Problem 1.

(1) *True.*

This is because if the dimension of V is m, then there is max m linear independent vectors. So for all linearly independent set of vectors $\in V$, there must be $\leq m$ elements in that set. This is a theorem we used in the lectures.

(2) *False.*

$$\begin{aligned}
\text{Let } U &= \mathbb{R}^2 \\
W &= \mathbb{R} \\
\vec{v_1} &= [1,0] \\
\vec{x} &= [x_1, x_2] \in U \\
T &: U \to W
\end{aligned}$$

Define $T(\vec{x}) = x_1$

$$\begin{aligned}
span(T(\vec{v_1})) &= span(1) \\
&= \mathbb{R} \\
&= U
\end{aligned}$$

$$\begin{aligned}
\text{but } span([1,0]) &\neq \mathbb{R}^2 \\
&= W
\end{aligned}$$

(3) *True.*

Given: $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is linearly independent.

Suppose to the contrary that $\vec{v_k} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k-1}\}$

This means there exists a linear combination of $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k-1}\}$ that equals $\vec{v}_k$

So $\exists\ a_1, \cdots, a_{k-1}$ *s.t.* $a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_{k-1}\vec{v}_{k-1} = \vec{v}_k$

This means that $\vec{v_k}$ is linearly dependent to $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k-1}\}$, which contradicts our *given.*

$\therefore$ Our supposition is wrong, and $\vec{v}_k \notin \text{Span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k-1}\}$

(4) *True.*
Let $\mathcal{B}_1 = \{x, x^2 + x, x^3 + x^2 + x, \cdots, x^n + \cdots + x^3 + x^2 + x\}$
Let $\mathcal{B}_2 = \{x^n, \cdots, x^3, x^2, x\}$

$$
\begin{aligned}
W &= \mathit{span}(\mathcal{B}_2) \\
&= \{b_1 x + b_2 x^2 + b_3 x^3 + \ldots + b_n x^n \mid b_1, \cdots, b_n \in \mathbb{F}\}
\end{aligned}
$$

$$
\begin{aligned}
V &= \mathit{span}(\mathcal{B}_1) \\
&= \{a_1 x + a_2(x^2 + x) + \ldots + a_n(x^n + \cdots + x^3 + x^2 + x) \mid a_1, \cdots, a_n \in \mathbb{F}\} \\
&= \{(a_1 + \ldots + a_n)x + (a_2 + \ldots + a_n)x^2 + \ldots + (a_n)x^n \mid a_1, \cdots, a_n \in \mathbb{F}\} \\
&= \{b_1 x + b_2 x^2 + b_3 x^3 + \ldots + b_n x^n \mid b_1, \cdots, b_n \in \mathbb{F}\} \\
&= \mathit{span}(\mathcal{B}_2) \\
&= W,
\end{aligned}
$$

as wanted.

Problem 2.

(1) *W.T.S.* $U + W$ is non-empty, closed under addition, and closed under scalar multiplication.

Non-empty:
Since we know U and W are subspaces, then there exists $\vec{0}$ for both subspaces. Therefore, we can choose $u = \vec{0} \in U$, and $w = \vec{0} \in W$ such that $\{u + w \mid u \in U, w \in W\}$. This means $\vec{0}$ is in $U + W$

Closed under addition:
Let $a, b \in U + W$
This means: $a = u_1 + w_1$
$b = u_2 + w_2$
Where: $u_1, u_2 \in U$
$w_1, w_2 \in W$

$$
\begin{aligned}
a + b &= (u_1 + w_1) + (u_2 + w_2) \\
&= (u_1 + u_2) + (w_1 + w_2) \\
&\in U + W
\end{aligned}
$$

since $(u_1 + u_2) \in U$ and $(w_1 + w_2) \in W$ which sastifies $\{u + w \mid u \in U, w \in W\}$

Since a and b are arbitrary elements in $U + W$, and $a + b \in U + W$, $U + W$ is closed under addition.

Closed under scalar multiplication:
Let $a \in U + W$, $\quad c \in \mathbb{R}$
This means $a = u_1 + w_1$

$$
\begin{aligned}
ca &= cu_1 + cw_1 \\
&\in U + W
\end{aligned}
$$

since $cu_1 \in U$ and $cw_1 \in W$ which sastifies $\{u + w \mid u \in U, w \in W\}$

Since a and b are arbitrary, and $ca \in U + W$, $U + W$ is closed under scalar multiplication.

