}(\gamma_0)c - \mathrm{E}(\gamma_0)c\bar{x} - \mathrm{E}(\gamma_1)x + \mathrm{E}(\gamma_1)c\bar{x} - c\gamma_0 + c\bar{x}\gamma_0 - c\bar{x}\gamma_1)]^2)$$

$$= ([[\gamma_0 - \mathrm{E}(\gamma_0)] - [c\gamma_0 - \mathrm{E}(\gamma_0)c] - x[\gamma_0 - \mathrm{E}(\gamma_0)] + x[\gamma_1 - \mathrm{E}(\gamma_1)] + c\bar{x}[\gamma_0 - \mathrm{E}(\gamma_0)] - c\bar{x}[\gamma_1 - \mathrm{E}(\gamma_1)]])]^2)$$

Now recall

1. $c = \frac{(1-x)(1-\bar{x})+x\bar{x}}{(1-\bar{x})^2+\bar{x}^2} = \frac{1-\bar{x}-x+2x\bar{x}}{1-2\bar{x}+2\bar{x}^2}$

and let

1. $a = (\gamma_0 - \mathrm{E}(\gamma_0))$
2. $b = (\gamma_1 - \mathrm{E}(\gamma_1))$

then

$$= [a - ca - xa + xb + c\bar{x}a - c\bar{x}b]^2)$$

$$= a^2c^2\bar{x}^2 - 2a^2c^2\bar{x} + a^2c^2 - 2a^2cx\bar{x} + 2a^2cx + 2a^2c\bar{x} -$$

$$2a^2c + a^2x^2 - 2a^2x + a^2 - 2abc^2\bar{x}^2 + 2abc^2\bar{x} + 4abcx\bar{x} -$$

$$2abcx - 2abc\bar{x} - 2abx^2 + 2abx + b^2c^2\bar{x}^2 - 2b^2cx\bar{x} + b^2x^2$$

But we know

1. $\mathrm{E}(a^2) = \mathrm{Var}(\gamma_0) = 1$
2. $\mathrm{E}(b^2) = \mathrm{Var}(\gamma_1) = 1$

3. $\mathrm{E}(ab) = \mathrm{Cov}(\gamma_0, \gamma_1) = 0$

So we can simply the above expression to the below expression using the linearity of $\mathrm{E}_{\gamma_{0,1}}$

$$J = c^2\bar{x}^2 - 2c^2\bar{x} + c^2 - 2cx\bar{x} + 2cx + 2c\bar{x} -$$

$$2c + x^2 - 2x + 1 + c^2\bar{x}^2 - 2cx\bar{x} + x^2$$

Now cleaning up and plugging in the expression for c

$$= c^2\bar{x}^2 - 2c^2\bar{x} + c^2 + c^2\bar{x}^2 - 2cx\bar{x} + 2cx + 2c\bar{x} - 2c - 2cx\bar{x} + x^2 - 2x + x^2 + 1$$

$$= (\frac{-\bar{x} + 2\bar{x}^2 + x - 2x\bar{x}}{1 - 2\bar{x} + 2\bar{x}^2})^2(\bar{x}^2 - 2\bar{x} + 1 + \bar{x}^2) -$$

$$(\frac{-\bar{x} + 2\bar{x}^2 + x - 2x\bar{x}}{1 - 2\bar{x} + 2\bar{x}^2})(2x\bar{x} + 2x + 2\bar{x} - 2 - 2x\bar{x}) +$$

$$(x^2 - 2x + x^2 + 1)$$

$$= \frac{(-\bar{x} + 2\bar{x}^2 + x - 2x\bar{x})^2}{1 - 2\bar{x} + 2\bar{x}^2} - (\frac{-\bar{x} + 2\bar{x}^2 + x - 2x\bar{x}}{1 - 2\bar{x} + 2\bar{x}^2})(2x\bar{x} + 2x + 2\bar{x} - 2 - 2x\bar{x}) + (x^2 - 2x + x^2 + 1)$$

$$= \frac{((4x^2\bar{x}^2 - 4x^2\bar{x} + x^2 - 4x\bar{x}^2 + 6x\bar{x} - 2x + \bar{x}^2 - 2\bar{x} + 1)}{(1 - 2\bar{x} + 2\bar{x}^2)} +$$

$$\frac{(-8x^2\bar{x}^2 + 8x^2\bar{x} - 2x^2 + 8x\bar{x}^2 - 12x\bar{x} + 4x - 2\bar{x}^2 + 4\bar{x} - 2)}{(1 - 2\bar{x} + 2\bar{x}^2)} +$$

