STATS 300A: Theory of Statistics I **Autumn 2016/17**

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14.1 Examples of UMP Tests

14.1.1 Example: Normal setting

Let X and Y be independent, $X \sim \mathcal{N}(\mu_x, \sigma^2)$, $Y \sim \mathcal{N}(\mu_y, \sigma^2)$ with σ^2 known. Testing $H_0 : \mu_y \leq \mu_x$ vs. $H_1 : \mu_y > \mu_x$.

As a first step, fix an alternative (θ_x, θ_y) with $\theta_y > \theta_x$. Let Λ put mass 1 on the point $(\bar{\theta}, \bar{\theta})$, where $\bar{\theta} = \dfrac{\theta_x + \theta_y}{2}$.

We then apply the Neyman-Pearson Lemma to the test H_Λ vs. the fixed alternative. In this setting, the likelihood ratio is given by

$$\begin{aligned}
\frac{\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x-\theta_x)^2\right\} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(y-\theta_y)^2\right\}}{\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x-\bar{\theta})^2\right\} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(y-\bar{\theta})^2\right\}} &\propto \exp\left\{\frac{1}{\sigma^2}(\theta_x X + \theta_y Y - \bar{\theta}X - \bar{\theta}Y)\right\} \\
&= \exp\left\{\frac{1}{\sigma^2}\left(\frac{\theta_x - \theta_y}{2}X + \frac{\theta_y - \theta_x}{2}Y\right)\right\} \\
&= \exp\left\{\frac{1}{\sigma^2}\left(\frac{\theta_y - \theta_x}{2}\right)(Y - X)\right\}.
\end{aligned}$$

Hence, the MP level α test φ rejects for large values of $Y - X$, i.e. if $\dfrac{Y - X}{\sqrt{2}\sigma} > Z_{1-\alpha}$.

Note that for any $\mu_y \leq \mu_x$, $Y - X$ is normally distributed with negative mean, hence

$$P_{\mu_x,\mu_y}\left(\frac{Y - X}{\sqrt{2}\sigma} > Z_{1-\alpha}\right) \leq \alpha,$$

i.e. φ is still level α for the composite H_0. In addition, φ does not depend on θ_x and θ_y. Hence, φ is UMP level α.

14.1.2 Example: Multivariate normal setting

(Details in TSH.)

Suppose X has multivariate normal distribution with mean vector $\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_d \end{pmatrix}$ and some known covariance matrix

Σ. Testing $H_0 : \sum_{i=1}^{d} c_i\mu_i \leq \delta$ vs. $H_1 : \sum_{i=1}^{d} c_i\mu_i > \delta$, where δ, $c_1, \ldots, c_d$ are fixed and known.

An argument similar to that of Section 14.1.1 shows that a UMP level α test exists, and it rejects for large values of $\sum_{i=1}^{d} c_i X_i$.

14.1.3 Example: General testing problem

Let $X_1, \ldots, X_n$ iid, $X_i \sim f_\theta$ where $\theta \in \mathbb{R}^d$. Let g be some real-valued function. Testing $H_0 : g(\theta) \leq 0$ vs. $H_1 : g(\theta) > 0$.

If θ_0 is such that $g(\theta_0) = 0$, then for θ near θ_0, we can use Taylor's approximation to get

$$g(\theta) \approx g(\theta_0) + \bigtriangledown g(\theta_0) \cdot (\theta - \theta_0),$$

where $\bigtriangledown g(\theta_0)$ and $\theta - \theta_0$ are $d \times 1$ vectors. Note that the RHS is a linear combination of components of θ! Thus, the setting can be reduced to "approximately" testing a linear combination of θ's components being ≤ 0 or > 0. This is similar to the setting of the previous section.

14.1.4 Example: Non-parametric setting (Sign Test)

Let $X_1, \ldots, X_n$ be iid on $\mathbb{R}$. Let $X_i \sim P \in \{\text{all CDFs on } \mathbb{R}\}$. Testing $H_0 : P\{X_i \leq u\} \geq p_0$ vs. $H_1 : P\{X_i \leq u\} < p_0$, where u and p_0 are fixed.

Parametrize P in the following way: Let P^- and P^+ be the conditional distribution of X given $X \leq u$ and $X > u$ respectively. Let $p = P\{X_i \leq u\}$. Then P can be identified with the triple (P^+, P^-, p).

