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Problem 1.

(1)
Suppose we differentiate a polynomial with degree n.
Then our matrix will have n rows and $n+1$ columns.
If both the domain and codomain are the same, then it'll be $n+1$ rows and $n+1$ columns.
For this question, we will have a 4×4 matrix

(2)

$T(1) = 0 \qquad T(x) = 1 \qquad T(x^2) = 2x \qquad T(x^3) = 3x^2$

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad [T(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad [T(x^3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & | & | \\ [T(1)]_{\mathcal{B}} & [T(x)]_{\mathcal{B}} & [T(x^2)]_{\mathcal{B}} & [T(x^3)]_{\mathcal{B}} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(3)

$A^4 x = \overbrace{T(T(T(T(x))))}^{\text{differentiate 4 times}} = 0$
since we differentiate a polynomial of degree 3, 4 times.

$$\therefore A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2.

(1)

Let $\vec{w} \in \text{Im}(T) \implies \exists\ \vec{v} \in V$ st $T(\vec{v}) = \vec{w}$

Since $\{T(b_1), \cdots, T(b_r)\}$ is a basis for $\text{Im}(T)$

$$\begin{aligned} \vec{w} &= \lambda_1 T(b_1) + \ldots + \lambda_r T(b_r) && [\text{where } \lambda_1, \cdots, \lambda_r \in \mathbb{F}] \\ &= T(\lambda_1 b_1 + \ldots + \lambda_r b_r) && [\text{by linearity}] \\ &= T(\vec{v}) && [\text{by given}] \end{aligned}$$

$\implies \vec{v} = \lambda_1 b_1 + \ldots + \lambda_r b_r + \vec{k}$ for some $\vec{k} \in \text{Ker}(T)$

Since $\{v_1, \cdots, v_d\}$ is a basis for $\text{Ker}(T)$

$\vec{k} = \alpha_1 v_1 + \ldots + \alpha_d v_d$ where $\alpha_1, \cdots, \alpha_d \in \mathbb{F}$

$\implies \vec{v} = \lambda_1 b_1 + \ldots + \lambda_r b_r + \alpha_1 v_1 + \ldots + \alpha_d v_d$

Since $\vec{v}$ is an arbitrary element in V,

and $\vec{v}$ is a linear combination of vectors in $\{b_1, \cdots, b_r, v_1, \cdots, v_d\}$

$\therefore \text{Span}(b_1, \cdots, b_r, v_1, \cdots, v_d) = V$ ■

(2)

WTS: if $\lambda_1 b_1 + \ldots + \lambda_r b_r + \alpha_1 v_1 + \ldots + \alpha_d v_d = 0$ for some $\alpha_1, \cdots, \alpha_d, \lambda_1, \cdots, \lambda_r \in \mathbb{F}$
$\implies \lambda_1 = \ldots = \lambda_r = \alpha_1 = \ldots = \alpha_d = 0$

$$\begin{aligned}
& \lambda_1 b_1 + \ldots + \lambda_r b_r + \alpha_1 v_1 + \ldots + \alpha_d v_d = 0 & \\
\implies\ & T(\lambda_1 b_1 + \ldots + \lambda_r b_r + \alpha_1 v_1 + \ldots + \alpha_d v_d) = T(0) & \text{[applying T to both sides]} \\
\implies\ & T(\lambda_1 b_1 + \ldots + \lambda_r b_r + \vec{k}) = 0 & \text{[since } k = \alpha_1 v_1 + \ldots + \alpha_d v_d\text{]} \\
\implies\ & \lambda_1 T(b_1) + \ldots + \lambda_r T(b_r) + T(\vec{k}) = 0 & \text{[by linearity]} \\
\implies\ & \lambda_1 T(b_1) + \ldots + \lambda_r T(b_r) = 0 & \text{[since } T(\vec{k}) = 0\text{]}
\end{aligned}$$

$\implies \lambda_1 = \ldots = \lambda_r = 0$ [since $T(b_1), \cdots, T(b_r)$ is a basis]

So we are left with:
$\alpha_1 v_1 + \ldots + \alpha_d v_d = 0$
$\implies \alpha_1 = \ldots = \alpha_d = 0$ [since $\{v_1, \cdots, v_d\}$ is a basis].

