

## Lecture 3: January 17

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### 3.1 Martingales

Recall these definitions from last lecture:

**Definition 3.1** Let  $(\Omega, \mathcal{F}, P)$  be a probability space. A **filtration** on this space is an increasing sequence of sub  $\sigma$ -algebras  $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ .

**Definition 3.2** A sequence of random variables  $Z_1, Z_2, \dots$  is **adapted** to a filtration  $\{\mathcal{F}_n\}$  if for all  $n$ ,  $Z_n$  is  $\mathcal{F}_n$ -measurable.

**Definition 3.3** An adapted sequence is called a **martingale** if:

1.  $\mathbb{E}|Z_n| < \infty$  for all  $n$ , and
2.  $\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = Z_n$  a.s. for all  $n$ .

#### 3.1.1 Examples of martingales

- Let  $Y_1, Y_2, \dots$  be independent random variables with  $\mathbb{E}Y_i = \mu_i$ , and let  $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ .

Let  $Z_n = \sum_{i=1}^n (Y_i - \mu_i)$ . Then  $\{Z_n\}$  is a martingale.

- Same setting as above. Assume further that  $\text{Var}(Y_i) = \sigma_i^2 < \infty$ . Let  $S_n = \sum_{i=1}^n (Y_i - \mu_i)$ , and let

$Z_n = S_n^2 - \sum_{i=1}^n \sigma_i^2$ . Then  $\{Z_n\}$  is a martingale:

$$\begin{aligned}
 \mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E} \left[ (S_n + (Y_{n+1} - \mu_{n+1}))^2 - \sum_{i=1}^{n+1} \sigma_i^2 \mid \mathcal{F}_n \right] \\
 &= \mathbb{E} \left[ S_n^2 + 2S_n(Y_{n+1} - \mu_{n+1}) + (Y_{n+1} - \mu_{n+1})^2 \mid \mathcal{F}_n \right] - \sum_{i=1}^{n+1} \sigma_i^2 \\
 &= S_n^2 + 2S_n \mathbb{E}[Y_{n+1} - \mu_{n+1}] + \mathbb{E}[(Y_{n+1} - \mu_{n+1})^2] - \sum_{i=1}^{n+1} \sigma_i^2 \\
 &= S_n^2 - 0 + \sigma_{n+1}^2 - \sum_{i=1}^{n+1} \sigma_i^2 \\
 &= Z_n.
 \end{aligned}$$

- Let  $Y_1, Y_2, \dots$  be independent non-negative random variables with  $\mathbb{E}Y_i = 1$  for all  $i$ . Let  $Z_n = \prod_{i=1}^n Y_i$ ,  $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ .

Since  $Z_n$  is non-negative,

$$\mathbb{E}|Z_n| = \mathbb{E}Z_n = \mathbb{E}\left[\prod_{i=1}^n Y_i\right] = \prod_{i=1}^n \mathbb{E}Y_i = 1,$$

i.e.  $Z_n \in L^1$  for all  $n$ . We also have

$$\begin{aligned}\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\prod_{i=1}^{n+1} Y_i \mid \mathcal{F}_n\right] \\ &= \mathbb{E}[Z_n Y_{n+1} \mid \mathcal{F}_n] \\ &= Z_n \mathbb{E}[Y_{n+1}] \\ &= Z_n.\end{aligned}$$

Hence,  $\{Z_n\}$  is a martingale.

- If  $X_1, X_2, \dots$  i.i.d. and  $M(\theta) = \mathbb{E}[e^{\theta X_i}] < \infty$  for some  $\theta$ , then  $Z_n = \frac{e^{\theta \sum_{i=1}^n X_i}}{M(\theta)^n}$  is a martingale. This can be seen from the previous example by setting  $Z_n = \prod_{i=1}^n Y_i$ , where  $Y_i = \frac{e^{\theta X_i}}{M(\theta)}$ . ( $\theta$  is usually chosen such that  $M(\theta) = 1$  and  $\theta \neq 0$ .)

