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Question 1.

(a)
If there is an interval $[a,\ b]$ such that

1. $g(x) \in [a,\ b] \qquad \forall x \in [a,\ b]$
2. $|g'(x)| \leq L < 1 \qquad \forall x \in [a,\ b]$

Then $g(x)$ has a unique fixed point in $[a,\ b]$

(b)
Suppose 1. and 2.
Start with any $x_0 \in [a,\ b]$ and iterate
$x_{k+1} = g(x_k) \qquad k = 1,\ 2,\ \cdots$

Then $x_k \in [a,\ b]$ by 1.

Moreover,

$$\begin{aligned} x_{k+1} - x_k &= g(x_k) - x_{k-1} \\ &= g'(\eta_k)(x_{k+1} - x_k) \end{aligned}$$

$$\text{for some } \eta_k \in [x_{k-1},\ x_k] \subset [a,\ b] \qquad \text{(condition 2)}$$

So

$$\begin{aligned} x_{k+1} - x_k &\leq L|x_k - x_{k-1}| \\ &\leq L^2|x_{k-1} - x_{k-2}| \\ &\leq \qquad \vdots \\ &\leq L^{k-1}|x_2 - x_1| \\ &\leq L^k|x_1 - x_0| \end{aligned}$$

L_k approaches 0 as k appraoches ∞. So $|x_1 - x_0|$ approaches 0 as well
$\therefore x_k$ converges to some point $\widetilde{x} \in [a,\ b]$

(c)

WTS: $g(\widetilde{x}) = \widetilde{x}$

This is equivalent as setting $f(x) = g(x) - x$ and showing $\widetilde{x}$ is a root of $f(x)$

$$\begin{aligned}
f(x_{k+1}) &= g(x_{k+1}) - x_{k+1} && \text{[by definition]} \\
&= g(x_{k+1}) - g(x_k) && \text{[since } x_{k+1} = g(x_k)\text{]} \\
&= g'(\eta)(x_{k+1} - x_k) \text{ for some } \eta \in [x_{k+1},\ x_k] && \text{[since } f(x) \text{ is differentiable by assumption,} \\
& && \text{we use MVT]}
\end{aligned}$$

Taking the limit of both sides...

$$\begin{aligned}
f(\widetilde{x}) &= \lim_{k->\infty} f\left(x_{k+1}\right) && [x_k \text{ converges to } \widetilde{x}] \\
&= \lim_{k->\infty} g'(\eta)(x_{k+1} - x_k) \text{ for some } \eta \in [x_{k+1},\ x_k] && \text{[limit both sides]} \\
&= g'(\eta) \lim_{k->\infty} (x_{k+1} - x_k) && [g'(\eta) \text{ does not depend on } x] \\
&= g'(\eta) \lim_{k->\infty} (\widetilde{x} - \widetilde{x}) && [x_k \text{ converges to } \widetilde{x}] \\
&= g'(\eta) \lim_{k->\infty} 0 && \text{[arithmetic]} \\
&= 0 && [g'(\eta) \text{ is not infinite}]
\end{aligned}$$

Since $\widetilde{x}$ is a root for $f(x)$

$\therefore \widetilde{x}$ is a fixed point.

(d)

Suppose $\widetilde{x_1},\ \widetilde{x_2}$ are fixed points of $g(x)$

WTS: $\widetilde{x_1} = \widetilde{x_2}$

$$\begin{aligned}
\widetilde{x_1} - \widetilde{x_2} &= g(\widetilde{x_1}) - g(\widetilde{x_2}) && \text{[since } g(x) = x\text{]} \\
&= g'(\eta)(\widetilde{x_1} - \widetilde{x_2}) \text{ for some } \eta \in [\widetilde{x_1},\ \widetilde{x_2}] && \text{[by MVT]}
\end{aligned}$$