Since $\lambda_1 b_1 + \ldots + \lambda_r b_r + \alpha_1 v_1 + \ldots + \alpha_d v_d = 0$ has only one solution which is the trivial solution, This means that the set $\{b_1, \cdots, b_r, v_1, \cdots, v_d\}$ is linearly independent. ■

(3)
Since $\{b_1, \cdots, b_r, v_1, \cdots, v_d\}$ is a basis for V,

$$\begin{aligned}
\text{Dim}(V) &= |\{b_1, \cdots, b_r, v_1, \cdots, v_d\}| \\
&= r + d \\
&= |\{T(b_1), \cdots, T(b_r)\}| + |\{v_1, \cdots, v_d\}| \\
&= \text{Dim}(\text{Im}(T)) + \text{Dim}(\text{Ker}(T))
\end{aligned}$$

$\implies \text{Dim}(V) = \text{Dim}(\text{Im}(T)) + \text{Dim}(\text{Ker}(T))$ ■

Problem 3.

Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \cdots$ be different basis for V

Let the matrix representation of T be $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

We know $A = [T]_{\mathcal{B}_1} = [T]_{\mathcal{B}_2} = [T]_{\mathcal{B}_3} = \dots$

By a theorem we learned in class, we know any different matrix representations of T are similar, so $A = C_1[T]_{\mathcal{B}_1}C_1^{-1} = C_2[T]_{\mathcal{B}_2}C_2^{-1} = C_3[T]_{\mathcal{B}_3}C_3^{-1} = \dots$

where C_i is an invertible 3×3 matrix

$\implies C_iA = AC_i$

or

$CA = AC \qquad \forall$ invertible matrix C

Choose $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then $AC = CA$

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & 2b & c \\ d & 2e & f \\ g & 2h & i \end{bmatrix}$$

$$\implies \begin{cases} 2b = b \\ 2d = d \\ 2f = f \\ 2h = h \end{cases} \implies \begin{cases} b = 0 \\ d = 0 \\ f = 0 \\ h = 0 \end{cases}$$

$$\therefore A = \begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{bmatrix}$$

Choose $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

then $AC = CA$

$$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{bmatrix} = \begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\implies \begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ 2g & 0 & 2i \end{bmatrix} = \begin{bmatrix} a & 0 & 2c \\ 0 & e & 0 \\ g & 0 & 2i \end{bmatrix}$$

$$\implies \begin{cases} 2g = g \\ 2c = c \end{cases} \implies \begin{cases} g = 0 \\ c = 0 \end{cases}$$

$$\therefore A = \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}$$

Choose $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then $AC = CA$

$$\implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} a & e & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix} = \begin{bmatrix} a & a & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}$$

$$\implies a = e$$

$$\therefore A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & i \end{bmatrix}$$

Choose $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then $AC = CA$

$$\implies \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & i \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} a & 0 & i \\ 0 & a & 0 \\ 0 & 0 & i \end{bmatrix} = \begin{bmatrix} a & 0 & a \\ 0 & a & 0 \\ 0 & 0 & i \end{bmatrix}$$

$\implies a = i$

$$\therefore A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the matrix representation of T is a scalar multiple of the identity matrix
$\implies T$ is a scalar multiple of the identity transformation $\blacksquare$

Problem 4.

