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Problem 1.

Let $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$ be an orthonormal basis for $\mathbb{R}^n$ such that
$B_i \subseteq \mathcal{B} \qquad \text{sp}(B_i) = V_i \qquad \text{for } i \in \{1, 2, \cdots, p\}$

Let $\vec{v}_i \in V_i \qquad \vec{v}_j \in V_j$ be arbitrary. Then
$\vec{v}_i \in \text{sp}(B_i) \qquad \vec{v}_j \in \text{sp}(B_j)$
Since $\vec{v}_i \perp \vec{v}_j \qquad \implies \qquad \langle \vec{v}_i,\ \vec{v}_j \rangle = 0$
But we know that $\vec{v}_i$ is a linear combination of vectors in B_i, and $\vec{v}_j$ is a linear combination of vectors in B_j
$\langle x,\ y \rangle = 0 \qquad \forall x \in B_i \qquad \forall y \in B_j$
We also know that $b \neq 0 \qquad \forall b \in \mathcal{B}$ since $\mathcal{B}$ is a basis.
This means that no 2 vectors in $\mathcal{B}$ is orthogonal to itself.
In order for the above statement to hold, we must have $B_i \cap B_j = \varnothing$
since V_i is mutually orthogonal to V_j.

$\therefore$ for a collection of mutually orthogonal subspaces $V_1, \cdots, V_p$, they must all strictly share the elements in $\mathcal{B}$. We know the dimension of R^n is n, and $V_1, \cdots, V_p$ are all spanned by a subset of vectors in $\mathcal{B}$.

$\therefore \dim V_1 + \ldots + \dim V_p \leq n$ ■

Problem 2.

(a)
Suppose U is an orthogonal $n \times n$ matrix.
$\implies UU^T = U^TU = I \qquad U^T = U^{-1}$

Define $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ to be
$\vec{v} \to U^T\vec{v}$
This is the inverse of T since

$$T^{-1} \circ T = U^TU \qquad\qquad T \circ T^{-1} = UU^T$$
$$= I \qquad\qquad = I$$

$\implies T$ is an isomorphism.

We are given that $T(W) \subseteq W$, and we know that T of a subspace is also a subspace. So it is enough to show: $\dim(W) = \dim\big(T(W)\big)$
since this implies $T(W) = W$

We know that T is an isomorphism, so T preserves dimension
$\implies \dim(W) = \dim\big(T(W)\big)$
$\therefore T(W) = W$ ■

(b)
We know by a theorem we learned in class that $\mathbb{R}^n = W \oplus W^\perp$
We also know from part (a) that $T(W)^\perp = W^\perp$
WTS: $T(W)^\perp = T(W^\perp)$

Let $\vec{v} \in T(W)^\perp \qquad \vec{w} \in W$

$$\begin{aligned}
\implies \quad & \vec{v} \cdot \vec{w} = 0 && \text{[since they're orthogonal]} \\
\implies \quad & \vec{v} \cdot U\vec{w} = 0 && \text{[since } U\vec{w} \in W\text{]} \\
\implies \quad & \vec{v}^T U\vec{w} = 0 && \text{[changing dot product to matrix multiplication]} \\
\implies \quad & \big(U^T\vec{v}\big)^T \vec{w} = 0 && \text{[transpose properties]} \\
\implies \quad & \big(U^{-1}\vec{v}\big)^T \vec{w} = 0 && \text{[since } U^T = T^{-1}\text{]} \\
\implies \quad & \big(T^{-1}(\vec{v})\big)^T \vec{w} = 0 && \text{[since } T^{-1}(\vec{x}) = U^{-1}(\vec{x})\text{]} \\
\implies \quad & \big(T^{-1}(\vec{v})\big) \cdot \vec{w} = 0 && \text{[changing dot product to matrix multiplication]} \\
\implies \quad & T^{-1}(\vec{v}) \in W^\perp && \text{[since } \vec{w} \perp T^{-1}(\vec{v})\text{]} \\
\implies \quad & \vec{v} \in T(W^\perp) && \text{[applying } T \text{ to both sides]}
\end{aligned}$$

Since $\vec{v}$ is arbitrary, we can say that $T(W^\perp) \subseteq W^\perp$
By the same arguments in (a), this implies $T(W^\perp) = W^\perp$ ■

Problem 3.

