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Question 1.

(a)

$$\begin{aligned}
A(n) &= n^2 A(n-1) \\
A(n-1) &= (n-1)^2 A(n-2) \\
A(n-2) &= (n-2)^2 A(n-3) \\
&\vdots \\
A(2) &= (2)^2 A(1) \\
A(1) &= 1
\end{aligned}$$

So by **Repeated Back Substitution,** we have:

$$\begin{aligned}
A(n) &= n^2(n-1)^2(n-2)^2 \cdots (2)^2(1)^2 \\
&= \big(n(n-1)(n-2)\cdots(2)(1)\big)^2 \\
&= (n!)^2
\end{aligned}$$

$$\therefore A(n) \in \Theta\big((n!)^2\big)$$

(b)

$$C(n) = \begin{cases} 8C\left\lfloor \dfrac{n}{2} \right\rfloor + 2n^3 + 4n & \text{if } n > 0 \\ 6 & \text{if } n = 0 \end{cases}$$

We can calculate $C(1)$ by plugging it in.

$$\begin{aligned} C(1) &= 8C\left\lfloor \frac{1}{2} \right\rfloor + 2(1)^3 + 4(1) \\ &= 8C(0) + 2 + 4 \\ &= 8(6) + 6 \\ &= 54 \end{aligned}$$

Now we can rewrite $C(n)$ to...

$$C(n) = \begin{cases} 8C\left\lfloor \dfrac{n}{2} \right\rfloor + 2n^3 + 4n & \text{if } n > 1 \\ 54 & \text{if } n = 1 \end{cases}$$

We can see that this sastisfies the **Generalized Master Theorem**, where

$a = 8$ $\quad b = 2$ $\quad c = 3$ $\quad d = 54$ $\quad f(n) = 2n^3 + 4n$

So we know that $f(n) \in \Theta(n^3)$, and $\log_2 8 = 3$
$\implies \log_b a = c$

By part (b) of **Generalized Master Theorem**, we have

$$\begin{aligned} T(n) &= \Theta(n^{\log_b a} \log_b n) \\ &= \Theta(n^3 \log_2 n) \end{aligned}$$

Problem 2.

Suppose...

$f(n) \in \mathcal{O}\big(g(n)\big) \qquad f(n) \geq 1 \qquad \log\big(g(n)\big) \geq 1 \qquad \forall n \in \mathbb{N}$

WTS: $\log\big(f(n)\big) \in \mathcal{O}\Big(\log\big(g(n)\big)\Big)$ **OR**

$\exists c' \in \mathbb{R}^+,\ \exists n_0' \in \mathbb{N},\ \forall n > n_0' \implies \log\big(f(n)\big) \leq c' \cdot \log\big(g(n)\big)$ **(★)**

Since $f(n) \in \mathcal{O}\big(g(n)\big)$ we have...

$\exists c \in \mathbb{R}^+,\ \exists n_0 \in \mathbb{N},\ \forall n > n_0 \implies f(n) \leq c \cdot g(n)$

$$
\begin{aligned}
f(n) &\leq c \cdot g(n) && \text{[by given]}\\
\log\big(f(n)\big) &\leq \log\big(c \cdot g(n)\big) && \text{[logging both sides since both sides } \geq 1]\\
&\leq \log(c) + \log\big(g(n)\big) && \text{[by log laws]}\\
&\leq \log(c)\log\big(g(n)\big) + \log\big(g(n)\big) && [\text{since } \log\big(g(n)\big) \geq 1]\\
&\leq \big(\log(c) + 1\big)\log\big(g(n)\big) && \text{[by factoring]}
\end{aligned}
$$

So choose:

$c' = \log(c) + 1 \qquad n_0' = n_0$

Then the predicate **(★)** holds ■

Problem 3.

Implementation:
for `RECENT`, I would use an `AVL Tree` and a `Doubly Linked List`.
I would also keep track of `size`, and the `head` and `tail` of the `Doubly Linked List`.

In the `AVL Tree`, the nodes would have an extra parameter:
`target`: a pointer to its node in the `Doubly Linked List`.

For every `ACCESS(x)`, it would insert `x` into both the `AVL Tree` and the `Doubly Linked List`.
If `x` is already in the `AVL Tree`, then use `target` to delete that node in the `Doubly Linked List`.
Then re-insert it back into the head of the `Doubly Linked List` by using `head`.