Since $U + W$ is non-empty, closed under addition, and closed under scalar multiplication, $U + W$ is a subspace of V ■

(2) $sp(\mathcal{U} \cup \mathcal{W}) = \{(a_1u_1 + \cdots + a_ru_r) + (b_1w_1 + \cdots + b_sw_s) \mid a_1, \cdots, a_r, b_1, \cdots, b_s \in \mathbb{R}\}$

If $sp(\mathcal{U}) = U$, then all elements in U can be written as a linear combination of elements in $\mathcal{U}$. The same can be said for $W = sp(\mathcal{W})$
This means: $U = \{a_1u_1 + \cdots + a_ru_r \mid a_1, \cdots, a_r \in \mathbb{R}\}$
$W = \{b_1w_1 + \cdots + b_sw_s \mid b_1, \cdots, b_s \in \mathbb{R}\}$

All elements in $U + W$ can be written as a linear combination of elements in $\mathcal{U}$ and $\mathcal{W}$.

$$\begin{aligned} U + W &= \{u + w \mid u \in U, w \in W\} \\ &= \{(a_1u_1 + \cdots + a_ru_r) + (b_1w_1 + \cdots + b_sw_s) \mid a_1, \cdots, a_r, b_1, \cdots, b_s \in \mathbb{R}\} \\ &= sp(\mathcal{U} \cup \mathcal{W}), \end{aligned}$$

as wanted. ■

(3)
W.T.S. $(a_1u_1 + \cdots + a_ru_r) + (b_1w_1 + \cdots + b_sw_s) = 0$ only has 1 unique solution, where $a_1 = \ldots = a_r = b_1 = \ldots = b_s = 0$

This is equivalent to saying $u + w = \vec{0}$ since
$U = \{a_1u_1 + \cdots + a_ru_r \mid a_1, \cdots, a_r \in \mathbb{R}\}$
$W = \{b_1w_1 + \cdots + b_sw_s \mid b_1, \cdots, b_s \in \mathbb{R}\}$

This means we *W.T.S.* $u = w = \vec{0}$

Given: $u + w = \vec{0}$
$w \in W$
$u \in U$
$U \cap W = \{0\}$

Suppose to the contrary that $u \neq 0$
Then w must be the inverse elemnt of u such that $u + w = 0$
This means $w = -u$
This contradicts our *given*, since $w = -u \notin W$
$\therefore$ Our supposition is wrong, and $u = 0$
Without loss of generality, this also applies if $w \neq 0$.

This means that $u = w = \vec{0}$
$\implies a_1 = \ldots = a_r = b_1 = \ldots = b_s = 0$
$\implies \mathcal{U} \cup \mathcal{W}$ are linearly independent. ■

(4)

$$\begin{aligned} Dim(U+W) &= |\{u_1, \cdots, u_r, w_1, \cdots, w_s\}| \\ &= r+s \end{aligned}$$

$$\begin{aligned} DimU + DimW &= |\{u_1, \cdots, u_r\}| + |\{w_1, \cdots, w_s\}| \\ &= r+s \end{aligned}$$

$\therefore$ *Dim*(U + W) = *Dim*U + *Dim*W

Problem 3.

(1) Proving V is finite-dimensional $\Longrightarrow V$ has finitely many elements
Suppose $\mathcal{B} = \{b_1, \cdots, b_n\}$ is the set of basis vectors for V.
This means $V = \{a_1 b_1 + \ldots + a_n b_n \mid a_1, \cdots, a_n \in \mathbb{F}\}$
Since $|\mathbb{F}|$ is finite, that means that there is a finite set of scalars for the set of basis vectors $\mathcal{B}$
This means the set of linear combinations is finite, which proves V has a finite number of elements.

Proving V has finitely many elements $\Longrightarrow V$ is finite-dimensional
Suppose V has finitely many elements
This means that the set of basis is $\mathcal{B}$ finite, since $\mathbb{F}$ is also finite. A linear combination of a finite set with a finite set of scalars is finite.
This means $dim(V) = |\mathcal{B}|$ which is a finite number. Also, $span(\mathcal{B}) = V$ which is a finite spanning set. ■

(2)
Suppose $\mathcal{B} = \{b_1, \cdots, b_n\}$ is the set of basis vectors for V.
This means $V = \{a_1 b_1 + \ldots + a_n b_n \mid a_1, \cdots, a_n \in \mathbb{F}\}$
For all n b_i vectors, there is $|\mathbb{F}|$ choices for scalars for each b_i.
This means there is
$$\begin{aligned} \underbrace{|\mathbb{F}| \times |\mathbb{F}| \times \ldots \times |\mathbb{F}|}_{\text{n-times}} &= |\mathbb{F}|^n \\ &= |\mathbb{F}|^{\dim(V)} \end{aligned}$$
number of elements in V

(3)
We know that $\mathcal{V} = \{(a_1, a_2, \ldots, a_n) \mid a_1, \cdots, a_n \in \mathbb{F}_3\}$
Where $\mathbb{F}_3 = \{0, 1, 2\}$
is the set of all tricolorings, where n is the number of line segments.