(a)

Let $B = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix}$ where $\lambda_{ij} \in \mathbb{F}$

Define $\overline{T} : V \longrightarrow \mathbb{F}^n$ as follows: $\overline{T}(\vec{v}) = [\vec{v}]_{\mathcal{B}}$

Define $T : V \longrightarrow V$ as follows: $T(\vec{v}) = \overline{T}^{-1}(B[\vec{v}]_{\mathcal{B}})$

Showing T is linear:

Let $\vec{a}, \vec{b} \in V \qquad r \in \mathbb{F}$

$$\begin{aligned} T\left(\vec{a} + r\vec{b}\right) &= \overline{T}^{-1}\left(B\left[\vec{a} + r\vec{b}\right]_{\mathcal{B}}\right) && \text{[by def of } T\text{]} \\ &= \overline{T}^{-1}\left(B\left[\vec{a}\right]_{\mathcal{B}} + rB\left[\vec{b}\right]_{\mathcal{B}}\right) && \text{[by linearity of matrix multiplication]} \\ &= \overline{T}^{-1}\left(B\left[\vec{a}\right]_{\mathcal{B}}\right) + r\overline{T}^{-1}\left(B\left[\vec{b}\right]_{\mathcal{B}}\right) && \text{[since } \overline{T} \text{ is an isomorphism]} \\ &= T\left(\vec{a}\right) + rT\left(\vec{b}\right) && \text{[by definition of } T\text{]} \end{aligned}$$

$\therefore T$ is linear.

WTS: $[T]_{\mathcal{B}} = B$

$$\begin{aligned} [T(f_i)]_{\mathcal{B}} &= \overline{T}\left(\overline{T}^{-1}(B[f_i]_{\mathcal{B}})\right) && \text{[by def of } \overline{T}\text{]} \\ &= B[f_i]_{\mathcal{B}} && \text{[since } \overline{T} \circ \overline{T}^{-1} = I\text{]} \\ &= \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{\text{th}} \text{ row} && \text{[since } f_i = 0f_1 + \ldots + f_i + \ldots + 0f_n\text{]} \\ &= \begin{bmatrix} \lambda_{1i} \\ \lambda_{2i} \\ \vdots \\ \lambda_{ni} \end{bmatrix} \text{ where } 1 \le i \le n. && \text{[by matrix multiplication]} \end{aligned}$$

$$\Longrightarrow [T]_{\mathcal{B}} = \begin{bmatrix} | & | & & | \\ \left[T(f_1)\right]_{\mathcal{B}} & \left[T(f_2)\right]_{\mathcal{B}} & \cdots & \left[T(f_n)\right]_{\mathcal{B}} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} = B$$

$\therefore B$ is the $\mathcal{B}$-matrix of T. ■

(b)
Let $T : V \longrightarrow V, \qquad T' : V \longrightarrow V, \qquad \mathcal{B} = (f_1, \cdots, f_n)$
Suppose $[T]_{\mathcal{B}} = [T']_{\mathcal{B}}$
WTS: $T = T'$

$$\text{Let } [T]_{\mathcal{B}} = [T']_{\mathcal{B}} = B = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} \qquad \text{where } \lambda_{ij} \in \mathbb{F}$$

Let $v \in V$ be arbitrary.
Suppose $T(v) = w \qquad T'(v) = w'$

$$\begin{aligned} T(v) &= \overline{T}^{-1}\big([T]_{\mathcal{B}}[v]_{\mathcal{B}}\big) && \text{[by definition of } T\text{]} \\ &= \overline{T}^{-1}\big(B[v]_{\mathcal{B}}\big) && \text{[since } [T]_{\mathcal{B}} = B\text{]} \\ &= \overline{T}^{-1}\big([T']_{\mathcal{B}}[v]_{\mathcal{B}}\big) && \text{[since } [T']_{\mathcal{B}}\text{]} \\ &= T'(v) && \text{[by definition of } T'\text{]} \end{aligned}$$

$\implies w = w'$

So $\forall v \in V,\ T(v) = T'(v)$
$\implies T = T'$ ■

(c)

Let $\mathcal{T}_{\mathcal{B}} : \mathcal{L}(V, V) \longrightarrow M_{n\times n}$

to be defined as: $\mathcal{T}_{\mathcal{B}}(T) = \begin{bmatrix} | & | & & | \\ \left[T(f_1)\right]_{\mathcal{B}} & \left[T(f_2)\right]_{\mathcal{B}} & \cdots & \left[T(f_n)\right]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$

