Lecture 13: November 10

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13.1 Families with Monotone Likelihood Ratio (MLR)

Definition 13.1 *Given a family of densities p_θ with $\theta \in \mathbb{R}$, the family has* ***monotone likelihood ratio (MLR)*** *in $T(x)$ if for all $\theta < \theta'$, $\dfrac{p_{\theta'}(x)}{p_\theta(x)}$ is a non-decreasing function of $T(x)$.*

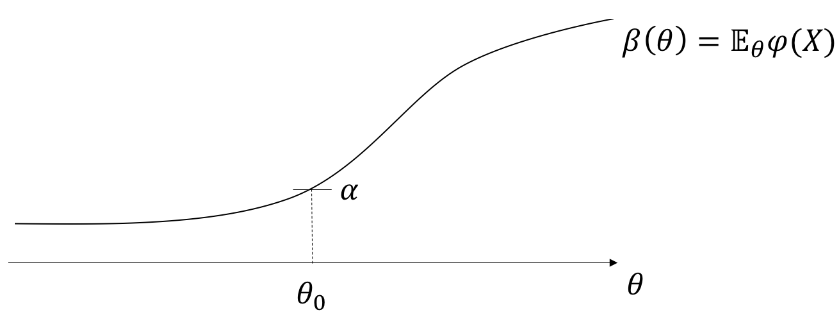
Some points we can make about families with MLR:

1. Rejecting for large values of likelihood ratio is equivalent to rejecting for large values of T.
2. For testing $\theta = \theta_0$ vs. $\theta > \theta_0$, there exists a UMP level α test of the form
$$\varphi(X) = \begin{cases} 1 & \text{if } T(X) > c, \\ \gamma & \text{if } T(X) = c, \\ 0 & \text{if } T(X) < c. \end{cases}$$
3. Class of examples with MLR: 1-parameter exponential families, i.e. families with densities of the form
$$p_\theta(x) \propto e^{\theta T(x)} \cdot h(x).$$

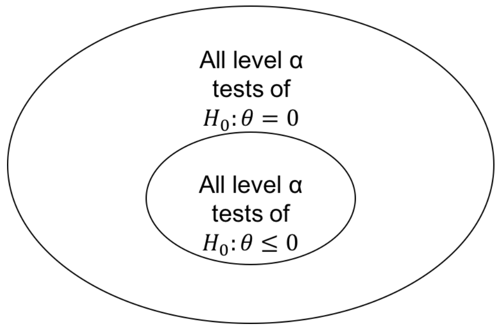
13.1.1 Example: Normal setting

Consider the normal setting from last lecture: $X_1, \ldots, X_n$ be i.i.d., $X_i \sim \mathcal{N}(\theta, \sigma^2)$ with σ known. For $H_0 : \theta = \theta_0$, $H_1 : \theta > \theta_0$, we showed that there exists a UMP level α test, φ^*.

Now consider the setting $H_0 : \theta \leq \theta_0$, $H_1 : \theta > \theta_0$. The power of φ^*, as a function of θ, has a graph that looks like this:



Because it's a monotone function, it means that φ^* is still a level α test for $H_0 : \theta \leq \theta_0$. Consider the hierarchy of tests:



Last lecture, we showed that with $H_1 : \theta > \theta_0$, φ^* is a UMP level α test for the bigger family. By analysis of the power function, φ^* is an element of the smaller family of tests. Hence, it is UMP level α test for $H_0 : \theta \leq \theta_0$, $H_1 : \theta > \theta_0$.

Another way to see this is as follows: If φ^* is not UMP level α for $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$, then there exists some level α test φ' such that $\mathbb{E}_\theta \varphi^*(X) < \mathbb{E}_\theta \varphi'(X)$ for some $\theta > \theta_0$. But if φ' is level α for the composite H_0, then it must be level α for the simple H_0 as well. But we showed last lecture that the MP level α test for simple H_0 vs. composite H_1 is φ^*. Contradiction!

13.1.2 Generalizing to MLR families

How can we generalize the example above (of finding a UMP test for $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$) to MLR families?

