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## Problem 1.

Let  $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$  be an orthonormal basis for  $\mathbb{R}^n$  such that  
 $B_i \subseteq \mathcal{B} \quad \text{sp}(B_i) = V_i \quad \text{for } i \in \{1, 2, \dots, p\}$

Let  $\vec{v}_i \in V_i \quad \vec{v}_j \in V_j$  be arbitrary. Then

$$\vec{v}_i \in \text{sp}(B_i) \quad \vec{v}_j \in \text{sp}(B_j)$$

$$\text{Since } \vec{v}_i \perp \vec{v}_j \implies \langle \vec{v}_i, \vec{v}_j \rangle = 0$$

But we know that  $\vec{v}_i$  is a linear combination of vectors in  $B_i$ , and  $\vec{v}_j$  is a linear combination of vectors in  $B_j$

$$\langle x, y \rangle = 0 \quad \forall x \in B_i \quad \forall y \in B_j$$

We also know that  $b \neq 0 \quad \forall b \in \mathcal{B}$  since  $\mathcal{B}$  is a basis.

This means that no 2 vectors in  $\mathcal{B}$  is orthogonal to itself.

In order for the above statement to hold, we must have  $B_i \cap B_j = \emptyset$   
 since  $V_i$  is mutually orthogonal to  $V_j$ .

$\therefore$  for a collection of mutually orthogonal subspaces  $V_1, \dots, V_p$ , they must all strictly share the elements in  $\mathcal{B}$ . We know the dimension of  $\mathbb{R}^n$  is  $n$ , and  $V_1, \dots, V_p$  are all spanned by a subset of vectors in  $\mathcal{B}$ .

$$\therefore \dim V_1 + \dots + \dim V_p \leq n$$

■

## Problem 2.

(a)

Suppose  $U$  is an orthogonal  $n \times n$  matrix.

$$\implies UU^T = U^T U = I \quad U^T = U^{-1}$$

Define  $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be

$$\vec{v} \mapsto U^T \vec{v}$$

This is the inverse of  $T$  since

$$T^{-1} \circ T = U^T U$$

$$= I$$

$$T \circ T^{-1} = U U^T$$

$$= I$$

$\implies T$  is an isomorphism.

We are given that  $T(W) \subseteq W$ , and we know that  $T$  of a subspace is also a subspace.

So it is enough to show:  $\dim(W) = \dim(T(W))$

since this implies  $T(W) = W$

We know that  $T$  is an isomorphism, so  $T$  preserves dimension

$$\implies \dim(W) = \dim(T(W))$$

$$\therefore T(W) = W$$

■

(b)

We know by a theorem we learned in class that  $\mathbb{R}^n = W \oplus W^\perp$

We also know from part (a) that  $T(W)^\perp = W^\perp$

$$\text{WTS: } T(W)^\perp = T(W^\perp)$$

$$\text{Let } \vec{v} \in T(W)^\perp \quad \vec{w} \in W$$

$$\implies \vec{v} \cdot \vec{w} = 0 \quad [\text{since they're orthogonal}]$$

$$\implies \vec{v} \cdot U\vec{w} = 0 \quad [\text{since } U\vec{w} \in W]$$

$$\implies \vec{v}^T U\vec{w} = 0 \quad [\text{changing dot product to matrix multiplication}]$$

$$\implies (U^T \vec{v})^T \vec{w} = 0 \quad [\text{transpose properties}]$$

$$\implies (U^{-1} \vec{v})^T \vec{w} = 0 \quad [\text{since } U^T = T^{-1}]$$

$$\implies (T^{-1}(\vec{v}))^T \vec{w} = 0 \quad [\text{since } T^{-1}(\vec{x}) = U^{-1}(\vec{x})]$$

$$\implies (T^{-1}(\vec{v})) \cdot \vec{w} = 0 \quad [\text{changing dot product to matrix multiplication}]$$

$$\implies T^{-1}(\vec{v}) \in W^\perp \quad [\text{since } \vec{w} \perp T^{-1}(\vec{v})]$$

$$\implies \vec{v} \in T(W^\perp) \quad [\text{applying } T \text{ to both sides}]$$