$$\frac{(4x^2\bar{x}^2 - 4x^2\bar{x} + 2x^2 - 4x\bar{x}^2 + 4x\bar{x} - 2x + 2\bar{x}^2 - 2\bar{x} + 1))}{(1 - 2\bar{x} + 2\bar{x}^2)}$$

$$= \frac{(x^2 - 2x\bar{x} + \bar{x}^2)}{(1 - 2\bar{x} + 2\bar{x}^2)}$$

Now we can use linearity of expectation with $\mathrm{E}_x(J)$ and get

$$= \frac{(p - 2p\bar{x} + \bar{x}^2)}{(1 - 2\bar{x} + 2\bar{x}^2)}$$

Solving the FOC we get

$$\frac{\partial J}{\partial \bar{x}} = \frac{2(\bar{x}^2 - \bar{x})(2p - 1)}{(1 - 2\bar{x} + 2\bar{x}^2)^2}$$

Which gives us roots $\bar{x} = 0$, $\bar{x} = 1$ and when $p = \frac{1}{2}$ any $\bar{x} \in [0, 1]$ is a root.

Now to check which is the minimizing one, we can use our second order condition to check when the second derivative is positive.

First,

$$\frac{\partial^2 J}{\partial \bar{x}^2} = \frac{-(2(2p-1)(4\bar{x}^3 - 6\bar{x}^2 + 1))}{(2(\bar{x}-1)\bar{x}+1)^3}$$

Examing the numerator we see that

$$\frac{-(2(2p-1)(4\bar{x}^3 - 6\bar{x}^2 + 1))}{(2(\bar{x}-1)\bar{x}+1)^3}$$

$$= -(2(2p-1)(2\bar{x}-1))\frac{(2\bar{x}^2 - 2\bar{x} - 1))}{(2(\bar{x}-1)\bar{x}+1)^3} =$$

Examing this we can see there is a root at $\bar{x} = \frac{1}{2}$ and graphing (or by inspection) we can see that for $p > \frac{1}{2}$ the second derivative is positive for $\bar{x} > \frac{1}{2}$ and if $p < \frac{1}{2}$ then the second derivative is positive for $\bar{x} < \frac{1}{2}$.

Combined with our critical points from the first derivative, this gives us our answer that for $p > \frac{1}{2}$ our error is minimized at the critical point $\bar{x} = 1$ and if $p < \frac{1}{2}$ our error is minimized at the critical point $\bar{x} = 0$. and if $p = \frac{1}{2}$ then our error has no minimum and we achieve the same error of 0.5 for all $\bar{x} \in [0, 1]$

2 Simpler Men V Women Solution

One way we can rewrite these equations is as the effect of women vs men.

$$\min \mathrm{E}_x \left(\mathrm{E}_{\gamma_{0,1}}[(\beta_i - \beta_{i,post})^2]\right)$$

let

$$\sqrt{J} = \beta_i - \beta_{i,post} = \underbrace{[(1-x)\gamma_0 + x\gamma_1]}_{\beta_i} - \underbrace{[((x - c\bar{x})\mathrm{E}(\gamma) + c\beta_{ATE})]}_{\beta_{i,post}}$$

$$= [(1-x)\gamma_0 + x\gamma_1] - [((1-x) - c(1-\bar{x}))\mathrm{E}(\gamma_0) + (x - c\bar{x})\mathrm{E}(\gamma_1) + c((1-\bar{x})\gamma_0 + \bar{x}\gamma_1)]$$

β_i $\beta_{i,post}$ and as a result, J are all a function of x. We can then describe
$\beta_i^{men} = \beta_i(x = 1)$ and similarly for other terms to get

So

$$\beta_i^{men} = \gamma_1$$

$$\beta_i^{post,men} = -c_{men}(1-\bar{x})\bar{\gamma_0} + (1 - c_{men}\bar{x})\bar{\gamma_1} + c_{men}[(1-\bar{x})\gamma_0 + \bar{x}\gamma_1]$$

$$\beta_i^{women} = \gamma_0$$

$$\beta_i^{post,women} = [1 - c_{wom}(1-\bar{x})]\mathrm{E}(\gamma_0) - c_{wom}\bar{x}\mathrm{E}(\gamma_1) + c_{wom}(1-\bar{x})\gamma_0 + c_{wom}\bar{x}\gamma_1$$

this can be broken down into

$$J = (\beta_i^{men} - \beta_i^{post,men}) + (\beta_i^{wom} - \beta_i^{post,wom})$$

