STATS 310B: Theory of Probability II **Winter 2016/17**

Lecture 15: February 28

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15.1 Time Homogeneous Markov Chains on Countable State Spaces

Let $\{X_n\}_{n\geq 0}$ be a time-homogeneous Markov chain taking values on a countable state space S. Recall the following definitions:

Definition 15.1 *For $x \in S$, the **first hitting time** of x is $T_x := \inf\{n \geq 1 : X_n = x\}$. (Note: Time 0 doesn't count.)*

Definition 15.2 *Let $\rho_{xy} := P(T_y < \infty \mid X_0 = x)$. In particular, ρ_{xx} is the probability of ever returning to x given that the chain starts at x.*

*We say that a state x is **recurrent** if $\rho_{xx} = 1$, and is **transient** otherwise.*

Definition 15.3 *Let $N(x) = \sum_{n=1}^{\infty} 1_{\{X_n = x\}}$, i.e. the **number of visits** to x (except time 0).*

Theorem 15.4 *The following are equivalent:*

(a) x is recurrent.

(b) $\mathbb{E}_x N(x) = \infty$, where $\mathbb{E}_x$ means $\mathbb{E}[\cdot \mid X_0 = x]$.

(c) $P_x(N(x) = \infty) = 1$.

Proof: $(a) \Rightarrow (c)$: x is recurrent implies that the chain returns to x a.s. By the strong Markov property, the chain after that returns again a.s., and so on.

More formally, let T_x^k be the time of the k^{th} return. Let $g(y) := P_y(T_x^1 < \infty)$. By definition of recurrence, $g(x) = 1$. Hence,

$$P_x\left(T_x^2 < \infty \mid \mathcal{F}_{T_x^1}\right) = g(X_{T_x^1}) = g(x) = 1,$$

so $P_x(T_x^2 < \infty) = 1$. Similarly, $P(T_x^k < \infty) = 1$ for all k, and so $P_x(N(x) = \infty) = 1$.

$(c) \Rightarrow (b)$: Trivial.

$(b) \Rightarrow (a)$: Note that we can write $N(x)$ as

$$N(x) = \sum_{k=1}^{\infty} 1_{\{T_x^k < \infty\}},$$

$$\Rightarrow \qquad \mathbb{E}_x N(x) = \sum_{k=1}^{\infty} P_x(T_x^k < \infty).$$

Note that $1 \le T_x^1 \le T_x^2 \le \ldots$, so

$$\begin{aligned} P_x(T_x^k < \infty) &= \mathbb{E}_x\left[P(T_x^k < \infty) \mid \mathcal{F}_{T_x^{k-1}}\right] && \text{(tower property)} \\ &= \mathbb{E}_x\left[g(X_{T_x^{k-1}}); T_x^{k-1} < \infty\right] \\ &= \mathbb{E}_x\left[g(x); T_x^{k-1} < \infty\right] \\ &= \rho_{xx} P(T_x^{k-1} < \infty), \\ \Rightarrow \qquad P_x(T_x^k < \infty) &= \rho_{xx}^k. \end{aligned}$$

This implies that

$$\mathbb{E}_x N(x) = \sum_{k=1}^{\infty} \rho_{xx}^k.$$

Since $\mathbb{E}_x N(x) = \infty$, we must have $\rho_{xx} = 1$, i.e. x is recurrent.

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15.2 Recurrence/Transience of Simple Random Walk (Pólya)

Theorem 15.5 *Let S_n be a simple symmetric random walk on $\mathbb{Z}^d$. Then 0 (or any other state) is recurrent if $d = 1$ or 2, and transient if $d \ge 3$.*

Proof: Let $p_n = P(S_n = 0 \mid S_0 = 0)$. By Theorem 15.4, 0 is recurrent if and only if $\mathbb{E}_0 N(0) = \sum_{n=1}^{\infty} p_n = \infty$.

Case 1: $d = 1$. Note that $p_n = 0$ if n is odd. Suppose that n is even. Then, since there must be exactly $n/2$ +1's and $n/2$ −1's, $p_n = \binom{n}{n/2} 2^{-n}$.

We can use Stirling's approximation, $n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$, to approximate p_n:

$$\begin{aligned} p_n = \binom{n}{n/2} 2^{-n} &= \frac{n!}{\left((n/2)!\right)^2} 2^{-n} \\ &\sim \frac{\sqrt{2\pi} n^{n+1/2} e^{-n}}{\left(\sqrt{2\pi}(n/2)^{n/2+1/2} e^{-n/2}\right)^2} 2^{-n} \\ &= \frac{2}{\sqrt{2\pi n}} \\ &= \sqrt{\frac{2}{\pi n}}. \end{aligned}$$

This implies that $\sum_{n=1}^{\infty} p_n = \infty$.

