STATS 310A: Theory of Probability I	**Autumn 2016/17**

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10.1 Integrals Continued

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $f : \Omega \to \mathbb{R}$ a measurable function. We defined the integral

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu$$

if the RHS is not $\infty - \infty$, where, for non-negative functions g,

$$\int g d\mu := \sup \sum \nu_i \mu(A_i)$$

where the sup is taken over all decompositions $\{A_i\}$ of Ω, $\nu_i = \inf\limits_{\omega \in A_i} g(\omega)$.

We proved the following proposition last lecture:

Proposition 10.1 *For non-negative measurable f and g:*

1. *If f is simple, i.e. $f = \sum\limits_{i=1}^n x_i \delta_{A_i}$, then $\int f d\mu = \sum\limits_{i=1}^n x_i \mu(A_i)$.*
2. *(Monotonicity) If $f \leq g$, then $\int f d\mu \leq \int g d\mu$.*
3. *(Monotone Convergence Theorem) If $f_n(\omega) \nearrow f(\omega)$ for all ω, then*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu = \int \lim_{n\to\infty} f_n d\mu.$$

4. *(Linearity) For real numbers α and β,*

$$\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta + \int g d\mu.$$

Note that all of the above hold if the hypotheses hold μ-a.e. For example, for property 2 above: if $0 \leq f \leq g$ μ-a.s., then $\int f d\mu \leq \int g d\mu$.

Proof: We prove the a.e. version of property 2. (The a.e. versions of the other properties can be proven in a similar manner.)

Let $G = \{\omega : f(\omega) \leq g(\omega)\}$. Then $\mu(G^c) = 0$, and for any partition $\{A_i\}$,

$$\begin{aligned}\sum_{i=1}^{n} \inf_{\omega \in A_i} f(\omega)\mu(A_i) &= \sum_{i=1}^{n} \inf_{\omega \in A_i} f(\omega)\mu(A_i \cap G) \\ &\leq \sum_{i=1}^{n} \inf_{\omega \in A_i \cap G} f(\omega)\mu(A_i \cap G) \\ &\leq \sum_{i=1}^{n} \inf_{\omega \in A_i \cap G} g(\omega)\mu(A_i \cap G) \\ &\leq \int g d\mu.\end{aligned}$$

Since this holds for all partitions $\{A_i\}$, we can take sup on the LHS to obtain the desired result.

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Important note: We can't always switch limits with integrals, i.e. if $f_n(\omega) \to f(\omega)$ for all ω, it does not follow that $\int f_n d\mu \to \int f d\mu$. E.g. take $(\Omega, \mathcal{F}, \mu) = ([0,1], \text{Borel sets}, \lambda)$. Let $f_n(\omega) = n^2 \delta_{(0,\frac{1}{n})}$. Then $f_n(\omega) \to 0$ for all ω, but $\int f_n d\mu = n \nearrow \infty$.

Theorem 10.2 (Fatou's Lemma) *Let $\{f_n\}$ be any sequence of non-negative measurable functions. Then*

$$\int \liminf f_n d\mu \leq \liminf \int f_n d\mu.$$

Proof: Let $g_n = \inf_{k \geq n} f_n$. Then $g_n \leq f_n$, so

$$\int g_n d\mu \leq \int f_n d\mu.$$

By definition of $\liminf$, g_n increases to $\liminf f_n$. Thus, by taking $\liminf$ on both sides of the above inequality and using the Monotone Convergence Theorem, we have

$$\int \liminf f_n d\mu = \int \lim g_n d\mu = \liminf \int g_n d\mu \leq \liminf \int f_n d\mu.$$

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Some remarks on Fatou's Lemma:

1. The f_n's can be any collection of non-negative measurable functions.
2. A case to demonstrate Fatou's Lemma: Take $(\Omega, \mathcal{F}, \mu) = ([0,1], \text{Borel sets}, \lambda)$. Let

$$f_n = \begin{cases} \delta_{(0,1/2)} & \text{if } n \text{ even,} \\ \delta_{(1/2,1)} & \text{if } n \text{ odd.} \end{cases}$$

   In this case, $\liminf f_n = 0$ but $\int f_n d\lambda = \frac{1}{2}$, so Fatou's Lemma holds.
3. The assumption of f being non-negative cannot be dropped. Take $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \text{Borel sets}, \lambda)$. Let $f_n = -\frac{1}{n}\delta_{(n,2n)}$. Then $f_n \to 0$ pointwise but $\int f_n d\mu = -1$ for all n. Fatou's Lemma does not hold in this case.

Theorem 10.3 (Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ a.s. and there exists an integrable function $g \geq 0$ such that $|f_n| \leq g$ a.s. Then f is measurable and integrable, and*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Proof: We have $f_n^+(\omega) \leq g(\omega)$, $f_n^-(\omega) \leq g(\omega)$, so they are both integrable. Let

$$f^* = \limsup f_n, \quad f_* = \liminf f_n.$$

We also have $f^* \leq g$ and $f_* \leq g$, and so they are integrable. In addition, $g + f_* \geq 0$ and $g - f^* \geq 0$. By the Monotone Convergence Theorem and Fatou's Lemma,

$$\begin{aligned}
\int (g + f_*) d\mu &= \int \liminf (g + f_n) d\mu \\
&\leq \liminf \int (g + f_n) d\mu \\
&= \int g d\mu + \liminf \int f_n d\mu, \\
\int f_* d\mu &\leq \liminf \int f_n d\mu.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int (g - f^*) d\mu &= \int \liminf (g - f_n) d\mu \\
&\leq \liminf \int (g - f_n) d\mu \\
&= \int g - \limsup \int f_n d\mu, \\
\limsup \int f_n d\mu &\leq \int f^* d\mu.
\end{aligned}$$