Taking the absolute values of both sides:

$$\begin{aligned}
|\widetilde{x_1} - \widetilde{x_2}| &= |g'(\eta)(\widetilde{x_1} - \widetilde{x_2})| && \text{[absolute value both sides]} \\
&= \Big|g'(\eta)\Big|\ |\widetilde{x_1} - \widetilde{x_2}| && \text{[splitting up the absolute value]}
\end{aligned}$$

Subtracting both sides:

$$\begin{aligned}
& & |\widetilde{x_1} - \widetilde{x_2}| - \Big|g'(\eta)\Big||\widetilde{x_1} - \widetilde{x_2}| &= 0 && \text{[subtracting both sides]} \\
&\iff & \Big(1 - |g'(\eta)|\Big)|\widetilde{x_1} - \widetilde{x_2}| &= 0 && \text{[factoring]} \\
&\iff & |\widetilde{x_1} - \widetilde{x_2}| &= 0 && \text{[dividing both sides since } \Big(1 - |g'(\eta)|\Big) \text{ is never zero]} \\
&\iff & \widetilde{x_1} &= \widetilde{x_2} && \text{[trivial]}
\end{aligned}$$

as wanted. ■

Question 2.

(a)

f has one root. This is calculated by setting $f(x) = 1 - \frac{1}{2x} = 0 \implies x = \frac{1}{2}$

$g\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{2}$
$\implies g\left(\frac{1}{2}\right) = \frac{1}{2}$
$\therefore \frac{1}{2}$ is a fixed point

To check if there are any other fixed points, solve

$$\begin{aligned} & 2x(1-x) = x \\ \iff & x - 2x^2 = 0 \\ \iff & x(1-2x) = 0 \\ \iff & x = 0,\ \frac{1}{2} \end{aligned}$$

$\therefore$ 0 is a fixed point which isn't a root of $f(x)$

(b)

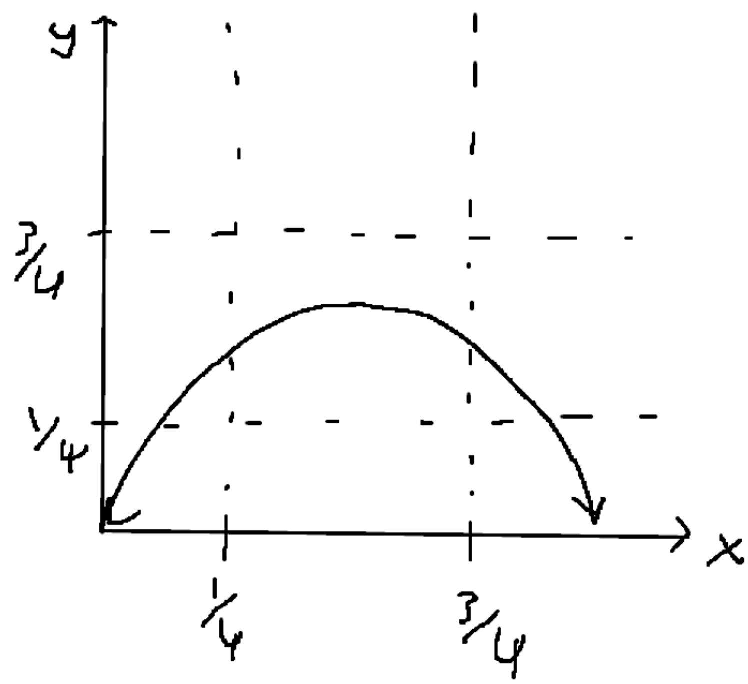
We can find the number of fixed points of $g(x)$ by using the Fixed Point Theorem

$g'(x) = 2 - 4x$

$$
\begin{aligned}
& & |g'(x)| &< 1 && \text{[by FPT]} \\
&\iff & |2-4x| &< 1 && \text{[substituting]} \\
&\iff & \begin{cases} 2-4x & \text{if } 2-4x > 0 \\ 4x-2 & \text{if } 2-4x \le 0 \end{cases} &< 1 && \text{[change absolute values to piecewise]} \\
&\iff & \begin{cases} 2-4x & \text{if } x < \frac{1}{2} \\ 4x-2 & \text{if } x \ge \frac{1}{2} \end{cases} &< 1 && \text{[simplifying]} \\
&\iff & x &\in \left[\frac{1}{4},\ \frac{3}{4}\right] && \text{[solving for } x\text{]}
\end{aligned}
$$

So condition 2 is satisfied with that range.

