

Lecture 17: March 7

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17.1 Renewal Theory

Let X_1, X_2, \dots be i.i.d. non-negative random variables with $P(X_1 = 0) < 1$. Let $\mu = \mathbb{E}X_1$ (possibly infinite), $S_0 = 0$, $S_n = \sum_{i=1}^n X_i$.

Definition 17.1 For $t \geq 0$ with $t \in \mathbb{R}$, let $N(t) := \sup\{n : S_n \leq t\}$. Then $\{N(t) : t \geq 0\}$ is called a *renewal process*.

We can think of this process as replacing lightbulbs, X_i is the lifetime of the i^{th} lightbulb, and when it dies, we replace it with a new lightbulb. S_n can be thought of as the time till the n^{th} lightbulb goes off.

Definition 17.2 $m(t) := \mathbb{E}[N(t)]$ is called the *renewal function*.

Before proving the Elementary Renewal Theorem, we introduce a few lemmas that we will use in the proof.

Lemma 17.3 Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. For any t , $N(t) + 1$ is a stopping time w.r.t. \mathcal{F}_n .

Proof:

$$\begin{aligned} \{N(t) + 1 = n\} &= \{N(t) = n - 1\} \\ &= \{S_1, S_2, \dots, S_{n-1} \leq t, S_n > t\} \\ &\in \mathcal{F}_n. \end{aligned}$$

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Lemma 17.4 For all t , $m(t) < \infty$.

Proof: Let $p = \mathbb{E}[e^{-X_1}]$. Since X_1 is non-negative, $1 - e^{-X_1} \geq 0$. If $p = 1$, then $\mathbb{E}[1 - e^{-X_1}] = 0$, which implies that $X_1 = 0$ a.s. But this is not true by assumption! Thus $p < 1$.

Note that

$$\begin{aligned} P(N(t) = n) &\leq P(S_n \leq t) \\ &= P(e^{-S_n} \geq e^{-t}) \\ &\leq e^t \mathbb{E}[e^{-S_n}] && \text{(Markov)} \\ &= e^t (\mathbb{E}[e^{-X_1}])^n \\ &= e^t p^n, \end{aligned}$$

so

$$\begin{aligned} m(t) &= \sum_{n=1}^{\infty} nP(N(t) = n) \\ &\leq e^t \mathbb{E}[X] \\ &< \infty. \end{aligned} \quad (\text{where } X \sim \text{Geom}(p))$$

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Corollary 17.5 *If $\mu < \infty$, then $\mathbb{E}[S_{N(t)+1}] = \mu(m(t) + 1)$.*

Proof: This follows directly from the 2 lemmas above and an application of Wald's equation for sums of i.i.d. random variables. ■

Theorem 17.6 (Elementary Renewal Theorem)

$$\lim_{t \rightarrow \infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

Proof: Note that by definition of $N(t)$, $S_{N(t)+1} > t$.

- If $\mu < \infty$, then by Corollary 17.5,

$$\begin{aligned} \mu(m(t) + 1) &= \mathbb{E}[S_{N(t)+1}] \geq t, \\ \liminf_{t \rightarrow \infty} \frac{m(t)}{t} &\geq \frac{1}{\mu}. \end{aligned}$$

- If $\mu = \infty$, then the above is trivially true as the LHS is the lim inf of a non-negative sequence.

It remains to show that $\limsup_{t \rightarrow \infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$. Let M be a positive real number. Define a new renewal process as follows:

$$\bar{X}_n = \begin{cases} X_n & \text{if } X_n \leq M, \\ M & \text{if } X_n > M. \end{cases}$$

Define $\bar{S}_n = \sum_{i=1}^n \bar{X}_i$, $\bar{N}(t) = \sup\{n : \bar{S}_n \leq t\}$, $\bar{m}(t) = \mathbb{E}[\bar{N}(t)]$. Define $\mu_M = \mathbb{E}\bar{X}_1 \leq M < \infty$.

