Lecture 16: March 2

Lecturer: Sourav Chatterjee *Scribes: Kenneth Tay*

16.1 Time Homogeneous Markov Chains on Countable State Spaces

16.1.1 Accessibility, Communication, Irreducibility

Let $X_0, X_1, \dots$ be a time homogeneous Markov chain on a countable state space S.

Let $p_{xy}^{(n)} := P(X_n = y \mid X_0 = x)$. We showed last lecture that a state x is recurrent if and only if $\displaystyle\sum_{n=1}^{\infty} p_{xx}^{(n)} = \infty$.

Definition 16.1 *A state y is said to be* ***accessible*** *from a state x if $p_{xy}^{(n)} > 0$ for some $n \geq 0$. If so, we write $x \to y$.*

Two states x and y are ***communicating*** *if x is accessible from y and y is accessible from x (i.e. $x \to y$ and $y \to x$). We write $x \leftrightarrow y$.*

Proposition 16.2 *Communication between states is an equivalence relation.*

Proof:

- $x \leftrightarrow x$ since $p_{xx}^{(0)} = 1$ by definition.
- If $x \leftrightarrow y$, it is clear that $y \leftrightarrow x$ as well.
- If $x \leftrightarrow y$ and $y \leftrightarrow z$, then $p_{xy}^{(m)} > 0$ and $p_{yz}^{(n)} > 0$ for some m and n, which implies, by Chapman-Kolmogorov, that
$$p_{xz}^{(m+n)} \geq p_{xy}^{(m)} p_{yz}^{(n)} > 0.$$
Thus $x \to z$. Similarly, $z \to x$.

■

Definition 16.3 *The Markov chain is said to be* ***irreducible*** *if the number of equivalence classes is 1.*

(**Note:** The concept of irreducibility is much more complicated in Markov chains with uncountable state spaces, because the concept of "accessibility" does not make sense.)

Definition 16.4 *A property of a state is called a* ***class property*** *if whenever a state has that property, every member of its equivalence class also has it.*

Proposition 16.5 *Recurrence (and therefore, transience) is a class property.*

Proof: Suppose x is recurrent. Since $x \leftrightarrow y$, $p_{xy}^{(m)} > 0$ and $p_{yx}^{(n)} > 0$ for some m and n. Thus, for any $k \geq 1$,

$$
\begin{aligned}
p_{yy}^{(m+n+k)} &\geq p_{yx}^{(n)} p_{xx}^{(k)} p_{xy}^{(m)}, && \text{(by Chapman-Kolmogorov)} \\
\Rightarrow \qquad \sum_{j=1}^{\infty} p_{yy}^{(j)} &\geq \sum_{k=1}^{\infty} p_{yy}^{(m+n+k)} \\
&\geq p_{yx}^{(n)} p_{xy}^{(m)} \sum_{k=1}^{\infty} p_{xx}^{(k)} \\
&= \infty,
\end{aligned}
$$

since x is recurrent. Hence, y is recurrent as well. ■

Lemma 16.6 *If the state space S is finite, then there exists at least one recurrent state.*

Proof: If a state x is transient, then we know that $\mathbb{E}_x N(x) < \infty$, where $N(x)$ is the number of returns to x. This in turn means that $N(x) < \infty$ a.s. if the chain starts at x.

If the chain starts somewhere else, then $N(x) < \infty$ a.s. as well, because either (i) x is never visited, or (ii) x is visited, then apply the strong Markov property and the first conclusion.

Therefore, if all states are transient, then each state is accessed finitely many times, no matter where the chain starts from. Since S is finite, this is impossible! Hence there exists at least one recurrent state.

■

Corollary 16.7 *If S is finite and the Markov chain is irreducible, then all states are recurrent.*

16.1.2 Periodicity

Definition 16.8 *The **period** of a state x is the greatest common divisor of all $n \geq 1$ such that $p_{xx}^{(n)} > 0$. It is denoted by $d(x)$.*

*A state is said to be **aperiodic** if its period is 1.*

For example, the period of 0 for the simple random walk on a lattice is 2.

Proposition 16.9 *If $x \leftrightarrow y$, then $d(x) = d(y)$.*

Proof: Suppose that $p_{xy}^{(m)} > 0$ and $p_{yx}^{(n)} > 0$. Suppose that $p_{xx}^{(s)} > 0$. Then

$$
\begin{aligned}
& p_{yy}^{(n+m)} \geq p_{yx}^{(n)} p_{xy}^{(m)} > 0 \quad \text{and} \quad p_{yy}^{(n+m+s)} \geq p_{yx}^{(n)} p_{xx}^{(s)} p_{xy}^{(m)} > 0, \\
\Rightarrow \quad & d(y) \mid n+m \quad \text{and} \quad d(y) \mid n+m+s, \\
\Rightarrow \quad & d(y) \mid s.
\end{aligned}
$$

This is true for all s such that $p_{xx}^{(s)} > 0$, which implies that $d(y) \mid d(x)$. Similarly, we can obtain $d(x) \mid d(y)$, and so $d(x) = d(y)$.

