

Lecture 12: February 16

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12.1 Ergodic Theorems

Theorem 12.1 (Maximal Ergodic Theorem) Let (Ω, \mathcal{F}, P) be a probability space, φ a measure-preserving transform with invariant σ -algebra \mathcal{I} , and X a random variable in L^1 .

Let $X_j(\omega) := X(\varphi^j(\omega))$, $S_k(\omega) = X_0(\omega) + X_1(\omega) + \dots + X_k(\omega)$, and $M_k(\omega) = \max\{0, S_0(\omega), S_1(\omega), \dots, S_k(\omega)\}$.

Then $\mathbb{E}[X; M_k > 0] \geq 0$.

Proof: (Due to Garsia.)

Note that $X_0(\omega) = X(\omega)$. If $j \leq k$, then $M_k(\varphi(\omega)) \geq S_j(\varphi(\omega))$, which implies that

$$X(\omega) + M_k(\varphi(\omega)) \geq X(\omega) + S_j(\varphi(\omega)) = S_{j+1}(\omega).$$

Thus, $X(\omega) \geq S_{j+1}(\omega) - M_k(\varphi(\omega))$ for $j = 0, 1, \dots, k$. Also, $X(\omega) \geq S_0(\omega) - M_k(\varphi(\omega))$ since $X = S_0$ and M_k is non-negative. Thus,

$$\begin{aligned} \mathbb{E}[X; M_k > 0] &\geq \int_{\{\omega: M_k(\omega) > 0\}} \left(\max\{S_0(\omega), \dots, S_k(\omega)\} - M_k(\varphi(\omega)) \right) dP(\omega) \\ &= \int_{\{\omega: M_k(\omega) > 0\}} \left(M_k(\omega) - M_k(\varphi(\omega)) \right) dP(\omega). \end{aligned}$$

Since φ is measure-preserving, $\int Y dP = \int Y \circ \varphi dP$ for all Y . Applying this to M_k , $\int_{\Omega} (M_k(\omega) - M_k(\varphi(\omega))) dP(\omega) = 0$. But, by M_k 's non-negativity, on the set $\{M_k \leq 0\}$, $M_k(\omega) - M_k(\varphi(\omega))$ is ≤ 0 . Thus,

$$\mathbb{E}[X; M_k > 0] \geq \int_{\{\omega: M_k(\omega) > 0\}} \left(M_k(\omega) - M_k(\varphi(\omega)) \right) dP(\omega) \geq 0.$$

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Theorem 12.2 (Birkhoff's Ergodic Theorem) Let (Ω, \mathcal{F}, P) be a probability space, φ a measure-preserving transform with invariant σ -algebra \mathcal{I} , and X a random variable in L^1 .

Then $\frac{1}{n} \sum_{m=0}^{n-1} X(\varphi^m(\omega)) \rightarrow \mathbb{E}[X | \mathcal{I}]$ a.s. and in L^1 .

In particular, if φ is ergodic, then $\frac{1}{n} \sum_{m=0}^{n-1} X(\varphi^m(\omega)) \rightarrow \mathbb{E}X$ a.s. and in L^1 ("space average is equal to time average").

Proof:

Step 1: We may assume that $\mathbb{E}[X \mid \mathcal{I}] = 0$ a.s.

Note that if Y is \mathcal{I} -measurable, then $Y \circ \varphi = Y$ a.s., so $Y \circ \varphi^j = Y$ a.s. for all j . $\mathbb{E}[X \mid \mathcal{I}]$ is \mathcal{I} -measurable. Thus, if $X' = X - \mathbb{E}[X \mid \mathcal{I}]$, then for all j , $X' \circ \varphi^j = X \circ \varphi^j - \mathbb{E}[X \mid \mathcal{I}]$, implying that

$$\frac{1}{n} \sum_{m=1}^{n-1} (X \circ \varphi^m - \mathbb{E}[X \mid \mathcal{I}]) = \frac{1}{n} \sum_{m=1}^{n-1} X' \circ \varphi^m.$$

Also, $\mathbb{E}[X' \mid \mathcal{I}] = 0$ a.s., so it suffices to prove the result for X' .

As in the set-up of the Maximal Ergodic Theorem, let $X_n = X \circ \varphi^n$, $S_n = X_0 + X_1 + \cdots + X_n$.

Step 2: $S_n/n \rightarrow 0$ a.s.

We will show that $\limsup \frac{S_n}{n+1} \leq 0$ a.s. A symmetric argument will show that $\liminf \frac{S_n}{n+1} \geq 0$ a.s., and so $\lim \frac{S_n}{n+1} = 0$ a.s.

Let $\bar{X} = \limsup \frac{S_n}{n+1}$. Note that \bar{X} is \mathcal{I} -measurable as $\bar{X} \circ \varphi = \bar{X}$. Take any $\varepsilon > 0$ and let $D = \{\omega : \bar{X}(\omega) > \varepsilon\}$. Then $D \in \mathcal{I}$.

Let $X^* = (X - \varepsilon)1_D$. Let $S_n^* = X^* + X^* \circ \varphi + \cdots + X^* \circ \varphi^n$, and let $M_n^* = \max\{0, S_0^*, \dots, S_n^*\}$.

Let $F_n = \{\omega : M_n^*(\omega) > 0\}$, and let $F = \bigcup_{n=0}^{\infty} F_n$.

