STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 10: February 8

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10.1 Martingale Strong Law of Large Numbers

We first prove 2 technical lemmas in analysis:

Lemma 10.1 *If a_n is a sequence of positive real numbers increasing to infinity, then* $\displaystyle\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{a_{n+1}} = \infty$.

Proof: Case 1: $\displaystyle\inf_n \frac{a_n}{a_{n+1}} = 0$**.**

Then

$$\frac{a_{n+1} - a_n}{a_{n+1}} = 1 - \frac{a_n}{a_{n+1}} \geq \frac{1}{2}$$

for infinitely many n.

Case 2: $\displaystyle\inf_n \frac{a_n}{a_{n+1}} = \delta > 0$**.**

Then

$$\begin{aligned}
\frac{a_{n+1} - a_n}{a_{n+1}} &= \frac{a_n}{a_{n+1}} \cdot \frac{a_{n+1} - a_n}{a_n} \\
&\geq \delta \cdot \frac{a_{n+1} - a_n}{a_n} \\
&= \delta \int_{a_n}^{a_{n+1}} \frac{1}{a_n} dt \\
&\geq \delta \int_{a_n}^{a_{n+1}} \frac{1}{t} dt, \\
\Rightarrow \quad \sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{a_{n+1}} &\geq \delta \int_{a_1}^{\infty} \frac{1}{t} dt = \infty.
\end{aligned}$$

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Lemma 10.2 *If a_n is a sequence of positive real numbers increasing to infinity, then* $\displaystyle\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{a_{n+1}^2} < \infty$.

Proof:

$$\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{a_{n+1}^2} = \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} \frac{1}{a_{n+1}^2} dt \leq \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} \frac{1}{t^2} dt = \int_{a_1}^{\infty} \frac{1}{t^2} dt < \infty.$$

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Theorem 10.3 (Martingale Strong Law of Large Numbers) *Let*

- $\{S_n, \mathcal{F}_n\}_{n\geq 1}$ *be a martingale with mean 0,*
- $X_n = S_n - S_{n-1}$ *(where $S_0 = 0$), with the assumption that $\mathbb{E}[X_n^2] < \infty$ for all n,*
- $\sigma_n^2 = \mathbb{E}[X_n^2 \mid \mathcal{F}_{n-1}]$,
- c_n *an $\mathcal{F}_{n-1}$-measurable random variable, with the assumption that $c_n > 0$ and increasing. Assume further that $\mathbb{E}\left(\frac{X_k^2}{c_1^2}\right) < \infty$ for all k.*

Then, on the set $\left\{\sum_{n=1}^{\infty} \frac{\sigma_n^2}{c_n^2} < \infty \text{ and } \lim_{n\to\infty} c_n = \infty\right\}$, $\frac{S_n}{c_n} \to 0$ *a.s.*

Proof: Let $Z_n = \left(\frac{S_n}{c_n}\right)^2$. Then

$$
\begin{aligned}
\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\frac{(S_n + X_{n+1})^2}{c_{n+1}^2}\Big|\mathcal{F}_n\right] \\
&= \frac{S_n^2}{c_{n+1}^2} + \frac{\sigma_{n+1}^2}{c_{n+1}^2} \qquad \text{(integrability conditions hold due to assumption on } c_n) \\
&= \frac{S_n^2}{c_n^2} + S_n^2\left(\frac{1}{c_{n+1}^2} - \frac{1}{c_n^2}\right) + \frac{\sigma_{n+1}^2}{c_{n+1}^2} \\
&= Z_n + \frac{\sigma_{n+1}^2}{c_{n+1}^2} - Z_n\left(\frac{c_{n+1}^2 - c_n^2}{c_{n+1}^2}\right),
\end{aligned}
$$

i.e. Z_n is an "almost supermartingale". By the almost supermartingale convergence theorem, on the set $\left\{\sum \frac{\sigma_n^2}{c_n^2} < \infty\right\}$, Z_n converges and $\sum Z_n \left(\frac{c_{n+1}^2 - c_n^2}{c_{n+1}^2}\right) < \infty$ a.s.

So, on the set $\left\{\sum \frac{\sigma_n^2}{c_n^2} < \infty \text{ and } c_n \to \infty\right\}$, by Lemma 10.1, $\sum \frac{c_{n+1}^2 - c_n^2}{c_{n+1}^2} = \infty$ a.s. This implies that $Z_n \to 0$ a.s. on this set.

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Corollary 10.4 *The theorem above holds when we drop the integrability condition on c_n, i.e. (A): $\mathbb{E}\left(\frac{X_k^2}{c_1^2}\right) < \infty$ for all k.*

Proof: Take any $\varepsilon > 0$ and let $c_n' = c_n + \varepsilon$. Then c_n' satisfies condition (A) and all other conditions as in Theorem 10.3.

Thus, on the set $B' = \left\{\sum \frac{\sigma_n^2}{(c_n')^2} < \infty \text{ and } c_n' \to \infty\right\}$, $\frac{S_n}{c_n'} \to 0$. But $B' = B = \left\{\sum \frac{\sigma_n^2}{c_n^2} < \infty \text{ and } c_n \to \infty\right\}$.