To use the procedure in the previous example, all we need is for the power function of the MP level α test

$$\varphi(X) = \begin{cases} 1 & \text{if } T(X) > c, \\ \gamma & \text{if } T(X) = c, \\ 0 & \text{if } T(X) < c. \end{cases}$$

to be non-decreasing in θ.

For MLR families, this is indeed the case (in fact, it is strictly increasing)! As before, define the power of φ by $\beta(\theta) := \mathbb{E}_\theta \varphi(X)$. Fix $\theta_1 < \theta_2$.

Think of φ as a test of $H_0 : \theta = \theta_1$ vs. $H_1 : \theta = \theta_2$ at level $\alpha' = \mathbb{E}_{\theta_1} \varphi(X)$. Last time, we proved that the power of an MP level α test is $> \alpha$. Hence,

$$\begin{aligned} \mathbb{E}_{\theta_2} \varphi(X) &> \alpha' = \mathbb{E}_{\theta_1} \varphi(X), \\ \beta(\theta_2) &> \beta(\theta_1), \end{aligned}$$

as required.

13.1.2.1 Example: Double exponential distribution

Let X have density $\frac{1}{2} \exp[-|x - \theta|]$. (Case of $n = 1$, i.e. only one observation.)

For the setting $H_0 : \theta = 0$, $H_1 : \theta > 0$, we claim that this family has MLR. Since, for $\theta > 0$,

$$|x| - |x - \theta| = \begin{cases} -\theta & \text{if } x < 0, \\ 2x - \theta & \text{if } 0 \le x < \theta, \\ \theta & \text{if } \theta \le x, \end{cases}$$

$|x| - |x - \theta|$ is non-decreasing in x, and so

$$\text{likelihood ratio} = \frac{e^{-|x-\theta|}}{e^{-|x|}} = \exp[|x| - |x - \theta|]$$

is also non-decreasing in x.

13.1.2.2 Example: Cauchy distribution

Let X have density $\frac{1}{\pi[1+(x-\theta)^2]}$.

In this case, no UMP test exists.

13.2 How to Find a UMP Test

- **Simple H_0 vs. simple H_1.**

  This case is completely solved using the Neyman-Pearson Lemma.

- **Simple H_0 vs. composite H_1.**

  Say we have $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \in \omega$, where ω is some subset of the parameter space Ω not containing θ_0.

  Fix $\theta' \in \omega$ and use the Neyman-Pearson Lemma to determine what a MP level α test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta'$ looks like. If there exists such a test that does not depend on θ', then it is UMP.

- **Composite H_0 vs. simple H_1.**

  This will be the subject of the next chapter.

- **Composite H_0 vs. composite H_1.**

  Say we have $H_0 : \theta \in \omega_0$ vs. $H_1 : \theta \in \omega_1$.

  Fix $\theta' \in \omega_1$ and determine the UMP test for $H_0 : \theta \in \omega_0$ vs. $H_1 : \theta = \theta'$. If this test does not depend on θ', then it is UMP for the original setting.

13.3 Composite H_0 vs. Simple H_1

Assume that $H_0 : X \sim f_\theta, \theta \in \omega$ and $H_1 : X \sim g$, where the f_θ's and g are densities w.r.t. some dominating measure μ. To find an MP level α test for this setting.

The intuition is to reduce this problem to the setting of simple H_0 vs. simple H_1. We do this by introducing a **mixture density**:

$$h_\Lambda(x) := \int_{\theta \in \omega} f_\theta(x) d\Lambda(\theta),$$

where Λ is some prior for θ.

Introduce a new hypothesis $H_\Lambda : X \sim h_\Lambda$. For testing H_Λ vs. H_1, we can use the Neyman-Pearson Lemma to get an MP level α test, φ_Λ, which rejects for large $\frac{g(x)}{h_\Lambda(x)}$.

