

## 15.1 UMPU Tests for Multi-parameter Exponential Families

Recall that if we are testing  $H_0 : \theta \in \Omega_0$  vs.  $H_1 : \theta \in \Omega_1$ , a UMPU test  $\varphi$  must satisfy 2 constraints:

1. the level constraint:  $\mathbb{E}_\theta \varphi \leq \alpha$  for all  $\theta \in \Omega_0$ , and
2. the unbiasedness constraint:  $\mathbb{E}_\theta \varphi \geq \alpha$  for all  $\theta \in \Omega_1$ .

For the rest of this lecture, we will assume that the background model is a multi-parameter exponential family with densities

$$p_{\theta, \vartheta}(x) \propto \exp \left[ \theta U(x) + \sum_{i=1}^d \vartheta_i T_i(x) \right] h(x).$$

We write densities in a slightly different form from the usual to highlight  $\theta$  as the parameter of interest. In this context, the  $\vartheta_i$ 's are **nuisance parameters**.

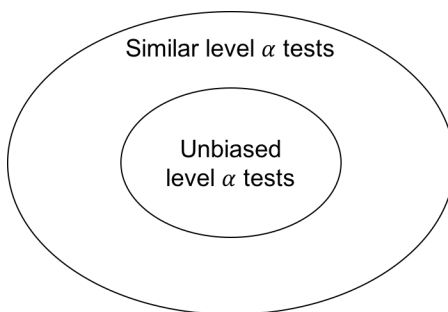
We will be testing  $H_0 : \theta \leq \theta_0$  vs.  $H_1 : \theta > \theta_0$  for some fixed  $\theta_0$ . (Note that in this set-up,  $H_0 : \theta = \theta_0$  is *not* a simple hypothesis:  $\vartheta_i$ 's could vary for fixed  $\theta = \theta_0$ .)

### 15.1.1 Similar Tests

**Definition 15.1** When testing  $H_0 : \theta \in \Omega_0$  vs.  $H_1 : \theta \in \Omega_1$ , define the **boundary parameter space** as  $\omega := \bar{\Omega}_0 \cap \bar{\Omega}_1$  (i.e. the intersection of the closures).

**Definition 15.2** A test  $\varphi$  is **similar** if it satisfies  $\mathbb{E}_\theta \varphi = \alpha$  for all  $\theta \in \omega$ .

Suppose we know that the power function of any test is continuous (as it is in exponential family models). Then a level  $\alpha$  test which is unbiased must also be similar. Hence, assuming continuity of power functions, unbiased tests form a subset of similar tests:



We will find UMP tests among all similar tests. If the UMP test we found is unbiased as well, then it will be UMPU.

**Definition 15.3** A test satisfying

$$\mathbb{E}_{\theta_0}[\varphi(X) \mid T(X) = t] = \alpha \quad (15.1)$$

for all  $t$  is said to have **Neyman structure** with respect to  $T$ .

It is clear that if a test function  $\varphi$  satisfies Equation 15.1 for all  $t$ , then  $\mathbb{E}_{\theta_0} \varphi(X) = \alpha$ .

The following theorem gives us a large class of functions which have Neyman structure:

**Theorem 15.4** If  $T$  is complete and sufficient for  $\omega$ , then every similar test has Neyman structure.

**Proof:** Suppose  $\varphi$  is a similar test, i.e.  $\mathbb{E}_{\theta} \varphi(X) - \alpha = 0$  for all  $\theta \in \omega$ .

Let  $\psi(t) = \mathbb{E}[\varphi(X) - \alpha \mid T(X) = t]$ . Then  $\mathbb{E}\psi(T) = 0$ . By completeness, we conclude that  $\psi(T) = 0$  almost surely. ■

(Remark: The proof still works if  $T$  is assumed only to be boundedly complete.)

