Worked Example of Change of Variables

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Assume that we have random variables $X_1, \dots, X_n \in \mathbb{R}$ with joint density f. Assume that we have $(U_1, \dots, U_n) = g(X_1, \dots, X_n)$, where g is bijective and differentiable. We wish to find the joint density of $U_1, \dots, U_n$. To do so:

1. Derive the inverse functions $h_1, \dots, h_n$, i.e. $X_i = h_i(U_1, \dots, U_n)$.
2. Compute the Jacobian J, where $J_{ij} = \dfrac{\partial h_i}{\partial u_j}$.
3. The joint density p of $U_1, \dots, U_n$ is given by

$$\begin{aligned} p(u_1, \dots, u_n) &= f(x_1, \dots, x_n) \cdot |\det J| \\ &= f\left[h_1(u_1, \dots, u_n), \dots, h_n(u_1, \dots, u_n)\right] \cdot |\det J|. \end{aligned}$$

Worked Example 1

Assume that X_i $(i = 1, 2, 3)$ are independent $\text{Gamma}(\alpha_i, 1)$. We wish to find the joint density of $S = X_1 + X_2 + X_3$, $p_1 = \dfrac{X_1}{S}$ and $p_2 = \dfrac{X_2}{S}$.

1. **Derive inverse functions.** We have $X_1 = Sp_1$, $X_2 = Sp_2$ and $X_3 = S - Sp_1 - Sp_2$.
2. **Compute Jacobian.** We have

$$J = \begin{pmatrix} p_1 & S & 0 \\ p_2 & 0 & S \\ 1 - p_1 - p_2 & -S & -S \end{pmatrix}.$$

3. **Plug into the change of variables formula.** Compute the determinant of the Jacobian:

$$\begin{aligned} \det J &= p_1 \begin{vmatrix} 0 & S \\ -S & -S \end{vmatrix} - S \begin{vmatrix} p_2 & S \\ 1 - p_1 - p_2 & -S \end{vmatrix} \\ &= p_1 S^2 - S[-Sp_2 - S(1 - p_1 - p_2)] \\ &= S^2. \end{aligned}$$

Since each X_i has density $\dfrac{1}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-x}$, we have

$$\begin{aligned} f_{S,p_1,p_2}(a, b, c) &= f(ab, ac, a - ab - ac) \cdot a^2 \\ &= \frac{1}{\Gamma(\alpha_1)} (ab)^{\alpha_1 - 1} e^{-ab} \frac{1}{\Gamma(\alpha_2)} (ac)^{\alpha_2 - 1} e^{-x} \frac{1}{\Gamma(\alpha_3)} (a - ab - ac)^{\alpha_3 - 1} e^{-(a - ab - ac)} \cdot a^2 \\ &= \left[\frac{1}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)} a^{\alpha_1 + \alpha_2 + \alpha_3 - 1} e^{-a} \right] \cdot \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} b^{\alpha_1 - 1} c^{\alpha_2 - 1} (1 - b - c)^{\alpha_3 - 1}. \end{aligned}$$

Thus, if we let $p_3 = \dfrac{X_3}{S}$, the above tells us that $S \sim \text{Gamma}(\alpha_1 + \alpha_2 + \alpha_3, 1)$, $(p_1, p_2, p_3) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3)$, and S is independent of (p_1, p_2, p_3).

Worked Example 2

Assume that $Z \sim \mathcal{N}(0, I_n)$. We wish to find the density of $X = AZ + \mu$, where A is an invertible $n \times n$ matrix.

Recall that the density of Z is $f(z) = (2\pi)^{-n/2} \exp\left[-\frac{z^T z}{2}\right]$.

1. **Derive inverse functions.** We have $Z = A^{-1}(X - \mu)$.
2. **Compute Jacobian.** We have
$$J = \frac{\partial Z}{\partial X} = A^{-1}.$$
3. **Plug into the change of variables formula.** Compute the determinant of the Jacobian:
$$\det J = \det\left(A^{-1}\right) = \frac{1}{\det A}.$$

   Thus, we have
$$\begin{aligned}
f_X(x) &= f_Z[A^{-1}(x-\mu)] \cdot \left|\frac{1}{\det A}\right| \\
&= (2\pi)^{-n/2} \exp\left[-\frac{[A^{-1}(x-\mu)]^T A^{-1}(x-\mu)}{2}\right] \cdot \left|\frac{1}{\det A}\right| \\
&= \frac{1}{(2\pi)^{n/2}|\det A|} \exp\left[-\frac{(x-\mu)^T (A^{-1})^T A^{-1}(x-\mu)}{2}\right].
\end{aligned}$$