STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 6: January 26

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6.1 Uniform Integrability

Definition 6.1 *A sequence of random variables $\{X_n\}$ is **uniformly integrable** if:*

1. $\mathbb{E}|X_n| < \infty$ *for all n, and*
2. $$\lim_{a\to\infty} \sup_n \mathbb{E}(|X_n|; |X_n| \geq a) = 0.$$

Recall the following lemma from last lecture:

Lemma 6.2 *If $\{X_n\}$ is uniformly integrable, then $\sup\limits_n \mathbb{E}|X_n| < \infty$.*

Lemma 6.3 *If $\{X_n\}$ is uniformly integrable, then for all $\varepsilon > 0$, there is $\delta > 0$ such that*

$$A \in \mathcal{F} \text{ with } P(A) < \delta \quad \Rightarrow \quad \mathbb{E}(|X_n|; A) < \varepsilon \text{ for all } n.$$

(**Note:** The reverse implication is also true, left as an exercise.)

Proof: Given $\varepsilon > 0$, we can find a so large that for all n,

$$\mathbb{E}(|X_n|; |X_n| \geq a) < \frac{\varepsilon}{2}.$$

Pick δ so small such that $a\delta < \frac{\varepsilon}{2}$. Then, for any $A \in \mathcal{F}$,

$$\begin{aligned} \mathbb{E}(|X_n|; A) &= \mathbb{E}(|X_n|; \{|X_n| < a\} \cap A) + \mathbb{E}(|X_n|; \{|X_n| \geq a\} \cap A) \\ &\leq aP(A) + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

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Lemma 6.4 *If $\{X_n\}$ is uniformly integrable and $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{L^1} X$.*

(**Note:** Actually we don't need almost sure convergence; $X_n \xrightarrow{P} X$ will do.)

Proof: Take any $\varepsilon > 0$. Find a so large that

$$\mathbb{E}(|X_n|; |X_n| \geq a) < \frac{\varepsilon}{2}.$$

Note that

$$
\begin{aligned}
\mathbb{E}|X| &\le \liminf_n \mathbb{E}|X_n| && \text{(by Fatou's Lemma)} \\
&\le \sup_n \mathbb{E}|X_n| \\
&< \infty. && \text{(by Lemma 6.2)}
\end{aligned}
$$

So X is integrable, and hence uniformly integrable. Thus, a can be chosen large enough so that $\mathbb{E}(|X|; |X| > a) < \frac{\varepsilon}{2}$ as well.

Consider the function

$$
\phi(x) = \begin{cases} -a & \text{if } x < -a \\ x & \text{if } -a \le x \le a \\ a & \text{if } x > a. \end{cases}
$$

We have the decomposition

$$
\mathbb{E}|X_n - X| \le \underbrace{\mathbb{E}|X_n - \phi(X_n)|}_{(1)} + \underbrace{\mathbb{E}|\phi(X_n) - \phi(X)|}_{(2)} + \underbrace{\mathbb{E}|\phi(X) - X|}_{(3)}.
$$

We can bound each component of the RHS:

- (2): $X_n \to X$ a.s. and ϕ is continuous implies that $\phi(X_n) \to \phi(X)$ a.s. In addition, ϕ is a bounded function, so by the Bounded Convergence Theorem, $\mathbb{E}|\phi(X_n) - \phi(X)| \to 0$.

- (1):
$$
\mathbb{E}|X_n - \phi(X_n)| \le \mathbb{E}(|X_n|; |X_n| \ge a) < \frac{\varepsilon}{2}.
$$

- (3): Similarly,
$$
\mathbb{E}|X - \phi(X)| \le \mathbb{E}(|X|, |X| \ge a) < \frac{\varepsilon}{2}.
$$

Thus, $\mathbb{E}|X_n - X| < 2\varepsilon$ for all large enough n.

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Corollary 6.5 *If $\{Z_n, \mathcal{F}_n\}$ is a uniformly integrable martingale (or sub/super-martingale), then there is a random variable Z which is finite a.s. and $\lim Z_n = Z$ a.s. and $Z_n \to Z$ in L^1.*

Proof: Since $\sup_n \mathbb{E}|Z_n| < \infty$, the Martingale Convergence Theorem establishes the existence of Z such that $Z_n \to Z$ almost surely. By Fatou's Lemma, $\mathbb{E}|Z| \le \liminf \mathbb{E}|Z_n| < \infty$, which implies $|Z| < \infty$ a.s.

Since $\{Z_n\}$ is uniformly integrable, we can apply Lemma 6.4 to get $Z_n \to Z$ in L^1.

