

Lecture 6: January 26

Lecturer: Sourav Chatterjee

Scribes: Kenneth Tay

6.1 Uniform Integrability

Definition 6.1 A sequence of random variables $\{X_n\}$ is **uniformly integrable** if:

1. $\mathbb{E}|X_n| < \infty$ for all n , and

2.

$$\lim_{a \rightarrow \infty} \sup_n \mathbb{E}(|X_n|; |X_n| \geq a) = 0.$$

Recall the following lemma from last lecture:

Lemma 6.2 If $\{X_n\}$ is uniformly integrable, then $\sup_n \mathbb{E}|X_n| < \infty$.

Lemma 6.3 If $\{X_n\}$ is uniformly integrable, then for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$A \in \mathcal{F} \text{ with } P(A) < \delta \quad \Rightarrow \quad \mathbb{E}(|X_n|; A) < \varepsilon \text{ for all } n.$$

(**Note:** The reverse implication is also true, left as an exercise.)

Proof: Given $\varepsilon > 0$, we can find a so large that for all n ,

$$\mathbb{E}(|X_n|; |X_n| \geq a) < \frac{\varepsilon}{2}.$$

Pick δ so small such that $a\delta < \frac{\varepsilon}{2}$. Then, for any $A \in \mathcal{F}$,

$$\begin{aligned} \mathbb{E}(|X_n|; A) &= \mathbb{E}(|X_n|; \{|X_n| < a\} \cap A) + \mathbb{E}(|X_n|; \{|X_n| \geq a\} \cap A) \\ &\leq aP(A) + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

■

Lemma 6.4 If $\{X_n\}$ is uniformly integrable and $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{L^1} X$.

(**Note:** Actually we don't need almost sure convergence; $X_n \xrightarrow{P} X$ will do.)

Proof: Take any $\varepsilon > 0$. Find a so large that

$$\mathbb{E}(|X_n|; |X_n| \geq a) < \frac{\varepsilon}{2}.$$

Note that

$$\begin{aligned}\mathbb{E}|X| &\leq \liminf_n \mathbb{E}|X_n| && \text{(by Fatou's Lemma)} \\ &\leq \sup_n \mathbb{E}|X_n| \\ &< \infty. && \text{(by Lemma 6.2)}\end{aligned}$$

So X is integrable, and hence uniformly integrable. Thus, a can be chosen large enough so that $\mathbb{E}(|X|; |X| > a) < \frac{\varepsilon}{2}$ as well.

Consider the function

$$\phi(x) = \begin{cases} -a & \text{if } x < -a \\ x & \text{if } -a \leq x \leq a \\ a & \text{if } x > a. \end{cases}$$

We have the decomposition

$$\mathbb{E}|X_n - X| \leq \underbrace{\mathbb{E}|X_n - \phi(X_n)|}_{(1)} + \underbrace{\mathbb{E}|\phi(X_n) - \phi(X)|}_{(2)} + \underbrace{\mathbb{E}|\phi(X) - X|}_{(3)}.$$

We can bound each component of the RHS:

- (2): $X_n \rightarrow X$ a.s. and ϕ is continuous implies that $\phi(X_n) \rightarrow \phi(X)$ a.s. In addition, ϕ is a bounded function, so by the Bounded Convergence Theorem, $\mathbb{E}|\phi(X_n) - \phi(X)| \rightarrow 0$.

- (1):

$$\mathbb{E}|X_n - \phi(X_n)| \leq \mathbb{E}(|X_n|; |X_n| \geq a) < \frac{\varepsilon}{2}.$$

- (3): Similarly,

$$\mathbb{E}|X - \phi(X)| \leq \mathbb{E}(|X|; |X| \geq a) < \frac{\varepsilon}{2}.$$

Thus, $\mathbb{E}|X_n - X| < 2\varepsilon$ for all large enough n . ■

Corollary 6.5 *If $\{Z_n, \mathcal{F}_n\}$ is a uniformly integrable martingale (or sub/super-martingale), then there is a random variable Z which is finite a.s. and $\lim Z_n = Z$ a.s. and $Z_n \rightarrow Z$ in L^1 .*

Proof: Since $\sup_n \mathbb{E}|Z_n| < \infty$, the Martingale Convergence Theorem establishes the existence of Z such that $Z_n \rightarrow Z$ almost surely. By Fatou's Lemma, $\mathbb{E}|Z| \leq \liminf_n \mathbb{E}|Z_n| < \infty$, which implies $|Z| < \infty$ a.s.