Since  $\vec{v}$  is arbitrary, we can say that  $T(W^\perp) \subseteq W^\perp$

By the same arguments in (a), this implies  $T(W^\perp) = W^\perp$

■

## Problem 3.

(a)

Let  $\mathcal{B} = \{b_1, \dots, b_n\}$  where  $\mathcal{B}$  is an orthonormal basis.

$$\implies \langle b_i, b_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Define  $T : V \rightarrow \mathbb{R}^n$  to be

$$T(\vec{v}) \rightarrow [\vec{v}]_{\mathcal{B}}$$

Which is an isomorphism.

$$\text{WTS: } \langle f, g \rangle = T(f) \cdot T(g)$$

Since  $\mathcal{B}$  is a basis, let

$$\begin{aligned} f &= \alpha_1 b_1 + \dots + \alpha_n b_n & g &= \lambda_1 b_1 + \dots + \lambda_n b_n \\ &= \sum_{i=1}^n \alpha_i b_i & &= \sum_{j=1}^n \lambda_j b_j \end{aligned} \quad \text{where } \alpha_i, \lambda_i \in \mathbb{R}$$

$$\begin{aligned} \langle f, g \rangle &= \left\langle \sum_{i=1}^n \alpha_i b_i, \sum_{j=1}^n \lambda_j b_j \right\rangle && [\text{by given}] \\ &= \sum_{i=1}^n \alpha_i \left\langle b_i, \sum_{j=1}^n \lambda_j b_j \right\rangle && [\text{since inner product is linear}] \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \lambda_j \langle b_i, b_j \rangle && [\text{since inner product is linear over reals}] \\ &= \sum_{i=1}^n \alpha_i \lambda_i \langle b_i, b_i \rangle && [\text{since } \langle b_i, b_j \rangle = 0 \text{ if } i \neq j] \\ &= \sum_{i=1}^n \alpha_i \lambda_i && [\text{since } \langle b_i, b_j \rangle = 1 \text{ if } i = j] \\ &= T(f) \cdot T(g) && [\text{by definition of dot product}] \end{aligned}$$

as wanted. ■

(b)

Suppose  $A$  is an  $n \times n$  matrix such that

$$\forall \vec{v} \in \mathbb{R}^n, \quad \vec{v} \neq 0 \implies \vec{v}^T A \vec{v} > 0 \quad (\star)$$

Consider  $\langle v, w \rangle = v^T A w$ WTS: (1)  $\langle v, v \rangle \geq 0$ 

$$(2) \langle v, v \rangle = 0 \iff v = 0$$

$$(3) \langle u + \lambda v, w \rangle = \langle u, w \rangle + \lambda \langle v, w \rangle$$

$$(4) \langle u, v \rangle = \langle v, u \rangle$$

(1)

$$\langle v, v \rangle = v^T A v \quad [\text{by given}]$$

$$\geq 0 \quad [\text{by } (\star)]$$

(2)

 $\implies$  direction :

$$\langle v, v \rangle = v^T A v \quad [\text{by given}]$$

$$= 0 \quad [\text{by given}]$$

By contrapositive of  $(\star)$ , we have  $\vec{v}^T A \vec{v} \leq 0 \implies \vec{v} = 0$  $\therefore v = 0$  in this case. $\Leftarrow$  direction :If  $v = 0$  we have:

$$\langle v, v \rangle = v^T A v \quad [\text{by given}]$$

$$= 0^T A 0 \quad [\text{by given}]$$

$$= 0 \quad [\text{since multiplying by } 0]$$

(3)

$$\begin{aligned}\langle u + \lambda v, w \rangle &= (u + \lambda v)^T A w && \text{[by given]} \\ &= (u^T + \lambda v^T) A w && \text{[by transpose properties]} \\ &= u^T A w + \lambda v^T A w && \text{[by expanding]} \\ &= \langle u, w \rangle + \lambda \langle v, w \rangle && \text{[by definition of the inner product]}\end{aligned}$$

(4)

$$\begin{aligned}\langle u, v \rangle &= u^T A v && \text{[by given]} \\ &= (u^T A v)^T && \text{[since the transpose of a scalar is the same]} \\ &= (A v)^T (u^T)^T && \text{[by transpose properties]} \\ &= v^T A^T u && \text{[by transpose properties]} \\ &= v^T A u && \text{[since } A^T = A\text{]} \\ &= \langle v, u \rangle && \text{[by definition of the inner product]}\end{aligned}$$

$\therefore \langle -, - \rangle$  is an inner product.

