STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 12: February 16

Lecturer: Sourav Chatterjee *Scribes: Kenneth Tay*

12.1 Ergodic Theorems

Theorem 12.1 (Maximal Ergodic Theorem) *Let $(\Omega, \mathcal{F}, P)$ be a probability space, φ a measure-preserving transform with invariant σ-algebra $\mathcal{I}$, and X a random variable in L^1.*

Let $X_j(\omega) := X(\varphi^j(\omega))$, $S_k(\omega) = X_0(\omega)+X_1(\omega)+\cdots+X_k(\omega)$, and $M_k(\omega) = \max\{0, S_0(\omega), S_1(\omega), \ldots, S_k(\omega)\}$.

Then $\mathbb{E}[X; M_k > 0] \geq 0$.

Proof: (Due to Garsia.)

Note that $X_0(\omega) = X(\omega)$. If $j \leq k$, then $M_k(\varphi(\omega)) \geq S_j(\varphi(\omega))$, which implies that

$$X(\omega) + M_k(\varphi(\omega)) \geq X(\omega) + S_j(\varphi(\omega)) = S_{j+1}(\omega).$$

Thus, $X(\omega) \geq S_{j+1}(\omega) - M_k(\varphi(\omega)$ for $j = 0, 1, \ldots, k$. Also, $X(\omega) \geq S_0(\omega) - M_k(\phi(\omega))$ since $X = S_0$ and M_k is non-negative. Thus,

$$\begin{aligned}
\mathbb{E}[X; M_k > 0] &\geq \int_{\{\omega: M_k(\omega)>0\}} \Big(\max\{S_0(\omega), \ldots, S_k(\omega)\} - M_k(\varphi(\omega)) \Big) dP(\omega) \\
&= \int_{\{\omega: M_k(\omega)>0\}} \Big(M_k(\omega) - M_k(\varphi(\omega)) \Big) dP(\omega).
\end{aligned}$$

Since φ is measure-preserving, $\displaystyle\int Y dP = \int Y \circ \varphi dP$ for all Y. Applying this to M_k, $\displaystyle\int_\Omega (M_k(\omega) - M_k(\varphi(\omega)))\, dP(\omega) =$ 0. But, by M_k's non-negativity, on the set $\{M_k \leq 0\}$, $M_k(\omega) - M_k(\varphi(\omega))$ is ≤ 0. Thus,

$$\mathbb{E}[X; M_k > 0] \geq \int_{\{\omega: M_k(\omega)>0\}} \Big(M_k(\omega) - M_k(\varphi(\omega)) \Big) dP(\omega) \geq 0.$$

■

Theorem 12.2 (Birkhoff's Ergodic Theorem) *Let $(\Omega, \mathcal{F}, P)$ be a probability space, φ a measure-preserving transform with invariant σ-algebra $\mathcal{I}$, and X a random variable in L^1.*

Then $\displaystyle\frac{1}{n}\sum_{m=0}^{n-1} X(\varphi^m(\omega)) \to \mathbb{E}[X \mid \mathcal{I}]$ a.s. and in L^1.

In particular, if φ is ergodic, then $\displaystyle\frac{1}{n}\sum_{m=0}^{n-1} X(\varphi^m(\omega)) \to \mathbb{E}X$ a.s. and in L^1 ("space average is equal to time average").

Proof:

Step 1: We may assume that $\mathbb{E}[X \mid \mathcal{I}] = 0$ a.s.

Note that if Y is $\mathcal{I}$-measurable, then $Y \circ \varphi = Y$ a.s., so $Y \circ \varphi^j = Y$ a.s. for all j. $\mathbb{E}[X \mid \mathcal{I}]$ is $\mathcal{I}$-measurable. Thus, if $X' = X - \mathbb{E}[X \mid \mathcal{I}]$, then for all j, $X' \circ \varphi^j = X \circ \varphi^j - \mathbb{E}[X \mid \mathcal{I}]$, implying that

$$\frac{1}{n}\sum_{m=1}^{n-1}\Big(X \circ \varphi^m - \mathbb{E}[X \mid \mathcal{I}]\Big) = \frac{1}{n}\sum_{m=1}^{n-1} X' \circ \varphi^m.$$

Also, $\mathbb{E}[X' \mid \mathcal{I}] = 0$ a.s., so it suffices to prove the result for X'.