So

$$\int \liminf f_n d\mu \leq \liminf \int f_n d\mu \leq \limsup \int f_n d\mu \leq \int \limsup f_n d\mu.$$

But since $f_n \to f$ a.s., the inequalities above are actually equalities, and so

$$\int f d\mu = \lim_n \int f_n d\mu.$$

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10.1.1 Applications of the Convergence Theorems

10.1.1.1 Proving f is finite a.s.

On $[0, 1]$, let $\{r_i\}_{i=1}^\infty$ be the rational numbers ordered in the "standard" way, e.g. $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \ldots$. Define

$$f(x) = \sum_{i=1}^\infty \frac{1}{i^2 |r_i - x|^{\frac{1}{2}}}.$$

Let $f_n(x) = \sum_{i=1}^{n} \frac{1}{i^2 |r_i - x|^{\frac{1}{2}}}$. We have

$$\begin{aligned}\int_0^1 f_n(x)dx &= \sum_{i=1}^{n} \frac{1}{i^2} \int_0^1 \frac{1}{|r_i - x|^2} dx \\ &\leq \sum \frac{c}{i^2}\end{aligned}$$

for some constant c, i.e. every f_n is integrable. Note that $f_n(x) \nearrow f(x)$. By Fatou's Lemma,

$$\int_0^1 f(x)dx \leq \liminf \sum_{i=1}^{n} \frac{c}{i^2} < \infty.$$

Therefore f is finite a.s. (even though it's infinite for all rational x)!

Take-home Problem: Find a single x for which $f(x)$ is finite. (Hint: $f\left(\frac{1}{\sqrt{2}}\right) < \infty$, use the Thue-Siegal-Roth Theorem.)

10.1.1.2 New measures from densities

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

Definition 10.4 *A* ***density*** *is a non-negative function f such that $\int f d\mu = 1$.*

Proposition 10.5 *For a density f, Define*

$$\nu(A) = \int_A f d\mu = \int \delta_A f d\mu.$$

Then ν is a probability measure on $(\Omega, \mathcal{F})$ such that $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Proof:

- $\nu(\emptyset) = \int \delta_\emptyset f d\mu = \int 0 d\mu = 0$. $\nu(\Omega) = \int f d\mu = 1$.
- If $A \subseteq B$, then $\delta_A(\omega) f(\omega) \leq \delta_B(\omega) f(\omega)$ for all ω, so $\nu(A) = \int \delta_A f d\mu \leq \int \delta_B f d\mu \leq \nu(B)$.
- If $\{A_i\}_{i=1}^{\infty}$ is a partition of A, then

$$\delta_A(\omega) = \delta_{\bigcup A_i}(\omega) = \sum_i \delta_{A_i}(\omega).$$

Integrating both sides and using the Dominated Convergence Theorem (both sides dominated by f),

$$\begin{aligned}\nu\left(\bigcup_i A_i\right) &= \int \delta_A f d\mu \\ &= \int \sum_i \delta_{A_i} f d\mu \\ &= \sum_i \int \delta_{A_i} f d\mu \\ &= \sum_i \nu(A_i).\end{aligned}$$

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10.2 Change of Measure

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let Ω' another space. Say we have a function $T : \Omega \to \Omega'$. We can use T to introduce a σ-algebra on Ω': let $\mathcal{F}' = \{B \subseteq \Omega' : T^{-1}B \in \mathcal{F}\}$. Then T becomes a measurable function from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$.

We can also use T to define a measure on $(\Omega', \mathcal{F}')$: Define a measure $\mu^{T^{-1}}$ on $(\Omega', \mathcal{F}')$ such that

$$\mu^{T^{-1}}(B) = \mu(T^{-1}(B)).$$

Theorem 10.6 *If $f : \Omega' \to \mathbb{R}$ is $\mu^{T^{-1}}$ integrable, then $f \circ T : \Omega \to \mathbb{R}$ is μ integrable, and*

$$\int_{\Omega'} f d\mu^{T^{-1}} = \int_{\Omega} f \circ T d\mu.$$

Proof: Step 1: Prove for indicator functions.

Take $f(\omega) = \delta_B(\omega)$ for some set $B \in \mathcal{F}'$. Then

$$\int_{\Omega'} f d\mu^{T^{-1}} = \mu^{T^{-1}}(B) = \mu(T^{-1}(B)) = \int_{\Omega} f \circ T d\mu.$$

Step 2: Prove for simple functions.

Since simple functions are finite linear combinations of indicator functions, by Step 1 and linearity of the integral, the theorem is true for simple functions too.

Step 3: Prove for general functions. For f^+ and f^-, there exist a sequences of simple functions $\{g_n\}$ and $\{h_n\}$ such that $g_n \nearrow f^+$, $h_n \nearrow f^-$. Since the theorem is true for each of the g_n and h_n's, it is true for f^+ and f^- by the Monotone Convergence Theorem, and hence it is true for $f = f^+ - f^-$.

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Example (coin tossing measure): Suppose $\Omega = \{0,1\}^n$, $\mu(x) = \dfrac{1}{2^n}$. Let $T : \Omega \to \mathbb{R}$ be the function $T(x) = \sum_{i=1}^{n} x_i$. Then $\mu^{T^{-1}}(\{j\}) = \dfrac{1}{2^n}\binom{n}{j}$.

So, for any function $f : \{0, 1, \ldots, n\} \to \mathbb{R}$, we have

$$\sum_{j=0}^{n} f(j) \frac{\binom{n}{j}}{2^n} = \int_{\Omega} f(x)\mu(dx).$$