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6.2 Lévy's Upward Convergence Theorem

Theorem 6.6 (Lévy's Upward Convergence Theorem) *Let $\{\mathcal{F}_n\}$ be a filtration and $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$.*

Let Y be a random variable such that $\mathbb{E}|Y| < \infty$.

Then $\mathbb{E}(Y \mid \mathcal{F}_n) \to \mathbb{E}(Y \mid \mathcal{F}_\infty)$ a.s. and in L^1.

Proof: Let $Z_n = \mathbb{E}[Y \mid \mathcal{F}_n]$.

Step 1: $\{Z_n\}$ is a martingale, $Z = \lim Z_n$ exists and is finite a.s.

By the tower property of expectation,

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] = \mathbb{E}[Y \mid \mathcal{F}_n] = Z_n.$$

By Jensen's inequality,

$$\begin{aligned}
\mathbb{E}|Z_n| &= \mathbb{E}(|\mathbb{E}(Y \mid \mathcal{F}_n)|) \\
&\leq \mathbb{E}(\mathbb{E}(|Y| | \mathcal{F}_n)) \\
&= \mathbb{E}|Y| \\
&< \infty,
\end{aligned}$$

i.e. $\sup \mathbb{E}|Z_n|$ is finite. Therefore $Z = \lim Z_n$ exists (Martingale Convergence Theorem) and is finite a.s. (Fatou's Lemma).

Step 2: $\{Z_n\}$ is uniformly integrable, and so $Z_n \to Z$ in L^1.

Take any $a > 0$. Then

$$\begin{aligned}
aP(|Z_n| \geq a) &\leq \mathbb{E}(|Z_n|; |Z_n| \geq a) \\
&= \mathbb{E}(|\mathbb{E}[Y \mid \mathcal{F}_n]|; |Z_n| \geq a) \\
&\leq \mathbb{E}(\mathbb{E}[|Y| | \mathcal{F}_n]; |Z_n| \geq a) && \text{(Jensen's Inequality)} \\
&= \mathbb{E}(|Y|; |Z_n| \geq a) \\
&\leq \mathbb{E}|Y| < \infty.
\end{aligned}$$

Therefore, for every $\delta > 0$ there is some $a > 0$ such that $P(|Z_n| \geq a) < \delta$ for all n.

The random variable Y is uniformly integrable, so by Lemma 6.3, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$P(A) < \delta \quad \Rightarrow \quad \mathbb{E}(|Y|; A) < \varepsilon.$$

Combining these 2 facts: for all $\varepsilon > 0$, there exists some $a > 0$ such that for all n, $\mathbb{E}(|Y|; |Z_n| \geq a) < \varepsilon$.

In the chain of inequalities above, we have $\mathbb{E}(|Z_n|; |Z_n| \geq a) \leq \mathbb{E}(|Y|; |Z_n| \geq a)$. Therefore, $\mathbb{E}(|Z_n|; |Z_n| \geq a) < \varepsilon$, which is what we need to show uniform integrability.

Havin shown uniform integrability, we can use Corollary 6.5 to conclude that $Z_n \to Z$ in L^1.

Step 3: $\mathbb{E}(Y; A) = \mathbb{E}(Z; A)$ for all $A \in \mathcal{F}_\infty$, so $Z = \mathbb{E}[Y \mid \mathcal{F}_\infty]$.

Take any $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. This means that $A \in \mathcal{F}_n$ for some n. For any $m \geq n$, $A \in \mathcal{F}_m$ and so

$$\mathbb{E}(Y; A) = \mathbb{E}(\mathbb{E}(Y \mid \mathcal{F}_m); A) = \mathbb{E}(Z_m; A).$$

Thus, for large m,

$$\begin{aligned}
|\mathbb{E}(Y; A) - \mathbb{E}(Z; A)| &= |\mathbb{E}(Z_m; A) - \mathbb{E}(Z; A)| \\
&= |\mathbb{E}(Z_m - Z)1_A| \\
&\leq \mathbb{E}(|Z_m - Z|1_A) && \text{(Jensen's inequality)} \\
&\leq \mathbb{E}|Z_m - Z| \\
&\to 0,
\end{aligned}$$

as $m \to \infty$ since $Z_n \to Z$ in L^1. Therefore, $\mathbb{E}(Y; A) = \mathbb{E}(Z; A)$ for all $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{E}(Y; A) = \mathbb{E}(Z; A)\}$. The above argument shows that $\bigcup_{n=1}^{\infty} \mathcal{F}_n \subseteq \mathcal{G}$. Since $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a π system and we can show that $\mathcal{G}$ is a λ system, by the $\pi - \lambda$ theorem, $\mathcal{F}_\infty \subseteq \mathcal{G}$.

