**STATS 310B: Theory of Probability II** **Winter 2016/17**

Lecture 17: March 7

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## 17.1 Renewal Theory

Let $X_1, X_2, \ldots$ be i.i.d. non-negative random variables with $P(X_1 = 0) < 1$. Let $\mu = \mathbb{E}X_1$ (possibly infinite), $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_n$.

**Definition 17.1** *For $t \geq 0$ with $t \in \mathbb{R}$, let $N(t) := \sup\{n : S_n \leq t\}$. Then $\{N(t) : t \geq 0\}$ is called a* ***renewal process****.*

We can think of this process as replacing lightbulbs, $X_i$ is the lifetime of the $i^{th}$ lightbulb, and when it dies, we replace it with a new lightbulb. $S_n$ can be thought of as the time till the $n^{th}$ lightbulb goes off.

**Definition 17.2** $m(t) := \mathbb{E}[N(t)]$ *is called the* ***renewal function****.*

Before proving the Elementary Renewal Theorem, we introduce a few lemmas that we will use in the proof.

**Lemma 17.3** *Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. For any $t$, $N(t) + 1$ is a stopping time w.r.t. $\mathcal{F}_n$.*

**Proof:**

$$\begin{aligned}
\{N(t) + 1 = n\} &= \{N(t) = n - 1\} \\
&= \{S_1, S_2, \ldots, S_{n-1} \leq t, S_n > t\} \\
&\in \mathcal{F}_n.
\end{aligned}$$

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**Lemma 17.4** *For all $t$, $m(t) < \infty$.*

**Proof:** Let $p = \mathbb{E}\left[e^{-X_1}\right]$. Since $X_1$ is non-negative, $1 - e^{-X_1} \geq 0$. If $p = 1$, then $\mathbb{E}\left[1 - e^{-X_1}\right] = 0$, which implies that $X_1 = 0$ a.s. But this is not true by assumption! Thus $p < 1$.

Note that

$$\begin{aligned}
P(N(t) = n) &\leq P(S_n \leq t) \\
&= P\left(e^{-S_n} \geq e^{-t}\right) \\
&\leq e^t \mathbb{E}\left[e^{-S_n}\right] && \text{(Markov)} \\
&= e^t \left(\mathbb{E}\left[e^{-X_1}\right]\right)^n \\
&= e^t p^n,
\end{aligned}$$

so

$$\begin{aligned} m(t) &= \sum_{n=1}^{\infty} nP(N(t)=n) \\ &\le e^t \mathbb{E}[X] && \text{(where } X \sim \text{Geom}(p)) \\ &< \infty. \end{aligned}$$

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**Corollary 17.5** *If $\mu < \infty$, then $\mathbb{E}[S_{N(t)+1}] = \mu(m(t)+1)$.*

**Proof:** This follows directly from the 2 lemmas above and an application of Wald's equation for sums of i.i.d. random variables. ■

**Theorem 17.6 (Elementary Renewal Theorem)**

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

**Proof:** Note that by definition of $N(t)$, $S_{N(t)+1} > t$.

- If $\mu < \infty$, then by Corollary 17.5,

$$\begin{aligned} \mu(m(t)+1) = \mathbb{E}\left[S_{N(t)+1}\right] &\ge t, \\ \liminf_{t\to\infty} \frac{m(t)}{t} &\ge \frac{1}{\mu}. \end{aligned}$$

- If $\mu = \infty$, then the above is trivially true as the LHS is the lim inf of a non-negative sequence.

It remains to show that $\limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu}$. Let $M$ be a positive real number. Define a new renewal process as follows:

$$\bar{X}_n = \begin{cases} X_n & \text{if } X_n \le M, \\ M & \text{if } X_n > M. \end{cases}$$

Define $\bar{S}_n = \sum_{i=1}^{n} \bar{X}_i$, $\bar{N}(t) = \sup\{n : \bar{S}_n \le t\}$, $\bar{m}(t) = \mathbb{E}[\bar{N}(t)]$. Define $\mu_M = \mathbb{E}\bar{X}_1 \le M < \infty$.

