STATS 300A: Theory of Statistics I **Autumn 2016/17**

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15.1 UMPU Tests for Multi-parameter Exponential Families

Recall that if we are testing $H_0 : \theta \in \Omega_0$ vs. $H_1 : \theta \in \Omega_1$, a UMPU test φ must satisfy 2 constraints:

1. the level constraint: $\mathbb{E}_\theta \varphi \leq \alpha$ for all $\theta \in \Omega_0$, and
2. the unbiasedness constraint: $\mathbb{E}_\theta \varphi \geq \alpha$ for all $\theta \in \Omega_1$.

For the rest of this lecture, we will assume that the background model is a multi-parameter exponential family with densities

$$p_{\theta,\vartheta}(x) \propto \exp\left[\theta U(x) + \sum_{i=1}^d \vartheta_i T_i(x)\right] h(x).$$

We write densities in a slightly different form from the usual to highlight θ as the parameter of interest. In this context, the ϑ_i's are **nuisance parameters**.

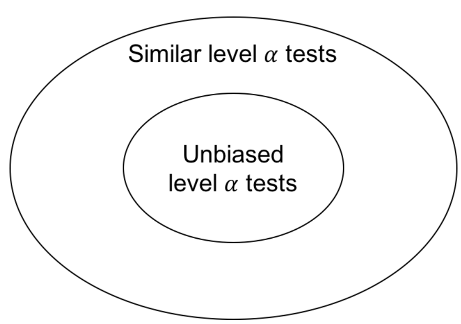
We will be testing $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ for some fixed θ_0. (Note that in this set-up, $H_0 : \theta = \theta_0$ is *not* a simple hypothesis: ϑ_i's could vary for fixed $\theta = \theta_0$.)

15.1.1 Similar Tests

Definition 15.1 *When testing $H_0 : \theta \in \Omega_0$ vs. $H_1 : \theta \in \Omega_1$, define the* ***boundary parameter space*** *as $\omega := \bar{\Omega}_0 \cap \bar{\Omega}_1$ (i.e. the intersection of the closures).*

Definition 15.2 *A test φ is* ***similar*** *if it satisfies $\mathbb{E}_\theta \varphi = \alpha$ for all $\theta \in \omega$.*

Suppose we know that the power function of any test is continuous (as it is in exponential family models). Then a level α test which is unbiased must also be similar. Hence, assuming continuity of power functions, unbiased tests form a subset of similar tests:



We will find UMP tests among all similar tests. If the UMP test we found is unbiased as well, then it will be UMPU.

Definition 15.3 *A test satisfying*

$$\mathbb{E}_{\theta_0}[\varphi(X) \mid T(X) = t] = \alpha \tag{15.1}$$

for all t is said to have ***Neyman structure*** *with respect to T.*

It is clear that if a test function φ satisfies Equation 15.1 for all t, then $\mathbb{E}_{\theta_0}\varphi(X) = \alpha$.

The following theorem gives us a large class of functions which have Neyman structure:

Theorem 15.4 *If T is complete and sufficient for ω, then every similar test has Neyman structure.*

Proof: Suppose φ is a similar test, i.e. $\mathbb{E}_\theta\varphi(X) - \alpha = 0$ for all $\theta \in \omega$.

Let $\psi(t) = \mathbb{E}[\varphi(X) - \alpha \mid T(X) = t]$. Then $\mathbb{E}\psi(T) = 0$. By completeness, we conclude that $\psi(T) = 0$ almost surely.

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(Remark: The proof still works if T is assumed only to be boundedly complete.)

Proposition 15.5 *Assume that we are in an exponential family model, i.e.*

$$p_{\theta,\vartheta}(x) \propto \exp\left[\theta U(x) + \sum_{i=1}^{d} \vartheta_i T_i(x)\right] h(x).$$

Then:

1. *With $\theta = \theta_0$ fixed, $(T_1, \ldots, T_d)$ is sufficient, and also complete if $\{\vartheta_1, \ldots, \vartheta_d\}$ contains a d-dimensional rectangle.*
2. *$T = (T_1, \ldots, T_d)$ has an exponential family of distributions.*
3. *The conditional distribution of $U \mid T = t$ is a 1-parameter exponential family. (By sufficiency, this exponential family does not depend on ϑ.)*

Proof: We only prove 3 for the discrete case. In this case,

$$\begin{aligned}
P_\theta\{U(X) = u \mid T(X) = t\} &= \frac{P_{\theta,\vartheta}(U = u, T = t)}{P_{\theta,\vartheta}(T = t)} \\
&= \frac{\displaystyle\sum_{x:U(x)=u,T(x)=t} C(\theta,\vartheta)\exp\left[\theta u + \sum \vartheta_i t_i\right] h(x)}{\displaystyle\sum_{x:T(x)=t} C(\theta,\vartheta)\exp\left[\theta u(x) + \sum \vartheta_i t_i\right] h(x)} \\
&= e^{\theta u} \cdot \frac{\displaystyle\sum_{x:U(x)=u,T(x)=t} h(x)}{\displaystyle\sum_{x:T(x)=t} h(x)}.
\end{aligned}$$

Since the fraction does not depend on θ, it is a 1-parameter exponential family.

