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Question 1.

(a)

$$\begin{aligned} p(-1) &= 4 \\ p(0) &= 6 \\ p(1) &= 12 \end{aligned} \iff \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix}$$

Eliminate 1st column:

$$P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_1 A = A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

Eliminate 2nd column:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_2 L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \qquad L_2 P_2 L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} L_2 P_2 L_1 P_1 A = U \iff& L_2 P_2 L_1 P_2 P_2 P_1 A = U \\ \iff& L_2 (P_2 L_1 P_2) P_2 P_1 A = U \\ \iff& L_2 \widetilde{L_1} P_2 P_1 A = U \\ \iff& P_2 P_1 A = \widetilde{L_1}^{-1} L_2^{-1} U \\ \iff& PA = LU \end{aligned}$$

$$\widetilde{L_1} = P_2 L_1 P_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad L = \widetilde{L_1}^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix}$$

$$P = P_2 P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} A\vec{x} = \vec{b} &\iff PA\vec{x} = P\vec{b} \\ &\iff LU\vec{x} = P\vec{b} \\ &\iff L(U\vec{x}) = P\vec{b} \\ &\iff L\vec{d} = P\vec{b} \end{aligned}$$

$$P\vec{b} = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$$

Forward solve $L\vec{d} = \vec{b}$ for $\vec{d}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$$

$d_1 = 4$

$d_1 + d_2 = 12$

$d_1 + \frac{1}{2}d_2 + d_3 = 6$

$d_1 = 4$

$d_2 = 8$

$d_3 = -2$

Backward solve $U\vec{x} = \vec{d}$ for $\vec{x}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -2 \end{bmatrix}$$

$x_3 = 2$

$x_2 = 4$

$x_1 = 6$

$\therefore p(x) = 6 + 4x + 2x^2$

(b)

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \qquad i = 0,\ 1,\ 2$$

$$\begin{aligned} x_0 &= -1 & y_0 &= 4 \\ x_1 &= 0 & y_1 &= 6 \\ x_2 &= 1 & y_2 &= 12 \end{aligned}$$

$i = 0$

$j = 1,\ 2$

$$\begin{aligned} l_0(x) &= \left(\frac{x-0}{-1-0}\right)\left(\frac{x-1}{-1-1}\right) \\ &= (-x)\left(\frac{x-1}{-2}\right) \\ &= \frac{1}{2}x(x-1) \end{aligned}$$

$i = 1$

$j = 0,\ 2$

$$\begin{aligned} l_1(x) &= \left(\frac{x-(-1)}{0-(-1)}\right)\left(\frac{x-1}{0-1}\right) \\ &= (x+1)(1-x) \end{aligned}$$

$i = 2$

$j = 0,\ 1$

$$\begin{aligned} l_2(x) &= \left(\frac{x-(-1)}{1-(-1)}\right)\left(\frac{x-0}{1-0}\right) \\ &= \left(\frac{x+1}{2}\right)(x) \\ &= \frac{1}{2}x(x+1) \end{aligned}$$

$$\begin{aligned} p(x) &= \sum_{i=0}^{n} l_i(x) y_i \\ &= l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2 \\ &= \frac{1}{2}x(x-1)(4) + (x+1)(1-x)(6) + \frac{1}{2}x(x+1)(12) \\ &= 2x^2 - 2x + (-6x^2) + 6 + 6x^2 + 6x \\ &= 2x^2 + 4x + 6 \end{aligned}$$

(c)

x_i	$y[x_i]$	$y[x_{i+1},\ x_i]$	$y[x_{i+2},\ x_{i+1},\ x_i]$
-1	4		
		$\frac{6-4}{1-(-1)} = 2$	
0	6		$\frac{6-2}{1-(-1)} = 2$
		$\frac{12-6}{1-0} = 6$	
1	12		

$$\begin{aligned}
p(x) &= y[x_0] + (x - x_0)y[x_1,\ x_0] + (x - x_0)(x - x_1)y[x_2,\ x_1,\ x_0] \\
&= 4 + \big(x - (-1)\big)(2) + \big(x - (-1)\big)(x - 0)(2) \\
&= 4 + 2x + 2 + 2x^2 + 2x \\
&= 6 + 4x + 2x^2
\end{aligned}$$

(d)

As all the equations are simplified from (a), (b), and (c), we can see that all polynomials are identical.

