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Question 1.

(a)

If there is an interval $[a, b]$ such that

1. $g(x) \in [a, b] \quad \forall x \in [a, b]$
2. $|g'(x)| \leq L < 1 \quad \forall x \in [a, b]$

Then $g(x)$ has a unique fixed point in $[a, b]$

(b)

Suppose 1. and 2.

Start with any $x_0 \in [a, b]$ and iterate

$$x_{k+1} = g(x_k) \quad k = 1, 2, \dots$$

Then $x_k \in [a, b]$ by 1.

Moreover,

$$\begin{aligned} x_{k+1} - x_k &= g(x_k) - x_{k-1} \\ &= g'(\eta_k)(x_{k+1} - x_k) \end{aligned}$$

for some $\eta_k \in [x_{k-1}, x_k] \subset [a, b]$ (condition 2)

So

$$\begin{aligned} x_{k+1} - x_k &\leq L|x_k - x_{k-1}| \\ &\leq L^2|x_{k-1} - x_{k-2}| \\ &\leq \vdots \\ &\leq L^{k-1}|x_2 - x_1| \\ &\leq L^k|x_1 - x_0| \end{aligned}$$

L^k approaches 0 as k approaches ∞ . So $|x_1 - x_0|$ approaches 0 as well

$\therefore x_k$ converges to some point $\tilde{x} \in [a, b]$

(c)

WTS: $g(\tilde{x}) = \tilde{x}$ This is equivalent as setting $f(x) = g(x) - x$ and showing \tilde{x} is a root of $f(x)$

$$\begin{aligned}
f(x_{k+1}) &= g(x_{k+1}) - x_{k+1} && \text{[by definition]} \\
&= g(x_{k+1}) - g(x_k) && \text{[since } x_{k+1} = g(x_k)\text{]} \\
&= g'(\eta)(x_{k+1} - x_k) \text{ for some } \eta \in [x_{k+1}, x_k] && \text{[since } f(x) \text{ is differentiable by assumption,} \\
&&& \text{we use MVT]}
\end{aligned}$$

Taking the limit of both sides...

$$\begin{aligned}
f(\tilde{x}) &= \lim_{k \rightarrow \infty} f(x_{k+1}) && [x_k \text{ converges to } \tilde{x}] \\
&= \lim_{k \rightarrow \infty} g'(\eta)(x_{k+1} - x_k) \text{ for some } \eta \in [x_{k+1}, x_k] && \text{[limit both sides]} \\
&= g'(\eta) \lim_{k \rightarrow \infty} (x_{k+1} - x_k) && [g'(\eta) \text{ does not depend on } x] \\
&= g'(\eta) \lim_{k \rightarrow \infty} (\tilde{x} - \tilde{x}) && [x_k \text{ converges to } \tilde{x}] \\
&= g'(\eta) \lim_{k \rightarrow \infty} 0 && \text{[arithmetic]} \\
&= 0 && [g'(\eta) \text{ is not infinite}]
\end{aligned}$$

Since \tilde{x} is a root for $f(x)$ $\therefore \tilde{x}$ is a fixed point.

(d)

Suppose $\widetilde{x}_1, \widetilde{x}_2$ are fixed points of $g(x)$ WTS: $\widetilde{x}_1 = \widetilde{x}_2$

$$\begin{aligned}\widetilde{x}_1 - \widetilde{x}_2 &= g(\widetilde{x}_1) - g(\widetilde{x}_2) && [\text{since } g(x) = x] \\ &= g'(\eta)(\widetilde{x}_1 - \widetilde{x}_2) \text{ for some } \eta \in [\widetilde{x}_1, \widetilde{x}_2] && [\text{by MVT}]\end{aligned}$$

Taking the absolute values of both sides:

$$\begin{aligned}|\widetilde{x}_1 - \widetilde{x}_2| &= |g'(\eta)(\widetilde{x}_1 - \widetilde{x}_2)| && [\text{absolute value both sides}] \\ &= |g'(\eta)| |\widetilde{x}_1 - \widetilde{x}_2| && [\text{splitting up the absolute value}]\end{aligned}$$

Subtracting both sides:

$$\begin{aligned}&|\widetilde{x}_1 - \widetilde{x}_2| - |g'(\eta)| |\widetilde{x}_1 - \widetilde{x}_2| = 0 && [\text{subtracting both sides}] \\ \iff &(1 - |g'(\eta)|) |\widetilde{x}_1 - \widetilde{x}_2| = 0 && [\text{factoring}] \\ \iff &|\widetilde{x}_1 - \widetilde{x}_2| = 0 && [\text{dividing both sides since } (1 - |g'(\eta)|) \text{ is never zero}] \\ \iff &\widetilde{x}_1 = \widetilde{x}_2 && [\text{trivial}]\end{aligned}$$

as wanted. ■

Question 2.

