STATS 310B: Theory of Probability II **Winter 2016/17**

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4.1 Application of Wald's Lemma to Random Walks

Consider the following setting:

- $X_1, X_2, \ldots$ be i.i.d. ± 1 valued random variables with $P\{X_i = 1\} = P\{X_i = -1\} = \frac{1}{2}$.
- $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $\mathcal{F}_0 =$ trivial σ-algebra.
- $S_n = \sum_{i=1}^n X_i$, $S_0 = 0$.
- $a < 0 < b$ are 2 integers, T is defined to be a stopping time $T = \inf\{k : S_k = a \text{ or } b\}$.

4.1.1 Symmetric Random Walk ($p = \frac{1}{2}$)

$\{S_n\}$ is a martingale w.r.t. $\{\mathcal{F}_n\}$. If we can show that $T < \infty$ a.s, then we can use Wald's Lemma (last lecture) to show that $P(S_T = a) = \dfrac{b}{b-a}$.

We now show that $\mathbb{E}T < \infty$. It then follows that $T < \infty$ a.s.

Proof: Define

$$T' := \min\{k : X_n = 1 \text{ for } n = (k-1)(b-a), (k-1)(b-a)+1, \ldots, (k-1)(b-a)+(b-a)\}.$$

(i.e. Partition $\mathbb{N}$ into blocks of size $b-a$, T' is the first time each step in the block is 1.) Then $T' \sim \text{Geom}(2^{-(b-a)})$, and so $\mathbb{E}T' < \infty$. However, note that $T \leq (b-a)T'$! Thus, $\mathbb{E}T \leq (b-a)\mathbb{E}T' < \infty$. ■

We can actually compute $\mathbb{E}T$ exactly:

Proposition 4.1 $\mathbb{E}T = -ab$.

Proof: Recall that $\{S_n^2 - n\}$ is also a martingale w.r.t. $\{\mathcal{F}_n\}$. Thus, by Wald's Lemma, for all n,

$$\mathbb{E}[S_{T\wedge n}^2 - T \wedge n] = 0,$$
$$\mathbb{E}[S_{T\wedge n}^2] = \mathbb{E}[T \wedge n].$$

Now, $S_{T\wedge n}^2 \to S_T^2$ and $S_{T\wedge n}$ is a uniformly bounded sequence of random variables, so by the Bounded Convergence Theorem, $\mathbb{E}[S_{T\wedge n}^2] \to \mathbb{E}[S_T^2]$. Also, $0 \leq T \wedge n \nearrow T$, so by the Monotone Convergence Theorem,

$\mathbb{E}[T \wedge n] \to \mathbb{E}T$. Hence,

$$\begin{aligned}\mathbb{E}T &= \mathbb{E}[S_T^2] \\ &= a^2 \frac{b}{b-a} + b^2 \frac{-a}{b-a} \\ &= -ab.\end{aligned}$$

■

4.1.1.1 Cautionary Tale

Consider the stopping time $T = \inf\{k : S_k = 1\}$. By Wald's Lemma, $\mathbb{E}[S_{T\wedge n}] = 0$ for all n. We can also show that $T < \infty$ a.s., so $S_{T\wedge n} \to S_T$.

However, $\mathbb{E}S_T = 1$ since $S_T = 1$ always! This happens because we do not have $\mathbb{E}S_{T\wedge n} \to \mathbb{E}S_T$: the Dominated Converge Theorem does not apply as $\{|S_{T\wedge n}|\}$ cannot be bounded above by an integrable random variable.

4.1.2 Biased Random Walk ($p \neq \frac{1}{2}$)

$\{S_n\}$ will no longer be a martingale. While $\{\sum_{i=1}^n X_i - \mathbb{E}X_i\} = \{S_n - n(2p-1)\}$ is a martingale, Wald's Lemma won't help us.

Instead, consider $M_n = \left(\frac{q}{p}\right)^{S_n}$, where $q = 1 - p$. $\{M_n\}$ is a martingale:

$$\begin{aligned}\mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}} \Big| \mathcal{F}_n\right] &= \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_n}\left(\frac{q}{p}\right)^{X_{n+1}} \Big| \mathcal{F}_n\right] \\ &= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{n+1}} \Big| \mathcal{F}_n\right] \\ &= \left(\frac{q}{p}\right)^{S_n} \left[\left(\frac{q}{p}\right)^1 p + \left(\frac{q}{p}\right)^{-1} q\right] \\ &= M_n.\end{aligned}$$