(a)

Let $\mathcal{B} = \{b_1, \cdots, b_n\}$ where $\mathcal{B}$ is an orthonormal basis.

$\implies \langle b_i,\ b_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Define $T : V \to \mathbb{R}^n$ to be

$T(\vec{v}) \to [\vec{v}]_{\mathcal{B}}$

Which is an isomorphism.

WTS: $\langle f,\ g\rangle = T(f) \cdot T(g)$

Since $\mathcal{B}$ is a basis, let

$$f = \alpha_1 b_1 + \ldots + \alpha_n b_n = \sum_{i=1}^{n} \alpha_i b_i \qquad g = \lambda_1 b_1 + \ldots + \lambda_n b_n = \sum_{j=1}^{n} \lambda_j b_j \qquad \text{where } \alpha_i,\ \lambda_i \in \mathbb{R}$$

$$\begin{aligned}
\langle f,\ g\rangle &= \left\langle \sum_{i=1}^{n} \alpha_i b_i,\ \sum_{j=1}^{n} \lambda_j b_j \right\rangle && \text{[by given]} \\
&= \sum_{i=1}^{n} \alpha_i \left\langle b_i,\ \sum_{j=1}^{n} \lambda_j b_j \right\rangle && \text{[since inner product is linear]} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \lambda_j \langle b_i,\ b_j\rangle && \text{[since inner product is linear over reals]} \\
&= \sum_{i=1}^{n} \alpha_i \lambda_i \langle b_i,\ b_i\rangle && \text{[since } \langle b_i,\ b_j\rangle = 0 \text{ if } i \neq j\text{]} \\
&= \sum_{i=1}^{n} \alpha_i \lambda_i && \text{[since } \langle b_i,\ b_j\rangle = 1 \text{ if } i = j\text{]} \\
&= T(f) \cdot T(g) && \text{[by definition of dot product]}
\end{aligned}$$

as wanted. ■

(b)
Suppose A is an $n{\times}n$ matrix such that
$\forall \vec{v} \in \mathbb{R}^n, \qquad \vec{v} \neq 0 \Longrightarrow \vec{v}^T A \vec{v} > 0 \qquad (\bigstar)$

Consider $\langle v,\ w\rangle = v^T A w$

WTS: (1) $\langle v,\ v\rangle \geq 0$
(2) $\langle v,\ v\rangle = 0 \iff v = 0$
(3) $\langle u + \lambda v,\ w\rangle = \langle u,\ w\rangle + \lambda\langle v,\ w\rangle$
(4) $\langle u,\ v\rangle = \langle v,\ u\rangle$

(1)
$$\begin{aligned} \langle v,\ v\rangle &= v^T A v && \text{[by given]} \\ &\geq 0 && \text{[by } (\bigstar)\text{]} \end{aligned}$$

(2)
$\Longrightarrow$ direction :
$$\begin{aligned} \langle v,\ v\rangle &= v^T A v && \text{[by given]} \\ &= 0 && \text{[by given]} \end{aligned}$$
By contrapositive of $(\bigstar)$, we have $\vec{v}^T A \vec{v} \leq 0 \Longrightarrow \vec{v} = 0$
$\therefore v = 0$ in this case.

$\Longleftarrow$ direction :
If $v = 0$ we have:
$$\begin{aligned} \langle v,\ v\rangle &= v^T A v && \text{[by given]} \\ &= 0^T A 0 && \text{[by given]} \\ &= 0 && \text{[since multiplying by 0]} \end{aligned}$$

(3)

$$
\begin{aligned}
\langle u + \lambda v,\ w\rangle &= (u + \lambda v)^T A w && \text{[by given]}\\
&= (u^T + \lambda v^T) A w && \text{[by transpose properties]}\\
&= u^T A w + \lambda v^T A w && \text{[by expanding]}\\
&= \langle u,\ w\rangle + \lambda \langle v,\ w\rangle && \text{[by definition of the inner product]}
\end{aligned}
$$

(4)

$$
\begin{aligned}
\langle u,\ v\rangle &= u^T A v && \text{[by given]}\\
&= \left(u^T A v\right)^T && \text{[since the transpose of a scalar is the same]}\\
&= (Av)^T \left(u^T\right)^T && \text{[by transpose properties]}\\
&= v^T A^T u && \text{[by transpose properties]}\\
&= v^T A u && \text{[since } A^T = A\text{]}\\
&= \langle v,\ u\rangle && \text{[by definition of the inner product]}
\end{aligned}
$$

$\therefore \langle -,\ -\rangle$ is an inner product.

