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Problem 1.

(1)

Suppose $A \in M_{n\times n}(\mathbb{C})$ is unitarily diagonalizbale

$D = PAP^{-1}$ D is a diagonal matrix P is a unitary matrix

WTS: $AA^* = A^*A$

$$\begin{aligned}
A &= P^{-1}DP && \text{[by given]} \\
A^* &= \left(P^{-1}DP\right)^* && \text{[by definition of } A\text{]} \\
&= P^*D^*P^{-1*} && \text{[by properties of } ^*\text{]} \\
&= P^{-1}D^*P && \text{[by properties of } ^*\text{]} \\
&= P^{-1}\overline{D^T}P && \text{[by definition of } ^*\text{]} \\
&= P^{-1}\overline{D}P && \text{[since diagonal matricies are symmetrical]}
\end{aligned}$$

$$\begin{aligned}
AA^* &= P^{-1}DPP^{-1}\overline{D}P && \text{[by given]} \\
&= P^{-1}D\overline{D}P && \text{[since } PP^{-1} = I\text{]} \\
&= P^{-1}\overline{D}DP && \text{[since diagonal matricies commute]} \\
&= P^{-1}\overline{D}PP^{-1}DP && \text{[since } PP^{-1} = I\text{]} \\
&= A^*A && \text{[by definition of } A \text{ and } A^*\text{]}
\end{aligned}$$

$\therefore A$ is normal ■

(2)
Suppose

$$A = UBU^* \qquad U \text{ is unitary or } U^* = U^{-1}$$

WTS: A is normal $\iff$ B is normal

Case 1: A is normal
$AA^* = A^*A$

WTS: $BB^* = B^*B$

$B = U^*AU$ [by given]
$B^* = U^*A^*U$ [by definition of B]

$$\begin{aligned}
BB^* &= U^*AUU^*A^*U && \text{[by definition of } B\text{]}\\
&= U^*AA^*U && \text{[since } UU^* = UU^{-1} = I\text{]}\\
&= U^*A^*AU && \text{[since } AA^* = A^*A\text{]}\\
&= U^*A^*UU^*AU && \text{[since } UU^* = UU^{-1} = I\text{]}\\
&= B^*B && \text{[by definition of } B\text{]}
\end{aligned}$$

$\therefore$ B is normal, as wanted.

Case 2: B is normal
$BB^* = B^*B$

WTS: $AA^* = A^*A$

$A = UBU^*$ [by given]
$A^* = UB^*U^*$ [by definition of B]

$$\begin{aligned}
AA^* &= UBU^*UB^*U^* && \text{[by definition of } A\text{]}\\
&= UBB^*U^* && \text{[since } UU^* = UU^{-1} = I\text{]}\\
&= UB^*BU^* && \text{[since } BB^* = B^*B\text{]}\\
&= UB^*U^*UBU^* && \text{[since } UU^* = UU^{-1} = I\text{]}\\
&= A^*A && \text{[by definition of } A\text{]}
\end{aligned}$$

$\therefore$ A is normal, as wanted.
$\therefore$ A is normal $\iff$ B is normal

(3)

$$\text{Let } B = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ 0 & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \cdots & \lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{nn} \end{bmatrix} \qquad B^* = \begin{bmatrix} \overline{\lambda_{11}} & 0 & 0 & \cdots & 0 \\ \overline{\lambda_{12}} & \overline{\lambda_{22}} & 0 & \cdots & 0 \\ \overline{\lambda_{13}} & \overline{\lambda_{23}} & \overline{\lambda_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\lambda_{1n}} & \overline{\lambda_{2n}} & \overline{\lambda_{3n}} & \cdots & \overline{\lambda_{nn}} \end{bmatrix}$$

Given: $BB^* = B^*B$
WTS: $\lambda_{ij} = 0 \qquad \text{if } i \neq j$

$$\begin{aligned} A &= BB^* \\ &= B^*B \\ &= \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ 0 & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \cdots & \lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{nn} \end{bmatrix} \begin{bmatrix} \overline{\lambda_{11}} & 0 & 0 & \cdots & 0 \\ \overline{\lambda_{12}} & \overline{\lambda_{22}} & 0 & \cdots & 0 \\ \overline{\lambda_{13}} & \overline{\lambda_{23}} & \overline{\lambda_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\lambda_{1n}} & \overline{\lambda_{2n}} & \overline{\lambda_{3n}} & \cdots & \overline{\lambda_{nn}} \end{bmatrix} \\ &= \begin{bmatrix} \overline{\lambda_{11}} & 0 & 0 & \cdots & 0 \\ \overline{\lambda_{12}} & \overline{\lambda_{22}} & 0 & \cdots & 0 \\ \overline{\lambda_{13}} & \overline{\lambda_{23}} & \overline{\lambda_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\lambda_{1n}} & \overline{\lambda_{2n}} & \overline{\lambda_{3n}} & \cdots & \overline{\lambda_{nn}} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ 0 & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \cdots & \lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{nn} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A_{11} &= \lambda_{11}\overline{\lambda_{11}} && \text{[from } BB^*\text{]} \\ &= |\lambda_{11}|^2 \\ &= \sum_{i=1}^{n} \lambda_{1i}\overline{\lambda_{1i}} && \text{[from } B^*B\text{]} \\ &= \sum_{i=1}^{n} |\lambda_{1i}|^2 \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^{n} |\lambda_{1i}|^2 = |\lambda_{11}|^2 \\ \implies & \sum_{i=2}^{n} |\lambda_{1i}|^2 = 0 \\ \implies & |\lambda_{1i}|^2 = 0 \\ \implies & |\lambda_{1i}| = 0 \\ \implies & \lambda_{1i} = 0 \end{aligned}$$