From what we found in (2), there must be $|\mathbb{F}|^{\dim(V)} = 3^n$ number of tricolorings. ■

Problem 4.

(1)

Let $f = a_0 + a_1x + a_2x^2$

$g = b_0 + b_1x + b_2x^2$

$r \in \mathbb{R}$

$W.T.S.\ T_{\vec{c}}(rf+g) = rT_{\vec{c}}(f) + T_{\vec{c}}(g)$

Where $T_{\vec{c}} : \mathcal{P}_2 \to \mathbb{R}^3$

$$\begin{aligned}
L.S. &= T_{\vec{c}}(rf+g) \\
&= \begin{bmatrix} rf(1)+g(1) \\ rf(2)+g(2) \\ rf(1)+g(1) \end{bmatrix} \\
&= \begin{bmatrix} ra_0+ra_1+ra_2+b_0+b_1+b_2 \\ ra_0+2ra_1+4ra_2+b_0+2b_1+4b_2 \\ ra_0+ra_1+ra_2+b_0+b_1+b_2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
R.S. &= rT_{\vec{c}}(f) + T_{\vec{c}}(g) \\
&= r\begin{bmatrix} f(1) \\ f(2) \\ f(1) \end{bmatrix} + \begin{bmatrix} g(1) \\ g(2) \\ g(1) \end{bmatrix} \\
&= r\begin{bmatrix} a_0+a_1+a_2 \\ a_0+2a_1+4a_2 \\ a_0+a_1+a_2 \end{bmatrix} + \begin{bmatrix} b_0+b_1+b_2 \\ b_0+2b_1+4b_2 \\ b_0+b_1+b_2 \end{bmatrix} \\
&= \begin{bmatrix} ra_0+ra_1+ra_2+b_0+b_1+b_2 \\ ra_0+2ra_1+4ra_2+b_0+2b_1+4b_2 \\ ra_0+ra_1+ra_2+b_0+b_1+b_2 \end{bmatrix}
\end{aligned}$$

Since $L.S. = R.S.$

$\therefore T_{\vec{c}}$ is linear for $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(2)

$$\begin{aligned}
ker(T_{\vec{c}}(f)) &= \begin{bmatrix} a_0+a_1+a_2 \\ a_0+2a_1+4a_2 \\ a_0+a_1+a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r_3-r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2-r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1-r_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

$$\implies \begin{cases} a_0 - 2a_2 = 0 \\ a_1 + 3a_2 = 0 \\ \qquad a_2 \in \mathbb{F} \end{cases}$$

$$\implies \begin{cases} a_0 = 2s \\ a_1 = -3s \\ a_2 = s \\ s \in \mathbb{F} \end{cases}$$

$$\implies ker(T_{\vec{c}}(f)) = span\left(\begin{bmatrix} 2 \\ -3x \\ x^2 \end{bmatrix}\right)$$

(3)
$C = \{[c_0, \cdots, c_n] \mid a_i \neq c_j,\ \forall i \neq j\}$
Let $f = [a_0, a_1, \cdots, a_n]$

$W.T.S.$ $T_{\vec{c}}(f)$ is an isomorphism.
It is enough to show $det(A) \neq 0$ where $T_{\vec{c}}(f) = Af \qquad \forall\ \vec{c} \in C$

$$dim(\mathbb{R}^{n+1}) = n + 1$$
$$dim(\mathcal{P}_n) = n + 1$$

$$T_{\vec{c}}(f) = \begin{bmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^n \\ 1 & c_2 & c_2^2 & \cdots & c_2^n \\ 1 & c_3 & c_3^2 & \cdots & c_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_{n+1} & c_{n+1}^2 & \cdots & c_{n+1}^n \end{bmatrix} f$$

A satisfies the definition of Vandermonde matrix.
The determinant of Vandermonde matrix is

$$\det(A) = \prod_{1 \leq i < j \leq n+1} (c_j - c_i)$$

By definition of C, all vectors in $\vec{c} \in C$ have different values for all parts of the vector.
This means $(c_j - c_i) \neq 0 \qquad \forall\ i, j$
The product of non-zero real numbers is non-zero $\implies \det(A) \neq 0$
$\therefore A$ is invertible $\implies T_{\vec{c}}(f)$ is an isomorphism ■

Problem 5.

(1) The **[I.H.]** states that a set of n vectors are linearly independent. This only means that the set $\{v_1, \cdots, v_n\}$ is linearly independent. So for the set $\{v_1, \cdots, v_n, v_{n+1}\}$, you can only claim that $\{v_1, \cdots, v_n\}$ is linearly independent. Not $\{v_2, \cdots, v_n, v_{n+1}\}$

(2) They did not state that $n < m$. This is because you cannot assume that n vectors is linearly independent if $n > m$. By the proof in **Problem 1.1**, you can only have max m linearly independent vectors.