(a)

f has one root. This is calculated by setting $f(x) = 1 - \frac{1}{2x} = 0 \quad \implies \quad x = \frac{1}{2}$

$$g\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

$$\implies g\left(\frac{1}{2}\right) = \frac{1}{2}$$

$\therefore \frac{1}{2}$ is a fixed point

To check if there are any other fixed points, solve

$$2x(1 - x) = x$$

$$\iff x - 2x^2 = 0$$

$$\iff x(1 - 2x) = 0$$

$$\iff x = 0, \frac{1}{2}$$

$\therefore 0$ is a fixed point which isn't a root of $f(x)$

(b)

We can find the number of fixed points of $g(x)$ by using the Fixed Point Theorem

$$g'(x) = 2 - 4x$$

$$|g'(x)| < 1 \quad [\text{by FPT}]$$

$$\iff |2 - 4x| < 1 \quad [\text{substituting}]$$

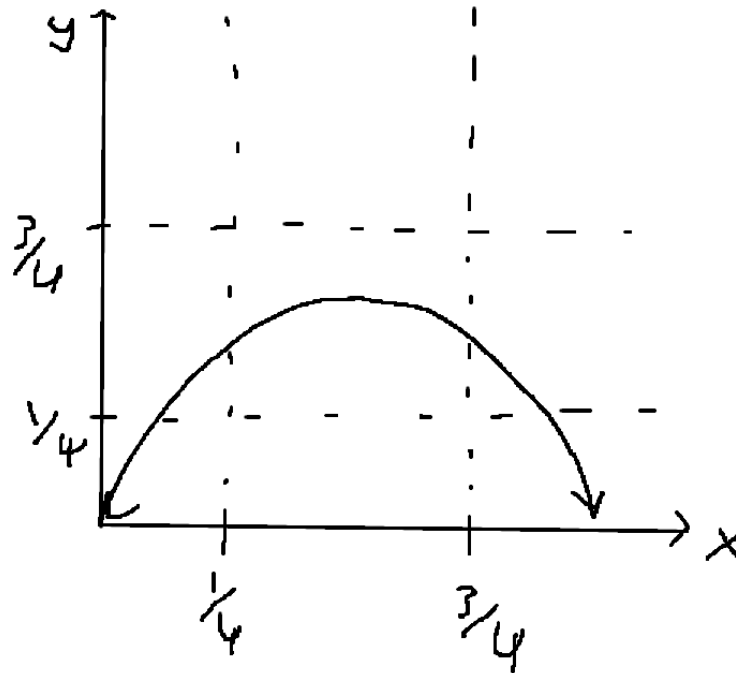
$$\iff \begin{cases} 2 - 4x & \text{if } 2 - 4x > 0 \\ 4x - 2 & \text{if } 2 - 4x \leq 0 \end{cases} < 1 \quad [\text{change absolute values to piecewise}]$$

$$\iff \begin{cases} 2 - 4x & \text{if } x < \frac{1}{2} \\ 4x - 2 & \text{if } x \geq \frac{1}{2} \end{cases} < 1 \quad [\text{simplifying}]$$

$$\iff x \in \left[\frac{1}{4}, \frac{3}{4}\right] \quad [\text{solving for } x]$$

So condition 2 is satisfied with that range.

For condition 1, we can plot the graph of $g(x)$



We can see that $g(x) \in \left[\frac{1}{4}, \frac{3}{4}\right] \quad \forall x \in \left[\frac{1}{4}, \frac{3}{4}\right]$
 \therefore condition 1 is satisfied.

\therefore the range $\left[\frac{1}{4}, \frac{3}{4}\right]$ guarantees convergence.

Question 3.

(a)

Let $f(x) = x + \ln(x) = 0$

Then $x = -\ln(x)$

\implies (1) is a valid formula

$$x = -\ln(x)$$

$$\iff \ln(x) = -x$$

$$\iff x = e^{-x}$$

\implies (2) is a valid formula

$$x = e^{-x}$$

$$\iff 2x = x + e^{-x}$$

$$\iff x = \frac{x + e^{-x}}{2}$$

\implies (3) is a valid formula

(b)

$$(1) \quad g'\left(\frac{1}{2}\right) = -\frac{1}{\frac{1}{2}} = -2$$

$$(2) \quad g'\left(\frac{1}{2}\right) = -e^{-\frac{1}{2}} \approx -0.6$$

$$(3) \quad g'\left(\frac{1}{2}\right) = \frac{1}{2}(1 - e^{-\frac{1}{2}}) \approx 0.2$$

\therefore (3) should be used since $g'\left(\frac{1}{2}\right)$ is closest to 0.

