Homework Set 1, Math 325, due Monday, January 30, 2023 in class. Hand in four out of these eight problems, with each problem being worth 25 points. References below are to Luenberger's text, *Optimization by Vector Space Methods*.

1. Let $u(t, x)$ be a $C^2(\mathbb{R}^2)$ solution of $u_{tt} - u_{xx} = 0$ (do not use d'Alembert's formula for this problem until the last question). Define the energy density to be $e(t, x) = \frac{1}{2}(u_t^2 + u_x^2)(t, x)$ and the momentum density $p(t, x) = u_t u_x(t, x)$. Show that $e_t - p_x = 0$. Apply the divergence theorem in the (t, x) plane to the vector field $\vec{v} = (e, -p)$ on a trapezoidal region which is bounded by four segments, with some interval $[a, b]$ as base on the $t = 0$ line, the characteristic $x - t = a$, $0 \le t \le T$ as left side, the characteristic $x + t = b$, $0 \le t \le T$ as right-hand side, and the top segment $[a + T, b - T]$ at time $t = T$. Here $2T < b - a$. Conclude that the energy $\int_{a+T}^{b-T} e(T, x)\, dx$ on the top segment is no larger than the energy $\int_a^b e(0, x)\, dx$ on the bottom segment. The heuristic behind this is that energy can only leave the trapezoid but not enter it due to transport along characteristics. Show important consequences of this fact: (i) finite propagation speed: if at time $t = 0$ the data vanish outside of some compact interval, then the solution has the same property for all times. Draw pictures to illustrate. (ii) Show that energy $E = \int_{-\infty}^{\infty} e(t, x)\, dx$ and momentum $P = \int_{-\infty}^{\infty} p(t, x)\, dx$ are meaningful and conserved (i.e., constant in time) for such data. (iii) Derive these exact same conclusions from d'Alembert's formula.

2. (i) Argue that a rotation in the plane about 0 by a fixed angle θ is necessarily a linear map. (ii) Compute its matrix relative to the standard basis. (iii) Relate this matrix to the map of complex numbers given by $z \mapsto e^{i\theta} z$. (iv) Show that the composition of two rotations is a rotation. Relate this fact to trigonometric identities. (v) Write down the matrix of a reflection about a line $\ell \subset \mathbb{R}^2$ passing through the origin. Argue first that a reflection is linear. (vii) Compute the determinant of such a reflection. How does it differ from the determinant of a rotation, and how do you interpret this? (viii) What kind of map is the composition of two reflections about possibly distinct lines ℓ and ℓ'?

3. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be an *isometry*, i.e., $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^d$ and $\|x\| = \sqrt{\sum_{j=1}^{d} x_j^2}$. (i) Assume in addition that $f(0) = 0$.

Prove that f is linear. In fact, show that it can be written as the product of at most d reflections about hyperplanes through the origin. What can you conclude without the assumption that $f(0) = 0$? (ii) Let A be the matrix of f relative to the standard basis. Show that $A^tA = \mathrm{Id}$, in other words A is an orthogonal matrix. (iii) What is the determinant of an orthogonal matrix? (iv) Show that in the plane every isometry is either a rotation or a reflection.

4. (i) Let A be an orthogonal real 3×3 matrix with determinant 1. Show that it is a rotation about some line $\ell \in \mathbb{R}^3$. Relate the angle of rotation about ℓ to the eigenvalues of A. Hint: Begin by showing that the characteristic polynomial has a real root which must be 1 or -1. Distinguish cases from there. (ii) Conclude that the composition of two rotations in $\mathbb{R}^3$ is a rotation. You should appreciate how difficult that is to visualize.

5. (i) Prove that every invertible linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ can be written in the form $T = PR$ where P is a symmetric and in fact, positive, linear transformation and R is an orthogonal transformation. Are P and R unique? This is called *polar decomposition*. (ii) Infer from this fact that the image of a sphere is an ellipsoid. (iii) work out the polar decomposition of a shearing transformation in $\mathbb{R}^2$ with a matrix as in class.

6. Demonstrate that the intersection of the double light cone $x_1^2 + x_2^2 - x_3^2 = 0$ in $\mathbb{R}^3$ with the real plane Π given by $a_1x_1 + a_2x_2 + x_3 = 1$ so that $a_1^2 + a_2^2 < 1$ is an ellipse (in that plane). What is the geometric significance of the condition $a_1^2 + a_2^2 < 1$? Do this by considering the projection of that intersection onto the (x_1, x_2)-plane. Note that this projection is an ellipse if and only if it is an ellipse in the plane Π (you can take that for granted). Also note that the ellipse cannot be centered at the origin. So you will need to translate. If the going gets too hard, use Mathematica or plug in concrete numbers like $a_1 = \frac{1}{2} = -a_2$.

7. Let $Tf = -f''$. Show that T is a linear transformation on the vector space $V = C^2([0, \pi])$ (twice continuously differentiable functions on $[0, \pi]$). Prove that V is infinite dimensional. Define a subspace $W \subset V$ via the boundary conditions $f(0) = f(\pi) = 0$. Show that T is symmetric on W relative to the inner product $\langle f, g \rangle = \int_0^\pi f(x)g(x)\,dx$.

In fact, show that T is a positive linear transformation. Exhibit an orthonormal family of eigenfunctions of T belonging to W and compute their eigenvalues. Note that I am not asking you to show that your orthonormal family is a *basis*, as we have not yet understood what this might mean in infinite dimensions. We will develop Hilbert spaces precisely in order to understand such questions, which will take us fully into functional analysis. See Chapter 3 of the text.

8. pages 43, 44, Section 2.16: problems 4, 5, 6, 9, 11, 12, 13. I object to the notation $D[a, b]$ in that latter problem. The standard notation is $C^1([a, b])$.