STATS 300A: Theory of Statistics I	**Autumn 2016/17**

Lecture 16: November 29

Lecturer: Joseph Romano *Scribes: Kenneth Tay*

16.1 Examples of UMPU Tests

16.1.1 Testing independence in a bivariate normal family

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be iid, $X_i \sim \mathcal{N}(\xi, \sigma^2)$, $Y_i \sim \mathcal{N}(\eta, \tau^2)$, and the (X_i, Y_i)'s having joint density

$$\propto \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{1}{\sigma^2}\sum_{i=1}^n (X_i-\xi)^2 - \frac{2\rho}{\sigma\tau}\sum_{i=1}^n (X_i-\xi)(Y_i-\eta) + \frac{1}{\tau^2}\sum_{i=1}^n (Y_i-\eta)^2\right]\right\},$$

where ρ is the correlation between the X_i's and Y_i's. Assume that all 5 parameters are unknown, and that we are testing $H_0 : \rho \le 0$ vs. $H_1 : \rho > 0$.

In canonical form (introduced in the previous lecture), we can write

$$\begin{aligned}
&U = \sum X_i Y_i, && \theta = \frac{2\rho}{\sigma\tau 2(1-\rho^2)}, && && \\
&\vartheta_1 = \xi, && \vartheta_2 = \eta, && \vartheta_3 = \sigma^2, && \vartheta_4 = \tau^2, \\
&T_1 = \sum X_i, && T_2 = \sum Y_i, && T_3 = \sum X_i^2, && T_4 = \sum Y_i^2.
\end{aligned}$$

Thus, a UMPU test exists, and it rejects if $U > C(T_1, T_2, T_3, T_4)$, where

$$P_{\rho=0}\left\{U > C(T_1, T_2, T_3, T_4) \mid T_1, \dots, T_4\right\} = \alpha.$$

Equivalently, if we let $\hat{\rho}$ denote the sample correlation, the UMPU test rejects if

$$\hat{\rho} := \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum (X_i - \bar{X})^2 \sum (X_i - \bar{X})^2}} > C'(T_1, \dots, T_4).$$

Note that for the reduced family with $\rho = 0$, the distribution of $\hat{\rho}$ does not change as $T_1, \dots, T_4$ vary. Hence, $\hat{\rho}$ is ancillary. Since $T_1, \dots, T_4$ are complete sufficient for the reduced family $\rho = 0$, by Basu's Theorem $\hat{\rho}$ is independent of $(T_1, \dots, T_4)$ under $\rho = 0$.

Thus, $C'(T_1, \dots, T_4)$ does not depend on $(T_1, \dots, T_4)$, i.e. the UMPU test rejects if $\hat{\rho} > \tilde{c}$, where $\tilde{c}$ is determined by $P_{\rho=0}\{\hat{\rho} > \tilde{c}\} = \alpha$.

Under $\rho = 0$, it is a fact that

$$\frac{\sqrt{n-2}\hat{\rho}}{\sqrt{1-\hat{\rho}^2}} \sim t_{n-2}.$$

Hence, the UMPU test rejects if $\dfrac{\sqrt{n-2}\hat{\rho}}{\sqrt{1-\hat{\rho}^2}} > t_{n-2}(1-\alpha)$.

16.1.2 One-sample randomization tests

Let $X_1, \dots, X_n$ iid, $X_i \sim f(x-\theta)$, where f is assumed to have a density and $f(\cdot)$ is assumed to be symmetric about 0. (No other assumptions on f.) We want to test $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$.

The unbiasedness of a test φ implies

$$\int \cdots \int \varphi(x_1, \dots, x_n) \prod_{i=1}^{n} f(x_i) dx_i = \alpha \quad \text{for all } f \text{ symmetric about 0.} \tag{16.1}$$

Let $\mathcal{F}_0 := \{\text{all densities about 0}\}$. Then $T :=$ set of ordered absolute values of X_i is complete sufficient for $\mathcal{F}_0$. Thus, any test satisfying Equation 16.1 must satisfy

$$\mathbb{E}(\varphi \mid T) = \alpha. \tag{16.2}$$

Under H_0, for any given T, there are 2^n equally likely datasets which give rise to that T. For any test statistic, we can calculate it for these 2^n datasets to obtain a null reference distribution. We then reject if the test statistic for the data was large relative to this reference distribution.

There is no one test statistic that results in a UMP test for the whole family $\mathcal{F}_0$. For a particular subfamily, how do we go about picking a test statistic?

