STATS 310C Notes

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Section 7.1: Definition, Canonical Construction and Law

- (Dfn 7.1.2) **Finite-dimensional distributions (f.d.d.)** of a stochastic process $\{X_t : t \in \mathbb{T}\}$ is the collection of joint laws of $X_{t_1}, \dots, X_{t_n}$ for all $n \in \mathbb{N}$, $t_1, \dots, t_n$ distinct values in $\mathbb{T}$.
- (Dfn 7.1.3) A collection of f.d.d. is **consistent** if for any $B_k \in \mathcal{B}$, distinct $t_k \in \mathbb{T}$ and finite n, $\mu_{t_1,\dots,t_n}(B_1 \times \ldots \times B_n) = \mu_{t_{\pi(1)},\dots,t_{\pi(n)}}(B_{\pi(1)} \times \ldots \times B_{\pi(n)})$ for all permutations π of $\{1, \dots, n\}$, and $\mu_{t_1,\dots,t_{n-1}}(B_1 \times \ldots \times B_{n-1}) = \mu_{t_1,\dots,t_n}(B_1 \times \ldots \times B_{n-1} \times \mathbb{R})$.
- (Dfn 7.1.5) $\mathbb{R}^{\mathbb{T}} :=$ {collection of all functions $x(t) : \mathbb{T} \mapsto \mathbb{R}$}. A **finite-dimensional rectangle** in $\mathbb{R}^{\mathbb{T}}$ is of the form $\{x(\cdot) : x(t_i) \in B_i, i = 1, \dots, n\}$. The **cylindrical σ-algebra** $\mathcal{B}^{\mathbb{T}}$ is the σ-algebra generated by the finite-dimensional rectangles.
- $\mathcal{B}_c$ is $\mathcal{B}^{\mathbb{T}}$ when $\mathbb{T} = \{1, 2, \dots\}$.
- (Dfn 7.1.6) $A \subseteq \mathbb{R}^{\mathbb{T}}$ has **countable representation** if $A = \{x(\cdot) \in \mathbb{R}^{\mathbb{T}} : (x(t_1), x(t_2), \dots) \in D\}$ for some $D \in \mathcal{B}_c$ and $\{t_1, t_2, \dots\} \subseteq \mathbb{T}$.
- (Lem 7.1.7) $A \in \mathcal{B}^{\mathbb{T}}$ iff A has countable representation.
- (Lem 7.1.8) **Consistent f.d.d. $\implies$ $\exists$ stochastic process with that f.d.d.:** For any consistent collection of f.d.d., there is a probability space and a stochastic process on it with that f.d.d.
- (Dfn 7.1.9) The **law** of a stochastic process is the probability measure $\mathcal{P}_X$ on $\mathcal{B}^{\mathbb{T}}$ such that for all $A \in \mathcal{B}^{\mathbb{T}}$, $\mathcal{P}_X(A) = \mathbb{P}\{\omega : X_\cdot(\omega) \in A\}$.
- (Ex 7.1.12, HW1) A continuous time stochastic process $\{X_t : t \geq 0\}$ has **independent increments** if $X_{t+h} - X_t$ is independent of $\sigma(X_s : 0 \leq s \leq t)$ for all $h > 0$, $t \geq 0$.

  If for all $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, $X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are mutually independent, then $\{X_t\}$ has independent increments. (Thus, this property is determined by the f.d.d.)

Section 7.2: Continuous and Separable Modifications

- (Dfn 7.2.2) Two stochastic processes $\{X_t\}$ and $\{Y_t\}$ are **versions** of each other if they have the same f.d.d. They are **modifications** if $\mathbb{P}(X_t \neq Y_t) = 0$ for all t. They are **indistinguishable** if $\{\omega : X_t(\omega) \neq Y_t(\omega)$ for some $t \in \mathbb{T}\}$ is a $\mathbb{P}$-null set.

  Modifications are versions, but converse fails.
- (Ex 7.2.3, HW1) **Right-continuous modification is unique:** Modifications which both have w.p. 1 right-continuous sample functions are indistinguishable. Hence, we can talk of **the** right-continuous modification.

- (Dfn 7.2.5) A function f on metric space (T, d) is **locally γ-Hölder continuous** if

$$\sup_{\{t \neq s, d(t,u) \vee d(s,u) < h_u\}} \frac{|f(t) - f(s)|}{d(t,s)^\gamma} \le c_u,$$

  for some $\gamma > 0$, some $c : \mathbb{T} \mapsto [0, \infty)$, and $h : \mathbb{T} \mapsto (0, \infty]$. It is **uniformly γ-Hölder continuous** if the same applies for constant $c < h = \infty$.

  A stochastic process is locally/uniformly γ-Hölder continuous if its sample functions have that property.

  Local γ-Hölder continuity implies continuity.

- (Thm 7.2.6) **Kolmogorov-Centsov continuity theorem:** Suppose $\{X_t : t \in \mathbb{T}\}$ with $\mathbb{T} = \mathbb{I}^r$, where $\mathbb{I}$ is a compact interval. If there exist positive constants α, β and finite c such that

$$\mathbb{E}[|X_t - X_s|^\alpha] \le c\|t - s\|^{r+\beta}$$

  for all $s, t \in \mathbb{T}$, then $\{X_t\}$ has a continuous modification which is locally γ-Hölder continuous for any $0 < \gamma < \beta/\alpha$.

- (Lemma 7.2.12) While the above provides a continuous modification on compact intervals, the proof can be modified to get one continuous modification valid on $[0, \infty)$.

- (Ex 7.2.13, HW1) A simpler condition than Kolmogorov-Centsov to show Hölder continuity.

- (Dfn 7.2.15) A function $x \in \mathbb{R}^{\mathbb{I}}$ is **$\mathbb{C}$-separable** if $\mathbb{C}$ is countable and for any t, there is a sequence $\{s_k\} \subseteq \mathbb{C}$ that converges to t such that $x(s_k) \to x(t)$. A continuous-time stochastic process is **$\mathbb{C}$-separable** if its sample functions are $\mathbb{C}$-separable.

- (Prop 7.2.16) **Can always work with a separable process:** Any continuous time stochastic process $\{X_t : t \in \mathbb{I}\}$ has a separable modification (consisting possibly of $\bar{\mathbb{R}}$-valued variables). (But note that separability does not imply measurability.)

- (Dfn 7.2.19) A stochastic process $\{X_t, t \in \mathbb{I}\}$ is **measurable** if $(t, \omega) \mapsto X_t(\omega)$ is measurable w.r.t. the joint σ-algebra.

- (Dfn 7.2.20) A stochastic process $\{X_t, t \in \mathbb{I}\}$ is **continuous in probability** if for any $t \in \mathbb{I}$ and $\varepsilon > 0$, $\lim_{s \to t} \mathbb{P}(|X_s - X_t| > \varepsilon) = 0$. This is a very mild condition which is determined by the f.d.d.

- (Prop 7.2.21) Any continuous in probability process has a separable modification (consisting possibly of $\bar{\mathbb{R}}$-valued variables) which is a measurable process.

Section 7.3: Gaussian and stationary processes

- (Dfn 7.3.1) A stochastic process is a **Gaussian process** iff it has multivariate normal f.d.d. A Gaussian process is **centered** if $\mathbb{E}[X_t] = 0$ for all t.

- (Dfn 7.3.2) A symmetric function c on a product set $\mathbb{T} \times \mathbb{T}$ is **non-negative definite/positive semidefinite** if for any finite n and $t_k \in \mathbb{T}$ and for any $a_k \in \mathbb{R}$, $\displaystyle\sum_{j=1}^n \sum_{k=1}^n a_j c(t_j, t_k) a_k \ge 0$.

- (Eg 7.3.3) The **auto-covariance function** $c(s, t) = \mathrm{Cov}(X_s, X_t)$ of a square-integrable stochastic process is non-negative definite.

- (Ex 7.3.4, HW2) The law of a Gaussian process is uniquely determined by its mean and auto-covariance functions. A Gaussian process exists for any mean function and PSD auto-covariance function.

- (Prop 7.3.5) Gaussian processes are closed w.r.t. L^2: i.e. if $\mathbb{E}[(X_t - X_t^{(k)})^2] \to 0$ as $k \to \infty$ for each fixed t, then X_t is also a Gaussian process, whose mean and auto-covariance functions are the pointwise limits of those for $X_t^{(k)}$.

- (Cor 7.3.6) **Condition for Gaussian SP to have indep increments:** A continuous time Gaussian stochastic process has independent increments iff $\text{Cov}(Y_t - Y_u, Y_s) = 0$ for all $s \leq u < t$. Equivalently, the Gaussian process has auto-covariance function of the form $c(t,s) = g(t \wedge s)$.

- (Dfn 7.3.7) A continuous time stochastic process is **stationary** if its law is invariant under any time shift θ_s, $s \geq 0$, or equivalently, $(X_{t_1}, \dots, X_{t_n}) \stackrel{d}{=} (X_{t_1+s}, \dots, X_{t_n+s})$ for all $s, t_1, \dots, t_n$.

- (Dfn 7.3.8) A square-integrable continuous time stochastic process of constant mean function and auto-covariance function of the form $c(t,s) = r(|t-s|)$ is **weakly stationary**.

  Any square-integrable stationary process is weakly stationary.

- (Ex 7.3.9, HW2) Any weakly stationary Gaussian process is also stationary. (Not true for general processes.)

- (Ex 7.3.10, HW2) Any centered, weakly stationary process of independent increments must be a modification of the trivial process (i.e. constant sample functions).

- (Dfn 7.3.11) A continuous time stochastic process $\{X_t\}$ has **stationary increments** if the law of $X_t - X_s$ depends only on $t - s$.

- (Ex 7.3.14) Every stationary process has stationary increments.

Section 8.1: Continuous time filtrations and stopping times

- (Dfn 8.1.4) **Left filtration** $\mathcal{F}_{t^-} = \sigma(\mathcal{F}_s, s < t)$, **right filtration** $\mathcal{F}_{t^+} = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$. A filtration $\{\mathcal{F}_t\}$ is **right-continuous** if $\mathcal{F}_t = \mathcal{F}_{t^+}$ for all $t \geq 0$.