(c)

Using newtons method:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(\tilde{x}) = \frac{f(\tilde{x})f''(\tilde{x})}{f'(\tilde{x})^2}$$

$$f(x) = x + \ln(x)$$

$$f'(x) = 1 + \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$\begin{aligned} g(x) &= x - \frac{1 - \ln(x)}{1 + \frac{1}{x}} \\ &= x - \frac{x - x \ln(x)}{x + 1} \\ &= \frac{x^2 + x \ln(x)}{x + 1} \end{aligned}$$

$$\begin{aligned} g'(x) &= \frac{f(\tilde{x})f''(\tilde{x})}{f'(\tilde{x})^2} = \frac{(x + \ln(x)) \left(-\frac{1}{x^2}\right)}{\left(1 + \frac{1}{x}\right)^2} \\ &= -\frac{x + \ln(x)}{(x + 1)^2} \end{aligned}$$

$$g'\left(\frac{1}{2}\right) \approx 0.09$$

$$\therefore g(x) = \frac{x^2 + x \ln(x)}{x + 1} \text{ is a better formula.}$$

Question 4.

$$x_{k+1} = 2x_k - x_k^2 y$$

$$\iff x_{k+1} = x_k - (x_k^2 y - x_k)$$

$$\text{So } \frac{f(x)}{f'(x)} = x_k^2 y - x_k$$

Since we are using Newton's method, we can assume it converges. Which means

$$\frac{f(x)}{f'(x)} = x_k^2 y - x_k = 0$$

$$\frac{f(x)}{f'(x)} = x_k^2 y - x_k = 0$$

$$\iff x_k(x_k y - 1) = 0$$

$$\text{So we have } x_k = 0, \frac{1}{y}$$

\therefore this fixed-point iteration is used to estimate $\frac{1}{y}$

Question 5.

(a)

Suppose we find a root α such that $f(\alpha) = 0$.

$$\begin{aligned} g(\alpha) &= \alpha - \frac{f(\alpha)^2}{f(\alpha + f(\alpha)) - f(\alpha)} \\ &= \alpha - \frac{0^2}{f(\alpha + 0) - 0} \\ &= \alpha - \frac{0}{0} \end{aligned}$$

all roots of $f(x)$ makes $g(x)$ diverge. However, taking the limit as $x \rightarrow \alpha$ will make $g(\alpha)$ converge to 0
 \therefore roots of $f(x)$ are not fixed points of $g(x)$

There are no other fixed point of $g(x)$. Since if the numerator vanishes, then so will the denominator which will make $g(x)$ diverge.

(b)

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} && \text{[by Newton's Method]} \\ &= x - \frac{f(x)}{\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}} && \text{[by derivative limit definition]} \\ &= x - \frac{f(x)}{\lim_{h \rightarrow f(x)} \frac{f(x+h) - f(x)}{h}} && \text{[since we're looking for } f(x) = 0\text{]} \\ &= x - \frac{f(x)}{\frac{f(x + f(x)) - f(x)}{f(x)}} && \text{[plugging in } f(x)\text{]} \\ &= x - \frac{f(x)^2}{f(x + f(x)) - f(x)} && \text{[arithmetic]} \end{aligned}$$

as wanted

(c)

Enough to show: $g'(x) = 0$

$$f'(x) = 2x - 10$$

$$f''(x) = 2$$

Newtons method:

$$\begin{aligned} g'(x) &= \frac{f(x)f''(x)}{f'(x)^2} \\ &= \frac{(x^2 - 10x + 24)(2)}{(2x - 10)^2} \\ &= \frac{2x^2 - 20x + 48}{4x^2 - 40x + 100} \end{aligned}$$

$$g'(4) = 0$$

$$g'(6) = 0$$

\therefore by RCT, Newton's method are quadratically convergent when x is near 4 and 6.

Steffensen's method:

$$\begin{aligned} g(x) &= x - \frac{(x^2 - 10x + 24)^2}{\left((x + x^2 - 10x + 24)^2 - 10(x + x^2 - 10x + 24) + 24\right) - (x^2 - 10x + 24)} \\ &= x - \frac{x^4 - 20x^3 + 148x^2 - 480x + 576}{\left((x^2 - 9x + 24)^2 - 10(x^2 - 9x + 24) + 24\right) - (x^2 - 10x + 24)} \\ &= x - \frac{x^4 - 20x^3 + 148x^2 - 480x + 576}{x^4 - 18x^3 + 129x^2 - 432x + 576 - 10x^2 + 90x + 240 - x^2 + 10x - 24} \\ &= x - \frac{x^4 - 20x^3 + 148x^2 - 480x + 576}{x^4 - 18x^3 + 118x^2 - 332x + 792} \\ g'(x) &= 1 - \frac{(4x^3 - 60x^2 + 296x - 480)(x^4 - 18x^3 + 118x^2 - 332x + 792) + (x^4 - 20x^3 + 148x^2 - 480x + 576)(-332 + 236x - 54x^2 + 4x^3)}{(x^4 - 18x^3 + 118x^2 - 332x + 792)^2} \end{aligned}$$

$$g'(4) = 0$$

$$g'(6) = 0$$

\therefore by RCT, Steffensen's method are quadratically convergent when x is near 4 and 6.

(d)

We don't need to compute $f'(x)$. $f'(x)$ may be expensive to calculate.