$$\begin{aligned} A_{22} &= \lambda_{22}\overline{\lambda_{22}} + \lambda_{12}\overline{\lambda_{12}} && \text{[from } BB^*\text{]} \\ &= |\lambda_{22}|^2 + 0 \\ &= |\lambda_{22}|^2 \\ &= \sum_{i=2}^{n} \lambda_{2i}\overline{\lambda_{2i}} && \text{[from } B^*B\text{]} \\ &= \sum_{i=2}^{n} |\lambda_{2i}|^2 \end{aligned}$$

$$\begin{aligned} & \sum_{i=2}^{n} |\lambda_{2i}|^2 = |\lambda_{22}|^2 \\ \implies & \sum_{i=3}^{n} |\lambda_{2i}|^2 = 0 \\ \implies & |\lambda_{2i}|^2 = 0 \\ \implies & |\lambda_{2i}| = 0 \\ \implies & \lambda_{2i} = 0 \end{aligned}$$

we can continue this pattern for all $i \in \{1, \cdots, n\}$
This means that for every $i \neq j,\ A_{ij} = 0$
$\therefore A$ is a diagonal matrix.

(4)

Suppose $A \in M_{n\times n}(\mathbb{C})$ is normal
WTS: $\exists U, D \in M_{n\times n}(\mathbb{C})$ such that $D = UAU^{-1},\ U$ is unitary, D is diagonal.

By Schur's diagonalizbale lemma, $\exists B,\ U \in M_{n\times n}(\mathbb{C})$ such that
$B = U^{-1}AU,\ U$ is unitary, B is an upper triangular matrix.

By (2), A is normal $\iff$ B is normal.
We know A is normal since it is given in the question, so this means B is normal.

by (3), we know that a normal upper triangular matrix is diagonal.
Since B sastifies these conditions, this means B is diagonal.

In the end, we have $B = U^{-1}AU,\ U$ is unitary, B is diagonal.
$\therefore$ Any normal matrix is unitarily diagonalizbale ■

Problem 2.

(1)
Suppose T is unitarily diagonalizbale with real eigenvalues. $\implies$ $[T]_{\mathcal{U}}$ is diagonal.
WTS: $[T]_{\mathcal{U}} = [T]_{\mathcal{U}}{}^*$

Since $[T]_{\mathcal{U}}$ is diagonal, let

$$[T]_{\mathcal{U}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \lambda_i \in \mathbb{C}$$

$$[T]_{\mathcal{U}}{}^* = \begin{bmatrix} \overline{\lambda_1} & 0 & \cdots & 0 \\ 0 & \overline{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\lambda_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Since the eigenvalues are real.
Since T is hermitian, T is self-adjoint. ■

(2)
Suppose T is an isometry. $\implies$ $\langle v,\ w\rangle = \big\langle T(v),\ T(w)\big\rangle$
WTS: $\forall v \neq 0$ such that $T(v) = \lambda v,\ \|\lambda\| = 1$

Let $v \in V$ be an arbitrary eigenvector.

$$\begin{aligned} \langle v,\ v\rangle &= \big\langle T(v),\ T(v)\big\rangle && \text{[since T is an isometry]} \\ &= \langle \lambda v,\ \lambda v\rangle && \text{[since } v \text{ us an eigenvector]} \\ &= \lambda\overline{\lambda}\,\langle v,\ v\rangle && \text{[by taking scalars out of inner product]} \\ &= |\lambda|\,\langle v,\ v\rangle && \text{[by definition of magnitude of a complex number]} \end{aligned}$$

We know that $\langle v,\ v\rangle > 0$ by the property of positive-definite, and that $v \neq 0$ since it is an eigenvector.
This means in order for $\langle v,\ v\rangle = |\lambda|\,\langle v,\ v\rangle$, $|\lambda|$ must equal 1. ■

(3)

Suppose T is unitarily diagonalizbale, and all eigenvalues of T has absolute value 1.
WTS: T is an isometry.

Let $\mathcal{U} = \{u_1, \cdots, u_n\}$ be an orthonormal basis of V.
So, for all $i \in \{1, \cdots, n\}$, u_i is an eigenvector.
Let $T(u_i) = \lambda_i u_i$

$$[T]_{\mathcal{U}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad |\lambda_i| = 1$$

Since for every column of $[T]_{\mathcal{U}}$, we have

$$\text{i}^{\text{th}} \text{ column of } T = \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i}^{\text{th}} \text{ row}$$

So $\left(\text{i}^{\text{th}} \text{ column of } T\right) \cdot \left(\text{j}^{\text{th}} \text{ column of } T\right)$

$$= \begin{cases} |\lambda_i|^2 & \text{if } i = j & \text{since } \lambda_i \cdot \lambda_i = |\lambda_i|^2 \\ 0 & \text{if } i \neq j & \text{since all other entires are 0} \end{cases}$$

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\therefore$ columns of $[T]_{\mathcal{U}}$ are orthonormal
$\implies [T]_{\mathcal{U}}$ is unitary
$\implies T$ is an isometry. $\blacksquare$

Problem 3.

(1)

(2)

(3)

Problem 4.

(1)

(2)

(3)