Showing $\mathcal{T}_{\mathcal{B}}$ is linear:

Let $A, B \in \mathcal{L}(V, V) \qquad r \in \mathbb{R}$

$\mathcal{T}_{\mathcal{B}}(A + rB)$

$$= \begin{bmatrix} | & | & & | \\ \left[(A+rB)(f_1)\right]_{\mathcal{B}} & \left[(A+rB)(f_2)\right]_{\mathcal{B}} & \cdots & \left[(A+rB)(f_n)\right]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \left[A(f_1)+rB(f_1)\right]_{\mathcal{B}} & \left[A(f_2)+rB(f_2)\right]_{\mathcal{B}} & \cdots & \left[A(f_n)+rB(f_n)\right]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \left[A(f_1)\right]_{\mathcal{B}} & \left[A(f_2)\right]_{\mathcal{B}} & \cdots & \left[A(f_n)\right]_{\mathcal{B}} \\ | & | & & | \end{bmatrix} + r \begin{bmatrix} | & | & & | \\ \left[B(f_1)\right]_{\mathcal{B}} & \left[B(f_2)\right]_{\mathcal{B}} & \cdots & \left[B(f_n)\right]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$$

$= \mathcal{T}_{\mathcal{B}}(A) + r\mathcal{T}_{\mathcal{B}}(B)$

$\therefore \mathcal{T}_{\mathcal{B}}$ is linear.

From part (a), we know that $\forall \mathcal{M} \in M_{n\times n}(\mathbb{R})\ \exists T \in \mathcal{L}(V, V)$ st $[T]_{\mathcal{B}} = \mathcal{M}$

This shows that $\mathcal{T}_{\mathcal{B}}$ is surjective since we have a T for every possible matrix.

From part (b), we know that the linear transformation T is unique.

This shows that $\mathcal{T}_{\mathcal{B}}$ is injective since no two T's will map to the same matrix.

$\therefore \mathcal{T}_{\mathcal{B}}$ is an isomorphism $\Longrightarrow \mathcal{L}(V, V) \cong M_{n\times n}$

We also know that $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n\times n}$ and an isomorphism composed with an isomorphism is an isomorphism.

$\therefore \mathcal{L}(V, V) \cong M_{n\times n} \cong \mathbb{R}^{n\times n}$

$\Longrightarrow \mathcal{L}(V, V) \cong \mathbb{R}^{n\times n}$ ∎

(d)
From (c), we know that $\mathcal{L}(V, V) \cong \mathbb{R}^{n\times n}$, so
$a_0 I + a_1 T + \ldots + a_m T^m = 0$
is equivalent to
$a_0 v_0 + a_1 v_1 + \ldots + a_m v_m = 0$

Choose $m = n \times n$

So we have $a_0 v_0 + a_1 v_1 + \ldots + a_{n\times n} v_{n\times n} = 0$
We know for a vector space of dimension $n \times n$, there are max $n \times n$ linearly independent vectors. The set $\{v_0, \cdots, v_{n\times n}\}$ has $n \times n + 1$ elements.
$\implies \{v_0, \cdots, v_{n\times n}\}$ is a linearly dependent relationship.
So $\exists a_i \in \{a_0, \cdots, a_{n\times n}\}$ st $a_i \neq 0$ ■

Problem 5.

(1)
4 Equiangular lines:
Suppose a cube is cenetered around the origin
Let 4 lines connect from the origin to the top 4 corners of the cube. Those 4 lines are equiangular.

6 Equiangular lines:
Suppose an icosohedron is centered around the origin
Let 6 lines connect from the origin to the top 6 verticies of the icosohedron. Those 6 lines are equiangular.

Dodecahedron:
We cannot construct equiangular lines using a dodecahedron.