- (Eg 8.1.5) **Interpolated filtration** $\mathcal{F}_t = \mathcal{G}_{\lfloor t \rfloor}$. Any interpolated filtration is right-continuous, but usually not left-continuous.

- (Eg 8.1.6) Sample path continuity for $\{X_t\}$ does not guarantee right-continuity of canonical filtration $\{\mathcal{F}_t^X\}$.

- (Dfn 8.1.7) An $\mathcal{F}_t$-adapted $\{X_t\}$ is **$\mathcal{F}_t$-progressively measurable** if $X_s(\omega) : [0,t] \times \Omega \mapsto \mathbb{R}$ is measurable w.r.t. $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$ for each $t \geq 0$.

- (Prop 8.1.8) **Adapted + right-continuous $\implies$ progressively measurable:** An $\mathcal{F}_t$-adapted stochastic process with right-continuous sample functions is also $\mathcal{F}_t$-progressively measurable.

- (Dfn 8.1.9) $\tau : \Omega \mapsto [0, \infty]$ is a **$\mathcal{F}_t$-stopping time** if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. The **stopped σ-algebra** is $\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$.

  $\mathcal{F}_{t^+}$-stopping times are **Markov times/optional times**.

  Every stopping time is a Markov time. The 2 ideas are the same for right-continuous filtrations.

- (Ex 8.1.10, HW2) τ is an $\mathcal{F}_t$-Markov time iff $\{\tau < t\} \in \mathcal{F}_t$ for all $t \geq 0$.

- (Ex 8.1.10, HW2) If $\tau_1, \tau_2, \dots$ are $\mathcal{F}_t$-stopping times, then so are $\tau_1 \wedge \tau_2$, $\tau_1 + \tau_2$ and $\sup_n \tau_n$. If they are $\mathcal{F}_t$-Markov times, then *in addition* $\inf_n \tau_n$, $\liminf_n \tau_n$ and $\limsup_n \tau_n$ are $\mathcal{F}_t$-Markov times.

- (Ex 8.1.10d, HW2) If τ_1 and τ_2 are $\mathcal{F}_t$-Markov times, then $\tau_1 + \tau_2$ is an $\mathcal{F}_t$-stopping time when either both are strictly positive, or alternatively when τ_1 is a strictly positive $\mathcal{F}_t$-stopping time.

- (Ex 8.1.11, HW3) Suppose θ and τ are $\mathcal{F}_t$-stopping times. Then
  - $\sigma(\tau) \subseteq \mathcal{F}_\tau$, $\mathcal{F}_\tau$ is a σ-algebra. If $\tau = t$ non-random, then $\mathcal{F}_\tau = \mathcal{F}_t$.
  - $\mathcal{F}_{\theta\wedge\tau} = \mathcal{F}_\theta \cap \mathcal{F}_\tau$. The events $\{\theta < \tau\}$, $\{\theta \le \tau\}$ and $\{\theta = \tau\}$ all belong to $\mathcal{F}_{\theta\wedge\tau}$.
  - For any integrable random variable Z, $\mathbb{E}[\mathbb{E}[Z \mid \mathcal{F}_\theta] \mid \mathcal{F}_\tau] = \mathbb{E}[Z \mid \mathcal{F}_{\theta\wedge\tau}]$.
  - If $\theta \le \xi$ and $\xi \in m\mathcal{F}_\theta$, then ξ is an $\mathcal{F}_t$-stopping time.

- (Prop 8.1.13) If $\{X_s\}$ is $\mathcal{F}_t$-progressively measurable, then for any $\mathcal{F}_t$-stopping time τ, $\{X_{s\wedge\tau}\}$ is also $\mathcal{F}_t$-progressively measurable. (Not true for adaptedness.)

  In particular, if $\tau < \infty$ or there exists $X_\infty \in m\mathcal{F}_\infty$, then $X_\tau \in m\mathcal{F}_\tau$.

- (Prop 8.1.15) Let $\{X_s, s \ge 0\}$ be an $\mathcal{F}_t$-adapted right-continuous stochastic process. The **first hitting time** $\tau_B(\omega) := \inf\{t \ge 0 : X_t(\omega) \in B\}$ is an $\mathcal{F}_t$-Markov time for an open set B. If B is a closed set and $\{X_s\}$ has continuous sample functions, then τ_B is an $\mathcal{F}_t$-stopping time.

- (Prop 8.1.16) Given an $\mathcal{F}_t$-Markov time τ, there is a decreasing sequence of $\mathcal{F}_t$-stopping times τ_ℓ such that $\tau_\ell \downarrow \tau$ and τ_ℓ is $\mathbb{Q}^{(2,\ell)}$-valued.

Section 8.2: Continuous time martingales

- (Ex 8.2.2) If $\{X_t\}$ is a submartingale with $\mathbb{E}X_t = \mathbb{E}X_0$ for all $t \ge 0$, then it is also a martingale.

- (Ex 8.2.3) If $\{X_t\}$ is a square-integrable martingale, then $\mathbb{E}[(X_t - X_s)^2 \mid \mathcal{F}_s] = \mathbb{E}[X_t^2 \mid \mathcal{F}_s] - X_s^2$ for all $0 \le s \le t$, which implies that $t \mapsto \mathbb{E}X_t^2$ is non-decreasing.

- (Prop 8.2.4) Any integrable stochastic process $\{X_t, t \ge 0\}$ of independent increments and constant mean function is a martingale.

- (Ex 8.2.6, HW3) If $\{X_t\}$ is square-integrable, having zero-mean independent increments, then $X_t^2 - \langle X\rangle_t$ is a martingale, where $\langle X\rangle_t = \mathbb{E}X_t^2 - \mathbb{E}X_0^2$ is non-random and non-decreasing.

- (Ex 8.2.6, HW3) If $\{X_t\}$ is square-integrable with $X_0 = 0$ and zero-mean, stationary independent increments, then $X_t^2 - t\mathbb{E}X_1^2$ is a martingale.

- (Thm 8.2.14) **Doob's inequality:** If $\{X_s\}$ is a right-continuous submartingale, then for $t \ge 0$ finite, $M_t = \sup\limits_{0\le s\le t} X_s$ and any $x > 0$,

$$\mathbb{P}(M_t \ge x) \le \frac{1}{x}\mathbb{E}[X_t 1_{\{M_t \ge x\}}] \le \frac{1}{x}\mathbb{E}[(X_t)_+].$$

- (Ex 8.2.15) For $p > 1$, in the setting above, $\mathbb{P}(M_t \ge x) \le \dfrac{1}{x^p}\mathbb{E}[(X_t)_+^p]$. If $\{X_t\}$ is a right-continuous martingale, we have $\mathbb{P}\left(\sup\limits_{0\le s\le t} |X_s| \ge x\right) \le \dfrac{1}{x^p}\mathbb{E}[|X_t|^p]$.

- (Cor 8.2.16) L^p **maximal inequalities:** For any $p >$, $t \geq 0$ and right-continuous submartingale $\{X_t\}$, we have $\mathbb{E}\left[\left(\sup_{0\leq u\leq t} X_u\right)_+^p\right] \leq q^p \mathbb{E}[(X_t)_+^p]$, where $q = p/(p-1)$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$).

  If $\{X_t\}$ is a right-continuous martingale, we have $\mathbb{E}\left[\left(\sup_{0\leq u\leq t} |X_u|\right)^p\right] \leq q^p \mathbb{E}[|X_t|^p]$.

- (Thm 8.2.20) **Doob's Convergence Theorem:** Suppose right-continuous supermartingale $\{X_t\}$ is such that $\sup_t \mathbb{E}[(X_t)_-] < \infty$. Then $X_t \stackrel{a.s.}{\to} X_\infty$ and $\mathbb{E}|X_\infty| \leq \liminf_t \mathbb{E}|X_t|$ is finite.

- (Dfn 8.2.22) A submartingale $\{X_t\}$ is **right closable**, or has a **last element** X_∞ if $X_\infty \in L^1(\Omega, \mathcal{F}_\infty, P)$ is such that for any $t \geq 0$, $\mathbb{E}[X_\infty \mid \mathcal{F}_t] \geq X_t$ a.s.

  For a supermartingale, we require $\mathbb{E}[X_\infty \mid \mathcal{F}_t] \leq X_t$ a.s. For a martingale, we require $\mathbb{E}[X_\infty \mid \mathcal{F}_t] = X_t$ a.s., i.e. $\{X_t\}$ is a Doob martingale of X_∞ w.r.t. $\{\mathcal{F}_t\}$.

- (Prop 8.2.23) For a right-continuous non-negative submartingale $\{X_t\}$, the following are equivalent:

  (a) $\{X_t\}$ is U.I.

  (b) $X_t \to X_\infty$ in L^1.

  (c) $X_t \stackrel{a.s.}{\to} X_\infty$ a last element of $\{X_t\}$.

  Without non-negativity, we still have $(a) \Leftrightarrow (b) \Rightarrow (c)$. A right-continuous martingale has all these properties iff it is a Doob martingale.

- (Prop 8.2.24) **Doob's L^p Martingale Convergence:** If right-continuous martingale $\{X_t\}$ is L^p-bounded for some $p > 1$, then $X_t \to X_\infty$ a.s. and in L^p.

- (Thm 8.2.25) Suppose $(X_t, \mathcal{F}_t)$ is a supermartingale with right-continuous filtration and $t \mapsto \mathbb{E}X_t$ is right-continuous. Then there exists an RCLL modification $\tilde{X}_t$ of X_t such that $(\tilde{X}_t, \mathcal{F}_t)$ is a supermartingale.

- (Thm 8.2.26) **Doob's Optional Stopping Theorem:** If $(X_t, \mathcal{F}_t)$ is a right-continuous submartingale with a last element $(X_\infty, \mathcal{F}_\infty)$, then for any $\mathcal{F}_t$-Markov times $\tau \geq \theta$, X_θ and X_τ are integrable and $\mathbb{E}[X_\tau] \geq \mathbb{E}[X_\theta]$, with equality in the case of a martingale.