Since $\{Z_n\}$ is uniformly integrable, we can apply Lemma 6.4 to get $Z_n \rightarrow Z$ in L^1 . ■

6.2 Lévy's Upward Convergence Theorem

Theorem 6.6 (Lévy's Upward Convergence Theorem) Let $\{\mathcal{F}_n\}$ be a filtration and $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$.

Let Y be a random variable such that $\mathbb{E}|Y| < \infty$.

Then $\mathbb{E}(Y \mid \mathcal{F}_n) \rightarrow \mathbb{E}(Y \mid \mathcal{F}_\infty)$ a.s. and in L^1 .

Proof: Let $Z_n = \mathbb{E}[Y \mid \mathcal{F}_n]$.

Step 1: $\{Z_n\}$ is a martingale, $Z = \lim Z_n$ exists and is finite a.s.

By the tower property of expectation,

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] = \mathbb{E}[Y \mid \mathcal{F}_n] = Z_n.$$

By Jensen's inequality,

$$\begin{aligned} \mathbb{E}|Z_n| &= \mathbb{E}(|\mathbb{E}(Y \mid \mathcal{F}_n)|) \\ &\leq \mathbb{E}(\mathbb{E}(|Y| \mid \mathcal{F}_n)) \\ &= \mathbb{E}|Y| \\ &< \infty, \end{aligned}$$

i.e. $\sup \mathbb{E}|Z_n|$ is finite. Therefore $Z = \lim Z_n$ exists (Martingale Convergence Theorem) and is finite a.s. (Fatou's Lemma).

Step 2: $\{Z_n\}$ is uniformly integrable, and so $Z_n \rightarrow Z$ in L^1 .

Take any $a > 0$. Then

$$\begin{aligned} aP(|Z_n| \geq a) &\leq \mathbb{E}(|Z_n|; |Z_n| \geq a) \\ &= \mathbb{E}(|\mathbb{E}[Y \mid \mathcal{F}_n]|; |Z_n| \geq a) \\ &\leq \mathbb{E}(\mathbb{E}(|Y| \mid \mathcal{F}_n); |Z_n| \geq a) && \text{(Jensen's Inequality)} \\ &= \mathbb{E}(|Y|; |Z_n| \geq a) \\ &\leq \mathbb{E}|Y| < \infty. \end{aligned}$$

Therefore, for every $\delta > 0$ there is some $a > 0$ such that $P(|Z_n| \geq a) < \delta$ for all n .

The random variable Y is uniformly integrable, so by Lemma 6.3, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$P(A) < \delta \quad \Rightarrow \quad \mathbb{E}(|Y|; A) < \varepsilon.$$

Combining these 2 facts: for all $\varepsilon > 0$, there exists some $a > 0$ such that for all n , $\mathbb{E}(|Y|; |Z_n| \geq a) < \varepsilon$.

In the chain of inequalities above, we have $\mathbb{E}(|Z_n|; |Z_n| \geq a) \leq \mathbb{E}(|Y|; |Z_n| \geq a)$. Therefore, $\mathbb{E}(|Z_n|; |Z_n| \geq a) < \varepsilon$, which is what we need to show uniform integrability.

Having shown uniform integrability, we can use Corollary 6.5 to conclude that $Z_n \rightarrow Z$ in L^1 .

Step 3: $\mathbb{E}(Y; A) = \mathbb{E}(Z; A)$ for all $A \in \mathcal{F}_\infty$, so $Z = \mathbb{E}[Y \mid \mathcal{F}_\infty]$.

Take any $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. This means that $A \in \mathcal{F}_n$ for some n . For any $m \geq n$, $A \in \mathcal{F}_m$ and so

$$\mathbb{E}(Y; A) = \mathbb{E}(\mathbb{E}(Y | \mathcal{F}_m); A) = \mathbb{E}(Z_m; A).$$

