STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 3: January 17

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3.1 Martingales

Recall these definitions from last lecture:

Definition 3.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A* ***filtration*** *on this space is an increasing sequence of sub σ-algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$*

Definition 3.2 *A sequence of random variables $Z_1, Z_2, \dots$ is* ***adapted*** *to a filtration $\{\mathcal{F}_n\}$ if for all n, Z_n is $\mathcal{F}_n$-measurable.*

Definition 3.3 *An adapted sequence is called a* ***martingale*** *if:*

1. $\mathbb{E}|Z_n| < \infty$ *for all n, and*
2. $\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = Z_n$ *a.s. for all n.*

3.1.1 Examples of martingales

- Let $Y_1, Y_2, \dots$ be independent random variables with $\mathbb{E}Y_i = \mu_i$, and let $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$. Let $Z_n = \displaystyle\sum_{i=1}^n (Y_i - \mu_i)$. Then $\{Z_n\}$ is a martingale.
- Same setting as above. Assume further that $\text{Var}(Y_i) = \sigma_i^2 < \infty$. Let $S_n = \displaystyle\sum_{i=1}^n (Y_i - \mu_i)$, and let $Z_n = S_n^2 - \displaystyle\sum_{i=1}^n \sigma_i^2$. Then $\{Z_n\}$ is a martingale:

$$\begin{aligned}
\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[(S_n + (Y_{n+1} - \mu_{n+1}))^2 - \sum_{i=1}^{n+1} \sigma_i^2 \middle| \mathcal{F}_n \right] \\
&= \mathbb{E}\left[S_n^2 + 2S_n(Y_{n+1} - \mu_{n+1}) + (Y_{n+1} - \mu_{n+1})^2 \mid \mathcal{F}_n \right] - \sum_{i=1}^{n+1} \sigma_i^2 \\
&= S_n^2 + 2S_n \mathbb{E}[Y_{n+1} - \mu_{n+1}] + \mathbb{E}[(Y_{n+1} - \mu_{n+1})^2] - \sum_{i=1}^{n+1} \sigma_i^2 \\
&= S_n^2 - 0 + \sigma_{n+1}^2 - \sum_{i=1}^{n+1} \sigma_i^2 \\
&= Z_n.
\end{aligned}$$

- Let $Y_1, Y_2, \ldots$ be independent non-negative random variables with $\mathbb{E}Y_i = 1$ for all i. Let $Z_n = \prod_{i=1}^{n} Y_i$, $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$.

  Since Z_n is non-negative,

$$\mathbb{E}|Z_n| = \mathbb{E}Z_n = \mathbb{E}\left[\prod_{i=1}^{n} Y_i\right] = \prod_{i=1}^{n} \mathbb{E}Y_i = 1,$$

  i.e. $Z_n \in L^1$ for all n. We also have

$$\begin{aligned}\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\prod_{i=1}^{n+1} Y_i \mid \mathcal{F}_n\right] \\ &= \mathbb{E}[Z_n Y_{n+1} \mid \mathcal{F}_n] \\ &= Z_n \mathbb{E}[Y_{n+1}] \\ &= Z_n.\end{aligned}$$

  Hence, $\{Z_n\}$ is a martingale.

- If $X_1, X_2, \ldots$ i.i.d. and $M(\theta) = \mathbb{E}[e^{\theta X_i}] < \infty$ for some θ, then $Z_n = \frac{e^{\theta \sum_{i=1}^{n} X_i}}{M(\theta)^n}$ is a martingale. This can be seen from the previous example by setting $Z_n = \prod_{i=1}^{n} Y_i$, where $Y_i = \frac{e^{\theta X_i}}{M(\theta)}$. ($\theta$ is usually chosen such that $M(\theta) = 1$ and $\theta \neq 0$.)

3.2 Stopping Times

Definition 3.4 *Let $\{\mathcal{F}_n\}$ be a filtration. A random variable $T : \Omega \to \{1, 2, \ldots\} \cup \{\infty\}$ is a* ***stopping time adapted to*** *$\{\mathcal{F}_n\}$ if $\{T = n\} \in \mathcal{F}_n$ for all n.*

Intuitively, a stopping time is a decision rule for stopping a particular process at a random time, and stopping the process at time n depends only on information available at time n.

3.2.1 Examples of stopping times

- Let $Y_1, Y_2, \ldots$ be independent random variables, $S_n = \sum_{i=1}^{n} Y_i$, $\mathcal{F}_n = \sigma(Y_1, \ldots Y_n)$. ($S_0 = \mathbb{E}Y_1$, $\mathcal{F}_0 =$ trivial σ-algebra.) Let $T = \inf\{k : S_k \geq a\}$ for some given a. (Usual convention: infimum of of an empty set is ∞.)

  T is a stopping time:

$$\{T = n\} = \{S_0 < a, S_1 < a, \ldots, S_{n-1} < a, S_n \geq a\} \in \mathcal{F}_n.$$

- Let $T' = \sup\{k : S_k \geq a\}$. Then T' is *not* a stopping time. (We will prove this later).

3.2.2 Properties of stopping times

- If T is a stopping time, then $\{T \leq n\} = \bigcup_{i=1}^{n} \{T = i\} \in \mathcal{F}_n$.
- $\{T \geq n\} = \{T \leq n-1\}^c \in \mathcal{F}_{n-1}$.
- Let T be a stopping time. Take any $n \in \mathbb{N} = \{1, 2, \dots\}$. Let

$$S = \min\{T, n\} = \begin{cases} T & \text{if } T \leq n, \\ n & \text{if } T > n. \end{cases}$$

  Then S is a stopping time, denoted by $T \wedge n$.