As in the set-up of the Maximal Ergodic Theorem, let $X_n = X \circ \varphi^n$, $S_n = X_0 + X_1 + \cdots + X_n$.

Step 2: $S_n/n \to 0$ a.s.

We will show that $\limsup \dfrac{S_n}{n+1} \le 0$ a.s. A symmetric argument will show that $\liminf \dfrac{S_n}{n+1} \ge 0$ a.s., and so $\lim \dfrac{S_n}{n+1} = 0$ a.s.

Let $\bar{X} = \limsup \dfrac{S_n}{n+1}$. Note that $\bar{X}$ is $\mathcal{I}$-measurable as $\bar{X} \circ \varphi = \bar{X}$. Take any $\varepsilon > 0$ and let $D = \{\omega : \bar{X}(\omega) > \varepsilon\}$. Then $D \in \mathcal{I}$.

Let $X^* = (X - \varepsilon)1_D$. Let $S_n^* = X^* + X^* \circ \varphi + \cdots + X^* \circ \varphi^n$, and let $M_n^* = \max\{0, S_0^*, \ldots, S_n^*\}$.

Let $F_n = \{\omega : M_n^*(\omega) > 0\}$, and let $F = \displaystyle\bigcup_{n=0}^{\infty} F_n$.

Step 2.1: $F = \left\{\omega : \sup_{k \ge 0} \dfrac{S_k^*(\omega)}{k+1} > 0.\right\}$.

$$\begin{aligned}
\omega \in F \quad &\Leftrightarrow \quad \omega \in F_n && \text{for some } n\\
&\Leftrightarrow \quad M_n^*(\omega) > 0 && \text{for some } n\\
&\Leftrightarrow \quad S_j^*(\omega) > 0 && \text{for some } 0 \le j \le n\\
&\Leftrightarrow \quad S_j^*(\omega)/(j+1) > 0 && \text{for some } 0 \le j \le n\\
&\Leftrightarrow \quad \sup_{k \ge 0} \frac{S_k^*(\omega)}{k+1} > 0.
\end{aligned}$$

Step 2.2: $F = D$.

If $\omega \in F$, then $\dfrac{S_k^*(\omega)}{k+1} > 0$ for some k, implying that $S_k^*(\omega) > 0$ for some k. Since $S_k^* = X_0^* + \cdots + X_k*$, it means that $X_j^*(\omega) > 0$ for some j. But by definition,

$$X_j^*(\omega) = X^*(\varphi^j(\omega)) = [X(\varphi^j(\omega)) - \varepsilon]1_D(\varphi^j(\omega)),$$

so, for some j,

$$\begin{aligned}
X_j^*(\omega) > 0 \quad &\Rightarrow \quad [X(\varphi^j(\omega)) - \varepsilon]1_D(\varphi^j(\omega)) > 0\\
&\Rightarrow \quad \varphi^j(\omega) \in D\\
&\Rightarrow \quad \omega \in \varphi^{-j}(D)\\
&\Rightarrow \quad \omega \in D. && \text{(since } D \in \mathcal{I})
\end{aligned}$$

On the other hand, if $\omega \in D$, then $\varphi^j(\omega) \in D$ for all j (by definition of D and $\bar{X}$). This implies that

$$\begin{aligned}
S_n^*(\omega) &= \sum_{i=0}^{n} X_i^*(\omega) = \sum_{i=0}^{n} X_i(\omega) - \varepsilon = S_n(\omega) - \varepsilon(n+1), \\
\sup \frac{S_n^*(\omega)}{n+1} &\geq \limsup \frac{S_n^*(\omega)}{n+1} \\
&= \limsup \frac{S_n(\omega)}{n+1} - \varepsilon \\
&> 0,
\end{aligned}$$

i.e. $\omega \in F$.