**Proposition 15.5** Assume that we are in an exponential family model, i.e.

$$p_{\theta, \vartheta}(x) \propto \exp \left[ \theta U(x) + \sum_{i=1}^d \vartheta_i T_i(x) \right] h(x).$$

Then:

1. With  $\theta = \theta_0$  fixed,  $(T_1, \dots, T_d)$  is sufficient, and also complete if  $\{\vartheta_1, \dots, \vartheta_d\}$  contains a  $d$ -dimensional rectangle.
2.  $T = (T_1, \dots, T_d)$  has an exponential family of distributions.
3. The conditional distribution of  $U \mid T = t$  is a 1-parameter exponential family. (By sufficiency, this exponential family does not depend on  $\vartheta$ .)

**Proof:** We only prove 3 for the discrete case. In this case,

$$\begin{aligned} P_{\theta} \{U(X) = u \mid T(X) = t\} &= \frac{P_{\theta, \vartheta}(U = u, T = t)}{P_{\theta, \vartheta}(T = t)} \\ &= \frac{\sum_{x: U(x)=u, T(x)=t} C(\theta, \vartheta) \exp \left[ \theta u + \sum \vartheta_i t_i \right] h(x)}{\sum_{x: T(x)=t} C(\theta, \vartheta) \exp \left[ \theta u(x) + \sum \vartheta_i t_i \right] h(x)} \\ &= e^{\theta u} \cdot \frac{\sum_{x: U(x)=u, T(x)=t} h(x)}{\sum_{x: T(x)=t} h(x)}. \end{aligned}$$

Since the fraction does not depend on  $\theta$ , it is a 1-parameter exponential family. ■

**Theorem 15.6** In the above  $(d+1)$ -parameter exponential family, assume that for fixed  $\theta = \theta_0$ , the family is of full-rank (i.e.  $(T_1, \dots, T_d)$  complete for  $\omega$ ).

For testing  $H_0 : \theta \leq \theta_0$  vs.  $H_1 : \theta > \theta_0$ , there exists UMPU level  $\alpha$  test of the form

$$\varphi(u, t) = \begin{cases} 1 & \text{if } u > c(t), \\ \gamma(t) & \text{if } u = c(t), \\ 0 & \text{if } u < c(t), \end{cases}$$

where  $u = U(x)$ ,  $t = T(x)$ ,  $c(t)$  and  $\gamma(t)$  are determined such that  $\mathbb{E}_{\theta_0}[\varphi(U, T) \mid T] = \alpha$ .

**Proof:** Fix any alternative  $(\theta', \vartheta')$  with  $\theta' > \theta_0$ . For any level  $\alpha$  test  $\varphi'$ , the power of the test

$$\mathbb{E}_{\theta', \vartheta'} \varphi'(U, T) = \mathbb{E}\{\mathbb{E}[\varphi'(U, T) \mid T]\}.$$

If we can find  $\varphi'$  that maximizes  $\mathbb{E}'_{\theta}[\varphi'(U, T) \mid T = t]$  for all  $t$ , we are done.

However, given  $T = t$ , we are in a 1-parameter exponential family model. Hence, we can construct the optimal condition test for each  $t$ . ■

## 15.1.2 Examples of UMPU Tests

### 15.1.2.1 Example: Poisson setting

Let  $X \sim \text{Poisson}(\lambda)$ ,  $Y \sim \text{Poisson}(\mu)$ ,  $X$  and  $Y$  independent. Testing  $H_0 : \mu \leq \lambda$ .

The joint density of  $X$  and  $Y$  is given by

$$\begin{aligned} \frac{e^{-\lambda} \lambda^x}{x!} \cdot \frac{e^{-\mu} \mu^y}{y!} &= \frac{e^{-(\lambda+\mu)}}{x! y!} \exp[x \log \lambda + y \log \mu] \\ &= \frac{e^{-(\lambda+\mu)}}{x! y!} \exp[y(\log \mu - \log \lambda) + (x+y) \log \lambda]. \end{aligned}$$

Let  $\theta = \log \frac{\mu}{\lambda}$ ,  $\vartheta = \log \lambda$ ,  $U = Y$ ,  $T = X + Y$ . Note that  $H_0 : \mu \leq \lambda \Leftrightarrow H_0 : \theta \leq 0$ . Using Theorem 15.6, we know that the UMPU test rejects  $H_0$  if  $Y$  is large conditional on  $X + Y = t$ .

