

Let $L = \{\langle M \rangle \mid \exists w \in \Sigma^* \text{ such that when } M \text{ runs on } w, M \text{ prints a blank on a non-blank cell}\}$

WTS: L is undecidable

$\implies HALT \leq_m L$

Consider the following TM F and the reduction it computes:

$F =$ “On input $\langle M, w \rangle$:

1. Construct a TM M_1 as follows:

$M_1 =$ “On input $\langle M, w \rangle$:

1. for $s = 1$ to ∞
2. run M on w for s steps
3. if M writes a blank on a non-blank cell then accept”

2. Construct a TM M_2 as follows:

$M_2 =$ “On input x :

1. \triangleright empty Part1
2. run M_1 on $\langle M, w \rangle$
3. accept”

3. return $\langle M_2 \rangle$ ”

We argue that $\langle M, w \rangle \in HALT \iff \langle M_2 \rangle \in L$

(\implies)

Suppose $\langle M, w \rangle \in HALT$

M halts on w

[definition of $HALT$]

$\implies M_2$ accepts $\langle M, w \rangle$ such that M prints a blank in a non-blank cell

[description of M_2]

$\implies \mathcal{L}(M_2) = \{\langle M \rangle \mid \exists w \in \Sigma^* \text{ such that when } M \text{ runs on } w, M \text{ prints a blank on a non-blank cell}\}$

$\implies \langle M_2 \rangle \in L$

[definition of L]

as wanted.

(\impliedby)

Suppose $\langle M, w \rangle \notin HALT$

M loops on w

[definition of $HALT$]

$\implies M_2$ loops on every input

[description of M_2]

$\implies \mathcal{L}(M_2) \neq \{\langle M \rangle \mid \exists w \in \Sigma^* \text{ such that when } M \text{ runs on } w, M \text{ prints a blank on a non-blank cell}\}$

$\implies \langle M_2 \rangle \notin L$

[definition of L]

as wanted.