STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 18: March 9

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18.1 Time Homogeneous Markov Chains on Countable State Spaces

Say we are in the setting of a time homogeneous Markov chain on a countable state space.

18.1.1 Positive and Null Recurrence

Recall the following from last lecture:

Theorem 18.1 *If x is a recurrent state, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{xx}^{(m)} = \frac{1}{\mu_{xx}}.$$

Corollary 18.2 *If x and y communicate ($x \leftrightarrow y$) and are recurrent, then*

$$\frac{1}{n} \sum_{m=1}^{n} p_{yx}^{(m)} = \frac{1}{\mu_{xx}}.$$

We have the following corollaries:

Corollary 18.3 *Positive recurrence and null recurrence are class properties.*

Proof: It suffices to show that null recurrence is a class property.

Suppose $x \leftrightarrow y$ and y is null recurrent. Then by Theorem 18.1,

$$\frac{1}{n} \sum_{m=1}^{n} p_{yy}^{(m)} \to 0 \text{ as } n \to \infty.$$

Suppose that $p_{xy}^{(k)} > 0$, $p_{yx}^{(l)} > 0$ for some k and l. (They exist since the 2 states communicate.) Then, by Chapman-Kolmogorov,

$$\begin{aligned} & p_{yy}^{(k+l+m)} \geq p_{yx}^{(l)} p_{xx}^{(m)} p_{xy}^{(k)} \qquad \text{for all } m, \\ \Rightarrow \qquad & \frac{1}{n} \sum_{m=1}^{n} p_{xx}^{(m)} \leq \frac{1}{p_{yx}^{(l)} p_{xy}^{(k)}} \frac{1}{n} \sum_{m=1}^{n} p_{yy}^{(k+l+m)}. \end{aligned}$$

Taking $n \to \infty$, we see that $\frac{1}{n} \sum_{m=1}^{n} p_{xx}^{(m)} \to 0$ i.e. x is null recurrent. ■

Corollary 18.4 *A finite state space chain cannot have a null recurrent state.*

Proof: Suppose x is a recurrent state. Observe that if $x \to y$, then $x \leftrightarrow y$. (Proof is left as an exercise.) Thus, the equivalence class of x is the same as the set of all states accessible from x. Call this set S'. Then,

$$\sum_{y \in S'} p_{xy}^{(m)} = 1 \qquad \text{for all } m,$$

$$\sum_{y \in S'} \left(\frac{1}{n} \sum_{m=1}^{n} p_{xy}^{(m)} \right) = 1 \qquad \text{for any } n.$$

Since S is finite, S' is finite, and so we can take $n \to \infty$ to get

$$\sum_{y \in S'} \frac{1}{\mu_{yy}} = 1.$$

In particular, $\mu_{yy} < \infty$ for some $y \in S'$, and so S' is a positive recurrent class. ∎

Corollary 18.5 *If S' is a positive recurrent equivalence class and $|S'| < \infty$, then*

$$\sum_{y \in S'} \frac{1}{\mu_{yy}} = 1.$$

In particular, if S is finite and the chain is irreducible (i.e. $|S| = |S'|$), then

$$\sum_{x \in S} \frac{1}{\mu_{xx}} = 1.$$

18.1.2 Stationary Distributions

Definition 18.6 *A probability measure π on S is called a* ***stationary distribution/invariant measure*** *for the chain if, for all $y \in S$,*

$$\sum_{x} \pi_x p_{xy} = \pi_y.$$

In other words, if $X_0 \sim \pi$, then $X_1 \sim \pi$ as well.

Proposition 18.7 *Simple random walk on $\mathbb{Z}$ has no stationary distribution.*

Proof: If π is a stationary distribution, then $\pi_x = \frac{1}{2}(\pi_{x-1} + \pi_{x+1})$ for all x, which implies that $\pi_x - \pi_{x-1} = \pi_{x+1} - \pi_x$ for all x. Either $\pi_x \to \infty$ as $x \to \infty$ (which is impossible), or π_x is the same for all x, which is also impossible.