Definition 13.2 *Let β_Λ be the power of the MP level α test φ_Λ when testing H_Λ vs. H_1. We say that Λ is **least favorable** if for any other Λ', $\beta_\Lambda \leq \beta_{\Lambda'}$.*

Theorem 13.3 *Let φ_Λ be the MP level α test for the H_Λ vs. simple H_1 setting.*

Suppose Λ is such that φ_Λ is level α for the composite H_0 vs. simple H_1 setting, i.e. $\sup_{\theta \in \omega} \mathbb{E}_\theta \varphi_\Lambda(X) \leq \alpha$. Then

1. *φ_Λ is MP for testing composite H_0 vs. simple H_1, and*
2. *Λ is least favorable.*

Proof: We will only prove 1.

Let φ^* be any other level α test for the orignal setting (i.e. composite H_0 vs. simple H_1), i.e.

$$\mathbb{E}_\theta \varphi^*(X) \leq \alpha \quad \text{for all } \theta \in \omega.$$

Then

$$\begin{aligned}
\alpha &\geq \int \left(\int \varphi^*(x) f_\theta(x) \mu(dx) \right) d\Lambda(\theta) \\
&= \int \varphi^*(x) \left(\int f_\theta(x) d\Lambda(\theta) \right) \mu(dx) \\
&= \int \varphi^*(x) h_\Lambda(x) \mu(dx),
\end{aligned}$$

i.e. φ^* is a level α test in the H_Λ vs. H_1 setting. Hence, by Neyman-Pearson, φ_Λ's power is larger than that of φ^*. ■

How do we utilize this theorem? From the H_Λ vs. H_1 setting, we have

$$\alpha = \int \varphi_\Lambda(x) h_\Lambda(x) \mu(dx) = \int \left[\mathbb{E}_\theta \varphi_\Lambda(x) \right] d\Lambda(\theta).$$

To satisfy the conditions of the theorem, we need $\mathbb{E}_\theta \varphi_\Lambda(X) \leq \alpha$ for all $\theta \in \omega$. Hence, we need to specify Λ such that

$$\Lambda\{\theta \in \omega : \mathbb{E}_\theta \varphi_\Lambda(X) = \alpha\} = 1.$$

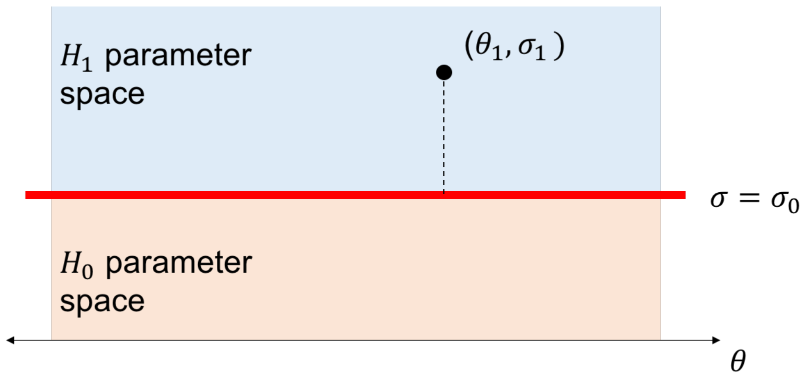
13.3.1 Example: 1-sided normal variance (part 1)

Let $X_1, \ldots, X_n$ iid, $X_i \sim \mathcal{N}(\theta, \sigma^2)$ with both θ and σ^2 unknown. Test $H_0 : \sigma \leq \sigma_0$ vs. $H_1 : \sigma > \sigma_0$ for some pre-specified σ_0.

First, pick a specific value in the H_1 parameter space, i.e. (θ_1, σ_1) with $\sigma_1 > \sigma_0$.

Next, restriction our attention to sufficient statistics (we showed last lecture that we can only do that). Let $(Y, U) = (\bar{X}, \sum(X_i - \bar{X})^2$. We know that Y and U are independent, and $U \sim \chi^2_{n-1}$.

Now we need to find an appropriate Λ. A first guess may be for Λ to concentrate all its mass at (θ_1, σ_0). It turns out that we need Λ to be supported on the whole line $\sigma = \sigma_0$.



