ASEN 5044 Statistical Estimation for Dynamical Systems
Fall 2024
Midterm Exam 2

Out: Thurs 11/07/2024(posted on Canvas and Gradescope)

Due: Thurs 11/14/2024, 11:59 pm (Gradescope)

This exam is open notes and open book. You may ask Prof. Ahmed and TAs for clarification, ***but you may not consult with each other (CU Honor Code applies and will be enforced)****. Show all your work and explain your reasoning for full credit.* ***Use computer software only if explicitly indicated in the problem statement.***

1. [25 pts] Suppose we have a set of scalar Gaussian random variables $\{x_1, x_2, ..., x_n\}$ that are independent and identically distributed (iid) with $x_i \sim \mathcal{N}(0, 1)$ for $i = 1, ..., n$. If $q = \sum_{i=1}^{n} x_i^2$, find $\mathbb{E}[q]$ and $\text{var}(q)$ (**Hint:** $\mathbb{E}[x_i^4] = 3\sigma^4$ if $x_i \sim \mathcal{N}(0, \sigma^2)$).

2. [25 pts] Suppose we want to estimate the static 1D position $x \in \mathbb{R}$ of some object. We receive a measurement from sensor A, $y_1^A = x + v_{1,A}$, where $v_{1,A}$ is AWGN with variance $\sigma_{v,A}^2 = 10$ m^2. We next have a choice of sensors to use for follow-up measurements. The options are: (i) use sensor B to obtain a single measurement $y_2^B = x + v_{2,B}$, where $v_{2,B}$ is AWGN with variance $\sigma_{v,B}^2 = 6$ m^2; **OR** (ii) use sensor C to obtain two measurements, $y_2^C = x + v_{2,C}$ and $y_3^C = x + v_{3,C}$, where $v_{2,C}$ and $v_{3,C}$ are AWGN with variance $\sigma_{v,C}^2 = 10$ m^2. **Answer the following:** if we wanted a batch estimate $\hat{x}$ with maximum certainty in the result, should we choose option (i) or option (ii)? Explain your logic and provide a careful mathematical justification (your answer should be based on the values given in order to assess both options, but do not use any computing software).

3. [50 pts] *(You may use computer software for parts b, c, and d of this problem only.)* Consider the coordinated turning aircraft problem from HW 7. Assume again that the dynamics are free of process noise,

$$x(k+1) = Fx(k), \tag{1}$$

for the F matrix and states as defined in HW 7 (for some known turning rate(s) Ω and discretization step ΔT), but now assume that noisy measurements of the form

$$y(k+1) = Hx(k+1) + v(k+1), \tag{2}$$

$$E[v(k)] = 0, \quad E[v(k)v^T(j)] = \delta(k,j)R(k) \tag{3}$$

are available with AWGN $v(k)$ and non-stationary noise covariance $R(k)$. For each part of this problem, it is desired to estimate the initial state $x(0) \in \mathbb{R}^n$ of a dynamical system consisting of either one or two turning aircraft at time $k = 0$ from noisy measurements $y(k) \in \mathbb{R}^p$ taken at time steps $k = 1, 2, ..., T$.

a. Derive the analytical expression for the *batch estimator* $\hat{x}(0)$ which minimizes

$$J(T) = \sum_{k=1}^{T} (y_k - \hat{y}_k)^T [R(k)]^{-1} (y_k - \hat{y}_k) \tag{4}$$

where $\hat{y}_k$ is the time k estimator-based predicted measurement (a function of $\hat{x}(0)$) and y_k is the actual measurement at time k. Be sure to fully and precisely define all matrices and vectors used by the estimator, for the case where x_k and y_k obey eqs. (1) and (2). **Hint:** If you're stuck, try thinking about the other (kind of big) hint buried in the first sentence of this part's problem statement.

b. Suppose a ground tracking station monitors aircraft A which is turning with $\Omega_A = 0.045$ rad/s , and converts 3D range and bearing data into 2D 'pseudo-measurements' $y_A(k)$ with the following DT measurement model (with the same ΔT value as in HW 7)

$$y_A(k) = Hx_A(k) + v_A(k),$$
$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; R_A(k) = \begin{bmatrix} 75 & 7.5 \\ 7.5 & 75 \end{bmatrix} + \begin{bmatrix} 12.5\sin(k/10) & 25.5\sin(k/10) \\ 25.5\sin(k/10) & 12.5\cos(k/10) \end{bmatrix}$$

where $R_A(k)$ has time-varying elements with units of m^2. Using the data posted in `'midterm2_problem3b.mat'`, *use your result from part (a)* to estimate $x_A(0)$, and report the final state estimation error covariance matrix (to 4 significant digits). Note that the data provided is for time steps $k \geq 1$, and each column k corresponds to a single y_k vector. **DO NOT SOLVE WITH A KALMAN FILTER (no credit will be given if you to attempt to do so).**

c. Suppose now there is also a second aircraft B which is turning with $\Omega_B = -0.045$ rad/s. The tracking station can only directly sense one aircraft at a time, and thus cannot sense B while it senses A. However, a transponder between A and B provides a noisy measurement $y_D(k)$ of the difference in their 2D positions as they turn, $r_A = [\xi_A, \eta_A]^T$ and $r_B = [\xi_B, \eta_B]^T$,

$$y_D(k) = r_A(k) - r_B(k) + v_D(k),$$
$$R_D = \begin{bmatrix} 8000 & 500 \\ 500 & 8000 \end{bmatrix},$$

where R_D has units of m^2 and $v_D(k) \sim \mathcal{N}(0, R_D)$ follows a stationary white noise process (ΔT again same as in HW 7).

Using the data posted in `'midterm2_problem3c.mat'` in the array 'yaugHist' (where each column contains a concatenated 4×1 data vector $[y_A^T(k), y_D^T(k)]^T$ that includes a new set of $y_A(k)$ measurements for time steps $k \geq 1$), compute an RLLS estimate for $x(0) = [x_A(0), x_B(0)]^T$. Be sure to explain/justify how you set up the RLLS estimator to estimate $x(0)$ and how you initialized RLLS to receive full credit. **DO NOT SOLVE WITH A KALMAN FILTER (no credit will be given if you to attempt to do so).**

d. In separate plots, use the results from part (c) to show the evolution of each aircraft's state estimate vs. k (use 4 subplots per aircraft); also on separate plots, show positive 2σ bounds for each estimated state vs. k (use 4 subplots per aircraft).

Advanced Questions *You are welcome to try these questions for extra credit (only given if all regular problems turned in on time as well). In either case, submit your responses for these questions separately as .pdf attachments via email to* `asen5044aq@gmail.com`, with subject line: 'ASEN 5044 Midterm # AQ#'. *Make sure your submission is clearly written, has your name, and is submitted separately from the rest of the assignment as a .pdf attachment. No credit given for guessing or hand-waving at answers – rigorous and careful mathematical reasoning is required for any credit to be given!*

AQ 11. Given the random vector $x \in \mathbb{R}^n$ and the constant non-random matrix $A \in \mathbb{R}^{n \times n}$ where $A = A^T$, find $\mathbb{E}[x^T A x]$.

AQ 12. Consider the standard KF equations for the prediction and measurement update derived in class. Let $e_{x,k} = x_k - \hat{x}_k^+$ be the state estimation error at time step k, where x_k is the true state and $\hat{x}_k^+$ is the state estimate following a measurement update. Prove that the state estimation errors $e_{x,k}$ do not form a white noise sequence (unlike the measurement innovations $e_{y,k} = y_k - \hat{y}_k^-$).