Let p_- and p_+ be the densities of P^- and P^+ respectively. Then the joint density of $X_1, \ldots, X_n$ at some point $x_1, \ldots, x_n$ is given by

$$p^m(1-p)^{n-m} p_-(x_{i1}) \ldots p_-(x_{im}) p_+(x_{j1}) \ldots p_+(x_{j(n-m)}),$$

if $x_{i1}, \ldots, x_{im} \leq u$ and $x_{j1}, \ldots, x_{j(n-m)} > u$.

Fix an alternative: (P_0, P_1, p) with $p < p_0$. Let Λ concentrate its mass on the point (P_0, P_1, p_0). The likelihood ratio for h_Λ vs. the fixed alternative is

$$\frac{p^m(1-p)^{n-m}}{p_0^m(1-p_0)^{n-m}}$$

where $m = \#\{X_i \leq u\}$. Note that the likelihood ratio is a decreasing function of m, so by the Neyman-Pearson Lemma, the MP level α test rejects for small values of m. Under the null hypothesis h_Λ, $m \sim \text{Binom}(n, p_0)$. So the MP level α test function φ is given by

$$\varphi(X) = \begin{cases} 1 & \text{if } m < k, \\ \gamma & \text{if } m = k, \\ 0 & \text{if } m > k, \end{cases}$$

where k and γ are determined by the binomial distribution so that

$$\sum_{j=0}^{k-1} \binom{n}{j} p_0^j (1-p_0)^{n-j} + \gamma \binom{n}{k} p_0^k (1-p_0)^{n-k} = \alpha.$$

Now, it is easy to see that the test is level α for the original composite null, and that the test does not depend on the fixed alternative. Hence, φ is UMP level α.

14.2 UMPU Tests

What can we do if no UMP test exists? As with estimation, we can either make restrictions on the tests allowed or change the optimality criterion. Unbiased tests is one way of doing this.

Definition 14.1 *Say we are testing $H_0 : \theta \in \omega_0$ vs. $H_1 : \theta \in \omega_1$. A test φ is* ***unbiased*** *at level α if:*

1. $\sup_{\theta\in\omega_0} \mathbb{E}_\theta\varphi \le \alpha$, *and*
2. $\inf_{\theta\in\omega_1} \mathbb{E}_\theta\varphi \ge \alpha$.

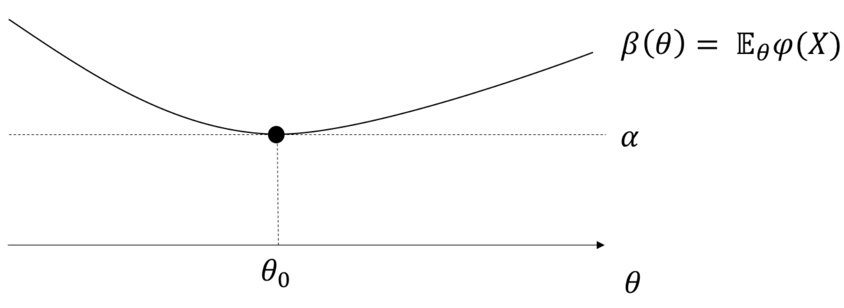
Definition 14.2 *A test is* ***uniformly most powerful unbiased (UMPU)*** *if it is uniformly most powerful among all unbiased tests.*

We will see that we can derive UMPU tests for multi-parameter exponential families.

14.2.1 Example: 1-parameter exponential family

Say $p_\theta(x) = C(\theta)e^{\theta T(x)}h(x)$. Testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.

Recall that for a 1-parameter exponential family, the power function is always continuous and the family has monotone likelihood ratio. Using the definition of an unbiased test, the test function φ must have a power function that looks like this:



Thus, we can rewrite the 2 constraints for a test φ to be unbiased as

$$\beta(\theta_0) = \alpha, \qquad \beta'(\theta_0) = 0. \tag{14.1}$$

Since

$$\beta(\theta) = \int \varphi(x)C(\theta)e^{\theta T(x)}h(x)\mu(dx),$$

for any test φ we can write

$$\begin{aligned}\beta'(\theta) &= \int \varphi(x)[C'(\theta)e^{\theta T(x)}h(x)]\mu(dx) + \int \varphi(x)\left[C(\theta)\cdot T(x)e^{\theta T(x)}\right]h(x)\mu(dx) \\ &= \frac{C'(\theta)}{C(\theta)}\mathbb{E}_\theta\varphi(X) + \mathbb{E}_\theta\left[\varphi(X)T(X)\right].\end{aligned}$$

