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Exercise 5.1

(a) From the slides we have the hint that equivalence relations induce partitions.
We look for the amount of relations over the equivalence set of M: $|R|\subseteq (\sim M), M = \{1, 2, 3\}$.
Since $(\sim M) = M_1 \times M_2$, where $M_1 \times M_2$ has to be reflexive, symetric and transitive, every possible subset of $(\sim M)$ has to fulfill $M_1 = M_2$.
This is equivalent to the different ways you can partition the set M into disjoint subsets.
$R_1 = \{\{1, 2\}, 3\}, R_2 = \{1, \{2, 3\}\}$ etc.., $|R|= 5$.

(b) $[a]_\sim = \{a\}$, $[b]_\sim = \{b, d\}$, $[c]_\sim = \{c, e\}$, $[d]_\sim = \{d, b\}$, $[e]_\sim = \{e, c\}$

Exercise 5.2

(a) Since the equivalence relation $\sim$ is reflexive, we have that for all $x \in S$ that $(x, x) \in\sim$. Therefore, for any equivalence class $[x]_\sim = \{y \in S \mid x \sim y\}$ there must exist a $y \in [x]_\sim$ with $x = y$. Each equivalence class of the set $E = \{[x]_\sim \mid x \in S\}$ has the latter property. Therefore, every element of S is in some equivalence class in E.

(b) I will show this by an indirect proof. Assume there is are $x, y, z \in S$ with $x \in [y]_\sim, x \in [z]_\sim$ and $[y]_\sim \neq [z]_\sim$. Because equivalence relations are symmetric, we have that for arbitrary s, q that $s \in [q]_\sim$ iff $q \in [s]_\sim$. Therefore $x \in [y]_\sim \leftrightarrow y \in [x]_\sim$ and $x \in [z]_\sim \leftrightarrow z \in [x]_\sim$. We have a contradicition, since we assumed $[y]_\sim \neq [z]_\sim$ which is equivalent to $[x]_\sim \neq [x]_\sim$, but it is the case that $[x]_\sim = [x]_\sim$. Thus every element of S is in at most one equivalence class in E.

Exercise 5.3

$S = \{(c, c), (d, d), (a, b)\}$
It is a partial order since not every element is related to another and its reflexive, antisymetric and transitive.
c and d are the minmial elements as there is no y with $y \preceq c, d$.

Exercise 5.4

(a) Let R be a total order over an arbitrary set S and let $x, y \in S$. If $x = y$, then the statements xRy and yRx are the same statements and thus count as only one statement. If $x \neq y$, then we have, because partial relations are antisymmetric, that if $xRy \in R$, then $yRx \notin R$ and vice versa. Therefore, either $xRy \in R$ xor $yRx \in R$ is true.

(b) I will disprove this statement by a counterexample. Let $S = \{a, b\}$. The only strict orders over S which exist are $R_1 = \{(a, b)\}$, $R_2 = \{(b, a)\}$.
Let's look at the strict order R_1. Here a is minimal, since there is no $y \in R_1$ with yR_1a. The element a is not maximal, since there exists $y \in R_1$ with aR_1y which is $y = b$.
An analogous argument can be made for R_2. Therefore, there does not exist a strict order over S where all $x \in S$ are both minimal and maximal. Thus the statement is false.

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Exercise 5.5

(a) $A^{-1} = \{\langle 2x, x\rangle | x \in \mathbb{N}_o\}$

(b) $B \setminus A^{-1} = \{\langle i * x, x\rangle | i, x \in \mathbb{N}_0, i \neq 2\}$

(c) $C \circ A = \{\langle 4, 2\rangle, \langle 8, 16\rangle, \langle 7, 14\rangle, \langle 2, 4\rangle, \langle 4, 9\rangle\}$

(d) $A \circ (A \circ A) = \{\langle x, 8 * x\rangle | x \in \mathbb{N}_0\}$

(e) $A^* = \{\langle x, 2 * y * x\rangle | x, y \in \mathbb{N}_0\}$

(f) $A \circ B = \{\langle x, x\rangle | 2 * x = i * x : i, x \in \mathbb{N}_0\}$