ASEN 6080 HW 1

- Ian Faber, 108577813

1. Derive the commonly used gravitational accelerations + associated partials for orbit determination.

a. Write the potential energy of a gravity field consisting of μ, J_2, + J_3. From this potential, derive an equation for the accelerations experienced by a spacecraft in orbit in the ECI frame.

We know that

$$U = \frac{\mu}{r} + U',$$

where U' accounts for any gravitational perturbations from mass distribution like so:

$$U' = \frac{\mu}{r} \sum_{l=2}^{\infty} \sum_{m=0}^{l} \left(\frac{R_i}{r}\right)^l P_{l,m}(\sin\phi)\left(C_{l,m}\cos(m\lambda) + S_{l,m}\sin(m\lambda)\right)$$

Including just J_2 + J_3 is equivalent to setting $l = 2, 3$ and $m = 0$, with $J_l = -C_{l,0}$:

$$U' = \frac{-\mu}{r}\left(\left(\frac{R_i}{r}\right)^2 P_2(\sin\phi) J_2 + \left(\frac{R_i}{r}\right)^3 P_3(\sin\phi) J_3\right).$$

Then,

$$U = \frac{\mu}{r}\left(1 - \left(\frac{R_i}{r}\right)^2 J_2 P_2(\sin\phi) - \left(\frac{R_i}{r}\right)^3 J_3 P_3(\sin\phi)\right)$$

We also know that

$$\vec{a} = \nabla U$$

$$= \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi} + \frac{1}{r\sin\phi}\frac{\partial U}{\partial \lambda}\hat{\lambda}$$

$$\frac{\partial U}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\mu}{r}\left(1 - \left(\frac{R_i}{r}\right)^2 J_2 P_2(\sin\phi) - \left(\frac{R_i}{r}\right)^3 J_3 P_3(\sin\phi)\right)\right)$$

$$= \frac{\mu}{r}\left(\frac{2R_i^2}{r^3}J_2P_2(\sin\phi) + \frac{3R_i^3}{r^4}J_3P_3(\sin\phi)\right) +$$

$$\left(1 - \left(\frac{R_i}{r}\right)^2 J_2P_2(\sin\phi) - \left(\frac{R_i}{r}\right)^3 J_3P_3(\sin\phi)\right)\left(-\frac{\mu}{r^2}\right)$$

$$= -\frac{\mu}{r^2} + \frac{\mu R_i^2}{r^4}J_2P_2(\sin\phi) + \frac{\mu R_i^3}{r^5}J_3P_3(\sin\phi)$$

$$+ \frac{2\mu R_i^2}{r^4}J_2P_2(\sin\phi) + \frac{3\mu R_i^3}{r^5}J_3P_3(\sin\phi)$$

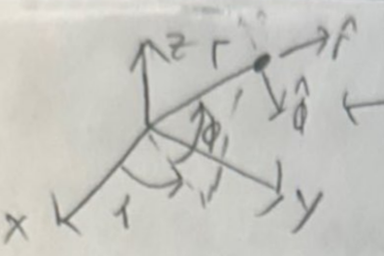
$$= -\frac{\mu}{r^2}\left(1 - \frac{3R_i^2}{r^2}J_2P_2(\sin\phi) - \frac{4R_i^3}{r^3}J_3P_3(\sin\phi)\right)$$

$$\frac{\partial U}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{\mu}{r}\left(1 + \left(\frac{R_i}{r}\right)^2 J_2P_2(\sin\phi) + \left(\frac{R_i}{r}\right)^3 J_3P_3(\sin\phi)\right)\right)$$

$$= \underline{0}$$

$$\frac{\partial U}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\frac{\mu}{r}\left(1 - \left(\frac{R_i}{r}\right)^2 J_2P_2(\sin\phi) - \left(\frac{R_i}{r}\right)^3 J_3P_3(\sin\phi)\right)\right)$$

$$P_l(\sin\phi) = \frac{1}{2^l}\sum_{j=0}^{l}\frac{(-1)^j(2l-2j)!}{j!(l-j)!(l-2j)!}(\sin\phi)^{l-2j}$$



$\leftarrow$ ϕ defined opposite of calc 3!

$$P_2(\sin\phi) = \frac{3\sin^2\phi - 1}{2}, \quad P_3(\sin\phi) = \frac{5\sin^3\phi - 3\sin\phi}{2}$$

$$\frac{\partial U}{\partial \phi} = \frac{\mu}{r}\frac{\partial}{\partial \phi}\left(-\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{3\sin^2\phi - 1}{2}\right) - \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{5\sin^3\phi - 3\sin\phi}{2}\right)\right)$$

$$= \frac{\mu}{r}\left(-\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{6\sin\phi\cos\phi}{2}\right) - \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{15\sin^2\phi\cos\phi - 3\cos\phi}{2}\right)\right)$$

$$= \frac{\mu}{r}\left(-3\left(\frac{R_i}{r}\right)^2 J_2 \sin\phi\cos\phi - \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{15\sin^2\phi - 3}{2}\right)\cos\phi\right)$$

$$= -\frac{\mu}{r}\left(3\left(\frac{R_i}{r}\right)^2 J_2 \sin\phi + \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{15\sin^2\phi - 3}{2}\right)\right)\cos\phi$$

then,

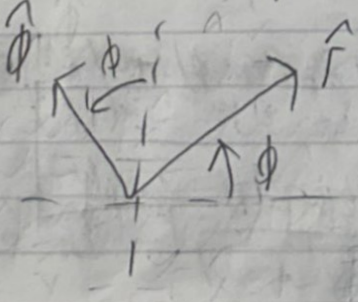
$$\vec{a} = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi} + \frac{1}{r\sin\phi}\frac{\partial U}{\partial \lambda}\hat{\lambda}$$

$$= -\frac{\mu}{r^2}\left(1 + \frac{3R_i^2}{r^2} J_2 P_2(\sin\phi) + \frac{4R_i^3}{r^3} J_3 P_3(\sin\phi)\right)\hat{r}$$

$$- \frac{\mu}{r^2}\left(3\left(\frac{R_i}{r}\right)^2 J_2 \sin\phi + \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{15\sin^2\phi - 3}{2}\right)\right)\cos\phi\,\hat{\phi}$$

$$+ 0\,\hat{\lambda}$$

$$\vec{r} = r\hat{r} \rightarrow \hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r}, \quad \phi = \sin^{-1}\left(\frac{z}{r}\right)$$



$$\hat{\phi} = \sin\phi\cos\lambda\,\hat{x} + \sin\phi\sin\lambda\,\hat{y} - \cos\phi\,\hat{z}$$

$$\cos\lambda = \frac{x}{\sqrt{x^2+y^2}},\quad \sin\lambda = \frac{y}{\sqrt{x^2+y^2}},\quad \cos\phi = \frac{\sqrt{x^2+y^2}}{r}$$

$$\hat{\phi} = \frac{xz}{r\sqrt{x^2+y^2}}\hat{x} + \frac{yz}{r\sqrt{x^2+y^2}}\hat{y} - \frac{\sqrt{x^2+y^2}}{r}\hat{z}$$

$$= \frac{(x\hat{x}+y\hat{y})z - (x^2+y^2)\hat{z}}{r\sqrt{x^2+y^2}}$$

substituting,

$$\vec{a} = -\frac{\mu}{r^2}\left(1 + \left(\frac{R_i}{r}\right)^2 J_2\left(\frac{9}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right) + \left(\frac{R_i}{r}\right)^3 J_3\left(10\left(\frac{z}{r}\right)^3 - 6\left(\frac{z}{r}\right)\right)\right)\frac{\vec{r}}{r}$$
$$-\frac{\mu}{r^2}\left(\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{3z}{r}\right) + \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\right)\cdot\frac{(x\hat{x}+y\hat{y})z - (x^2+y^2)\hat{z}}{r^2}$$

$$\ddot{x} = -\frac{\mu}{r^2}\frac{x}{r} - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{9}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\frac{x}{r} - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^3 J_3\left(10\left(\frac{z}{r}\right)^3 - 6\left(\frac{z}{r}\right)\right)\frac{x}{r}$$
$$-\frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^2 J_2\left(3\left(\frac{z}{r}\right)^2\right)\frac{x}{r} - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{15}{2}\left(\frac{z}{r}\right)^3 - \frac{3z}{2r}\right)\frac{x}{r}$$

$$\ddot{y} = -\frac{\mu}{r^2}\frac{y}{r} - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{9}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\frac{y}{r} - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^3 J_3\left(10\left(\frac{z}{r}\right)^3 - 6\left(\frac{z}{r}\right)\right)\frac{y}{r}$$
$$-\frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^2 J_2\left(3\left(\frac{z}{r}\right)^2\right)\frac{y}{r} - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{15}{2}\left(\frac{z}{r}\right)^3 - \frac{3}{2}\left(\frac{z}{r}\right)\right)\frac{y}{r}$$

$$\cos^2\phi = \frac{x^2+y^2}{r^2} = 1 - \sin^2\phi = 1 - \left(\frac{z}{r}\right)^2$$

$$\ddot{z} = -\frac{\mu}{r^2}\frac{z}{r} - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{9}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\frac{z}{r} - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^3 J_3\left(10\left(\frac{z}{r}\right)^3 - 6\left(\frac{z}{r}\right)\right)\frac{z}{r}$$
$$-\frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{3z}{r}\right)\left(\left(\frac{z}{r}\right)^2 - 1\right) - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\left(\left(\frac{z}{r}\right)^2 - 1\right)$$

Finally,

$$\ddot{x} = -\frac{\mu x}{r^3}\left(1 + \left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right) + \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3 - \frac{15}{2}\left(\frac{z}{r}\right)\right)\right)$$

$$\ddot{y} = -\frac{\mu y}{r^3}\left(1 + \left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right) + \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3 - \frac{15}{2}\left(\frac{z}{r}\right)\right)\right)$$

$$\ddot{z} = -\frac{\mu z}{r^3}\left(1 + \left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{9}{2}\right)\right) - \frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^4 - 15\left(\frac{z}{r}\right)^2 + \frac{3}{2}\right)$$

where $r = \sqrt{x^2+y^2+z^2}$

b. Derive the partials of $\vec{a}$ wrt $(\vec{r}, \vec{v})$ and μ, J_2, and J_3.

$$X = [x, y, z, \dot{x}, \dot{y}, \dot{z}, \mu, J_2, J_3]^T$$

$$\dot{X} = [\dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}, \dot{\mu}, \dot{J}_2, \dot{J}_3]^T$$

$$A = \frac{\partial \dot{X}}{\partial X} = \begin{bmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y & \partial\dot{x}/\partial z & \partial\dot{x}/\partial\dot{x} & \partial\dot{x}/\partial\dot{y} & \partial\dot{x}/\partial\dot{z} & \partial\dot{x}/\partial\mu & \partial\dot{x}/\partial J_2 & \partial\dot{x}/\partial J_3 \\ \partial\dot{y}/\partial x & | & | & | & | & | & | & | & | \\ \partial\dot{z}/\partial x & | & | & | & | & | & | & | & | \\ \partial\ddot{x}/\partial x & | & | & | & | & | & | & | & | \\ \partial\ddot{y}/\partial x & | & | & | & | & | & | & | & | \\ \partial\ddot{z}/\partial x & | & | & | & | & | & | & | & | \\ \partial\dot{\mu}/\partial x & | & | & | & | & | & | & | & | \\ \partial\dot{J}_2/\partial x & | & | & | & | & | & | & | & | \\ \partial\dot{J}_3/\partial x & | & | & | & | & | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \partial\ddot{x}/\partial x & \partial\ddot{x}/\partial y & \partial\ddot{x}/\partial z & 0 & 0 & 0 & \partial\ddot{x}/\partial\mu & \partial\ddot{x}/\partial J_2 & \partial\ddot{x}/\partial J_3 \\ \partial\ddot{y}/\partial x & \partial\ddot{y}/\partial y & \partial\ddot{y}/\partial z & 0 & 0 & 0 & \partial\ddot{y}/\partial\mu & \partial\ddot{y}/\partial J_2 & \partial\ddot{y}/\partial J_3 \\ \partial\ddot{z}/\partial x & \partial\ddot{z}/\partial y & \partial\ddot{z}/\partial z & 0 & 0 & 0 & \partial\ddot{z}/\partial\mu & \partial\ddot{z}/\partial J_2 & \partial\ddot{z}/\partial J_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

since μ, J_2, + J_3 are constant, $\partial\mu/\partial\bullet = \partial J_2/\partial\bullet = \partial J_3/\partial\bullet = 0$, where $\bullet$ is any element of $\bar{X}$.

