

Exercise 2.1

We examine the following theorem:

$$\sum_{i=0}^n i = \frac{n*(n+1)}{2} \quad \forall \quad n \in \mathbb{N}_0$$

Proof. Mathematical induction over n :

Basis $n - 1 = 0$:

$$0 = \frac{0 * 1}{2}$$

$$0 = 0$$

Induction Hypothesis:

$$\sum_{i=0}^k i = \frac{k*(k+1)}{2} \text{ for } k = n - 1$$

Inductive Step: $n - 1 \rightarrow n$

$$\begin{aligned} \sum_{i=0}^{n-1} i &= \sum_{i=0}^n i - n \stackrel{IH}{=} \frac{n*(n+1)}{2} - n \\ &\iff \frac{n*(n+1)}{2} - \frac{2n}{2} \\ &\iff \frac{n*(n+1) - 2n}{2} \\ &\iff \frac{n*(n+1-2)}{2} \\ &\iff \frac{n*(n-1)}{2} \\ &\iff \frac{n-1*(n-1+1)}{2} \end{aligned}$$

□

Exercise 2.2

We will prove the following statement by structural induction:

Theorem 1. *For all binary trees B it holds that $edges(B) = 2 \cdot leaves(B) - 2$.*

Proof. **Base Case:**

$$edges(\square) = 0 = 2 - 2 = 2 \cdot leaves(\square) - 2.$$

\leadsto Statement is true for the base case

Induction Hypothesis: Assume that for a composite tree $\langle L, \circ, R \rangle$ the statement is true for the subtrees L and R .

Inductive Step: Consider a composite tree $B = \langle L, \circ, R \rangle$.

$$\begin{aligned} edges(B) &= edges(L) + edges(R) + 2 \\ &\stackrel{IH}{=} 2 \cdot leaves(L) - 2 + 2 \cdot leaves(R) - 2 + 2 \\ &= 2 \cdot (leaves(L) + leaves(R)) - 2 \\ &= 2 \cdot leaves(B) - 2 \end{aligned}$$

□

Exercise 2.3

A set of words 'S' is defined as follows:

- 'baa' is in S.
- 'c' is in S.
- If x and y are in S, then so is xyx.
- If x is in S, then so is 'b'x'b'.

Theorem 2. *All words in S have odd length.*

We prove by structural induction:

Proof. **Basis:**

'baa' and 'c' are in S, their length is 3 and 1, both of these numbers are odd.

Induction Hypothesis:

There are words $x, y \in S$ that are of odd length.

Inductive Step:

case1: We apply the transformation xyx to two words x and y. Considering only the length of the words we can rewrite this as $2 \cdot x + 1 \cdot y$. Using the Hypothesis that x and y are odd words we can reinterpret this as $2 \cdot \text{odd} + 1 \cdot \text{odd} = 3 \cdot \text{odd}$. Since 3 is an odd number we know $\text{odd} \cdot \text{odd} = \text{odd}$.

case2: We apply the transformation 'b' x 'b' to the word x. Using the Hypothesis that x is of odd length and 'b' being odd length of 1 then we can rewrite the length as $2 \cdot \text{odd} + 1 \cdot \text{odd} = 3 \cdot \text{odd}$. Since 3 is an odd number we know $\text{odd} \cdot \text{odd} = \text{odd}$.

□

Exercise 2.4

- (a) The set builder notation is wrong because it does not specify that x is a natural number and it also does not define n . The correct notation would be $\{x \mid x \in \mathbb{N}, x < 20, x \bmod 2 = 1\}$.
- (b) The notation is wrong since x is undefined and the 6 alone is not a set. The correct notation would be $\{x \mid x \in \mathbb{N}, x \neq 6\}$.

Exercise 2.5

- (a) We can first find out what the union of A and B is: $A \cup B = U \setminus (A \cup B)^c = \{1, 3, 5, 6, 8, 9, 10\}$. With the latter and $A \cap B = \{1, 3\}$ it follows that $\{1, 3\} \subset A \subset A \cup B$ and $\{1, 3\} \subset B \subset A \cup B$. The following sets satisfy the properties: $A = \{1, 3, 5, 6, 8, 9\}$, $B = \{1, 3, 10\}$.
- (b) It is not possible to satisfy all conditions at the same time. If $A \cap B = \emptyset$, it means that A, B have no elements in common. However, if $A \subset B$, it means that every element of A is an element of B , which contradicts the first condition of A and B having no common elements. Therefore, A, B under the given conditions do not exist.
- (c) The following set A, B satisfy the required properties: $A = B = \{6, 7, 8\}$.