What is the conditional distribution of  $Y$  given  $X + Y = t$  under  $\theta = 0$ ? It is  $Y \mid X + Y = t \sim \text{Binom}(t, \frac{1}{2})$ . Hence, the UMPU test is given by

$$\varphi(y, t) = \begin{cases} 1 & \text{if } y > c(t), \\ \gamma(t) & \text{if } y = c(t), \\ 0 & \text{if } y < c(t), \end{cases}$$

where  $c(t)$  and  $\gamma(t)$  are determined by

$$\sum_{j=c(t)+1}^t \binom{t}{j} \left(\frac{1}{2}\right)^t + \gamma(t) \binom{t}{c(t)} \left(\frac{1}{2}\right)^t = \alpha.$$

### 15.1.2.2 Example: Binomial setting

Let  $X \sim \text{Binom}(m, p_1)$ ,  $Y \sim \text{Binom}(n, p_2)$ ,  $X$  and  $Y$  independent. Testing  $H_0 : p_2 \leq p_1$ .

If we let  $q_i = 1 - p_i$ , the joint pmf of  $X$  and  $Y$  is given by

$$\begin{aligned} \binom{m}{x} p_1^x q_1^{m-x} \binom{n}{y} p_2^y q_2^{n-y} &= \binom{m}{x} \binom{n}{y} q_1^m q_2^n \exp \left[ x \log \frac{p_1}{q_1} + y \log \frac{p_2}{q_2} \right] \\ &= \binom{m}{x} \binom{n}{y} q_1^m q_2^n \exp \left[ y \log \left( \frac{p_2/q_2}{p_1/q_1} \right) + (x+y) \log \frac{p_1}{q_1} \right]. \end{aligned}$$

Let  $\theta = \log \left( \frac{p_2/q_2}{p_1/q_1} \right)$ ,  $\vartheta = \log \frac{p_1}{q_1}$ ,  $U = Y$ ,  $T = X + Y$ . Note that  $H_0 : p_2 \leq p_1 \Leftrightarrow H_0 : \theta \leq 0$ . Using Theorem 15.6, we know that the UMPU test rejects  $H_0$  if  $Y$  is large conditional on  $X + Y = t$ .

Note that  $Y \mid X + Y = t$  has a hypergeometric distribution. With this information, we can compute  $c(t)$  and  $\gamma(t)$  in a similar manner to the previous example.

### 15.1.2.3 Example: Normal setting, testing variance

Let  $X_1, \dots, X_n$  iid,  $X_i \sim \mathcal{N}(\xi, \sigma^2)$ , both parameters unknown. Testing  $H_0 : \sigma \geq \sigma_0$  vs.  $H_1 : \sigma < \sigma_0$ .

We have

$$\begin{aligned} \text{joint density of } X \text{ \& } Y &\propto \exp \left[ -\frac{1}{2\sigma^2} \sum X_i^2 + \frac{\xi}{\sigma^2} \sum X_i \right] \\ &= \exp \left[ -\frac{1}{2\sigma^2} \sum X_i^2 + \frac{n\xi}{\sigma^2} \bar{X} \right]. \end{aligned}$$

