# 201yams-bootstrap

B-MAT-400

# Experiment

- Sample space $\Omega$: set of all possible outcomes
  - Example: rolling a dice

$$\Omega = \{1,2,3,4,5,6\}$$

- Event: subset of the sample set
  - Example: rolling an odd number

$$A = \{1,3,5\}$$

- An event is realized if the outcome of the experiment is an element of the event
- Probability of the event: $P(A)$

# Events

- For an event $A$
  - $A^C$ is the complimentary event ($A$ is not realized)
- For two events $A$ and $B$
  - $A \cup B$ is the event where $A$ or $B$ are realized
  - $A \cap B$ is the event where $A$ and $B$ are realized
- Example: dice roll
  - $\Omega = \{1,2,3,4,5,6\}, A = \{1,3,5\}, B = \{1,2,3\}$
  - $A^C = \{2,4,6\}$
  - $A \cup B = \{1,2,3,5\}$
  - $A \cap B = \{1,3\}$

# Probabilities

- $0 \leq P(A) \leq 1$
- $P(\Omega) = 1$
- $P(\emptyset) = 0$
- $P(A^C) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If $A$ and $B$ are independent, $P(A \cap B) = \mathrm{P(A)P(B)}$
  - Example: the chance of two coin flips being heads is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
- If $A$ and $B$ are mutually exclusive, $P(A \cap B) = P(\emptyset) = 0$
- If $A$ and $B$ are mutually exclusive, $P(A \cup B) = P(A) + P(B)$

# Discrete uniform distribution

- $\Omega$ is finite
- Every outcome ω are equally likely to be observed
- $P(\omega) = \frac{1}{|\Omega|}$ ($|\Omega|$ is the number of elements in $\Omega$)
- Example: the probability of each result of a dice roll is $\frac{1}{6}$
- For an event $A$, $P(A) = \frac{|A|}{|\Omega|}$
- Example: the probability of rolling an odd number is $\frac{3}{6}$

# Permutations

- If $E = \{1, \ldots, n\}$ is a finite set, a permutation of $E$ is a specific order of its element.
- Eexample: the permutations of $\{1,2,3\}$ are:
$$\{1,2,3\}, \{1,3,2\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}, \{3,2,1\}$$
- The number of permutations of a set of $n$ elements is $n!$
$$n! = 1 \times \cdots \times n$$
- Examples:
  - $0! = 1$
  - $1! = 1$
  - $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$

# Combinations

- A subset of $k$ elements taken from a set of $n$ elements
- Example: if $E = \{1,2,3,4,5\}$, the combinations of 3 elements are:

$$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}$$
$$\{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}$$

- The number of combinations of k-elements of a set of n elements is:

$$C_n^k = \frac{n!}{k!\,(n-k)!}$$

- With the above example:

$$C_5^3 = \frac{5!}{3!\,2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = \frac{5 \times 4}{2} = 10$$

# Bernouilli trial

- Experiment with exactly two possible outcomes, “success” and “failure”
- If $p$ is the probability of success, $1 - p$ is the probability of failure
- Example: rolling a 6 (event $A$)
  - $P(A) = \frac{1}{6}$
  - $P(A^C) = \frac{5}{6}$

# Binomial experiment

- $n$ Bernouilli trials, probability of $k$ successes?

$$P(k) = C_n^k p^k (1-p)^{n-k}$$

- Example: rolling exactly three 6 from 5 dice:

$$P(3) = C_5^3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 = 3.22\%$$

# Yams (Yahtzee)

- Goal: Scoring points by rolling 5 dice to make certain combinations
- Up to 3 rolls per turn
- After each roll, the player can keep any number of dice and roll the other ones
- Combinations:
  - Pair
  - Three of a kind
  - Four of a kind
  - Full house
  - Straight
  - Yahtzee

# Example: rolling a three of a kind of 6

- A three of a kind of 6 can be:
  - Exactly three 6
  - Exactly four 6
  - Exactly five 6
- Using the binomial experiments, and noticing that those 3 events are mutually exclusive:

$$
\begin{aligned}
P &= P(3) + P(4) + P(5) \\
&= C_5^3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + C_5^4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right) + C_5^5 \left(\frac{1}{6}\right)^5 \\
&= 3.55\%
\end{aligned}
$$

# Example: rolling a 6 full of 5

- The number of possible outcomes from rolling 5 dice is $6^5$
- The number of combinations giving a 6 full of 5 is the number of combinations of three (or two!) dice taken from 5:
$$C_5^3 = C_5^2 = 10$$
- The probability of getting a 6 full of 5 is:
$$P = \frac{C_5^3}{6^5} = 0.13\%$$

# Example: rolling a 6 straight

- The number of possible outcomes from rolling 5 dice is $6^5$
- The number of outcomes being a 6 straight is the number of permutations of $\{2,3,4,5,6\}$, which is $5!$
- The probability of rolling a 6 straight is:

$$P = \frac{5!}{6^5} = 1.54\%$$

# Exercises

- Implement two functions that return:
  - The number of permutation in a set of n element
  - The number of combination of k elements in a set of n elements
- Implement a function that returns the probability of rolling a specific combination by rolling 5 dice.
- Implement a function that takes 5 rolled dice and a combination, and returns the number of dice to reroll