Lecture 9: February 6

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9.1 Martingales with Bounded Increments

Lemmas we proved from last time:

Lemma 9.1 *If $\{Z_n, \mathcal{F}_n\}$ is any martingale and τ is a stopping time, then $\{Z_{\tau\wedge n}, \mathcal{F}_n\}$ is also a martingale.*

Lemma 9.2 *If Z_n is a supermartingale with $\mathbb{E}|Z_1| < \infty$ that is uniformly bounded below by a constant, then $\lim Z_n$ exists and is finite a.s.*

[Same for submartingale bounded above by a constant.]

These will help us prove the following theorem:

Theorem 9.3 *Let $\{Z_n, \mathcal{F}_n\}$ be a martingale. Suppose that there exists a constant c such that for all n, $|Z_n - Z_{n-1}| \leq c$ a.s.*

With probability 1, either $\lim Z_n$ exists and is finite, or $\limsup Z_n = \infty$ and $\liminf Z_n = -\infty$, i.e.

$$P\left(\{\lim Z_n \text{ exists and is finite}\} \cup \{\limsup Z_n = \infty \text{ and } \liminf Z_n = -\infty\}\right) = 1.$$

Proof:

Take any $b \in \mathbb{R}$. Let $\tau = \inf\{n : Z_n \geq b\}$. Then $\{Z_{\tau\wedge n}, \mathcal{F}_n\}$ is a martingale. Moreover, $Z_{\tau\wedge n} \leq b + c$. Thus, $\lim Z_{\tau\wedge n}$ exists and is finite a.s.

If $\tau = \infty$, then $\tau \wedge n = n$ for all n, thus $\lim Z_{\tau\wedge n} = \lim Z_n$. Therefore, $\lim Z_n$ exists and is finite a.s. on the set $\{\tau = \infty\}$, i.e.

$$P\left(\{\lim Z_n \text{ exists and is finite}\} \cup \{\tau = \infty\}\right) = P(\tau = \infty).$$

But $\{\tau = \infty\} = \{Z_n < b \text{ for all } n\}$. Therefore, $\lim Z_n$ exists and is finite a.s. on $\{Z_n < b \text{ for all } n\}$. Taking the union over $b = 1, 2, \ldots$, this implies that $\lim Z_n$ exists and is finite a.s. on $\{\sup Z_n < \infty\} = \{\limsup Z_n < \infty\}$.

Similarly, by a symmetric argument, $\lim Z_n$ exists and is finite a.s. on $\{\inf Z_n > -\infty\} = \{\liminf Z_n > -\infty\}$. This completes the proof.

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Corollary 9.4 (Lévy's form of the Borel-Cantelli Lemma) *Let $\{\mathcal{F}_n\}_{n\geq 1}$ be a filtration and events $A_n \in \mathcal{F}_n$ for all n.*

Then

$$\sum_{n=1}^\infty 1_{A_n} = \infty \text{ if and only if (a.s.) } \sum_{n=1}^\infty P(A_n \mid \mathcal{F}_{n-1}) = \infty.$$

(A if and only if B (a.s.) means $P(A \triangle B) = 0$.)

Proof: Let $Z_n = \sum_{k=1}^{n} (1_{A_k} - P(A_k \mid \mathcal{F}_{k-1}))$. Then Z_n is a martingale with uniformly bounded increments. By the theorem above, this implies that

$$P\left(\{\lim Z_n \text{ exists and is finite}\} \cup \{\limsup Z_n = \infty \text{ and } \liminf Z_n = -\infty\}\right) = 1.$$

- If $\lim Z_n$ exists and is finite, then $\sum 1_{A_n} = \infty \Leftrightarrow \sum P(A_n \mid F_{n-1}) = \infty$.
- If $\limsup Z_n = \infty$ and $\liminf Z_n = -\infty$, then $\sum 1_{A_n} = \infty \;\Rightarrow\; \sum P(A_n \mid F_{n-1}) = \infty$ since otherwise, $\liminf Z_n = -\infty$ would be violated.

  Similarly, $\sum P(A_n \mid F_{n-1}) = \infty \;\Rightarrow\; \sum 1_{A_n} = \infty$, since otherwise $\limsup Z_n = \infty$ would be violated.

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9.2 "Almost Supermartingales"

Definition 9.5 *Let $\{\mathcal{F}_n\}_{n\geq 1}$ be a filtration. Let Z_n be a sequence of non-negative random variables adapted to this filtration with $\mathbb{E}Z_n < \infty$ for all n.*

Suppose that ξ_n and ζ_n are two other non-negative adapted sequences such that $\mathbb{E}\xi_n < \infty$, $\mathbb{E}\zeta_n < \infty$ for all n, and $\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] \leq Z_n + \xi_n - \zeta_n$ a.s. for all n.

*Then Z_n is called an **almost supermartingale**.*

Below is the convergence theorem for almost supermartingales:

Theorem 9.6 *Let Z_n, ξ_n, ζ_n, $\mathcal{F}_n$ be as above. Then on the set $\left\{\sum_{n=1}^{\infty} \xi_n < \infty\right\}$, $\lim Z_n$ exists and is finite and $\sum \zeta_n < \infty$ a.s.*

Proof: Let $Y_n = Z_n - \sum_{k=1}^{n-1} (\xi_k - \zeta_k)$. It can be checked that $\{Y_n\}$ is a supermartingale.

