Homework Set 2, Math 325, due Wednesday, February 8, 2023 in class. Do five out of seven problems. 20 points each.

1. Show that the open unit ball of ℓ^p for $1 < p < \infty$ is uniformly convex, i.e., if x, y are distinct and lie on the boundary of the unit ball, then the open line segment connecting them lies inside of it. What can you say about $p = 1$ and $p = \infty$?

2. Show that ℓ^∞ is a Banach space.

3. (i) We say a set $A \subset X$ is precompact in a normed space X if any sequence in A admits a convergent subsequence. Show that such a set A is compact iff it is closed. (ii) Show that a precompact set is bounded, i.e., contained in a ball relative to the norm. (iii) Show that the closed unit ball in ℓ^p is not compact for any $1 \leq p \leq \infty$.

4. Characterize precompact subsets of $\mathrm{C}(I)$ where I is an interval as in the previous problem. Conclude that the closed unit ball in $\mathrm{C}(I)$ is not compact.

5. Let $X = \mathrm{Mat}(n \times m, \mathbb{R})$ where m, n are positive integers. All matrices in this problem are over $\mathbb{R}$. (i) Show that $\langle A, B \rangle := \mathrm{trace}(A^t B)$ defines an inner product on X. (ii) Show that any map $A \mapsto RAS$ where R, S are orthogonal square matrices of dimensions n resp. m, is an orthogonal transformation on X relative to this inner product. (iii) Let C, D be symmetric matrices of dimensions n resp. m. Show that $T : A \mapsto CAD$ is a symmetric linear transformation on X relative to this inner product. By the spectral theorem it is diagonalizable with real eigenvalues and an associated orthonormal eigenbasis in X. Exhibit this diagonalization in terms of the eigenvalues and eigenvectors of C, resp. D.

6. Prove Hölder and Minkowski (or triangle) inequalities in the Lebesgue spaces $L^p([0,1])$ with $1 \leq p < \infty$. Use the same basic convexity inequality as in class, i.e., for all $a, b > 0$,

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

with dual p, q. Discuss the case of equality.

7. Let $K \subset \mathbb{R}^d$ be compact. Show that K is the closed unit ball of some norm $\|\cdot\|$ on $\mathbb{R}^d$ iff

   - K has non-empty interior
   - K is balanced: if $x \in K$ then $-x \in K$
   - K is convex.

   Note that due to our assumption of finite dimension the notion of openness needed in the first bullet point is well-defined (does not depend on the choice of norm). *Hint:* For the hard direction define $\|x\| := \inf\{t > 0 \mid x \in tK\}$.