STATS 310A: Theory of Probability I **Autumn 2016/17**

Lecture 4: October 5

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4.1 The $\pi - \lambda$ Theorem

Recall:

- $\mathcal{P}$ is a π-system if it is closed under finite intersections.
- $\mathcal{L}$ is a λ-system if
  1. $\Omega \in \mathcal{L}$,
  2. $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$,
  3. $\mathcal{L}$ closed under countable disjoint unions.

Fact 0: If $A \subseteq B$ are in $\mathcal{L}$, then $B - A := B \cap A^c \in \mathcal{L}$.

Fact 1: If $\mathcal{L}$ is also a π-system, then $\mathcal{L}$ is a σ-algebra.

Proof: We just need to prove closure under countable unions.

If $A_1, A_2, \ldots \in \mathcal{L}$, then $A_i' := A_i \cap (A_1 \cup \ldots \cup A_{i-1})^c$ is in $\mathcal{L}$ as well. The A_i''s are disjoint and $\bigcup A_i' = \bigcup A_i$. Hence, we can use the last property of a λ-system to obtain $\bigcup A_i \in \mathcal{L}$. ∎

Theorem 4.1 ($\pi - \lambda$ Theorem) *If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system and $\mathcal{P} \subseteq \mathcal{L}$, then*

$$\sigma(\mathcal{P}) \subseteq \mathcal{L}.$$

Proof: Let $\mathcal{L}_0$ be the λ-system generated by $\mathcal{P}$ (i.e. the intersection of all λ-systems containing $\mathcal{P}$). If we show that $\mathcal{L}_0$ is a π-system, we can use Fact 1 to complete the proof.

Let $A \in \mathcal{L}_0$. Define

$$\mathcal{L}_A = \{B : A \cap B \in \mathcal{L}_0\},$$

i.e. all sets whose intersection with A lies in $\mathcal{L}_0$. We claim that $\mathcal{L}_A$ is a λ-system:

- Certainly $\Omega \in \mathcal{L}_A$ since $\Omega \cap A = A \in \mathcal{L}_0$.
- If $B_1 \subseteq B_2$, $B_1, B_2 \in \mathcal{L}_A$, then $A \cap B_1 \subseteq A \cap B_2 \in \mathcal{L}_0$. Since $\mathcal{L}_0$ is a λ-system, by Fact 0,

  $$\begin{aligned} B_2 \cap A - (B_1 \cap A) &\in \mathcal{L}_0, \\ (B_2 - B_1) \cap A &\in \mathcal{L}_0, \end{aligned}$$

  so $B_2 - B_1$ in $\mathcal{L}_A$.

  Take $B_2 = \Omega$, then $B_2 - B_1 = B_1^c$ is in $\mathcal{L}_A$.

- If $B_1, B_2, \ldots \in \mathcal{L}_A$ disjoint, then $A \cap B_1, \ldots A \cap B_n \in \mathcal{L}_0$ are disjoint. Since $\mathcal{L}_0$ is a λ-system,

$$\bigcup_n (A \cap B_n) \in \mathcal{L}_0,$$
$$A \cap \left(\bigcup_n B_n\right) \in \mathcal{L}_0,$$
$$\bigcup_n B_n \in \mathcal{L}_A.$$

In particular, if $A \in \mathcal{P} \subseteq \mathcal{L}_0$, then $\mathcal{L}_A$ is a λ-system, and if $B \in \mathcal{P}$, we have $A \cap B \in \mathcal{P}$, so $B \in \mathcal{L}_A$. So $\mathcal{P} \subseteq \mathcal{L}_A$, and so $\mathcal{L}_0 \subseteq \mathcal{L}_A$ (since $\mathcal{L}_0$ is the smallest λ-system containing $\mathcal{P}$).

Now, if $B \in \mathcal{L}_0$, then $B \in \mathcal{L}_A$, i.e. $B \cap A \in \mathcal{L}_0$, so $A \in \mathcal{L}_B$.

Since A was an arbitrary set in $\mathcal{P}$, we have $\mathcal{P} \subseteq \mathcal{L}_B$, so $\mathcal{L}_0 \subseteq \mathcal{L}_B$.

Thus, if B and C are contained in $\mathcal{L}_0$, then $B \cap C \in \mathcal{L}_0$, so $\mathcal{L}_0$ is a π-system.