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Theorem 15.6 *In the above $(d+1)$-parameter exponential family, assume that for fixed $\theta = \theta_0$, the family is of full-rank (i.e. $(T_1, \ldots, T_d)$ complete for ω).*

For testing $H_0 : \theta \le \theta_0$ vs. $H_1 : \theta > \theta_0$, there exists UMPU level α test of the form

$$\varphi(u,t) = \begin{cases} 1 & \text{if } u > c(t), \\ \gamma(t) & \text{if } u = c(t), \\ 0 & \text{if } u < c(t), \end{cases}$$

where $u = U(x)$, $t = T(x)$, $c(t)$ and $\gamma(t)$ are determined such that $\mathbb{E}_{\theta_0}[\varphi(U,T) \mid T] = \alpha$.

Proof: Fix any alternative (θ', ϑ') with $\theta' > \theta_0$. For any level α test φ', the power of the test

$$\mathbb{E}_{\theta',\vartheta'}\varphi'(U,T) = \mathbb{E}\{\mathbb{E}[\varphi'(U,T) \mid T]\}.$$

If we can find φ' that maximizes $\mathbb{E}'_\theta[\varphi'(U,T) \mid T = t]$ for all t, we are done.

However, given $T = t$, we are in a 1-parameter exponential family model. Hence, we can construct the optimal condition test for each t.

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15.1.2 Examples of UMPU Tests

15.1.2.1 Example: Poisson setting

Let $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, X and Y independent. Testing $H_0 : \mu \le \lambda$.

The joint density of X and Y is given by

$$\begin{aligned} \frac{e^{-\lambda}\lambda^x}{x!} \cdot \frac{e^{-\mu}\mu^y}{y!} &= \frac{e^{-(\lambda+\mu)}}{x!\,y!} \exp[x \log\lambda + y \log\mu] \\ &= \frac{e^{-(\lambda+\mu)}}{x!\,y!} \exp[y(\log\mu - \log\lambda) + (x+y)\log\lambda]. \end{aligned}$$

Let $\theta = \log\dfrac{\mu}{\lambda}$, $\vartheta = \log\lambda$, $U = Y$, $T = X + Y$. Note that $H_0 : \mu \le \lambda \Leftrightarrow H_0 : \theta \le 0$. Using Theorem 15.6, we know that the UMPU test rejects H_0 if Y is large conditional on $X + Y = t$.

What is the conditional distribution of Y given $X + Y = t$ under $\theta = 0$? It is $Y \mid X + Y = t \sim \text{Binom}\left(t, \frac{1}{2}\right)$. Hence, the UMPU test is given by

$$\varphi(y,t) = \begin{cases} 1 & \text{if } y > c(t), \\ \gamma(t) & \text{if } y = c(t), \\ 0 & \text{if } y < c(t), \end{cases}$$

where $c(t)$ and $\gamma(t)$ are determined by

$$\sum_{j=c(t)+1}^{t} \binom{t}{j} \left(\frac{1}{2}\right)^t + \gamma(t) \binom{t}{c(t)} \left(\frac{1}{2}\right)^t = \alpha.$$

15.1.2.2 Example: Binomial setting

Let $X \sim \text{Binom}(m, p_1)$, $Y \sim \text{Binom}(n, p_2)$, X and Y independent. Testing $H_0 : p_2 \leq p_1$.

If we let $q_i = 1 - p_i$, the joint pmf of X and Y is given by

$$\begin{aligned}
\binom{m}{x} p_1^x q_1^{m-x} \binom{n}{y} p_2^x q_2^{m-x} &= \binom{m}{x}\binom{n}{y} q_1^m q_2^m \exp\left[x \log \frac{p_1}{q_1} + y \log \frac{p_2}{q_2}\right] \\
&= \binom{m}{x}\binom{n}{y} q_1^m q_2^m \exp\left[y \log \left(\frac{p_2/q_2}{p_1/q_1}\right) + (x+y) \log \frac{p_1}{q_1}\right].
\end{aligned}$$

Let $\theta = \log\left(\frac{p_2/q_2}{p_1/q_1}\right)$, $\vartheta = \log \frac{p_1}{q_1}$, $U = Y$, $T = X + Y$. Note that $H_0 : p_2 \leq p_1 \Leftrightarrow H_0 : \theta \leq 0$. Using Theorem 15.6, we know that the UMPU test rejects H_0 if Y is large conditional on $X + Y = t$.

