STATS 310A: Theory of Probability I **Autumn 2016/17**

Lecture 3: October 3

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3.1 Extending a Measure

Start with a set Ω, $\mathcal{F}_0$ an algebra of subsets, P probability on $\mathcal{F}_0$ (i.e. countably additive if the countable union happens to be in $\mathcal{F}_0$ as well).

For all $E \subseteq \Omega$, define

$$P^*(E) := \inf \sum_{i=1}^{\infty} P(A_i)$$

where $A_i \in \mathcal{F}_0$ and $E \subseteq \bigcup A_i$.

We showed that $P^*(\emptyset) = 0$, $P^*(\Omega) = 1$, and for any $E_i \subseteq \Omega$, countable sub-additivity holds:

$$P^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P^*(E_i).$$

Definition 3.1 *Let $\mathcal{M}$ be all $A \subseteq \Omega$ such that for every $E \subseteq \Omega$,*

$$P^*(E) = P^*(E \cap A) + P^*(E \cap A^c).$$

We will prove a number of propositions/facts about $\mathcal{M}$. In proving these facts, we will often use a **key trick**: to show that

$$P^*(E) = P * (E \cap A) + P * (E \cap A^c),$$

we only need to show that

$$P^*(E) \geq P^*(E \cap A) + P * (E \cap A^c).$$

(The other direction is true by countable sub-additivity.)

Proposition 3.2 *$\mathcal{M}$ is a field.*

Proof: Clearly $\Omega \in \mathcal{M}$. If A in $\mathcal{M}$, then

$$\begin{aligned} P^*(E) &= P^*(E \cap A) + P^*(E \cap A^c) \\ &= P^*(E \cap A^c) + P^*(E \cap (A^c)^c) \end{aligned}$$

for all E. So A^c in $\mathcal{M}$ as well.

Say A and B are in $\mathcal{M}$. Then for any E,

$$\begin{aligned} P^*(E) &= P^*(E \cap A) + P^*(E \cap A^c) \\ &= P^*(E \cap A \cap B) + P^*(E \cap A \cap B^c) + P^*(E \cap A^c \cap B) + P^*(E \cap A^c \cap B^c) \\ &\geq P^*(E \cap A \cap B) + P^*((E \cap A \cap B^c) \cup (E \cap A^c \cap B) \cup (E \cap A^c \cap B^c)) \\ &= P * (E \cap (A \cap B)) + P^*(E \cap (A \cap B)^c). \end{aligned}$$

Thus by the key trick, we have $A \cap B \in \mathcal{M}$. Therefore $\mathcal{M}$ is a field. ∎

Proposition 3.3 *If $\{A_i\}_{i\in I}$ (I finite or countable) are disjoint sets in $\mathcal{M}$, then for any E,*

$$P^*\left(E \cap \bigcup_{i\in I} A_i\right) = \sum_{i\in I} P^*(E\cap A_i).$$

Proof: If $|I|=1$, there is nothing to prove.

If $|I|=2$,

$$\begin{aligned} P^*(E\cap(A_1\cup A_2)) &= P^*(E\cap(A_1\cup A_2)\cap A_1) + P^*(E\cap(A_1\cup A_2)\cap A_1^c) \\ &= P^*(E\cap A_1) + P^*(E\cap A_2). \end{aligned}$$

Thus the proposition is true for $n=2$. Hence, by induction, the proposition is true for any finite number of sets.

If $|I|=\infty$, let $A=\bigcup_{i=1}^{\infty} A_i$ disjoint, and define $F_n := \bigcup_{i=1}^{n} A_i$. Then

$$\begin{aligned} P^*(E\cap A) &\geq P^*(E\cap F_n) \\ &= \sum_{i=1}^{n} P^*(E\cap A_i) \end{aligned}$$

for any n. Letting $n\to\infty$, we have

$$P^*(E\cap A) \geq \sum_{i=1}^{\infty} P^*(E\cap A_i).$$

The other direction is immediately obtained from sub-additivity. ■

Proposition 3.4 *$\mathcal{M}$ is a σ-algebra, and P^* is countably additive on $\mathcal{M}$.*

Proof: To show that $\mathcal{M}$ is a σ-algebra, we just need to show closure under countable unions.

Let $A_i \in \mathcal{M}$, $1\leq i\leq\infty$. **Trick**: Let $A_1' = A_1$, $A_2' = A_2\cap A_1^c$, and

$$A_n' = A_n \cap \left(\bigcup_{1}^{n-1} A_i\right)^c.$$

(i.e. create a sequence of disjoint sets from A_i's).

The A_i's are disjoint and they are in $\mathcal{M}$, and $\bigcup A_i' = \bigcup A_i$. Hence, without loss of generality, we can assume that A_i are disjoint!

