ASEN 5044 Statistical Estimation for Dynamical Systems
Fall 2022
Midterm Exam 1

Out: Thursday 10/03/2024 (posted on Canvas + Gradescope)

Due: Thursday 10/10/2024 11:59 pm (Gradescope)

This exam is open notes and open book. You may ask the TAs and Prof. Ahmed for clarification only, ***but students may not consult with each other (CU Honor Code applies and will be enforced)****. Show all your work and explain your reasoning for full credit.* ***You may not use computer software (e.g. Matlab, Python) to solve any problems below, unless otherwise explicitly indicated in the problem statement*** *(simple hand calculators are ok and permitted for any question).*

1. [40 pts - 5 parts] The *inverted pendulum on a cart* is a classical problem in control theory. This system model was originally developed to study the dynamics and control of vertical rocket stabilization before and during launch. This dynamics model is also the basis for the famous Segway human transport vehicle. Figure 1 shows the basic set up of the inverted pendulum system, which consists of a cart with a massless rod of length l and mass m concentrated at the end. The cart has mass M, and rolls along the ground without any resistance according to a forcing input $u(t) = P(t)$. The system has the following non-linear equations of motion (EOMs), where z is the horizontal translation (in m), θ is the angular displacement (in rads), and $g = 9.81$ m/s^2,

$$(M + m)\ddot{z} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = P, \tag{1}$$

$$l\ddot{\theta} - g\sin\theta = \ddot{z}\cos\theta. \tag{2}$$

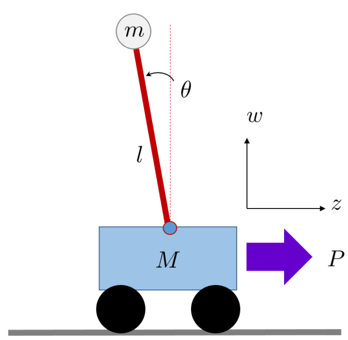

Figure 1: Inverted pendulum on a cart.

Answer the following questions (you can use software to assist with calculations for parts (c)-(e) only, but must still show and implement your own work and code for full credit; **symbolic computing tools are NOT permitted for any part**):

(a) Put the non-linear EOMs into standard state space form using the state vector $x = [z, \dot{z}, \theta, \dot{\theta}]^T$, and show that the upright stationary state for the pendulum, i.e. where $\dot{z} = 0$, $\theta = 0$, and $\dot{\theta} = 0$ for $P(t) = 0$ N for all $t \geq 0$, is an equilibrium state .

(b) Suppose a sensor is attached to this system which reports the pendulum bob's horizontal displacement relative to the vertical, such that $y = z - l\sin\theta$. Linearize the equations of motion and this output equation about the stationary upright state, and put the result into LTI $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ form. What elements comprise the corresponding state vector $\bar{x}(t)$? **Note:** you must do and show the analytical derivation of the linearized model to receive credit. **Hint:** if $\ddot{z} = \mathcal{F}_2(x, u, t)$ and $\ddot{\theta} = \mathcal{F}_4(x, u, t)$, then you're on the right track if you can use the quotient rule for derivatives to show (i.e. on your own, for full credit):

$$\frac{\partial \mathcal{F}_2}{\partial \theta} = \frac{[M + m(s\theta)^2][mg(1 - 2(s\theta)^2) - ml\dot{\theta}^2 c\theta] - [mgc\theta s\theta - ml\dot{\theta}^2 s\theta + P][2ms\theta c\theta]}{[M + m(s\theta)^2]^2},$$

$$\frac{\partial \mathcal{F}_4}{\partial \theta} = \frac{[M + m(s\theta)^2][(\frac{g}{l})(M + m)c\theta - m\dot{\theta}^2(1 - 2[s\theta]^2) - (\frac{P}{l})s\theta]}{[M + m(s\theta)^2]^2} - \frac{[(\frac{g}{l})(M + m)s\theta - m\dot{\theta}^2 c\theta s\theta + (\frac{P}{l})c\theta][2ms\theta c\theta]}{[M + m(s\theta)^2]^2}$$

$$\frac{\partial \mathcal{F}_2}{\partial \dot{\theta}} = \frac{-2\dot{\theta} m l s\theta}{M + m(s\theta)^2},$$

$$\frac{\partial \mathcal{F}_4}{\partial \dot{\theta}} = \frac{-2\dot{\theta} m c\theta s\theta}{M + m(s\theta)^2},$$

where $s\theta = \sin\theta$ and $c\theta = \cos\theta$. Of course, you still need to find other relevant partial derivatives as well!

(c) Suppose $l = 1.85$ m, $m = 2$ kg, and $M = 4$ kg. Transform the CT LTI linearized dynamics near equilibrium into an equivalent set of DT LTI model parameter matrices using $\Delta T = 0.05$ sec, and assess the stability and observability of the DT system near the equilibrium.