Then $\bar{S}_{\bar{N}(t)+1} \leq t + M$. By Corollary 17.5,

$$\begin{aligned} \mu_M(\bar{m}(t) + 1) &\leq \mathbb{E}[\bar{S}_{\bar{N}(t)+1}] \leq t + M, \\ \limsup_{t \rightarrow \infty} \frac{\bar{m}(t)}{t} &\leq \frac{1}{\mu_M}. \end{aligned}$$

Note that $\bar{S}_n \leq S_n$ for all n , which implies that $\bar{N}(t) \geq N(t)$ for all t , and so

$$\limsup_{t \rightarrow \infty} \frac{m(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_M}.$$

As $M \rightarrow \infty$, $0 \leq \bar{X}_1 \nearrow X_1$. Thus, by the Monotone Convergence Theorem, $\mu_M \rightarrow \mu$. This gives us the desired result. ■

17.2 Time Homogeneous Markov Chains on Countable State Spaces

Recall that $p_{xy}^{(n)} = P(X_n = y \mid X_0 = x)$, $\mu_{xy} = \mathbb{E}[\text{time to hit } y \text{ starting from } x \text{ after time } 0]$.

Theorem 17.7 *If x is a recurrent state, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n p_{xx}^{(m)} = \frac{1}{\mu_{xx}}.$$

Proof: Suppose that the chain starts at x . Then the return times to x form a renewal process with i.i.d. increments (by the Strong Markov Property). Hence,

$$\frac{1}{n} \sum_{k=1}^n p_{xx}^{(k)} = \frac{1}{n} \mathbb{E}_x[\text{no. of returns to } x \text{ by time } n]$$

is just $\frac{m(n)}{n}$ for this renewal process. Also, the inter-arrival times for this renewal process have expected value μ_{xx} .

Thus, we have the result by applying the Elementary Renewal Theorem. ■

Corollary 17.8 *If x and y communicate ($x \leftrightarrow y$) and are recurrent, then*

$$\frac{1}{n} \sum_{m=1}^n p_{yx}^{(m)} = \frac{1}{\mu_{xx}}.$$

Proof:

Step 1: $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n p_{yx}^{(m)} \leq \frac{1}{\mu_{xx}}.$

Since $x \leftrightarrow y$ and are recurrent, the chain starting at y is sure to hit x eventually. (You can try to prove this.)

With the Strong Markov property, this implies that

$$\mathbb{E}_y[\text{no. of visits to } x \text{ by time } n] \leq \mathbb{E}_x[\text{no. of visits to } x \text{ by time } n] + 1.$$

Applying Theorem 17.7 gives

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n p_{yx}^{(m)} \leq \frac{1}{\mu_{xx}}.$$

This completes the proof if $\mu_{xx} = \infty$. Assume from now on that $\mu_{xx} < \infty$.

Step 2: If T_x is the first hitting time of x , then $\mu_{yx} = \mathbb{E}_y[T_x] < \infty$.

Since $x \leftrightarrow y$, there exists $x_0 = x, x_1, \dots, x_k = y$ such that $p_{x_0 x_1} p_{x_1 x_2} \dots p_{x_{k-1} x_k} > 0$. Without loss of generality, $x_1, x_2, \dots, x_{k-1} \neq x$. Then

$$\begin{aligned} \mu_{xx} &= \mathbb{E}_x[T_x] \\ &\geq \mathbb{E}_x[T_x; X_1 = x_1, \dots, X_{k-1} = x_{k-1}, X_k = y] \\ &= \mathbb{E}_x[T_x \mid X_1 = x_1, \dots, X_{k-1} = x_{k-1}, X_k = y] P_x(X_1 = x_1, \dots, X_{k-1} = x_{k-1}, X_k = y) \\ &= (\mu_{yx} + k) P_x(X_1 = x_1, \dots, X_{k-1} = x_{k-1}, X_k = y). \end{aligned}$$

Since the probability on the RHS is positive, it follows that $\mu_{yx} < \infty$.

Step 3: $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n p_{yx}^{(m)} \geq \frac{1}{\mu_{xx}}.$

$$\begin{aligned} \sum_{m=1}^n p_{yx}^{(m)} &= \mathbb{E}_y[\text{no. of visits to } x \text{ by time } n] \\ &\geq \mathbb{E}_y[\text{no. of visits to } x \text{ in time interval } [T_x + 1, T_x + n] - T_x] \\ &= \sum_{m=1}^n p_{xx}^{(m)} - \mu_{yx}. \end{aligned} \quad (\text{by Strong Markov Property})$$

Since $\mu_{yx} < \infty$, this shows that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n p_{yx}^{(m)} \geq \frac{1}{\mu_{xx}}.$$

This completes the proof. ■