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## Question 1.

(a)

Let  $\mathcal{L}_k$  be a Gauss Transform. Then it has the form

$$\mathcal{L}_k = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_n & 0 & \cdots & 1 \end{bmatrix}$$

Then let  $m_k =$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -a_{k+1} \\ \vdots \\ -a_n \end{bmatrix}$$

$$\text{Now } -m_k e_k^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_{k+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_n & 0 & \cdots & 0 \end{bmatrix}$$

So  $\mathcal{L}_k = I - m_k e_k^T$

(b)

By definition, the inverse of a Gauss Transform is simply flipping the signs of the non-identity values. So if we invert the signs of  $-m_k$ , we have

$$m_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix}$$

Then

$$I + m_k e_k^T = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -a_{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -a_n & 0 & \cdots & 1 \end{bmatrix} = \mathcal{L}_k^{-1}$$

(c)

$$\text{WTS: } (L_k L_j)^{-1} = (I + m_k e_k^T) + (m_j e_j^T) - I$$

$$\begin{aligned}
 LHS &= (L_k L_j)^{-1} && \text{[given]} \\
 &= L_j^{-1} L_k^{-1} && \text{[by inverse laws]} \\
 &= (I + m_j e_j^T)(I + m_k e_k^T) && \text{[by (b)]} \\
 &= I + m_j e_j^T + m_k e_k^T + m_j e_j^T m_k e_k^T && \text{[expanding]} \\
 &= I + m_j e_j^T + m_k e_k^T + 0 && \text{[since } j < k \text{ and } e_j^T m_k = 0] \\
 &= I + m_j e_j^T + m_k e_k^T + I - I && \text{[} I - I = 0 \text{]} \\
 &= (I + m_k e_k^T) + (I + m_j e_j^T) - I && \text{[rearranging]} \\
 &= RHS && \text{[given]}
 \end{aligned}$$

as wanted

(d)

WTS:  $\widetilde{\mathcal{L}}_k = \mathcal{L}_k$  with multipliers  $i$  and  $j$  swapped

$$\begin{aligned}
LHS &= \widetilde{\mathcal{L}}_k && \text{[given]} \\
&= P_i \mathcal{L}_k P_i && \text{[given]} \\
&= P_i (I - m_k e_k^T) P_i && \text{[by (a)]} \\
&= (P_i - P_i m_k e_k^T) P_i && \text{[expanding]} \\
&= P_i P_i - P_i m_k e_k^T P_i && \text{[expanding]} \\
&= I - P_i m_k e_k^T P_i && \text{[since } P_i P_i = I\text{]} \\
&= \mathcal{L}_k \text{ with multipliers } i \text{ and } j \text{ swapped} && \text{[since } m_k e_k^T \text{ multipliers got swapped]} \\
&= RHS && \text{[given]}
\end{aligned}$$

as wanted

## Question 2.

$$\begin{array}{lll} PA = LU & \iff & \det(PA) = \det(LU) & [\det \text{ both sides}] \\ & \iff & \det(P) \det(A) = \det(L) \det(U) & [\text{by det laws}] \\ & \iff & \det(A) = \det(L) \det(U) & [\text{since } \det(P) = 1] \\ & \iff & \det(A) = 1 \times \det(U) & [\text{since } L \text{ is a unit lower triangle}] \\ & \iff & \det(A) = \prod_{i=1}^n a_{ii} & [\text{since } U \text{ is an upper triangle}] \end{array}$$

This is much more efficient than just finding the determinant of  $A$  since the determinant of a triangular matrix is just the product of the diagonal.

## Question 3.

(a)

$$A = \begin{bmatrix} 3 & 3 & 9 & 6 \\ 4 & 4 & 4 & 4 \\ 1 & 1 & 5 & 5 \\ 2 & 2 & 4 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 21 \\ 24 \\ 10 \\ 16 \end{bmatrix}$$

Eliminate 1<sup>st</sup> column:

$$P_1 = P_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ -1/4 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 1 \end{bmatrix},$$

$$P_1 A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 3 & 3 & 9 & 6 \\ 1 & 1 & 5 & 5 \\ 2 & 2 & 4 & 6 \end{bmatrix}$$

$$L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

Eliminate 3<sup>rd</sup> column:

$$P_3 = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}$$

$$P_3 L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

$$L_3 P_3 L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b)

$$\det(A) = \prod_{i=1}^n a_{ii} = 0$$

So there's either infinite, or no solutions to  $A\vec{x} = \vec{b}$

(c)

(d)

It's much more efficient to factor first if we have multiple  $A\vec{x} = \vec{b}_i$  equations to solve



## Question 4.