∎

Proposition 18.8 *Suppose $p > \frac{1}{2}$. Consider a random walk on $\mathbb{Z}$ where*

$$X_{n+1} = \begin{cases} 1 \text{ w.p. } p \text{ and } -1 \text{ w.p. } 1-p & \text{if } S_n < 0, \\ 1 \text{ w.p. } 1-p \text{ and } -1 \text{ w.p. } p & \text{if } S_n > 0, \\ 1 \text{ w.p. } 1/2 \text{ and } -1 \text{ w.p. } 1/2 & \text{if } S_n = 0. \end{cases}$$

This random walk has a stationary distribution.

(Proof left as an exercise.)

Theorem 18.9 *If S is finite and the chain is irreducible, then $\pi_x = \dfrac{1}{\mu_{xx}}$ is the unique stationary distribution.*

Proof: Take any $y \in S$. Because the chain is irreducible,

$$\begin{aligned}
\sum_x \pi_x p_{xy} &= \sum_x \frac{1}{\mu_{xx}} p_{xy} \\
&= \lim_{n\to\infty} \sum_x \left(\frac{1}{n} \sum_{m=1}^{n} p_{yx}^{(m)} \right) p_{xy} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left(\sum_x p_{yx}^{(m)} p_{xy} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{yy}^{(m+1)} \\
&= \frac{1}{\mu_{yy}} \\
&= \pi_y .
\end{aligned}$$

To prove uniqueness: If π is any stationary distribution, then

$$\begin{aligned}
\sum_x \pi_x p_{xy} &= \pi_y && \text{for all } y, \\
\pi P &= \pi && \text{where } \pi \text{ row vector, } P \text{ transition matrix,} \\
\pi P^n &= \pi && \text{for all } n, \\
\sum_x \pi_x p_{xy}^{(n)} &= \pi_y && \text{for all } y,\ n, \\
\sum_x \pi_x \left(\frac{1}{n} \sum_{m=1}^{n} p_{xy}^{(m)} \right) &= \pi_y, && \text{(by taking averages of the above)} \\
\sum_x \pi_x \frac{1}{\mu_{yy}} &= \pi_y . && \text{(by taking } n \to \infty\text{)}
\end{aligned}$$

Since $\sum \pi_x = 1$, this shows that $\pi_y = \dfrac{1}{\mu_{yy}}$. ■

Theorem 18.10 *If S is finite and the chain is irreducible and aperiodic, then $\pi_x = \dfrac{1}{\mu_{xx}}$ is the unique stationary distribution. Moreover,*

$$\lim_{n\to\infty} p_{xy}^{(n)} = \frac{1}{\mu_{yy}} .$$

18.2 Concentration Inequalities

18.2.1 Motivating Example: First-Passage Percolation

Consider the lattice $\mathbb{Z}^d$.

- On each edge e, we have a non-negative random variable X_e, called the weight of the edge. Assume that the X_e are i.i.d.
- The weight of a path is equal to the sum of edge weights along the path.
- The first-passage time from x to y, denoted T_{xy}, is the minimum of the weights of all paths from x to y.

Question: What is the behavior of T_{xy} if $\|x-y\|$ is large? In particular, consider $T_n = T_{0,ne_1}$, where $e_1 = (1, 0, \dots, 0)$. How does T_n behave?

Assume that $\mathbb{E}X_e < \infty$. We can show that $\mu = \lim_{n\to\infty} \dfrac{\mathbb{E}T_n}{n}$ exists. Moreover, if $P(X_e = 0) = 0$, then $\mu > 0$.

(A simple upper bound of μ is $\mathbb{E}X_e$, but usually μ is a factor smaller than $\mathbb{E}X_e$.)

Question: What is the order of Var T_n? In particular, does $\dfrac{T_n}{n} \xrightarrow{P} \mu$?

We will show that Var $T_n \le Cn$ for some constant C, under some additional assumptions on the law of X_e.

It is known that Var $T_n \le \dfrac{Cn}{\log n}$. Below are a series of related open questions, in increasing strength of claim:

- Var $T_n = o\left(\dfrac{n}{\log n}\right)$.
- $T_n \le Cn^{\alpha}$ for $\alpha < 1$.
- $T_n \le Cn^{2/3}$.