For condition 1, we can plot the graph of $g(x)$



We can see that $g(x) \in \left[\frac{1}{4},\ \frac{3}{4}\right] \quad \forall x \in \left[\frac{1}{4},\ \frac{3}{4}\right]$

$\therefore$ condition 1 is satisfied.

$\therefore$ the range $\left[\frac{1}{4},\ \frac{3}{4}\right]$ guarantees convergence.

Question 3.

(a)

Let $f(x) = x + \ln(x) = 0$

Then $x = -\ln(x)$

$\Longrightarrow$ (1) is a valid formula

$$\begin{aligned} & x = -\ln(x) \\ \iff \quad & \ln(x) = -x \\ \iff \quad & x = e^{-x} \end{aligned}$$

$\Longrightarrow$ (2) is a valid formula

$$\begin{aligned} & x = e^{-x} \\ \iff \quad & 2x = x + e^{-x} \\ \iff \quad & x = \frac{x + e^{-x}}{2} \end{aligned}$$

$\Longrightarrow$ (3) is a valid formula

(b)

$$(1) \qquad g'\left(\frac{1}{2}\right) = -\frac{1}{\frac{1}{2}} \qquad = -2$$

$$(2) \qquad g'\left(\frac{1}{2}\right) = -e^{-\frac{1}{2}} \qquad \approx -0.6$$

$$(3) \qquad g'\left(\frac{1}{2}\right) = \frac{1}{2}(1 - e^{-\frac{1}{2}}) \approx 0.2$$

$\therefore$ (3) should be used since $g'\left(\frac{1}{2}\right)$ is closest to 0.

(c)

Using newtons method:

$$g(x) = x - \frac{f(x)}{f'(x)}$$
$$g'(\tilde{x}) = \frac{f(\tilde{x})f''(\tilde{x})}{f'(\tilde{x})^2}$$

$$\begin{aligned} f(x) &= x + \ln(x) \\ f'(x) &= 1 + \frac{1}{x} \\ f''(x) &= -\frac{1}{x^2} \end{aligned}$$

$$\begin{aligned} g(x) &= x - \frac{1 - \ln(x)}{1 + \frac{1}{x}} \\ &= x - \frac{x - x\ln(x)}{x+1} \\ &= \frac{x^2 + x\ln(x)}{x+1} \end{aligned}$$

$$\begin{aligned} g'(x) = \frac{f(\tilde{x})f''(\tilde{x})}{f'(\tilde{x})^2} &= \frac{\left(x + \ln(x)\right)\left(-\frac{1}{x^2}\right)}{\left(1 + \frac{1}{x}\right)^2} \\ &= -\frac{x + \ln(x)}{(x+1)^2} \end{aligned}$$

$$g'\left(\frac{1}{2}\right) \approx 0.09$$
$\therefore g(x) = \dfrac{x^2 + x\ln(x)}{x+1}$ is a better formula.

Question 4.

$$\begin{aligned} &x_{k+1} = 2x_k - {x_k}^2 y \\ \iff\quad &x_{k+1} = x_k - ({x_k}^2 y - x_k) \end{aligned}$$

So $\dfrac{f(x)}{f'(x)} = {x_k}^2 y - x_k$

Since we are using Newton's method, we can assume it converges. Which means $\dfrac{f(x)}{f'(x)} = {x_k}^2 y - x_k = 0$

$$\begin{aligned} &\frac{f(x)}{f'(x)} = {x_k}^2 y - x_k = 0 \\ \iff\quad &x_k(x_k y - 1) = 0 \end{aligned}$$

So we have $x_k = 0,\ \dfrac{1}{y}$

$\therefore$ this fixed-point iteration is used to estimate $\dfrac{1}{y}$

Question 5.