(e)

We can use the method in (c) since all we have to do is to add to the table

x_i	$y[x_i]$	$y[x_{i+1},\ x_i]$	$y[x_{i+2},\ x_{i+1},\ x_i]$	$y[x_{i+3},\ x_{i+2},\ x_{i+1},\ x_i]$
-1	4			
		$\frac{6-4}{1-(-1)} = 2$		
0	6		$\frac{6-2}{1-(-1)} = 2$	
		$\frac{12-6}{1-0} = 6$		$\frac{-1-2}{2-(-1)} = -1$
1	12		$\frac{4-6}{2-0} = -1$	
		$\frac{16-12}{2-1} = 4$		
2	16			

$$
\begin{aligned}
p(x) &= y[x_0] + (x - x_0)y[x_1,\ x_0] \\
&\qquad + (x - x_0)(x - x_1)y[x_2,\ x_1,\ x_0] \\
&\qquad + (x - x_0)(x - x_1)(x - x_2)y[x_3,\ x_2,\ x_1,\ x_0] \\
&= 4 + \Big(x - (-1)\Big)(2) + \Big(x - (-1)\Big)(x - 0)(2) + \Big(x - (-1)\Big)(x - 0)(x - 1)(-1) \\
&= 4 + 2x + 2 + 2x^2 + 2x - x^3 + x \\
&= 6 + 5x + 2x^2 - x^3
\end{aligned}
$$

(f)

Equation of line from $(-1,\ 4)$ to $(0,\ 6)$:

$$\begin{aligned} \text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{6-4}{0-(-1)} \\ &= 2 \end{aligned}$$

$$\begin{aligned} y - y_1 = \text{slope}_1(x - x_1) &\iff y - 4 = 2\Big(x - (-1)\Big) \\ &\iff y = 2(x+1) + 4 \\ &\iff y = 2x + 6 \end{aligned}$$

Equation of line from $(0,\ 6)$ to $(1,\ 12)$:

$$\begin{aligned} \text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{12-6}{1-0} \\ &= 6 \end{aligned}$$

$$\begin{aligned} y - y_1 = \text{slope}_1(x - x_1) &\iff y - 6 = 6(x - 0) \\ &\iff y = 6x + 6 \end{aligned}$$

Equation of line from $(1,\ 12)$ to $(2,\ 16)$:

$$\begin{aligned} \text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{16-12}{2-1} \\ &= 4 \end{aligned}$$

$$\begin{aligned} y - y_1 = \text{slope}_1(x - x_1) &\iff y - 12 = 4(x - 1) \\ &\iff y = 4x + 8 \end{aligned}$$

$\therefore$ the equation of the linear spline is:

$$\begin{cases} y = 2x + 6 & \text{if } -1 \le x < 0 \\ y = 6x + 6 & \text{if } 0 \le x < 1 \\ y = 4x + 8 & \text{if } 1 \le x \le 2 \end{cases}$$

Question 2.

(a)
When we solve using the Vandermonde method, we get a polynomial of the form:

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{n-1}x^{n-1} + a_nx^n$$

We can factor this polynomial into the form:

$$p(x) = a_0 + x\Big(a_1 + x\Big(a_2 + x\Big(a_3 + \ldots + x(a_{n-1} + xa_n)\cdots\Big)\Big)\Big)$$

There are $n-1$ additions, and $n-1$ multiplications, so we have
$2n + \mathcal{O}(1)$ flops.

(b)

For divided-difference, we get a polynomial of the form:

$$\begin{aligned}p(x) = y[x_0] &+ (x - x_0)y[x_1,\ x_0] \\ &+ (x - x_0)(x - x_1)y[x_2,\ x_1,\ x_0] \\ &+ (x - x_0)(x - x_1)(x - x_2)y[x_3,\ x_2,\ x_1,\ x_0] \\ &\vdots \\ &+ (x - x_0)(x - x_1)\cdots(x - x_{n-3})(x - x_{n-2})y[x_{n-1},\ x_{n-2},\ \cdots,\ x_1,\ x_0] \\ &+ (x - x_0)(x - x_1)\cdots(x - x_{n-2})(x - x_{n-1})y[x_n,\ x_{n-1},\ \cdots,\ x_1,\ x_0]\end{aligned}$$

We can also factor this to be of the form:

$$\begin{aligned}p(x) = y[x_0] + (x - x_0)\Bigg(y[x_1,\ x_0] \\ &+ (x - x_1)\Bigg(y[x_2,\ x_1,\ x_0] \\ &+ (x - x_2)\Bigg(y[x_3,\ x_2,\ x_1,\ x_0] \\ &\vdots \\ &+ (x - x_{n-2})\Big(y[x_{n-1},\ x_{n-2},\ \cdots,\ x_1,\ x_0] \\ &+ (x - x_{n-1})(y[x_n,\ x_{n-1},\ \cdots,\ x_1,\ x_0])\Big)\cdots\Bigg)\Bigg)\Bigg)\end{aligned}$$

Assuming $y[]$ has already been computed, then we only have $n-1$ additions, $n-1$ subtractions, and $n-1$ multiplications.