(2)

$$\begin{aligned}
\text{Let } A &= \big(1 - cos^2(\theta)\big)\mathrm{id}_n + cos^2(\theta)J_n \\
&= sin^2(\theta)\mathrm{id}_n + cos^2(\theta)J_n \\
&= \begin{bmatrix} sin^2(\theta)+cos^2(\theta) & cos^2(\theta) & \cdots & cos^2(\theta) \\ cos^2(\theta) & sin^2(\theta)+cos^2(\theta) & \cdots & cos^2(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ cos^2(\theta) & cos^2(\theta) & \cdots & sin^2(\theta)+cos^2(\theta) \end{bmatrix} \\
&= \begin{bmatrix} 1 & cos^2(\theta) & \cdots & cos^2(\theta) \\ cos^2(\theta) & 1 & \cdots & cos^2(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ cos^2(\theta) & cos^2(\theta) & \cdots & 1 \end{bmatrix}
\end{aligned}$$

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad x_i \in \mathbb{F}$

Suppose $A\vec{x} = \vec{0}$
WTS: $\vec{x} = \vec{0}$

$$(\clubsuit): \quad \begin{bmatrix} 1 & cos^2(\theta) & \cdots & cos^2(\theta) \\ cos^2(\theta) & 1 & \cdots & cos^2(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ cos^2(\theta) & cos^2(\theta) & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\implies \begin{cases} (1) & x_1 + x_2cos^2(\theta) + \ldots + x_ncos^2(\theta) = 0 \\ (2) & x_1cos^2(\theta) + x_2 + \ldots + x_ncos^2(\theta) = 0 \\ \vdots & \vdots \qquad\qquad \vdots \\ (n) & x_1cos^2(\theta) + x_2cos^2(\theta) + \ldots + x_n = 0 \end{cases}$$

Let $i, j \in \{1, \cdots, n\} \qquad i \neq j$
Subtract $(i) - (j)$

$$\begin{aligned} (i) - (j) &= x_i + x_jcos^2(\theta) - x_icos^2(\theta) - x_j \\ &= x_i\big(1 - cos^2(\theta)\big) + x_j\big(cos^2(\theta) - 1\big) \\ &= x_i\big(1 - cos^2(\theta)\big) - x_j\big(1 - cos^2(\theta)\big) \\ &= \big(1 - cos^2(\theta)\big)(x_i - x_j) \\ &= 0 \\ &\implies x_i = x_j \qquad \text{since } 1 - cos^2(\theta) \neq 0 \text{ for } \theta \in (0, \pi/2] \end{aligned}$$

$$\implies x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \lambda \in \mathbb{F}$$

$$(\clubsuit): \quad \lambda \begin{bmatrix} 1 & cos^2(\theta) & \cdots & cos^2(\theta) \\ cos^2(\theta) & 1 & \cdots & cos^2(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ cos^2(\theta) & cos^2(\theta) & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The only way to sastify ($\clubsuit$) is if $\lambda = \vec{x} = 0$
$\therefore A$ is invertible. $\blacksquare$

(3)
Suppose $L_1, \cdots, L_n$ are equiangular.
Let $v_i = (a_i, b_i, c_i) \qquad a_i, b_i, c_i \in \mathbb{R} \qquad i, j \in \{1, \cdots, n\} \qquad i \neq j$

Then $\big| \langle v_i, v_j \rangle \big| = cos(\theta) \qquad \theta \in \left(0, \dfrac{\pi}{2}\right]$
$\implies |a_i a_j + b_i b_j + c_i c_j| = cos(\theta)$
So $(a_i a_j + b_i b_j + c_i c_j)^2 = cos^2(\theta)$
$\implies a_i^2 a_j^2 + b_i^2 b_j^2 + c_i^2 c_j^2 + 2a_i a_j b_i b_j + 2a_i a_j c_i c_j + 2b_i b_j c_i c_j = cos^2(\theta)$

$$v_i v_i^t = \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} (a_i, b_i, c_i) = \begin{bmatrix} a_i^2 & a_i b_i & a_i c_i \\ a_i b_i & b_i^2 & b_i c_i \\ a_i c_i & b_i c_i & c_i^2 \end{bmatrix}$$

