Variants of squarability problem

17. listopadu 2015

In all versions let $\mathcal{R}$ denote a set of axis-parallel rectangles in $\mathbb{R}^2$ and S be a mapping from $\mathcal{R}$ to axis-parallel squares in $\mathbb{R}^2$ satisfying certain restrictions (note that S might not exist). Then $\mathrm{S}(\mathcal{R})$ is a set of squares obtained from $\mathcal{R}$ in a way specific to the particular variant and $\mathrm{S}(R)$ is the square representing the rectangle $R \in \mathcal{R}$. In each variant we explain the restrictions placed on the input set of rectangles $\mathcal{R}$ and on the output set of squares $\mathrm{S}(\mathcal{R})$.

There are four intersection types: Corner intersection, side-piercing, cross intersection and containment. We say $\mathcal{R}$ and $\mathrm{S}(\mathcal{R})$ are *combinatorially equivalent* if the intersection types are preserved and these intersection happen exactly on the same sides (and corners). For example, if $R_1, R_2 \in \mathcal{R}$ have corner intersection that is in upper left corner on R_1 and lower right corner of R_2, the same must hold for $\mathrm{S}(R_1)$ and $\mathrm{S}(R_2)$.

In all the variants here, we assume that the input set $\mathcal{R}$ contains no two rectangles with side-piercing or cross intersections.

1 Preserve order of all sides. The output $\mathrm{S}(\mathcal{R})$ has to be combinatorially equivalent to $\mathcal{R}$ and the respective order of sides on both axes has to be preserved. On a chosen axis, we can contruct the sequence of sides of rectangles $\mathcal{R}$ from left to right as they appear, i.e. every rectangle will appear exactly twice. Then the same sequence of sides has to be realized in $\mathrm{S}(\mathcal{R})$.
2 Combinatorial equivalence. The output $\mathrm{S}(\mathcal{R})$ has to be combinatorially equivalent.

We say that $\mathcal{R}$ and $\mathrm{S}(\mathcal{R})$ are *order equivalent*, if the cyclical order of intersecting rectangles around each rectangle is preserved in their images. Moreover, they are *fixed-north-order equivalent* if the sequences of sides of intersecting rectangles constructed for each rectangle by starting in the upper right corner and walking counter-clockwise around the border of the rectangle and recording every side are also preserved in their images.

3 Fixed north. We require that $\mathcal{R}$ and $\mathrm{S}(\mathcal{R})$ are fixed-north-order equivalent.
4 Preserve order. We require that $\mathcal{R}$ and $\mathrm{S}(\mathcal{R})$ are order equivalent.
5 Intersection types. The output $\mathrm{S}(\mathcal{R})$ has to keep the intersection types.

We say that $\mathcal{R}$ and $\mathrm{S}(\mathcal{R})$ are *intersection-pattern equivalent* if the following holds: $R_1 \cap R_2 \neq \emptyset$ if and only if $\mathrm{S}(R_1) \cap \mathrm{S}(R_2) \neq \emptyset$ for all $R_1, R_2 \in \mathcal{R}$. Note that both combinatorial equivalence and order equivalence imply intersection-pattern equivalence.

6 Keep intersections, forbid side-piercing. We require intersection-pattern equivalence. Squares in the output set $\mathrm{S}(\mathcal{R})$ must only have corner intersections or containment.
7 Keep intersections, allow side-piercing. We only require intersection-pattern equivalence.