STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 8: February 2

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8.1 The Secretary Problem

This is an application of stochastic optimization for finite horizon which we discussed last lecture.

8.1.1 Set-up

There are N candidates for a job interview, with ranks $r_1, \ldots, r_N$ (all distinct). The candidates are interviewed one by one. At each time point, you know the relative ranks of the candidates interviewed so far. At the end of each interview, you can either offer the job to the candidate or let them go (no recalls).

Assumption: $(r_1, \ldots, r_N)$ is a uniform random permutation of $\{1, \ldots, N\}$.

Let $\mathcal{F}_n$ be the σ-algebra generated by information up to time n (i.e. $1_{\{r_i < r_j\}}$ for all $1 \leq i \leq j \leq n$).

8.1.2 Goal

Goal: Find a stopping time T w.r.t. this filtration that maximizes $P(r_T = 1)$.

The problem is that $1_{\{r_n = 1\}}$ is not $\mathcal{F}_n$-measurable, so we cannot apply Snell's dynamic programming result directly.

8.1.3 Solution

Step 1: Find an adapted sequence $\{X_n\}$ such that $P(r_T = 1) = \mathbb{E}X_T$.

Let Y_n = rank of the n^{th} candidate among the first n. Then $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$. Let

$$X_n = \begin{cases} 0 & \text{if } n^{th} \text{ candidate is not the best among the first } n \text{ candidates, i.e. } Y_n \neq 1, \\ \dfrac{n}{N} & \text{otherwise.} \end{cases}$$

It can be checked that $X_n = P(r_n = 1 \mid \mathcal{F}_n)$. Therefore, for any stopping time T taking values in $\{1, \ldots, N\}$,

$$\begin{aligned}
\mathbb{E}X_T &= \mathbb{E}\left[\sum_{n=1}^{N} X_n 1_{\{T=n\}}\right] \\
&= \sum_{n=1}^{N} \mathbb{E}\left[P(r_n = 1 \mid \mathcal{F}_n) 1_{\{T=n\}}\right] \\
&= \sum_{n=1}^{N} \mathbb{E}\left[1_{\{r_n=1\}} 1_{\{T=n\}}\right] && \text{(definition of conditional expectation)} \\
&= \mathbb{E}\left[\sum_{n=1}^{N} 1_{\{r_T=1\}} 1_{\{T=n\}}\right] \\
&= \mathbb{E}[1_{\{r_T=1\}}] \\
&= P(r_T = 1).
\end{aligned}$$

It can also be checked that X_n is $\mathcal{F}_n$-measurable.

Step 2: The Y_i's are independent, so X_m is independent of $\mathcal{F}_n$ if $m > n$.

Note that

$$P(Y_n = k) = \begin{cases} \frac{1}{n} & \text{if } 1 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, there is a 1-to-1 correspondence between $(Y_1, \dots, Y_N)$ and $(r_1, \dots, r_N)$. That is, for any $(y_1, \dots, y_N)$ such that $1 \le y_n \le n$ for all n, there exists a permutation $(\pi_1, \dots, \pi_N)$ so that $(Y_1 = y_1, \dots, Y_N = y_1) \Leftrightarrow (r_1 = \pi_1, \dots, r_N = \pi_N)$. Thus, if $1 \le y_n \le n$ for all n, then

$$P(Y_1 = y_1, \dots, Y_N = y_N) = \frac{1}{n!} = P(Y_1 = y_1) \dots P(Y_N = y_N).$$

So the Y_i's are independent! Since X_n is a function of Y_n, $(X_{n+1}, \dots, X_N)$ is independent of $\mathcal{F}_n$.

Step 3: Let $\{V_n\}$ and τ be as in the dynamic programming solution. Then τ has a very specific form.

Set $V_N = X_N$, $V_n = \max\{X_n, \mathbb{E}[V_{n+1} \mid \mathcal{F}_n]\}$ for $n < N$.

Claim: $\mathbb{E}[V_{n+1} \mid \mathcal{F}_n]$ is a constant v_n for each $n < N$.

Proof: We use backward induction. $\mathbb{E}[V_N \mid \mathcal{F}_{N-1}]$ is a constant since $V_N = X_N$ is independent of $\mathcal{F}_{N-1}$.

Suppose the statement is true for some n (i.e. $v_n = \mathbb{E}[V_{n+1} \mid \mathcal{F}_n]$ is a constant). Then $V_n = \max\{X_n, v_n\}$ is a function of X_n only, so it is independent of $\mathcal{F}_{n-1}$, i.e. $\mathbb{E}[V_n \mid \mathcal{F}_{n-1}]$ is a constant.

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(**Note:** The claim applies as long as the reward process $\{X_n\}$ are independent.)

If, in addition, we set $v_N = -\infty$, then the optimal stopping time (by Snell) is

$$\begin{aligned}\tau &= \min\{n : X_n = V_n\} \\ &= \min\{n : X_n = \max\{X_n, v_n\}\} \\ &= \min\{n : X_n \geq v_n\} \\ &= \min\left\{n : \frac{n}{N} 1_{\{Y_n = 1\}} \geq v_n\right\}.\end{aligned}$$

Claim: $v_1 \geq v_2 \geq \ldots \geq v_N$.