Note that $Y \mid X + Y = t$ has a hypergeometric distribution. With this information, we can compute $c(t)$ and $\gamma(t)$ in a similar manner to the previous example.

15.1.2.3 Example: Normal setting, testing variance

Let $X_1, \ldots, X_n$ iid, $X_i \sim \mathcal{N}(\xi, \sigma^2)$, both parameters unknown. Testing $H_0 : \sigma \geq \sigma_0$ vs. $H_1 : \sigma < \sigma_0$.

We have

$$\begin{aligned}
\text{joint density of } X \text{ \& } Y &\propto \exp\left[-\frac{1}{2\sigma^2}\sum X_i^2 + \frac{\xi}{\sigma^2}\sum X_i\right] \\
&= \exp\left[-\frac{1}{2\sigma^2}\sum X_i^2 + \frac{n\xi}{\sigma^2}\bar{X}\right].
\end{aligned}$$

Let $\theta = -\frac{1}{2\sigma^2}$, $\vartheta = \frac{n\xi}{\sigma^2}$, $U = \sum X_i^2$, $T = \bar{X}$. Note that $H_0 : \sigma \geq \sigma_0 \Leftrightarrow H_0 : \theta \geq \theta_0 = -\frac{1}{2\sigma_0^2}$. Using Theorem 15.6, we know that the UMPU test rejects H_0 if $\sum X_i^2 \leq c(\bar{X})$, where $c(\bar{X})$ satisfies

$$\begin{aligned}
P_{\sigma_0}\left\{\sum X_i^2 \leq c(\bar{X}) \mid \bar{X}\right\} &= \alpha, \\
P_{\sigma_0}\left\{\frac{\sum (X_i - \bar{X})^2}{\sigma_0^2} \leq c'(\bar{X}) \mid \bar{X}\right\} &= \alpha, \\
P_{\sigma_0}\left\{\frac{\sum (X_i - \bar{X})^2}{\sigma_0^2} \leq c''\right\} &= \alpha,
\end{aligned}$$

since $\bar{X}$ is independent of $\sum (X_i - \bar{X})^2$.

Since $\frac{\sum (X_i - \bar{X})^2}{\sigma_0^2}$ has χ^2_{n-1} distribution, c'' will be equal to the α quantile of χ^2_{n-1}.

15.1.2.4 Example: Normal setting, testing mean

Let $X_1, \ldots, X_n$ iid, $X_i \sim \mathcal{N}(\xi, \sigma^2)$, both parameters unknown. Testing $H_0 : \xi \leq 0$.

As with the previous example, we have

$$\text{joint density of } X \ \& \ Y \propto \exp\left[-\frac{1}{2\sigma^2}\sum X_i^2 + \frac{n\xi}{\sigma^2}\bar{X}\right].$$

Let $\theta = \frac{n\xi}{\sigma^2}$, $\vartheta = -\frac{1}{2\sigma^2}$, $U = \bar{X}$, $T = \sum X_i^2$. Note that H_0 remains the same. Using Theorem 15.6, we know that the UMPU test rejects H_0 if $\bar{X} > c\left(\sum X_i^2\right)$, with

$$P_0\left\{\bar{X} > c\left(\sum X_i^2\right) \mid \sum X_i^2\right\} = \alpha.$$

To find c, we use the following general fact:

Proposition 15.7 *Suppose we are in an exponential family model with densities* $\propto \exp\left[\theta U + \sum \vartheta_i T_i\right]$, *and that we are testing* $\theta \leq \theta_0$.

Suppose there exists a function $V = h(U, T)$ *which is independent of* T *when* $\theta = \theta_0$ *and which is increasing in* u *for fixed* $T = t$.

Then the UMPU test is given by

$$\varphi = \begin{cases} 1 & \text{if } V > c_0, \\ \gamma & \text{if } V = c_0, \\ 0 & \text{if } V < c_0, \end{cases} \qquad \mathbb{E}_{\theta_0}\varphi = \alpha.$$

In our case, let

$$V = \frac{\bar{X}}{\sqrt{\sum(X_i - \bar{X})^2}} = \frac{U}{\sqrt{T - nU^2}}.$$

When $\theta = 0$, V is ancillary as its distribution does not depend on σ. By Basu's Theorem, V is independent of $\sum X_i^2$. Also, it is clear that V is increasing in U when T is fixed.

Hence, we can apply Proposition 15.7 to conclude that the UMPU test rejects if V is greater than some constant, or equivalently, reject if

$$\frac{\sqrt{n}\bar{X}}{\sqrt{\sum(X_i - \bar{X})^2/(n-1)}} > \tilde{c}.$$

Under $\xi = 0$, the LHS has a t_{n-1} distribution. Hence, $\tilde{c}$ is the $1 - \alpha$ quantile of the t_{n-1} distribution.