

## Lecture 4: January 19

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## 4.1 Application of Wald's Lemma to Random Walks

Consider the following setting:

- $X_1, X_2, \dots$  be i.i.d.  $\pm 1$  valued random variables with  $P\{X_i = 1\} = P\{X_i = -1\} = \frac{1}{2}$ .
- $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ ,  $\mathcal{F}_0 =$  trivial  $\sigma$ -algebra.
- $S_n = \sum_{i=1}^n X_i$ ,  $S_0 = 0$ .
- $a < 0 < b$  are 2 integers,  $T$  is defined to be a stopping time  $T = \inf\{k : S_k = a \text{ or } b\}$ .

4.1.1 Symmetric Random Walk ( $p = \frac{1}{2}$ )

$\{S_n\}$  is a martingale w.r.t.  $\{\mathcal{F}_n\}$ . If we can show that  $T < \infty$  a.s., then we can use Wald's Lemma (last lecture) to show that  $P(S_T = a) = \frac{b}{b-a}$ .

We now show that  $\mathbb{E}T < \infty$ . It then follows that  $T < \infty$  a.s.

**Proof:** Define

$$T' := \min\{k : X_n = 1 \text{ for } n = (k-1)(b-a), (k-1)(b-a)+1, \dots, (k-1)(b-a) + (b-a)\}.$$

(i.e. Partition  $\mathbb{N}$  into blocks of size  $b-a$ ,  $T'$  is the first time each step in the block is 1.) Then  $T' \sim \text{Geom}(2^{-(b-a)})$ , and so  $\mathbb{E}T' < \infty$ . However, note that  $T \leq (b-a)T'$ . Thus,  $\mathbb{E}T \leq (b-a)\mathbb{E}T' < \infty$ . ■

We can actually compute  $\mathbb{E}T$  exactly:

**Proposition 4.1**  $\mathbb{E}T = -ab$ .

**Proof:** Recall that  $\{S_n^2 - n\}$  is also a martingale w.r.t.  $\{\mathcal{F}_n\}$ . Thus, by Wald's Lemma, for all  $n$ ,

$$\begin{aligned}\mathbb{E}[S_{T \wedge n}^2 - T \wedge n] &= 0, \\ \mathbb{E}[S_{T \wedge n}^2] &= \mathbb{E}[T \wedge n].\end{aligned}$$

Now,  $S_{T \wedge n}^2 \rightarrow S_T^2$  and  $S_{T \wedge n}$  is a uniformly bounded sequence of random variables, so by the Bounded Convergence Theorem,  $\mathbb{E}[S_{T \wedge n}^2] \rightarrow \mathbb{E}[S_T^2]$ . Also,  $0 \leq T \wedge n \nearrow T$ , so by the Monotone Convergence Theorem,

$\mathbb{E}[T \wedge n] \rightarrow \mathbb{E}T$ . Hence,

$$\begin{aligned}\mathbb{E}T &= \mathbb{E}[S_T^2] \\ &= a^2 \frac{b}{b-a} + b^2 \frac{-a}{b-a} \\ &= -ab.\end{aligned}$$

■

#### 4.1.1.1 Cautionary Tale

Consider the stopping time  $T = \inf\{k : S_k = 1\}$ . By Wald's Lemma,  $\mathbb{E}[S_{T \wedge n}] = 0$  for all  $n$ . We can also show that  $T < \infty$  a.s., so  $S_{T \wedge n} \rightarrow S_T$ .

However,  $\mathbb{E}S_T = 1$  since  $S_T = 1$  always! This happens because we do not have  $\mathbb{E}S_{T \wedge n} \rightarrow \mathbb{E}S_T$ : the Dominated Convergence Theorem does not apply as  $\{|S_{T \wedge n}|\}$  cannot be bounded above by an integrable random variable.

#### 4.1.2 Biased Random Walk ( $p \neq \frac{1}{2}$ )

$\{S_n\}$  will no longer be a martingale. While  $\{\sum_{i=1}^n X_i - \mathbb{E}X_i\} = \{S_n - n(2p-1)\}$  is a martingale, Wald's Lemma won't help us.

Instead, consider  $M_n = \left(\frac{q}{p}\right)^{S_n}$ , where  $q = 1 - p$ .  $\{M_n\}$  is a martingale:

$$\begin{aligned}\mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}} \middle| \mathcal{F}_n\right] &= \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_n} \left(\frac{q}{p}\right)^{X_{n+1}} \middle| \mathcal{F}_n\right] \\ &= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{n+1}} \middle| \mathcal{F}_n\right] \\ &= \left(\frac{q}{p}\right)^{S_n} \left[\left(\frac{q}{p}\right)^1 p + \left(\frac{q}{p}\right)^{-1} q\right] \\ &= M_n.\end{aligned}$$