all we need to solve for are the remaining non-trivial partials.

$$\frac{\partial \ddot{x}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-\mu x}{r^3}\left(1+\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2-\frac{3}{2}\right)+\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3-\frac{15}{2}\left(\frac{z}{r}\right)\right)\right)\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{\mu x}{r^3}\right)+\frac{\partial}{\partial x}\left(-\frac{\mu x}{r^3}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2-\frac{3}{2}\right)\right)+\frac{\partial}{\partial x}\left(-\frac{\mu x}{r^3}\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3-\frac{15}{2}\left(\frac{z}{r}\right)\right)\right)$$

$$\frac{\partial}{\partial x}\left(-\frac{\mu x}{r^3}\right) = -\mu\left(\frac{r^3(1)-x\frac{\partial}{\partial x}(r^3)}{r^6}\right) = -\mu\left(\frac{r^3-3xr^2\frac{\partial}{\partial x}(r)}{r^6}\right)$$

$$\frac{\partial}{\partial x}(r) = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}.$$

$$\frac{\partial}{\partial x}\left(-\frac{\mu x}{r^3}\right) = -\mu\left(\frac{r^3-3x^2r}{r^6}\right) = \underline{-\mu\left(\frac{r^2-3x^2}{r^5}\right)}$$

$$\frac{\partial}{\partial x}\left(-\frac{\mu x}{r^3}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2-\frac{3}{2}\right)\right) = \frac{-\mu R_i^2 J_2}{2}\frac{\partial}{\partial x}\left(\frac{x}{r^5}\left(15\left(\frac{z}{r}\right)^2-3\right)\right)$$

$$= \frac{-\mu R_i^2 J_2}{2}\frac{\partial}{\partial x}\left(\frac{15xz^2}{r^7}-\frac{3x}{r^5}\right)$$

$$= \frac{-\mu R_i^2 J_2}{2}\left(\frac{15z^2r^7-15xz^2(7r^6\frac{\partial}{\partial x}(r))}{r^{14}}-\frac{3r^5-15xr^4\frac{\partial}{\partial x}(r)}{r^{10}}\right)$$

$$= \underline{\frac{-\mu R_i^2 J_2}{2}\left(\frac{15z^2r^7-105x^2z^2r^5}{r^{14}}-\frac{3r^5-15x^2r^3}{r^{10}}\right)}$$

$$\frac{\partial}{\partial x}\left(\frac{-\mu x}{r^3}\left(\frac{R_1}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3-\frac{15}{2}\left(\frac{z}{r}\right)\right)\right)=\frac{-\mu R_1^3 J_3}{2}\frac{\partial}{\partial x}\left(\frac{x}{r^6}\left(35\left(\frac{z}{r}\right)^3-15\left(\frac{z}{r}\right)\right)\right)$$

$$=\frac{-\mu R_1^3 J_3}{2}\frac{\partial}{\partial x}\left(\frac{35xz^3}{r^9}-\frac{15xz}{r^7}\right)$$

$$=\frac{-\mu R_1^3 J_3}{2}\left(\frac{35z^3r^9-35xz^3(9r^8\frac{\partial}{\partial x}(r))}{r^{18}}-\frac{15zr^7-15xz(7r^6\frac{\partial}{\partial x}(r))}{r^{14}}\right)$$

$$=\frac{-\mu R_1^3 J_3}{2}\left(\frac{35z^3r^9-315x^2z^3r^7}{r^{18}}-\frac{15zr^7-105x^2zr^5}{r^{14}}\right)$$

$$\frac{\partial \ddot{x}}{\partial x}=-\mu\left(\frac{r^2-3x^2}{r^5}+\frac{R_1^2J_2}{2}\left(\frac{15z^2r^7-105x^2z^2r^5}{r^{14}}-\frac{3r^5-15x^2r^3}{r^{10}}\right)\right.$$
$$\left.+\frac{R_1^3J_3}{2}\left(\frac{35z^3r^9-315x^2z^3r^7}{r^{18}}-\frac{15zr^7-105x^2zr^5}{r^{14}}\right)\right)$$

Following a similar process due to similar form,

$$\frac{\partial \ddot{y}}{\partial y}=-\mu\left(\frac{r^2-3y^2}{r^5}+\frac{R_1^2J_2}{2}\left(\frac{15z^2r^7-105y^2z^2r^5}{r^{14}}-\frac{3r^5-15y^2r^3}{r^{10}}\right)\right.$$
$$\left.+\frac{R_1^3J_3}{2}\left(\frac{35z^3r^9-315y^2z^3r^7}{r^{18}}-\frac{15zr^7-105y^2zr^5}{r^{14}}\right)\right)$$

$$=-\mu\left(\frac{1}{r^3}-\frac{3y^2}{r^5}+\frac{R_1^2J_2}{2}\left(\frac{15z^2}{r^7}-\frac{105y^2z^2}{r^9}-\frac{3}{r^5}+\frac{15y^2}{r^7}\right)\right.$$
$$\left.+\frac{R_1^3J_3}{2}\left(\frac{35z^3}{r^9}-\frac{315y^2z^3}{r^{11}}-\frac{15z}{r^7}+\frac{105y^2z}{r^9}\right)\right)$$

$c^2 \lambda m,$

$$\frac{\partial \ddot{x}}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\mu x}{r^3}\right) + \frac{\partial}{\partial y}\left(-\frac{\mu x}{r^3}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\right)$$

$$+ \frac{\partial}{\partial y}\left(-\frac{\mu x}{r^3}\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3 - \frac{15}{2}\left(\frac{z}{r}\right)\right)\right)$$

$$\frac{\partial}{\partial y}\left(-\frac{\mu x}{r^3}\right) = -\mu x \frac{\partial}{\partial y}\left(r^{-3}\right) = -\mu x\left(-3r^{-4}\frac{\partial}{\partial y}(r)\right)$$

$$\frac{\partial}{\partial y}(r) = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{y}{\sqrt{x^2+y^2+z^2}} = y/r$$

$$\frac{\partial}{\partial y}\left(-\frac{\mu x}{r^3}\right) = \underline{\frac{3\mu x y}{r^5}}$$

$$\frac{\partial}{\partial y}\left(-\frac{\mu x}{r^3}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\right) = \frac{-\mu x R_i^2 J_2}{2}\frac{\partial}{\partial y}\left(\frac{1}{r^5}\left(15\left(\frac{z}{r}\right)^2 - 3\right)\right)$$

$$= \frac{-\mu x R_i^2 J_2}{2}\frac{\partial}{\partial y}\left(\frac{15z^2}{r^7} - \frac{3}{r^5}\right)$$

$$= \frac{-\mu x R_i^2 J_2}{2}\left(15z^2\left(-7r^{-8}\frac{\partial}{\partial y}(r)\right) - 3\left(-5r^{-6}\frac{\partial}{\partial y}(r)\right)\right)$$

$$= \frac{-\mu x R_i^2 J_2}{2}\left(\frac{-105z^2 y}{r^9} + \frac{15y}{r^7}\right)$$

$$= \underline{\frac{-\mu R_i^2 J_2 x y}{2}\left(\frac{-105z^2}{r^9} + \frac{15}{r^7}\right)}$$

Scanned with CamScanner

$$\frac{\partial}{\partial y}\left(-\frac{\mu x}{r^3}\left(\frac{R_1}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3-\frac{15}{2}\left(\frac{z}{r}\right)\right)\right)=-\frac{\mu x R_1^3 J_3}{2}\frac{\partial}{\partial y}\left(\frac{1}{r^6}\left(35\left(\frac{z}{r}\right)^3-15\left(\frac{z}{r}\right)\right)\right)$$

$$=-\frac{\mu x R_1^3 J_3}{2}\frac{\partial}{\partial y}\left(\frac{35z^3}{r^9}-\frac{15z}{r^7}\right)$$

$$=-\frac{\mu x R_1^3 J_3}{2}\left(35z^3\left(-9r^{-10}\frac{\partial}{\partial y}(r)\right)-15z\left(-7r^{-8}\frac{\partial}{\partial y}(r)\right)\right)$$

$$=-\frac{\mu x R_1^3 J_3}{2}\left(\frac{-315z^3y}{r^{11}}+\frac{105zy}{r^9}\right)$$

$$=-\frac{\mu R_1^3 J_3 xy}{2}\left(\frac{-315z^3}{r^{11}}+\frac{105z}{r^9}\right)$$

$$\frac{\partial \ddot{x}}{\partial y}=\mu\left(\frac{3xy}{r^5}-\frac{R_1^2J_2xy}{2}\left(\frac{15}{r^7}-\frac{105z^2}{r^9}\right)-\frac{R_1^3J_3xy}{2}\left(\frac{105z}{r^9}-\frac{315z^3}{r^{11}}\right)\right)$$

Following a similar process due to similar form

$$\frac{\partial \ddot{y}}{\partial x}=\mu\left(\frac{3xy}{r^5}-\frac{R_1^2J_2xy}{2}\left(\frac{15}{r^7}-\frac{105z^2}{r^9}\right)-\frac{R_1^3J_3xy}{2}\left(\frac{105z}{r^9}-\frac{315z^3}{r^{11}}\right)\right)$$

Then,

$$\frac{\partial \ddot{x}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{-\mu x}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{-\mu x}{r^3}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{-\mu x}{r^3}\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3 - \frac{15}{2}\left(\frac{z}{r}\right)\right)\right)$$

$$\frac{\partial}{\partial z}\left(\frac{-\mu x}{r^3}\right) = \underline{\frac{3\mu x z}{r^5}}$$ ← By similarity, with $\frac{\partial}{\partial z}(r) = \frac{z}{r}$

$$\frac{\partial}{\partial z}\left(\frac{-\mu x}{r^3}\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)\right) = -\frac{\mu x R_i^2 J_2}{2}\frac{\partial}{\partial z}\left(\frac{1}{r^5}\left(15\left(\frac{z}{r}\right)^2 - 3\right)\right)$$

$$= -\frac{\mu x R_i^2 J_2}{2}\frac{\partial}{\partial z}\left(\frac{15 z^2}{r^7} - \frac{3}{r^5}\right)$$

$$= \frac{-\mu x R_i^2 J_2}{2}\left(\frac{r^7(30z) - 15z^2\left(7r^6 \frac{\partial}{\partial z}(r)\right)}{r^{14}} - 3\left(-5r^{-6}\frac{\partial}{\partial z}(r)\right)\right)$$

$$= \frac{-\mu x R_i^2 J_2}{2}\left(\frac{30 z r^7 - 105 z^3 r^5}{r^{14}} + \frac{15 z}{r^7}\right)$$

$$= \frac{-\mu R_i^2 J_2 x z}{2}\left(\frac{45 r^7 - 105 z^2 r^5}{r^{14}}\right)$$

$$= \underline{\frac{-\mu R_i^2 J_2 x z}{2}\left(\frac{45}{r^7} - \frac{105 z^2}{r^9}\right)}$$