(a)

Suppose we find a root α such that $f(\alpha) = 0$.

$$\begin{aligned} g(\alpha) &= \alpha - \frac{f(\alpha)^2}{f\Big(\alpha + f(\alpha)\Big) - f(\alpha)} \\ &= \alpha - \frac{0^2}{f\Big(\alpha + 0\Big) - 0} \\ &= \alpha - \frac{0}{0} \end{aligned}$$

all roots of $f(x)$ makes $g(x)$ diverge. However, taking the limit as $x \to \alpha$ will make $g(\alpha)$ converge to 0
$\therefore$ roots of $f(x)$ are not fixed points of $g(x)$

There are no other fixed point of $g(x)$. Since if the numerator vanishes, then so will the denominator which will make $g(x)$ diverge.

(b)

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} && \text{[by Newton's Method]} \\ &= x - \frac{f(x)}{\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}} && \text{[by derivative limit definition]} \\ &= x - \frac{f(x)}{\lim\limits_{h\to f(x)} \dfrac{f(x+h) - f(x)}{h}} && \text{[since we're looking for } f(x) = 0\text{]} \\ &= x - \frac{f(x)}{\dfrac{f\Big(x + f(x)\Big) - f(x)}{f(x)}} && \text{[plugging in f(x)]} \\ &= x - \frac{f(x)^2}{f\Big(x + f(x)\Big) - f(x)} && \text{[arithmetic]} \end{aligned}$$

as wanted

(c)
Enough to show: $g'(x) = 0$

$f'(x) = 2x - 10$
$f''(x) = 2$

Newtons method:

$$\begin{aligned} g'(x) &= \frac{f(x)f''(x)}{f'(x)^2} \\ &= \frac{(x^2 - 10x + 24)(2)}{(2x-10)^2} \\ &= \frac{2x^2 - 20x + 48}{4x^2 - 40x - 100} \end{aligned}$$

$g'(4) = 0$
$g'(6) = 0$
$\therefore$ by RCT, Newton's method are quadratically convergent when x is near 4 and 6.

Steffensen's method:

$$\begin{aligned} g(x) &= x - \frac{(x^2 - 10x + 24)^2}{\Big((x + x^2 - 10x + 24)^2 - 10(x + x^2 - 10x + 24) + 24\Big) - (x^2 - 10x + 24)} \\ &= x - \frac{x^4 - 20x^3 + 148x^2 - 480x + 576}{\Big((x^2 - 9x + 24)^2 - 10(x^2 - 9x + 24) + 24\Big) - (x^2 - 10x + 24)} \\ &= x - \frac{x^4 - 20x^3 + 148x^2 - 480x + 576}{x^4 - 18x^3 + 129x^2 - 432x + 576 - 10x^2 + 90x + 240 - x^2 + 10x - 24} \\ &= x - \frac{x^4 - 20x^3 + 148x^2 - 480x + 576}{x^4 - 18x^3 + 118x^2 - 332x + 792} \end{aligned}$$

$$g'(x) = 1 - \frac{(4x^3 - 60x^2 + 296x - 480)(x^4 - 18x^3 + 118x^2 - 332x + 792) + (x^4 - 20x^3 + 148x^2 - 480x + 576)(-332 + 236x - 54x^2 + 4x^3)}{(x^4 - 18x^3 + 118x^2 - 332x + 792)^2}$$

$g'(4) = 0$
$g'(6) = 0$
$\therefore$ by RCT, Steffensen's method are quadratically convergent when x is near 4 and 6.

(d)
We don't need to compute $f'(x)$. $f'(x)$ may be expensive to calculate.