- (Cor 8.2.27) If $(X_t, \mathcal{F}_t)$ is a right-continuous submartingale with a last element $(X_\infty, \mathcal{F}_\infty)$, then for any $\mathcal{F}_t$-Markov times $\tau \geq \theta$, $\mathbb{E}[X_\tau \mid \mathcal{F}_{\theta^+}] \geq X_\theta$ w.p. 1 (with equality in the case of a martingale).

  If θ is a $\mathcal{F}_t$-stopping time, then we also have $\mathbb{E}[X_\tau \mid \mathcal{F}_\theta] \geq X_\theta$ w.p. 1 (with equality in the case of a martingale).

- (Remark 8.2.28) In Thm 8.2.26 and Cor 8.2.27, if τ is a bounded Markov time, then we don't need the existence of a last element for the statements to be true.

- (Cor 8.2.29) **Stopped processes:** If η is an $\mathcal{F}_t$-stopping time and $(X_t, \mathcal{F}_t)$ is a right-continuous submartingale, then $\{X_{t\wedge\eta}\}$ is also a right-continuous submartingale.

- (Ex 8.2.30, HW4) **Optional Stopping Theorem for stopped processes:** If $(X_t, \mathcal{F}_t)$ is a right-continuous submartingale and $u \geq 0$ a non-random constant, then for any $\mathcal{F}_t$-stopping times $\tau \geq \theta$, we have $\mathbb{E}[X_{u\wedge\tau} \mid \mathcal{F}_\theta] \geq X_{u\wedge\theta}$ w.p. 1 (equality for a martingale). This implies that $\mathbb{E}[X_{u\wedge\tau}] \geq \mathbb{E}[X_{u\wedge\theta}]$ (equality for martingale).

  If $X_{u\wedge\tau}$ is U.I., then we can also conclude that X_θ and X_τ are integrable and $\mathbb{E}[X_\tau \mid \mathcal{F}_\theta] \geq X_\theta$ a.s. (equality for martingale).

- (Dfn 8.2.37) The **standard d-dimensional Brownian motion** is the $\mathbb{R}^d$-valued stochastic process such that its d components are mutually independent, standard one-dimensional Wiener processes. It is a martingale and a centered Gaussian process of continuous sample functions and stationary, independent increments.

- (Dfn 8.2.39) Fix a right-continuous filtration $\{\mathcal{F}_t\}$. $\mathcal{M}_2 :=$ the vector space of all square integrable $\{\mathcal{F}_t\}$-martingales which have $X_0 = 0$ and right-continuous sample functions.

  $\mathcal{M}_2^c :=$ the linear subspace of $\mathcal{M}_2$ with continuous sample functions.

- (Dfn 8.2.40) An **$\{\mathcal{F}_t\}$-increasing process** is an $\mathcal{F}_t$-adapted, integrable stochastic process $\{A_t\}$ of right-continuous, non-decreasing sample functions starting at $A_0 = 0$.

- (Dfn 8.2.41) q^{th} **variation:** For any finite partition $\pi = \left\{a = s_0^{(\pi)} < \cdots < s_k^{(\pi)} = b\right\}$ of $[a,b]$, let $\|\pi\|$ denote the length of the longest interval in π, and let the q^{th} **variation of f on partition** π be

$$V_{(\pi)}^{(q)}(f) = \sum_{i=1}^{k} \left| f(s_i^{(\pi)}) - f(s_{i-1}^{(\pi)}) \right|^q .$$

  The q^{th} **variation of f on** $[a,b]$ is $V^{(q)}(f) = \lim_{\|\pi\| \to 0} V_{(\pi)}^{(q)}(f)$.

- (Lem 8.2.43) If a martingale of continuous sample functions has finite total variation on each compact interval, then it is indistinguishable from a constant.

- (Lem 8.2.44) Suppose $X \in \mathcal{M}_2^c$. For any partition $\pi = \{0 = s_0 < s_1 < \dots\}$ of $[0,\infty)$ with a finite number of points on each compact interval, $M_t^{(\pi)} = X_t^2 - V_t^{(\pi)}(X)$ is an $\mathcal{F}_t$-martingale of continuous sample paths, where

$$V_t^{(\pi)}(X) := \sum_{i=1}^{k} (X_{s_i} - X_{s_{i-1}})^2 + (X_t - X_{s_k})^2, \qquad \text{for all } t \in [s_k, s_{k+1}).$$

- (Thm 8.2.45) **Special case of Doob-Meyer Decomposition:** For $X \in \mathcal{M}_2^c$, the continuous modification of $V^{(2)}(X)_t$ is the unique $\mathcal{F}_t$-increasing process $A_t = \langle X \rangle_t$ of continuous sample paths, such that $M_t = X_t^2 - A_t$ is an $\mathcal{F}_t$-martingale of continuous sample paths. Any two such decompositions of X_t^2 as the sum of a martingale and increasing process are indistinguishable.

- (Ex 8.2.46, HW8) Suppose $\{X_t\}$ of continuous sample paths has an a.s. finite r^{th} variation for each fixed $t > 0$. Then for each $t > 0$, $V^{(q)}(X)_t \overset{a.s.}{=} 0$ if $q > r$. If $0 < q < r$, $V^{(q)}(X)_t \overset{a.s.}{=} \infty$ a.e. ω such that $V^{(r)}(X)_t > 0$.

- (Ex 8.2.46, HW8) If $X \in \mathcal{M}_2^c$ and $\widetilde{A}_t$ has continuous sample paths and finite total variation on compact intervals, then the quadratic variation of $X_t + \widetilde{A}_t$ is $\langle X \rangle_t$.

- (Ex 8.2.46, HW8) If $\{X_t\}$ is locally γ-Hölder continuous on $[0,T]$ for some $\gamma > 1/2$, then its quadratic variation on this interval is 0.

- (Dfn 8.2.47) An $\mathcal{F}_t$-progressively measurable stochastic process $\{Y_t\}$ is of **class DL** if the collection $\{Y_{u \wedge \theta} : \theta \text{ an } \mathcal{F}_t\text{-stopping time}\}$ is U.I. for each finite, non-random u.

  - (Ex 8.2.50) Every non-negative right-continuous submartingale is of class DL.

- (Thm 8.2.48) **Doob-Meyer Decomposition:** A right-continuous $\mathcal{F}_t$-submartingale $\{Y_t\}$ admits the decomposition $Y_t = M_t + A_t$ with M_t a continuous $\mathcal{F}_t$-martingale and A_t an $\mathcal{F}_t$-increasing process, iff $\{Y_t\}$ is of class DL.

- For uniqueness, we must require A_t to be a natural process. Every continuous increasing process is a natural process, and a natural process is an increasing process.

- (Remark) For each $X \in \mathcal{M}_2$, we can associate with it a unique natural process $\langle X \rangle_t$, called the **predictable quadratic variation**, such that $X_t^2 - \langle X \rangle_t$ is a right-continuous martingale. When $X \notin \mathcal{M}_2^c$, it is no longer the case that the predictable quadratic variation must match the quadratic variation.

- (Eg 8.2.53) **Lévy's martingale characterization of Brownian motion:** Any $X \in \mathcal{M}_2^c$ of quadratic variation $\langle X \rangle_t = t$ must be a standard Brownian Markov process.

- (Eg 8.2.53) For any square integrable stochastic process with $X_0 = 0$ and zero-mean, stationary and independent increments, we have $\langle X \rangle_t = t\mathbb{E}[X_1^2]$.

- (Dfn 8.2.56) For any pair $X, Y \in \mathcal{M}_2$, the **bracket of** X **and** Y is $\langle X, Y \rangle_t := \dfrac{1}{4}\left[\langle X+Y \rangle_t - \langle X-Y \rangle_t\right]$. X and Y are **orthogonal** if for any $t \geq 0$, $\langle X, Y \rangle_t = 0$ a.s.

- (Ex 8.2.57, HW9) **Properties of bracket:**
  - $XY - \langle X, Y \rangle$ is a martingale.
  - $\langle c_1 X_1 + c_2 X_2, Y \rangle = c_1 \langle X_1, Y \rangle + c_2 \langle X_2, Y \rangle$.
  - $\langle X, Y \rangle = \langle Y, X \rangle$.
  - $|\langle X, Y \rangle|^2 \leq \langle X \rangle \langle Y \rangle$.
  - $V(\langle X, Y \rangle)_t - V(\langle X, Y \rangle)_s \leq \dfrac{1}{2}[\langle X \rangle_t - \langle X \rangle_s + \langle Y \rangle_t - \langle Y \rangle_s]$ for all $0 \leq s < t < \infty$.

Section 8.3: Markov and Strong Markov processes

- (Dfn 8.3.1) A collection $\{p_{s,t}(\cdot, \cdot), t \geq s \geq 0\}$ of **transition probabilities** on a measurable space $(\mathbb{S}, \mathcal{S})$ is **consistent** if it satisfies the **Chapman-Kolmogorov equations**

$$p_{t_1,t_3}(x, B) = p_{t_1,t_2} p_{t_2,t_3}(x, B) = \int p_{t_1,t_2}(x, dy) p_{t_2,t_3}(y, B) \qquad \text{for all } x \in \mathbb{S}, B \in \mathcal{S}, t_3 \geq t_2 \geq t_1 \geq 0.$$

  This collection is a **Markov semi-group** if $p_{s,t} = p_{t-s}$ for all $t \geq s \geq 0$.

- (Dfn 8.3.1) An $\mathcal{F}_t$-adapted $\{X_t\}$ taking values in $(\mathbb{S}, \mathcal{S})$ is an $\mathcal{F}_t$**-Markov process** if for any $t \geq s \geq 0$ and $B \in \mathcal{S}$, $\mathbb{P}(X_t \in B \mid \mathcal{F}_s) \overset{a.s.}{=} p_{s,t}(X_s, B)$.