From part (a), we know that  
 if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is defined as  $T : \vec{v} \rightarrow [\vec{v}]_{\mathcal{B}}$ ,  
 then  $\langle v, w \rangle = T(v) \cdot T(w)$  (♥)

Choose  $B$  = the change of basis matrix from  $v \rightarrow [\vec{v}]_{\mathcal{B}}$   
 $\implies B$  is invertible

$$\begin{aligned}
 \langle v, w \rangle &= v^T A w && \text{[by given]} \\
 &= T(v) \cdot T(w) && \text{[by (♥)]} \\
 &= Bv \cdot Bw && \text{[by our choice of } B] \\
 &= (Bv)^T Bw && \text{[changing dot product to matrix multiplication]} \\
 &= v^T B^T B w && \text{[by transpose properties]} \\
 \implies v^T A w &= v^T B^T B w
 \end{aligned}$$

Since the above equation holds  $\forall v, w \in \mathbb{R}^n$ , choose  $w = e_j$        $v = e_i$        $i, j \in \{1, 2, \dots, n\}$

$$\begin{aligned}
 v^T A w &= v^T (A e_j) && \text{[by given]} \\
 &= v^T (j^{\text{th}} \text{ column of } A) && \text{[by matrix multiplication]} \\
 &= e_i^T (j^{\text{th}} \text{ column of } A) && \text{[by given]} \\
 &= i^{\text{th}} \text{ row of } (j^{\text{th}} \text{ column of } A) && \text{[by matrix multiplication]} \\
 &= A_{ij} && \text{[matrix notation]}
 \end{aligned}$$

A similar argument holds for the matrix  $B^T B$

$$\implies A_{ij} = B^T B_{ij}$$

Since  $i, j$  are arbitrary, this means all entries in the matrix are equal.

$\therefore A = B^T B$ , as wanted ■

## Problem 4.

(a)

Let  $\vec{w} \in V$  be arbitrary

$$\vec{w} = (w_1, \dots, w_n)$$

Let  $\vec{v}_1, \vec{v}_2 \in V$   $r \in \mathbb{F}$ 

$$\text{WTS: } \langle \vec{v}_1 + r\vec{v}_2, \vec{w} \rangle = \langle \vec{v}_1, \vec{w} \rangle + r\langle \vec{v}_2, \vec{w} \rangle$$

$$\begin{aligned} \langle \vec{v}_1 + r\vec{v}_2, \vec{w} \rangle &= \langle \vec{v}_1, \vec{w} \rangle + \langle r\vec{v}_2, \vec{w} \rangle && [\text{by linearity of inner product}] \\ &= \langle \vec{v}_1, \vec{w} \rangle + r\langle \vec{v}_2, \vec{w} \rangle && [\text{by linearity of inner product}] \end{aligned}$$

$\therefore$  the function  $\vec{v} \mapsto \langle \vec{v}, \vec{w} \rangle$  is linear ■

(b)

Let  $\mathcal{U} = \{u_1, \dots, u_n\}$  be an orthonormal basis for  $V$ 

$$\text{WTS: } \exists \vec{w} \in V \text{ such that } T(\vec{v}) = \langle \vec{v}, \vec{w} \rangle$$