## 3.2 Stopping Times

**Definition 3.4** Let  $\{\mathcal{F}_n\}$  be a filtration. A random variable  $T : \Omega \rightarrow \{1, 2, \dots\} \cup \{\infty\}$  is a **stopping time adapted to  $\{\mathcal{F}_n\}$**  if  $\{T = n\} \in \mathcal{F}_n$  for all  $n$ .

Intuitively, a stopping time is a decision rule for stopping a particular process at a random time, and stopping the process at time  $n$  depends only on information available at time  $n$ .

### 3.2.1 Examples of stopping times

- Let  $Y_1, Y_2, \dots$  be independent random variables,  $S_n = \sum_{i=1}^n Y_i$ ,  $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ . ( $S_0 = \mathbb{E}Y_1$ ,  $\mathcal{F}_0 =$  trivial  $\sigma$ -algebra.) Let  $T = \inf\{k : S_k \geq a\}$  for some given  $a$ . (Usual convention: infimum of an empty set is  $\infty$ .)

$T$  is a stopping time:

$$\{T = n\} = \{S_0 < a, S_1 < a, \dots, S_{n-1} < a, S_n \geq a\} \in \mathcal{F}_n.$$

- Let  $T' = \sup\{k : S_k \geq a\}$ . Then  $T'$  is *not* a stopping time. (We will prove this later).

### 3.2.2 Properties of stopping times

- If  $T$  is a stopping time, then  $\{T \leq n\} = \bigcup_{i=1}^n \{T = i\} \in \mathcal{F}_n$ .
- $\{T \geq n\} = \{T \leq n-1\}^c \in \mathcal{F}_{n-1}$ .
- Let  $T$  be a stopping time. Take any  $n \in \mathbb{N} = \{1, 2, \dots\}$ . Let

$$S = \min\{T, n\} = \begin{cases} T & \text{if } T \leq n, \\ n & \text{if } T > n. \end{cases}$$

Then  $S$  is a stopping time, denoted by  $T \wedge n$ .

**Proof:**

If  $k < n$ ,  $\{S = k\} = \{T = k\} \in \mathcal{F}_k$ .

If  $k > n$ ,  $\{S = k\} = \emptyset \in \mathcal{F}_k$ .

If  $k = n$ ,  $\{S = k\} = \{T \geq n\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$ . ■

### 3.2.3 Stopped $\sigma$ -algebras and random variables

**Definition 3.5** Let  $T$  be a stopping time w.r.t. some filtration  $\{\mathcal{F}_n\}$ . The **stopped  $\sigma$ -algebra of  $T$** , denoted by  $\mathcal{F}_T$ , is the set of all  $A \in \mathcal{F}$  such that for all  $n \in \mathbb{N}$ ,  $A \cap \{T = n\} \in \mathcal{F}_n$ .

You can think of  $\mathcal{F}_T$  as “all events that are dependent only on information up to the stopping time”. It is easy to verify that  $\mathcal{F}_T$  is indeed a  $\sigma$ -algebra. This  $\sigma$ -algebra is *not* the same as the  $\sigma$ -algebra generated by  $T$ .

Example of a random variable in  $\mathcal{F}_T$

Let  $X_1, X_2, \dots$  be i.i.d.  $\pm 1$  valued random variables, and let  $S_n = \sum X_i$ ,  $S_0 = 0$ . Let  $a < 0 < b$  be 2 integers, and let  $T = \inf\{k : S_k = a \text{ or } b\}$ . Then the random variable  $S_T$  is  $\mathcal{F}_T$ -measurable.

**Proof:**  $S_T$  can be take 2 values ( $a$  or  $b$ ). It suffices to show that  $\{S_T = a\} \in \mathcal{F}_T$ . But

$$\{S_T = a\} \cap \{T = n\} = \{S_n = a\} \cap \{T = n\} \in \mathcal{F}_n.$$
■

**Definition 3.6** Let  $\{\mathcal{F}_n\}$  be a filtration,  $\{Z_n\}$  a sequence of random variables adapted to this filtration, and  $T$  a stopping time such that  $T < \infty$  a.s..

