Exercise 3.1

$A, B \subseteq \{1, ..., 10\}$
$|B| = 3$, so we choose arbitrarily $B = \{1, 2, 3\}$.
$|\mathcal{P}(A)| = 16 \iff |A| = \log_2(16) = 4$
$|A \cup B| = 5$ so to reach sum of 5, A has to share 2 Elements with B while having 4 in total.
$A = \{2, 3, 4, 5\}$ satisfies these conditions.

Exercise 3.2

(a) $\{e1, e4, e6, e7\} = 01001011$

(b) $01110110 = \{e1, e2, e3, e5, e6\}$

(c) $A \cup B \iff$ A bitwise OR B
$A \cap B \iff$ A bitwise AND B
$\neg A \iff$ bitwise NOT A

Exercise 3.3

We will refute the statement for all sets A, B it holds that $|A| < |A \cup B|$. Let $A = B$, then we have $A = A \cup B$. In this case it holds that $|A| = |A \cup B|$, which contradicts the statement.

Exercise 3.4

Let $g\colon \{n \in \mathbb{N} \mid n \bmod 2 = 0\} \to \mathbb{Z}\colon g(n) = -\frac{n}{2}$. The function $g(x)$ is a bijective function because it maps every even natural number to every negated natural number (which is injective and surjective).
Let $s\colon \{n \in \mathbb{N} \mid n \bmod 2 \neq 0\} \to \mathbb{Z}\colon g(n) = \frac{n+1}{2}$. The function $s(x)$ is a bijective function because it maps every odd natural number to a natural number (it is a subset of the natural numbers).
The union of two bijective functions is bijective, therefore

$$f(n) = \begin{cases} -\frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{otherwise,} \end{cases} \quad \text{for } n \in \mathbb{N},$$

is a bijective function from $\mathbb{N}$ to $\mathbb{Z}$ and thus $\mathbb{Z}$ is countable.

Exercise 3.5

Proof. We want to show that $\mathbb{Q}$ is countable. We already know that $\mathbb{Q}_+$ is countable.
We want to form a bijection from $\mathbb{Q}_+$ to $\mathbb{Q}_-$ to show that $|\mathbb{Q}_+| = |\mathbb{Q}_-|$ and hence $\mathbb{Q}_-$ is countable too. We define $f : \mathbb{Q}_+ \to \mathbb{Q}_-$ as $f(x) = -x$.
We can define $\mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_-$, and we know that the union of two countable sets is countable. □

Exercise 3.6

I will first proof that the set S containing all finite strings that consist only of symbols from the set $\Sigma = \{✈, ✿, ♥\}$ is countable.
For each $n \in \mathbb{N}$, there are 3^n strings of length n, which is finite. We can then enumerate all strings by length, starting with strings of length 0, then length 1, and so on. Within each length n, we enumerate the 3^n strings in some specific order. By mapping each string to a unique natural number based on its position in the complete enumeration, starting from 0, we establish a one-to-one correspondence between S and $\mathbb{N}$. Thus, the set S is countable.
Since all strings in the set of the tarradiddles T consist of symbols from Σ we have that $T \subseteq S$. Because the subset of a countable set is countable, T is countable.