  It is a **homogeneous** $\mathcal{F}_t$**-Markov process of semi-group** $\{p_u\}$ if for any $u, s \geq 0$ and $B \in \mathcal{S}$, $\mathbb{P}(X_{s+u} \in B \mid \mathcal{F}_s) \overset{a.s.}{=} p_u(X_s, B)$.

- Define the map $f \mapsto (p_{s,t} f) : b\mathcal{S} \mapsto b\mathcal{S}$ by $(p_{s,t} f)(x) = \displaystyle\int p_{s,t}(x, dy) f(y)$ for each x. Then for $t \geq s \geq 0$, $\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = (p_{s,t} f)(X_s)$.

- (Thm 8.3.2) Given consistent transition probabilities, you can get a Markov process with those transitions. The law of a Markov process is just its f.d.d. We use $\mathbb{P}_x$ to denote the law of the Markov process given that it starts at non-random x.

- (Ex 8.3.4, HW4) **Closure of Markov processes (useful for showing that a transformation of a Markov process is still Markov:** Let $(X_t, \mathcal{F}_t^X)$ be a Markov process on state space $(\mathbb{S}, \mathcal{S})$. Let $u : [0, \infty) \mapsto [0, \infty)$ be an invertible strictly increasing function. For each t, let $\Phi_t : (\mathbb{S}, \mathcal{S}) \mapsto (\tilde{\mathbb{S}}, \tilde{\mathcal{S}})$ be an invertible measurable mapping such that the inverse is measurable as well.

- If $Y_t = \Phi_t(X_{u(t)})$, then $(Y_t, \mathcal{F}_t^Y)$ is a Markov process on state space $(\tilde{\mathbb{S}}, \tilde{\mathcal{S}})$.
- If X_t is a homogeneous Markov process, then so is $Z_t = \Phi_0(X_t)$.

- (Prop 8.3.5) **Independent increments $\implies$ Markov process:** If a real-valued $\{X_t\}$ has independent increments, then $(X_t, \mathcal{F}_t^X)$ is a Markov process with transition probabilities $p_{s,t}(y, B) = P_{X_t - X_s}(\{z : y + z \in B\})$. If $\{X_t\}$ has stationary independent increments, then this Markov process is homogeneous.

- (Remark) **Generator of a Markov process** is the operator $\mathbf{L} = \lim_{s \downarrow 0} \frac{p_s - p_0}{s}$.

- (Dfn 8.3.7) $(W_t, \mathcal{F}_t)$ is a **Brownian Markov process** if it has continuous sample paths and is a homogeneous $\mathcal{F}_t$-Markov process with Brownian semi-group. If in addition $W_0 = 0$, we call it a **standard Brownian Markov process**.

- (Dfn 8.3.8) A probability measure ν on $(\mathbb{S}, \mathcal{S})$ is an **invariant measure** for a semi-group of transition probabilities $\{p_u, u \geq 0\}$, if the induced law $\mathbb{P}_\nu(\cdot) = \int_{\mathbb{S}} P_x(\cdot)\nu(dx)$ (induced law on the space of trajectories) is invariant under any time shift θ_s, $s \geq 0$.

- (Ex 8.3.9, HW5) A probability measure ν on $(\mathbb{S}, \mathcal{S})$ is an invariant measure for $\{p_u\}$ iff $\nu p_t = \nu$ for any $t \geq 0$.

- (Prop 8.3.11) **Markov Property:** Suppose $\{X_t\}$ is a homogeneous $\mathcal{F}_t$-Markov process on $(\mathbb{S}, \mathcal{S})$. Let P_x denote the family of laws associated with its semi-group. Then, $x \mapsto \mathbb{E}_x[h]$ is measurable on $(\mathbb{S}, \mathcal{S})$ for any $h \in b\mathcal{S}^{[0,\infty)}$, and futher for any $s \geq 0$, almost surely

$$\mathbb{E}[h \circ \theta_s(X_\cdot(\omega)) \mid \mathcal{F}_s] = \mathbb{E}[h \circ (X_{\cdot+s}(\omega)) \mid \mathcal{F}_s] = \mathbb{E}_{X_s}[h].$$

(**Note:** If we take $h(x(\cdot)) = I_B(x(u))$, the above reduces to $\mathbb{P}(X_{s+u} \in B \mid \mathcal{F}_s) = \mathbb{P}_{X_s}(X_u \in B) = p_u(X_s, B)$. In order to get the above, it suffices to check that this relation holds for all u, s and B!)

- (Dfn 8.3.13) **Strong Markov process:** An $\mathcal{F}_t$-progressive measurable, homogeneous Markov process $\{X_t\}$ on $(\mathbb{S}, \mathcal{S})$ has the **strong Markov property** if for any bounded $h(s, x(\cdot))$ measurable on the product σ-algebra $\mathcal{U} = \mathcal{B}_{[0,\infty)} \times \mathcal{S}^{[0,\infty)}$ and any $\mathcal{F}_t$-Markov time τ, we have (almost surely)

$$I_{\{\tau<\infty\}}\mathbb{E}[h(\tau, X_{\tau+\cdot}(\omega)) \mid \mathcal{F}_{\tau^+}] = I_{\{\tau<\infty\}} g_h(\tau, X_\tau),$$

where $g_h(s, x) = \mathbb{E}_x[h(s, \cdot)]$ is bounded and measurable on $\mathcal{B}_{[0,\infty)} \times \mathcal{S}$.

- (Cor 8.3.14) If $(X_t, \mathcal{F}_t)$ is a strong Markov process and τ is an $\mathcal{F}_t$-stopping time, then for any $h \in b\mathcal{U}$, a.s.

$$I_{\{\tau<\infty\}}\mathbb{E}[h(\tau, X_{\tau+\cdot}(\omega)) \mid \mathcal{F}_\tau] = I_{\{\tau<\infty\}} g_h(\tau, X_\tau).$$

In particular, $\{X_t\}$ is a homogeneous $\mathcal{F}_{t^+}$-Markov process and for any $s \geq 0$ and $h \in b\mathcal{S}^{[0,\infty)}$, $\mathbb{E}[h(X_\cdot) \mid \mathcal{F}_{s^+}] = \mathbb{E}[h(X_\cdot) \mid \mathcal{F}_s]$.

- (Prop 8.3.15) **Sufficient condtion for strong Markov property:** An $\mathcal{F}_t$-progressive measurable, homogeneous Markov process $\{X_t\}$ with semi-group $\{p_u\}$ has the strong Markov property if for any $u \geq 0$, $B \in \mathcal{S}$ and bounded $\mathcal{F}_t$-Markov times τ, we have $\mathbb{P}(X_{\tau+u} \in B \mid \mathcal{F}_{\tau^+}) = p_u(X_\tau, B)$.

- (Ex 8.3.17, HW5) If a stopping time takes on only countably many values, then any homogeneous Markov process has the strong Markov property for this stopping time.

- (Dfn 8.3.18) A **Feller semi-group** is a Markov semi-group $\{p_u\}$ on $(\mathbb{R}, \mathcal{B})$ such that $p_t : C_b(\mathbb{R}) \mapsto C_b(\mathbb{R})$ for any $t \geq 0$, i.e. $x \mapsto (p_t f)(x)$ is continuous for any fixed bounded, continuous function f and $t \geq 0$.

- (Prop 8.3.19) **Right-continuous + Feller $\implies$ strong Markov process:** Any right-continuous homogeneous Markov process with a Feller semi-group is a strong Markov process.

- (Ex 8.3.20, HW5) A real-valued process of stationary and independent increments has Feller semi-group. Hence, if it is also right-continuous, then it is a strong Markov process.

- (Eg 8.3.21) Example of a Markov process which is not a strong Markov process.

- (Dfn 8.3.23) A function $x : \mathbb{R}_+ \mapsto \mathbb{S}$ is a **step function** if it is constant on each of the intervals $[s_{k-1}, s_k)$ for some countable (possibly finite) set of isolated points $0 = s_0 < s_1 < \dots$. (Note that a step function is right-continuous.)

  A stochastic process $\{X_t\}$ is a **pure jump process** if its sample functions are step functions.

  A **Markov (pure) jump process** is a homogeneous Markov process which, starting at any non-random $X_0 = x \in \mathbb{S}$, is also a pure jump process. (Sometimes called **continuous-time Markov chains**.)

- (Prop 8.3.24) Any Markov jump process is a strong Markov process.

- (Eg 8.3.25) Example of a Markov jump process with non-Feller semi-group.

- (Prop 8.3.26) Suppose $\{X_t\}$ is a right-continuous, homogeneous Markov process.

  - **Time to jump has exponential distribution:** There is a measurable $\lambda : \mathbb{S} \mapsto [0, \infty]$ such that for all $y \in \mathbb{S}$, under $\mathbb{P}_y$, the $\mathcal{F}_t^X$-Markov time $\tau = \inf\{t \geq 0 : X_t \neq X_0\}$ has the exponential distribution of parameter λ_y. (Note: λ_y can be 0 or ∞.)
  - **When you jump is independent of where you jump to:** If $\lambda_y > 0$, then τ is $\mathbb{P}_y$-a.s. finite and $\mathbb{P}_y$-independent of the $\mathbb{S}$-valued random variable X_τ.
  - If $(X_t, t \geq 0)$ is a strong Markov process and $0 < \lambda_y < \infty$, then P_y-a.s. $X_\tau \neq y$.
  - If $(X_t, t \geq 0)$ is a Markov jump process, then τ is a strictly positive $\mathcal{F}_t^X$-stopping time.

- (Dfn 8.3.27) For a Markov jump process $\{X_t, t \geq 0\}$, $p(x, A)$ and $\{\lambda_x\}$ are the **jump transition probability** and **jump rates** if

$$p(x, A) = \begin{cases} \mathbb{P}_x(X_\tau \in A) & \text{if } A \in \mathcal{S} \text{ and } x \in \mathbb{S} \text{ s.t. } \lambda_x > 0, \\ I\{x \in A\} & \text{if } A \in \mathcal{S} \text{ and } x \in \mathbb{S} \text{ s.t. } \lambda_x = 0. \end{cases}$$

  More generally, a pair (λ, p) with $\lambda : \mathbb{S} \mapsto \mathbb{R}_+$ and $p(\cdot, \cdot)$ transition probability on $(\mathbb{S}, \mathcal{S})$ such that $p(x, \{x\}) = I\{\lambda_x = 0\}$ is called **jump parameters**.