Take some $a > 0$, and let $\tau = \inf\left\{n : \sum_{k=1}^{n} \xi_k \geq a\right\}$. Then $Y_{\tau\wedge n}$ is a supermartingale. Moreover, it is uniformly bounded below:

$$\begin{aligned} Y_{\tau\wedge n} &= Z_{\tau\wedge n} - \sum_{k=1}^{\tau\wedge n-1} \xi_k + \sum_{k=1}^{\tau\wedge n-1} \zeta_k \\ &\geq - \sum_{k=1}^{\tau\wedge n-1} \xi_k \\ &> -a. \end{aligned}$$

So $\lim Y_{\tau\wedge n}$ exists and is finite a.s. This implies that $\lim Y_n$ exists and is finite a.s. on the set $\{\tau = \infty\} = \{\sum_{k=1}^{n} \xi_k < a \text{ for all } n\}$. Taking the union over $a = 1, 2, \dots$, we get $\lim Y_n$ exists and is finite a.s. on the set $\{\sum_{n=1}^{\infty} \xi_n < \infty\}$.

Suppose that $\sum_{n=1}^{\infty} \xi_n < \infty$. Then with probability 1, $\lim Y_n$ exists and is finite. But $Y_n = Z_n - \sum_{k=1}^{n-1} \xi_k + \sum_{k=1}^{n-1} \zeta_k$, so

$$
\begin{aligned}
& \lim \left(Z_n + \sum_{k=1}^{n-1} \zeta_k \right) \text{ exists and is finite,} \\
\Rightarrow \quad & \sup_n \sum_{k=1}^{n-1} \zeta_k < \infty, \\
\Rightarrow \quad & \sum_{k=1}^{\infty} \zeta_k < \infty, \\
\Rightarrow \quad & \lim Z_n \text{ exists and is finite a.s.}
\end{aligned}
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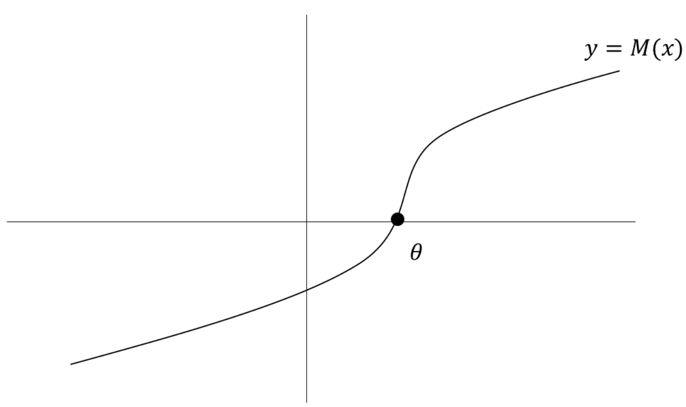
9.2.1 Application: A General Form of Stochastic Gradient Descent

Assume that we have a measurable function $y = M(x)$ with the following conditions:

- $M(x)$ is positive if $x > \theta$ $M(x)$ negative if $x < \theta$,
- $|M(x)| \leq m$ for all x,
- For all $\varepsilon > 0$, $\inf_{\varepsilon < x < 1/\varepsilon} M(x+\theta) > 0$ and $\sup_{-1/\varepsilon < x < -\varepsilon} M(x+\theta) < 0$.



Given x, we can't generate $M(x)$ exactly, but we can generate a random variable Y with mean $M(x)$ and variance $\leq \sigma^2$. **Goal: Estimate root θ.**

Consider the following procedure:

1. Choose a sequence of predetermined non-negative real numbers $\{a_n\}$.

2. Start with some arbitrary X_1.
3. (Loop) Given X_n, generate Y_n with mean $M(X_n)$ and variance $\leq \sigma^2$. Set $X_{n+1} = X_n - a_n Y_n$.

Theorem 9.7 *For the procedure above, if* $\sum a_n = \infty$ *and* $\sum a_n^2 < \infty$*, then* $\lim_{n\to\infty} X_n = \theta$ *a.s.*

Proof: Define $Z_n = (X_n - \theta)^2$. Then

$$\begin{aligned}
Z_{n+1} &= (X_n - a_n Y_n - \theta)^2 \\
&= [(X_n - \theta) - a_n(Y_n - M(X_n)) - a_n M(X_n)]^2, \\
\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &\leq Z_n + a_n^2\sigma^2 + a_n^2 m^2 - 2a_n(X_n - \theta)M(X_n) && \text{(since } \mathbb{E}[Y_n \mid \mathcal{F}_n] = M(X_n)) \\
&= Z_n + a_n^2(\sigma^2 + m^2) - 2a_n|X_n - \theta||M(X_n)|. && \text{(by properties of } M)
\end{aligned}$$

We have put Z_n is the form of an almost supermartingale. Since $\sum a_n^2 < \infty$, by Theorem 9.6, $\lim Z_n$ exists and is finite a.s., and $\sum a_n|X_n - \theta||M(X_n)|$ is finite a.s.

Let C be the limit of $|X_n - \theta|$ (which exists as Z_n has a limit a.s.). If $C \neq 0$, then by $\sum a_n = \infty$ and the properties of the function M, $\sum a_n|X_n - \theta||M(X_n)| = \infty$. This contradicts what we have already. Thus, $X_n \to \theta$ a.s.

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