Proof: For $n < N$, $v_n = \mathbb{E}[V_{n+1} \mid \mathcal{F}_n] = \mathbb{E}(V_{n+1})$. Thus,

$$v_{n-1} = \mathbb{E}V_n = \mathbb{E}[\max\{X_n, v_n\}] \geq v_n.$$

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Consider the graphs of $y = n/N$ vs. n and and $y = v_n$ vs. n. The first is strictly increasing from $\frac{1}{N}$ while the second is non-increasing and ends up at ∞. Hence, there will be some first point, say n^*, where the first graph is above the second.

Thus, τ is of the form $\min\{n : n \geq n^* \text{ and } Y_n = 1\}$ (and if there is no such n, $\tau = N$), where $n^* = \min\{n : \frac{n}{N} \geq v_n\}$. By backward induction, we can show that n^* depends only on N.

Step 4: Optimize $P(r_T = 1)$ over all stopping times of the form $\min\{n : n \geq n^*$ and $Y_n = 1\}$.

For each $1 \leq j \leq N$, let $T_j = \min\{n : n \geq j, Y_n = 1\}$, with the convention that $\min \emptyset = N$. Let $f(j) = P(r_{T_j} = 1)$. We have to maximize $f(j)$.

Clearly $f(1) = \dfrac{1}{N}$. For $j \geq 2$,

$$\begin{aligned}f(j) &= P(r_{T_j} = 1) \\ &= \sum_{k=j}^{N} P(r_k = 1 \text{ and } T_j = k) \\ &= \sum_{k=j}^{N} P(r_k = 1) P(T_j = k \mid r_k = 1) \\ &= \sum_{k=j}^{N} \frac{1}{N} P(\text{the best candidate before } k \text{ was among than the first } j-1 \mid r_k = 1)\end{aligned}$$

Given that $r_k = 1$, all other candidates are distributed uniformly in $\{1, \ldots, N\} \setminus \{k\}$. Thus,

$$f(j) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{k=j}^{N} \frac{j-1}{k-1} & \text{if } j \geq 2, \\ \dfrac{1}{N} & \text{if } j = 1. \end{cases}$$

With this formula for $f(j)$, compare the sizes of $f(j)$ and $f(j+1)$.

$$\begin{aligned} & f(j) > f(j+1) \\ \Leftrightarrow \quad & \frac{1}{N}\sum_{k=j}^{N}\frac{j-1}{k-1} > \frac{1}{N}\sum_{k=j+1}^{N}\frac{j}{k-1} \\ \Leftrightarrow \quad & 1 - \sum_{k=j+1}^{N}\frac{1}{k-1} > 0. \end{aligned}$$

(The above holds true even if $j = 1$).

Thus f is unimodal (i.e. increases at first, then decreases). Thus, the first j that satisfies the above gives the optimal j.

For large N,

$$\sum_{k=j+1}^{N}\frac{1}{k-1} \approx \log\frac{N}{j},$$

so the optimal j^* satisfies

$$\log N/j^* \approx 1, \quad j^* \approx N/e.$$

8.2 Martingales with Bounded Increments

First, two important lemmas for martingales:

Lemma 8.1 *If $\{Z_n, \mathcal{F}_n\}$ is any martingale and τ is a stopping time, then $\{Z_{\tau\wedge n}, \mathcal{F}_n\}$ is also a martingale.*

Proof:

$$Z_{\tau\wedge n} = \sum_{k=1}^{n} Z_k 1_{\{\tau=k\}} + Z_n 1_{\{\tau>n\}}.$$

This shows that it is $\mathcal{F}_n$-measurable and taking expectations of the absolute values, that it is integrable. Finally,

$$\begin{aligned} \mathbb{E}[Z_{\tau\wedge(n+1)} \mid \mathcal{F}_n] &= \mathbb{E}\left[\sum_{k=1}^{n} Z_k 1_{\{\tau=k\}} + Z_{n+1} 1_{\{\tau>n\}} \middle| \mathcal{F}_n\right] \\ &= \sum_{k=1}^{n} Z_k 1_{\{\tau=k\}} + 1_{\{\tau>n\}}\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] \\ &= Z_{\tau\wedge n}. \end{aligned}$$

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Lemma 8.2 *If Z_n is a supermartingale with $\mathbb{E}|Z_1| < \infty$ that is uniformly bounded below by a constant, then $\lim Z_n$ exists and is finite a.s.*

[Same for submartingale bounded above by a constant.]

Proof: Suppose that $Z_n \geq c$ a.s. for $c \in \mathbb{R}$. Then $Z_n - c$ is a non-negative supermartingale, so by the Martingale Convergence Theorem, $\lim Z_n - c$ exists a.s. in $[0, \infty]$.

But by Fatou's Lemma,

$$\mathbb{E}[\lim(Z_n - c)] \leq \liminf \mathbb{E}[Z_n - c] \leq \mathbb{E}[Z_1 - c] < \infty,$$

since a supermartingale has non-increasing expectation.

$\blacksquare$

Theorem 8.3 *Let $\{Z_n, \mathcal{F}_n\}$ be a martingale. Suppose that there exists a constant c such that for all n, $|Z_n - Z_{n-1}| \leq c$ a.s.*

With probability 1, either $\lim Z_n$ exists and is finite, or $\limsup Z_n = \infty$ and $\liminf Z_n = -\infty$.