From part (a), we know that
if $T : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $T : \vec{v} \to [\vec{v}]_{\mathcal{B}}$,
then $\langle v,\ w\rangle = T(v) \cdot T(w) \qquad (\heartsuit)$

Choose $B =$ the change of basis matrix from $v \to [\vec{v}]_{\mathcal{B}}$
$\implies B$ is invertible

$$\begin{aligned}
\langle v,\ w\rangle &= v^T A w && \text{[by given]}\\
&= T(v) \cdot T(w) && \text{[by } (\heartsuit)\text{]}\\
&= Bv \cdot Bw && \text{[by our choice of } B\text{]}\\
&= (Bv)^T Bw && \text{[changing dot product to matrix multiplication]}\\
&= v^T B^T B w && \text{[by transpose properties]}
\end{aligned}$$

$\implies v^T A w = v^T B^T B w$

Since the above equation holds $\forall v, w \in \mathbb{R}^n$, choose $w = e_j \qquad v = e_i \qquad i, j \in \{1, 2, \cdots, n\}$

$$\begin{aligned}
v^T A w &= v^T (A e_j) && \text{[by given]}\\
&= v^T (j^{\text{th}} \text{ column of } A) && \text{[by matrix multiplication]}\\
&= {e_i}^T (j^{\text{th}} \text{ column of } A) && \text{[by given]}\\
&= i^{\text{th}} \text{ row of } (j^{\text{th}} \text{ column of } A) && \text{[by matrix multiplication]}\\
&= A_{ij} && \text{[matrix notation]}
\end{aligned}$$

A similar argument holds for the matrix $B^T B$

$\implies A_{ij} = B^T B_{ij}$
Since i, j are arbitrary, this means all entries in the matrix are equal.
$\therefore A = B^T B$, as wanted $\blacksquare$

Problem 4.

(a)
Let $\vec{w} \in V$ be arbitrary
$\vec{w} = (w_1, \cdots, w_n)$
Let $\vec{v}_1, \vec{v}_2 \in V \qquad r \in \mathbb{F}$
WTS: $\langle \vec{v}_1 + r\vec{v}_2,\ \vec{w} \rangle = \langle \vec{v}_1,\ \vec{w} \rangle + r\langle \vec{v}_2,\ \vec{w} \rangle$

$$\begin{aligned}
\langle \vec{v}_1 + r\vec{v}_2,\ \vec{w} \rangle &= \langle \vec{v}_1,\ \vec{w} \rangle + \langle r\vec{v}_2,\ \vec{w} \rangle && \text{[by linearity of inner product]} \\
&= \langle \vec{v}_1,\ \vec{w} \rangle + r\langle \vec{v}_2,\ \vec{w} \rangle && \text{[by linearity of inner product]}
\end{aligned}$$

$\therefore$ the function $\vec{v} \ \mapsto \ \langle \vec{v},\ \vec{w} \rangle$ is linear ■

(b)
Let $\mathcal{U} = \{u_1, \cdots, u_n\}$ be an orthonormal basis for V
WTS: $\exists \vec{w} \in V$ such that $T(\vec{v}) = \langle \vec{v},\ \vec{w} \rangle$