  Proof:

  If $k < n$, $\{S = k\} = \{T = k\} \in \mathcal{F}_k$.

  If $k > n$, $\{S = k\} = \emptyset \in \mathcal{F}_k$.

  If $k = n$, $\{S = k\} = \{T \geq n\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$. ■

3.2.3 Stopped σ-algebras and random variables

Definition 3.5 *Let T be a stopping time w.r.t. some filtration $\{\mathcal{F}_n\}$. The **stopped σ-algebra of** T, denoted by $\mathcal{F}_T$, is the set of all $A \in \mathcal{F}$ such that for all $n \in \mathbb{N}$, $A \cap \{T = n\} \in \mathcal{F}_n$.*

You can think of $\mathcal{F}_T$ as "all events that are dependent only on information up to the stopping time". It is easy to verify that $\mathcal{F}_T$ is indeed a σ-algebra. This σ-algebra is *not* the same as the σ-algebra generated by T.

Example of a random variable in $\mathcal{F}_T$

Let $X_1, X_2, \dots$ be i.i.d. ± 1 valued random variables, and let $S_n = \sum X_i$, $S_0 = 0$. Let $a < 0 < b$ be 2 integers, and let $T = \inf\{k : S_k = a \text{ or } b\}$. Then the random variable S_T is $\mathcal{F}_T$-measurable.

Proof: S_T can be take 2 values (a or b). It suffices to show that $\{S_T = a\} \in \mathcal{F}_T$. But

$$\{S_T = a\} \cap \{T = n\} = \{S_n = a\} \cap \{T = n\} \in \mathcal{F}_n.$$

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Definition 3.6 *Let $\{\mathcal{F}_n\}$ be a filtration, $\{Z_n\}$ a sequence of random variables adapted to this filtration, and T a stopping time such that $T < \infty$ a.s..*

*The **stopped random variable** Z_T is defined as*

$$Z_T(\omega) := Z_{T(\omega)}(\omega),$$

that is,

$$Z_T = \sum_{n=1}^{\infty} Z_n 1_{\{T=n\}}.$$

Fact: Z_T is always $\mathcal{F}_T$-measurable.

3.3 Wald's Lemma (for Bounded Stopping Times)

Theorem 3.7 *Let $\{\mathcal{F}_n\}$ be a filtration, $\{Z_n\}$ a martingale adapted to this filtration, T a stopping time w.r.t. this filtration.*

Suppose that there exists some $N \in \mathbb{N}$ such that $T \leq N$ a.s. Then $\mathbb{E}Z_T = \mathbb{E}Z_1$.

Proof:

First, observe that when $m > n$, $\mathbb{E}[Z_m \mid \mathcal{F}_n] = Z_n$ by the tower property of conditional expectation.

Since $1_{\{T=n\}}$ is $\mathcal{F}_n$-measurable, by definition of conditional expectation, for any $n \leq N$,

$$\mathbb{E}[Z_N 1_{\{T=n\}}] = \mathbb{E}[\mathbb{E}[Z_N \mid \mathcal{F}_n] 1_{\{T=n\}}] = \mathbb{E}[Z_n 1_{\{T=n\}}].$$

Hence

$$\begin{aligned}
\mathbb{E}Z_T &= \mathbb{E}\left[\sum_{n=1}^{N} Z_n 1_{\{T=n\}}\right] \\
&= \sum_{n=1}^{N} \mathbb{E}\left[Z_n 1_{\{T=n\}}\right] \\
&= \sum_{n=1}^{N} \mathbb{E}\left[Z_N 1_{\{T=n\}}\right] \\
&= \mathbb{E}\left[Z_N \sum_{n=1}^{N} 1_{\{T=n\}}\right] \\
&= \mathbb{E}Z_N \\
&= \mathbb{E}Z_1.
\end{aligned}$$

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3.3.1 Example (Symmetric random walk)

Let $X_1, X_2, \ldots$ be i.i.d. ± 1 valued random variables with $P\{X_i = 1\} = P\{X_i = 1\} = \frac{1}{2}$. Let $S_n = \sum_{i=1}^{n} X_i$, $S_0 = 0$. $\{S_n\}$ is a martingale.

Let $a < 0 < b$ be 2 integers, and let $T = \inf\{k : S_k = a \text{ or } b\}$. We will show later that $T < \infty$ a.s.

We can use Wald's Lemma to show that $P(S_T = a) = \frac{b}{b-a}$.

Proof:

For any $n \in \mathbb{N}$, $T \wedge n$ is a bounded stopping time. Hence, by Wald's Lemma, $\mathbb{E}S_{T\wedge n} = \mathbb{E}S_1 = 0$.

Since T is finite a.s., for almost all $\omega \in \Omega$,

$$\begin{aligned}
S_{T\wedge n}(\omega) &= S_{(T\wedge n)(\omega)}(\omega) \\
&= S_{T(\omega)\wedge n}(\omega) \\
&\to S_{T(\omega)}(\omega)
\end{aligned}$$

as $n \to \infty$, i.e. $S_{T\wedge n} \stackrel{a.s.}{\to} S_T$. Moreover, $a \leq S_{T\wedge n} \leq b$ a.s. Hence, by the Bounded Convergence Theorem,

$$\mathbb{E}S_T = \lim_{n\to\infty} \mathbb{E}\left[S_{T\wedge n}\right] = 0.$$

Since S_T can only take on values a or b,

$$\begin{aligned}
\mathbb{E}S_T &= aP(S_T = a) + bP(S_T = b) \\
&= aP(S_T = a) + b[1 - P(S_T = a)], \\
P(S_T = a) &= \frac{b}{b-a}.
\end{aligned}$$

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