Then $\bar{S}_{\bar{N}(t)+1} \le t + M$. By Corollary 17.5,

$$\begin{aligned} \mu_M(\bar{m}(t)+1) \le \mathbb{E}\left[\bar{S}_{\bar{N}(t)+1}\right] &\le t + M, \\ \limsup_{t\to\infty} \frac{\bar{m}(t)}{t} &\le \frac{1}{\mu_M}. \end{aligned}$$

Note that $\bar{S}_n \le S_n$ for all $n$, which implies that $\bar{N}(t) \ge N(t)$ for all $t$, and so

$$\limsup \frac{m(t)}{t} \le \limsup \frac{\bar{m}(t)}{t} \le \frac{1}{\mu_M}.$$

As $M \to \infty$, $0 \le \bar{X}_1 \nearrow X_1$. Thus, by the Monotone Convergence Theorem, $\mu_M \to \mu$. This gives us the desired result.

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## 17.2 Time Homogeneous Markov Chains on Countable State Spaces

Recall that $p_{xy}^{(n)} = P(X_n = y \mid X_0 = x)$, $\mu_{xy} = \mathbb{E}[\text{time to hit } y \text{ starting from } x \text{ after time } 0]$.

**Theorem 17.7** *If $x$ is a recurrent state, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{xx}^{(m)} = \frac{1}{\mu_{xx}}.$$

**Proof:** Suppose that the chain starts at $x$. Then the return times to $x$ form a renewal process with i.i.d. increments (by the Strong Markov Property). Hence,

$$\frac{1}{n} \sum_{k=1}^{n} p_{xx}^{(k)} = \frac{1}{n} \mathbb{E}_x[\text{no. of returns to } x \text{ by time } n]$$

is just $\dfrac{m(n)}{n}$ for this renewal process. Also, the inter-arrival times for this renewal process have expected value $\mu_{xx}$.

Thus, we have the result by applying the Elementary Renewal Theorem. ■

**Corollary 17.8** *If $x$ and $y$ communicate ($x \leftrightarrow y$) and are recurrent, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{yx}^{(m)} = \frac{1}{\mu_{xx}}.$$

**Proof:**

**Step 1:** $\limsup_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{yx}^{(m)} \le \frac{1}{\mu_{xx}}$**.**

Since $x \leftrightarrow y$ and are recurrent, the chain starting at $y$ is sure to hit $x$ eventually. (You can try to prove this.)

With the Strong Markov property, this implies that

$$\mathbb{E}_y[\text{no. of visits to } x \text{ by time } n] \le \mathbb{E}_x[\text{no. of visits to } x \text{ by time } n] + 1.$$

Applying Theorem 17.7 gives

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{yx}^{(m)} \le \frac{1}{\mu_{xx}}.$$

This completes the proof if $\mu_{xx} = \infty$. Assume from now on that $\mu_{xx} < \infty$.

**Step 2: If $T_x$ is the first hitting time of $x$, then $\mu_{yx} = \mathbb{E}_y[T_x] < \infty$.**

Since $x \leftrightarrow y$, there exists $x_0 = x, x_1, \ldots, x_k = y$ such that $p_{x_0 x_1} p_{x_1 x_2} \cdots p_{x_{k-1} x_k} > 0$. Without loss of generality, $x_1, x_2, \ldots, x_{k-1} \ne x$. Then

$$\begin{aligned}
\mu_{xx} &= \mathbb{E}_x[T_x] \\
&\ge \mathbb{E}_x[T_x; X_1 = x_1, \ldots, X_{k-1} = x_{k-1}, X_k = y] \\
&= \mathbb{E}_x[T_x \mid X_1 = x_1, \ldots, X_{k-1} = x_{k-1}, X_k = y] P_x(X_1 = x_1, \ldots, X_{k-1} = x_{k-1}, X_k = y) \\
&= (\mu_{yx} + k) P_x(X_1 = x_1, \ldots, X_{k-1} = x_{k-1}, X_k = y).
\end{aligned}$$

Since the probability on the RHS is positive, it follows that $\mu_{yx} < \infty$.

**Step 3:** $\liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{yx}^{(m)} \geq \frac{1}{\mu_{xx}}$**.**

$$
\begin{aligned}
\sum_{m=1}^{n} p_{yx}^{(m)} &= \mathbb{E}_y[\text{no. of visits to } x \text{ by time } n] \\
&\geq \mathbb{E}_y[\text{no. of visits to } x \text{ in time interval } [T_x+1, T_x+n] - T_x] \\
&= \sum_{m=1}^{n} p_{xx}^{(m)} - \mu_{yx}. && \text{(by Strong Markov Property)}
\end{aligned}
$$

Since $\mu_{yx} < \infty$, this shows that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_{yx}^{(m)} \geq \frac{1}{\mu_{xx}}.$$

This completes the proof. ■