So in total, we have $3n + \mathcal{O}(1)$ flops.

Question 3.

(a)

Suppose we are given $p(x)$ of the form $p(x) = \sum_{i=0}^{n} b_i(x-c)^i$

By the binomial theorem, $(x-y)^i = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(-y)^k = \sum_{k=0}^{n} \binom{n}{k} x^k(-y)^{n-k}$

Then

$$
\begin{aligned}
p(x) &= \sum_{i=0}^{n} b_i(x-c)^i \\
&= \sum_{i=0}^{n} b_i \sum_{k=0}^{i} \binom{i}{k} x^{i-k}(-c)^k \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} b_i \binom{i}{k} x^{i-k}(-c)^k \\
&= \sum_{k=0}^{0} b_0 \binom{0}{k} x^{0-k}(-c)^k \\
&\quad + \sum_{k=0}^{1} b_1 \binom{1}{k} x^{1-k}(-c)^k \\
&\quad + \sum_{k=0}^{2} b_2 \binom{2}{k} x^{2-k}(-c)^k \\
&\quad \vdots \\
&\quad + \sum_{k=0}^{n-1} b_{n-1} \binom{n-1}{k} x^{(n-1)-k}(-c)^k \\
&\quad + \sum_{k=0}^{n} b_n \binom{n}{k} x^{n-k}(-c)^k \\
&= b_0 \binom{0}{0} x^{0-0}(-c)^0 \\
&\quad + b_1 \binom{1}{0} x^{1-0}(-c)^0 + b_1 \binom{1}{1} x^{1-1}(-c)^1 \\
&\quad + b_2 \binom{2}{0} x^{2-0}(-c)^0 + b_2 \binom{2}{1} x^{2-1}(-c)^1 + b_2 \binom{2}{2} x^{2-2}(-c)^2 \\
&\quad \vdots \\
&\quad + b_{n-1} \binom{n-1}{0} x^{(n-1)-0}(-c)^0 + b_{n-1} \binom{n-1}{1} x^{(n-1)-1}(-c)^1 + \ \dots \ + b_{n-1} \binom{n-1}{n-2} x^{(n-1)-(n-2)}(-c)^{n-2} + b_{n-1} \binom{n-1}{n-1} x^{(n-1)-(n-1)}(-c)^{n-1} \\
&\quad + b_n \binom{n}{0} x^{n-0}(-c)^0 + b_n \binom{n}{1} x^{n-1}(-c)^1 + \ \dots \ + b_n \binom{n}{n-1} x^{n-(n-1)}(-c)^{n-1} + b_{n-1} \binom{n}{n} x^{n-n}(-c)^n
\end{aligned}
$$

We can notice the last element of all sums are x^0, 2^{nd} last element (if it has one) are x^1, and so on. We can factor out all the x's with the same power.

$$\begin{aligned}
p(x) = x^0 &\left(b_0\binom{0}{0}(-c)^0 + b_1\binom{1}{1}(-c)^1 + b_2\binom{2}{2}(-c)^2 + \ \dots\ + b_{n-1}\binom{n-1}{n-1}(-c)^{n-1} + b_n\binom{n}{n}(-c)^n\right) \\
&+ x^1\left(b_1\binom{1}{0}(-c)^0 + b_2\binom{2}{1}(-c)^1 + b_3\binom{3}{2}(-c)^2 + \ \dots\ + b_{n-1}\binom{n-1}{n-2}(-c)^{n-2} + b_n\binom{n}{n-1}(-c)^{n-1}\right) \\
&+ x^2\left(b_2\binom{2}{0}(-c)^0 + b_3\binom{3}{1}(-c)^1 + b_4\binom{4}{2}(-c)^2 + \ \dots\ + b_{n-1}\binom{n-1}{n-3}(-c)^{n-3} + b_n\binom{n}{n-2}(-c)^{n-2}\right) \\
&\vdots \\
&+ x^{n-1}\left(b_{n-1}\binom{n-1}{0}(-c)^0 + b_n\binom{n}{1}(-c)^1\right) \\
&+ x^n\left(b_n\binom{n}{0}(-c)^0\right)
\end{aligned}$$

so

$$p(x) = \sum_{i=0}^{n}\left(\sum_{k=i}^{n} b_k\binom{k}{k-i}(-c)^{k-i}\right)x^i$$

$$\implies p(x) = \sum_{i=0}^{n} a_i x^i \text{ where } a_i = \sum_{k=i}^{n} b_k\binom{k}{k-i}(-c)^{k-i}$$

(b)

When calculating the reciprocal condition of the Vandermode matrix for values of c, we get the following table:

c	Reciprocal condition
0	4.2535e-07
0.5	1.9436e-06
1	7.5962e-06
1.5	2.6885e-05
2	5.3226e-05
2.5	0.0001131
3	0.00030227
3.5	0.00016034
4	0.00014415
4.5	3.5049e-05
5	4.8742e-05
5.5	1.9436e-06
6	4.2535e-07

We can see that for $c = 3$, we get the biggest reciprocal condition $\Longrightarrow$ it minimizes the condition of the Vandermonde matrix.

