

(\Rightarrow)

Suppose a language L is decidable.

Let M be a decider TM such that $\mathcal{L}(M) = L$

Let $S = x_1, x_2, \dots$ be the sequence of words of L in shortlex order.

Here's an enumerator that prints L in shortlex order

$E =$ "Ignore the input

1. for $i = 1$ to ∞
2. run M on x_i
3. if M accepts x_i
4. print x_i "

(\Leftarrow)

Suppose enumerator E enumerates L .

Here's a TM that is a decider for L .

$M =$ "On input w

1. run E
2. if E prints w
3. accept
4. if the output of $E > |w|$ "
5. reject"

To prove M is a decider, we will look at 2 cases:

Case 1: L is unbounded

M never loops since $|x_i|$ is increasing as i increases.

Since Σ is finite, x_i will eventually increase as you can't have an infinite list of words with equal length.

Line 4 will eventually be true which leads to M rejecting.

Case 2: L is bounded

If the length of words in L is bounded by some $n \in \mathbb{N}$, then S is finite.

So even if $|w| > n$, E will eventually stop printing which leads to

M rejecting.