$$\frac{\partial}{\partial z}\left(\frac{-\mu x}{r^3}\left(\frac{R_1}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3-\frac{15}{2}\left(\frac{z}{r}\right)\right)\right)=\frac{-\mu x R_1^3 J_3}{2}\frac{\partial}{\partial z}\left(\frac{1}{r^6}\left(35\left(\frac{z}{r}\right)^3-15\left(\frac{z}{r}\right)\right)\right)$$

$$=\frac{-\mu x R_1^3 J_3}{2}\frac{\partial}{\partial z}\left(\frac{35z^3}{r^9}-\frac{15z}{r^7}\right)$$

$$=\frac{-\mu x R_1^3 J_3}{2}\left(\frac{r^9(105z^2)-35z^3(9r^8\frac{\partial}{\partial z}(r))}{r^{18}}-\frac{r^7(15)-15z(7r^6\frac{\partial}{\partial z}(r))}{r^{14}}\right)$$

$$=\frac{-\mu x R_1^3 J_3}{2}\left(\frac{105z^2r^9-315z^4r^7}{r^{18}}-\frac{15r^7-105z^2r^5}{r^{14}}\right)$$

$$=\frac{-\mu x R_1^3 J_3}{2}\left(\frac{105z^2}{r^9}-\frac{315z^4}{r^{11}}-\frac{15}{r^7}+\frac{105z^2}{r^9}\right)$$

$$=\frac{-\mu x R_1^3 J_3}{2}\left(\frac{210z^2}{r^9}-\frac{315z^4}{r^{11}}-\frac{15}{r^7}\right)$$

$$\frac{\partial \ddot{x}}{\partial z}=\mu\left(\frac{3xz}{r^5}-\frac{R_1^2J_2xz}{2}\left(\frac{45}{r^7}-\frac{105z^2}{r^9}\right)-\frac{R_1^3J_3x}{2}\left(\frac{210z^2}{r^9}-\frac{315z^4}{r^{11}}-\frac{15}{r^7}\right)\right)$$

similarly,

$$\frac{\partial \ddot{y}}{\partial z}=\mu\left(\frac{3yz}{r^5}-\frac{R_1^2J_2yz}{2}\left(\frac{45}{r^7}-\frac{105z^2}{r^9}\right)-\frac{R_1^3J_3y}{2}\left(\frac{210z^2}{r^9}-\frac{315z^4}{r^{11}}-\frac{15}{r^7}\right)\right)$$

Q2 con)

$$\frac{\partial \ddot{z}}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\mu z}{r^3}\right) + \frac{\partial}{\partial x}\left(-\frac{\mu z}{r^3}\left(\frac{R_1}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{9}{2}\right)\right)$$

$$+\frac{\partial}{\partial x}\left(-\frac{\mu}{r^2}\left(\frac{R_1}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^4 - 15\left(\frac{z}{r}\right)^2 + \frac{3}{2}\right)\right)$$

$$\frac{\partial}{\partial x}\left(-\frac{\mu z}{r^3}\right) = \underline{\frac{3\mu x z}{r^5}}$$ ← By similarity with $\frac{\partial}{\partial x}(r) = \frac{x}{r}$

$$\frac{\partial}{\partial x}\left(-\frac{\mu z}{r^3}\left(\frac{R_1}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{9}{2}\right)\right) = -\frac{\mu z R_1^2 J_2}{2}\frac{\partial}{\partial x}\left(\frac{1}{r^3}\left(\frac{15z^2}{r^2} - 9\right)\right)$$

$$= -\frac{\mu z R_1^2 J_2}{2}\frac{\partial}{\partial x}\left(\frac{15z^2}{r^7} - \frac{9}{r^5}\right)$$

$$= -\frac{\mu z R_1^2 J_2}{2}\left(15z^2\left(-7r^{-8}\frac{\partial}{\partial x}(r)\right) - 9\left(-5r^{-6}\frac{\partial}{\partial x}(r)\right)\right)$$

$$= -\frac{\mu z R_1^2 J_2}{2}\left(-\frac{105 z^2 x}{r^9} + \frac{45x}{r^7}\right)$$

$$= \underline{-\frac{\mu R_1^2 J_2 x z}{2}\left(\frac{45}{r^7} - \frac{105 z^2}{r^9}\right)}$$

$$\frac{\partial}{\partial x}\left(-\frac{\mu}{r^2}\left(\frac{R_1}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^4-15\left(\frac{z}{r}\right)^2+\frac{3}{2}\right)\right)=-\frac{\mu R_1^3 J_3}{2}\frac{\partial}{\partial x}\left(\frac{1}{r^5}\left(35\left(\frac{z}{r}\right)^4-30\left(\frac{z}{r}\right)^2+3\right)\right)$$

$$=-\frac{\mu R_1^3 J_3}{2}\frac{\partial}{\partial x}\left(\frac{35z^4}{r^9}-\frac{30z^2}{r^7}+\frac{3}{r^5}\right)$$

$$=-\frac{\mu R_1^3 J_3}{2}\left(35z^4\left(-9r^{-10}\frac{\partial}{\partial x}(r)\right)-30z^2\left(-7r^{-8}\frac{\partial}{\partial x}(r)\right)+3\left(-5r^{-6}\frac{\partial}{\partial x}(r)\right)\right)$$

$$=-\frac{\mu R_1^3 J_3}{2}\left(\frac{-315z^4x}{r^{11}}+\frac{210z^2x}{r^9}-\frac{15x}{r^7}\right)$$

$$=\frac{-\mu R_1^3 J_3 x}{2}\left(\frac{210z^2}{r^9}-\frac{315z^4}{r^{11}}-\frac{15}{r^7}\right)$$

$$\frac{\partial \ddot{z}}{\partial x}=\mu\left(\frac{3xz}{r^5}-\frac{R_1^2 J_2 xz}{2}\left(\frac{45}{r^7}-\frac{105z^2}{r^9}\right)-\frac{R_1^3 J_3 x}{2}\left(\frac{210z^2}{r^9}-\frac{315z^4}{r^{11}}-\frac{15}{r^7}\right)\right)$$

$=\frac{\partial \ddot{x}}{\partial z}$!!!!! cool!

similarly,

$$\frac{\partial \ddot{z}}{\partial y}=\mu\left(\frac{3yz}{r^5}-\frac{R_1^2 J_2 yz}{2}\left(\frac{45}{r^7}-\frac{105z^2}{r^9}\right)-\frac{R_1^3 J_3 x}{2}\left(\frac{210z^2}{r^9}-\frac{315z^4}{r^{11}}-\frac{15}{r^7}\right)\right)$$

$=\frac{\partial \ddot{y}}{\partial z}$!!!!!!! wow!

Finally,

$$\frac{\partial \ddot{z}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{-\mu z}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{-\mu z}{r^3}\left(\frac{R_1}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{9}{2}\right)\right)$$

$$+ \frac{\partial}{\partial z}\left(-\frac{\mu}{r^2}\left(\frac{R_1}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^4 - 15\left(\frac{z}{r}\right)^2 + \frac{3}{2}\right)\right)$$

$$\frac{\partial}{\partial z}\left(-\frac{\mu z}{r^3}\right) = -\mu\left(\frac{r^2 - 3z^2}{r^5}\right) = -\mu\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{-\mu z}{r^3}\left(\frac{R_1}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{9}{2}\right)\right) = \frac{-\mu R_1^2 J_2}{2}\frac{\partial}{\partial z}\left(\frac{z}{r^5}\left(15\left(\frac{z}{r}\right)^2 - 9\right)\right)$$

$$= \frac{-\mu R_1^2 J_2}{2}\frac{\partial}{\partial z}\left(\frac{15 z^3}{r^7} - \frac{9z}{r^5}\right)$$

$$= \frac{-\mu R_1^2 J_2}{2}\left(\frac{r^7(45z^2) - 15z^3\left(7r^6 \frac{\partial}{\partial z}(r)\right)}{r^{14}} - \frac{r^5(9) - 9z\left(5r^4 \frac{\partial}{\partial z}(r)\right)}{r^{10}}\right)$$

$$= \frac{-\mu R_1^2 J_2}{2}\left(\frac{45z^2}{r^7} - \frac{105 z^4}{r^9} - \frac{9}{r^5} + \frac{45 z^2}{r^7}\right)$$

$$\frac{\partial}{\partial z}\left(-\frac{\mu}{r^2}\left(\frac{R_1}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^4 - 15\left(\frac{z}{r}\right)^2 + \frac{3}{2}\right)\right)$$

$$= -\frac{\mu R_1^3 J_3}{2}\frac{\partial}{\partial z}\left(\frac{1}{r^5}\left(35\left(\frac{z}{r}\right)^4 - 30\left(\frac{z}{r}\right)^2 + 3\right)\right)$$

$$= -\frac{\mu R_1^3 J_3}{2}\frac{\partial}{\partial z}\left(\frac{35 z^4}{r^9} - \frac{30 z^2}{r^7} + \frac{3}{r^5}\right)$$

$$= -\frac{\mu R_i^3 J_3}{2}\left(\frac{r^9(140z^3) - 35z^4\left(9r^8 \frac{\partial}{\partial z}(r)\right)}{r^{18}} - \frac{r^7(60z) - 30z^2\left(7r^6\frac{\partial}{\partial z}(r)\right)}{r^{14}} + 3\left(-5r^{-6}\frac{\partial}{\partial z}(r)\right)\right)$$

$$= -\frac{\mu R_i^3 J_3}{2}\left(\frac{140z^3}{r^9} - \frac{315z^5}{r^{11}} - \frac{60z}{r^7} + \frac{210z^3}{r^9} - \frac{15z}{r^7}\right)$$

$$= -\frac{\mu R_i^3 J_3 z}{2}\left(\frac{350z^2}{r^9} - \frac{315z^4}{r^{11}} - \frac{75}{r^7}\right)$$

$$\frac{\partial \ddot{z}}{\partial z} = -\mu\left(\frac{r^2 - 3z^2}{r^5} + \frac{R_i^2 J_2}{2}\left(\frac{90z^2}{r^7} - \frac{105z^4}{r^9} - \frac{9}{r^5}\right) + \frac{R_i^3 J_3 z}{2}\left(\frac{350z^2}{r^9} - \frac{315z^4}{r^{11}} - \frac{75}{r^7}\right)\right)$$

now, the constants!

$$\frac{\partial \ddot{x}}{\partial \mu} = -\frac{x}{r^3}\left(1 + \left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right) + \left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3 - \frac{15}{2}\left(\frac{z}{r}\right)\right)\right)$$

$$\frac{\partial \ddot{x}}{\partial J_2} = -\frac{\mu x}{r^3}\left(\frac{R_i}{r}\right)^2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2 - \frac{3}{2}\right)$$

$$\frac{\partial \ddot{x}}{\partial J_3} = -\frac{\mu x}{r^3}\left(\frac{R_i}{r}\right)^3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3 - \frac{15}{2}\left(\frac{z}{r}\right)\right)$$

$$\frac{\partial \ddot{y}}{\partial \mu} = -\frac{y}{r^3}\left(1+\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2-\frac{3}{2}\right)+\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3-\frac{15}{2}\left(\frac{z}{r}\right)\right)\right)$$

$$\frac{\partial \ddot{y}}{\partial J_2} = -\frac{\mu y}{r^3}\left(\frac{R_i}{r}\right)^2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2-\frac{3}{2}\right)$$

$$\frac{\partial \ddot{y}}{\partial J_3} = -\frac{\mu y}{r^3}\left(\frac{R_i}{r}\right)^3\left(\frac{35}{2}\left(\frac{z}{r}\right)^3-\frac{15}{2}\left(\frac{z}{r}\right)\right)$$

$$\frac{\partial \ddot{z}}{\partial \mu} = -\frac{z}{r^3}\left(1+\left(\frac{R_i}{r}\right)^2 J_2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2-\frac{9}{2}\right)\right)-\frac{1}{r^2}\left(\frac{R_i}{r}\right)^3 J_3\left(\frac{35}{2}\left(\frac{z}{r}\right)^4-15\left(\frac{z}{r}\right)^2+\frac{3}{2}\right)$$

$$\frac{\partial \ddot{z}}{\partial J_2} = -\frac{\mu z}{r^3}\left(\frac{R_i}{r}\right)^2\left(\frac{15}{2}\left(\frac{z}{r}\right)^2-\frac{9}{2}\right)$$

$$\frac{\partial \ddot{z}}{\partial J_3} = -\frac{\mu}{r^2}\left(\frac{R_i}{r}\right)^3\left(\frac{35}{2}\left(\frac{z}{r}\right)^4-15\left(\frac{z}{r}\right)^2+\frac{3}{2}\right)$$

phew!!