Step 2.1: $F = \left\{ \omega : \sup_{k \geq 0} \frac{S_k^*(\omega)}{k+1} > 0 \right\}$.

$$\begin{aligned} \omega \in F &\Leftrightarrow \omega \in F_n && \text{for some } n \\ &\Leftrightarrow M_n^*(\omega) > 0 && \text{for some } n \\ &\Leftrightarrow S_j^*(\omega) > 0 && \text{for some } 0 \leq j \leq n \\ &\Leftrightarrow S_j^*(\omega)/(j+1) > 0 && \text{for some } 0 \leq j \leq n \\ &\Leftrightarrow \sup_{k \geq 0} \frac{S_k^*(\omega)}{k+1} > 0. \end{aligned}$$

Step 2.2: $F = D$.

If $\omega \in F$, then $\frac{S_k^*(\omega)}{k+1} > 0$ for some k , implying that $S_k^*(\omega) > 0$ for some k . Since $S_k^* = X_0^* + \cdots + X_k^*$, it means that $X_j^*(\omega) > 0$ for some j . But by definition,

$$X_j^*(\omega) = X^*(\varphi^j(\omega)) = [X(\varphi^j(\omega)) - \varepsilon]1_D(\varphi^j(\omega)),$$

so, for some j ,

$$\begin{aligned} X_j^*(\omega) > 0 &\Rightarrow [X(\varphi^j(\omega)) - \varepsilon]1_D(\varphi^j(\omega)) > 0 \\ &\Rightarrow \varphi^j(\omega) \in D \\ &\Rightarrow \omega \in \varphi^{-j}(D) \\ &\Rightarrow \omega \in D. \end{aligned} \quad (\text{since } D \in \mathcal{I})$$

On the other hand, if $\omega \in D$, then $\varphi^j(\omega) \in D$ for all j (by definition of D and \bar{X}). This implies that

$$\begin{aligned} S_n^*(\omega) &= \sum_{i=0}^n X_i^*(\omega) = \sum_{i=0}^n X_i(\omega) - \varepsilon = S_n(\omega) - \varepsilon(n+1), \\ \sup \frac{S_n^*(\omega)}{n+1} &\geq \limsup \frac{S_n^*(\omega)}{n+1} \\ &= \limsup \frac{S_n(\omega)}{n+1} - \varepsilon \\ &> 0, \end{aligned}$$

i.e. $\omega \in F$.

Step 2.3: Use the Maximum Ergodic Theorem to conclude that $P(D) = 0$.

Note that $E|X^*| < E|X| + \varepsilon < \infty$. The Maximum Ergodic Theorem implies that $\mathbb{E}[X^*; F_n] \geq 0$ for all n . Thus,

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \mathbb{E}[X^* 1_{F_n}] = \mathbb{E} \left[\lim_{n \rightarrow \infty} X^* 1_{F_n} \right] && \text{(Dominated Convergence Theorem)} \\ &= \mathbb{E}[X^*; F] \\ &= \mathbb{E}[X^*; D] && \text{(since } F = D) \\ &= \mathbb{E}[(X - \varepsilon) 1_D] \\ &= \mathbb{E}[\mathbb{E}[(X - \varepsilon) 1_D \mid \mathcal{I}]] \\ &= \mathbb{E}[(\mathbb{E}[X \mid \mathcal{I}] - \varepsilon) 1_D] \\ &= -\varepsilon P(D), \end{aligned}$$

hence $P(D) = 0$, as required.

Step 3: $S_n/n \rightarrow 0$ in L^1 .

Take any $M > 0$. Let $X'_M := X 1_{\{|X| \leq M\}}$, $X''_M = X - X'_M$.

Then, by the a.s. part of Birkhoff's Ergodic Theorem (Step 2), $\frac{1}{n} \sum_{m=0}^{n-1} X'_M(\varphi^m(\omega)) \rightarrow \mathbb{E}[X'_M \mid \mathcal{I}]$ a.s. By the Dominated Convergence Theorem, this convergence also holds in L^1 . If we can show that $\frac{1}{n} \sum_{m=0}^{n-1} X''_M(\varphi^m(\omega)) \rightarrow \mathbb{E}[X''_M \mid \mathcal{I}]$ in L^1 , we would be done.

Since φ is measure-preserving, $\mathbb{E}[X''_M \circ \varphi^m] = \mathbb{E}[X''_M]$ for all m , and so

$$\mathbb{E} \left| \frac{1}{n} \sum_{m=0}^{n-1} X''_M \circ \varphi^m \right| \leq \frac{1}{n} \sum_{m=0}^{n-1} \mathbb{E}|X''_M \circ \varphi^m| = \mathbb{E}|X''_M|.$$

By Jensen's inequality, we also have $\mathbb{E}|\mathbb{E}[X''_M \mid \mathcal{I}]| \leq \mathbb{E}|X''_M|$. Putting the two together, we get

$$\mathbb{E} \left| \frac{1}{n} \sum_{m=0}^{n-1} X''_M \circ \varphi^m - \mathbb{E}[X''_M \mid \mathcal{I}] \right| \leq 2\mathbb{E}|X''_M|$$

for all n , and so

$$\limsup_{n \rightarrow \infty} \mathbb{E} \left| \frac{1}{n} \sum_{m=0}^{n-1} X''_M \circ \varphi^m - \mathbb{E}[X''_M \mid \mathcal{I}] \right| \leq 2\mathbb{E}|X''_M|. \quad (12.1)$$

In the above, M is arbitrary. As $M \rightarrow \infty$, $|X_M''| \rightarrow 0$ a.s. Since $|X_M''| < |X|$ for all M and $\mathbb{E}|X| < \infty$, we can use the Dominated Convergence Theorem to get $\lim \mathbb{E}|X_M''| = 0$. Thus, letting $M \rightarrow \infty$ in Equation 12.1, we get the desired result. ■