- (Thm 8.3.28) **Canonical construction of Markov jump process:** Suppose (λ, p) are jump parameters on a $\mathcal{B}$-isomorphic space $(\mathbb{S}, \mathcal{S})$. Define

  - $\{Z_n, n \geq 0\}$ a homogeneous Markov chain of transition probability $p(\cdot, \cdot)$ and initial state $Z_0 = x \in \mathbb{S}$.
  - For each $y \in \mathbb{S}$, let $\{\tau_j(y) \geq 1\}$ be i.i.d. random variables independent of $\{Z_n\}$ and each having $\text{Exp}(\lambda_y)$ distribution.
  - $T_0 = 0$, $T_k = \displaystyle\sum_{j=1}^{k} \tau_j(Z_{j-1})$ for $k \geq 1$.
  - $X_t = Z_k$ for all $t \in [T_k, T_{k+1})$, $k \geq 0$.

Assuming $\mathbb{P}_x(T_\infty < \infty) = 0$ for all $x \in \mathbb{S}$ ("non-explosion condition"), then $\{X_t\}$ is the unique Markov jump process with the given jump parameters.

Conversely, (λ, p) are the parameters of a Markov jump process iff $\mathbb{P}_x(T_\infty < \infty) = 0$ for all $x \in \mathbb{S}$.

- (Remark 8.3.29) When the jump rates are all equal, i.e. $\lambda_x = \lambda$, the jump times T_k are those of a Poisson process N_t of rate λ, which is independent of the Markov chain $\{Z_n\}$. Hence, we can write the Markov jump process as $X_t = Z_{N_t}$.

- (Ex 8.3.30, HW9) Suppose (λ, p) are jump parameters on $(\mathbb{S}, \mathcal{S})$.

  - **Rewriting of non-explosion condition:** $\mathbb{P}_x(T_\infty < \infty) = 0$ iff $\mathbb{P}_x\left(\sum_n \lambda_{Z_n}^{-1} < \infty\right) = 0$.
  - **Sufficient condition on jump parameters to get MJP:** If $\lambda \in b\mathcal{S}$ (i.e. bounded), then the jump parameters correspond to a well-defined unique Markov jump process.

- (Eg 8.3.31) **Birth processes:** Birth processes are Markov jump processes which are also counting processes. (Generalization of Poisson processes: jump times can depend on the state.) They have state space $\mathbb{S} = \{0, 1, \dots\}$ and jump transition probability $p(x, x+1) = 1$.

- (Dfn 8.3.32) The linear operator $\mathbf{L} : b\mathcal{S} \mapsto m\mathcal{S}$ such that $(\mathbf{L}h)(x) = \lambda_x \int (h(y) - h(x)) p(x, dy) = \lambda_x \mathbb{E}_x[h(X_\tau) - h(X_0)]$ is the **generator** of the Markov jump process with parameters (λ, p).

  In particular, $(\mathbf{L}I_{\{x\}^c})(x) = \lambda_x$, and for any $B \subseteq \{x\}^c$, $(\mathbf{L}I_B)(x) = \lambda_x p(x, B)$.

- (Ex 8.3.33, HW9) Let $\{X_t\}$ be a Markov jump process of semi-group $p_t(\cdot, \cdot)$ and jump parameters (λ, p).

  - **Kolmogorov backward equation:** $t \mapsto (\mathbf{L}p_t h)(x)$ is continuous and $t \mapsto (p_t h)(x)$ is differentiable for any $x \in \mathbb{S}$, $h \in b\mathcal{S}$, $t \geq 0$. We also have $\partial_t (p_t h)(x) = (\mathbf{L}p_t h)(x)$ for all $t \geq 0, x \in \mathbb{S}, h \in b\mathcal{S}$.
  - **Kolmogorov forward equation:** If $\sup_{s \in \mathbb{S}} \lambda_x$ is finite, then $\mathbf{L} : b\mathcal{S} \mapsto b\mathcal{S}$, and for all $t \geq 0, x \in \mathbb{S}, h \in b\mathcal{S}$, we have $\partial_t (p_t h)(x) = (p_t(\mathbf{L}h))(x)$.
  - A Markov semi-group $p_t(\cdot, \cdot)$ corresponds to a Markov jump process only if for any $x \in \mathbb{S}$, the limit $\lim\limits_{t \downarrow 0} \dfrac{1 - p_t(x, \{x\})}{t} = \lambda_x$ exists, is finite and $\mathcal{S}$-measurable.

- (Dfn 8.3.35) **Compound Poisson process:** A compound Poisson process is a real-valued Markov jump processes with a constant jump rate λ, whose jump transition probability is of the form $p(x, B) = P_\xi(\{z : x + z \in B\})$ for some law P_ξ on $(\mathbb{R}, \mathcal{B})$.

  It is of the form $X_t = S_{N_t}$, $S_n = S_0 + \displaystyle\sum_{k=1}^{n} \xi_k$ is an i.i.d. random walk which is independent of Poisson process N_t with rate λ. (A random walk sampled at Poisson process times.)

- (Prop 8.3.36) A compound Poisson process X_t has stationary, independent increments and the characteristic function of its Markov semi-group $p_t(x, \cdot)$ is $\mathbb{E}_x[e^{i\theta X_t}] = \exp\left[i\theta x + \lambda t(\Phi_\xi(\theta) - 1)\right]$, where $\Phi_\xi(\cdot)$ is the characteristic function for ξ.

- (Prop 8.3.38) Partitioning of compound Poisson processes.

Section 9.1: Brownian transformations, hitting times and maxima

- (Ex 9.1.1, HW5) Let $\{W_t, t \geq 0\}$ be a standard Wiener process. The following are also standard Wiener processes:
  - **Symmetry:** $\widetilde{W}_t^{(1)} = -W_t$.
  - **Time-homogeneity:** $\widetilde{W}_t^{(2)} = W_{T+t} - W_T$, $t \geq 0$ with $T > 0$ a non-random constant.
  - **Time-reversal:** $\widetilde{W}_t^{(3)} = W_T - W_{T-t}$ for $t \in [0, T]$, with $T > 0$ a non-random constant. Morever, $\widetilde{W}_t^{(2)}$ and $\widetilde{W}_t^{(3)}$ are independent.
  - **Scaling:** $\widetilde{W}_t^{(4)} = \alpha^{-1/2} W_{\alpha t}$, where $\alpha > 0$ is a non-random cosntant.
  - **Time-inversion:** $\widetilde{W}_t^{(5)} = tW_{1/t}$ for $t > 0$ and $\widetilde{W}_0^{(5)} = 0$.
  - **Averaging:** $\widetilde{W}_t^{(6)} = \displaystyle\sum_{k=1}^{n} c_k W_t^{(k)}$, where $W_t^{(k)}$ are independent copies of the standard Wiener process and $\sum c_k^2 = 1$.
- (Remark after Ex 9.1.1) If $L_{a,b} = \sup\{t \geq 0 : W_t \notin (-at, bt)\}$, by time inversion we can show that $L_{a,b}$ is a.s. finite and $\mathbb{P}(W_{L_{a,b}} = bL_{a,b}) = a/(a+b)$.
- (Cor 9.1.2) If $(W_t, \mathcal{F}_t)$ is a Brownian Markov process, then it is a homogeneous $\mathcal{F}_{t^+}$-Markov process, and for any $s \geq 0$ and bounded Borel-measurable functional $h : ([0, \infty)) \mapsto \mathbb{R}$, a.s. $\mathbb{E}[h(W_\cdot) \mid \mathcal{F}_{s^+}] = \mathbb{E}[h(W_\cdot) \mid \mathcal{F}_s]$.
- (Prop 9.1.4) **Blumenthal's 0-1 Law:** Let P_x denote the law of the Wiener process starting at $W_0 = x$. Then $P_x(A) \in \{0, 1\}$ for each $A \in \mathcal{F}_{0^+}^W$ and $x \in \mathbb{R}$. If A is in the tail algebra for $\{W_t\}$, then either $P_x(A) = 0$ for all x or $P_x(A) = 1$ for all x.
- (Cor 9.1.5) Let $\tau_{0^+} = \inf\{t \geq 0 : W_t > 0\}$, $\tau_{0^-} = \inf\{t \geq 0 : W_t < 0\}$ and $T_0 = \inf\{t \geq 0 : W_t = 0\}$. Then $P_0(\tau_{0^+} = 0) = P_0(\tau_{0^-} = 0) = P_0(T_0 = 0) = 1$, and w.p. 1 the standard Wiener process changes sign infinitely many times in any time interval $[0, \varepsilon]$, $\varepsilon > 0$.
- (Cor 9.1.5) For any $x \in \mathbb{R}$, with P_x-probability 1,
$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{t}} = \infty, \quad \liminf_{t\to\infty} \frac{W_t}{\sqrt{t}} = -\infty, \quad W_{u_n} = 0 \text{ for some } u_n(\omega) \nearrow \infty.$$
- (Cor 9.1.6) **Regeneration:** If $(W_t, \mathcal{F}_t)$ is a Brownian Markov process and τ is an a.s. finite $\mathcal{F}_t$-Markov time, then $\{W_{\tau+t} - W_\tau\}$ is a standard Wiener process independent of $\mathcal{F}_{\tau^+}$.
- (Ex 9.1.8, HW6) Let $\tau_b = \inf\{t \geq 0 : W_t \geq b\}$ and $\tau_{b^+} = \inf\{t \geq 0 : W_t > b\}$. Then $P_0(\tau_b \neq \tau_{b^+}) = 0$.
- (Prop 9.1.10) **Reflection principle:** Let $M_t = \sup_{s\in[0,t]} W_s$, $T_b = \inf\{t \geq 0 : W_t = b\}$. Then for any $t, b > 0$,
$$P(M_t \geq b) = P(\tau_b \leq t) = P(T_b \leq t) = 2P(W_t \geq b).$$
Further,
$$f_{T_b}(t) = \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t} = \frac{b}{\sqrt{t^3}} \phi\left(\frac{b}{\sqrt{t}}\right), \qquad f_{M_t}(b) = \frac{2}{\sqrt{2\pi t}} e^{-b^2/2t}.$$
- (Ex 9.1.12) For any $u > 0$ and $a_1 < a_2 \leq b$, $\mathbb{P}(T_b < u, a_1 < W_u < a_2) = \mathbb{P}(2b - a_2 < W_u < 2b - a_1)$.
- (Ex 9.1.12) **Joint density of** (M_t, W_t) is given by $f_{W_t,M_t}(a, b) = \dfrac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left[-\dfrac{(2b-a)^2}{2t}\right]$.