■

16.1.3 Positive & Null Recurrence

Definition 16.10 *For any recurrent state x, let $\mu_{xx} :=$ expected time of first return to x starting from x.*

*A recurrent state x is called **positive recurrent** if $\mu_{xx} < \infty$. If $\mu_{xx} = \infty$, it is called **null recurrent**.*

Note: Positive recurrence is a class property.

16.1.3.1 Example: Simple symmetric random walk

Before stating the example, we first prove Wald's equation for sums of i.i.d. random variables:

Proposition 16.11 (Wald's equation for sums of i.i.d. random variables) *Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}X_1 = \mu$, $\mathbb{E}|X_1| < \infty$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let τ be a stopping time w.r.t. $\{\mathcal{F}_n\}$. Let $S_n = \sum_{i=1}^{n} X_i$.*

If $\mathbb{E}\tau < \infty$, then $\mathbb{E}[S_\tau] = \mu \mathbb{E}\tau$.

Proof: Observe that $S_n - \mu n$ is a martingale, which implies that $\mathbb{E}[S_{\tau\wedge n}] = \mu\mathbb{E}[\tau \wedge n]$ (Wald's Lemma for bounded stopping times).

By the Monotone Convergence Theorem, $\mathbb{E}[\tau \wedge n]$ increases to $\mathbb{E}\tau$. Note that

$$|S_{\tau\wedge n}| = \left|\sum_{i=1}^{\tau\wedge n} X_i\right| \leq \sum_{i=1}^{\tau\wedge n} |X_i| \leq \sum_{i=1}^{\tau} |X_i| = \sum_{i=1}^{\infty} |X_i| 1_{\{\tau \geq i\}}.$$

Let $Z = \sum_{i=1}^{\tau} |X_i| = \sum_{i=1}^{\infty} |X_i| 1_{\{\tau\geq i\}}$. Note that $\{\tau \geq i\} = \{\tau < i\}^c = \{\tau \leq i-1\}^c \in \mathcal{F}_{i-1}$, so $|X_i|$ and $1_{\{\tau \geq i\}}$ are independent. Hence,

$$\begin{aligned}
\mathbb{E}Z &= \sum_{i=1}^{\infty} \mathbb{E}[|X_i| 1_{\{\tau\geq i\}}] && \text{(Monotone Convergence Theorem)} \\
&= \sum_{i=1}^{\infty} \mathbb{E}|X_i| P(\tau \geq i) && \text{(by independence)} \\
&= \mathbb{E}|X_1| \sum_{i=1}^{\infty} P(\tau \geq i) \\
&= \mathbb{E}|X_1| \mathbb{E}\tau < \infty.
\end{aligned}$$

As such, $S_{\tau\wedge n}$ is bounded above by an integrable Z, so by the Dominated Convergence Theorem, $\mathbb{E}[S_{\tau\wedge n}] \to \mathbb{E}[S_\tau]$. This completes the proof.

■

Let us now describe the example. Let S_n be a simple symmetric random walk on $\mathbb{Z}$, $S_0 = 0$. Let $T = \inf\{n \geq 1 : S_n = 0\}$.

Proposition 16.12 *$\mathbb{E}T = \infty$, i.e. 0 is null recurrent.*

Proof: Note that

$$\begin{aligned}\mathbb{E}[T] &= \mathbb{E}[\mathbb{E}[T \mid S_1]] \\ &= \frac{1}{2}\mathbb{E}[T \mid S_1 = 1] + \frac{1}{2}\mathbb{E}[T \mid S_1 = -1].\end{aligned}$$

Consider $\tau = \inf\{n : S_n = 1\}$ with $S_0 = 0$. If $\mathbb{E}\tau < \infty$, then by Wald's equation, $\mathbb{E}[S_\tau] = 0$. However, by definition, we must have $S_\tau = 1$. Contradiction! Therefore $\mathbb{E}\tau = \infty$.

The above implies that both $\mathbb{E}[T \mid S_1 = 1] = \mathbb{E}[T \mid S_1 = -1]\infty$, and so $\mathbb{E}[T] = \infty$.

■

Theorem 16.13 *If x is a recurrent state, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{xx}^{(m)} = \frac{1}{\mu_{xx}}.$$

(**Note:** This statement is true for both positive recurrent and null recurrent states.)

To prove this, we will need some tools from renewal theory.

16.2 Digression: Renewal Theory

Let $X_1, X_2, \ldots$ be i.i.d. non-negative random variables with $P(X_1 = 0) < 1$. Let $\mu = \mathbb{E}X_1$ (possibly infinite), $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_n$.

Definition 16.14 *For $t \geq 0$ with $t \in \mathbb{R}$, let $N(t) := \sup\{n : S_n \leq t\}$. Then $\{N(t) : t \geq 0\}$ is called a **renewal process**.*

We can think of this process as replacing lightbulbs, X_i is the lifetime of the i^{th} lightbulb, and when it dies, we replace it with a new lightbulb. S_n can be thought of as the time till the n^{th} lightbulb goes off.

Definition 16.15 *$m(t) := \mathbb{E}[N(t)]$ is called the **renewal function**.*

Theorem 16.16 (Elementary Renewal Theorem)

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$