In particular, consider the test function φ^* which rejects all the time. Clearly $\beta'(\theta) = 0$ for all θ, which implies that

$$\begin{aligned}0 &= \frac{C'(\theta)}{C(\theta)}\cdot 1 + \mathbb{E}_\theta[T(X)], \\ \frac{C'(\theta)}{C(\theta)} &= -\mathbb{E}_\theta[T(X)].\end{aligned}$$

Hence, the constraints in (14.1) are equivalent to

$$\beta(\theta_0) = \alpha, \qquad \mathrm{Cov}_{\theta_0}(T(X), \varphi(X)) = 0.$$

It's plausible (not a proof!) that a UMPU test rejects if $T > C_1$ or $T < C_2$, where C_1 and C_2 are determined so that both constraints are satisfied.

14.2.2 Example: Normal setting

Let $X_1, \dots, X_n$ iid, $X_i \sim \mathcal{N}(0, \sigma^2)$. Testing $H_0 : \sigma = \sigma_0$ vs. $H_1 : \sigma \neq \sigma_0$. $T = \sum X_i^2$ is a sufficient statistic, and under H_0, $\frac{T}{\sigma_0^2} \sim \chi_n^2$.

One could consider the *equitailed test*, where we reject if $T < C_n(\alpha)$ or $T > C_n(1-\alpha)$. (Here, C_n represents the quantile function for χ_n^2.) However, the equitailed test is not unbiased.

Let the density of χ_n^2 be f_n. From the previous example, we want to find C_1 and C_2 such that

$$\begin{aligned}\mathbb{E}_{\sigma_0}\varphi &= \alpha, \\ \int_{C_1}^{C_2} f_n(t)dt &= 1-\alpha,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_{\sigma_0}[T\varphi] &= \mathbb{E}_{\sigma_0}T\cdot\mathbb{E}_{\sigma_0}\varphi = n\alpha, \\ \int_{C_1}^{C_2} tf_n(t)dt &= n(1-\alpha).\end{aligned}$$

We have the following fact: $tf_n(t) = nf_{n+2}(t)$. Thus, the second constraint reduces to

$$\int_{C_1}^{C_2} f_{n+2}(t)dt = 1-\alpha.$$

For large n, this test will be close to the equitailed test.

14.2.3 How to find φ that satisfies constraints

We first present a lemma in a general setting:

Lemma 14.3 *Let $F_1, \ldots, F_{m+1}$ be real-valued functions on a space U. Suppose that we want to maximize $F_{m+1}(u)$ subject to constraints $F_i(u) = c_i$ for $i = 1, \ldots, m$, with c_i fixed.*

A sufficient condition for u_0 to maximize $F_{m+1}(u)$ subject to the constraints is it maximizes $F_{m+1}(u) - \sum_{i=1}^{m} k_i F_i(u)$ for some $k_1, \ldots, k_m$.

Proof: If u is any other value that satisfies the constraints, then

$$\begin{aligned} F_{m+1}(u_0) - \sum k_i F_i(u_0) &\geq F_{m+1}(u) - \sum k_i F_i(u), \\ F_{m+1}(u_0) - \sum k_i c_i &\geq F_{m+1}(u) - \sum k_i c_i, \\ F_{m+1}(u_0) &\geq F_{m+1}(u). \end{aligned}$$

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Let's apply the lemma to the hypothesis testing setting.

Corollary 14.4 *Let U be the family of test functions φ. For $i = 1, \ldots, m$, let the i^{th} constraint be $F_i(\varphi) = \int \varphi f_i = c_i$.*

Then a sufficient condition for φ to maximize $\int \varphi f_{m+1} d\mu$ subject to $\int \varphi f_i = c_i$, $i = 1, \ldots, n$ is

$$\varphi(X) = \begin{cases} 1 & \text{if } f_{m+1}(X) > \sum_{i=1}^{m} k_i f_i(X), \\ \gamma & \text{if } f_{m+1}(X) = \sum_{i=1}^{m} k_i f_i(X), \\ 0 & \text{if } f_{m+1}(X) < \sum_{i=1}^{m} k_i f_i(X) \end{cases}$$

for some $k_1, \ldots, k_m$.