- $M_t = \max_{0 \le s \le t} W_s$ has the same distribution as $\sqrt{t}|Z|$.

- The processes $\{M_t - W_t\}$ and $\{|W_t|\}$ have the same distributions.

- (Ex 9.1.15, HW6) **Brownian motion absorbed at zero** $\{W_{t\wedge T_0}, \mathcal{F}_t\}$ is a homogeneous Markov process whose transition probabilities are

$$p_{-,t}(x,B) = \begin{cases} 1 & \text{if } x = 0, B = \{0\}, \\ p_t(x,B) - p_t(x,-B) & \text{if } x > 0, B \subseteq (0,\infty), \\ 2p_t(x,(-\infty,0]) & \text{if } x > 0, B = \{0\}. \end{cases}$$

- (Ex 9.1.15, HW6) **Reflected Brownian motion** $\{|W_t|, \mathcal{F}_t\}$ is a homogeneous Markov process whose transition probabilities are $p_{+,t}(x,B) = p_t(x,B) + p_t(x,-B)$ for $x \ge 0$ and $B \subseteq [0,\infty)$.

- (Ex 9.1.16, HW6) $Y_t = M_t - W_t$ is an $\mathcal{F}_t$-Markov process with the same transition probabilities (and hence the same law) as reflected Brownian motion.

Section 9.2: Weak convergence and invariance principles

- (Dfn 9.2.1) **Convergence in distribution:** $\{X_n(t), t \ge 0\}$ of continuous sample functions **converge in distribution** to a stochastic process $\{X_\infty(t), t \ge 0\}$ if the corresponding laws converge weakly in the topological space $C([0,\infty))$ with the topology of uniform convergence on compact subsets of $[0,\infty)$.

  That is, if $g(X_n(\cdot)) \xrightarrow{d} g(X_\infty(\cdot))$ whenever $g : C([0,\infty)) \mapsto \mathbb{R}$ is Borel-measurable and is such that w.p. 1, the sample function of $X_\infty(\cdot)$ is not in the set D_g of points of discontinuity of g ($D_g = \{X_\infty(\cdot) \in C([0,\infty)), \exists\, X_n(\cdot) \to X_\infty(\cdot) \text{ and } g(X_n) \not\to g(X_\infty)\}$).

- (Thm 9.2.2) **Donsker's Invariance Principle:** If $\{\xi_k\}$ are i.i.d. with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 = 1$, then for $S(\cdot)$ where $S(t) = \sum_{k=1}^{\lfloor t \rfloor} \xi_k + (t - \lfloor t \rfloor)\xi_{\lfloor t \rfloor + 1}$, the stochastic processes $\widehat{S}_n(\cdot) = n^{-1/2}S(n\cdot)$ converges in distribution to the standard Wiener process.

- (Prop 9.2.4) If the laws of $\{X_n(\cdot)\}$ of continuous sample functions are uniformly tight in $C([0,\infty))$ and the f.d.d. of $\{X_n(\cdot)\}$ converge weakly to the f.d.d. of $\{X_\infty(\cdot)\}$, then $X_n(\cdot) \xrightarrow{d} X_\infty(\cdot)$.

- (Thm 9.2.5) Arzelà-Ascoli Theorem.

- (Eg 9.2.9, Ex 9.2.10, Eg 9.2.11) Examples of applying Donsker's invariance principle for different functionals.

- (Cor 9.2.13) **Glivenko-Cantelli:** Suppose $\{X_k, X\}$ i.i.d. and $x \mapsto F_X(x)$ is continuous. Then if $D_n = \sup_{x\in\mathbb{R}} |F_X(x) - F_n(x)|$, as $n \to \infty$ we have $n^{1/2}D_n \xrightarrow{d} \sup_{t\in[0,1]} |\widehat{B}_t|$, where $\widehat{B}_t$ is the standard Brownian bridge.

- (Lem 9.2.16) Let $\{W(t)\}$ be a standard Wiener process and let $k \mapsto T_{n,k}$ be non-decreasing, such that $T_{n,\lfloor nt \rfloor} \xrightarrow{P} t$ as $n \to \infty$, for each fixed $t \in [0,\ell]$. (Think of the $T_{n,k}$'s as sampling times.)

  Then for the norm $\|x(\cdot)\| = \sup_{t\in[0,\ell]} |x(t)|$ and $\widehat{S}_n(t) = S_n(nt)$, where $S_n(t) = W(T_{n,\lfloor t \rfloor}) + (t - \lfloor t \rfloor)[W(T_{n,\lfloor t \rfloor + 1}) - W(T_{n,\lfloor t \rfloor})]$, we have $\|\hat{S}_n - W\| \xrightarrow{P} 0$. In fact, $\hat{S}_n(\cdot) \xrightarrow{d} W(\cdot)$ in $C([0,\infty))$.

- (Thm 9.2.19) **Skorohod's Representation:** Suppose $(W_t, \mathcal{F}_t)$ is a Brownian Markov process such that $W_0 = 0$. Given the law P_X of an integrable X such that $\mathbb{E}X = 0$, there exists an a.s. finite $\mathcal{F}_t$-stopping time τ such that $W_\tau \overset{d}{=} X$, $\mathbb{E}\tau = \mathbb{E}[X^2]$ and $\mathbb{E}[\tau^2] \le 2\mathbb{E}[X^4]$.

- (Thm 9.2.20) **Strassen's Martingale Representation:** Suppose the probability space contains a discrete-time martingale $\{M_\ell, \mathcal{F}_\ell\}$ such that $M_0 = 0$, as well as a standard Wiener process $\{W(t)\}$, independent of $\mathcal{F}_\infty$ and $\{M_\ell\}$. Then

  - The filtrations $\mathcal{F}_{k,t} = \sigma(\mathcal{F}_k, \mathcal{F}_t^W)$ are such that $(W(t), \mathcal{F}_{k,t})$ is a Brownian Markov process for any $1 \le k \le \infty$.
  - There exist non-decreasing a.s. finite $\mathcal{F}_{k,t}$-stopping times $\{T_k\}$, starting with $T_0 = 0$, where $\tau_k = T_k - T_{k-1}$ and the filtration $\mathcal{H}_k = \mathcal{F}_{k,T_k}$ are such that w.p. 1, $\mathbb{E}[\tau_k \mid \mathcal{H}_{k-1}] = \mathbb{E}[D_k^2 \mid \mathcal{F}_{k-1}]$ and $\mathbb{E}[\tau_k^2 \mid \mathcal{H}_{k-1}] \le 2\mathbb{E}[D_k^4 \mid \mathcal{F}_{k-1}]$, where $D_k = M_k - M_{k-1}$ are the martingale differences, for all $k \ge 1$.
  - The discrete time process $\{W(T_\ell)\}$ has the same f.d.d. as $\{M_\ell\}$.

- (Cor 9.2.21) **Skorohod's Representation for random walks:** Suppose ξ_1 integrable and of zero mean. The random walk $S_n = \sum_{k=1}^n \xi_k$ for i.i.d. $\{\xi_k\}$ can be represented as $S_n = W(T_n)$ for $T_0 = 0$, i.i.d. $\tau_k = T_k - T_{k-1} \ge 0$ such that $\mathbb{E}[\tau_1] = \mathbb{E}[\xi_1^2]$, and standard Wiener process W. Also, each T_k is a stopping time for $\mathcal{F}_t^W$.

- (Thm 9.2.22) **Lindeberg's Martingale CLT:** Suppose that for any $n \ge 1$ fixed, $(M_{n,\ell}, \mathcal{F}_{n,\ell})$ is a discrete-time L^2 martingale starting at 0, with martingale differences $D_{n,k} = M_{n,k} - M_{n,k-1}$ and predictable compensators $\langle M_n \rangle_\ell = \sum_{k=1}^{\ell} \mathbb{E}[D_{n,k}^2 \mid \mathcal{F}_{n,k-1}]$. If

  (a) For any fixed $t \in [0,1]$, $\langle M_n \rangle_{\lfloor nt \rfloor} \xrightarrow{P} t$ as $n \to \infty$, and

  (b) For each $\varepsilon > 0$, $g_n(\varepsilon) = \sum_{k=1}^{n} \mathbb{E}[D_{n,k}^2 I\{|D_{n,k}| \ge \varepsilon\} \mid \mathcal{F}_{n,k-1}] \xrightarrow{P} 0$ as $n \to \infty$,

  then as $n \to \infty$, the linearly interpolated, time-scaled $\widehat{S}_n(t) = M_{n,\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) D_{n,\lfloor nt \rfloor + 1}$ converges in distribution on $C([0,1])$ to the standard Wiener process $\{W_t, t \in [0,1]\}$.

- (Cor 9.2.23) **CLT for a single martingale:** Suppose $(M_\ell, \mathcal{F}_\ell)$ is an L^2 martingale starting at 0. If

  (a) $\langle M \rangle_n / n \xrightarrow{P} 1$ as $n \to \infty$, and

  (b) For each $\varepsilon > 0$, $\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[(M_k - M_{k-1})^2; |M_k - M_{k-1}| \ge \varepsilon\sqrt{n}] \to 0$ as $n \to \infty$,

  then as $n \to \infty$, the linearly interpolated, time-scaled $\widehat{S}_n(t) = n^{-1/2}[M_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)(M_{\lfloor nt \rfloor + 1} - M_{\lfloor nt \rfloor})]$ converges in distribution on $C([0,1])$ to the standard Wiener process $\{W_t, t \in [0,1]\}$.