The **stopped random variable  $Z_T$**  is defined as

$$Z_T(\omega) := Z_{T(\omega)}(\omega),$$

that is,

$$Z_T = \sum_{n=1}^{\infty} Z_n 1_{\{T=n\}}.$$

**Fact:**  $Z_T$  is always  $\mathcal{F}_T$ -measurable.

### 3.3 Wald's Lemma (for Bounded Stopping Times)

**Theorem 3.7** Let  $\{\mathcal{F}_n\}$  be a filtration,  $\{Z_n\}$  a martingale adapted to this filtration,  $T$  a stopping time w.r.t. this filtration.

Suppose that there exists some  $N \in \mathbb{N}$  such that  $T \leq N$  a.s. Then  $\mathbb{E}Z_T = \mathbb{E}Z_1$ .

**Proof:**

First, observe that when  $m > n$ ,  $\mathbb{E}[Z_m \mid \mathcal{F}_n] = Z_n$  by the tower property of conditional expectation.

Since  $1_{\{T=n\}}$  is  $\mathcal{F}_n$ -measurable, by definition of conditional expectation, for any  $n \leq N$ ,

$$\mathbb{E}[Z_N 1_{\{T=n\}}] = \mathbb{E}[\mathbb{E}[Z_N \mid \mathcal{F}_n] 1_{\{T=n\}}] = \mathbb{E}[Z_n 1_{\{T=n\}}].$$

Hence

$$\begin{aligned} \mathbb{E}Z_T &= \mathbb{E}\left[\sum_{n=1}^N Z_n 1_{\{T=n\}}\right] \\ &= \sum_{n=1}^N \mathbb{E}[Z_n 1_{\{T=n\}}] \\ &= \sum_{n=1}^N \mathbb{E}[Z_N 1_{\{T=n\}}] \\ &= \mathbb{E}\left[Z_N \sum_{n=1}^N 1_{\{T=n\}}\right] \\ &= \mathbb{E}Z_N \\ &= \mathbb{E}Z_1. \end{aligned}$$

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#### 3.3.1 Example (Symmetric random walk)

Let  $X_1, X_2, \dots$  be i.i.d.  $\pm 1$  valued random variables with  $P\{X_i = 1\} = P\{X_i = -1\} = \frac{1}{2}$ . Let  $S_n = \sum_{i=1}^n X_i$ ,  $S_0 = 0$ .  $\{S_n\}$  is a martingale.

Let  $a < 0 < b$  be 2 integers, and let  $T = \inf\{k : S_k = a \text{ or } b\}$ . We will show later that  $T < \infty$  a.s.

We can use Wald's Lemma to show that  $P(S_T = a) = \frac{b}{b-a}$ .

**Proof:**

For any  $n \in \mathbb{N}$ ,  $T \wedge n$  is a bounded stopping time. Hence, by Wald's Lemma,  $\mathbb{E}S_{T \wedge n} = \mathbb{E}S_1 = 0$ .

Since  $T$  is finite a.s., for almost all  $\omega \in \Omega$ ,

$$\begin{aligned} S_{T \wedge n}(\omega) &= S_{(T \wedge n)(\omega)}(\omega) \\ &= S_{T(\omega) \wedge n}(\omega) \\ &\rightarrow S_{T(\omega)}(\omega) \end{aligned}$$

as  $n \rightarrow \infty$ , i.e.  $S_{T \wedge n} \xrightarrow{a.s.} S_T$ . Moreover,  $a \leq S_{T \wedge n} \leq b$  a.s. Hence, by the Bounded Convergence Theorem,

$$\mathbb{E}S_T = \lim_{n \rightarrow \infty} \mathbb{E}[S_{T \wedge n}] = 0.$$

Since  $S_T$  can only take on values  $a$  or  $b$ ,

$$\begin{aligned}\mathbb{E}S_T &= aP(S_T = a) + bP(S_T = b) \\ &= aP(S_T = a) + b[1 - P(S_T = a)], \\ P(S_T = a) &= \frac{b}{b - a}.\end{aligned}$$

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