Additionally, on B, $\frac{S_n}{c_n'} \to 0 \Leftrightarrow \frac{S_n}{c_n} \to 0$. Therefore on B, $\frac{S_n}{c_n} \to 0$ a.s.

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Corollary 10.5 *Let $X_n \in \{0,1\}$ for all n, X_n $\mathcal{F}_n$-measurable. Let $S_n = \sum_{i=1}^{n} X_i$, and $p_n = P(X_n = 1 \mid \mathcal{F}_{n-1})$.*

Then $\lim_{n\to\infty} \frac{S_n}{\sum_{i=1}^n p_i} = 1$ *a.s. on the set* $\left\{\sum_{i=1}^{\infty} p_i = \infty\right\}$.

(**Note:** This is a stronger version of what we can get from Lévy's version of the Second Borel-Cantelli Lemma.)

Proof: Apply Theorem 10.3. Here, S_n is replaced by $S_n - \sum_{i=1}^{n} p_i$, X_n is replaced by $X_n - p_n$, $\sigma_n^2 = p_n(1-p_n)$, $c_n = \sum_{i=1}^{n} p_i$.

On the set $\left\{\sum_{n=1}^{\infty} \frac{p_n(1-p_n)}{\left(\sum_{i=1}^n p_i\right)^2} < \infty \text{ and } \sum_{n=1}^{\infty} p_n = \infty\right\}$, we have $\frac{S_n}{\sum_{i=1}^n p_i} \to 1$ a.s. But on the set $\{\sum_{n=1}^\infty p_n = \infty\}$, the first condition automatically holds by Lemma 10.2, taking $a_n = \sum_{i=1}^{n} p_i$ and observing that $p_n(1-p_n) \le p_n$.

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10.2 Square-Integrable Martingales

Definition 10.6 *Let $\{Z_n, \mathcal{F}_n\}_{n\ge 1}$ be a martingale. It is called **square-integrable** is $\mathbb{E}Z_n^2 < \infty$ for all n.*

Theorem 10.7 *If $\{Z_n\}$ is a square-integrable martingale and $\sup_n \mathbb{E}Z_n^2 < \infty$, then there exists Z such that $Z_n \to Z$ a.s. and in L^2.*

Proof: Existence of Z such that $Z_n \to Z$ a.s. follows from the Martingale Convergence Theorem.

We will now show that $\{Z_n\}$ is Cauchy in L^2. Let $X_n = Z_n - Z_{n-1}$, $X_1 = Z_1$. Then for all n and m, $X_n, X_m \in L^2 \Rightarrow X_n X_m \in L^1$. Thus, if $n > m$,

$$\begin{aligned} & \mathbb{E}[X_n X_m \mid \mathcal{F}_m] = X_m \mathbb{E}[X_n \mid \mathcal{F}_m] = 0, \\ \Rightarrow \quad & \mathbb{E}[X_n X_m] = 0, \\ \Rightarrow \quad & \mathbb{E}Z_n^2 = \sum_{i=1}^{n} \mathbb{E}X_i^2. \end{aligned}$$

Since $\sup \mathbb{E}Z_n^2 < \infty$, we have $\sum_{i=1}^n \mathbb{E}X_i^2 < \infty$, so $\lim_{n,m\to\infty, m\le n} \sum_{i=m+1}^{n} \mathbb{E}X_i^2 = 0$. But $\sum_{i=m+1}^{n} \mathbb{E}X_i^2 = \mathbb{E}(Z_n - Z_m)^2$. Therefore $\{Z_n\}$ is Cauchy in L^2.

Since L^2 is complete, there is a Z' such that $Z_n \to Z'$ in L^2. But $Z = Z'$ a.s. since any sequence that converges in L^2 has a subsequence that converges a.s. to the same limit. ∎

Definition 10.8 *A sequence $\{Y_n\}$ is called a* ***predictable process*** *if Y_n is $\mathcal{F}_{n-1}$-measurable for all n.*

Definition 10.9 *Let Z_n be a square-integrable martingale, and let $X_n = Z_n - Z_{n-1}$. Define $\sigma_n^2 = \mathbb{E}[X_n^2 \mid \mathcal{F}_{n-1}]$. It can be checked that $Z_n^2 - \sum_{i=1}^{n} \sigma_i^2$ is a martingale.*

$\left\{ \sum_{i=1}^{n} X_i^2 \right\}$ *is called the* ***quadratic variation*** *of $\{Z_n\}$.*

$\left\{ \sum_{i=1}^{n} \sigma_i^2 \right\}$ *is called the* ***predictable quadratic variation*** *of $\{Z_n\}$.*

We have a strengthening of the previous result:

Theorem 10.10 *For a square-integrable martingale $\{Z_n\}$, $\lim Z_n$ exists and is finite a.s. on the set $\left\{ \sum_{n=1}^{\infty} \sigma_n^2 < \infty \right\}$.*