Since $\mathcal{U}$ is an orthonormal basis, then $\vec{v} = \langle v_1,\ u_1 \rangle u_1 + \langle v_2,\ u_2 \rangle u_2 + \ldots + \langle v_n,\ u_n \rangle u_n$

$$\begin{aligned}
T(\vec{v}) &= T\big(\langle \vec{v},\ u_1 \rangle u_1 + \ldots + \langle \vec{v},\ u_n \rangle u_n\big) && \text{[by given]} \\
&= T\big(\langle \vec{v},\ u_1 \rangle u_1\big) + \ldots + T\big(\langle \vec{v},\ u_n \rangle u_n\big) && \text{[by linearity of } T\text{]} \\
&= \langle \vec{v},\ u_1 \rangle T(u_1) + \ldots + \langle \vec{v},\ u_n \rangle T(u_n) && \text{[since } \langle \vec{v},\ u_i \rangle \text{ is a scalar]} \\
&= \left\langle \vec{v},\ \overline{T(u_1)} u_1 \right\rangle + \ldots + \left\langle \vec{v},\ \overline{T(u_n)} u_n \right\rangle && \text{[bringing a scalar into the second slot]} \\
&= \left\langle \vec{v},\ \overline{T(u_1)} u_1 + \ldots + \overline{T(u_n)} u_n \right\rangle && \text{[by linearity of inner product]}
\end{aligned}$$

So choose $\vec{w} = \overline{T(u_1)} u_1 + \ldots + \overline{T(u_n)} u_n \qquad \vec{w} \in V$
$\therefore T(\vec{v}) = \langle \vec{v},\ \vec{w} \rangle$

WTS: $\vec{w}$ is unique.

Suppose $T(\vec{v}) = \langle \vec{v},\ \vec{w}_1 \rangle = \langle \vec{v},\ \vec{w}_1 \rangle \qquad \forall \vec{v} \in V$
Enough to prove: $\vec{w}_2 = \vec{w}_2$

$$\begin{aligned}
\langle \vec{v},\ \vec{w}_1 \rangle = \langle \vec{v},\ \vec{w}_1 \rangle &\implies \langle \vec{v},\ \vec{w}_1 \rangle - \langle \vec{v},\ \vec{w}_1 \rangle = 0 \\
&\implies \langle \vec{v},\ \vec{w}_1 - \vec{w}_2 \rangle = 0
\end{aligned}$$

Since this holds $\forall \vec{v} \in V$, choose $\vec{v} = \vec{w}_1 - \vec{w}_2$
$\implies \ \langle \vec{w}_1 - \vec{w}_2,\ \vec{w}_1 - \vec{w}_2 \rangle = 0$
Since an inner product is zero-definite, then this means
$\vec{w}_1 - \vec{w}_2 = 0 \qquad \implies \qquad \vec{w}_1 = \vec{w}_2$ ■

(c)

Let $t \in \mathbb{R}$ be arbitrary.
Define an inner product:
$\langle f,\ g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$

Define $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$ to be
$T\big(p(x)\big) = p(t)$

By the Riesz Representation Theorem we proved in (b), we know that
for all polynomials $p \in \mathcal{P}_3(\mathbb{R})$, we have a unique polynomial $q_t \in \mathcal{P}_3(\mathbb{R})$ such that
$T\big(p(x)\big) = \langle p,\ q_t\rangle$
$\implies p(t) = \int_0^1 p(x)q_t(x)\,\mathrm{d}x$
as wanted. ■

(d)

From part (b), we know that $\vec{w} = \overline{T(u_1)}u_1 + \ldots + \overline{T(u_n)}u_n$
The orthonormal basis for $\mathcal{P}_3(\mathbb{R})$ is
$\mathcal{U} = \left\{1,\ 2\sqrt{3}\left(x - \frac{1}{2}\right),\ 6\sqrt{5}\left(x^2 - x - \frac{1}{6}\right),\ 20\sqrt{7}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right)\right\}$