We have $\mathcal{P} \subseteq \sigma(P) \subseteq \mathcal{L}_0$. ■

4.1.1 Applications to Independence

Definition 4.2 *$\{\mathcal{F}_i\}_{i \in I}$ in $(\Omega, \mathcal{F}, \mathcal{P})$ are **independent** if for every k, $\{A_i\}_{i=1}^k$, $A_i \in \mathcal{F}_i$, we have*

$$P\left(\bigcap_{i=1}^n A_i\right) = \prod_{i=1}^n P(A_i).$$

Proposition 4.3 *Let $\{\mathcal{A}_i\}_{i \in I}$ be π-systems which are independent. Then, if $\mathcal{F}_i = \sigma(\mathcal{A}_i)$, the $\{\mathcal{F}_i\}$ are independent.*

Proof: We will only prove this proposition for finite $I = \{1, 2, \ldots, n\}$.

Let $\mathcal{B}_i := \mathcal{A}_i \cup \{\Omega\}$. These are still independent π-systems.

Define

$$\mathcal{L} = \left\{ B_1 \in \mathcal{F} : P(B_1 \cap \ldots \cap B_n) = \prod_{i=1}^n P(B_i) \quad \forall B_2, \ldots, B_n \in \mathcal{B}_2, \ldots, \mathcal{B}_n, \right\},$$

i.e. collection of sets in $\mathcal{F}$ which are independent of the other $\{\mathcal{B}_i\}$. Clearly $\mathcal{B}_1 \subseteq \mathcal{L}$.

We claim that $\mathcal{L}$ is a λ-system:

- $\Omega \in \mathcal{B}_1 \subseteq \mathcal{L}$.

- Let $B_1 \in \mathcal{L}$. Then

$$\begin{aligned}
P\{B_2 \cap \ldots \cap B_n\} &= P\left\{(B_1 \cup B_1^c) \cap B_2 \cap \ldots \cap B_n\right\} \\
&= P\{B_1 \cap B_2 \cap \ldots \cap B_n\} + P\{B_1^c \cap B_2 \cap \ldots \cap B_n\} \\
&= \prod_{i=1}^n P(B_i) + P\{B_1^c \cap B_2 \cap \ldots \cap B_n\}, \\
\prod_{i=2}^n P(B_i) - \prod_{i=1}^n P(B_i) &= P\{B_1^c \cap B_2 \cap \ldots \cap B_n\}, \\
P\{B_1^c \cap B_2 \cap \ldots \cap B_n\} &= P(B_1^c) \cdot \prod_{i=2}^n P(B_i),
\end{aligned}$$

  hence $B_1^c \in \mathcal{L}$.

- If $B_1^j, j = 1, 2, \ldots$ are disjoint sets in $\mathcal{L}$, then

$$\begin{aligned}
P\left(\left(\bigcup_j B_1^j\right) \cap B_2 \cap \ldots \cap B_n\right) &= P\left(\bigcup_j (B_1^j \cap B_2 \cap \ldots \cap B_n)\right) \\
&= \sum_j P(B_1^j \cap B_2 \cap \ldots \cap B_n) \\
&= \sum_j P(B_1^j) \prod_{i=2}^n P(B_i) \\
&= P\left(\bigcup_j B_1^j\right) \prod_{i=2}^n P(B_i),
\end{aligned}$$

  i.e. $\bigcup_j B_1^j \in \mathcal{L}$.

Since $\mathcal{L}$ is a λ-system containing the π-system $\mathcal{B}_1$, we have $\sigma(\mathcal{B}_1) = \sigma(\mathcal{A}_1) \subseteq \mathcal{L}$. Thus, $\sigma(\mathcal{A}_1)$ is independent of $B_2, \ldots, B_n$.

Repeat this argument with

$$\mathcal{L} = \left\{ B_2 \in \mathcal{F} : P(B_1 \cap \ldots \cap B_n) = \prod_{i=1}^n P(B_i), B_1 \in \sigma(\mathcal{A}_1), B_i \in \mathcal{B}_i \text{ for } i = 3, \ldots \right\},$$

and so on. We obtain the result that $\sigma(\mathcal{A}_1), \ldots, \sigma(A_n)$ are independent. ■

4.2 0-1 Laws

4.2.1 Borel-Cantelli Lemmas

Definition 4.4 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A_i, 1 \le i < \infty$ be measurable sets. Define the event*

$$\begin{aligned} A_i \text{ i.o.} &:= \{\omega : \omega \in \text{infinitely many } A_i\} \\ &= \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m. \end{aligned}$$

For example, in the coin tossing example, if $A_i = \{d_i = 1\}$, then A_i i.o. $= \{\omega : \omega$ has infinitely many 1s$\}$.