Hence, with T as before, by Wald's Lemma,

$$\mathbb{E}\left(\frac{q}{p}\right)^{S_{T\wedge n}} = \mathbb{E}M_0 = 1.$$

Using a similar argument as that for the symmetric random walk, $T < \infty$ a.s., so $S_{T\wedge n} \to S_T$ a.s. and $|S_{T\wedge n}|$ is uniformly bounded, so

$$\begin{aligned}\mathbb{E}\left[\left(\frac{q}{p}\right)^{S_T}\right] &= 1, \\ \left(\frac{q}{p}\right)^a P\{S_T = a\} + \left(\frac{q}{p}\right)^b (1 - P\{S_T = a\}) &= 1, \\ P\{S_T = a\} &= \frac{1 - (q/p)^b}{(q/p)^a - (q/p)^b}.\end{aligned}$$

4.2 Submartingales and Supermartingales

Note that we can define $\mathbb{E}X$ whenever at least one of $\mathbb{E}X^+$ and $\mathbb{E}X^-$ is finite. Similarly, we can define $\mathbb{E}[X \mid \mathcal{G}]$ whenever at least one of $\mathbb{E}X^+$ and $\mathbb{E}X^-$ is finite.

Definition 4.2 *$\{Z_n, \mathcal{F}_n\}_{n=1}^{\infty}$ is a **submartingale** if:*

1. *$\mathbb{E}Z_n^+ < \infty$ for all n, and*
2. *$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] \geq Z_n$ a.s. for all n.*

Definition 4.3 *$\{Z_n, \mathcal{F}_n\}_{n=1}^{\infty}$ is a **supermartingale** if:*

1. *$\mathbb{E}Z_n^- < \infty$ for all n, and*
2. *$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] \leq Z_n$ a.s. for all n.*

Proposition 4.4 *Let $\{Z_n, \mathcal{F}_n\}_{n=1}^{\infty}$ be a martingale. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Assume that $\mathbb{E}|\phi(Z_n)| < \infty$ for all n.*

Then $\{\phi(Z_n), \mathcal{F}_n\}_{n=1}^{\infty}$ is a submartingale.

Proof: By Jensen's inequality,

$$\mathbb{E}[\phi(Z_{n+1}) \mid \mathcal{F}_n] \geq \phi(\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]) = \phi(Z_n).$$

∎

Proposition 4.5 *Let $\{Z_n, \mathcal{F}_n\}_{n=1}^{\infty}$ be a <u>submartingale</u>. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex and <u>non-decreasing</u> function. Assume that $\mathbb{E}|\phi(Z_n)| < \infty$ for all n.*

Then $\{\phi(Z_n), \mathcal{F}_n\}_{n=1}^{\infty}$ is again a submartingale.

Proof: By Jensen's inequality,

$$\mathbb{E}[\phi(Z_{n+1}) \mid \mathcal{F}_n] \geq \phi(\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]) \geq \phi(Z_n).$$

(Need ϕ to be non-decreasing for the last inequality.) ∎

4.2.1 Examples

- If $\{Z_n\}$ is a martingale, then $\{|Z_n|\}$ is a submartingale.
- If $\{Z_n\}$ is a submartingale, $\{|Z_n|\}$ may not be a submartingale, but $\{Z_n^+\}$ is a submartingale.
- If $\{Z_n\}$ is a martingale, then $\{Z_n^2\}$ is a submartingale.
- If $\{Z_n\}$ is a submartingale, $\{Z_n^2\}$ may not be a submartingale.

4.3 (Sub)martingale Convergence Theorem

Theorem 4.6 *Let* $\{Z_n, \mathcal{F}_n\}_{n=1}^{\infty}$ *be a submartingale. Suppose* $\sup_n \mathbb{E}Z_n^+ < \infty$*.*

Then there is a random variable Z *taking values in* $[-\infty, \infty)$ *such that* $Z_n \to Z$ *a.s.*

(**Note:** Martingales are submartingales, and so the theorem above applies as well. For a martingale, we will have Z finite a.s.)

Main Idea of Proof:

We want to show that $\lim Z_n$ exists, and is $< \infty$ a.s. Let's consider the probability that $\lim Z_n$ does not exist. Note that

$$\{\lim Z_n \text{ does not exist in } [-\infty, \infty]\} = \{\exists\, a, b \in \mathbb{Q}, a < b \text{ such that } \liminf Z_n \leq a, \text{ and } \limsup Z_n \geq b\},$$

hence

$$P\{\lim Z_n \text{ does not exist in } [-\infty, \infty]\} \leq \sum_{a,b \in \mathbb{Q}} P\{\liminf Z_n \leq a, \text{ and } \limsup Z_n \geq b\}.$$

We will show that for each (a, b) with $a < b$, the probability on the RHS is equal to 0.

Once we know that $Z = \lim Z_n$ exists a.s., then $Z_n^+ \to Z^+$ a.s., so by Fatou's Lemma and the fact that $\sup_n \mathbb{E}Z_n^+ < \infty$,

$$\mathbb{E}Z^+ \leq \liminf \mathbb{E}Z_n^+ < \infty.$$