STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 13: February 21

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13.1 Application of Birkhoff's Ergodic Theorem

Proposition 13.1 *Let $\Omega = [0,1)$, $\varphi(x) = x + \theta \pmod 1$, where θ is irrational. Take any subinterval $[a,b)$ of $[0,1)$. Then, for any $x \in [0,1)$,*

$$\frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(x) \in [a,b)\} \to b-a.$$

(**Note:** For almost surely any x is a direct application of Birkhoff's Theorem. What we are claiming here is stronger, i.e. all x.)

Proof: We know that φ is measure-preserving and ergodic, so by Birkhoff's Ergodic Theorem,

$$\frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(x) \in [a,b)\} \to b-a$$

for almost all $x \in [0,1)$.

Take any $k \geq 1$. Let $A_k = \left[a + \frac{1}{k}, b - \frac{1}{k}\right)$, and let $A = [a,b)$. We will work with k so large that A_k is non-empty. Birkhoff's Ergodic Theorem says that

$$\frac{1}{n}\sum_{m=0}^{n-1} 1\left\{\varphi^m(x) \in A_k\right\} \to b-a-\frac{2}{k}$$

for almost every x. Let $\Omega_k \subseteq [0,1)$ be a set of measure 1 on which this convergence takes place, and let $\mathcal{G} = \bigcap_{k=1}^{\infty} \Omega_k$. Then $P(G) = 1$, which implies that G is dense in $[0,1)$.

Now take any $x \in [0,1)$. Since G is dense, we can find $y \in G$ such that $|x-y| \leq \dfrac{1}{k}$. Noting that $\varphi^m(\omega) = \omega + m\theta \pmod 1$,

$$\begin{aligned}
\varphi^m(y) \in A_k \quad &\Rightarrow \quad y + m\theta \in \left[N + a + \frac{1}{k}, N + b - \frac{1}{k}\right) \qquad \text{for some integer } N \\
&\Rightarrow \quad x + m\theta \in [N+a, N+b), \\
&\Rightarrow \quad \varphi^m(x) \in A, \\
\Rightarrow \quad \frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(y) \in A_k\} \quad &\leq \quad \frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(x) \in A\}.
\end{aligned}$$

But, by Birkhoff, since $y \in G \subseteq \Omega_k$, the LHS $\dfrac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(y) \in A_k\} \to b - a - \dfrac{2}{k}$. Thus,

$$\liminf \frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(x) \in A\} \geq b - a - \frac{2}{k}.$$

Since this is true for all k, we have

$$\liminf \frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(x) \in A\} \geq b - a.$$

Note that A^c is the union of 2 half-open intervals, and so by the same argument as before, we obtain

$$\begin{aligned} &\liminf \frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(x) \in A^c\} \geq |A^c|, \\ \Rightarrow \quad &\limsup \frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(x) \in A\} \leq b - a, \end{aligned}$$

and so

$$\lim \frac{1}{n}\sum_{m=0}^{n-1} 1\{\varphi^m(x) \in A\} \leq b - a,$$

as required.

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13.2 von Neumann's Ergodic Theorem

Theorem 13.2 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, φ a measure-preserving transform with invariant σ-algebra $\mathcal{I}$, and X a random variable in L^2. Then*

$$\frac{1}{n}\sum_{m=0}^{n-1} X \circ \varphi^m \xrightarrow{L^2} \mathbb{E}[X \mid \mathcal{I}].$$

Note: While more specific than Birkhoff's Ergodic Theorem, the proof of von Neumann's Ergodic Theorem is more "transparent", i.e. we can see what is going on.

Proof: Let $W = \{X \circ \varphi - X : X \in L^2\}$. W is a subspace of L^2. Let $\bar{W}$ be the closure of W. Let $V = \{X \in L^2 : X \circ \varphi = X \text{ a.s.}\}$. V is also a subspace.

Claim: X can be decomposed into $Y + Z$, where $Y \in \bar{W}$, $Z \in V$, and $Y \perp Z$.