Proposition 4.5 (1st Borel-Cantelli Lemma) *If $\sum_{i=1}^{\infty} P(A_i) < \infty$, then $P(A_i \text{ i.o.}) = 0$.*

Proof: For any n,

$$A_i \text{ i.o.} \subseteq \bigcup_{m=n}^{\infty} A_m.$$

Choose n large so that $\sum_{m=n}^{\infty} P(A_m) < \varepsilon$. Then

$$P(A_i \text{ i.o.}) \le \sum_{m=n}^{\infty} P(A_m) < \varepsilon.$$

■

Example (Coin flipping): Let $(\Omega, \mathcal{F}, P)$ be the unit interval with length. Look at the longest head run starting at n, denoted by l_n:

$$l_n(\omega) = k \quad \Leftrightarrow \quad d_n(w) = d_{n+1}(\omega) = \cdots = d_{n+k-1}(\omega) = 1, d_{n+k}(\omega) = 0.$$

It is clear that

$$P(l_n(\omega) = k) = \frac{1}{2^{k+1}}.$$

Pick integers $r_1, r_2, \ldots$. Let $A_n = \{\omega : l_n(\omega) \ge r_n\}$. Then

$$\begin{aligned} P(A_n) &= \sum_{k=0}^{\infty} P(l_n = r_n + k) \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{r_n+k+1}} \\ &= \frac{1}{2^{r_n}}. \end{aligned}$$

By the 1st Borel-Cantelli Lemma, if $\sum \frac{1}{2^{r_n}} < \infty$, then $P(A_n \text{ i.o.}) = 0$. (For example, $r_n = n$, $r_n = (1+\varepsilon)\log_2 n$.)

Proposition 4.6 (2nd Borel-Cantelli Lemma) *If* $\sum_{i=1}^{\infty} P(A_i) = \infty$ ***and*** A_i *are independent, then* $P(A_i \text{ i.o.}) = 1$.

Proof: Recall this fact: $1 - x \le e^{-x}$ for all $x \in \mathbb{R}$. Hence,

$$\begin{aligned}
\{A_n \text{ i.o.}\}^c &= \left[\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right]^c \\
&= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c, \\
P\left(\bigcap_{m=n}^{\infty} A_m^c\right) &= \prod_{i=n}^{\infty} P(A_i^c) \\
&= \prod_{i=n}^{\infty} (1 - P(A_i)) \\
&\le \exp\left[-\sum_{i=n}^{\infty} P(A_n)\right].
\end{aligned}$$

For any n, the exponent is $-\infty$. Hence $P\{A_n \text{ i.o.}\} = 1$. ■

Definition 4.7 *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence of real numbers,* $\limsup x_n := l$ *if for all* $\varepsilon > 0$,

1. $x_n \ge l - \varepsilon$ *infinitely often, and*
2. $x_n < l + \varepsilon$ *for all sufficiently large* n.

Example: $0, 1, 0, 1, \ldots$ has $\limsup x_n = 1$.

Theorem 4.8 *For the longest head run,*

$$\limsup \frac{l_n(\omega)}{\log_2 n} = 1$$

for almost all ω.

Proof: We proved that with probability 1, $l_n(\omega) < (1+\varepsilon)\log_2 n$ for sufficiently large n (from the 1st Borel-Cantelli Lemma.

$l_n(\omega) > (1-\varepsilon)\log_2 n$ i.o. follows from the 2nd Borel-Cantelli Lemma. ■

4.2.2 Kolmogorov's 0-1 Law

Definition 4.9 *For a sequence of events $A_1, A_2, \ldots$, define the* ***tail field*** *of the A_i's as*

$$\mathcal{T} := \bigcap_{i=1}^{\infty} \sigma(A_i, A_{i+1}, \ldots).$$

Intuitively, this is the set of events that don't depend on any finite number of the A_i's.

Example: $A_i = \{d_i = 1\}$ in coin tossing. Then $A = \left\{\frac{S_n}{n} \text{ converges}\right\}$ is a tail set, i.e. is in $\mathcal{T}$.

Theorem 4.10 (Kolmogorov 0-1 Law) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A_i, 1 \leq i < \infty$ be sets in $\mathcal{F}$ which are independent. Then any tail set has probability 0 or 1.*

Proof: Take $A \in \mathcal{T}$. Then $A \subseteq \sigma(A_i, A_{i+1}, \ldots)$, which means that A is independent of $A_1, \ldots, A_{i-1}$.

But this is true for every i! So A is independent of $\sigma(A_1, A_2, \ldots)$.

However, $A \in \sigma(A_1, A_2, \ldots)$, so $A \perp A$, i.e. is independent of itself.

This means that $P(A \cap A) = P(A)P(A)$, i.e. $P(A) = 0$ or 1. ∎