Since  $\mathcal{U}$  is an orthonormal basis, then  $\vec{v} = \langle v_1, u_1 \rangle u_1 + \langle v_2, u_2 \rangle u_2 + \dots + \langle v_n, u_n \rangle u_n$

$$\begin{aligned} T(\vec{v}) &= T(\langle \vec{v}, u_1 \rangle u_1 + \dots + \langle \vec{v}, u_n \rangle u_n) && [\text{by given}] \\ &= T(\langle \vec{v}, u_1 \rangle u_1) + \dots + T(\langle \vec{v}, u_n \rangle u_n) && [\text{by linearity of } T] \\ &= \langle \vec{v}, u_1 \rangle T(u_1) + \dots + \langle \vec{v}, u_n \rangle T(u_n) && [\text{since } \langle \vec{v}, u_i \rangle \text{ is a scalar}] \\ &= \left\langle \vec{v}, \overline{T(u_1)}u_1 \right\rangle + \dots + \left\langle \vec{v}, \overline{T(u_n)}u_n \right\rangle && [\text{bringing a scalar into the second slot}] \\ &= \left\langle \vec{v}, \overline{T(u_1)}u_1 + \dots + \overline{T(u_n)}u_n \right\rangle && [\text{by linearity of inner product}] \end{aligned}$$

So choose  $\vec{w} = \overline{T(u_1)}u_1 + \dots + \overline{T(u_n)}u_n$   $\vec{w} \in V$

$$\therefore T(\vec{v}) = \langle \vec{v}, \vec{w} \rangle$$

WTS:  $\vec{w}$  is unique.

$$\text{Suppose } T(\vec{v}) = \langle \vec{v}, \vec{w}_1 \rangle = \langle \vec{v}, \vec{w}_2 \rangle \quad \forall \vec{v} \in V$$

Enough to prove:  $\vec{w}_1 = \vec{w}_2$

$$\begin{aligned} \langle \vec{v}, \vec{w}_1 \rangle &= \langle \vec{v}, \vec{w}_2 \rangle \implies \langle \vec{v}, \vec{w}_1 \rangle - \langle \vec{v}, \vec{w}_2 \rangle = 0 \\ &\implies \langle \vec{v}, \vec{w}_1 - \vec{w}_2 \rangle = 0 \end{aligned}$$

Since this holds  $\forall \vec{v} \in V$ , choose  $\vec{v} = \vec{w}_1 - \vec{w}_2$

$$\implies \langle \vec{w}_1 - \vec{w}_2, \vec{w}_1 - \vec{w}_2 \rangle = 0$$

Since an inner product is zero-definite, then this means

$$\vec{w}_1 - \vec{w}_2 = 0 \implies \vec{w}_1 = \vec{w}_2 \quad \blacksquare$$



(c)

Let  $t \in \mathbb{R}$  be arbitrary.

Define an inner product:

$$\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$$

Define  $T : \mathcal{P}_3(\mathbb{R}) \rightarrow \mathbb{R}$  to be

$$T(p(x)) = p(t)$$

By the Riesz Representation Theorem we proved in (b), we know that

for all polynomials  $p \in \mathcal{P}_3(\mathbb{R})$ , we have a unique polynomial  $q_t \in \mathcal{P}_3(\mathbb{R})$  such that

$$T(p(x)) = \langle p, q_t \rangle$$

$$\implies p(t) = \int_0^1 p(x)q_t(x) \, dx$$

as wanted. ■

(d)

From part (b), we know that  $\vec{w} = \overline{T(u_1)}u_1 + \dots + \overline{T(u_n)}u_n$ The orthonormal basis for  $\mathcal{P}_3(\mathbb{R})$  is

$$\mathcal{U} = \left\{ 1, 2\sqrt{3} \left( x - \frac{1}{2} \right), 6\sqrt{5} \left( x^2 - x - \frac{1}{6} \right), 20\sqrt{7} \left( x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20} \right) \right\}$$

$$\vec{w} = \overline{T(u_1)}u_1 + \overline{T(u_2)}u_2 + \overline{T(u_3)}u_3 + \overline{T(u_4)}u_4 \quad [\text{by given}]$$

$$= T(u_1)u_1 + T(u_2)u_2 + T(u_3)u_3 + T(u_4)u_4 \quad [\text{since real polynomials}]$$

$$= u_1 \left( \frac{1}{2} \right) u_1 + u_2 \left( \frac{1}{2} \right) u_2 + u_3 \left( \frac{1}{2} \right) u_3 + u_4 \left( \frac{1}{2} \right) u_4 \quad [\text{by definition of } T]$$