c. implement the partials from 1b. into a function and verify that it matches the provided test data.

see PDF for code and output, it matches!

```matlab
function A = DynamicsPartials_MuJ2J3(X, Ri)
% Function that outputs the acceleration partials matrix for orbital
% dynamics including contributions from Mu, J2, and J3 according to ASEN
% 6080 HW 1 Problem 1
%   Inputs:
%       - X: System state in SI units
%           [(x,y,z) -> km, (xDot, yDot, zDot) -> km/s, mu -> km^3/s^2,
%            (J2, J3) -> n.d.]
%           X = [x; y; z; xDot; yDot; zDot; mu; J2; J3]
%       - Ri: Reference radius [km]
%
%   Outputs:
%       - A: Acceleration partials matrix
%
%   By: Ian Faber, 01/22/2025
%

    % Extract states
x = X(1);
y = X(2);
z = X(3);

mu = X(7);
J2 = X(8);
J3 = X(9);

    % Define range
r = sqrt(x^2 + y^2 + z^2);

    % Define non-trivial partials
delXddDelX = -mu*( ((r^2 - 3*x^2)/r^5) + ((Ri^2*J2)/2)*((((15*(x^2 + z^2))/
r^7) - ((105*x^2*z^2)/r^9) - (3/r^5)) + ((Ri^3*J3*z)/2)*(((35*(z^2 +
(3*x^2)))/r^9) - ((315*x^2*z^2)/r^11) - (15/r^7)) );
delYddDelY = -mu*( ((r^2 - 3*y^2)/r^5) + ((Ri^2*J2)/2)*((((15*(y^2 + z^2))/
r^7) - ((105*y^2*z^2)/r^9) - (3/r^5)) + ((Ri^3*J3*z)/2)*(((35*(z^2 +
(3*y^2)))/r^9) - ((315*y^2*z^2)/r^11) - (15/r^7)) );

delXddDelY = mu*( ((3*x*y)/r^5) - ((Ri^2*J2*x*y)/2)*((15/r^7) - ((105*z^2)/
r^9)) - ((Ri^3*J3*x*y)/2)*(((105*z)/r^9) - ((315*z^3)/r^11)) );
delYddDelX = delXddDelY;

delXddDelZ = mu*( ((3*x*z)/r^5) - ((Ri^2*J2*x*z)/2)*((45/r^7) - ((105*z^2)/
r^9)) - ((Ri^3*J3*x)/2)*(((210*z^2)/r^9) - ((315*z^4)/r^11) - (15/r^7)) );
delYddDelZ = mu*( ((3*y*z)/r^5) - ((Ri^2*J2*y*z)/2)*((45/r^7) - ((105*z^2)/
r^9)) - ((Ri^3*J3*y)/2)*(((210*z^2)/r^9) - ((315*z^4)/r^11) - (15/r^7)) );

delZddDelX = delXddDelZ;
delZddDelY = delYddDelZ;

delZddDelZ = -mu*( ((r^2 - 3*z^2)/r^5) + ((Ri^2*J2)/2)*((((90*z^2)/r^7) -
((105*z^4)/r^9) - (9/r^5)) + ((Ri^3*J3*z)/2)*(((350*z^2)/r^9) - ((315*z^4)/
r^11) - (75/r^7)) );
```

```
delXddDelMu = (-x/r^3)*(1 + ((Ri/r)^2)*J2*((15/2)*(z/r)^2 - (3/2)) + ((Ri/
r)^3)*J3*((35/2)*(z/r)^3 - (15/2)*(z/r)));
delXddDelJ2 = (-mu*x/r^3)*((Ri/r)^2)*((15/2)*(z/r)^2 - (3/2));
delXddDelJ3 = (-mu*x/r^3)*((Ri/r)^3)*((35/2)*(z/r)^3 - (15/2)*(z/r));

delYddDelMu = (-y/r^3)*(1 + ((Ri/r)^2)*J2*((15/2)*(z/r)^2 - (3/2)) + ((Ri/
r)^3)*J3*((35/2)*(z/r)^3 - (15/2)*(z/r)));
delYddDelJ2 = (-mu*y/r^3)*((Ri/r)^2)*((15/2)*(z/r)^2 - (3/2));
delYddDelJ3 = (-mu*y/r^3)*((Ri/r)^3)*((35/2)*(z/r)^3 - (15/2)*(z/r));

delZddDelMu = (-z/r^3)*(1 + ((Ri/r)^2)*J2*((15/2)*(z/r)^2-(9/2))) - (1/
r^2)*((Ri/r)^3)*J3*((35/2)*(z/r)^4 - 15*(z/r)^2 + (3/2));
delZddDelJ2 = (-mu*z/r^3)*((Ri/r)^2)*((15/2)*(z/r)^2-(9/2));
delZddDelJ3 = -(mu/r^2)*((Ri/r)^3)*((35/2)*(z/r)^4 - 15*(z/r)^2 + (3/2));

    % Create matrix blocks for convenience
dynamicsBlock = [
                    delXddDelX, delXddDelY, delXddDelZ;
                    delYddDelX, delYddDelY, delYddDelZ;
                    delZddDelX, delZddDelY, delZddDelZ
                ];

constantsBlock = [
                    delXddDelMu, delXddDelJ2, delXddDelJ3;
                    delYddDelMu, delYddDelJ2, delYddDelJ3;
                    delZddDelMu, delZddDelJ2, delZddDelJ3
                 ];

    % Construct A matrix
A = [
        zeros(3,3), eye(3), zeros(3,3);
        dynamicsBlock, zeros(3,3), constantsBlock;
        zeros(3,9)
    ];

end
```

```
A =

   1.0e+10  *

  Columns 1 through 3

                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
  -0.073316273741784  -0.479226920185805   0.756743028208603
  -0.479226920185805   0.535527724733773  -1.377226825305907
   0.756743028208603  -1.377226825305907  -0.462211450991990
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
```

```
  Columns 4 through 6

   0.000000000100000                   0                   0
                   0   0.000000000100000                   0
                   0                   0   0.000000000100000
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0

  Columns 7 through 9

                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
  -0.391067159312613  -0.000050683493002  -1.109870644452823
   0.711718723826872   0.000092240910799   2.019897861204512
  -0.632704122700506   0.000525359280250  -1.795099627614160
                   0                   0                   0
                   0                   0                   0
                   0                   0                   0
```

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2. Given the following:

$a = 10{,}000$ km, $e = 0.001$, $i = 40°$, $\Omega = 80°$,
$\omega = 40°$, $\nu_0 = 0°$

include only μ + J_2 and integrate the system for 15 orbits.

σ^2 km, integrate a second trajectory with

$$\delta x_0 = [1 \text{ km}, 0, 0, 0, 10 \text{ m/s}, 0, 0]$$

where J_2 is the last state. Compare the 2 trajectories.

Then, use the STM evaluated at the reference trajectory to propagate

$$\delta x(t) = \Phi(t, t_0)\,\delta x_0$$

and compare to the integrated perturbations

a. compare your integrator to the truth values provided.

First, need to convert the given orbital elements into cartesian, then integrate.

we know that

$$x = r\cos\nu\, Q_{11} + r\sin\nu\, Q_{12}$$
$$y = r\cos\nu\, Q_{21} + r\sin\nu\, Q_{22}$$
$$z = r\cos\nu\, Q_{31} + r\sin\nu\, Q_{32}$$

$$\dot{x} = \dot{x}^* Q_{11} + \dot{y}^* Q_{12}$$
$$\dot{y} = \dot{x}^* Q_{21} + \dot{y}^* Q_{22}$$
$$\dot{z} = \dot{x}^* Q_{31} + \dot{y}^* Q_{32}$$

where

$$Q_{11} = \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i$$
$$Q_{12} = -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i$$
$$Q_{21} = \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i$$
$$Q_{22} = -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i$$
$$Q_{31} = \sin\omega\sin i$$
$$Q_{32} = \cos\omega\sin i$$

and

$$\dot{x}^* = V_r\cos\nu - V_\theta\sin\nu$$
$$\dot{y}^* = V_r\sin\nu + V_\theta\cos\nu$$

$$V_r = \frac{he}{p}\sin\nu, \quad V_\theta = \frac{h}{r}$$

also,

$$p = a(1-e^2), \quad h = \sqrt{\mu a(1-e^2)}$$

Plugging in the provided orbital elements,

$x_0 = -3515.49$ km $\quad \dot{x}_0 = -4.3577$ km/s

$y_0 = 8390.72$ km $\quad \dot{y}_0 = -3.3586$ km/s

$z_0 = 4127.63$ km $\quad \dot{z}_0 = 3.1119$ km/s

my plotted trajectory looks like a proper J_2-perturbed orbit, which makes me feel confident that my integrator works.

b. Implement your STM computation as a function callable by ode45.

we know that

$$\dot{\Phi}(t,t_0) = \left[\frac{\partial f}{\partial X}\right]_{X=X^*} \Phi(t,t_0) = A\,\Phi(t,t_0)$$

where

$$\Phi(t,t_0) = e^{A(t-t_0)}$$

Further, we know that

$$\dot{X} = f(X,t)$$

which we already coded in 2a.

see PDF for code, the results match!

```matlab
function dX = orbitEOM_MuJ2(t,X,mu,Ri)
% Function for the equations of motion for a body in orbit around a central
% body including mu and J2 in the 2 body problem
%   Inputs:
%       - t: Current integration time
%       - X: State vector arranged as follows:
%            [X; Y; Z; Xdot; Ydot; Zdot; J2]
%       - mu: Gravitational parameter for the central body of interest
%       - Ri: Reference radius
%
%   Outputs:
%       - dX: Rate of change vector for the provided state
%             [Xdot; Ydot; Zdot; Xddot; Yddot; Zddot; J2dot]
%
%   By: Ian Faber, 01/23/2025
%

x = X(1);
y = X(2);
z = X(3);
xDot = X(4);
yDot = X(5);
zDot = X(6);
J2 = X(7);

r = sqrt(x^2 + y^2 + z^2);

xDDot = -(mu*x/(r^3))*(1 + (((Ri/r)^2)*J2*((15/2)*(z/r)^2-(3/2))));
yDDot = -(mu*y/(r^3))*(1 + (((Ri/r)^2)*J2*((15/2)*(z/r)^2-(3/2))));
zDDot = -(mu*z/(r^3))*(1 + (((Ri/r)^2)*J2*((15/2)*(z/r)^2-(9/2))));

J2Dot = 0;

dX = [xDot; yDot; zDot; xDDot; yDDot; zDDot; J2Dot];


end
```

