STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 7: January 31

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7.1 Stochastic Optimization (Finite Horizon)

Suppose we have a filtration $\{\mathcal{F}_n\}_{1\leq n\leq N}$. Let $\{X_n\}_{1\leq n\leq N}$ be a sequence of random variables such that X_n is $\mathcal{F}_n$-measurable, $\mathbb{E}|X_n|< \infty$ for all n, with the interpretation that X_n is the "reward" that you get if you stop at time n.

Goal: Maximize $\mathbb{E}X_T$ over all stopping times T.

Solution: Dynamic Programming (due to Snell).

Theorem 7.1 *Define an adapted sequence $\{V_n\}$ as follows: Let $V_N = X_N$. For all $n < N$, define*

$$V_n := \max\{X_n, \mathbb{E}[V_{n+1} \mid \mathcal{F}_n]\}.$$

Let $\tau = \inf\{X_k = V_k\}$. (Note that $\tau \leq N$.) Then τ is the solution, i.e.

$$\mathbb{E}X_\tau = \sup_T \mathbb{E}X_T,$$

where the sup *over all stopping times T.*

Proof: It is easy to see that V_n is $\mathcal{F}_n$-measurable, so τ is indeed a stopping time.

By definition, $V_n \geq \mathbb{E}[V_{n+1} \mid \mathcal{F}_n]$, and so it is a supermartingale. Hence, for any stopping time T taking values in $\{1,\dots,N\}$, we can apply Wald's Lemma for supermartingales and bounded stopping times and use the definition of V to get

$$\mathbb{E}X_T \leq \mathbb{E}V_T \leq \mathbb{E}V_1.$$

Now, since $X_\tau = V_\tau$, we have $\mathbb{E}X_\tau = \mathbb{E}V_\tau$. Thus, the proof will be complete if we can show that $\mathbb{E}V_\tau = \mathbb{E}V_1$.

Since $V_\tau = V_{\tau\wedge N}, V_1 = V_{\tau \wedge 1}$, $\mathbb{E}V_\tau = \mathbb{E}V_1 \Leftrightarrow \mathbb{E}V_{\tau\wedge N} = \mathbb{E}V_{\tau\wedge 1}$. This will follow if we can show that $\{V_{\tau\wedge n}, \mathcal{F}_n\}$ is a martingale.

$$\begin{aligned} V_{\tau\wedge n} &= 1_{\{\tau\leq n\}} V_\tau + 1_{\{\tau > n\}} V_n \\ &= \sum_{k=1}^n 1_{\{\tau = k\}} V_k + 1_{\{\tau>n\}} V_n. \end{aligned}$$

Each term in the first sum is $\mathcal{F}_k$-measurable while the second term is $\mathcal{F}_n$-measurable, so it is $\mathcal{F}_n$-measurable.

Also, $\mathbb{E}|V_{\tau\wedge n}|< \infty$ (exercise). Take any $1 \le k \le n$, then

$$\begin{aligned}\mathbb{E}[V_{\tau\wedge(k+1)} \mid \mathcal{F}_k] &= \mathbb{E}[1_{\{\tau\le k\}}V_\tau + 1_{\{\tau>k\}}V_{k+1} \mid \mathcal{F}_k] \\ &= \mathbb{E}\left[\sum_{j=1}^{k} 1_{\{\tau=j\}}V_j + 1_{\{\tau>k\}}V_{k+1} \mid \mathcal{F}_k\right] \\ &= 1_{\{\tau\le k\}}V_\tau + 1_{\{\tau>k\}}\mathbb{E}[V_{k+1} \mid \mathcal{F}_k].\end{aligned}$$

Recall that τ is the first n such that $X_n = V_n$, and that $V_n = \max\{X_n, \mathbb{E}[V_{n+1} \mid \mathcal{F}_n]\}$. Thus, for each n, X_n is either equal to V_n or strictly less than V_n. If $X_n < V_n$, then $V_n = \mathbb{E}[V_{n+1} \mid \mathcal{F}_n]$. So if $k < \tau$, then $X_k < V_k$, and therefore $V_k = \mathbb{E}[V_{k+1} \mid \mathcal{F}_k]$.

$$\begin{aligned}\mathbb{E}[V_{\tau\wedge(k+1)} \mid \mathcal{F}_k] &= 1_{\{\tau\le k\}}V_\tau + 1_{\{\tau>k\}}\mathbb{E}[V_{k+1} \mid \mathcal{F}_k] \\ &= 1_{\{\tau\le k\}}V_\tau + 1_{\{\tau>k\}}V_k \\ &= V_{\tau\wedge k}.\end{aligned}$$

This proves that $\{V_{\tau\wedge k}\}$ is a martingale. ■

7.1.1 Application: Secretary Problem

There are N candidates for a job interview, with ranks $r_1, \ldots, r_N$ (all distinct). The candidates are interviewed one by one. At each time point, you know the relative ranks of the candidates interviewed so far. At the end of each interview, you can either offer the job to the candidate or let them go (no recalls).

Assumption: $(r_1, \ldots, r_N)$ is a uniform random permutation of $\{1, \ldots, N\}$.

Goal: Let $\mathcal{F}_n$ be the σ-algebra generated by information up to time n (i.e. $1_{\{r_i<r_j\}}$ for all $1 \le i \le j \le n$). Find a stopping time T w.r.t. this filtration that maximizes $P(r_T = 1)$.

The problem is that $1_{\{r_n=1\}}$ is not $\mathcal{F}_n$-measurable, so we cannot apply Snell's result directly.

Solution: Let

$$X_n = \begin{cases} 0 & \text{if } n^{th} \text{ candidate is not the best among the first } n \text{ candidates,} \\ \frac{n}{N} & \text{otherwise.} \end{cases}$$

It can be checked that $X_n = P(r_n = 1 \mid \mathcal{F}_n)$. Therefore, for any stopping time T taking values in $\{1, \ldots, N\}$,

$$
\begin{aligned}
\mathbb{E}X_T &= \mathbb{E}\left[\sum_{n=1}^{N} X_n 1_{\{T=n\}}\right] \\
&= \sum_{n=1}^{N} \mathbb{E}\left[P(r_n = 1 \mid \mathcal{F}_n) 1_{\{T=n\}}\right] \\
&= \sum_{n=1}^{N} \mathbb{E}\left[1_{\{r_n=1\}} 1_{\{T=n\}}\right] && \text{(definition of conditional expectation)} \\
&= \mathbb{E}\left[\sum_{n=1}^{N} 1_{\{r_T=1\}} 1_{\{T=n\}}\right] \\
&= \mathbb{E}[1_{\{r_T=1\}}] \\
&= P(r_T = 1).
\end{aligned}
$$