STATS 310A: Theory of Probability I **Autumn 2016/17**

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9.1 Maximum of Normal Random Variables

This is a typical problem we encounter in probability: Let $X_1, \dots, X_n$ be iid, $X_i \sim \mathcal{N}(0,1)$ for all i. Let $M_n = \max_{1 \le i \le n} X_i$. Find the limiting behavior of M_n.

$$\begin{aligned} P\{M_n \le x\} &= P\{X_1 \le x\}^n \\ &= \exp[n \log P\{X_1 \le x\}] \\ &= \exp[n \log(1 - (1 - \Phi(x)))], \end{aligned}$$

where $\Phi(x) = P\{X_1 \le x\}$. Since $\log(1-x) \sim -x$ for small x, we have the estimate

$$P\{M_n \le x\} \sim \exp[-n(1 - \Phi(x))]$$

for large x.

For $0 < x < \infty$, we can bound the tail of the Gaussian distribution as follows:

$$\frac{x}{1+x^2} e^{-x^2/2} \le \int_x^\infty e^{-t^2/2} dt \le \frac{1}{x} e^{-x^2/2}.$$

For large x, the above implies that $1 - \Phi(x) \sim \dfrac{e^{-x^2/2}}{\sqrt{2\pi} x}$, so

$$\begin{aligned} P\{M_n \le x\} &\sim \exp[-n(1 - \Phi(x))] \\ &\sim \exp\left[-\frac{n}{\sqrt{2\pi} x} e^{-x^2/2} \right]. \end{aligned}$$

Let $x = \sqrt{2 \log n - \log \log n + y}$. Then $x \sim \sqrt{2 \log n}$, and so:

$$\begin{aligned} \frac{e^{-x^2/2}}{x} &= \frac{\exp[-\log n + \frac{1}{2} \log \log n - y/2]}{x}, \\ P\{M_n \le x\} &\sim \exp\left[-\frac{n}{\sqrt{2\pi}\sqrt{2 \log n}} \exp\left[-\log n + \frac{1}{2} \log \log n - y/2 \right] \right] \\ &= \exp\left[-\frac{e^{-y/2}}{2\sqrt{\pi}} \right]. \end{aligned}$$

This is a Gumbel distribution.

9.2 Building the Integral

Let $(\Omega, \mathcal{F})$ be a measurable space, μ a measure on this space. Let $f : \Omega \to \mathbb{R} \cup \pm\infty$ be a function. We want to define

$$\int f d\mu = \int_\Omega f(\omega)\mu(d\omega)$$

when we can.

Conventions:

- $0 \cdot \infty = \infty \cdot 0 = 0$.
- If $A = \emptyset$, $\inf\{\omega \in A\} = \infty$.

Definition 9.1 *A **simple function** is a function which takes on finitely many values, i.e. there exist $x_i \in \mathbb{R} \cup \pm\infty$ and a partition of Ω, $A_i \in \mathcal{F}$, such that*

$$f(\omega) = \sum_{i=1}^{n} x_i \delta_{A_i}(\omega).$$

The following proposition links a general function to a sequence of simple functions. This will prove very useful in proving facts about the integral.

Proposition 9.2 *Let $f : \Omega \to \mathbb{R} \cup \infty$ be a non-negative measurable function. Then there exists a sequence of simple functions f_n such that $f_n(\omega) \nearrow f(\omega)$ for all ω.*

Proof: We construct the sequence of f_n explicitly:

$$f_n(\omega) = \begin{cases} \dfrac{k-1}{2^n} & \text{if } \dfrac{k-1}{2^n} \le f(\omega) < \dfrac{k}{2^n}, 1 \le k \le n2^n, \\ n & \text{if } f(\omega) \ge n. \end{cases}$$

∎

We can now construct the integral:

Definition 9.3

- *For $f \ge 0$, define*

$$\int f d\mu := \sup \sum_{i=1}^{n} \nu_i \mu(A_i),$$

where the sup is taken over all finite measurable partitions $\{A_i\}_{i=1}^n$ of Ω, and $\nu_i = \inf_{\omega \in A_i} f(\omega)$.

