STATS 310B: Theory of Probability II — Winter 2016/17

Lecture 19: March 14

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19.1 Concentration Inequalities

Theorem 19.1 (Efron-Stein Inequality (1981)) *Let $X_1, \dots, X_n$ be independent random variables. Let $X_1', \dots, X_n'$ be another set of independent random variables, independent of $X_1, \dots, X_n$, such that X_i' has the same distribution as X_i for all i.*

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that $\mathbb{E}[W^2] < \infty$, where $W = f(X_1, \dots, X_n)$. Then,

$$\text{Var } W \leq \frac{1}{2} \sum_{i=1}^n \mathbb{E}\left[(\Delta_i f)^2\right],$$

where $\Delta_i f = f(X_1, \dots, X_n) - f(X_1, \dots, X_i', \dots, X_n)$.

Proof: Let $\mathcal{F}_i = \sigma(X_1, \dots, X_i)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $M_i = \mathbb{E}[W \mid \mathcal{F}_i]$. Then $\{M_i\}_{i=0}^n$ is a martingale.

Note that $M_n = W$ and $M_0 = \mathbb{E}W$, implying that

$$\begin{aligned}
\text{Var } W &= \mathbb{E}(M_n - M_0)^2 \\
&= \mathbb{E}\left[\sum_{i=1}^n (M_i - M_{i-1})\right]^2 \\
&= \sum_{i=1}^n \mathbb{E}\left[M_i - M_{i-1}\right]^2,
\end{aligned}$$

since $\mathbb{E}[(M_i - M_{i-1})(M_j - M_{j-1})] = 0$ for all $i \neq j$ (this is a property of martingales).

For each i, let μ_i be the law of X_i (i.e. for all $B \in \mathcal{B}(\mathbb{R})$, $\mu_i(B) = P(X_i \in B)$). Then,

$$\begin{aligned}
M_i &= \int_{\mathbb{R}^{n-i}} f(X_1, \dots, X_i, x_{i+1}, \dots, x_n) d\mu_{i+1}(x_{i+1}) \dots d\mu_n(x_n) \\
&=: g_i(X_1, \dots, X_i).
\end{aligned}$$

Thus,

$$\begin{aligned}
M_{i-1} &= \mathbb{E}[M_i \mid \mathcal{F}_{i-1}], \\
\mathbb{E}(M_i - M_{i-1})^2 &= \mathbb{E}[\mathbb{E}[(M_i - M_{i-1})^2 \mid \mathcal{F}_{i-1}]] \\
&= \mathbb{E}\left[\text{Var } (M_i \mid \mathcal{F}_{i-1})\right] \\
&= \mathbb{E}\left[\text{Var } (g_i(X_1, \dots, X_i) \mid X_1, \dots, X_{i-1})\right].
\end{aligned}$$

Using the fact that for any X, X' i.i.d., Var $X = \frac{1}{2}\mathbb{E}(X-X')^2$,

$$\begin{aligned}
\mathbb{E}(M_i - M_{i-1})^2 &= \frac{1}{2}\mathbb{E}\Big[\mathbb{E}\left[(g_i(X_1,\dots,X_i) - g_i(X_1,\dots,X_{i-1},X_i'))^2 \mid X_1,\dots,X_{i-1})\right]\Big] \\
&= \frac{1}{2}\mathbb{E}\left[(g_i(X_1,\dots,X_i) - g_i(X_1,\dots,X_{i-1},X_i'))^2\right] \\
&= \frac{1}{2}\mathbb{E}\left[\left(\int f(X_1,\dots,X_i,x_{i+1},\dots,x_n) - f(X_1,\dots,X_i',x_{i+1},\dots,x_n) d\mu_{i+1}(x_{i+1})\dots d\mu_n(x_n)\right)^2\right] \\
&\le \frac{1}{2}\mathbb{E}\int \left[f(X_1,\dots,X_i,x_{i+1},\dots,x_n) - f(X_1,\dots,X_i',x_{i+1},\dots,x_n)\right]^2 d\mu_{i+1}(x_{i+1})\dots d\mu_n(x_n) \\
&= \frac{1}{2}\mathbb{E}\left[(f(X_1,\dots,X_n) - f(X_1,\dots,X_i',\dots,X_n))^2\right] \\
&= \frac{1}{2}\mathbb{E}\left[(\Delta_i f)^2\right].
\end{aligned}$$

Summing up over all i, we get the desired inequality.