Let  $\theta = -\frac{1}{2\sigma^2}$ ,  $\vartheta = \frac{n\xi}{\sigma^2}$ ,  $U = \sum X_i^2$ ,  $T = \bar{X}$ . Note that  $H_0 : \sigma \geq \sigma_0 \Leftrightarrow H_0 : \theta \geq \theta_0 = -\frac{1}{2\sigma_0^2}$ . Using Theorem 15.6, we know that the UMPU test rejects  $H_0$  if  $\sum X_i^2 \leq c(\bar{X})$ , where  $c(\bar{X})$  satisfies

$$\begin{aligned} P_{\sigma_0} \left\{ \sum X_i^2 \leq c(\bar{X}) \mid \bar{X} \right\} &= \alpha, \\ P_{\sigma_0} \left\{ \frac{\sum (X_i - \bar{X})^2}{\sigma_0^2} \leq c'(\bar{X}) \mid \bar{X} \right\} &= \alpha, \\ P_{\sigma_0} \left\{ \frac{\sum (X_i - \bar{X})^2}{\sigma_0^2} \leq c'' \right\} &= \alpha, \end{aligned}$$

since  $\bar{X}$  is independent of  $\sum (X_i - \bar{X})^2$ .

Since  $\frac{\sum (X_i - \bar{X})^2}{\sigma_0^2}$  has  $\chi_{n-1}^2$  distribution,  $c''$  will be equal to the  $\alpha$  quantile of  $\chi_{n-1}^2$ .

### 15.1.2.4 Example: Normal setting, testing mean

Let  $X_1, \dots, X_n$  iid,  $X_i \sim \mathcal{N}(\xi, \sigma^2)$ , both parameters unknown. Testing  $H_0 : \xi \leq 0$ .

As with the previous example, we have

$$\text{joint density of } X \text{ \& } Y \propto \exp \left[ -\frac{1}{2\sigma^2} \sum X_i^2 + \frac{n\xi}{\sigma^2} \bar{X} \right].$$

Let  $\theta = \frac{n\xi}{\sigma^2}$ ,  $\vartheta = -\frac{1}{2\sigma^2}$ ,  $U = \bar{X}$ ,  $T = \sum X_i^2$ . Note that  $H_0$  remains the same. Using Theorem 15.6, we know that the UMPU test rejects  $H_0$  if  $\bar{X} > c(\sum X_i^2)$ , with

$$P_0 \left\{ \bar{X} > c \left( \sum X_i^2 \right) \mid \sum X_i^2 \right\} = \alpha.$$

To find  $c$ , we use the following general fact:

**Proposition 15.7** Suppose we are in an exponential family model with densities  $\propto \exp [\theta U + \sum \vartheta_i T_i]$ , and that we are testing  $\theta \leq \theta_0$ .

Suppose there exists a function  $V = h(U, T)$  which is independent of  $T$  when  $\theta = \theta_0$  and which is increasing in  $u$  for fixed  $T = t$ .

Then the UMPU test is given by

$$\varphi = \begin{cases} 1 & \text{if } V > c_0, \\ \gamma & \text{if } V = c_0, \\ 0 & \text{if } V < c_0, \end{cases} \quad \mathbb{E}_{\theta_0} \varphi = \alpha.$$

In our case, let

$$V = \frac{\bar{X}}{\sqrt{\sum (X_i - \bar{X})^2}} = \frac{U}{\sqrt{T - nU^2}}.$$

When  $\theta = 0$ ,  $V$  is ancillary as its distribution does not depend on  $\sigma$ . By Basu's Theorem,  $V$  is independent of  $\sum X_i^2$ . Also, it is clear that  $V$  is increasing in  $U$  when  $T$  is fixed.

Hence, we can apply Proposition 15.7 to conclude that the UMPU test rejects if  $V$  is greater than some constant, or equivalently, reject if

$$\frac{\sqrt{n}\bar{X}}{\sqrt{\sum (X_i - \bar{X})^2 / (n-1)}} > \tilde{c}.$$

Under  $\xi = 0$ , the LHS has a  $t_{n-1}$  distribution. Hence,  $\tilde{c}$  is the  $1 - \alpha$  quantile of the  $t_{n-1}$  distribution.

$x$

$x$