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Exercise 2.1

We examine the following theorem:

$\sum_{i=0}^{n} i = \frac{n*(n+1)}{2} \quad \forall \quad n \in \mathbb{N}_0$

Proof. Mathematical induction over n:

Basis $n - 1 = 0$ **:**

$0 = \dfrac{0*1}{2}$
$0 = 0$

Induction Hypothesis:

$\sum_{i=0}^{k} i = \frac{k*(k+1)}{2}$ for $k = n - 1$

Inductive Step: $n - 1 \to n$

$$\begin{aligned}
\sum_{i=0}^{n-1} i = \sum_{i=0}^{n} i - n &\overset{IH}{=} \frac{n*(n+1)}{2} - n \\
\iff \quad & \frac{n*(n+1)}{2} - \frac{2n}{2} \\
\iff \quad & \frac{n*(n+1) - 2n}{2} \\
\iff \quad & \frac{n*(n+1-2)}{2} \\
\iff \quad & \frac{n*(n-1)}{2} \\
\iff \quad & \frac{n-1*((n-1)+1)}{2}
\end{aligned}$$

□

Exercise 2.2

We will prove the following statement by structural induction:

Theorem 1. *For all binary trees B it holds that $edges(B) = 2 \cdot leaves(B) - 2$.*

Proof. **Base Case:**

$$edges(\square) = 0 = 2 - 2 = 2 * leaves(\square) - 2.$$
$$\rightsquigarrow \text{Statement is true for the base case}$$

Induction Hypothesis: Assume that for a composite tree $\langle L, \circ, R\rangle$ the statement is true for the subtrees L and R.

Inductive Step: Consider a composite tree $B = \langle L, \circ, R\rangle$.

$$\begin{aligned} edges(B) &= edges(L) + edges(B) + 2 \\ &\stackrel{IH}{=} 2 \cdot leaves(L) - 2 + 2 \cdot leaves(R) - 2 + 2 \\ &= 2 \cdot (leaves(L) + leaves(R)) - 2 \\ &= 2 \cdot leaves(B) - 2 \end{aligned}$$

□

Exercise 2.3

A set of words 'S' is defined as follows:

- 'baa' is in S.
- 'c' is in S.
- If x and y are in S, then so is xyx.
- If x is in S, then so is 'b'x'b'.

Theorem 2. *All words in S have odd length.*

We prove by structural induction:

Proof. **Basis:**
'baa' and 'c' are in S, their length is 3 and 1, both of these numbers are odd.

Induction Hypothesis:
There are words $x, y \in S$ that are of odd length.

Inductive Step:

case1: We apply the transformation xyx to two words x and y. Considering only the length of the words we can rewrite this as 2*x + 1*y. Using the Hypothesis that x and y are odd words we can reinterpreted this as 2*odd + 1*odd = 3 * odd. Since 3 is an odd number we know odd*odd = odd.

case2: We apply the transformation 'b' x 'b' to the word x. Using the Hypothesis that x is of odd length and 'b' being odd length of 1 then we can rewrite the length as 2*odd + 1* odd = 3*odd. Since 3 is an odd number we know odd*odd = odd.

□

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Exercise 2.4

(a) The set builder notation is wrong because it does not specify that x is a natural number and it also does not define n. The correct notation would be $\{x \mid x \in \mathbb{N}, x < 20, x \mod 2 = 1\}$.

(b) The notation is wrong since x is undefined and the 6 alone is not a set. The correct notation would be $\{x \mid x \in \mathbb{N}, x \neq 6\}$.

Exercise 2.5

(a) We can first find out what the union of A and B is: $A \cup B = U \setminus (A \cup B)^c = \{1, 3, 5, 6, 8, 9, 10\}$. With the latter and $A \cap B = \{1, 3\}$ it follows that $\{1, 3\} \subset A \subset A \cup B$ and $\{1, 3\} \subset B \subset A \cup B$. The following sets satisfy the properties: $A = \{1, 3, 5, 6, 8, 9\}, B = \{1, 3, 10\}$.

(b) It is not possible to satisfy all conditions at the same time. If $A \cap B = \emptyset$, it means that A, B have no elements in common. However, if $A \subset B$, it means that every element of A is an element of B, which contradicts the first condition of A and B having no common elements. Therefore, A, B under the given conditions do not exist.

(c) The following set A, B satisfy the required properties: $A = B = \{6, 7, 8\}$.