Thus, for large m ,

$$\begin{aligned} |\mathbb{E}(Y; A) - \mathbb{E}(Z; A)| &= |\mathbb{E}(Z_m; A) - \mathbb{E}(Z; A)| \\ &= |\mathbb{E}(Z_m - Z)1_A| \\ &\leq \mathbb{E}(|Z_m - Z|1_A) && \text{(Jensen's inequality)} \\ &\leq \mathbb{E}|Z_m - Z| \\ &\rightarrow 0, \end{aligned}$$

as $m \rightarrow \infty$ since $Z_n \rightarrow Z$ in L^1 . Therefore, $\mathbb{E}(Y; A) = \mathbb{E}(Z; A)$ for all $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{E}(Y; A) = \mathbb{E}(Z; A)\}$. The above argument shows that $\bigcup_{n=1}^{\infty} \mathcal{F}_n \subseteq \mathcal{G}$. Since $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a π system and we can show that \mathcal{G} is a λ system, by the $\pi - \lambda$ theorem, $\mathcal{F}_{\infty} \subseteq \mathcal{G}$.

We conclude that $\mathbb{E}(Y; A) = \mathbb{E}(Z; A)$ for all $A \in \mathcal{F}_{\infty}$, so $Z = \mathbb{E}(Y | \mathcal{F}_{\infty})$. ■

6.2.1 Examples

- **(Kolmogorov 0-1 Theorem)** If X_1, X_2, \dots are independent, A is tail measurable, then A is independent of $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ for all n . So $\mathbb{E}[1_A | \mathcal{F}_n] = P(A)$ for all n .

However, by Lévy's Theorem, $\mathbb{E}[1_A | \mathcal{F}_n] \rightarrow 1_A$ a.s.! Therefore $P(A) \rightarrow 1_A$ a.s., which means that $P(A) = 0$ or 1 .

- Suppose we have independent X_1, X_2, \dots , and suppose Y depends on infinitely many X_i (and so is \mathcal{F}_{∞} -measurable). Then Lévy's Theorem says that $\mathbb{E}(Y | \mathcal{F}_n) \rightarrow Y$ in L^1 and a.s. That is, if you take conditional expectation of Y with a large number of X_i 's, you get something "close" to Y .

There is no equivalent statement when Y depends on a very large (but finite) number of X_i .

6.3 Radon-Nikodym Theorem for Probability Measures

Definition 6.7 Let (Ω, \mathcal{F}) be a measurable space. Let P and Q be two probability measures on this space. We say that Q is **absolutely continuous w.r.t. P** , and write $Q \ll P$, if

$$P(A) = 0 \quad \Rightarrow \quad Q(A) = 0.$$

Lemma 6.8 Assume that $Q \ll P$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that $P(A) < \delta \Rightarrow Q(A) < \varepsilon$.

Proof: Suppose not. Then there is some $\varepsilon > 0$ such that for all n , there is some set A_n such that

$$P(A_n) < 2^{-n} \text{ and } Q(A_n) > \varepsilon.$$

Let $A = \limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Then

$$\begin{aligned} P(A) &= \lim_{n \rightarrow \infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} P(A_k) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} 2^{-k} = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} Q(A) &= \lim_{n \rightarrow \infty} Q\left(\bigcup_{k=n}^{\infty} A_k\right) \\ &\geq \limsup_{n \rightarrow \infty} Q(A_n) \\ &\geq \varepsilon. \end{aligned}$$

Contradiction! Thus the lemma is true. ■

Theorem 6.9 (Radon-Nikodym) *Let (Ω, \mathcal{F}) be a measurable space and suppose that \mathcal{F} is countably generated. Let P and Q be two probability measures on this space such that $Q \ll P$.*

Then, there exists a non-negative random variable L on this space such that for all $A \in \mathcal{F}$,

$$Q(A) = \int_A dQ = \int_A L dP.$$

We write $L := \frac{dQ}{dP}$.

Proof: \mathcal{F} is countably generated implies that there exist events A_1, A_2, \dots such that $\mathcal{F} = \sigma(A_1, A_2, \dots)$.

Let $\mathcal{F}_n = \sigma(A_1, \dots, A_n)$, and $\mathcal{F} = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$.

Take any n . Let B_{n1}, \dots, B_{nk_n} be the partition generated by A_1, \dots, A_n , i.e. all non-empty sets of the form $C_1 \cap \dots \cap C_n$, where $C_i = A_i$ or A_i^c . Then $\mathcal{F}_n = \{\text{all possible unions of } B_{n1}, \dots, B_{nk_n}\}$.

Step 1: Define a P -martingale $\{L_n, \mathcal{F}_n\}$ such that $Q(A) = \int_A L_n dP$ for $A \in \mathcal{F}_n$.