Set $F_n = \bigcup_1^n A_i$. Then for every $E\subseteq\Omega$,

$$\begin{aligned} P^*(E) &= P^*(E\cap F_n) + P^*(E\cap F_n^c) \\ &\geq \sum_{i=1}^{n} P^*(E\cap A_i) + P^*(E\cap A^c). \end{aligned}$$

Letting n go to infinity and using Prop 3.3,

$$
\begin{aligned}
P^*(E) &\geq \sum_{i=1}^{\infty} P^*(E \cap A_i) + P^*(E \cap A^c) \\
&= P^*(E \cap A) + P^*(E \cap A^c).
\end{aligned}
$$

Hence $A \in \mathcal{M}$.

To show that P^* is countably additive on $\mathcal{M}$, use Prop 3.3 with $E = \Omega$. ■

Proposition 3.5 *$\mathcal{F}_0 \subseteq \mathcal{M}$.*

Proof: Given E and $\varepsilon > 0$, I can choose A_i, $i = 1, 2, \ldots$, such that $A_i \in \mathcal{F}_0$ for all i and

$$
E \subseteq \bigcup A_i, \qquad \sum_{1}^{\infty} P(A_i) \leq P^*(E) + \varepsilon.
$$

For each n, let $B_n = A \cap A_n$, $C_n = A^c \cap A_n$. Note that B_n and C_n are in $\mathcal{F}_0$, and

$$
\bigcup B_n \supseteq A \cap E, \qquad \bigcup C_n \supseteq A^c \cap E.
$$

Hence,

$$
\begin{aligned}
P^*(E \cap A) + P^*(E \cap A^c) &\leq \sum P(B_n) + \sum P(C_n) \\
&= \sum_{n=1}^{\infty} P(A_n) \\
&= P^*(E) + \varepsilon.
\end{aligned}
$$

Let $\varepsilon \to 0$, and we are done. ■

Proposition 3.6 *$P^*(A) = P(A)$ for $A \in \mathcal{F}_0$, i.e. the restrction of P^* on $\mathcal{F}_0$ is equal to P.*

Proposition 3.7 *$\mathcal{M} \supseteq \sigma(\mathcal{F}_0)$, so P^* is countably additive on $\sigma(\mathcal{F}_0)$ and extends P. This extension is unique.*

The proof for Prop 3.6 is straightforward. To prove Prop 3.7, we need the $\pi - \lambda$ theorem.

3.2 π-Systems and λ-Systems

Definition 3.8 *A collection of subsets $\mathcal{P}$ of Ω is a **π-system** if it is closed under finite intersections.*

*A collection of subsets $\mathcal{L}$ of Ω is a **λ-system** if*

- *$\Omega \in \mathcal{L}$,*

- *$A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$,*
- *$\mathcal{L}$ closed under countable disjoint unions.*

Example: Take $\Omega = \{1, 2, 3, 4\}$, $\mathcal{L} = \{\emptyset, \Omega, 12, 13, 14, 23, 24, 34\}$. $\mathcal{L}$ is a λ-system but not a π-system.

Theorem 3.9 ($\pi - \lambda$ Theorem) *If a π-system $\mathcal{P}$ is contained in a λ-system $\mathcal{L}$, then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.*

We will postpone the proof till next time. Here are 2 quick applications of the $\pi - \lambda$ Theorem:

Corollary 3.10 *P^* is unique on $\sigma(\mathcal{F}_0)$.*

Proof: Let Q^* be another extension of P. Consider the collection of sets on which P^* and Q^* agree, i.e.

$$\mathcal{G} := \{A \in \sigma(\mathcal{F}_0) : P^*(A) = Q^*(A)\}.$$

$\mathcal{G}$ is a λ-system (just check the definition), and clearly $\mathcal{G} \supseteq \mathcal{F}$. Also, $P^*(A) = Q^*(A)$ on $A \in \mathcal{F}_0$, which is a π-system. Hence, they must agree on $\sigma(\mathcal{F}_0)$. ■

Corollary 3.11 *The Lebesgue measure on the Borel sets of $(0, 1]$ is the only probability extending length on intervals.*

3.3 Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Two sets $A_1, A_2 \in \mathcal{F}$ are **independent** if

$$P(A_1 \cap A_2) = P(A_1)P(A_2).$$

A family of σ-algebras $\{\mathcal{F}_i\}_{i \in I}$ are **independent** if for every $k, i_1, i_2, \ldots, i_k$, and $A_{i_k} \in \mathcal{F}_{i_k}$,

$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j})$$

Example: $\Omega = (0, 1]$, P is length on Borel sets, and $\omega = \sum_{i=1}^{\infty} \frac{d_i(\omega)}{2^i}$. Let $A_i := \{d_i = 1\}$, $\mathcal{F}_i := \sigma(A_i)$. Then the $\mathcal{F}_i$'s are independent.