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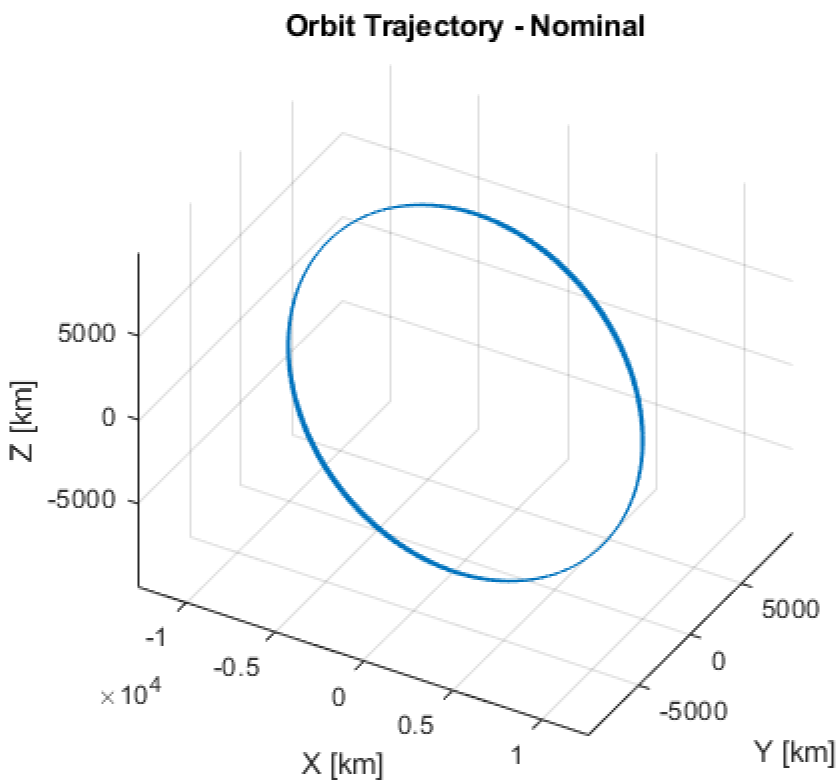
ASEN 6080 HW 1 Problem 2 Script

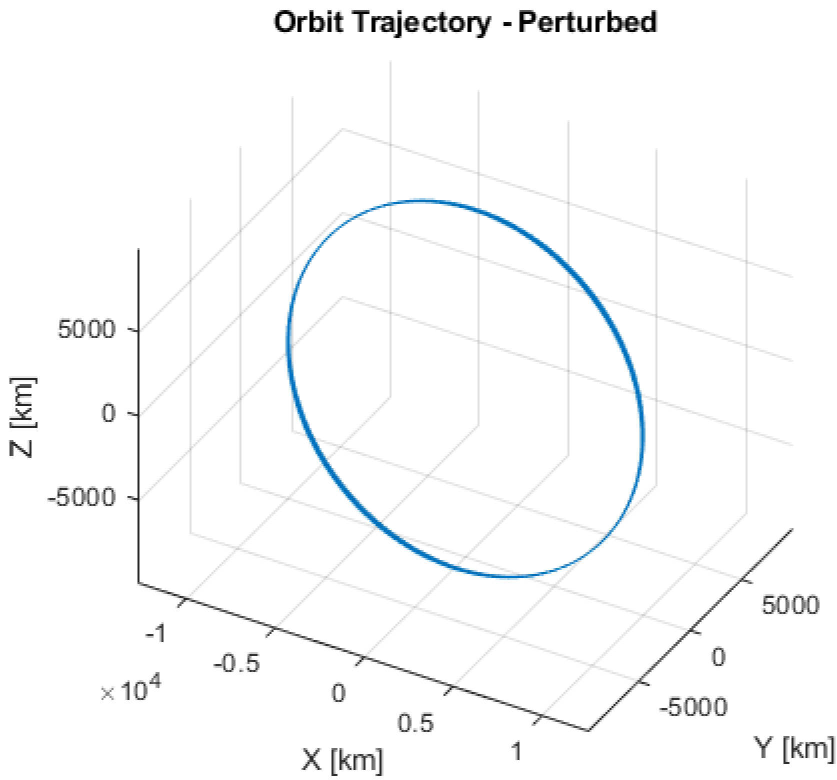
By: Ian Faber

Housekeeping

Part 2a

[redacted]





Part 2b

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted] [redacted] [redacted]

```matlab
function dPhiX = STMEOM_J2(t,XPhi,mu,Ri)
% Function for the equations of motion for an STM of the 2 body problem
% including J2 contributions
%   Inputs:
%       - t: Current integration time
%       - XPhi: State transition matrix and state vector (7^2 + 7)x1:
%               [X; Phi]
%       - mu: Gravitational parameter for the central body of interest
%       - Ri: Reference radius
%
%   Outputs:
%       - dPhiX: Rate of change vector for the provided STM and state.
%                Note: dPhi is 7^2x1, not 7x7!
%                [dPhi; dX]
%
%   By: Ian Faber, 01/23/2025
%

x = XPhi(1);
y = XPhi(2);
z = XPhi(3);
xDot = XPhi(4);
yDot = XPhi(5);
zDot = XPhi(6);
J2 = XPhi(7);
Phi = XPhi(8:56);

Phi = reshape(Phi, 7, 7); % Phi is converted into a 49x1 vector...

X_a = [x; y; z; xDot; yDot; zDot; mu; J2];

r = sqrt(x^2 + y^2 + z^2);

A = DynamicsPartials_MuJ2(X_a, Ri);

dPhi_mat = A*Phi;
dPhi = reshape(dPhi_mat,49,1); % Need to convert back to a column vector...

X0 = [x;y;z;xDot;yDot;zDot;J2];
dX = orbitEOM_MuJ2(t,X0,mu,Ri);

dPhiX = [dX; dPhi];

end
```

```
XDot =

   1.0e+12 *

   0.000000000000876
  -0.000000000000243
   0.000000000000167
```

```
  -3.413686753568604
   3.207050117068763
  -0.357531958249240
                   0


PhiDot_mat =

   1.0e+13 *

  Columns 1 through 3

  -0.000000000000028   0.000000000000140   0.000000000000079
   0.000000000000060  -0.000000000000137   0.000000000000093
   0.000000000000178  -0.000000000000029  -0.000000000000049
   0.374936725526471   0.220415198396986   0.712850905227827
  -0.359462275935478  -1.330425361600812  -0.464009296480646
   1.912845832957822   0.261325351717644   0.070365033084850
                   0                   0                   0

  Columns 4 through 6

  -0.000000000000054   0.000000000000093   0.000000000000020
  -0.000000000000016  -0.000000000000148   0.000000000000043
   0.000000000000061  -0.000000000000056  -0.000000000000127
  -0.973504410324571  -0.308043951672802   1.337366787539103
   0.882180120948528  -0.310126639387312  -1.708438336510815
  -0.326422999680118  -0.614276201454800  -0.012194092462935
                   0                   0                   0

  Column 7

   0.000000000000004
   0.000000000000028
   0.000000000000006
   0.622263144368798
  -0.471646032233606
  -1.110000530701040
                   0
```

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c. Plot the 2 $\delta x(t)$ time histories on the same plot.

See PDF for plot

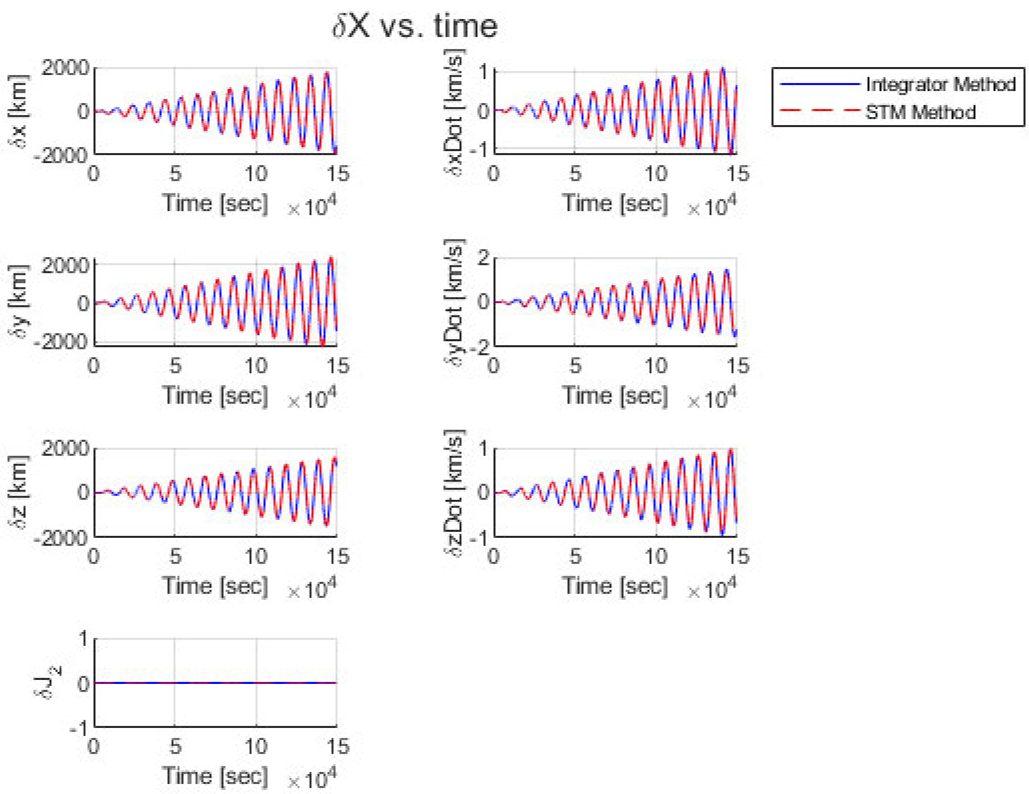
d. Plot the difference in the 2 deviation vectors vs. time and discuss.

See PDF for plot

It seems that using the STM to predict perturbed orbits works well on short time scales, but deteriorates after about 5 orbits or so. While after 15 orbits the position difference is only on the order of 200 km and velocity difference is on the order of 0.2 km/s, for some situations (i.e. docking, precise measurements, etc.) this could be a massive hindrance.

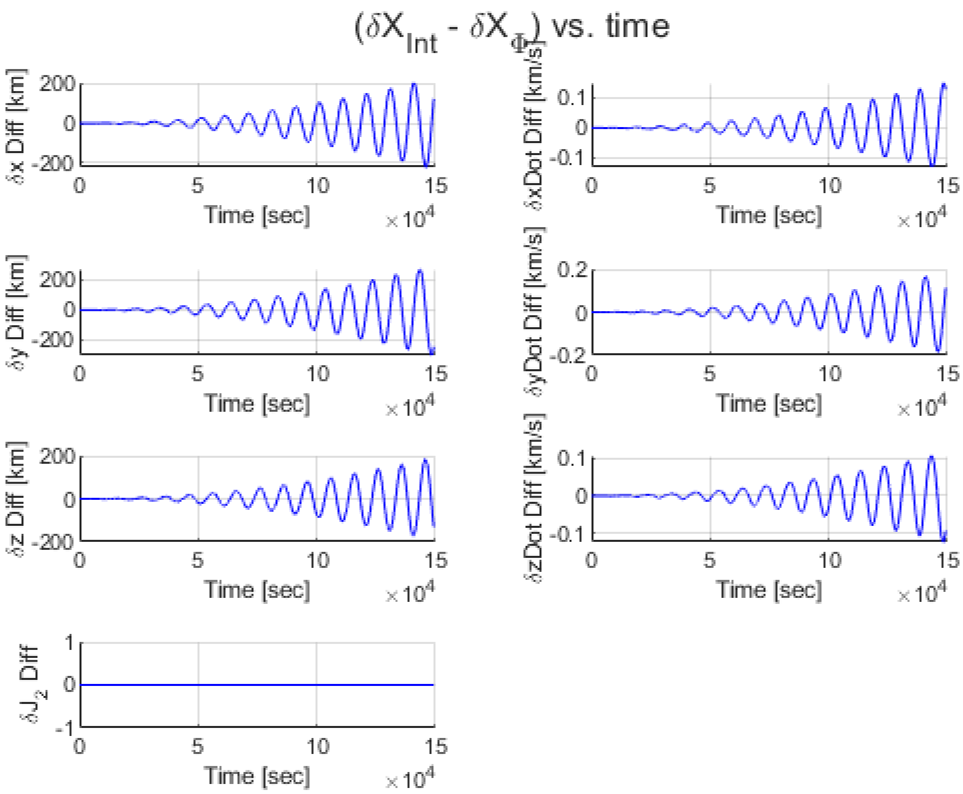
e. Investigate how well the STM works in other orbit regions.