Suppose we want to maximize power when $f = \mathcal{N}(\mu, \sigma^2)$. The conditional probability density function at $(x_1, \dots, x_n)$, where $(x_1, \dots, x_n)$ has absolute values $\{t_1, \dots, t_n\}$, is

$$\begin{aligned}
&\propto \prod_{i=1}^{n} f_\mu(x_i) \\
&= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\} \\
&\propto \exp\left\{-\frac{1}{2\sigma^2}\sum X_i^2 + \frac{\mu}{\sigma^2}\sum X_i - \frac{n\mu^2}{\sigma^2}.\right\}
\end{aligned}$$

Conditionally, $\sum X_i^2$ is constant. Hence, the UMPU test is equivalent to rejecting for large values of $\sum X_i$. Since the t-statistic

$$\hat{t} = \frac{\sqrt{n}\bar{X}}{\sqrt{(\sum X_i^2 - n\bar{X}^2)/(n-1)}}$$

is an increasing function of $barX$, the UMPU test is equivalent to rejecting for large values of the t-statistic.

16.2 Invariant Tests

We try to motivate invariant tests through 2 examples.

16.2.1 Example: Normal setting

Let $X_1, \dots, X_n$ be mutually independent, with $X_i \sim \mathcal{N}(\theta_i, 1)$. Testing $H_0 : \theta_1 = \cdots = \theta_n = 0$ vs. $H_1 :$ not all 0.

Consider our data as a vector $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$. Let $Y = OX$, where O is an orthogonal matrix.

Geometrically speaking, Y is simply a rotation of the data X. Hence, it is reasonable that a test's result for data X should be the same as that for data Y. In this context, we say that a test φ is **invariant** if $\varphi(X) = \varphi(OX)$ for any orthogonal matrix O.

Now, think of data X as living in n-dimensional space. Note that the invariance condition means that φ must give the same result for 2 datasets if their distances from the origin are the same. This implies that for a test φ to be invariant, it must be a function of $T := \sum_{i=1}^{n} X_i^2$.

In general, $\sum_{i=1}^{n} X_i^2$ has a non-central chi-squared distribution:

$$\sum_{i=1}^{n} X_i^2 \sim \chi_n^2 \left(\sum_{i=1}^{n} \theta_i^2 \right).$$

Let $\psi^2 = \sum_{i=1}^{n} \theta_i^2$. Then the original testing setting is equivalent to testing $\psi^2 = 0$ vs. $\psi^2 > 0$ based on T. Note that the family of distributions of T has monotone likelihood ratio in T, hence the UMPU test rejects if $T > c_n(1-\alpha)$, where $c_n(1-\alpha)$ is the $(1-\alpha)$-quantile of χ_n^2.

16.2.2 Example: Non-parametric setting with symmetry

Let $X_1, \ldots X_n$ be iid on $(0,1)$. Testing $H_0 : X_i \sim \text{Unif}(0,1)$ vs. $H_1 : X_i$'s have density $f(x)$ or $f(1-x)$, where f is fixed.

Let $Y_i = 1 - X_i$. Then testing based on the Y's is the same problem as testing based on the X's, so an invariant test φ should give the same result, i.e.

$$\varphi(x_1, \ldots, x_n) = \varphi(1 - x_1, \ldots, 1 - x_n).$$

Note now that if φ is invariant, the power of φ at $f(x)$ or $f(1-x)$ is the same, which implies that it is the same at $\frac{f(x) + f(1-x)}{2}$ as well.

But now we are testing a simple null vs. a simple alternative! We can use the Neyman-Pearson Lemma to obtain an invariant test which rejects for large values of $\prod_{i=1}^{n} \frac{f(X_i) + f(1-X_i)}{2}$.

16.2.3 General Set-up

Assume we have data $X \sim P_\theta$, $\theta \in \Omega$. Let $\mathcal{S}$ be the sample space for X. Testing $H_0 : \theta \in \Omega_0$ vs. $H_1 : \theta \in \Omega_1$.

Let's assume that there is a group of 1-to-1 transformations G which act on the data such that for any $g \in G$,

$$X \sim P_\theta \quad \Rightarrow \quad gX \sim P_{\theta'}$$

for some $\theta' \in \Omega$. If this is the case, we write $\theta' = \bar{g}\theta$, and we say that g induces a transformation on the parameter space.

We also assume that Ω_0 and Ω_1 are *preserved* in the sense that

$$\bar{g}\theta \in \Omega_0 \text{ iff } \theta \in \Omega_0,$$
$$\bar{g}\theta \in \Omega_1 \text{ iff } \theta \in \Omega_1.$$

We write $\bar{g}\Omega_i = \Omega_i$.

Definition 16.1 *A test φ is* ***invariant*** *if $\varphi(x) = \varphi(gx)$ for all $g \in G$.*