Hence, with  $T$  as before, by Wald's Lemma,

$$\mathbb{E}\left(\frac{q}{p}\right)^{S_{T \wedge n}} = \mathbb{E}M_0 = 1.$$

Using a similar argument as that for the symmetric random walk,  $T < \infty$  a.s., so  $S_{T \wedge n} \rightarrow S_T$  a.s. and  $|S_{T \wedge n}|$  is uniformly bounded, so

$$\begin{aligned}\mathbb{E}\left[\left(\frac{q}{p}\right)^{S_T}\right] &= 1, \\ \left(\frac{q}{p}\right)^a P\{S_T = a\} + \left(\frac{q}{p}\right)^b (1 - P\{S_T = a\}) &= 1, \\ P\{S_T = a\} &= \frac{1 - (q/p)^b}{(q/p)^a - (q/p)^b}.\end{aligned}$$

## 4.2 Submartingales and Supermartingales

Note that we can define  $\mathbb{E}X$  whenever at least one of  $\mathbb{E}X^+$  and  $\mathbb{E}X^-$  is finite. Similarly, we can define  $\mathbb{E}[X \mid \mathcal{G}]$  whenever at least one of  $\mathbb{E}X^+$  and  $\mathbb{E}X^-$  is finite.

**Definition 4.2**  $\{Z_n, \mathcal{F}_n\}_{n=1}^\infty$  is a **submartingale** if:

1.  $\mathbb{E}Z_n^+ < \infty$  for all  $n$ , and
2.  $\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] \geq Z_n$  a.s. for all  $n$ .

**Definition 4.3**  $\{Z_n, \mathcal{F}_n\}_{n=1}^\infty$  is a **supermartingale** if:

1.  $\mathbb{E}Z_n^- < \infty$  for all  $n$ , and
2.  $\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] \leq Z_n$  a.s. for all  $n$ .

**Proposition 4.4** Let  $\{Z_n, \mathcal{F}_n\}_{n=1}^\infty$  be a martingale. Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function. Assume that  $\mathbb{E}|\phi(Z_n)| < \infty$  for all  $n$ .

Then  $\{\phi(Z_n), \mathcal{F}_n\}_{n=1}^\infty$  is a submartingale.

**Proof:** By Jensen's inequality,

$$\mathbb{E}[\phi(Z_{n+1}) \mid \mathcal{F}_n] \geq \phi(\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]) = \phi(Z_n).$$

■

**Proposition 4.5** Let  $\{Z_n, \mathcal{F}_n\}_{n=1}^\infty$  be a submartingale. Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a convex and non-decreasing function. Assume that  $\mathbb{E}|\phi(Z_n)| < \infty$  for all  $n$ .

Then  $\{\phi(Z_n), \mathcal{F}_n\}_{n=1}^\infty$  is again a submartingale.

**Proof:** By Jensen's inequality,

$$\mathbb{E}[\phi(Z_{n+1}) \mid \mathcal{F}_n] \geq \phi(\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]) \geq \phi(Z_n).$$

(Need  $\phi$  to be non-decreasing for the last inequality.)

■

### 4.2.1 Examples

- If  $\{Z_n\}$  is a martingale, then  $\{|Z_n|\}$  is a submartingale.
- If  $\{Z_n\}$  is a submartingale,  $\{|Z_n|\}$  may not be a submartingale, but  $\{Z_n^+\}$  is a submartingale.
- If  $\{Z_n\}$  is a martingale, then  $\{Z_n^2\}$  is a submartingale.
- If  $\{Z_n\}$  is a submartingale,  $\{Z_n^2\}$  may not be a submartingale.

### 4.3 (Sub)martingale Convergence Theorem

**Theorem 4.6** Let  $\{Z_n, \mathcal{F}_n\}_{n=1}^\infty$  be a submartingale. Suppose  $\sup_n \mathbb{E}Z_n^+ < \infty$ .

Then there is a random variable  $Z$  taking values in  $[-\infty, \infty)$  such that  $Z_n \rightarrow Z$  a.s.

(**Note:** Martingales are submartingales, and so the theorem above applies as well. For a martingale, we will have  $Z$  finite a.s.)

**Main Idea of Proof:**

We want to show that  $\lim Z_n$  exists, and is  $< \infty$  a.s. Let's consider the probability that  $\lim Z_n$  does not exist. Note that

$$\{\lim Z_n \text{ does not exist in } [-\infty, \infty]\} = \{\exists a, b \in \mathbb{Q}, a < b \text{ such that } \liminf Z_n < a, \text{ and } \limsup Z_n > b\},$$

hence

$$P\{\lim Z_n \text{ does not exist in } [-\infty, \infty]\} \leq \sum_{a, b \in \mathbb{Q}} P\{\liminf Z_n < a, \text{ and } \limsup Z_n > b\}.$$

We will show that for each  $(a, b)$  with  $a < b$ , the probability on the RHS is equal to 0.

Once we know that  $Z = \lim Z_n$  exists a.s., then  $Z_n^+ \rightarrow Z^+$  a.s., so by Fatou's Lemma and the fact that  $\sup_n \mathbb{E}Z_n^+ < \infty$ ,

$$\mathbb{E}Z^+ \leq \liminf \mathbb{E}Z_n^+ < \infty.$$