OPTIMAL ASSIGNEMENTS

1. Model

There is a finite set of objects A and the capacity of each object $a \in A$ is denoted by $c_a > 0$. We assume that there is one object $\phi \in A$ with an infinite capacity that represents the outside option. There is a finite set of Θ of agents types and a commonly known distribution $f \in \Delta(\Theta)$ over agents types. The utility of an agent of type θ for object a is denoted by $u_{\theta a}$ and is privately known. We denote the utility vector for agent of type $\theta \in \Theta$ by $u_\theta \in R^{|A|}$.

An *assignment* provides each type a probability distribution over objects. Given an allocation $m \in \Delta(A)^{|\Theta|}$, we denote by $m_{\theta a}$ the probability that an agent of type θ is assigned object a.

2. Optimal mechanism

This is the programm describing the optimal mechanism.

$$
\begin{aligned}
& \max_{m_{\theta a}} \sum_{\theta \in \Theta} f_\theta \sum_{a \in A} u_{\theta a} m_{\theta a} \\
\text{s.t.} \quad & \sum_{a \in A} u_{\theta a} m_{\theta a} - \sum_{a \in A} u_{\theta a} m_{\theta' a} \geq 0 \qquad \forall \theta, \theta' \in \Theta \quad \text{IC} \\
& \sum_{\theta \in \Theta} f_\theta m_{\theta a} \leq c_a \qquad \forall a \in A \quad \text{Capacity} \\
& \sum_{a \in A} m_{\theta a} \leq 1 \quad \forall \theta \in \Theta \quad \text{Feasibility} \\
& m_{\theta a} \geq 0 \qquad \forall \theta \in \Theta, a \in A
\end{aligned}
$$

The optimal ordinal mechanism is the following:

$$
\begin{aligned}
&\max_{m_{\theta a}} \sum_{\theta\in\Theta} f_\theta \sum_{a\in A} u_{\theta a} m_{\theta a} \\
\text{s.t.}\quad & \sum_{a\in A} u_{\theta a} m_{\theta a} - \sum_{a\in A} u_{\theta a} m_{\theta' a} \geq 0 \qquad \forall \theta,\theta' \in \Theta \quad \text{IC} \\
& \sum_{\theta\in\Theta} f_\theta m_{\theta a} \leq c_a \qquad \forall a \in A \quad \text{Capacity} \\
& \sum_{a\in A} m_{\theta a} \leq 1 \quad \forall \theta\in\Theta \quad \text{Feasibility} \\
m_\theta = m_{\theta'} \quad \text{if} \quad u_\theta \quad & is equivalent to \quad u_{\theta'} \quad \text{ordinal constraint} \\
& m_{\theta a} \geq 0 \qquad \forall \theta\in\Theta, a\in A
\end{aligned}
$$

3. Optimal Efficient mechanism

We in addition to the other constraints need to impose efficiency constraints given in the main file. Unfortunately, this seems make it into a non-linear program. Itai conjectures that one can use some integer program.

4. Price mechanism

This is an lower bound on the optimal price mechanism where we allow the agent to deviate and purchase more than one unit of a single object

$$
\begin{aligned}
&\max_{m_{\theta a}, p_a} \sum_{\theta\in\Theta} f_\theta \sum_{a\in A} u_{\theta a} m_{\theta a} \\
\text{s.t.}\quad & \frac{u_{\theta a'}}{p_{a'}} - \sum_{a\in A} u_{\theta a} m_{\theta a} \leq 0 \qquad \forall \theta, a' \in \Theta \quad \text{IC} \\
& \sum_{\theta\in\Theta} f_\theta m_{\theta a} \leq c_a \quad \forall a\in A \quad \text{Capacity} \\
& \sum_{a\in A} m_{\theta a} \leq 1 \qquad \forall \theta\in\Theta \\
& m_{\theta a} \geq 0 \qquad \forall \theta\in\Theta, a\in A \\
& p_a \geq 0 \qquad \forall a \in A
\end{aligned}
$$

We can have an upper bound on the welfare in the optimal price mechanism by considering the program

$$
\begin{aligned}
&\max_{m_{\theta a}, p_a} \sum_{\theta \in \Theta} f_\theta \sum_{a \in A} u_{\theta a} m_{\theta a} \\
\text{s.t.} \quad & \min\left\{1, \frac{u_{\theta a'}}{p_{a'}}\right\} - \sum_{a \in A} u_{\theta a} m_{\theta a} \leq 0 && \forall \theta, a' \in \Theta \quad \text{IC} \\
& \sum_{\theta \in \Theta} f_\theta m_{\theta a} \leq c_a && \forall a \in A \quad \text{Capacity} \\
& \sum_{a \in A} m_{\theta a} \leq 1 && \forall \theta \in \Theta \\
& m_{\theta a} \geq 0 && \forall \theta \in \Theta, a \in A \\
& p_a \geq 0 && \forall a \in A
\end{aligned}
$$

Here we just restricted the agent to investing all his money into a single object. We imposed the same restriction in the above program, but as there was no capacity constraint this is never binding. We note here that this is not a quadratic program and it is unclear how to solve it.

5. Raffle mechanism

Let $1/p_a$ be the probability of getting object a when entering the raffle for object a. Note, that the probability of getting object a can be smaller one, ie $p_a > 1$ only if the object is completely depleted $\sum_\theta f_\theta m_{\theta a} = 1$. This leads to the following quadratic program

$$
\begin{aligned}
&\max_{m_{\theta a}, p_a} \sum_{\theta \in \Theta} f_\theta \sum_{a \in A} u_{\theta a} m_{\theta a} \\
\text{s.t.} \quad & u_{\theta a'} - p_{a'} \sum_{a \in A} u_{\theta a} m_{\theta a} \leq 0 \qquad \forall \theta, a' \in \Theta \quad \text{IC} \\
& \sum_{\theta \in \Theta} f_\theta m_{\theta a} \leq c_a \quad \forall a \in A \quad \text{Capacity} \\
& \sum_{a \in A} m_{\theta a} \leq 1 \qquad \forall \theta \in \Theta \\
& m_{\theta a} \geq 0 \qquad \forall \theta \in \Theta, a \in A \\
& p_a \geq 1 \qquad \forall a \in A \\
(p_a - 1) & \left(\sum_\theta f_\theta m_{\theta a} - c_a \right) = 0
\end{aligned}
$$

the last constraint is equivalent to $(p_a - 1) \sum_\theta f_\theta m_{\theta a} - p_a = 1$.

Proposition: Comparison of the above programs means that the Raffle mechanism is always dominated by the optimal price mechanism as it contains the constraints of the lower bound on price mechanisms.

6. Logistic Type Space

Each object has a utility v_a and $u_{a\theta} = v_a + F^{-1}(\theta_a)$ where θ_a is a vector determining the quantiles of the shocks for each object. For example $\theta = (1/4, 1, 1/2, 1/4)$ describes an agent whose utilty for the first object is at the 25% quantile of the taste shock distribution F.

7. Program

Read utilities from a file

For each of the programs create a separate julia file

Each program writes out the optimal allocation and value achieved in some standarized format