(d) Suppose the system is run from some unknown initial condition $x(0) = \bar{x}(0)$ (without any process or sensor noise), using a ZOH DT state feedback control law of the form $u(k) = \bar{u}(k) = -K_c \bar{x}(k)$ for $k = 0, 1, 2, \ldots$, where K_c is a 1×4 gain matrix. The resulting DT non-linear sensor outputs $y(k)$ are stored in the data array 'yNLhist' for the discrete time steps $k = 0, 1, 2, \ldots$ specified in 'thist', contained in `midterm1problem1data.mat` along with gain matrix K_c used. Use the DT matrices from (c) to estimate $\bar{x}(0)$ from 'yNLhist' (show all work and explain your reasoning; report to 4 significant digits).

(e) Generate and plot the predicted sequence of measurements $y(k)$ produced by the linearized DT dynamics model using the state feedback control law over the entire time interval in 'thist', using your estimated $\bar{x}(0)$ as the initial condition. On the same plot, also plot all the original data from 'yNLhist' (using a different marker/color/line scheme to distinguish) and compare the results. How well do the outputs predicted by the linearized DT closed-loop model agree with the actual measurements in 'yNLhist' (explain why/why not)?

2. [20 pts - 4 parts] Two 6-sided dice are tossed together; let R_1 and R_2 denote the outcome of the first die and second die, respectively. Note, that in this problem, we are **not** summing the outcome of the dice together.

(a) What is the joint probability distribution $P(R_1, R_2)$ for R_1 and R_2, if the tosses for each die are independent and the outcomes of each die are all equally probable?

(b) Now let $X = |R_1 - R_2|$ and $Y = R_1^{R_2}$ (e.g. if $R_1 = 2$ and $R_2 = 5$, then $X = 3$ and $Y = 2^5 = 32$). Find the joint probability distribution table for X and Y.

(c) Find the marginal probability distributions for X and Y in (b).

(d) Are X and Y independent (justify)?

3. [20 pts - 3 parts] A random variable X has the pdf $p(x) = k(2 - x^6)$ for $-1 \leq x \leq 1$ and $p(x) = 0$ elsewhere.

(a) Find k, $\mathbb{E}[x]$ and $\mathrm{var}(x)$.

(b) Find the cumulative distribution function (cdf) of X, $P_X(\zeta)$.

(c) Find $P(|X| < 0.65)$.

4. [20 pts - 3 parts] You are responsible for the autonomous hazard detection and avoidance subsystem for a new exploration robot that will land and 'hop' around Europa, one of Jupiter's icy moons (Forbes article) . This subsystem uses a lidar-based sensing suite to scan the moon's surface as the vehicle transits between destinations and assess whether intended landing sites contain obstacles. The robot uses this information to take one of two actions A: $A = 0 \rightarrow$ continue hop descent to target surface site; or $A = 1 \rightarrow$ fire thrusters to redirect hop away from surface target to a known safe landing site.

Since conditions on Europa are uncertain, $P(H = 1) = 0.5$ is assumed at the start of each hop, where $H = 1$ denotes a hazard is present and $H = 0$ denotes no hazard is present. The lidar system returns $L = 1$ if an object is sensed, and $L = 0$ otherwise. The probabilities for L given H (i.e. the sensing suite's performance characteristics) are:

$P(L\|H)$	$H = 0$	$H = 1$
$L = 0$	0.92	0.05
$L = 1$	0.08	0.95

For the robot to select A with uncertain surface conditions H and noisy sensor data L, you program it to use *the principle of maximum expected utility*, which says to take whatever action A^* maximizes the expected value of the *utility (total reward/penalty) function* $U(H, A)$, i.e. $A^* = \arg\max_{A \in \{0,1\}} E_H[U(H, A)]$[1]. The rewards/penalties U for each A and H are:

(a) On a typical hop, which A would the robot select *before* (i.e. prior to) taking any kind of reading L?

(b) If the robot records $L = 0$ during a hop, what should the probability for $H = 1$ now be? Does the selected A now change? (**Hint:** consider what the robot learns – how should the calculation of $E_H[U(H, A)]$ be updated accordingly, for either value of A?)

[1]for example, using made up numbers for illustration purposes only: if $E_H[U(H, A = 0)] = -1$ and $E_H[U(H, A = 1)] = +2$, then $A^* = 1$, i.e. the value for input argument A which maximizes $E_H[U(H, A)]$; note $U(H, A)$ is a function of both the (non-random) action A and random variable H, whereas $E_H[U(H, A)]$ is just a function of A.

A	H	$U(A,H)$
0	0	0
0	1	-1000
1	0	-10
1	1	100

(c) Suppose the robot records two consecutive lidar readings L_1 and L_2 on the same hop before selecting A, such that $P(L_1, L_2|H) = P(L_1|H)P(L_2|H)$, where $P(L_1|H)$ and $P(L_2|H)$ are both given by the same $P(L|H)$ table above. What action will be taken for each of the following observation cases: (i) $(L_1 = 0, L_2 = 0)$; and (ii) $(L_1 = 1, L_2 = 0)$?

