Lecture 6: October 12

Lecturer: Persi Diaconis *Scribes: Kenneth Tay*

6.1 General Measures

Definition 6.1 *Let Ω be a set, $\mathcal{F}$ a field of subsets of Ω. Then $\mu : \mathcal{F} \to \mathbb{R}$ is a* ***measure*** *if*

1. *(non-trivial)* $\mu(\emptyset) = 0$,
2. *(non-negative)* $\mu(A) \in [0, \infty]$, *and*
3. *(countably additive) If* $A_i \in \mathcal{F}$ *for* $i = 1, 2, \ldots$ *with* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ *and* A_i*'s disjoint, then* $\mu(\bigcup A_i) = \sum \mu(A_i)$.

Example: On $\mathbb{R}$, let $\mu(\emptyset) = 0$, $\mu(A) = \infty$ for all other A. μ is a measure (but not a probability measure, or even a finite measure).

Why are we introducing this notion?

1. To have densities with respect to some measure on the real line (e.g. Lebesgue measure).
2. It's basically "free": most of the work done for probability measures translates directly to general measures.

- If $\mu(\Omega) < \infty$, then you can scale it to be a probability measure. So it's essentially the same as probabilities that we have been talking about.
- Say A_n are in $\mathcal{F}$, $A_n \nearrow A \in \mathcal{F}$, then $\mu(A_n) \nearrow \mu(A)$.

  Proof: "Make them disjoint": Let $B_n = A_n \cap \left(\bigcup_{i=1}^{n-1} A_i \right)^c$. Then the B_n's are disjoint and $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i = A_n$ for all n. Hence,

$$\mu(A) = \mu\left(\bigcup B_n\right) = \sum \mu(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu(A_n).$$

  ■

- **One point of caution**: For probability measures, if $A \subseteq B$, then $\mu(B - A) = \mu(B) - \mu(A)$. If $A \downarrow \emptyset$, then $\mu(A_n) \downarrow 0$.

  However, this is not always true for infinite measures! For example, consider the Lebesgue measure on $\mathbb{R}$. $A = (-\infty, 0]$, $B = (-\infty, 1]$. Then $\mu(B - A) = 1$ but $\mu(B) - \mu(A) = \infty - \infty$. Also, $A_n = [n, \infty) \downarrow \emptyset$ but $\mu(A_n) = \infty$ for all n.

Definition 6.2 *μ is **σ-finite** if there exist $B_i \in \mathcal{F}$, $i = 1, 2, \ldots$ such that $\bigcup_{i=1}^{\infty} B_i = \Omega$ and $\mu(B_i) < \infty$ for all i.*

Some examples to illustrate σ-finiteness:

- $\Omega = \mathbb{N} = \{0, 1, \ldots\}$, $\mu(i) = 1$ for all i (i.e. counting measure). Then μ is σ-finite.
- Lebesgue measure on $\mathbb{R}$ is σ-finite.
- On $[0, 1]$, let $\mu(A) =$ number of points in A. Then μ is not σ-finite.

Theorem 6.3 *Let Ω be some set, $\mathcal{P}$ a π-system of subsets of Ω. Let μ_1 and μ_2 be measures defined on $\sigma(\mathcal{P})$. If*

1. *$\mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{P}$, and*
2. *there exist $B_j \in \mathcal{P}$ such that $\Omega = \bigcup_{i=j}^{\infty} B_j$ and $\mu_i(B_j) < \infty$ for $i = 1, 2$ and all j,*

then $\mu_1(A) = \mu_2(A)$ for all $A \in \sigma(\mathcal{P})$.

Proof: Fix some $B \in \mathcal{P}$ such that $\mu_i(B) < \infty$ for $i = 1, 2$.

For $i = 1, 2$, define a new measure $\nu_i(F) = \mu_i(F \cap B)$ for all $F \in \sigma(\mathcal{P})$. The ν_i's are finite measures, and $\nu_1 = \nu_2$ on $\mathcal{P}$. Thus, by the $\pi - \lambda$ theorem, $\nu_1 = \nu_2$ on $\sigma(\mathcal{P})$.

