STATS 310A: Theory of Probability I **Autumn 2016/17**

Lecture 2: September 28

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2.1 Strong Law of Large Numbers

Recall the set-up from the last lecture:

- $\Omega = (0, 1]$ is the sample space.
- For intervals $(a, b]$ in Ω, we assign probabilities: $p((a, b]) = b - a$.
- For each $\omega \in \Omega$, write $\omega = 0.d_1(\omega)d_2(\omega)\ldots$ (binary expansion).
- Let $r_i = 2d_i - 1$. The r_i's take on the value $+1$ or -1.
- Let $S_n := \sum_{i=1}^n r_i$.

Theorem 2.1 (Strong Law of Large Numbers) *Let*

$$B = \left\{ \omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = 0 \right\}.$$

Then B^c is negligible, i.e. for every $\varepsilon > 0$, there exist a covering of $B^c \subset \bigcup_{i=1}^\infty (a_i, b_i]$ with $\sum(b_i - a_i) < \varepsilon$.

Proof: For every $\varepsilon > 0$,

$$\begin{aligned} P\left\{ \left|\frac{S_n}{n}\right| > \varepsilon \right\} &= P\left\{ \omega : \left|\frac{S_n}{n}\right|^4 > \varepsilon^4 \right\} \\ &\leq \frac{\int_0^1 |S_n|^4}{n^4 \varepsilon^4} \end{aligned}$$

by Markov's inequality. It can also be shown that $\int |S_n|^4 \leq 3n^2$. Applying this result, we have

$$P\left\{ \left|\frac{S_n}{n}\right| > \varepsilon \right\} \leq \frac{3}{n^2 \varepsilon^4}.$$

Choose a sequence $\varepsilon_n \downarrow 0$ such that $\sum \frac{1}{n^2 \varepsilon^4} < \infty$. Then for any fixed m,

$$B \supseteq \bigcap_{n=m}^\infty \left\{ \left|\frac{S_n}{n}\right| < \varepsilon_n \right\}.$$

Taking complements on both sides,

$$\bigcup_{n=m}^\infty \left\{ \left|\frac{S_n}{n}\right| \geq \varepsilon_n \right\} \supseteq B^c.$$

Note that for any fixed n, each set of the LHS can be expressed as a disjoint union of intervals, i.e. $\left\{\left|\frac{S_n}{n}\right| \geq \varepsilon_n\right\} = \bigcup_{i=1}^{k_n} I_{n,i}$ where the I's are disjoint intervals. So

$$B^c \subseteq \bigcup_{\substack{n=1 \\ i=1,\ldots k_n}}^{\infty} I_{n,i}$$

and

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} |I_{n,i}| \leq \sum_{n=m}^{\infty} \frac{1}{n^2\varepsilon^4}.$$

Choose m large so that $\displaystyle\sum_{n=m}^{\infty} \frac{1}{n^2\varepsilon^4} < \varepsilon$. Hence, B^c is contained in a set of intervals length $< \varepsilon$. ■

Note: There is an explicit expression for B:

$$B = \bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\left\{\left|\frac{S_n}{n}\right| < \frac{1}{k}\right\}.$$

2.2 Assigning Probabilities

We want to assign probabilities to a large variety of sets, e.g.

- $\mathbb{R}^n$,
- $C[0,1]$ (all continuous functions on the unit interval),
- $P[0,1]$ (all probability distributions on $[0,1]$).

To do so, we will need a few more definitions.

Definition 2.2 *A* ***field*** $\mathcal{F}_0$ *is a collection of subsets* $A \subseteq \Omega$ *such that:*

1. $\Omega \in \mathcal{F}_0$.
2. *(Closed under complements)* $A \in \mathcal{F}_0 \Rightarrow A^c \in \mathcal{F}_0$.
3. *(Closed under finite unions)* $A_1, \ldots, A_n \in \mathcal{F}_0 \Rightarrow \bigcup_{i=1}^n A_i \in \mathcal{F}_0$.

Here is an example of a field: Let $\Omega = (0,1]$. Then $\mathcal{F}_0 = \{\text{disjoint unions of}(a_i, b_i] \subseteq (0,1]\}$ is a field.

Definition 2.3 *A* ***σ-field*** *is a field which is closed under countable unions.*

Here are some examples of σ-fields:

- $\mathcal{F} = \{\emptyset, \Omega\}$.

- All subsets of Ω.
- The intersection of any collection of σ-fields is also a σ-field.
- The fact above has an important consequence. If $\mathcal{C}$ is any collection of subsets of Ω, there is a smallest σ-field that contains $\mathcal{C}$:
$$\mathcal{F}_\mathcal{C} = \bigcap \{\text{all } \sigma\text{-fields containing } \mathcal{C}\}\,.$$
- Let $\mathcal{C}$ = all sets of the form $(a_i, b_i]$. Then $\mathcal{F}_\mathcal{C}$ is a σ-field. We call the sets in $\mathcal{F}_\mathcal{C}$ **Borel sets**.