Let Y be the projection of X onto $\bar{W}$, and let $Z = X - Y$. Then, by definition of projections, Y is certainly in $\bar{W}$, and Z is orthogonal to any element of $\bar{W}$. In particular, Z is orthogonal to $Z \circ \varphi - Z$, i.e. $\mathbb{E}[Z(Z \circ \varphi - Z)] = 0$. Also, because φ is measure-preserving, $\mathbb{E}[(Z \circ \varphi)^2] = \mathbb{E}[Z^2]$. Combining these 2 facts, we get

$$\begin{aligned}
\mathbb{E}[(Z\circ\varphi - Z)^2] &= \mathbb{E}[(Z\circ\varphi)^2 - 2Z(Z\circ\varphi) + Z^2] \\
&= \mathbb{E}[2Z^2 - 2Z(Z\circ\varphi)] \\
&= 0, \\
\Rightarrow \qquad Z\circ\varphi &= Z \text{ a.s.},
\end{aligned}$$

i.e. $Z \in V$. Thus we have the desired decomposition.

With this decomposition, we can rewrite the sum in the theorem as follows:

$$\begin{aligned}
\frac{1}{n}\sum_{m=0}^{n-1} X\circ\varphi^m &= \frac{1}{n}\sum_{m=0}^{n-1} Y\circ\varphi^m + \frac{1}{n}\sum_{m=0}^{n-1} Z\circ\varphi^m \\
&= \frac{1}{n}\sum_{m=0}^{n-1} Y\circ\varphi^m + Z. && \text{(since } Z\circ\varphi = Z \text{ a.s.)}
\end{aligned}$$

Take any $\varepsilon > 0$. There exists $Y' \in W$ such that $\|Y - Y'\|_2 \le \varepsilon$. Since φ is measure-preserving,

$$\begin{aligned}
\|Y\circ\varphi^m - Y'\circ\varphi^m\|_2 &\le \varepsilon && \text{for all } m, \\
\left\|\frac{1}{n}\sum_{m=0}^{n-1} Y\circ\varphi^m - \frac{1}{n}\sum_{m=0}^{n-1} Y'\circ\varphi^m\right\| &\le \varepsilon. && \text{(triangle inequality for } L^2 \text{ norm)}
\end{aligned}$$

But $Y' = U\circ\varphi - U$ for some U, so

$$\begin{aligned}
\frac{1}{n}\sum_{m=0}^{n-1} Y'\circ\varphi^m &= \frac{1}{n}\sum_{m=0}^{n-1} U\circ\varphi^{m+1} - U\circ\varphi^m = \frac{1}{n}(U\circ\varphi^n - U), \\
\left\|\frac{1}{n}\sum_{m=0}^{n-1} Y'\circ\varphi^m\right\| &\xrightarrow{L^2} 0. && \text{(since } \|U\circ\varphi^n\|_2 = \|U\|_2\text{)}
\end{aligned}$$

As a result, $\frac{1}{n}\sum_{m=0}^{n-1} Y\circ\varphi^m \to 0$ in L^2 as well.

It remains to show that $Z = \mathbb{E}[X \mid \mathcal{I}]$. We will do this by showing that Z satisfies the definition of conditional expectation.

Note that $V = L^2(\mathcal{I})$. Take any $S \in V = L^2(\mathcal{I})$. Using the decomposition from before, we have

$$\mathbb{E}[XS] = \mathbb{E}[(X - Z + Z)S] = \mathbb{E}[ZS] + \mathbb{E}[YS].$$

Since $Y \in \bar{W}$, there exist $Y_n \in W$ such that $Y_n \to W$ in L^2. Since there exist U_n such that $Y_n = U_n\circ\varphi - U_n$,

$$\begin{aligned}
\mathbb{E}[Y_n S] &= \mathbb{E}[(U_n\circ\varphi)S] - \mathbb{E}[U_n S] \\
&= \mathbb{E}[(U_n\circ\varphi)(S\circ\varphi)] - \mathbb{E}[U_n S] \\
&= \mathbb{E}[(U_n S)\circ\varphi] - \mathbb{E}[U_n S] \\
&= 0.
\end{aligned}$$

Thus, $\mathbb{E}[YS] = 0$, i.e. $\mathbb{E}XS = \mathbb{E}ZS$. By definition of conditional expectation, $Z = \mathbb{E}[X \mid \mathcal{I}]$.

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