

## Problem 8. (Page 72 #8 in course notes)

### Step 1:

Loop invariants

(a)  $x$  is even and  $\geq 0 \implies x = 2k, k \in \mathbb{N}$

(b)  $y \geq -2$

### Step 2:

BASIS:

$y = 0, x = 0$

So  $x$  is even and  $\geq 0$ , and  $y \geq -2$ , as wanted

INDUCTION STEP: Consider an arbitrary iteration of the loop

Suppose LI holds before the loop iteration [IH]

WTP: LI holds after the iteration

$x \neq 0$  [line 1]

$\implies x \geq 2$  since 2 is the next even number after 0. (♣)

Case 1:  $y \geq 1$

$y' = y - 3$  [line 3]

$\geq 1 - 3$  [line 2]

$= -2$

as wanted for LI(a)

$x' = x + 2$  [line 3]

$= 2k + 2$  [IH]

$= 2(k + 1)$  which is even

$\geq 0$  [IH]

as wanted for LI(b)

Case 2:  $y < 1$

$y' = y$  [line 3]

$\geq -2$  [IH]

as wanted for LI(a)

$x' = x - 2$  [line 3]

$= 2k - 2$  [IH]

$= 2(k - 1)$  which is even

Also,

$x' = x - 2$  [line 3]

$\geq 2 - 2$  (♣)

$= 0$

as wanted for LI(b)

**Step 4:**

Let  $e = y + x + 2$

**Step 5:**

(A) Proving  $e \geq 0$

$$\begin{aligned} e &= y + x + 2 \\ &\geq -2 + x + 2 && [\text{by LI(a)}] \\ &= x \\ &\geq 0 && [\text{By LI(b)}] \end{aligned}$$

(B) Consider an arbitrary iteration.

Case 1:  $y \geq 1$

$$\begin{aligned} y' &= y - 3 && [\text{line 3}] \\ x' &= x + 2 && [\text{line 3}] \end{aligned}$$

$$\begin{aligned} e' &= y' + x' + 2 \\ &= y - 3 + x + 2 + 2 \\ &= y + x + 2 - 1 \\ &= e - 1 && [\text{definition of } e] \\ &< e \end{aligned}$$

Case 2:  $y < 1$

$$\begin{aligned} y' &= y \\ x' &= x - 2 && [\text{line 5}] \end{aligned}$$

$$\begin{aligned} e' &= y' + x' + 2 \\ &= y + x - 2 + 2 \\ &= e - 2 && [\text{definition of } e] \\ &< e \end{aligned}$$

as wanted. ■

If line 3 were changed to  $y := y - 1$

**Step 1:**

Loop invariants

(a)  $x$  is even and  $\geq 0 \implies x = 2k, k \in \mathbb{N}$

(b)  $y \geq 0$

**Step 2:**

BASIS:

$y = 0, x = 0$

So  $x$  is even and  $\geq 0$ , and  $y \geq 0$ , as wanted

INDUCTION STEP: Consider an arbitrary iteration of the loop

Suppose LI holds before the loop iteration [IH]

WTP: LI holds after the iteration

$x \neq 0$  [line 1]

$\implies x \geq 2$  since 2 is the next even number after 0. (♣)

Case 1:  $y \geq 1$

$y' = y - 1$  [line 3]

$\geq 1 - 1$  [line 2]

$= 0$

as wanted for LI(a)

$x' = x + 2$  [line 3]

$= 2k + 2$  [IH]

$= 2(k + 1)$  which is even

$\geq 0$  [IH]

as wanted for LI(b)

Case 2:  $y < 1$

$y' = y$  [line 3]

$\geq 0$  [IH]

as wanted for LI(a)

$x' = x - 2$  [line 3]

$= 2k - 2$  [IH]

$= 2(k - 1)$  which is even

Also,

$x' = x - 2$  [line 3]

$\geq 2 - 2$  (♣)

$= 0$

as wanted for LI(b)

**Step 4:**

Let  $e = 3y + x$

**Step 5:**

(A) Proving  $e \geq 0$

$$\begin{aligned} e &= 3y + x \\ &\geq 3(0) + 0 && \text{[by LI(a) and LI(b)]} \\ &= 0 \end{aligned}$$

(B) Consider an arbitrary iteration.

Case 1:  $y \geq 1$

$$\begin{aligned} y' &= y - 1 && \text{[line 3]} \\ x' &= x + 2 && \text{[line 3]} \end{aligned}$$

$$\begin{aligned} e' &= 3y' + x' \\ &= 3(y - 1) + x + 2 \\ &= 3y + x - 1 \\ &= e - 1 && \text{[definition of } e\text{]} \\ &< e \end{aligned}$$

Case 2:  $y < 1$

$$\begin{aligned} y' &= y \\ x' &= x - 2 && \text{[line 5]} \end{aligned}$$

$$\begin{aligned} e' &= 3y' + x' \\ &= 3y + x - 2 \\ &= e - 2 && \text{[definition of } e\text{]} \\ &< e \end{aligned}$$

as wanted. ■

## Problem 9. (Page 73 #9 in course notes)

### Step 1:

Loop invariants

(a)  $y = x^2$

(b)  $x \geq 0$

### Step 2:

BASIS:

$$y = 0, x = 0$$

So  $x \geq 0$ , and  $y = x^2$ , as wanted

INDUCTION STEP: Consider an arbitrary iteration of the loop

Suppose LI holds before the loop iteration [IH]

WTP: LI holds after the iteration

$$y \neq 0 \text{ [line 2]} \implies x \geq 1 \quad (\clubsuit)$$

since  $y = x^2$  and the next natural number after 0 is 1

$$x' = x - 1 \quad [\text{line 3}]$$

$$\geq 1 - 1 \quad (\clubsuit)$$

$$= 0$$

as wanted for LI(b)

$$y' = y - 2x - 1 \quad [\text{line 4}]$$

$$= x^2 - 2x - 1$$

$$= (x - 1)^2$$

$$= (x')^2 \quad [\text{since } x' = x - 1]$$

as wanted for LI(a)

### Step 4:

Let  $e = x$

### Step 5:

(A) Proving  $e \geq 0$

$$e = x$$

$$\geq 0 \quad [\text{by LI(b)}]$$

(B) Consider an arbitrary iteration.

$$x' = x - 1 \quad [\text{line 3}]$$

$$e' = x'$$

$$= x - 1$$

$$= e - 1 \quad [\text{definition of } e]$$

$$< e$$

as wanted. ■