Let $\lambda_1 \begin{bmatrix} a_1^2 & a_1 b_1 & a_1 c_1 \\ a_1 b_1 & b_1^2 & b_1 c_1 \\ a_1 c_1 & b_1 c_1 & c_1^2 \end{bmatrix} + \cdots + \lambda_n \begin{bmatrix} a_n^2 & a_n b_n & a_n c_n \\ a_n b_n & b_n^2 & b_n c_n \\ a_n c_n & b_n c_n & c_n^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_i \in \mathbb{R}$
WTS: $\lambda_i = 0 \qquad \forall i \in \{1, \cdots, n\}$

$$\lambda_1 \begin{bmatrix} a_1^2 & a_1 b_1 & a_1 c_1 \\ a_1 b_1 & b_1^2 & b_1 c_1 \\ a_1 c_1 & b_1 c_1 & c_1^2 \end{bmatrix} + \cdots + \lambda_n \begin{bmatrix} a_n^2 & a_n b_n & a_n c_n \\ a_n b_n & b_n^2 & b_n c_n \\ a_n c_n & b_n c_n & c_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 a_1^2 & \lambda_1 a_1 b_1 & \lambda_1 a_1 c_1 \\ \lambda_1 a_1 b_1 & \lambda_1 b_1^2 & \lambda_1 b_1 c_1 \\ \lambda_1 a_1 c_1 & \lambda_1 b_1 c_1 & \lambda_1 c_1^2 \end{bmatrix} + \cdots + \begin{bmatrix} \lambda_n a_n^2 & \lambda_n a_n b_n & \lambda_n a_n c_n \\ \lambda_n a_n b_n & \lambda_n b_n^2 & \lambda_n b_n c_n \\ \lambda_n a_n c_n & \lambda_n b_n c_n & \lambda_n c_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 a_1^2 + \ldots + \lambda_n a_n^2 & \lambda_1 a_1 b_1 + \ldots + \lambda_n a_n b_n & \lambda_1 a_1 c_1 + \ldots + \lambda_n a_n c_n \\ \lambda_1 a_1 b_1 + \ldots + \lambda_n a_n b_n & \lambda_1 b_1^2 + \ldots + \lambda_n b_n^2 & \lambda_1 b_1 c_1 + \ldots + \lambda_n b_n c_n \\ \lambda_1 a_1 c_1 + \ldots + \lambda_n a_n c_n & \lambda_1 b_1 c_1 + \ldots + \lambda_n b_n c_n & \lambda_1 c_1^2 + \ldots + \lambda_n c_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} (1) & \sum_{i=1}^{n} \lambda_i a_i^2 = 0 \\ (2) & \sum_{i=1}^{n} \lambda_i b_i^2 = 0 \\ (3) & \sum_{i=1}^{n} \lambda_i c_i^2 = 0 \\ & \sum_{i=1}^{n} \lambda_i a_i b_i = 0 \\ & \sum_{i=1}^{n} \lambda_i a_i c_i = 0 \\ & \sum_{i=1}^{n} \lambda_i b_i c_i = 0 \end{cases} \implies \begin{cases} (4) & \sum_{i=1}^{n} \lambda_i a_i^2 a_j^2 = 0 \\ (5) & \sum_{i=1}^{n} \lambda_i b_i^2 b_j^2 = 0 \\ (6) & \sum_{i=1}^{n} \lambda_i c_i^2 c_j^2 = 0 \\ (7) & \sum_{i=1}^{n} \lambda_i a_i b_i a_j b_j = 0 \\ (8) & \sum_{i=1}^{n} \lambda_i a_i c_i a_j c_j = 0 \\ (9) & \sum_{i=1}^{n} \lambda_i b_i c_i b_j c_j = 0 \end{cases} \quad \text{for some } j \in \{1, \cdots, n\}$$

Adding (1) + (2) + (3) results in:

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i a_i^2 + \sum_{i=1}^{n} \lambda_i b_i^2 + \sum_{i=1}^{n} \lambda_i c_i^2 &= \sum_{i=1}^{n} \lambda_i \big(a_i^2 + b_i^2 + c_i^2\big) && \text{[sum rules]} \\ &= \sum_{i=1}^{n} \lambda_i && \text{[since } \vec{v_i} \text{ is a unit vector]} \\ &= 0 && \text{[since all equations = 0]} \end{aligned}$$

Adding (4) + (5) + (6) + 2×(7) + 2×(8) + 2×(9) results in

$$\begin{aligned} &\sum_{i=1}^{n} \lambda_i a_i^2 a_j^2 + \lambda_i b_i^2 b_j^2 + \lambda_i c_i^2 c_j^2 + 2\lambda_i a_i b_i a_j b_j + 2\lambda_i a_i c_i a_j c_j + 2\lambda_i b_i c_i b_j c_j \\ &= \left(\sum_{i=1}^{n} \lambda_i cos^2(\theta)\right) - \lambda_j \left(a_j^4 + b_j^4 + c_j^4 + 2a_j^2 b_j^2 + 2a_j^2 c_j^2 + 2b_j^2 c_j^2\right) \\ &= \left(\sum_{i=1}^{n} \lambda_i cos^2(\theta)\right) - \lambda_j \left(a_j^2 + b_j^2 + c_j^2\right)^2 \\ &= \left(\sum_{i=1}^{n} \lambda_i cos^2(\theta)\right) - \lambda_j \qquad \text{[since } \vec{v_j} \text{ is a unit vector]} \\ &= 0 \qquad \text{[since all equations add to 0]} \end{aligned}$$

$$\implies \lambda_j = \sum_{i=1}^{n} \lambda_i cos^2(\theta)$$

$$\implies \lambda_1 = \lambda_2 = \ldots = \lambda_j = \ldots = \lambda_n$$

Since $\sum_{i=1}^{n} \lambda_i = 0$
and $\lambda_1 = \lambda_2 = \ldots = \lambda_j = \ldots = \lambda_n$

$$\begin{aligned} \text{Then } \sum_{i=1}^{n} \lambda_j &= \lambda_j \sum_{i=1}^{n} 1 && \text{[since } j \text{ doesn't depend on } i\text{]} \\ &= n\lambda_j && \text{[by summing 1 } n\text{-times]} \\ &= 0 && \text{[by given]} \end{aligned}$$

$\therefore \lambda_j = 0 \qquad \forall j \in \{1, \cdots, n\}$
as wanted. ∎

(4)
A basis for symmetric matricies are:

$$\mathcal{B} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & d \\ 0 & 0 & 0 \\ d & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f \end{bmatrix} \right\}$$

$|\mathcal{B}| = 6$

Because the matrix $v_i v_i^T$ for $i \in \{1, \cdots, \#\text{of equiangular lines}\}$ are linearly independent, and they're all symmetric, then the max number of equiangular lines is the dimension of symmetric matricies.

$\therefore$ The largest number of equiangular lines in $\mathbb{R}^3$ is 6. ∎

(5)
The dimension of an $n \times n$ symmetric matrix is:
A symmetric $n \times n$ matrix is of the form:

$$A = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{12} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1n} & \lambda_{2n} & \cdots & \lambda_{nn} \end{bmatrix}$$

So the number of basis matricies that can be made is determined by the number of elemnts in an upper triangle matrix. This is because the bottom triangle is determined by the top part of the triangle.

$$\begin{aligned} \dim(A) &= \sum_{n=1}^{n} n && \text{[sum of a triangle]} \\ &= \frac{n(n+1)}{2} && \text{[by sum rules]} \\ &= \frac{(n+1)(n)(n-1)(n-2)\cdots(2)(1)}{(2!)(n-1)(n-2)\cdots(2)(1)} && \text{[multiplying both top and bottom]} \\ &= \binom{n+1}{2} && \text{[by combination roles]} \end{aligned}$$

So there are max $\binom{n+1}{2}$ number of equiangular lines. ■