let

$$W^{men} = (\beta_i^{men} - \beta_i^{post,men})$$

$$= (1 - \bar{x}c_{men})(\gamma_1 - \bar{\gamma_1}) - c_{men}(1-\bar{x})(\gamma_0 - \bar{\gamma_0})$$

$$W^{wom} = (\beta_i^{wom} - \beta_i^{post,wom})$$

$$= [1 - c_{wom}(1-\bar{x})](\gamma_0 - \bar{\gamma_0}) - c_{wom}\bar{x}(\gamma_1 - \bar{\gamma_1})$$

and recall that

$$c_i = \frac{(1-x)(1-\bar{x}) + x * \bar{x}}{(1-\bar{x})^2 + \bar{x}^2}$$

so if you are a man, then

$$c_{men} = \frac{\bar{x}}{(1-\bar{x})^2 + \bar{x}^2}$$

and likewise if you are a woman, then

$$c_{woman} = \frac{(1-\bar{x})}{(1-\bar{x})^2 + \bar{x}^2}$$

and

$$W_{men}^2 = \left(\frac{(1-\bar{x})^4}{((1-\bar{x})^2+\bar{x}^2)^2}\right)\mathrm{Var}(\gamma_1) + \frac{\bar{x}^2(1-\bar{x})^2}{((1-\bar{x})^2+\bar{x}^2)^2}\mathrm{Var}(\gamma_0)$$

$$W_{women}^2 = \left(\frac{(\bar{x})^4}{((1-\bar{x})^2+\bar{x}^2)^2}\right)\mathrm{Var}(\gamma_0) + \frac{\bar{x}^2(1-\bar{x})^2}{((1-\bar{x})^2+\bar{x}^2)^2}\mathrm{Var}(\gamma_1)$$

so then

$$\mathrm{E}_x J = pW_{men}^2 + (1-p)W_{wom}^2$$

$$= p\left(\frac{(1-\bar{x})^4\mathrm{Var}(\gamma_1) + (\bar{x}^2(1-\bar{x})^2)\mathrm{Var}(\gamma_0)}{((1-\bar{x})^2+\bar{x}^2)^2}\right) + (1-p)\left(\frac{(\bar{x})^4\mathrm{Var}(\gamma_0) + (\bar{x}^2(1-\bar{x})^2)\mathrm{Var}(\gamma_1)}{((1-\bar{x})^2+\bar{x}^2)^2}\right)$$

so using the fact that $\mathrm{Var}(\gamma_0) = \mathrm{Var}(\gamma_1) = 1$ we get

$$\mathrm{E}_x J = p\left(\frac{(1-\bar{x})^4 + (\bar{x}^2(1-\bar{x})^2)}{((1-\bar{x})^2+\bar{x}^2)^2}\right) + (1-p)\left(\frac{(\bar{x})^4 + (\bar{x}^2(1-\bar{x})^2)}{((1-\bar{x})^2+\bar{x}^2)^2}\right)$$

$$\mathrm{E}_x J = p\left(\frac{(1-\bar{x})^2}{(1-\bar{x})^2+\bar{x}^2}\right) + (1-p)\left(\frac{(\bar{x})^2}{(1-\bar{x})^2+\bar{x}^2}\right)$$

$$= p\frac{(1-\bar{x})^2}{(1-\bar{x})^2+\bar{x}^2} + (1-p)\frac{\bar{x}^2}{(1-\bar{x})^2+\bar{x}^2}$$

In other words for individual i we have that

$$\mathrm{E}(J) = p(1 - c_{men}\bar{x}) + (1-p)(1 - c_{women}(1-\bar{x}))$$

$$\mathrm{E}(J) = p\left(1 - \frac{\bar{x}^2}{(1-\bar{x})^2+\bar{x}^2}\right) + (1-p)\left(1 - \frac{(1-\bar{x})^2}{(1-\bar{x})^2+\bar{x}^2}\right)$$

$$\mathrm{E}(J) = p\left(\frac{(1-\bar{x})^2}{(1-\bar{x})^2+\bar{x}^2}\right) + (1-p)\left(\frac{\bar{x}^2}{(1-\bar{x})^2+\bar{x}^2}\right)$$

$$\mathrm{E}(J) = p\left(c_{wom}\right) + (1-p)\left(c_{men}\right)$$

Interestingly this implies a FOC of

$$\frac{\partial \mathrm{E}(J)}{\partial \bar{x}} = p\left(\frac{2\bar{x}(1-\bar{x})}{((1-\bar{x})^2+\bar{x}^2)^2}\right) - (1-p)\left(\frac{2\bar{x}(1-\bar{x})}{((1-\bar{x})^2+\bar{x}^2)^2}\right) = 0$$

$$\frac{\partial \mathrm{E}(J)}{\partial \bar{x}} = (2p-1)\left(\frac{2\bar{x}(1-\bar{x})}{((1-\bar{x})^2+\bar{x}^2)^2}\right) = 0$$

meaning we've retrieved our original FOC.