We conclude that $\mathbb{E}(Y; A) = \mathbb{E}(Z; A)$ for all $A \in \mathcal{F}_\infty$, so $Z = \mathbb{E}(Y \mid \mathcal{F}_\infty)$.

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6.2.1 Examples

- **(Kolmogorov 0-1 Theorem)** If $X_1, X_2, \ldots$ are independent, A is tail measurable, then A is independent of $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for all n. So $\mathbb{E}[1_A \mid \mathcal{F}_n] = P(A)$ for all n.

  However, by Lévy's Theorem, $\mathbb{E}[1_A \mid \mathcal{F}_n] \to 1_A$ a.s.! Therefore $P(A) \to 1_A$ a.s., which means that $P(A) = 0$ or 1.

- Suppose we have independent $X_1, X_2, \ldots$, and suppose Y depends on infinitely many X_i (and so is $\mathcal{F}_\infty$-measurable). Then Lévy's Theorem says that $\mathbb{E}(Y \mid \mathcal{F}_n) \to Y$ in L^1 and a.s. That is, if you take conditional expectation of Y with a large number of X_i's, you get something "close" to Y.

  There is no equivalent statement when Y depends on a very large (but finite) number of X_i.

6.3 Radon-Nikodym Theorem for Probability Measures

Definition 6.7 *Let $(\Omega, \mathcal{F})$ be a measurable space. Let P and Q be two probability measures on this space. We say that Q* ***is absolutely continuous w.r.t.*** *P, and write $Q << P$, if*

$$P(A) = 0 \quad \Rightarrow \quad Q(A) = 0.$$

Theorem 6.8 (Radon-Nikodym) *Let $(\Omega, \mathcal{F})$ be a measurable space and suppose that $\mathcal{F}$ is countably generated. Let P and Q be two probability measures on this space such that $Q << P$.*

Then, there exists a non-negative random variable L on this space such that for all $A \in \mathcal{F}$,

$$Q(A) = \int_A dQ = \int_A L dP.$$

We write $L := \frac{dQ}{dP}$.

Proof: (First Part of Proof)

$\mathcal{F}$ is countably generated implies that there exist events $A_1, A_2, \ldots$ such that $F = \sigma(A_1, A_2, \ldots)$. Let $F_n = \sigma(A_1, \ldots, A_n)$, and $\mathcal{F} = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$.

Take any n. Let $B_{n1}, \ldots, B_{nk_n}$ be the partition generated by $A_1, \ldots, A_n$, i.e. all non-empty sets of the form $C_1 \cap \ldots \cap C_n$, where $C_i = A_i$ or A_i^c. Then $\mathcal{F}_n = \{$all possible unions of $B_{n1}, \ldots, B_{nk_n}\}$. Define

$$L_n(\omega) := \begin{cases} \frac{Q(B_{ni})}{P(B_{ni})} & \text{if } \omega \in B_{ni} \text{ and } P(B_{ni}) > 0, \\ 0 & \text{if } \omega \in B_{ni} \text{ and } P(B_{ni}) = 0. \end{cases}$$

For any $A \in \mathcal{F}_n$, there is an index set I such that $A = \bigcup_{i \in I} B_{ni}$. Let $I' = \{i \in I : P(B_{ni}) > 0\}$. Then

$$\begin{aligned} Q(A) &= \sum_{i \in I} Q(B_{ni}) = \sum_{i \in I'} Q(B_{ni}) \\ &= \sum_{i \in I'} \frac{Q(B_{ni})}{P(B_{ni})} P(B_{ni}) \\ &= \sum_{i \in I'} \int_{B_{ni}} L_n dP = \sum_{i \in I} \int_{B_{ni}} L_n dP \\ &= \int_A L_n dP. \end{aligned}$$

Note that L_n is $\mathcal{F}_n$-measurable, non-negative, and

$$\mathbb{E}_P[L_n] = \int_\Omega L_n dP = Q(\Omega) = 1 < \infty.$$

In addition, for any $A \in \mathcal{F}_{n-1}$,

$$\begin{aligned} \mathbb{E}_P(L_n; A) &= \int_A L_n dP && (\text{since } A \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n) \\ &= Q(A) \\ &= \int_A L_{n-1} dP && (\text{since } A \in \mathcal{F}_{n-1}) \\ &= \mathbb{E}_P(L_{n-1}; A). \end{aligned}$$

Thus $L_{n-1} = \mathbb{E}_P(L_n \mid \mathcal{F}_{n-1})$, and therefore, under P, $\{L_n, \mathcal{F}_n\}$ is a martingale. ■