STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 5: January 24

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5.1 (Sub)martingale Convergence Theorem

We will develop the lemmas needed to prove the (sub)martingale convergence theorem in this lecture.

Lemma 5.1 *Let $\{Z_n, \mathcal{F}_n\}_{n=1}^\infty$ be a submartingale with $\mathbb{E}Z_1 \geq 0$. Let $X_1 = Z_1$, $X_n = Z_n - Z_{n-1}$ for $n \geq 2$. Let $U_1, U_2, \ldots$ be $\{0,1\}$-valued random variables such that for all n, U_n is $\mathcal{F}_{n-1}$-measurable ($\mathcal{F}_0$ = trivial σ-algebra).*

Let $\tilde{Z}_n = \sum_{i=1}^n U_i X_i$. Then $\mathbb{E}\tilde{Z}_n \leq \mathbb{E}Z_n$.

Proof: Since U_n is $\mathcal{F}_{n-1}$-measurable,

$$\begin{aligned}
\mathbb{E}\tilde{Z}_n &= \sum_{i=1}^n \mathbb{E}[U_i X_i] \\
&= \sum_{i=1}^n \mathbb{E}[X_i 1_{\{U_i = 1\}}] \\
&= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[X_i \mid \mathcal{F}_{i-1}] 1_{\{U_i=1\}}].
\end{aligned}$$

Note that $\mathbb{E}[X_i \mid \mathcal{F}_{i-1}] = \mathbb{E}[Z_i \mid \mathcal{F}_{i-1}] - Z_{i-1}$ is a non-negative random variable since $\{Z_n\}$ is a submartingale. Thus

$$\begin{aligned}
\mathbb{E}\tilde{Z}_n &= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[X_i \mid \mathcal{F}_{i-1}] 1_{\{U_i=1\}}] \\
&\leq \sum_{i=1}^n \mathbb{E}[\mathbb{E}[X_i \mid \mathcal{F}_{i-1}]] \\
&= \sum_{i=1}^n \mathbb{E}X_i \\
&= \mathbb{E}Z_n.
\end{aligned}$$

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5.1.1 Upcrossings

Let $\{Z_n, \mathcal{F}_n\}_{n=1}^{\infty}$ be a non-negative submartingale. Take any $b > 0$. For $i \in \mathbb{N}$, define

$$\tau_i = \begin{cases} 0 & \text{if } i = 0, \\ \inf\{n : n > \tau_{i-1}, Z_n = 0\} & \text{if } i \text{ is odd}, \\ \inf\{n : n > \tau_{i-1}, Z_n > b\} & \text{if } i \text{ is even}. \end{cases}$$

Definition 5.2 *If i is odd, the interval $\{\tau_i + 1, \tau_i + 2, \ldots \tau_{i+1}\}$ is called an **upcrossing of** $[0, b]$ **by the submartingale** $\{Z_n\}$.*

Lemma 5.3 *Let β_n be the number of completed upcrossings by time n. Then*

$$b \cdot \mathbb{E}\beta_n \leq \mathbb{E}Z_n.$$

Proof: Let $U_i = I\{i \in \text{some upcrossing}\}$. Note that U_i is $\mathcal{F}_{i-1}$-measurable.

Let $X_1 = Z_1$, $X_n = Z_n - Z_{n-1}$ for $n \geq 2$.

For any complete upcrossing A, we have $\sum_{i \in A} X_i \geq b$. If A is an incomplete upcrossing, $\sum_{i \in A} X_i \geq 0$. Therefore $\sum_{i=1}^{n} U_i X_i \geq b\beta_n$.

By Lemma 5.1,

$$b\mathbb{E}\beta_n \leq \mathbb{E}\left[\sum_{i=1}^{n} U_i X_i\right] \leq \mathbb{E}Z_n.$$

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5.1.2 Proof of (Sub)martingale Convergence Theorem

Theorem 5.4 ((Sub)martingale Convergence Theorem) *Let $\{Z_n, \mathcal{F}_n\}_{n=1}^{\infty}$ be a submartingale. Suppose $\sup_n \mathbb{E}Z_n^+ < \infty$.*

Then there is a random variable Z taking values in $[-\infty, \infty)$ such that $Z_n \to Z$ a.s.