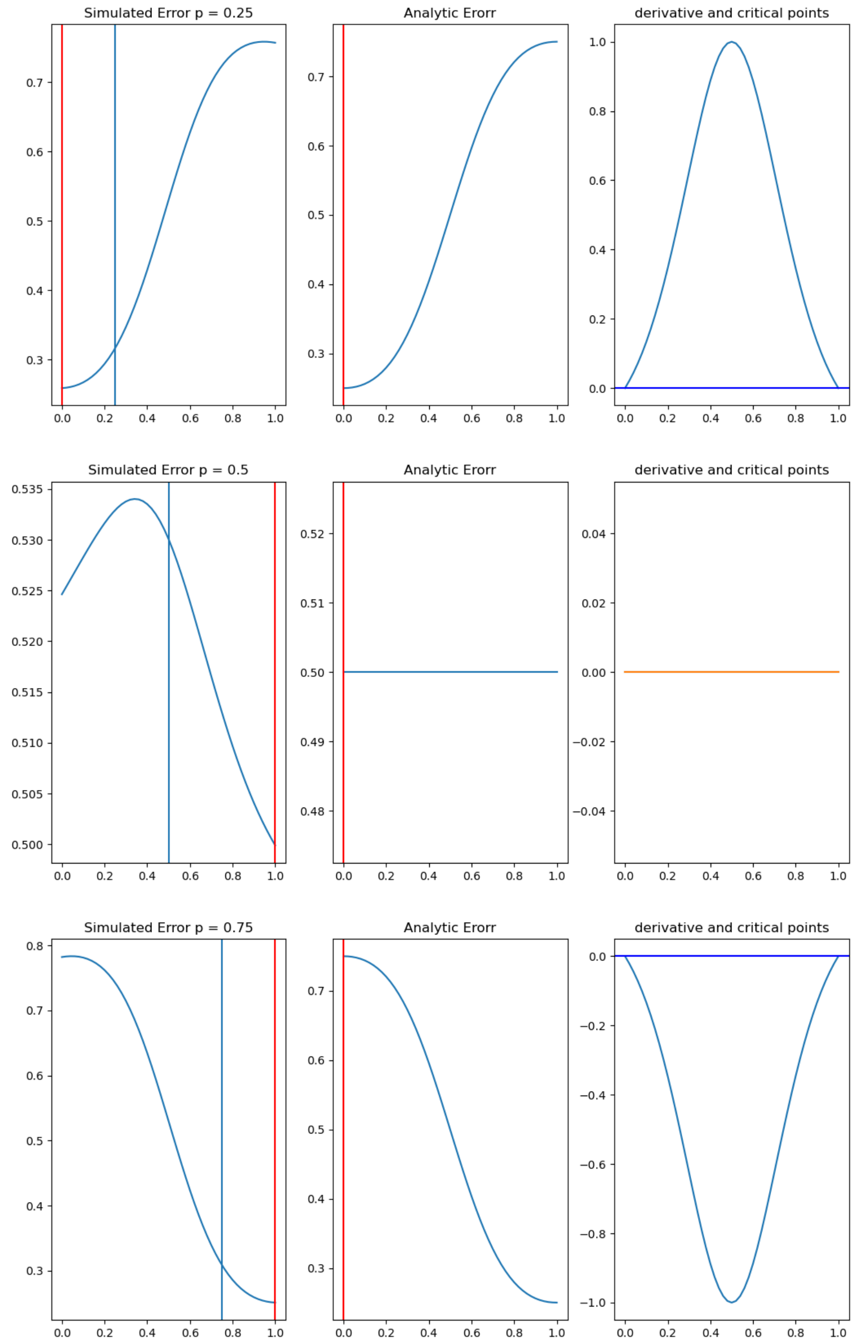

Figure 1: First column contains the simulated error for different population proportions with the population proportion shown by the blue vertical line and point of minimum error shown with the red vertical line. The second column has the analytic error computed in the math above as well as a vertical line showing the first critical point. The third column shows the first derivative with respect to $\bar{x}$.