Case 2: $d = 2$.

If we rotate the lattice by 45°, the random walk remains the same, but the possible moves from $(0,0)$ are $(1,1)$, $(1,-1)$, $(-1,1)$ or $(-1,-1)$. Viewing the random walk in this way, the coordinates of S_n are performing independent simple random walks on $\mathbb{Z}$!

Therefore $p_n \sim \left(\sqrt{\frac{2}{\pi n}}\right)^2 = \frac{2}{\pi n}$, and hence $\sum_{n=1}^{\infty} p_n = \infty$.

Case 3: $d \geq 3$**.**

We will show that $p_n \leq C n^{-d/2}$, where C is a constant that does not depend on n. This will imply that $\sum_{n=1}^{\infty} p_n < \infty$.

We can write $S_n = X_1 + \cdots + X_n$, where $X_i \overset{iid}{\sim} \text{Unif}\{(1, 0 \ldots, 0), (-1, 0 \ldots, 0) \ldots, (0, \ldots, 0, 1)\}$.

Let's compute the characteristic function of S_n. Letting $t = (t_1, \ldots, r_d) \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{E}\left[e^{it\cdot X_1}\right] &= \frac{1}{2d}\sum_{j=1}^{d}(e^{it_j} + e^{-it_j}) \\
&= \frac{1}{d}\sum_{j=1}^{d}\cos t_j, \\
\mathbb{E}[e^{it\cdot S_n}] &= \left(\frac{1}{d}\sum_{j=1}^{d}\cos t_j\right)^d.
\end{aligned}$$

Note that for $x \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} e^{itx}dt = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Thus, if X is an integer-valued random variable,

$$P(X = 0) = \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2}\mathbb{E}[e^{itX}]dt.$$

This generalizes to d-dimensions: for $x \in \mathbb{Z}^d$,

$$\begin{aligned}
\frac{1}{(2\pi)^d}\int_{[-\pi/2, 3\pi/2]^d} e^{it\cdot x}dt &= \prod_{j=1}^{d}\left(\int_{-\pi/2}^{3\pi/2} e^{it_j x_j}dt_j\right) \\
&= 1_{\{x_1 = x_2 = \cdots = x_d = 0\}},
\end{aligned}$$

so if X is a $\mathbb{Z}^d$-valued random variable, then

$$P(X = 0) = \frac{1}{(2\pi)^d}\int_{[-\pi/2, 3\pi/2]^d}\mathbb{E}[e^{it\cdot X}]dt.$$

Using this identity for S_n, we get

$$\begin{aligned}
P(S_n = 0) &= \frac{1}{(2\pi)^d}\int_{[-\pi/2, 3\pi/2]^d}\left(\frac{1}{d}\sum_{j=1}^{d}\cos t_j\right)^n dt \\
&\leq \frac{1}{(2\pi)^d}\int_{[-\pi/2, 3\pi/2]^d}\left(\frac{1}{d}\sum_{j=1}^{d}|\cos t_j|\right)^n dt
\end{aligned}$$

Note that the graph of $y = |\cos x|$ is the same on the intervals $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, so

$$\frac{1}{(2\pi)^d} \int_{[-\pi/2, 3\pi/2]^d} \left(\frac{1}{d} \sum_{j=1}^{d} |\cos t_j| \right)^n dt = \frac{2^d}{(2\pi)^d} \int_{[-\pi/2, \pi/2]^d} \left(\frac{1}{d} \sum_{j=1}^{d} |\cos t_j| \right)^n dt.$$

Look at the graph $y = |\cos x|$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. It is an easy fact that there is some $c > 0$ such that $|\cos x| \leq 1 - cx^2$ for all x in this interval. With this bound, we get

$$\begin{aligned}
P(S_n = 0) &\leq \frac{2^d}{(2\pi)^d} \int_{[-\pi/2, \pi/2]^d} \left(\frac{1}{d} \sum_{j=1}^{d} (1 - ct_j^2) \right)^n dt \\
&= \frac{1}{\pi^d} \int_{[-\pi/2, \pi/2]^d} \left(1 - \frac{c \sum_{j=1}^{d} t_j^2}{d} \right)^n dt \\
&\leq \frac{1}{\pi^d} \int_{[-\pi/2, \pi/2]^d} \exp\left(-cn \sum t_j^2 / d \right) dt \qquad \text{(since } 1 - x \leq e^{-x}\text{)} \\
&\leq \frac{1}{\pi^d} \int_{\mathbb{R}^d} \exp\left(-cn \sum t_j^2 / d \right) dt \\
&= \frac{1}{\pi^d} \left(\int_{\mathbb{R}} \exp\left(-cnt^2 / d \right) dt \right)^d \\
&\sim \mathit{Constant} \cdot n^{-d/2},
\end{aligned}$$

as required. ■