Definition 2.4 *Let $(\Omega, \mathcal{F})$ be a measurable space. A **probability** on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0,1]$ such that:*

1. *$P(\emptyset) = 0, P(\Omega) = 1$.*
2. *If $A \in \mathcal{F}$, then $P(A^c) = 1 - P(A)$.*
3. *(Countable additivity) If $A_i \in \mathcal{F}$ disjoint for $i = 1, 2, \ldots$, then $P\left(\bigcup A_i\right) = \sum P(A_i)$.*

2.2.1 Densities for subsets of $\mathbb{N}$

Definition 2.5 *Let $\Omega = \mathbb{N} = \{0, 1, \ldots\}$. $A \subseteq \Omega$ has **density** l if*
$$\#\{i \leq n : i \in A\}/(n+1) \to l \text{ as } n \to \infty.$$

Here are some examples to give some intuition on what values densities take:

- $A = \{0, 2, 4, \ldots\}$ has density $\frac{1}{2}$.
- $A = \{1, 3, 5, \ldots\}$ has density $\frac{1}{2}$.
- $A = \{0, 3, 6, \ldots\}$ has density $\frac{1}{3}$.
- $A = \{1, 4, 7, \ldots\}$ has density $\frac{1}{3}$.
- The set $A = \{\text{all primes}\}$ has zero density as $\#\{\text{primes } < n\} \sim \frac{n}{\log n}$.
- The squarefree numbers have density $\frac{6}{\pi^2}$.

However, not all sets have density! For example, take $A = \{i : i \text{ begins with } 1\}$. $A = \{1\} \cup \{10, 11, \ldots, 19\} \cup \{100, \ldots 199\} \cup \ldots$ The size of $\#\{m : 1 \leq m \leq n, m \in A\}$ oscillates between $\frac{1}{2}$ and $\frac{1}{9}$, and does not have a density.

Consider $\mathcal{C} = \{A \in \mathbb{N} \text{ which have a density}\}$. Is $\mathcal{C}$ a field? Let's check.

- $\Omega \in \mathcal{C}$,
- closed under complements,
- closed under finite disjoint unions,
- BUT we can find 2 sets $A, B \in \mathcal{C}$ but $A \cap B \notin \mathcal{C}$. Notice also that $\{i\}$ has density 0, but $\bigcup_{i=0}^{\infty} \{i\}$ has density 1. So the measure is not countably additive.

2.2.2 Extending probabilities on fields to σ-fields

Given a probability on $(\Omega, \mathcal{F}_0)$ where $\mathcal{F}_0$ is a field, we want to extend P to $\mathcal{F}_{\mathcal{F}_0}$, i.e. the σ-algebra generated by $\mathcal{F}_0$.

Definition 2.6 *With $(\Omega, \mathcal{F}_0)$, let P be a probability on $\mathcal{F}_0$. For every $A \subseteq \Omega$, define*

$$P^*(A) := \inf \sum_{i=1}^{\infty} P(B_i)$$

*where $A \subseteq \cup B_i$, $B_i \in \mathcal{F}_0$. (*inf *is taken over all possible coverings.) $P^*(A)$ is called the* ***outer measure*** *of A.*

Elementary properties of outer measure:

1. $P^*(\emptyset) = 0$, $P^*(\Omega) = 1$.
2. P^* is countably sub-additive, i.e. if A_i are any (countable number of) sets in Ω, then $P^*(\cup A_i) \leq \sum P^*(A_i)$.

Let's prove the second property above.

Proof: Using the definition of $P^*(A_i)$ as an infimum: for any $\varepsilon > 0$, there exists some covering $\{B_{ij}\}$ of A_i $(j = 1, \ldots \infty)$ with $B_{ij} \in \mathcal{F}_0$ such that

$$P^*(A_i) + \varepsilon/2^i \geq \sum_j P(B_{ij}).$$

Now,

$$\bigcup_i A_i \subseteq \bigcup_{i,j} B_{ij},$$

so

$$P^*\left(\bigcup A_i\right) \leq \sum \left[\sum P(B_{ij}) + \varepsilon/2^i\right] \leq \sum P^*(A_i) + \varepsilon.$$

Since ε was arbitrary, we are done.

■

Aside: What have we learnt about length and area since the Greeks? Two things:

1. You can't consistently assign length to all subsets of $(0, 1]$.
2. One more idea (due to Carathéodory) makes a reasonable theory. On $(\Omega, \mathcal{F}_0)$, with P a probability on F_0, let

   $$\mathcal{M} = \{A : \forall E \subseteq \Omega, P*(E) = P*(A \cap E) + P*(A^c \cap E)\}.$$

   We will show that:

   (a) $\mathcal{M}$ is a σ-field containing $\mathcal{F}_0$.
   (b) P^* is countably additive on $\mathcal{M}$.
   (c) $P^*(A) = P(A)$ for $A \in \mathcal{F}_0$.
   (d) P^* is unique.