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Example: Let $f(x_1,\dots,x_n) = \sum_{i=1}^n x_i$. Then $\Delta_i f = X_i - X_i'$, and the Efron-Stein Inequality gives

$$\text{Var} \sum_{i=1}^n X_i \le \frac{1}{2}\sum_{i=1}^n \mathbb{E}(X_i - X_i')^2 = \sum_{i=1}^n \text{Var } X_i = \text{Var} \sum_{i=1}^n X_i,$$

i.e. equality is achieved.

19.1.1 Application to First-Passage Percolation

Consider the first-passage percolation model on the lattice $\mathbb{Z}^d$:

- On each edge e, we have a non-negative random variable X_e, called the weight of the edge. Assume that the X_e are i.i.d.
- The weight of a path is equal to the sum of edge weights along the path.
- The first-passage time from x to y, denoted T_{xy}, is the minimum of the weights of all paths from x to y.
- $T_n := T_{0,ne_1}$, where $e_1 = (1,0,\dots,0)$, i.e. the first-passage time from 0 to $(n,0,\dots,0)$.

Theorem 19.2 *Assume that there exist* $0 < a < b$ *such that* $P(a \le X_e \le b) = 1$.

Then Var $T_n \le Cn$, *where* C *depends only on* a, b *and* d.

Proof: The straight line path from 0 to $(n,0,\dots,0)$ has weight $\le nb$. On the other hand, any path of length m has weight $\ge am$. Thus, T_n is actually a minimum over paths of length $\le \frac{nb}{a}$, i.e. T_n is a function of finitely many edge weights, and so we can apply the Efron-Stein Inequality.

Let $T_n^e :=$ new first passage time if X_e is replaced by an independent copy X_e'. Let $\Delta_e T_n := T_n - T_n^e$.

First, note that

$$\begin{aligned} & T_n^e \le T_n + |X_e' - X_e| \le T_n + b, \\ & T_n \le T_n^e + |X_e' - X_e| \le T_n^e + b, \\ \Rightarrow \quad & |\Delta_e T_n| \le b. \end{aligned}$$

If $X_e' > X_e$ and e is not in some optimal path in the old environment, then $\Delta_e T_n = 0$. Similarly, if $X_e' < X_e$ and e does not belong to some optimal path in the new environment, $\Delta_e T_n = 0$ as well. Thus, if

$$\begin{aligned} A &= \{e \text{ belongs to all optimal paths in old environment}\}, \\ B &= \{e \text{ belongs to all optimal paths in new environment}\}, \end{aligned}$$

then $(\Delta_e T_n)^2 \le b^2 1_{A \cup B}$. This implies that

$$\begin{aligned} \mathbb{E}\left[(\Delta_e T_n)^2\right] &\le b^2[P(A) + P(B)] \\ &= 2b^2 P(A), && (\because A \text{ and } B \text{ have the same law}) \\ \frac{1}{2}\sum_e \mathbb{E}\left[(\Delta_e T_n)^2\right] &\le b^2 \sum_e P(e \in \text{ all optimal paths}) \\ &= b^2 \mathbb{E}[\text{no. of edges belonging to all optimal paths}] \\ &\le b^2 \frac{nb}{a} \\ &= \frac{nb^3}{a}. \end{aligned}$$

By the Efron-Stein Inequality, we have the desired bound with $C = \dfrac{b^3}{a}$.

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Theorem 19.3 *Suppose that $\mathbb{E}[X_e] < \infty$. Then $\lim_{n\to\infty} \dfrac{\mathbb{E}T_n}{n}$ exists. Moreover, if $P(X_e > 0) = 1$, then the limit is positive.*

Proof: To show existence of limit: Take any n, m. Let T_m' be the first-passage time from ne_1 to $(n+m)e_1$. Then T_m' has the same distribution as T_m. Also,

$$\begin{aligned} T_{n+m} &\le T_n + T_m', \\ \mathbb{E}T_{n+m} &\le \mathbb{E}T_n + \mathbb{E}T_m' \\ &= \mathbb{E}T_n + \mathbb{E}T_m. \end{aligned}$$

By the Subadditive Lemma (see below), the limit exists.

To show $\mu := \lim \dfrac{\mathbb{E}T_n}{n} > 0$: A **self-avoiding** path is a path that does not visit any vertex more than once. Note that any optimal path is self-avoiding. Also, any optimal path from 0 to ne_1 has length $\ge n$, and hence, contains a self-avoiding path of length n starting at 0. Thus, $T_n \ge S_n$, where S_n is the minimum weight of a self-avoiding path of length n starting at 0.