Consider the joint density of (Y, U). Under H_Λ, it is

$$C_0 u^{(n-3)/2} \exp\left[-\frac{u}{2\sigma_0^2}\right] \int \exp\left[-\frac{n}{2\sigma_0^2}(y-\theta)^2\right] d\Lambda(\theta),$$

for some constant C_0, and under the fixed alternative, it is

$$C_1 u^{(n-3)/2} \exp\left[-\frac{u}{2\sigma_1^2}\right] \exp\left[-\frac{n}{2\sigma_1^2}(y-\theta_1)^2\right],$$

for some constant C_1. Note that Λ only affects the distribution of Y. Hence, the least favorable Λ should be such that the density of Y under h_Λ is as close as possible as the density of Y under the alternative $\mathcal{N}(\theta_1, \sigma_1^2)$.

In this case, it turns out that we can make the density of Y exactly the same in both settings! Under H_Λ,

$$Y \sim \mathcal{N}\left(0, \frac{\sigma_0^2}{n}\right) * \Lambda.$$

If we take $\Lambda = \mathcal{N}\left(\theta_1, \frac{\sigma_1^2 - \sigma_0^2}{n}\right)$, then $Y \sim \mathcal{N}\left(\theta_1, \frac{\sigma_1^2}{n}\right)$, which is its distribution under the fixed alternative.

For this choice of Λ, the likelihood ratio becomes

$$\exp\left[-\frac{u}{2\sigma_1^2} + \frac{u}{2\sigma_0^2}\right]$$

which is equivalent to u. Therefore, the MP level α test for H_Λ vs. fixed alternative (θ_1, σ_1^2) is Reject if U is large, or more precisely, reject when

$$\frac{U}{\sigma_0^2} > C_{n-1}(1-\alpha),$$

the $(1-\alpha)$ quantile of the χ^2_{n-1} distribution.

Let us check that this test is level α for the original composite null. For $\sigma \leq \sigma_0$,

$$\begin{aligned}
\mathbb{E}_{\theta,\sigma}\varphi_\Lambda(X) &= P_{\theta,\sigma}\left\{\frac{\sum(X_i-\bar{X})^2}{\sigma_0^2} > C_{n-1}(1-\alpha)\right\} \\
&= P_{\theta,\sigma}\left\{\frac{\sum(X_i-\bar{X})^2}{\sigma^2} > \frac{\sigma_0^2}{\sigma^2}\cdot C_{n-1}(1-\alpha)\right\} \\
&\leq P_{\theta,\sigma}\left\{\frac{\sum(X_i-\bar{X})^2}{\sigma^2} > C_{n-1}(1-\alpha)\right\} \\
&= \alpha.
\end{aligned}$$

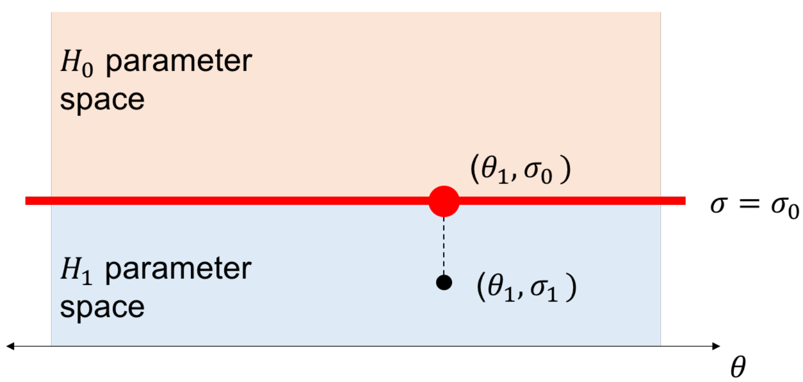
We can now make 3 conclusions:

1. The test level α is MP for composite H_0 vs. fixed alternative.
2. Λ is least favorable.
3. It is UMP for the composite H_1 because the test does not depend on the fixed alternative (θ_1, σ_1^2).

13.3.2 Example: Example: 1-sided normal variance (part 2)

Let's say that we are testing $H_0 : \sigma \geq \sigma_0$ vs. $H_1 : \sigma < \sigma_0$ instead.

1. In this case, the Λ which is least favorable puts all of its mass at $(\theta 1, \sigma_0)$.



2. However, the test depends on the fixed alternative; hence, no UMP test exists.