Step 2.3: Use the Maximum Ergodic Theorem to conclude that $P(D) = 0$.

Note that $E|X^*| < \mathbb{E}|X| + \varepsilon < \infty$. The Maximum Ergodic Theorem implies that $\mathbb{E}[X^*; F_n] \geq 0$ for all n. Thus,

$$\begin{aligned}
0 \leq \lim_{n\to\infty} \mathbb{E}[X^* 1_{F_n}] &= \mathbb{E}\left[\lim_{n\to\infty} X^* 1_{F_n}\right] && \text{(Dominated Convergence Theorem)} \\
&= \mathbb{E}[X^*; F] \\
&= \mathbb{E}[X^*; D] && \text{(since } F = D\text{)} \\
&= \mathbb{E}[(X - \varepsilon) 1_D] \\
&= \mathbb{E}[\mathbb{E}[(X - \varepsilon) 1_D \mid \mathcal{I}]] \\
&= \mathbb{E}[(\mathbb{E}[X \mid \mathcal{I}] - \varepsilon) 1_D] \\
&= -\varepsilon P(D),
\end{aligned}$$

hence $P(D) = 0$, as required.

Step 3: $S_n/n \to 0$ in L^1.

Take any $M > 0$. Let $X_M^{'} := X 1_{\{|X| \leq M\}}$, $X_M'' = X - X_M'$.

Then, by the a.s. part of Birkhoff's Ergodic Theorem (Step 2), $\frac{1}{n}\sum_{m=0}^{n-1} X_M'(\varphi^m(\omega)) \to \mathbb{E}[X_M' \mid \mathcal{I}]$ a.s. By the Dominated Convergence Theorem, this convergence also holds in L^1. If we can show that $\frac{1}{n}\sum_{m=0}^{n-1} X_M''(\varphi^m(\omega)) \to \mathbb{E}[X_M'' \mid \mathcal{I}]$ in L^1, we would be done.

Since φ is measure-preserving, $\mathbb{E}|X_M'' \circ \varphi^m| = \mathbb{E}|X_M''|$ for all m, and so

$$\mathbb{E}\left|\frac{1}{n}\sum_{m=0}^{n-1} X_M'' \circ \varphi^m\right| \leq \frac{1}{n}\sum_{m=0}^{n-1} \mathbb{E}|X_M'' \circ \varphi^m| = \mathbb{E}|X_M''|.$$

By Jensen's inequality, we also have $\mathbb{E}|\mathbb{E}[X_M'' \mid \mathcal{I}]| \leq \mathbb{E}|X_M''|$. Putting the two together, we get

$$\mathbb{E}\left|\frac{1}{n}\sum_{m=0}^{n-1} X_M'' \circ \varphi^m - \mathbb{E}[X_M'' \mid \mathcal{I}]\right| \leq 2\mathbb{E}|X_M''|$$

for all n, and so

$$\limsup_{n\to\infty} \mathbb{E}\left|\frac{1}{n}\sum_{n=0}^{n-1} X_M'' \circ \varphi^m - \mathbb{E}[X_M'' \mid \mathcal{I}]\right| \leq 2\mathbb{E}|X_M''|. \tag{12.1}$$

In the above, M is arbitrary. As $M \to \infty$, $|X''_M| \to 0$ a.s. Since $|X''_M| < |X|$ for all M and $\mathbb{E}|X| < \infty$, we can use the Dominated Convergence Theorem to get $\lim \mathbb{E}|X''_M| = 0$. Thus, letting $M \to \infty$ in Equation 12.1, we get the desired result.

$\blacksquare$