We will show that $\liminf_{n\to\infty} \frac{\mathbb{E}S_n}{n} > 0$. Let $\mathcal{S}_n$ be the set of all self-avoiding paths of length n starting at 0. For any $p \in \mathcal{S}_n$, let X_p be the weight of the path p.

For any $\theta > 0$,

$$\begin{aligned}
\mathbb{E}\left[e^{-\theta X_p}\right] &= \mathbb{E}\left[\prod_{e\in p} e^{-\theta X_e}\right] \\
&= \prod_{e\in p} \mathbb{E}\left[e^{-\theta X_e}\right] && \text{(since } p \text{ self-avoiding)} \\
&= \varphi(\theta)^n, && \text{(since } p \text{ has } n \text{ edges)}
\end{aligned}$$

where $\phi(\theta) = \mathbb{E}\left[e^{-\theta X_e}\right]$. For any $c > 0$,

$$\begin{aligned}
P(X_p \leq cn) &= P\left(e^{-\theta X_p} \geq e^{-\theta cn}\right) \\
&\leq e^{\theta cn}\mathbb{E}\left[e^{-\theta X_p}\right] \\
&= e^{\theta cn}\varphi(\theta)^n.
\end{aligned}$$

Take any $K > 0$. We know (by the Dominated Convergence Theorem) that $\lim_{\theta\to\infty} \varphi(\theta) = 0$ since $P(X_e = 0) = 0$. Find θ so large that $\varphi(\theta) \leq e^{-2K}$. Then, find c so small that $\theta c \leq K$. We have

$$\begin{aligned}
P(X_p \leq cn) &\leq e^{\theta cn}\varphi(\theta)^n \\
&\leq e^{Kn}e^{-2Kn} \\
&= e^{-Kn}.
\end{aligned}$$

We have thus shown the following: Given any $K > 0$, we can find c small enough so that $P(X_p \leq cn) \leq e^{-Kn}$.

Note that $|\mathcal{S}_n| \leq$ no. of paths of length $n = (2d)^n$. Thus,

$$\begin{aligned}
P(S_n \leq cn) &= P\left(\min_{p\in\mathcal{S}_n} X_p \leq cn\right) \\
&\leq \sum_{p\in\mathcal{S}_n} P(X_p \leq cn) \\
&\leq |\mathcal{S}_n|e^{-Kn} \\
&\leq (2d)^n e^{-Kn}.
\end{aligned}$$

If K is chosen such that $K > \log(2d)$, then $P(S_n \leq cn) \to 0$ as $n \to \infty$.

On the other hand, if $\frac{\mathbb{E}S_n}{n} \to 0$ on a subsequence, then by Markov's inequality, $p(S_n \leq cn) \to 1$ through that subsequence. Contradiction!

Thus $\liminf_{n\to\infty} \frac{\mathbb{E}S_n}{n} > 0$, as required. ■

Lemma 19.4 (Subadditive Lemma) *Let $\{a_n\}$ be a sequence of real numbers such that $a_{n+m} \leq a_n + a_m$ for all n and m.*

Then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\inf_{n\geq 1} \frac{a_n}{n}$.

Proof: Take any $m \geq 1$ and any $n \geq m$. Write $n = qm + r$, where $0 \leq r \leq m-1$. By subadditivity, $a_n \leq qa_m + a_r$, which implies that

$$\begin{aligned}\frac{a_n}{n} &\leq \frac{q}{n}a_m + \frac{a_r}{n}\\ &\leq \frac{q}{n}a_m + \frac{\max_{0\leq r\leq m-1} a_r}{n}.\end{aligned}$$

As $n \to \infty$, $\dfrac{q}{n} \to \dfrac{1}{m}$ and $\dfrac{\max_{0\leq r\leq m-1} a_r}{n} \to 0$. Hence,

$$\begin{aligned}& \limsup \frac{a_n}{n} \leq \frac{a_m}{m} && \text{for all } m,\\ \Rightarrow\quad & \limsup \frac{a_n}{n} \leq \inf_m \frac{a_m}{m}\\ & \qquad\qquad \leq \liminf \frac{a_n}{n},\\ \Rightarrow\quad & \lim \frac{a_n}{n} = \inf_m \frac{a_m}{m}.\end{aligned}$$

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