Advanced Questions *You are welcome to try these questions for extra credit (only given if all regular problems turned in on time as well). Submit your responses for these questions separately as .pdf attachments via email to* `asen5044aq@gmail.com`, with subject line: 'ASEN 5044 Midterm # AQ#'. *Make sure your submission is clearly written, has your name, and is submitted separately from the rest of the assignment as a .pdf attachment. No credit given for guessing or hand-waving at answers – rigorous and careful mathematical reasoning is required for any credit to be given!*

AQ5. Consider a Bayesian analysis of the following fundamental question in astrobiology: what is the probability of abiogenesis (i.e. the original evolution of living organisms from inorganic substances) occurring on an Earth-like planet, given that life emerged sometime in Earth's past? The probability of life arising n times in time t, for $t_{min} < t < t_{max}$ could be modeled as

$$P[\lambda, n, t] = Poisson[\lambda, n, t] = e^{-\lambda(t-t_{min})} \frac{\{\lambda(t - t_{min})\}^n}{n!},$$

where λ defines the probability per unit time of abiogenesis occurring on an 'Earth-like' planet, t_{min} is the time following planet formation after which conditions are suitable for the possibility of life emerging, and t_{max} is the time after which conditions on the planet will no longer allow life to arise (e.g. following the death of its parent star). The difficulty with using this model to estimate the abundance of life in the universe is that λ is unknown. But while t_{min}, and t_{max} are also generally not precisely known, these time values can be reasonably constrained and thus used as evidence for narrowing down likely values of $\lambda > 0$, based on the fact that life most definitely arose on Earth within a limited period of time (i.e. early enough to allow us to ponder abiogenesis).

(a) Show that, under the above model, the probability of life occurring at least once within some time t of planet formation is given by,

$$P_{life} = 1 - e^{-\lambda(t-t_{min})}.$$

(b) Suppose t_{emerge} is the age of the Earth from when the earliest extant evidence of life remains, and also suppose t_{req} is maximum age that the Earth could have had at the origin of life in order for humanity to have a chance of showing up by the present, where $t_{min} < t_{emerge} < t_{req}$. Define two binary events E and R with the following interpretations: $E = 1$ means 'life emerged at least once within time t_{emerge}' ($E = 0$ otherwise), and $R = 1$ means 'life emerged at least once within time t_{req}' ($R = 0$

otherwise). Show (for some given set of values λ, t_{min}, t_{emerge}, and t_{req}) that it must therefore follow that

$$P(E = 1|R = 1, \lambda, t_{min}) = \frac{1 - e^{-\lambda(t_{emerge} - t_{min})}}{1 - e^{-\lambda(t_{req} - t_{min})}}.$$

(c) (software permitted) Suppose you distill your current knowledge on the origin and evolution of intelligent life into an a priori pdf $p(y)$ for $y = \log_{10} \lambda$. You then read a newly published study about abiogenesis on Earth which shows $t_{min} = 0.61$ GigaYears (GYr), $t_{emerge} = 0.78$ GYr, and $t_{req} = 1.4$ GYr. You recognize this means $P(E = 1|R = 1, \lambda, t_{min})$ can be easily transformed into an observation likelihood function $P(E = 1|R = 1, y, t_{min})$, thus letting you derive an updated pdf $p(y|E = 1, R = 1, t_{min})$ for y. Derive the likelihood function $P(E = 1|R = 1, y, t_{min})$ in terms of y, and then use it to numerically evaluate and plot the pdf $p(y|E = 1, R = 1, t_{min})$ vs. y using a grid-based approximation on the interval $-3 \leq y \leq 3$ for each of the following possible $p(y)$ cases (use 10^4 evenly spaced grid points for y in this interval, and label carefully):

Case 1: $p(y) = \mathcal{U}[-3, 3]$ (i.e. uniform on $y = \log \lambda$ for $-3 \leq y \leq 3$),

Case 2: $p(y) = \dfrac{|\ln(10) \cdot 10^y|}{(10^3 - 10^{-3})}$ (i.e. uniform on $10^y = \lambda$ for $10^{-3} \leq \lambda \leq 10^3$),

Case 3: $p(y) = \dfrac{|-\ln(10) \cdot 10^{-y}|}{(10^3 - 10^{-3})}$ (i.e. uniform on $10^{-y} = \lambda^{-1}$ for $10^{-3} \leq \lambda^{-1} \leq 10^3$).

(d) For which of the 3 cases evaluated above does the updated pdf show the greatest change from the a priori pdf? Discuss the implications for the problem choosing $p(y)$ – is any one of the choices above 'better' than the others (explain)?