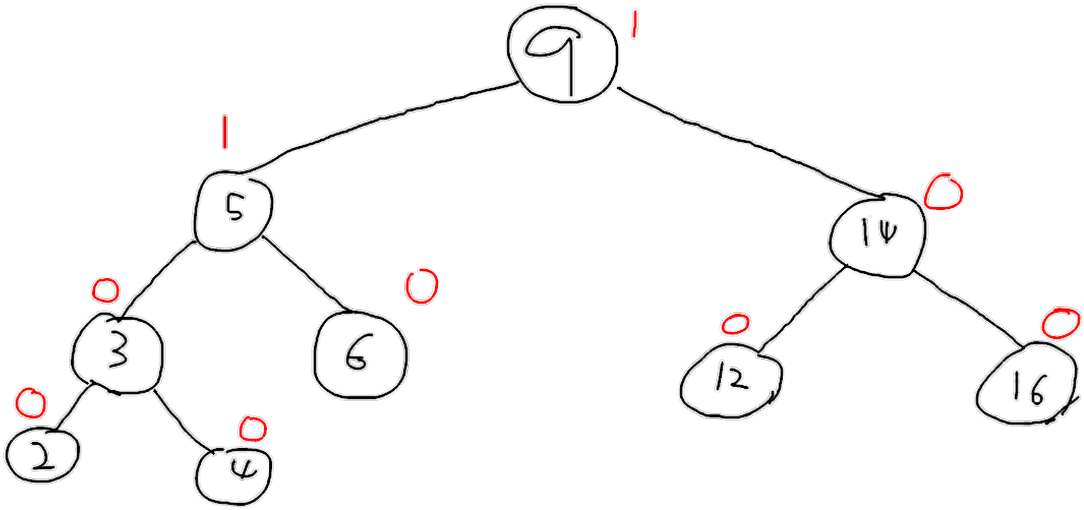
If `size > m`, then delete both `tail`, and its corresponding node in the `AVL Tree`. Then continue to insert `x`.

Justification:
This data structure uses $\mathcal{O}(m \log n)$ space since we have 2 data structures of size `m`. This means that we will only have about `2m` nodes to store, which according to this piazza post, takes about $2(m \log n) = \mathcal{O}(m \log n)$ space

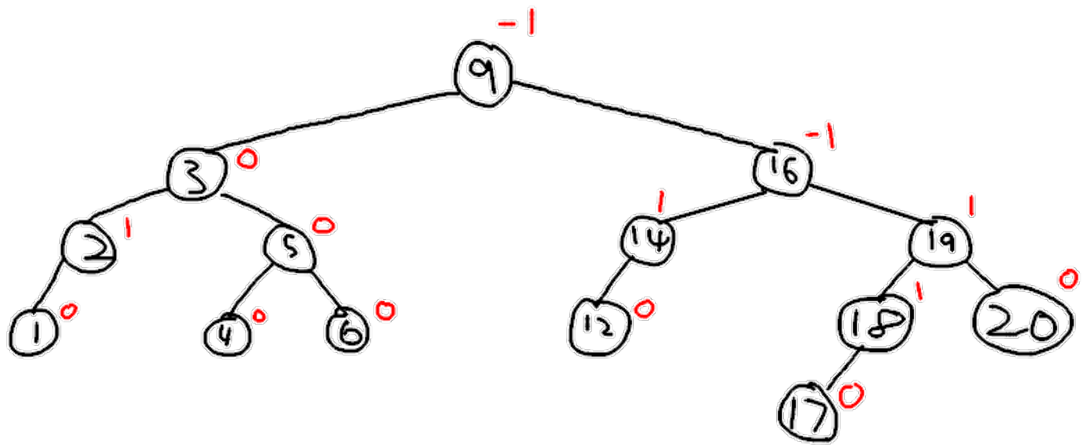
This data structure takes worst case $\mathcal{O}(\log m)$ time complexity. This is because for every `ACCESS(x)`, we need to search the `AVL Tree` which takes $\mathcal{O}(\log m)$ time. It could also delete and insert in the `Doubly Linked List` after searching, which would take $\mathcal{O}(1)$ since we keep a pointer to `size`, `head`, and `tail`. Lastly, we would also need to delete from both the `AVL Tree` and the `Doubly Linked List` if `size` gets too large, which would take $\mathcal{O}(\log m)$ and $\mathcal{O}(1)$ time respectively.
Overall, the worst case time complexity is $\mathcal{O}(\log m)$.

Problem 4.

Tree after inserting 5, 4, 6, 9, 12, 16, 14, 2, 3



Tree after inserting 1, 20, 19, 18, 17



Tree after deleting 1, 2, 3, 4, 5, 12, 6