- *For general f, define auxiliary functions*

$$f_+(\omega) := \max(f(\omega), 0), \qquad f_-(\omega) := \max(-f(\omega), 0).$$

*(Note that $f = f_+ - f_-$, $|f| = f_+ + f_-$.) If $\int f_+ d\mu$ and $\int f_- d\mu$ are not both ∞, we say that f is **integrable** (w.r.t. μ), and*

$$\int f d\mu := \int f_+ d\mu - \int f_- d\mu.$$

This is the Lebesgue integral for general measure μ.

9.2.1 Properties of the integral

Proposition 9.4 *If f is simple, i.e.* $f = \sum_{i=1}^{n} x_i \delta_{A_i}$*, then* $\int f d\mu = \sum_{i=1}^{n} x_i \mu(A_i)$*.*

Proof: Let $f = \sum_{i=1}^{n} x_i \delta_{A_i}$. Let $\{B_i\}_{i=1}^m$ be a measurable partition of Ω, and let $\beta_i = \inf_{\omega \in B_i} f(\omega)$. Note that if $A_i \cap B_j \neq \emptyset$, then $\beta_j \leq x_i$. Thus,

$$\begin{aligned} \sum_{j=1}^{m} \beta_j \mu(B_j) &= \sum_{i,j} \beta_j \mu(B_j \cap A_i) \\ &\leq \sum_{i,j} x_i \mu(B_j \cap A_i) \\ &= \sum_{i=1}^{n} x_i \mu(A_i). \end{aligned}$$

Taking sups on both sides, we get

$$\int f d\mu \leq \sum_{i=1}^{n} x_i \mu(A_i).$$

The other direction is obvious since $\{A_i\}$ is an admissible partition. ∎

Proposition 9.5 *(Monotonicity) If $f \leq g$, then $\int f d\mu \leq \int g d\mu$.*

Proposition 9.6 (Monotone Convergence Theorem) *If $f_n(\omega) \nearrow f(\omega)$ for all ω, then*

$$\lim_{n \to \infty} \int f_n d\mu = \int f d\mu = \int \lim_{n \to \infty} f_n d\mu.$$

Proof: Since $f_n \nearrow f$, we have $\int f_n d\mu \leq \int f d\mu$ for all n. Taking $n \to \infty$, we have

$$\lim_{n \to \infty} \int f_n d\mu \leq \int f d\mu.$$

For the other direction, we need to show that

$$\int f d\mu \leq \lim \int f_n d\mu,$$

or equivalently, that

$$\sum_{i=1}^{m} v_i \mu(A_i) \leq \lim \int f_n d\mu$$

for all partitions $\{A_i\}_{i=1}^m$, where $\nu_i = \inf_{\omega \in A_i} f(\omega)$. Call the LHS of the inequality above S.

Case 1: Suppose $S < \infty$ and $0 < \nu_i < \infty$, $0 < \mu(A_i) < \infty$ for all i.

Choose $\varepsilon > 0$ such that $\varepsilon < \nu_i$ for all i. Let $A_{n_i} := \{\omega : \omega \in A_i \text{ and } f_n(\omega) \geq \nu_i - \varepsilon\}$. Since $f_n \nearrow f$, we have $A_{n_i} \nearrow A_i$, which implies $\mu(A_{n_i}) \nearrow \mu(A_i)$. Since

$$\int f_n d\mu \geq \sum_{i=1}^{m} (\nu_i - \varepsilon)\mu(A_{n_i})$$

for any n, let $n \to \infty$:

$$\begin{aligned}\lim_{n\to\infty} \int f_n d\mu &\geq \sum_{i=1}^{m} (\nu_i - \varepsilon)\mu(A_i) \\ &= \sum_{i=1}^{m} \nu_i \mu(A_i) - \varepsilon \sum_{i=1}^{m} \mu(A_i).\end{aligned}$$

As $\varepsilon \to 0$, the second term goes to 0, and what remains is the inequality we wanted to prove.

Case 2: $S < \infty$, some ν_i or $\mu(A_i)$ is 0 or ∞.

Assume that $0 < \nu_i < \infty$ and $0 < \mu(A_i) < \infty$ for $1 \leq i \leq n_0$, then the rest have to be of the form $0 \cdot \infty$ of $0 \cdot 0$. Use the same argument as in Case 1 with the partition $\bigcup_{i=1}^{n_0} A_i$ and its complement.