- (Thm 9.2.27) **Kinchin's LIL:** Set $h(t) = \sqrt{2t \log\log(1/t)}$ for $t < 1/e$ and $\tilde{h}(t) = t h(1/t)$. Then, for standard Wiener processes W_t and $\widetilde{W}_t$, we have

$$\limsup_{t \downarrow 0} \frac{W_t}{h(t)} = 1, \qquad \liminf_{t \downarrow 0} \frac{W_t}{h(t)} = -1,$$

$$\limsup_{t \to \infty} \frac{\widetilde{W}_t}{\tilde{h}(t)} = 1, \qquad \liminf_{t \to \infty} \frac{\widetilde{W}_t}{\tilde{h}(t)} = -1.$$

- (Prop 9.2.9) **Hartman-Wintner's LIL for random walks:** Suppose $S_n = \sum_{k=1}^{n} \xi_k$, where ξ_k's are i.i.d. with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 = 1$. Then w.p. 1, $\limsup\limits_{n\to\infty} \dfrac{S_n}{\sqrt{2n\log\log n}} = 1$.

Section 9.3: Brownian path: regularity, local maxima and level sets

- (Dfn 9.3.1) For a continuous $f : [0,\infty) \mapsto \mathbb{R}$ and $\gamma \in (0,1]$, the **upper and lower (right) γ-derivatives** at $s \geq 0$ are the $\overline{\mathbb{R}}$-valued $D^\gamma f(s) = \limsup\limits_{u\downarrow 0} u^{-\gamma}[f(s+u)-f(s)]$ and $D_\gamma f(s) = \liminf\limits_{u\downarrow 0} u^{-\gamma}[f(s+u)-f(s)]$. (These always exist.)

  The **Dini derivatives** correspond to the case of $\gamma = 1$. A continuous function is **differentiable from the right** at s if $D^1 f(s) = D_1 f(s)$ is finite.

- (Prop 9.3.2) **Nowhere differentiability of BM:** With probability 1, the sample function of a Wiener process is nowhere differentiable. More precisely, for $\gamma = 1$ and any $T \leq \infty$,

$$\mathbb{P}(\{\omega : -\infty < D_\gamma W_t(\omega) \leq D^\gamma W_t(\omega) < \infty \text{ for some } t \in [0,T]\}) = 0.$$

  (Ex 9.3.3, HW8) The above actually holds for any $\gamma > 1/2$.

- (Thm 9.3.5) **Lévy's modulus of continuity:** For $\delta \in (0,1]$, set $g(\delta) = \sqrt{2\delta\log(1/\delta)}$. For a Wiener process $\{W_t, t \in [0,T]\}$ for $0 < T < \infty$, almost surely $\limsup\limits_{\delta\downarrow 0} \dfrac{osc_{T,\delta}(W.)}{g(\delta)} = 1$, where $osc_{t,\delta}(W.) = \sup\limits_{0\leq h\leq\delta} \sup\limits_{0\leq s\leq s+h\leq t} |W_{s+h} - W_s|$.

- (Dfn) For non-random $b \in \mathbb{R}$, the **level set** of the standard Wiener process is defined by $\mathcal{Z}_\omega(b) = \{t \geq 0 : W_t(\omega) = b\}$. The **zero set** is $\mathcal{Z}_\omega = \mathcal{Z}_\omega(0)$.

- (Prop 9.3.7, Cor 9.3.8) Fix $b \in \mathbb{R}$. For almost every $\omega \in \Omega$, the level set $\mathcal{Z}_\omega(b)$ of the standard Wiener process is closed, unbounded, of zero Lebesgue measure and having no isolated points.

- (Remark 9.3.9) For almost every ω, the sample path of the Wiener process is monotone in no interval.

- (Dfn 9.3.11) Suppose $f : [0,\infty) \mapsto \mathbb{R}$. $t \geq 0$ is a **point of local maximum** if there is a neighborhood such that $f(t) \geq f(s)$ for s in that neighborhood. It is a **strict point of local maximum** if $f(t) > f(s)$ for any point in a neighborhood. It is a **point of increase** if there exists $\delta > 0$ s.t. $f((t-h)_+) \leq f(t) \leq f(t+h)$ for all $h \in (0,\delta]$.

- (Prop 9.3.13) For almost every $\omega \in \Omega$, the set of points of local maximum for the Wiener sample path is a countable dense subset of $[0,\infty)$, and all local maxima are strict.

- (Thm 9.3.14) Almost every sample path of the Wiener process has no point of increase (or decrease).

Other Stuff

Brownian Motion Facts

- (Dfn 7.3.12) $\{W_t, t \geq 0\}$ is called a **Brownian motion/Wiener process** starting at $x \in \mathbb{R}$ if it is a Gaussian process with mean function $m(t) = x$, auto-covariance $\mathrm{Cov}(W_t, W_s) = t \wedge s$, and its sample functions are continuous.

  If $x = 0$, it is called **standard Brownian motion**.

- (Ex 7.3.13c, HW2) For any finite T, $\{W_t, t \in [0,T]\}$ can be viewed as the random variable $W_\cdot : (\Omega, \mathcal{F}) \mapsto (C([0,T]), \|\cdot\|_\infty)$, which is measurable w.r.t. the Borel σ-algebra on $C([0,T])$, and is a.s. locally γ-Hölder continuous for any $\gamma < 1/2$.

  Kinchin's LIL tells us that Brown motion sample paths are not γ-Hölder continuous for any $\gamma \geq 1/2$.
- (Ex 7.3.13d, HW2) Brownian motion is non-stationary but has stationary, independent increments.
- (Ex 7.3.16a, HW2) For $s < t$, $W_s \mid W_t \sim \mathcal{N}\left(\frac{s}{t}W_t, \frac{s(t-s)}{t}\right)$. (Proof: Look at joint density function of W_s and W_t.)
- (Ex 7.3.16b, HW2) $\lim_{t\to\infty} \frac{W_t}{t} = 0$ (a.s. limit).
- (Ex 8.2.35d, HW4) $\limsup_{t\to\infty} W_t = \infty$ w.p. 1 and $\liminf_{t\to-\infty} W_t = -\infty$ w.p. 1.
- (Ex 8.2.7, HW3) Important collection of martingales based on Brownian motion and $u_0(t,y,\theta) = \exp(\theta y - \theta^2 t/2)$.
  - For any $\theta \in \mathbb{R}$, $\{u_0(t, B_t, \theta)\} = \{\exp(\theta B_t - \theta^2 t/2)\}$ is a martingale.
  - Let $u_{k+1}(t,y,\theta) = \frac{\partial}{\partial\theta} u_k(t,y,\theta)$. For any $k \geq 0$, $u_k(t, B_t, \theta)$ is a martingale.
  - Evaluating at $\theta = 0$, we get that $B_t^2 - t$, $B_t^3 - 3tB_t$, $B_t^4 - 6tB_t^2 + 3t^2$ and $B_t^6 - 15tB_t^4 + 45t^2B_t^2 - 15t^3$ are all martingales.
- $(x + B_t)^2 - (t + x^2)$ is a martingale.
- (Ex 8.2.36, HW4) **Exit times:** For $a, b > 0$, let $\tau_{a,b} = \inf\{t \geq 0 : W_t \notin (-a, b)\}$.
  - $\tau_{a,b}$ is an a.s. finite $\mathcal{F}_t^W$-stopping time, and $\mathbb{P}\left(W_{\tau_{a,b}} = -a\right) = \frac{b}{a+b}$.
  - For all $s \geq 0$, $\mathbb{E}[e^{-s\tau_{a,b}}] = \frac{\sinh(a\sqrt{2s}) + \sinh(b\sqrt{2s})}{\sinh[(a+b)\sqrt{2s}]}$.
  - $\mathbb{E}\tau_{a,b} = ab$ and Var $\tau_{a,b} = \frac{ab}{3}(a^2 + b^2)$.
- (Eg 8.2.51) **Quadratic variation:** A quadratic variation for the standard Brownian motion is $\langle W \rangle_t = t$. By Ex 8.2.46, this implies that the total variation of the Brownian sample path is a.s. infinite on any interval of positive length.
- (Eg 8.3.6) Brownian motion is a homogeneous Markov process with semi-group $p_t(x, B) = \int_B \frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}} dy$.
- (Ex 8.3.20, HW5) Any Brownian Markov process is a strong Markov process.
- **Wald's Lemma for Brownian motion:** If $\{W_t\}$ is a standard Brownian motion and τ is a Markov time s.t. $\mathbb{E}\tau < \infty$, then $\mathbb{E}[B_\tau] = 0$ and $\mathbb{E}[B_\tau^2] = \mathbb{E}\tau$. If $\{B_{t\wedge\tau}\}$ is dominated by an integrable random variable, then we still have $\mathbb{E}[B_\tau] = 0$.
- (Karatzas & Shreve Prop 2.8.15, p100) Define the **last exit time** $\theta_t = \sup\{0 \leq s \leq t : W_s = M_t\}$. Then for $a \in \mathbb{R}$, $b \geq a^+$, $0 < s < t$, we have

$$\mathbb{P}(W_t \in da, M_t \in db, \theta_t \in ds) = \frac{b(b-a)}{\pi\sqrt{s^3(t-s)^3}} \exp\left[-\frac{b^2}{2s} - \frac{(b-a)^2}{2(t-s)}\right] da\, db\, ds.$$

- (Mörters & Peres Thm 2.37) Let $W_1(t)$ and $W_2(t)$ be independent Brownian motions. Let $T_a = \inf\{t : W_1(t) = a\}$. Then $W_2(T_a)$ has density $\frac{a}{\pi(a^2 + x^2)}$.

k-dimensional Brownian Motion Facts

- (Ex 8.2.38, HW4) Let $R_t = \|\underline{W}(t)\|_2$ be the Euclidean distance of the process from the origin. Let $\theta_b = \inf\{t \geq 0 : R_t \geq b\}$ be the first hitting time of a sphere of radius $b > 0$ centered at the origin. Then $M_t = R_t^2 - kt$ is an $\mathcal{F}_t^W$-martingale of continuous sample functions, and θ_b is an a.s. finite $\mathcal{F}_t^W$ stopping time with $\mathbb{E}[\theta_b] = b^2/k$.

