Lecture 10: October 27

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10.1 Examples of Minimax Estimators

10.1.1 Binomial setting, weighted squared loss

$X \sim \text{Binom}(n,p)$. We showed last week that if the loss function is given by $L(d,p) = \dfrac{(d-p)^2}{p(1-p)}$, then the estimator $\frac{X}{n}$ is Bayes w.r.t. the prior $U(0,1) = \text{Beta}(1,1)$. We also showed last week that a Bayes estimator with constant risk is minimax.

Since $\frac{X}{n}$ has constant risk $\frac{1}{n}$ and is Bayes, it is minimax.

10.1.2 Binomial setting, squared error loss

We know that $\dfrac{X+a}{n+a+b}$ is Bayes w.r.t. the prior $\text{Beta}(a,b)$. If we can find values of a and b such that this estimator has constant risk, then it will be minimax. The risk of the estimator is given by

$$\begin{aligned}
\mathbb{E}_p\left[\left(\frac{X+a}{n+a+b} - p\right)^2\right] &= \frac{1}{(n+a+b)^2}\mathbb{E}_p\left[[X+a-p(n+a+b)]^2\right] \\
&= \frac{1}{(n+a+b)^2}\mathbb{E}_p\left[[(X-np)+(a-pa-pb)]^2\right] \\
&= \frac{1}{(n+a+b)^2}[np(1-p)+(a-pa-pb)^2].
\end{aligned}$$

To make the risk constant, we need the coefficients of p^2 and p in the second factor of the product on the RHS to be zero, i.e.

$$\begin{cases} -n+(a+b)^2 &= 0, \\ n-2a(a+b) &= 0. \end{cases}$$

Hence, if $a = b = \dfrac{\sqrt{n}}{2}$, the estimator $\dfrac{X+\frac{\sqrt{n}}{2}}{n+\sqrt{n}}$ has constant risk and is Bayes w.r.t. to the $\text{Beta}(a,b)$ prior, so is minimax.

10.1.3 Normal location model

We will use the following general fact:

Proposition 10.1 *If an estimator $\hat{\theta}$ has constant risk and is admissible, then it is minimax.*

Proof: If it is not minimax, then there is an estimator δ with better worst case risk, i.e.

$$\sup_{\theta} R(\theta, \delta) < \sup_{\theta} R(\theta, \hat{\theta}) = R(\theta, \hat{\theta}),$$

which means that δ dominates $\hat{\theta}$, making $\hat{\theta}$ inadmissible. ■

Let $X_1, \ldots, X_n$ iid, with $X_i \sim \mathcal{N}(\theta, 1)$. Under squared error loss, we showed last week that $\bar{X}$ is admissible and has constant risk. Thus, by the proposition above, it is minimax.

Note: The converse of the proposition is not true: A minimax estimator with constant risk need not be admissible.