Case 3: $S = \infty$.

We must have some i_0 with $\nu_{i_0} \cdot \mu(A_{i_0}) = \infty$, so either $\nu_{i_0} = \infty$ and $\mu(A_{i_0}) > 0$, or vice versa.

We can choose $x, y > 0$ so that $0 < x < \nu_{i_0}$, $0 < y < \mu(A_{i_0})$. Let $A_n = \{\omega : f_n(\omega) > x\}$. Note that $A_n \nearrow A = \{\omega : f(\omega) > x\}$, so $\mu(A_n) \nearrow \mu(A)$. Considering the partition $\{A_n, A_n^c\}$, we have

$$\int f_n d\mu \geq x\mu(A_n), \quad \Rightarrow \quad \lim_{n\to\infty} \int f_n d\mu \geq xy.$$

Since either $\mu(A_{i_0}) = \infty$ or $\nu_{i_0} = \infty$, we can let one of x, y go to infinity. So $\lim \int f_n d\mu = \infty$, as required. ■

Proposition 9.7 *(Linearity) For real numbers α and β,*

$$\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta + \int g d\mu.$$

Proof: Assume first that f and g are simple functions, i.e.

$$f = \sum_{i=1}^{n} x_i \delta_{A_i}, \qquad g = \sum_{j=1}^{m} y_j \delta_{B_j}.$$

Then,

$$\begin{aligned}\int (\alpha f + \beta g) d\mu &= \sum_{i,j} (\alpha x_i + \beta y_j)\mu(A_i \cap B_j) \\ &= \sum_{i,j} \alpha x_i \mu(A_i \cap B_j) + \sum_{i,j} \beta y_j \mu(A_i \cap B_j) \\ &= \sum_{i=1}^{n} \alpha x_i \mu(A_i) + \sum_{j=1}^{m} \beta y_j \mu(B_j) \\ &= \alpha \int f d\mu + \beta \int g d\mu.\end{aligned}$$

For general f and g, let f_n be simple and increasing to f, and let g_n be simple and increasing to g. Then we can use the Monotone Convergence Theorem repeatedly to obtain

$$\begin{aligned}\int(\alpha f+\beta g)d\mu &= \int \lim_{n\to\infty}(\alpha f_n+\beta g_n)d\mu \\ &= \lim_{n\to\infty}\int(\alpha f_n+\beta g_n)d\mu \\ &= \lim_{n\to\infty}\alpha\int f_n d\mu+\lim_{n\to\infty}\beta\int g_n d\mu \\ &= \alpha\int\lim_{n\to\infty}f_n d\mu+\beta\int\lim_{n\to\infty}g_n d\mu \\ &= \alpha\int f d\mu+\beta\int g d\mu.\end{aligned}$$

■

Some remarks:

- The Lebesgue integral generalizes the Riemann integral.
- If $(\Omega,\mathcal{F}) = ([0,1],\text{Borel sets})$, the function $f(x) = 1$ if x rational, $f(x) = 0$ if x irrational is not Riemann integrable but is Lebesgue integrable, with $\int f d\lambda = 0$.
- $L^2(\mu)$ is complete for this integral.
- However, the Riemann integral is still useful, e.g.
  - stochastic calculus,
  - $f : \Omega \to V$ a vector space,
  - indefinite integrals (e.g. $\int_0^\infty \frac{\sin x}{x}dx$ does not exist as a Lebesgue integral but does exist as a Riemann integral),
  - Cauchy principle values and "renormalization".

9.3 Class Problem

(a)

$$\frac{x}{1+x^2}e^{-x^2/2} \le \int_x^\infty e^{-t^2/2}dt \le \frac{1}{x}e^{-x^2/2}$$

for $0 < x < \infty$.

(b) Let $X_1, X_2, \ldots$ iid, $X_i \sim \mathcal{N}(0,1)$. Let $Y_n =$ closest integer to X_n, and let $M_n = \max_{1\le i\le n} Y_i$. Find explicit a_n, p_n, $a_n \in \{1,2,3\ldots,\}$, $p_n \in (0,1)$ such that

$$P\{M_n = a_n\} \sim p_n, \qquad P\{M_n = a_n+1\} \sim 1-p_n.$$

(c) Show that $\limsup p_n \ne \liminf p_n$.