Define

$$L_n(\omega) := \begin{cases} \frac{Q(B_{ni})}{P(B_{ni})} & \text{if } \omega \in B_{ni} \text{ and } P(B_{ni}) > 0, \\ 0 & \text{if } \omega \in B_{ni} \text{ and } P(B_{ni}) = 0. \end{cases}$$

For any $A \in \mathcal{F}_n$, there is an index set I such that $A = \bigcup_{i \in I} B_{ni}$. Let $I' = \{i \in I : P(B_{ni}) > 0\}$. Then

$$\begin{aligned} Q(A) &= \sum_{i \in I} Q(B_{ni}) = \sum_{i \in I'} Q(B_{ni}) \\ &= \sum_{i \in I'} \frac{Q(B_{ni})}{P(B_{ni})} P(B_{ni}) \\ &= \sum_{i \in I'} \int_{B_{ni}} L_n dP = \sum_{i \in I} \int_{B_{ni}} L_n dP \\ &= \int_A L_n dP. \end{aligned}$$

Note that L_n is \mathcal{F}_n -measurable, non-negative, and

$$\mathbb{E}_P[L_n] = \int_{\Omega} L_n dP = Q(\Omega) = 1 < \infty.$$

In addition, for any $A \in \mathcal{F}_{n-1}$,

$$\begin{aligned} \mathbb{E}_P(L_n; A) &= \int_A L_n dP && (\text{since } A \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n) \\ &= Q(A) \\ &= \int_A L_{n-1} dP && (\text{since } A \in \mathcal{F}_{n-1}) \\ &= \mathbb{E}_P(L_{n-1}; A). \end{aligned}$$

Thus $L_{n-1} = \mathbb{E}_P(L_n \mid \mathcal{F}_{n-1})$, and therefore, under P , $\{L_n, \mathcal{F}_n\}$ is a martingale.

Step 2: $L_n \rightarrow L$ P -a.s. for some P -a.s. finite L .

Since $\{L_n, \mathcal{F}_n\}$ is a non-negative supermartingale, By the Martingale Convergence Theorem, there exists an L such that $L_n \rightarrow L$ P -a.s., and by Fatou's Lemma,

$$\mathbb{E}_P L \leq \liminf \mathbb{E}_P L_n = 1 < \infty,$$

i.e. L is finite P -a.s.

Step 3: $\{L_n\}$ is uniformly integrable under P , so $L_n \rightarrow L$ in L^1 .

For any $a > 0$ and n ,

$$\begin{aligned} aP(L_n \geq a) &\leq \mathbb{E}_P(L_n; L_n \geq a) \\ &= Q(L_n \geq a) && (\text{since } \{L_n \geq a\} \in \mathcal{F}_n) \\ &\leq 1, \\ P(L_n \geq a) &\leq \frac{1}{a}. \end{aligned}$$

Take any $\varepsilon > 0$. By Lemma 6.8, we can find $\delta > 0$ such that $P(A) < \delta \Rightarrow Q(A) < \varepsilon$. Let $a = \frac{1}{\delta}$. Then

$$\begin{aligned} P(L_n \geq a) &\leq \frac{1}{a} = \delta, \\ \Rightarrow \quad \varepsilon &\geq Q(L_n \geq a) \\ &= \mathbb{E}_P(L_n; L_n \geq a). \end{aligned}$$

This proves the uniform integrability of L_n under P .

By Lemma 6.4, L_n is a uniformly integrable martingale under P , so $L_n \rightarrow L$ in L^1 .

Step 4: For any $A \in \mathcal{F}$, $Q(A) = \int_A L dP$.

Take any $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Then $A \in \mathcal{F}_n$ for some n , so $A \in \mathcal{F}_m$ for all $m \geq n$. Thus $Q(A) = \int_A L_m dP$ for all $m \geq n$.

But since

$$\begin{aligned} \left| \int_A L_n dP - \int_A L dP \right| &= \left| \int_A (L_n - L) dP \right| \\ &\leq \int_A |L_n - L| dP \\ &\leq \int_{\Omega} |L_n - L| dP \\ &\rightarrow 0, \end{aligned}$$

we conclude that $Q(A) = \int_A L dP$.

As in the proof of Lévy's Upward Convergence Theorem, we can apply $\pi - \lambda$ Theorem to conclude that the same holds for all $A \in \mathcal{F}$.

■