$$= (1)u_1 + 2\sqrt{3} \left( \frac{1}{2} - \frac{1}{2} \right) u_2$$

$$+ \left( 6\sqrt{5} \left( \left( \frac{1}{2} \right)^2 - \frac{1}{2} + \frac{1}{6} \right) \right) u_3 \quad [\text{plugging in numbers}]$$

$$+ \left( 20\sqrt{7} \left( \left( \frac{1}{2} \right)^3 - \frac{3}{2} \left( \frac{1}{2} \right)^2 + \frac{3}{5} \left( \frac{1}{2} \right) - \frac{1}{20} \right) \right) u_4$$

$$= (1) + 0 + \left( \frac{-\sqrt{5}}{2} \right) \left( 6\sqrt{5} \left( x^2 - x + \frac{1}{6} \right) \right) + 0 \quad [\text{by algebra}]$$

$$= 1 + (-15) \left( x^2 - x + \frac{1}{6} \right) \quad [\text{by algebra}]$$

$$= -15x^2 + 15x - \frac{3}{2} \quad [\text{by algebra}]$$

## Problem 5.

(a)

No.

This is because the graph looks like a sinusoidal wave, so a straight line wouldn't be an accurate model of the graph. It would be better to use a sin function as a model for the graph.

(b)

To do this, we would modify the matrix  $A$  to be:

$$A = \begin{bmatrix} \sin(x_1) & 1 \\ \sin(x_2) & 1 \\ \vdots & \vdots \\ \sin(x_n) & 1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

To match the function to our new model.

(c)

$$\text{Let } D = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix} \mu_i \in \mathbb{R}^+ \quad \text{Let } \langle \vec{v}, \vec{w} \rangle = \vec{v}^T D \vec{w}$$

WTS: (1)  $\langle v, v \rangle \geq 0$

(2)  $\langle v, v \rangle = 0 \iff v = 0$

(3)  $\langle u + \lambda v, w \rangle = \langle u, w \rangle + \lambda \langle v, w \rangle$

(4)  $\langle u, v \rangle = \langle v, u \rangle$

(1)

$$\begin{aligned} \langle v, v \rangle &= v^T D v && \text{[by given]} \\ &= D v^T v && \text{[since diagonal matrices commute]} \\ &= D(v \cdot v) && \text{[changing matrix multiplication to dot product]} \\ &\geq 0 && \text{[by dot product properties]} \end{aligned}$$

(2)

 $\implies$  direction :

$$\langle v, v \rangle = D(v \cdot v) \quad [\text{by (1)}]$$

$$= 0 \quad [\text{by given}]$$

Since  $D$  is a positive matrix, then the only solution to this equation is if  $v = 0 \therefore v = 0$ .

 $\Leftarrow$  direction :If  $v = 0$  we have:

$$\langle v, v \rangle = v^T D v \quad [\text{by given}]$$

$$= 0^T D 0 \quad [\text{by given}]$$

$$= 0 \quad [\text{since multiplying by 0}]$$

(3)

$$\langle u + \lambda v, w \rangle = (u + \lambda v)^T D w \quad [\text{by given}]$$

$$= (u^T + \lambda v^T) D w \quad [\text{by transpose properties}]$$

$$= u^T D w + \lambda v^T D w \quad [\text{by expanding}]$$

$$= \langle u, w \rangle + \lambda \langle v, w \rangle \quad [\text{by definition of the inner product}]$$

(4)

$$\langle u, v \rangle = u^T D v \quad [\text{by given}]$$

$$= (u^T D v)^T \quad [\text{since the transpose of a scalar is the same}]$$

$$= (D v)^T (u^T)^T \quad [\text{by transpose properties}]$$

$$= v^T D^T u \quad [\text{by transpose properties}]$$

$$= v^T D u \quad [\text{since } D^T = D]$$

$$= \langle v, u \rangle \quad [\text{by definition of the inner product}]$$

$\therefore \langle -, - \rangle$  is an inner product.

WTS:  $\text{proj}_{\text{col}(A)} v = A(A^T D A)^{-1} A^T D v$

Showing

(d)

(e)