Problem 8. (Page 72 #8 in course notes)

Step 1:

Loop invariants

(a) x is even and $\geq 0 \implies x = 2k,\ k \in \mathbb{N}$

(b) $y \geq -2$

Step 2:

Basis:

$y = 0,\ x = 0$

So x is even and ≥ 0, and $y \geq -2$, as wanted

Induction Step: Consider an arbitrary iteration of the loop

Suppose LI holds before the loop iteration [IH]

WTP: LI holds after the iteration

$x \neq 0$ [line 1]

$\implies x \geq 2$ since 2 is the next even number after 0. **(♣)**

Case 1: $y \geq 1$

$$\begin{aligned} y' &= y - 3 && \text{[line 3]} \\ &\geq 1 - 3 && \text{[line 2]} \\ &= -2 \end{aligned}$$

as wanted fo LI(a)

$$\begin{aligned} x' &= x + 2 && \text{[line 3]} \\ &= 2k + 2 && \text{[IH]} \\ &= 2(k+1) && \text{which is even} \\ &\geq 0 && \text{[IH]} \end{aligned}$$

as wanted for LI(b)

Case 2: $y < 1$

$$\begin{aligned} y' &= y && \text{[line 3]} \\ &\geq -2 && \text{[IH]} \end{aligned}$$

as wanted fo LI(a)

$$\begin{aligned} x' &= x - 2 && \text{[line 3]} \\ &= 2k - 2 && \text{[IH]} \\ &= 2(k-1) && \text{which is even} \end{aligned}$$

Also,

$$\begin{aligned} x' &= x - 2 && \text{[line 3]} \\ &\geq 2 - 2 && (\clubsuit) \\ &= 0 \end{aligned}$$

as wanted for LI(b)

Step 4:

Let $e = y + x + 2$

Step 5:

(A) Proving $e \geq 0$

$$\begin{aligned} e &= y + x + 2 \\ &\geq -2 + x + 2 && \text{[by LI(a)]} \\ &= x \\ &\geq 0 && \text{[By LI(b)]} \end{aligned}$$

(B) Consider an arbitrary interation.

Case 1: $y \geq 1$

$y' = y - 3$ [line 3]
$x' = x + 2$ [line 3]

$$\begin{aligned} e' &= y' + x' + 2 \\ &= y - 3 + x + 2 + 2 \\ &= y + x + 2 - 1 \\ &= e - 1 && \text{[definition of } e\text{]} \\ &< e \end{aligned}$$

Case 2: $y < 1$

$y' = y$
$x' = x - 2$ [line 5]

$$\begin{aligned} e' &= y' + x' + 2 \\ &= y + x - 2 + 2 \\ &= e - 2 && \text{[definition of } e\text{]} \\ &< e \end{aligned}$$

as wanted. ■

If line 3 were changed to $y := y - 1$

Step 1:

Loop invariants

(a) x is even and $\geq 0 \implies x = 2k,\ k \in \mathbb{N}$

(b) $y \geq 0$

Step 2:

Basis:

$y = 0,\ x = 0$

So x is even and ≥ 0, and $y \geq 0$, as wanted

Induction Step: Consider an arbitrary iteration of the loop

Suppose LI holds before the loop iteration [IH]

WTP: LI holds after the iteration

$x \neq 0$ [line 1]

$\implies x \geq 2$ since 2 is the next even number after 0. **(♣)**

Case 1: $y \geq 1$

$$
\begin{aligned}
y' &= y - 1 && \text{[line 3]} \\
&\geq 1 - 1 && \text{[line 2]} \\
&= 0
\end{aligned}
$$

as wanted fo LI(a)

$$
\begin{aligned}
x' &= x + 2 && \text{[line 3]} \\
&= 2k + 2 && \text{[IH]} \\
&= 2(k+1) && \text{which is even} \\
&\geq 0 && \text{[IH]}
\end{aligned}
$$

as wanted for LI(b)

Case 2: $y < 1$

$$
\begin{aligned}
y' &= y && \text{[line 3]} \\
&\geq 0 && \text{[IH]}
\end{aligned}
$$

as wanted fo LI(a)

$$
\begin{aligned}
x' &= x - 2 && \text{[line 3]} \\
&= 2k - 2 && \text{[IH]} \\
&= 2(k-1) && \text{which is even}
\end{aligned}
$$

Also,

$$
\begin{aligned}
x' &= x - 2 && \text{[line 3]} \\
&\geq 2 - 2 && (\clubsuit) \\
&= 0
\end{aligned}
$$

as wanted for LI(b)

Step 4:

Let $e = 3y + x$

Step 5:

(A) Proving $e \geq 0$

$$\begin{aligned} e &= 3y + x \\ &\geq 3(0) + 0 && \text{[by LI(a) and LI(b)]} \\ &= 0 \end{aligned}$$

(B) Consider an arbitrary interation.

Case 1: $y \geq 1$

$y' = y - 1$ [line 3]
$x' = x + 2$ [line 3]

$$\begin{aligned} e' &= 3y' + x' \\ &= 3(y-1) + x + 2 \\ &= 3y + x - 1 \\ &= e - 1 && \text{[definition of } e\text{]} \\ &< e \end{aligned}$$

Case 2: $y < 1$

$y' = y$
$x' = x - 2$ [line 5]

$$\begin{aligned} e' &= 3y' + x' \\ &= 3y + x - 2 \\ &= e - 2 && \text{[definition of } e\text{]} \\ &< e \end{aligned}$$

as wanted. ■

Problem 9. (Page 73 #9 in course notes)

Step 1:

Loop invariants

(a) $y = x^2$

(b) $x \geq 0$

Step 2:

BASIS:

$y = 0,\ x = 0$

So $x \geq 0$, and $y = x^2$, as wanted

INDUCTION STEP: Consider an arbitrary iteration of the loop

Suppose LI holds before the loop iteration [IH]

WTP: LI holds after the iteration

$y \neq 0$ [line 2] $\implies x \geq 1$ ($\clubsuit$)

since $y = x^2$ and the next natural number after 0 is 1

$$\begin{array}{rcll} x' & = & x - 1 & \text{[line 3]} \\ & \geq & 1 - 1 & (\clubsuit) \\ & = & 0 & \end{array}$$

as wanted fo LI(b)

$$\begin{array}{rcll} y' & = & y - 2x - 1 & \text{[line 4]} \\ & = & x^2 - 2x - 1 & \\ & = & (x-1)^2 & \\ & = & (x')^2 & \text{[since } x' = x - 1\text{]} \end{array}$$

as wanted fo LI(a)

Step 4:

Let $e = x$

Step 5:

(A) Proving $e \geq 0$

$$\begin{array}{rcll} e & = & x & \\ & \geq & 0 & \text{[by LI(b)]} \end{array}$$

(B) Consider an arbitrary interation.

$x' = x - 1$ [line 3]

$$\begin{array}{rcll} e' & = & x' & \\ & = & x - 1 & \\ & = & e - 1 & \text{[definition of } e\text{]} \\ & < & e & \end{array}$$

as wanted. ■