Alex Lutsch
Ephraim Siegfried

Exercise 2.1

We examine the following theorem:

$\sum\limits_{i=0}^{n} i = \frac{n*(n+1)}{2} \quad \forall \quad n \in \mathbb{N}_0$

Proof. Mathematical induction over n:

Basis $n - 1 = 0$ **:**

$0 = \dfrac{0 * 1}{2}$

$0 = 0$

Induction Hypothesis:

$\sum\limits_{i=0}^{k} i = \frac{k*(k+1)}{2}$ for $k = n - 1$

Inductive Step: $n - 1 \rightarrow n$

$$\begin{aligned}
\sum_{i=0}^{n-1} i = \sum_{i=0}^{n} i - n &\overset{\text{IH}}{=} \frac{n*(n+1)}{2} - n \\
\iff &\quad \frac{n*(n+1)}{2} - \frac{2n}{2} \\
\iff &\quad \frac{n*(n+1) - 2n}{2} \\
\iff &\quad \frac{n*(n+1-2)}{2} \\
\iff &\quad \frac{n*(n-1)}{2} \\
\iff &\quad \frac{n - 1 * ((n-1)+1)}{2}
\end{aligned}$$

□

Exercise 2.2

We will prove the following statement by structural induction:

Theorem 1. *For all binary trees B it holds that $\mathit{edges}(B) = 2 \cdot \mathit{leaves}(B) - 2$.*

Proof. **Base Case:**

$$edges(\square) = 0 = 2 - 2 = 2 * leaves(\square) - 2.$$
$$\rightsquigarrow \text{Statement is true for the base case}$$

Induction Hypothesis: Assume that for a composite tree $\langle L, \circ, R\rangle$ the statement is true for the subtrees L and R.

Inductive Step: Consider a composite tree $B = \langle L, \circ, R\rangle$.

$$\begin{aligned} edges(B) &= edges(L) + edges(B) + 2 \\ &\overset{IH}{=} 2 \cdot leaves(L) - 2 + 2 \cdot leaves(R) - 2 + 2 \\ &= 2 \cdot (leaves(L) + leaves(R)) - 2 \\ &= 2 \cdot leaves(B) - 2 \end{aligned}$$

□

Exercise 2.3

Exercise 2.4

(a) The set builder notation is wrong because it does not specify that x is a natural number and it also does not define n. The correct notation would be $\{x \mid x \in \mathbb{N}, x < 20, x \mod 2 = 1\}$.

(b) The notation is wrong since x is undefined and the 6 alone is not a set. The correct notation would be $\{x \mid x \in \mathbb{N}, x \neq 6\}$.

Exercise 2.5

(a) We can first find out what the union of A and B is: $A \cup B = U \setminus (A \cup B)^c = \{1, 3, 5, 6, 8, 9, 10\}$. With the latter and $A \cap B = \{1, 3\}$ it follows that $\{1, 3\} \subset A \subset A \cup B$ and $\{1, 3\} \subset B \subset A \cup B$. The following sets satisfy the properties: $A = \{1, 3, 5, 6, 8, 9\}, B = \{1, 3, 10\}$.

(b) It is not possible to satisfy all conditions at the same time. If $A \cap B = \emptyset$, it means that A, B have no elements in common. However, if $A \subset B$, it means that every element of A is an element of B, which contradicts the first condition of A and B having no common elements. Therefore, A, B under the given conditions do not exist.

(c) The following set A, B satisfy the required properties: $A = B = \{6, 7, 8\}$.