Exercise 8.1

(a) I will disprove the statement with a counterexample. Let $G = (\{a, b, c\}, \{\langle a, b\rangle, \langle b, c\rangle\})$. It holds that $\langle a, b\rangle \in S_G^1 = \{\langle a, b\rangle, \langle b, c\rangle\}$, but it is not the case that $\langle a, b\rangle \in S_G^2 = \{\langle a, c\rangle\}$.

(b) ($\rightarrow$) If n_1, n_2 are mutually reachable, then there exist a walk from n_1 to n_2 and a walk from n_2 to n_1. We can combine these walks and walk from n_1 to n_1, which is a cycle containing n_1, n_2. ($\leftarrow$) If there exists a cycle involving n_1 and n_2, there exists a walk $\pi = \langle n_0, \ldots n_1, \ldots, n_0\rangle$. We can decompose this walk by walking from n_1 to n_2, which means n_1 reaches n_2, and also walk from n_2 to n_1, which means n_2 reaches n_1. Hence n_1 and n_2 are mutually reachable.

(c) If n_1 and n_2 are in the same strongly connected component, then they are mutually reachable. This implies that they are in a component, where for all vertices u, v in the component a walk (u, v) OR (v, u) exists, which means a reachability relation exists between u, v. This also means that they are in the same equivalence class of the reachability relation, which implies that they are in a weakly connected component.

Exercise 8.2

(a) We will prove this by contradiction. Assume a topological order exits for a Graph $G = (N, A)$ and G is cyclic.
We look at three arcs $u, v, w \in A$ which form a cycle and propose a topological order $f(u) < f(v) < f(w)$.
But as these arcs form a cycle it also holds that $f(w) < f(u)$ which contradicts our assumption of a topological order.
Hence G has to be asyclic if a topological order exists for G.

(b) If $G = (N, A)$ is asyclic, then a node $v \in N$ exists which has $outdeg(v) = 0$.
We now remove the node $v \in N$ from G and the resulting G_{-1} will also by asyclic as removing nodes doesn't introduce cycles.
As G_{-1} is asyclic we can keep finding nodes with $outdeg(v) = 0$ and remove them until no nodes are left. Every $u \in indegree(N)$ has to be removed before the node is removed and since each node is removed in an order where $v < v_{-1}$ the arcs fulfill $f(u) < f(u_{-1})$ which is a topological order.

Exercise 8.3

(a) Let $L = \{v \in V \mid v \text{ is a leaf }\}$. If we consider $|V| = 3$, we have a (root) vertex $r \in V$ which is connected to exactly two leaf vertices, since G is a tree. If we want to add a new vertex n, we can only connect n to r, since connecting it to another vertex, i.e. a leaf, would result in a path in G which is longer than 2. All the newly added vertices are leaves since they are only connected to r and therefore have degree 1. To calculate the number of leaves we can therefore count all vertices which are not the root, i.e. $|L| = |V| - 1$.

(b) Let $L = \{v \in V \mid v \text{ is a leaf }\}$. If the longest path in G has length $|V| - 1$, then G is a path graph, i.e. a graph with two leaves and where the other vertices have vertex degree 2. To justify this, consider the following: The longest path has to visit all vertices since $|V| - 1 = |E|$. If the graph had more than two leaves, then the longest path would not visit all vertices, since a path in a tree can only visit two leaves at most. Therefore the graph can only have 2 leaves, i.e. $|L| = 2$.

Exercise 8.4