Now consider the cover of Ω given to us as an assumption. We can make the sets in the cover disjoint (i.e. $A_i = B_i \cap \left(\bigcup_{j=1}^{i-1} B_j \right)^c$. For every $F \in \sigma(\mathcal{P})$,

$$
\begin{aligned}
\mu_1(F) &= \mu_1\left(F \cap \left(\bigcup_{i=1}^{\infty} A_i \right) \right) \\
&= \sum_{i=1}^{\infty} \mu_1(F \cap A_i) && \text{(by countable additivity)} \\
&= \sum_{i=1}^{\infty} \mu_2(F \cap A_i) \\
&= \mu_2(F).
\end{aligned}
$$

■

The condition of μ being σ-finite is necessary: Take $\Omega = (0, 1]$, $\mathcal{F} =$ all subsets. Define

$$
\begin{aligned}
&\mu_1(\emptyset) = 0, \quad \mu_1(A) = \infty \text{ for all other } A, \\
&\mu_2(A) = \# \text{ of points in } A.
\end{aligned}
$$

These are measures, and they agree on intervals $(a, b]$, which is a π-system. But for any point $\{x\}$, the 2 measures don't agree.

6.2 Outer Measures

Definition 6.4 *Let Ω be any set. An **outer measure** μ^* is a set function defined on all subsets of Ω such that*

1. *(non-negative)* $\mu^*(A) \in [0, \infty]$,
2. *(non-trivial)* $\mu^*(\emptyset) = 0$,
3. *(monotonicity) If* $A \subseteq B$, *then* $\mu^*(A) \leq \mu^*(B)$, *and*
4. *(countable sub-additivity)* $\mu^*(\bigcup A_n) \leq \sum \mu^*(A_n)$.

Here are 2 examples of outer measures:

- Let $\rho(A)$ be any non-negative set function defined on any collection of subsets $\mathcal{A}$ with $\emptyset \in \mathcal{A}$ and $\rho(\emptyset) = 0$. Define
$$\mu^*(B) = \inf \sum_{i=1}^{\infty} \rho(A_i),$$
where the infimum is taken over all $A_i \in \mathcal{A}$ such that $B \subseteq \bigcup A_i$. Then μ^* is an outer measure.

  Proof: We only have to show countable sub-additivity (the other 3 conditions are obvious).

  Let A_i, $1 \leq i < \infty$ be any sets. If $\mu^*(A_i) = \infty$ for some i, we are done as the RHS is ∞.

  If not, for every $\varepsilon > 0$, all i, there exist sets A_{ij} so that $A_{ij} \in \mathcal{A}$, $A_i \subseteq \bigcup_j A_{ij}$ and
$$\sum_{j=1}^{\infty} \rho(A_{ij}) \leq \mu^*(A_i) + \frac{\varepsilon}{2^i}.$$

  Because $\bigcup A_i \subseteq \bigcup A_{ij}$, we have
$$\begin{aligned} \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho(A_{ij}) \\ &\leq \sum_{i=1}^{\infty} \mu^*(A_i) + \frac{\varepsilon}{2^i} \\ &= \sum_{i=1}^{\infty} \mu^*(A_i) + \varepsilon. \end{aligned}$$

  Since this is true for every ε, we are done. ■

- **(γ-Hausdorff measure on a metric space)** Let $\Omega, d(x, y)$ be a metric space. Let $\mathcal{A} = \{$all balls $B_\varepsilon(x) = \{y : d(x, y) < \varepsilon\}\}$ (ε and x allowed to vary).

  For $0 \leq \gamma < \infty$, define $\rho_\gamma(\text{ball } B) = \text{diam}(B)^\gamma$. Then $\mu^*_{\gamma,d}$ is the $d - \gamma$ Hausdorff measure.

The following theorem is the general version of the Extension Theorem for probability measures. The proof of this is, word-for-word, symbol-by-symbol, the same as our proof of the Extension Theorem.