As eccentricity increases, the STM differs more and more from the fully integrated solution.



Part 2d

[redacted]



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3. Given the following measurement equations:

$$\rho = |\vec{R} - \vec{R}_s|$$

$$\dot{\rho} = \frac{(\vec{R}-\vec{R}_s)\cdot(\vec{V}-\vec{V}_s)}{\rho}$$

where $\vec{R}$ is the s/c position vector, R_s is the station position vector, $\vec{V}$ is the s/c position vector, and $\vec{V}_s$ is the station velocity vector, derive the measurement partials commonly used in orbit determination.

a. Derive the partials of ρ + $\dot{\rho}$ wrt the spacecraft state, X.

we know that $\tilde{H}_{sc} = \frac{\partial \vec{y}}{\partial X}$, where

$\vec{y} = \begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix}$, then,

$$\frac{\partial \vec{y}}{\partial X} = \begin{bmatrix} \partial\rho/\partial x & \partial\rho/\partial y & \partial\rho/\partial z & \partial\rho/\partial\dot{x} & \partial\rho/\partial\dot{y} & \partial\rho/\partial\dot{z} \\ \partial\dot{\rho}/\partial x & \partial\dot{\rho}/\partial y & \partial\dot{\rho}/\partial z & \partial\dot{\rho}/\partial\dot{x} & \partial\dot{\rho}/\partial\dot{y} & \partial\dot{\rho}/\partial\dot{z} \end{bmatrix}$$

$$= \begin{bmatrix} \partial\rho/\partial x & \partial\rho/\partial y & \partial\rho/\partial z & 0 & 0 & 0 \\ \partial\dot{\rho}/\partial x & \partial\dot{\rho}/\partial y & \partial\dot{\rho}/\partial z & \partial\dot{\rho}/\partial\dot{x} & \partial\dot{\rho}/\partial\dot{y} & \partial\dot{\rho}/\partial\dot{z} \end{bmatrix}$$

rewriting ρ:

$$\rho = |\vec{R} - \vec{R}_s| = \sqrt{(x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2}$$

then,

$$\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{(x-x_s)^2+(y-y_s)^2+(z-z_s)^2}\right)$$

$$= \frac{\cancel{2}(x-x_s)}{\cancel{2}\sqrt{(x-x_s)^2+(y-y_s)^2+(z-z_s)^2}} = \frac{x-x_s}{\rho}$$

similarly,

$$\frac{\partial \rho}{\partial y} = \frac{y-y_s}{\rho}, \quad \frac{\partial \rho}{\partial z} = \frac{z-z_s}{\rho}$$

Rewriting $\dot{\rho}$:

$$\dot{\rho} = \frac{(\vec{R}-\vec{R}_s)\cdot(\vec{V}-\vec{V}_s)}{\rho}$$

$$= \frac{(x-x_s)(\dot{x}-\dot{x}_s)+(y-y_s)(\dot{y}-\dot{y}_s)+(z-z_s)(\dot{z}-\dot{z}_s)}{\rho}$$

$$\frac{\partial \dot{\rho}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{(x-x_s)(\dot{x}-\dot{x}_s)+(y-y_s)(\dot{y}-\dot{y}_s)+(z-z_s)(\dot{z}-\dot{z}_s)}{\rho}\right)$$

$$= \frac{\rho(\dot{x}-\dot{x}_s)-\left((x-x_s)(\dot{x}-\dot{x}_s)+(y-y_s)(\dot{y}-\dot{y}_s)+(z-z_s)(\dot{z}-\dot{z}_s)\right)\frac{\partial}{\partial x}(\rho)}{\rho^2}$$

$$= \frac{\rho^2(\dot{x}-\dot{x}_s)-(x-x_s)\left((x-x_s)(\dot{x}-\dot{x}_s)+(y-y_s)(\dot{y}-\dot{y}_s)+(z-z_s)(\dot{z}-\dot{z}_s)\right)}{\rho^3}$$

similarly,

$$\frac{\partial \dot{\rho}}{\partial y} = \frac{\rho^2(\dot{y}-\dot{y}_s) - (y-y_s)\left((x-x_s)(\dot{x}-\dot{x}_s)+(y-y_s)(\dot{y}-\dot{y}_s)+(z-z_s)(\dot{z}-\dot{z}_s)\right)}{\rho^3}$$

$$\frac{\partial \dot{\rho}}{\partial z} = \frac{\rho^2(\dot{z}-\dot{z}_s) - (z-z_s)\left((x-x_s)(\dot{x}-\dot{x}_s)+(y-y_s)(\dot{y}-\dot{y}_s)+(z-z_s)(\dot{z}-\dot{z}_s)\right)}{\rho^3}$$

$$\frac{\partial \dot{\rho}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{(x-x_s)(\dot{x}-\dot{x}_s)+(y-y_s)(\dot{y}-\dot{y}_s)+(z-z_s)(\dot{z}-\dot{z}_s)}{\rho}\right)$$

$$= \frac{\rho(x-x_s) - \cancel{(x-x_s)(\dot{x}-\dot{x}_s)\frac{\partial}{\partial \dot{x}}(\rho)}^{0}}{\rho^2}$$

$$= \frac{x-x_s}{\rho}$$

similarly,

$$\frac{\partial \dot{\rho}}{\partial \dot{y}} = \frac{y-y_s}{\rho}, \quad \frac{\partial \dot{\rho}}{\partial \dot{z}} = \frac{z-z_s}{\rho}$$

b. implement this matrix as a function.

see PDF for code, output matches!

```matlab
function Htilde = MeasurementPartials_RngRngRate_sc(X,X_s)
% Function that outputs the measurement partials matrix for orbital
% measurements using range and range rate for spacecraft
%   Inputs:
%       - X: Spacecraft state arranged as follows:
%            [x; y; z; xDot; yDot; zDot; J2]
%       - X_s: Station state arranged as follows:
%            [x_s; y_s; z_s; xDot_s; yDot_s; zDot_s]
%
%   Outputs:
%       - Htilde: Measurement partials matrix
%
%   By: Ian Faber, 01/24/2025
%

x = X(1);
y = X(2);
z = X(3);
xDot = X(4);
yDot = X(5);
zDot = X(6);

x_s = X_s(1);
y_s = X_s(2);
z_s = X_s(3);
xDot_s = X_s(4);
yDot_s = X_s(5);
zDot_s = X_s(6);

rho = norm(X(1:3) - X_s(1:3));

delRhoDelX =  (x-x_s)/rho;
delRhoDelY =  (y-y_s)/rho;
delRhoDelZ =  (z-z_s)/rho;

delRhoDotDelX = (rho^2*(xDot - xDot_s) - (x-x_s)*((x-x_s)*(xDot-xDot_s) + (y-
y_s)*(yDot-yDot_s) + (z-z_s)*(zDot-zDot_s)))/(rho^3);
delRhoDotDelY = (rho^2*(yDot - yDot_s) - (y-y_s)*((x-x_s)*(xDot-xDot_s) + (y-
y_s)*(yDot-yDot_s) + (z-z_s)*(zDot-zDot_s)))/(rho^3);
delRhoDotDelZ = (rho^2*(zDot - zDot_s) - (z-z_s)*((x-x_s)*(xDot-xDot_s) + (y-
y_s)*(yDot-yDot_s) + (z-z_s)*(zDot-zDot_s)))/(rho^3);

delRhoDotDelXDot = delRhoDelX;
delRhoDotDelYDot = delRhoDelY;
delRhoDotDelZDot = delRhoDelZ;

Htilde = [
            delRhoDelX, delRhoDelY, delRhoDelZ, zeros(1,3);
            delRhoDotDelX, delRhoDotDelY, delRhoDotDelZ, delRhoDotDelXDot,
delRhoDotDelYDot, delRhoDotDelZDot
         ];
```

```
end
```

```
Htilde_xsc =

  Columns 1 through 3

   0.411675406210137  -0.728665623223756   0.454315409890463
   0.106678749088795  -0.089553413359551  -0.191228169768138

  Columns 4 through 6

                   0                   0                   0
   0.411675406210137  -0.728665623223756   0.454315409890463
```

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c. Derive the partials of the measurement equations with respect to R_s. Do we also need partials with respect to $\vec{V}_s$?

We know that $\tilde{H}_{station} = \frac{\partial \vec{y}}{\partial \vec{R}_s}$.

While $\vec{V}_s$ is important for the measurements of the satellite, we only care about estimating the position of each ground station, as we can model their velocities precisely from the motion of Earth once we know where they are.

Since $\vec{y} = \begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix}$,

$$\frac{\partial \vec{y}}{\partial \vec{R}_s} = \begin{bmatrix} \partial\rho/\partial x_s & \partial\rho/\partial y_s & \partial\rho/\partial z_s \\ \partial\dot{\rho}/\partial x_s & \partial\dot{\rho}/\partial y_s & \partial\dot{\rho}/\partial z_s \end{bmatrix}$$

$$\frac{\partial \rho}{\partial x_s} = \frac{\partial}{\partial x_s}\left(\sqrt{(x-x_s)^2+(y-y_s)^2+(z-z_s)^2}\right)$$

$$= \frac{-\cancel{2}(x-x_s)}{\cancel{2}\sqrt{(x-x_s)^2+(y-y_s)^2+(z-z_s)^2}} = \frac{x_s - x}{\rho}$$

Similarly,

$$\frac{\partial \rho}{\partial y_s} = \frac{y_s - y}{\rho}, \quad \frac{\partial \rho}{\partial z_s} = \frac{z_s - z}{\rho}$$

then,

$$\frac{\partial \dot{\rho}}{\partial x_s} = \frac{\partial}{\partial x_s}\left(\frac{(x-x_s)(\dot{x}-\dot{x}_s) + (y-y_s)(\dot{y}-\dot{y}_s) + (z-z_s)(\dot{z}-\dot{z}_s)}{\rho} \right)$$

$$= \frac{\rho(-(\dot{x}-\dot{x}_s)) - ((x-x_s)(\dot{x}-\dot{x}_s) + (y-y_s)(\dot{y}-\dot{y}_s) + (z-z_s)(\dot{z}-\dot{z}_s))\frac{\partial}{\partial x_s}(\rho)}{\rho^2}$$

$$= \frac{\rho(\dot{x}_s-\dot{x}) - ((x-x_s)(\dot{x}-\dot{x}_s) + (y-y_s)(\dot{y}-\dot{y}_s) + (z-z_s)(\dot{z}-\dot{z}_s))\left(\frac{x_s-x}{\rho}\right)}{\rho^2}$$

$$= \frac{\rho^2(\dot{x}_s-\dot{x}) + (x-x_s)((x-x_s)(\dot{x}-\dot{x}_s) + (y-y_s)(\dot{y}-\dot{y}_s) + (z-z_s)(\dot{z}-\dot{z}_s))}{\rho^3}$$

similarly,

$$\frac{\partial \dot{\rho}}{\partial y_s} = \frac{\rho^2(\dot{y}_s-\dot{y}) + (y-y_s)((x-x_s)(\dot{x}-\dot{x}_s) + (y-y_s)(\dot{y}-\dot{y}_s) + (z-z_s)(\dot{z}-\dot{z}_s))}{\rho^3}$$

$$\frac{\partial \dot{\rho}}{\partial z_s} = \frac{\rho^2(\dot{z}_s-\dot{z}) + (z-z_s)((x-x_s)(\dot{x}-\dot{x}_s) + (y-y_s)(\dot{y}-\dot{y}_s) + (z-z_s)(\dot{z}-\dot{z}_s))}{\rho^3}$$

d. implement this matrix as a function.