Proof: Recall that $\{Z_n, \mathcal{F}_n\}_{n=1}^{\infty}$ is a submartingale with $\sup_n \mathbb{E}Z_n^+ < \infty$. Let $M = \sup_n \mathbb{E}Z_n^+ < \infty$.

In the previous lecture, we reduced the problem to showing that for any $a < b$ with $a, b \in \mathbb{Q}$,

$$P\{\liminf Z_n < a, \text{ and } \limsup Z_n > b\} = 0.$$

Fix some $a < b$. Let $Z_n^* := (Z_n - a)^+$. Since $f(x) = (x - a)^+$ is a non-decreasing convex function, $\{Z_n^*\}$ is a non-negative submartingale.

If $\liminf Z_n < a$ and $\limsup Z_n > b$, then Z_n^* has infinitely many upcrossings of $[0, b - a]$. Let β_n be the number of complete upcrossings of $[0, b-a]$ by $\{Z_n^*\}$ within time n, and let β be the total number of complete crossings by $\{Z_n^*\}$. Then, $\beta_n \nearrow \beta$ and $\mathbb{E}\beta = \lim \mathbb{E}\beta_n$ by the Monotone Convergence Theorem.

By Lemma 5.3,

$$\begin{aligned}\mathbb{E}\beta_n &\leq \frac{1}{(b-a)}\mathbb{E}Z_n^* \\ &= \frac{\mathbb{E}(Z_n - a)^+}{b-a} \\ &\leq \frac{\mathbb{E}Z_n^+ + |a|}{b-a} \\ &\leq \frac{M + |a|}{b-a}\end{aligned}$$

for all n. Hence, $\mathbb{E}\beta < \infty$ and so $\beta < \infty$ a.s.. This implies that

$$P\{\liminf Z_n < a, \text{ and } \limsup Z_n > b\} = 0,$$

as required. ∎

Corollary 5.5 *If* $\{Z_n\}$ *is a non-negative martingale, then* $\lim Z_n$ *exists a.s. in* $[0, \infty]$.

Proof: $\{-Z_n\}$ is a submartingale, and $(-Z_n)^+ = 0$ a.s., implying that $\sup \mathbb{E}[(-Z_n)^+] = 0$.

Thus, $\lim(-Z_n)$ exists a.s. in $[-\infty, \infty)$. ∎

5.2 Pólya's Urn Model

An urn contains one black and one white ball. Each time, you pick a ball at random and put it back into the urn, together with another ball of the same color. (Number of balls at time n is $n+2$.)

Let B_n be the proportion of black balls at time n, $\mathcal{F}_n$ be the σ-algebra generated by all events up to time n. Then

$$\begin{aligned}\mathbb{E}[B_{n+1} \mid \mathcal{F}_n] &= (B_{n+1} \mid \text{black ball picked})P\{\text{black ball picked}\} \\ &\quad + (B_{n+1} \mid \text{white ball picked})P\{\text{white ball picked}\} \\ &= \frac{(n+2)B_n + 1}{n+3} \cdot B_n + \frac{(n+2)B_n}{n+3} \cdot (1 - B_n) \\ &= \frac{(n+2)B_n^2 + B_n + (n+2)B_n - (n+2)B_n^2}{n+3} \\ &= B_n.\end{aligned}$$

Thus, $\{B_n, \mathcal{F}_n\}$ is a non-negative martingale. By the Martingale Convergence Theorem, $B = \lim B_n$ exists. In fact, $B \sim \text{Unif}[0, 1]$.

5.3 Uniform Integrability

Definition 5.6 *A sequence of random variables* $\{X_n\}$ *is* ***uniformly integrable*** *if:*

1. $\mathbb{E}|X_n| < \infty$ *for all* n, *and*

2.
$$\lim_{a\to\infty} \sup_n \mathbb{E}(|X_n|; |X_n| \geq a) = 0.$$

A single random variable X with $\mathbb{E}|X| < \infty$ is uniformly integrable: the conditions hold by the Dominated Convergence Theorem.

Lemma 5.7 *If $\{X_n\}$ is uniformly integrable, then $\sup_n \mathbb{E}|X_n| < \infty$.*

Proof: Choose a such that $\mathbb{E}(|X_n|; |X_n| \geq a) \leq 1$ for all n. Then

$$\mathbb{E}|X_n| \leq a + \mathbb{E}(|X_n|; |X_n| \geq a) \leq a + 1.$$

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