Theorem 6.5 *If μ^* is an outer measure on subsets of Ω, then μ^* is a measure on*
$$\mathcal{M}^*(\mu^*) = \{A \subseteq \Omega : \forall\, E \subseteq \Omega, \mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E)\}.$$

6.2.1 Semi-Rings

Definition 6.6 *Let Ω be a set. A class of subsets of Ω, denoted by $\mathcal{R}$, is a **semi-ring** if:*

1. $\emptyset \in \mathcal{R}$,
2. *$\mathcal{R}$ is closed under finite intersections, and*
3. *If $A, B \in \mathcal{R}$, $A \subseteq B$, then $B - A = \bigcup_{i=1}^{n} C_i$ with $C_i \in \mathcal{R}$, C_i's disjoint.*

Here are 3 examples of semi-rings:

- Finite subsets of $[0, 1]$.
- ∞ rectangles in $\mathbb{R}^d$, sides parallel to the axes (must extend to infinity in at least 1 direction).
- Finite rectangles in $\mathbb{R}^d$.

Semi-rings are important because of the following extension theorem:

Theorem 6.7 *Let Ω be a set, $\mathcal{R}$ a semi-ring of subsets of Ω, $\mu : \mathcal{R} \to [0, \infty]$ such that*

1. $\mu(\emptyset) = 0$,
2. *μ finitely additive on $\mathcal{R}$, and*
3. *μ countably sub-additive.*

Then, μ has an extension to a measure on $\sigma(\mathcal{R})$. This measure is unique if μ is σ-finite.

Proof: Define

$$\mu^*(A) = \inf \sum_{i=1}^{\infty} \mu(A_i)$$

where $A \subseteq \bigcup A_i$, $A_i \in \mathcal{R}$ for all A_i. This is an outer measure.

Step 1: $\mathcal{R} \in \mathcal{M}^*(\mu^*)$.

Take $A \in \mathcal{R}$. We only need to show that, for all E,

$$\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(A^c \cap E).$$

(The other direction is obvious from sub-additivity). If $\mu^*(E)$ is ∞, then we are done. If not, for every $\varepsilon > 0$ there exist $A_i \in \mathcal{R}$, $E \subseteq \bigcup A_i$, such that $\sum \mu(A_i) \leq \mu^*(E) + \varepsilon$.

Define $B_n = A \cap A_n$. Then $A^c \cap A_n = A_n - B_n$, by definition of semi-ring, is in $\mathcal{R}$. So $A^c \cap A_n = \bigcup_{k=1}^{m_n} C_{nk}$ with $C_{nk} \in \mathcal{R}$.

We have the following relations for A_n, $A \cap E$ and $A^c \cap E$:

$$\begin{aligned} A_n &= B_n \cup \left(\bigcup_{k=1}^{m_n} C_{nk} \right), \\ A \cap E &\subseteq \bigcup B_i, \\ A^c \cap E &= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m_n} C_{nk}. \end{aligned}$$

Hence,

$$\begin{aligned} \mu^*(A \cap E) + \mu^*(A^c \cap E) &\leq \sum_{n=1}^{\infty} \mu(B_n) + \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \mu(C_{nk}) \\ &= \sum_{n=1}^{\infty} \mu(B_n) + \sum_{n=1}^{\infty} \mu(A_n - B_n) \\ &= \sum_{n=1}^{\infty} \mu(A_n) \\ &\leq \mu^*(E) + \varepsilon. \end{aligned}$$

Let $\varepsilon \to 0$, then we are done, i.e. $\mathcal{R} \in \mathcal{M}^*(\mu^*)$.

Step 2: $\mu^* |_{\mathcal{R}} = \mu$**.**

Consider $A \in \mathcal{R}$. Since A covers itself, by the definition of μ^* we have $\mu(A) \geq \mu^*(A)$.

For the other direction: if $A \subseteq \bigcup A_i$, $A_i \in \mathcal{R}$, by finite sub-additivity of μ, we have

$$\mu(A) \leq \sum \mu(A_i).$$

Taking the infimum over all covers of A, we get $\mu(A) \leq \mu^*(A)$. Therefore we have $\mu(A) = \mu^*(A)$ for all $A \in \mathcal{R}$.

Step 3: Finishing up.

By Theorem 6.5, μ^* is a measure on $\mathcal{M}^*(\mu^*)$. Recall that $\mathcal{M}^*(\mu^*)$ is a σ-field. Hence, $\mathcal{R} \subseteq \mathcal{M}^*(\mu^*) \Rightarrow \sigma(\mathcal{R}) \subseteq \mathcal{M}^*(\mu^*)$.

Thus, μ^* is a measure on $\sigma(\mathcal{R})$ which agrees with μ on $\mathcal{R}$. ∎