See PDF for code, output matches!

```matlab
function Htilde = MeasurementPartials_RngRngRate_stat(X,X_s)
% Function that outputs the measurement partials matrix for orbital
% measurements using range and range rate for ground stations
%   Inputs:
%       - X: Spacecraft state arranged as follows:
%            [x; y; z; xDot; yDot; zDot; J2]
%       - X_s: Station state arranged as follows:
%            [x_s; y_s; z_s; xDot_s; yDot_s; zDot_s]
%
%   Outputs:
%       - Htilde: Measurement partials matrix
%
%   By: Ian Faber, 01/24/2025
%

x = X(1);
y = X(2);
z = X(3);
xDot = X(4);
yDot = X(5);
zDot = X(6);

x_s = X_s(1);
y_s = X_s(2);
z_s = X_s(3);
xDot_s = X_s(4);
yDot_s = X_s(5);
zDot_s = X_s(6);

rho = norm(X(1:3) - X_s(1:3));

delRhoDelXs =  (x_s-x)/rho;
delRhoDelYs =  (y_s-y)/rho;
delRhoDelZs =  (z_s-z)/rho;

delRhoDotDelXs = (rho^2*(xDot_s - xDot) + (x-x_s)*((x-x_s)*(xDot-xDot_s) +
(y-y_s)*(yDot-yDot_s) + (z-z_s)*(zDot-zDot_s)))/(rho^3);
delRhoDotDelYs = (rho^2*(yDot_s - yDot) + (y-y_s)*((x-x_s)*(xDot-xDot_s) +
(y-y_s)*(yDot-yDot_s) + (z-z_s)*(zDot-zDot_s)))/(rho^3);
delRhoDotDelZs = (rho^2*(zDot_s - zDot) + (z-z_s)*((x-x_s)*(xDot-xDot_s) +
(y-y_s)*(yDot-yDot_s) + (z-z_s)*(zDot-zDot_s)))/(rho^3);

Htilde = [
            delRhoDelXs, delRhoDelYs, delRhoDelZs;
            delRhoDotDelXs, delRhoDotDelYs, delRhoDotDelZs
         ];

end
```

```
Htilde_xstat =

  -0.411675406210137   0.728665623223756  -0.454315409890463
```

```
-0.106678749088795    0.089553413359551    0.191228169768138
```

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e. Imagine 2 stations are taking measurements, and 1 station has a bias in its range measurements. How should this bias be modeled in the measurement equations? What partials would be needed to estimate it, and how do they appear in $\tilde{H}$?

If there's a bias in range, then the measurements look as follows:

$$\vec{y}_{bias} = \begin{bmatrix} \rho_{bias} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} \rho + b \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$= \vec{y} + \vec{b}$$

We can then say that

$$\vec{y} = \vec{y}_{bias} - \vec{b} = \vec{y}^* + \vec{y} = \vec{y}^* + \tilde{H}x$$

or

$$\vec{y} = \vec{y}^* + \tilde{H}x + \vec{b},$$

where $\vec{x} = [\delta\vec{x}, \delta\vec{v}]^T$. If we introduce $\vec{x}_{bias} = [\delta\vec{x}, \delta\vec{v}, \vec{b}]^T$, then we can say that

$$\vec{y} = \vec{y}^* + [\tilde{H} \;\; I]\, \vec{x}_{bias},$$

where $\tilde{H}_{bias} = [\tilde{H} \;\; I]$.

4. In this problem, measurements will be simulated of the orbit in problem 2.

There are 3 stations on a spherical Earth at the following locations:

station	latitude (deg)	longitude (deg)
1	-35.398333	148.981944
2	40.427222	355.749444
3	35.247164	243.205

Each with elevation masks of 10°.

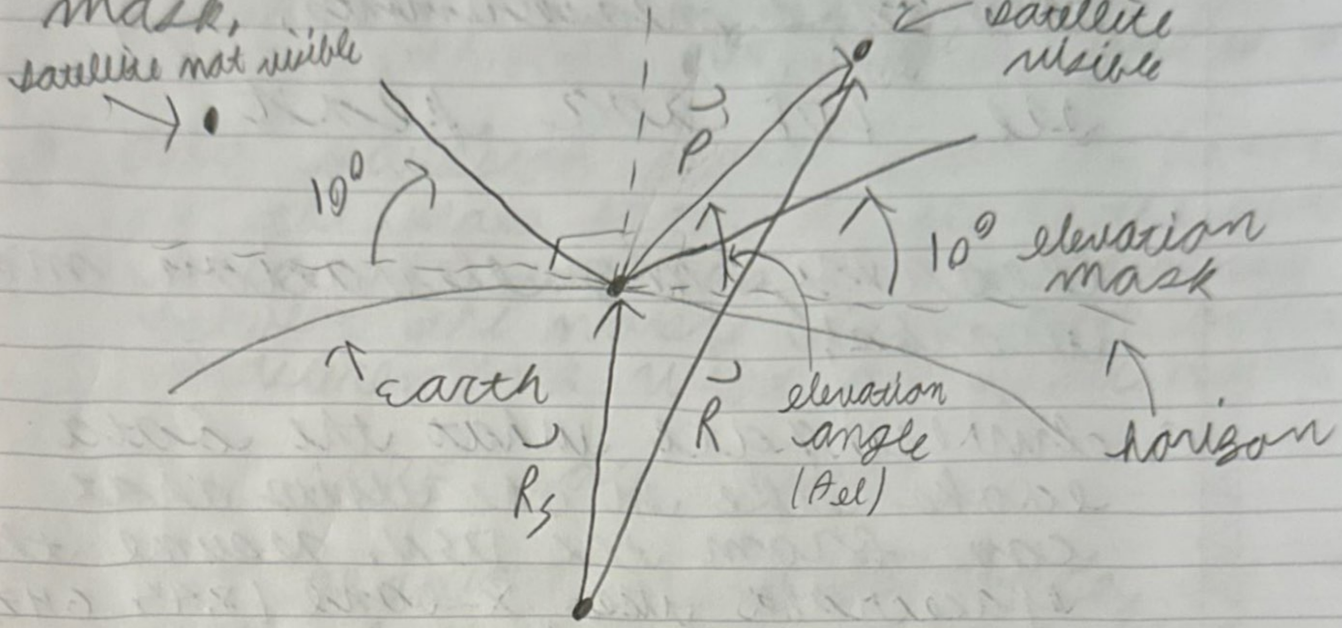
If Earth is modeled with a constant spin around the z-axis with period 24 hours + $\theta_0 = 122^\circ$, do the following:

a. Produce plots of range and range-rate over the same 15 orbit period as problem 2, generating measurements every 10 seconds. When is the first + last measurement, in seconds?

See PDF for plots

We have equations for $\rho + \dot{\rho}$, but we also need elevation angle.

For a measurement to be valid, the spacecraft needs to be visible by the station, i.e. within line of sight. This happens when the satellite lies within the station's elevation mask.



from vector math,

$$\vec{R_s} + \vec{P} = \vec{R} \rightarrow \vec{P} = \vec{R} - \vec{R_s},$$ as the measurement eqn. implies.

then, elevation angle is as follows;

$$\vec{P} \cdot \vec{R_s} = |\vec{P}||\vec{R_s}| \cos(90 - \theta_{el})$$

$$= |\vec{P}||\vec{R_s}| \sin(\theta_{el})$$

$$\rightarrow \theta_{el} = \sin^{-1}\left(\frac{\vec{P} \cdot \vec{R_s}}{|\vec{P}||\vec{R_s}|}\right)$$

First measurement: 0 seconds
Last measurement: 143,370 seconds

b. Plot the elevation angle of the spacecraft wrt the stations as they take measurements.

See PDF for plot

Clearly, my elevation mask works!

c. Investigate what the data looks like in the units that come from the DSN. Assume the spacecraft uses X-band (8.44 GHz) using the equations below:

$$F_{shift} = -\frac{2\dot{\rho}}{c} F_{T,ref}$$

$$RU = \frac{221}{749} \frac{\tau_{delay}}{2} F_{up} = \frac{221}{749} \frac{\rho}{c} F_{T,ref}$$

Reproduce your range + range-rate plots in terms of RU and F_{shift}. Does this change the data's appearance? What if $F_{T,ref}$ isn't constant?

See PDF for plot

ASEN 6080 HW 1 Problem 4

By: Ian Faber

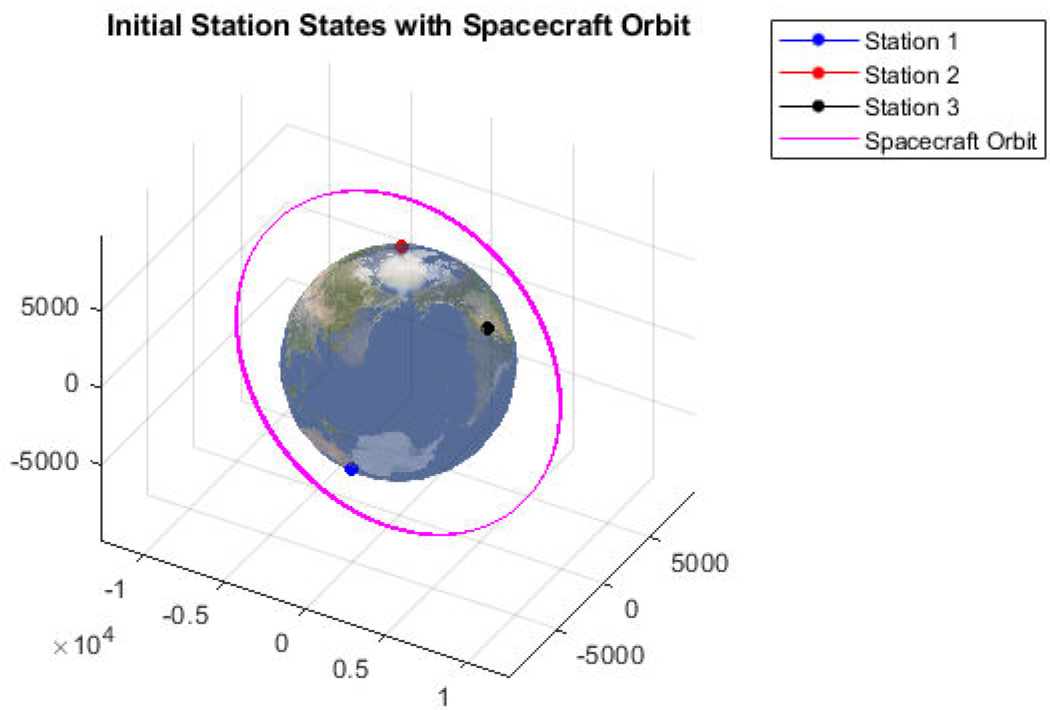
[redacted]