To check more accurately, we can use finer values of `c`. So we have:

c	Reciprocal condition
2.5	0.0001131
2.55	0.00011994
2.6	0.00012769
2.65	0.00013636
2.7	0.00014597
2.75	0.00015654
2.8	0.00016809
2.85	0.00018069
2.9	0.00019448
2.95	0.00020968
3	0.00030227
3.05	0.00028431
3.1	0.0002672
3.15	0.00025084
3.2	0.00023527
3.25	0.00022053
3.3	0.00020671
3.35	0.00019392
3.4	0.00018223
3.45	0.00017171
3.5	0.00016034

Question 4.

x_i	$y[x_i]$	$y[x_{i+1},\ x_i]$	$y[x_{i+2},\ x_{i+1},\ x_i]$	$y[x_{i+3},\ \cdots,\ x_i]$	$y[x_{i+4},\ \cdots,\ x_i]$	$y[x_{i+5},\ \cdots,\ x_i]$	$y[x_{i+6},\ \cdots,\ x_i]$
-1	4						
		$\frac{7-4}{0-(-1)} = 3$					
0	7		$\frac{6-3}{0-(-1)} = 3$				
		$\frac{y'(0)}{1!} = 6$		$\frac{15-3}{1-(-1)} = 6$			
0	7		$\frac{21-6}{1-0} = 15$		$\frac{20-6}{1-(-1)} = 7$		
		$\frac{28-7}{1-0} = 21$		$\frac{35-15}{1-0} = 20$		$\frac{15-7}{1-(-1)} = 4$	
1	28		$\frac{56-21}{1-0} = 35$		$\frac{35-20}{1-0} = 15$		$\frac{7-4}{2-(-1)} = 1$
		$\frac{y'(1)}{1!} = 56$		$\frac{70-35}{1-0} = 35$		$\frac{29-15}{2-0} = 7$	
1	28		$\frac{y''(1)}{2!} = 70$		$\frac{93-35}{2-0} = 29$		
		$\frac{y'(1)}{1!} = 56$		$\frac{163-70}{2-1} = 93$			
1	28		$\frac{219-56}{2-1} = 163$				
		$\frac{247-28}{2-1} = 219$					
2	247						

$$\begin{aligned}
p(x) = y[x_0] &+ (x-x_0)y[x_1,\ x_0] \\
&+ (x-x_0)(x-x_1)y[x_2,\ x_1,\ x_0] \\
&+ (x-x_0)(x-x_1)(x-x_2)y[x_3,\ x_2,\ x_1,\ x_0] \\
&+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)y[x_4,\ x_3,\ x_2,\ x_1,\ x_0] \\
&+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)y[x_5,\ x_3,\ x_2,\ x_1,\ x_0] \\
&+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)y[x_6,\ x_5,\ x_3,\ x_2,\ x_1,\ x_0] \\
= 4 &+ \Big(x-(-1)\Big)(3) \\
&+ \Big(x-(-1)\Big)(x)(3) \\
&+ \Big(x-(-1)\Big)(x)(x)(6) \\
&+ \Big(x-(-1)\Big)(x)(x)(x-1)(7) \\
&+ \Big(x-(-1)\Big)(x)(x)(x-1)(x-1)(4) \\
&+ \Big(x-(-1)\Big)(x)(x)(x-1)(x-1)(x-1)(1) \\
&= 4 + 3(x+1) + 3x(x+1) + 6x^2(x+1) + 7x^2(x+1)(x-1) + 4x^2(x+1)(x-1)^2 + x^2(x+1)(x-1)^3 \\
&= x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 7
\end{aligned}$$

$$\begin{aligned}
p'(x) &= 6x^5 + 10x^4 + 12x^3 + 12x^2 + 10x + 6 \\
p''(x) &= 30x^4 + 40x^3 + 36x^2 + 24x + 10
\end{aligned}$$

$$p(-1) = 4 \quad p(0) = 7 \quad p(1) = 28 \quad p(2) = 247$$
$$p'(0) = 6 \quad p'(1) = 56$$
$$p''(1) = 140$$