Brownian Motion with Drift Facts

- We write it as $Z_t^{(r,\sigma)} = \sigma W_t + rt + x$, with non-random drift $r \in \mathbb{R}$ and diffusion coefficient $\sigma > 0$.
- (Ex 8.3.10, HW5) $Z_t^{(r,\sigma)}$ is a homogeneous Markov process.
- Since $\exp\left[\theta W_t - \dfrac{\theta^2 t}{2}\right]$ is a martingale for all θ, so is $\exp\left[\theta(W_t + rt) - \dfrac{(\theta^2 + 2r\theta)t}{2}\right]$.
- **Hitting times:** Let $Z_t^{(r)} = W_t + rt$. For $b > 0$, let $\tau_b^{(r)} = \inf\{t \geq 0 : Z_t^{(r)} \geq b\}$.
  - (Ex 8.2.35, HW4) $\tau_b^{(r)}$ is an $\mathcal{F}_t^W$-stopping time.
  - (Karatzas & Shreve p197) $\tau_b^{(r)}$ has density
  $$\mathbb{P}\left(\tau_b^{(r)} \in dt\right) = \frac{b}{\sqrt{2\pi t^3}} \exp\left[-\frac{(b+rt)^2}{2t}\right] dt.$$
  - (Karatzas & Shreve p197) $\mathbb{P}\left(\tau_b^{(r)} < \infty\right) = \exp[-rb - |rb|]$.

  Case 1: $r > 0$**.**
  - $\mathbb{P}\left(\tau_b^{(r)} < \infty\right) = 1$.
  - $\mathbb{E}[\tau_b^{(r)}] < \infty$.

  Case 2: $r = 0$**.**
  - (Ex 8.2.35, HW4) $\mathbb{E}\left[\exp\left(-s\tau_b^{(0)}\right)\right] = \exp\left[-b\sqrt{2s}\right]$.
  - (Ex 8.2.35, HW4) $\mathbb{P}\left(\tau_b^{(0)} < \infty\right) = 1$.
  - (Prob Qual 2011-6, Session 11)$\mathbb{E}[\tau_b^{(0)}] = \infty$.

  Case 3: $r < 0$**.**
  - (Ex 8.2.35, HW4) $\mathbb{E}\left[\exp\left(-s\tau_b^{(r)}\right)\right] = \exp\left[-b(\sqrt{r^2+2s} - r)\right]$.
  - (Ex 8.2.35, HW4) $\mathbb{P}\left(\tau_b^{(r)} < \infty\right) = e^{2rb} < 1$. Hence, $\mathbb{E}[\tau_b^{(r)}] = \infty$.
- (Ex 8.2.36, HW4) **Exit times from** $(-a, b)$**:** Now consider $Z_t^{(r)}$ for all $r \in \mathbb{R}$. For $a, b > 0$, let $\tau_{a,b}^{(r)} = \inf\{t \geq 0 : Z_t^{(r)} \notin (-a, b)\}$. Then $\tau_{a,b}^{(r)}$ is an a.s. finite $\mathcal{F}_t^W$-stopping time, and for $r \neq 0$,
  $$\mathbb{P}\left(Z^{(r)}_{\tau_{a,b}^{(r)}}) = -a\right) = 1 - \mathbb{P}\left(Z^{(r)}_{\tau_{a,b}^{(r)}}) = b\right) = \frac{1 - \exp(-2rb)}{\exp(2ra) - \exp(-2rb)}.$$
  When $r = 0$, we have $\mathbb{P}\left(W_{\tau_{a,b}^{(0)}} = -a\right) = \dfrac{b}{a+b}$.

- (Eg 8.3.6) $\{Z_t^{(r)}\}$ is a homogeneous Markov process with semi-group $p_t(x+rt,B)$, where $p_t(x,B) = \int_B \frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}}dy$.

Brownian Bridge Facts

- The standard Brownian bridge is a Gaussian process with mean function 0 and auto-covariance function $c(s,t) = s \wedge t - st$.
- (Ex 7.3.15a) We can write the standard Brownian bridge as $\widehat{B}_t = W_t - \min(t,1)W_1$.
- (Ex 7.3.16c,d, HW2) We can also write it as $\widetilde{B}_t = (1-t)W_{t/(1-t)}$ with $\widetilde{B}_0 = 0$, or $W_t \mid W_1 = 0$, $t \in [0,1]$.
- (Ex 8.3.10, HW5) Brownian bridge is a Markov process with stationary increments, but is not a homogeneous Markov process.
- **Useful transformation for stopping times:** By time inversion, we have $\left\{\sup\limits_{t\in[0,1]}\{W_t - tW_1\} \geq b\right\} = \left\{\sup\limits_{s\geq 1}\{\widetilde{W}_s - \widetilde{W}_1 - sb\} \geq 0\right\}$, where $\widetilde{W}_t = tW_{1/t}$.
- (Ex 9.2.14, HW7) $\mathbb{P}\left(\sup\limits_{t\in[0,1]}\widehat{B}_t \geq b\right) = \exp(-2b^2)$.

  For any non-random $a,c > 0$, $\mathbb{P}\left(\inf\limits_{t\in[0,1]}\widehat{B}_t \leq -a \text{ or } \sup\limits_{t\in[0,1]}\widehat{B}_t \geq c\right) = \sum\limits_{n\geq 1}(-1)^{n-1}(p_n + r_n)$, where $p_{2n} = r_{2n} = \exp[-2(na+nc)^2]$, $r_{2n+1} = \exp[-2(na+nc+c)^2]$ and $p_{2n+1} = \exp[-2(na+nc+a)^2]$.
- (Ex 9.2.14, HW7) Let $F_{KS}(\cdot)$ be the distribution function for $\sup\limits_{t\in[0,1]}|\widehat{B}_t|$. Then $F_{KS}(b) = 1 - 2\sum\limits_{n=1}^{\infty}(-1)^{n-1}e^{-2n^2b^2}$.
- (Lemma used in HW7 Ex 9.2.14) For $b > 0$, let $P_{b,\varepsilon} = \mathbb{P}(W_t \geq b \text{ for some } t \in [0,1] \mid |W_1| < \varepsilon)$. Then $P_{b,\varepsilon} = \frac{\mathbb{P}(|W_1 - 2b| < \varepsilon)}{\mathbb{P}(|W_1| < \varepsilon)}$, and $\lim\limits_{\varepsilon\downarrow 0} P_{b,\varepsilon} = \exp(-2b^2)$.

Fractional Brownian Motion Facts

- (Ex 7.3.17, HW2) For $H \in (0,1)$, fractional Brownian motion of Hurst parameter H is the centered Gaussian stochastic process with auto-covariance function $c(s,t) = \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right]$.
- (Ex 7.3.17b) Fractional Brownian motion exists and has a continuous modification that is locally γ-Hölder continuous for any $0 < \gamma < H$.
- (Ex 7.3.17c) When $H = 1/2$, fBM is the standard Wiener process.
- (Ex 7.3.17d) For any non-random $b > 0$, $\{b^{-H}X_{bt}, t \geq 0\}$ is also an fBM with Hurst parameter H.
- (Ex 7.3.17e) The increments of fBM are stationary for all values of H. The increments of fBM are independent only when $H = 1/2$.

Geometric Brownian Motion Facts

- (Ex 8.3.10) This is defined by $Y_t = e^{W_t}$.
- (Ex 8.3.10, HW5) Geometric Brownian motion is a homogeneous Markov process whose increments are neither independent nor stationary.

Ornstein-Uhlenbeck Process Facts

- (Ex 7.3.15b) The Ornstein-Uhlenbeck process is given by $U_t = e^{-t/2} W_{e^t}$.
- The OU process is a stationary process.
- (Ex 8.3.10, HW5) The OU process is a homogeneous Markov process.
- (Qual 2010 Qn5) The OU is a homogeneous, zero-mean, Gaussian Markov process such that $\mathbb{E}[U_t \mid U_0] = e^{-t/2} U_0$ and $\text{Var}\,(U_t \mid U_0) = 1 - e^{-t}$.
- (Qual 2010 Qn5) $\mathbb{E}[U_t] = 0$, $\text{Cov}(U_t, U_s) = \exp\left(-\frac{|s-t|}{2}\right)$.
- (Qual 2010 Qn5) The transition kernel is $p_t(x,y) = \frac{1}{\sqrt{2\pi(1-e^{-t})}} \exp\left[-\frac{1}{2(1-e^{-t})}(y - e^{-t/2}x)^2\right]$.

Poisson process

- (Dfn 3.4.8) The Poisson process of rate $\lambda > 0$, denoted N_t, is the unique counting process with gaps between jump times being i.i.d. $\text{Exp}(\lambda)$ variables.
- The k^{th} arrival time T_k has distribution $\text{Gam}(k, \lambda)$ (shape-rate).
- As $t \to \infty$, $N_t/t \overset{a.s.}{\to} \lambda$.
- (Prop 3.4.9) The Poisson process has independent increments, and $N_t - N_s \sim \text{Pois}(\lambda(t-s))$.
- (Eg 8.2.5) The **compensated Poisson process** $M_t = N_t - \lambda t$ is a martingale.
- (Eg 8.2.53) $M_t^2 - \lambda t$ is a right-continuous martingale, so $\langle M \rangle_t = \lambda t$.
- (Eg 8.3.6) The **Poisson process with drift** is $N_t^{(r)} = N_t + rt + x$. This is a homogeneous Markov process with semi-group $q_t(x + rt, B)$, where $q_t(x, B) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} I_B(x+k)$.
- (Ex 8.3.20, HW5) The Poisson process is a strong Markov process.