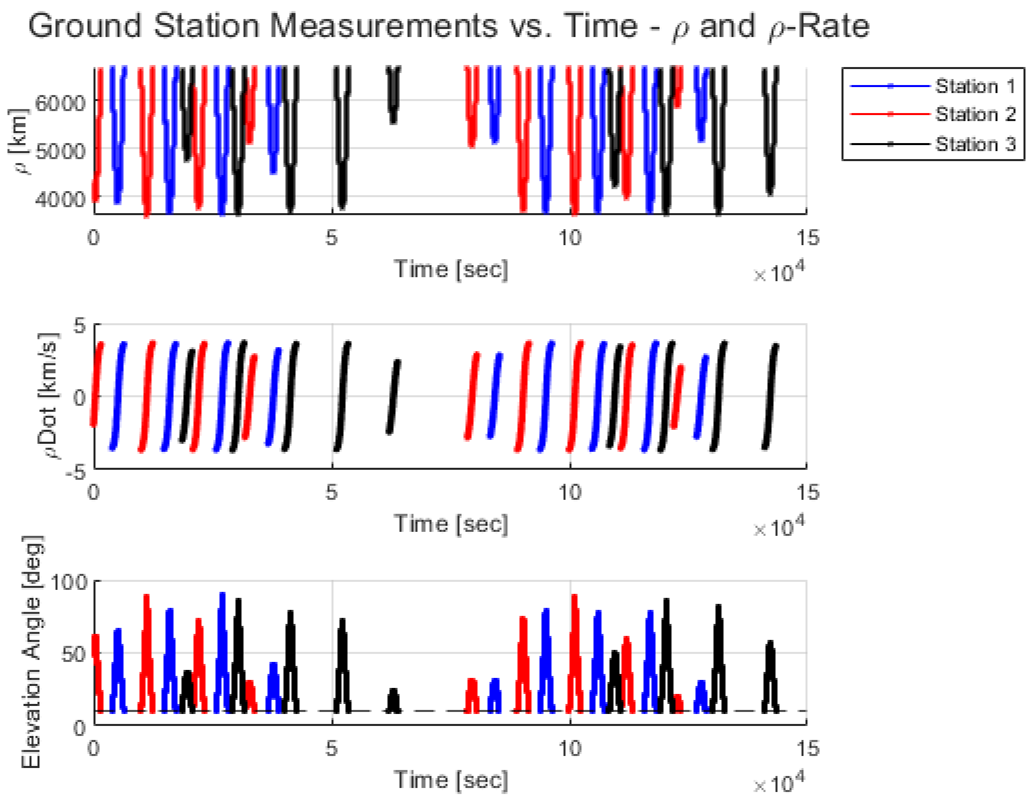
[redacted]

[redacted]

Part 4a and 4b

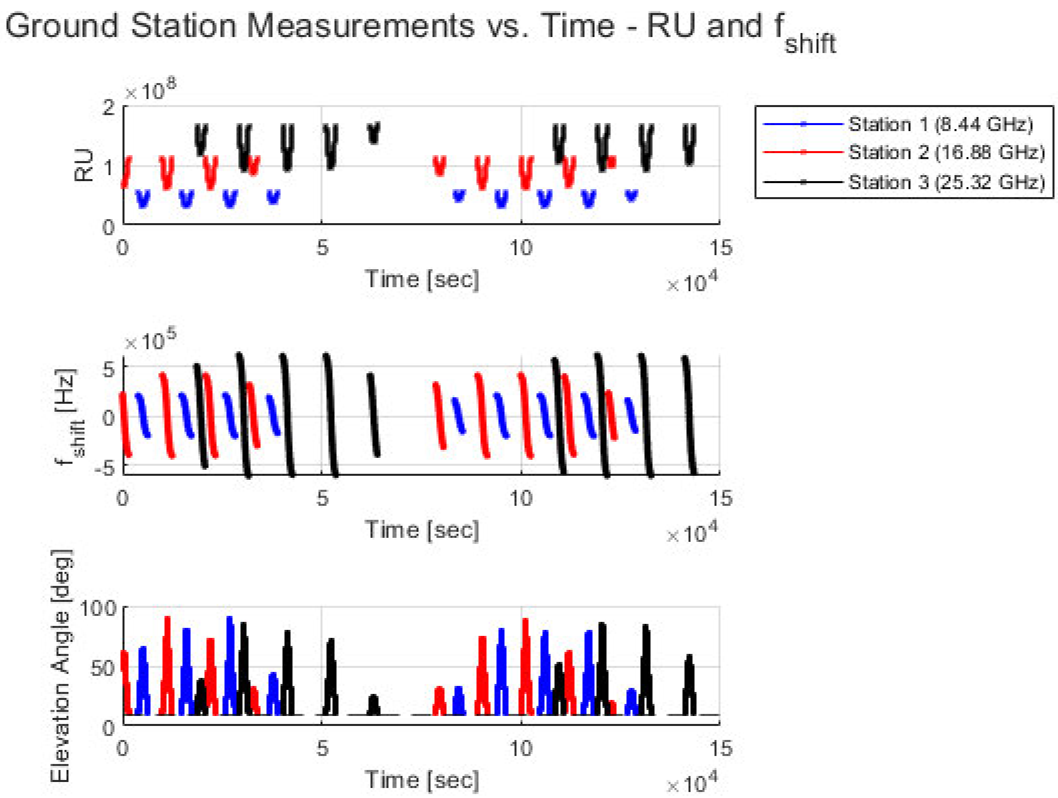
[redacted]





Part 4c

[redacted]



Ground Station Measurements vs. Time - RU and f_shift
RU
×10^8
Time [sec]
×10^4
f_shift [Hz]
×10^5
Elevation Angle [deg]
Station 1 (8.44 GHz)
Station 2 (16.88 GHz)
Station 3 (25.32 GHz)

switching to RR and doppler shift, the data looks identical. However, if $F_{T,ref}$ changes then the data gets shifted up or down based on the scale of $F_{T,ref}$. The visibility remains the same, though. also, F_{shift} is flipped rel. to $\dot{\rho}$.

d. add Gaussian noise with $\sigma = 0.5$ mm/s to the plots from a. plot the new data, as well as the difference between the noisy data and original measurements. What do you notice?

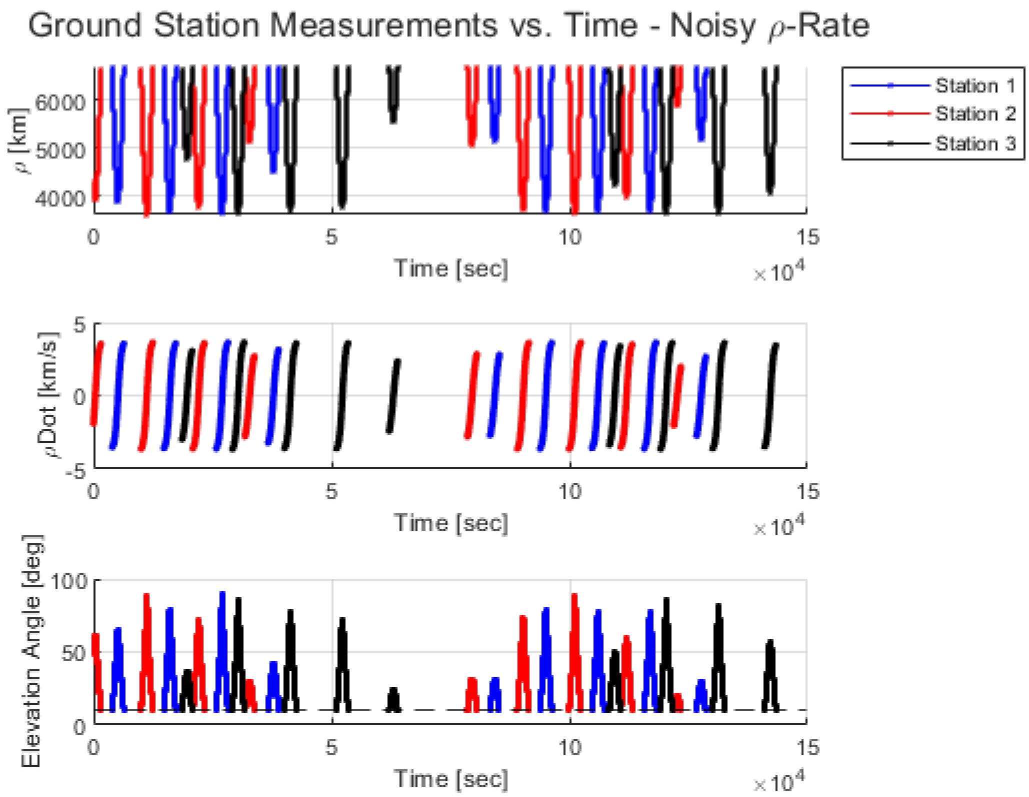
See PDF for plots

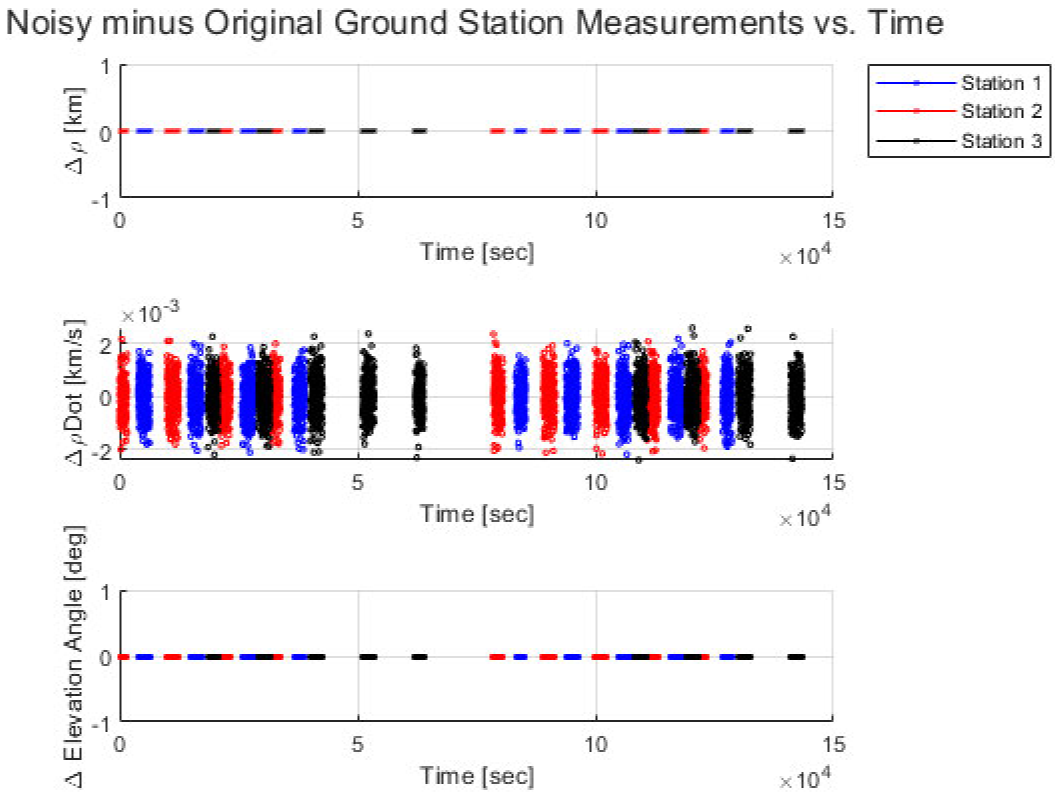
In plotting the difference, I noticed that the result is just the noise itself, centered at 0 km/s with $\sigma = 0.5$ mm/s.

e. see what happens to the measurements in different orbit regimes

At the provided GEO scenario, it appears that range stays roughly around the semi-major axis, while range rate stays relatively small. Further, station 1 never sees the satellite, while stations 2/3 can almost always see it when it is above the equator,

Part 4d


Ground Station Measurements vs. Time - Noisy ρ-Rate
ρ [km]
6000
5000
4000
0
5
10
15
Time [sec]
×10^4
Station 1
Station 2
Station 3
ρDot [km/s]
5
0
-5
Elevation Angle [deg]
100
50
0



[redacted]

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meanwhile, for the provided hyperbolic orbit, the range measurements seem almost continuous and the spacecraft is always visible — sometimes by multiple stations.

See videos for proof!