$$\begin{aligned}
\vec{w} &= \overline{T(u_1)}u_1 + \overline{T(u_2)}u_2 + \overline{T(u_3)}u_3 + \overline{T(u_4)}u_4 && \text{[by given]}\\
&= T(u_1)u_1 + T(u_2)u_2 + T(u_3)u_3 + T(u_4)u_4 && \text{[since real polynomials]}\\
&= u_1\left(\frac{1}{2}\right)u_1 + u_2\left(\frac{1}{2}\right)u_2 + u_3\left(\frac{1}{2}\right)u_3 + u_4\left(\frac{1}{2}\right)u_4 && \text{[by definition of } T\text{]}\\
&= (1)u_1 + 2\sqrt{3}\left(\frac{1}{2} - \frac{1}{2}\right)u_2\\
&\qquad + \left(6\sqrt{5}\left(\left(\frac{1}{2}\right)^2 - \frac{1}{2} + \frac{1}{6}\right)\right)u_3 && \text{[plugging in numbers]}\\
&\qquad + \left(20\sqrt{7}\left(\left(\frac{1}{2}\right)^3 - \frac{3}{2}\left(\frac{1}{2}\right)^2 + \frac{3}{5}\left(\frac{1}{2}\right) - \frac{1}{20}\right)\right)u_4\\
&= (1) + 0 + \left(\frac{-\sqrt{5}}{2}\right)\left(6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right) + 0 && \text{[by algebra]}\\
&= 1 + (-15)\left(x^2 - x + \frac{1}{6}\right) && \text{[by algebra]}\\
&= -15x^2 + 15x - \frac{3}{2} && \text{[by algebra]}
\end{aligned}$$

Problem 5.

(a)
No.
This is because the graph looks like a sinusoidal wave, so a straight line wouldn't be an accurate model of the graph. It would be better to use a sin function as a model for the graph.

(b)
To do this, we would modify the matrix A to be:

$$A = \begin{bmatrix} \sin(x_1) & 1 \\ \sin(x_2) & 1 \\ \vdots & \vdots \\ \sin(x_n) & 1 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

To match the function to our new model.

(c)

$$\text{Let } D = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix} \mu_i \in \mathbb{R}^+ \qquad \text{Let } \langle \vec{v},\ \vec{w} \rangle = \vec{v}^T D \vec{w}$$

WTS: (1) $\langle v,\ v \rangle \geq 0$
(2) $\langle v,\ v \rangle = 0 \iff v = 0$
(3) $\langle u + \lambda v,\ w \rangle = \langle u,\ w \rangle + \lambda \langle v,\ w \rangle$
(4) $\langle u,\ v \rangle = \langle v,\ u \rangle$

(1)

$$\begin{aligned} \langle v,\ v \rangle &= v^T D v && \text{[by given]} \\ &= D v^T v && \text{[since diagonal matricies commute]} \\ &= D(v \cdot v) && \text{[changing matrix multiplication to dot product]} \\ &\geq 0 && \text{[by dot product properties]} \end{aligned}$$

(2)
$\implies$ direction :

$$\begin{aligned}\langle v,\ v\rangle &= D(v\cdot v) && \text{[by (1)]}\\ &= 0 && \text{[by given]}\end{aligned}$$

Since D is a positive matrix, then the only solution to this equation is if $v = 0 \therefore v = 0$.

$\impliedby$ direction :
If $v = 0$ we have:

$$\begin{aligned}\langle v,\ v\rangle &= v^T Dv && \text{[by given]}\\ &= 0^T D0 && \text{[by given]}\\ &= 0 && \text{[since multiplying by 0]}\end{aligned}$$

(3)

$$\begin{aligned}\langle u+\lambda v,\ w\rangle &= (u+\lambda v)^T Dw && \text{[by given]}\\ &= (u^T+\lambda v^T)Dw && \text{[by transpose properties]}\\ &= u^T Dw + \lambda v^T Dw && \text{[by expanding]}\\ &= \langle u,\ w\rangle + \lambda\langle v,\ w\rangle && \text{[by definition of the inner product]}\end{aligned}$$

(4)

$$\begin{aligned}\langle u,\ v\rangle &= u^T Dv && \text{[by given]}\\ &= \left(u^T Dv\right)^T && \text{[since the transpose of a scalar is the same]}\\ &= (Dv)^T\left(u^T\right)^T && \text{[by transpose properties]}\\ &= v^T D^T u && \text{[by transpose properties]}\\ &= v^T Du && \text{[since } D^T = D\text{]}\\ &= \langle v,\ u\rangle && \text{[by definition of the inner product]}\end{aligned}$$

$\therefore \langle -,\ -\rangle$ is an inner product.

WTS: $\text{proj}_{\text{col}(A)} v = A(A^T DA)^{-1}A^T Dv$

Showing

(d)

(e)