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## Question 1.

(a)

$$\begin{aligned} p(-1) &= 4 \\ p(0) &= 6 \\ p(1) &= 12 \end{aligned} \iff \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix}$$

Eliminate 1<sup>st</sup> column:

$$P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_1 A = A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

Eliminate 2<sup>nd</sup> column:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2 L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

$$L_2 P_2 L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} L_2 P_2 L_1 P_1 A &= U &\iff L_2 P_2 L_1 P_2 P_2 P_1 A &= U \\ &&\iff L_2 (P_2 L_1 P_2) P_2 P_1 A &= U \\ &&\iff L_2 \widetilde{L}_1 P_2 P_1 A &= U \\ &&\iff P_2 P_1 A &= \widetilde{L}_1^{-1} L_2^{-1} U \\ &&\iff PA &= LU \end{aligned}$$

$$\widetilde{L}_1 = P_2 L_1 P_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L = \widetilde{L}_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix}$$

$$P = P_2 P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \iff PA\vec{x} = P\vec{b}$$

$$\iff LU\vec{x} = P\vec{b}$$

$$\iff L(U\vec{x}) = P\vec{b}$$

$$\iff L\vec{d} = P\vec{b}$$

$$P\vec{b} = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$$

Forward solve  $L\vec{d} = \vec{b}$  for  $\vec{d}$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$$

$$d_1 = 4$$

$$d_1 + d_2 = 12$$

$$d_1 + \frac{1}{2}d_2 + d_3 = 6$$

$$d_1 = 4$$

$$d_2 = 8$$

$$d_3 = -2$$

Backward solve  $U\vec{x} = \vec{d}$  for  $\vec{x}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -2 \end{bmatrix}$$

$$x_3 = 2$$

$$x_2 = 4$$

$$x_1 = 6$$

$$\therefore p(x) = 6 + 4x + 2x^2$$

(b)

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} \quad i = 0, 1, 2$$

$$x_0 = -1 \quad y_0 = 4$$

$$x_1 = 0 \quad y_1 = 6$$

$$x_2 = 1 \quad y_2 = 12$$

$$i = 0$$

$$i = 1$$

$$i = 2$$

$$j = 1, 2$$

$$j = 0, 2$$

$$j = 0, 1$$

$$l_0(x) = \left( \frac{x-0}{-1-0} \right) \left( \frac{x-1}{-1-1} \right)$$

$$l_1(x) = \left( \frac{x-(-1)}{0-(-1)} \right) \left( \frac{x-1}{0-1} \right)$$

$$l_2(x) = \left( \frac{x-(-1)}{1-(-1)} \right) \left( \frac{x-0}{1-0} \right)$$

$$= (-x) \left( \frac{x-1}{-2} \right)$$

$$= (x+1)(1-x)$$

$$= \left( \frac{x+1}{2} \right) (x)$$

$$= \frac{1}{2}x(x-1)$$

$$= \frac{1}{2}x(x+1)$$

$$p(x) = \sum_{i=0}^n l_i(x)y_i$$

$$= l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2$$

$$= \frac{1}{2}x(x-1)(4) + (x+1)(1-x)(6) + \frac{1}{2}x(x+1)(12)$$

$$= 2x^2 - 2x + (-6x^2) + 6 + 6x^2 + 6x$$

$$= 2x^2 + 4x + 6$$

(c)

$x_i$	$y[x_i]$	$y[x_{i+1}, x_i]$	$y[x_{i+2}, x_{i+1}, x_i]$
-1	4	$\frac{6-4}{1-(-1)} = 2$	$\frac{6-2}{1-(-1)} = 2$
0	6		
1	12	$\frac{12-6}{1-0} = 6$	

$$\begin{aligned}
p(x) &= y[x_0] + (x - x_0)y[x_1, x_0] + (x - x_0)(x - x_1)y[x_2, x_1, x_0] \\
&= 4 + (x - (-1))(2) + (x - (-1))(x - 0)(2) \\
&= 4 + 2x + 2 + 2x^2 + 2x \\
&= 6 + 4x + 2x^2
\end{aligned}$$

(d)

As all the equations are simplified from (a), (b), and (c), we can see that all polynomials are identical.

(e)

We can use the method in (c) since all we have to do is to add to the table

$x_i$	$y[x_i]$	$y[x_{i+1}, x_i]$	$y[x_{i+2}, x_{i+1}, x_i]$	$y[x_{i+3}, x_{i+2}, x_{i+1}, x_i]$
-1	4	$\frac{6-4}{1-(-1)} = 2$		
0	6			
1	12	$\frac{12-6}{1-0} = 6$	$\frac{6-2}{1-(-1)} = 2$	$\frac{-1-2}{2-(-1)} = -1$
2	16	$\frac{16-12}{2-1} = 4$	$\frac{4-6}{2-0} = -1$	

$$\begin{aligned}
 p(x) &= y[x_0] + (x - x_0)y[x_1, x_0] \\
 &\quad + (x - x_0)(x - x_1)y[x_2, x_1, x_0] \\
 &\quad + (x - x_0)(x - x_1)(x - x_2)y[x_3, x_2, x_1, x_0] \\
 &= 4 + (x - (-1))(2) + (x - (-1))(x - 0)(2) + (x - (-1))(x - 0)(x - 1)(-1) \\
 &= 4 + 2x + 2 + 2x^2 + 2x - x^3 + x \\
 &= 6 + 5x + 2x^2 - x^3
 \end{aligned}$$

(f)

Equation of line from  $(-1, 4)$  to  $(0, 6)$ :

$$\begin{aligned}
 \text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\
 &= \frac{y_2 - y_1}{x_2 - x_1} \\
 &= \frac{6 - 4}{0 - (-1)} \\
 &= 2
 \end{aligned}
 \qquad
 \begin{aligned}
 y - y_1 &= \text{slope}_1(x - x_1) \iff y - 4 = 2(x - (-1)) \\
 &\iff y = 2(x + 1) + 4 \\
 &\iff y = 2x + 6
 \end{aligned}$$

Equation of line from  $(0, 6)$  to  $(1, 12)$ :

$$\begin{aligned}
 \text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\
 &= \frac{y_2 - y_1}{x_2 - x_1} \\
 &= \frac{12 - 6}{1 - 0} \\
 &= 6
 \end{aligned}
 \qquad
 \begin{aligned}
 y - y_1 &= \text{slope}_1(x - x_1) \iff y - 6 = 6(x - 0) \\
 &\iff y = 6x + 6
 \end{aligned}$$

Equation of line from  $(1, 12)$  to  $(2, 16)$ :

$$\begin{aligned}
 \text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\
 &= \frac{y_2 - y_1}{x_2 - x_1} \\
 &= \frac{16 - 12}{2 - 1} \\
 &= 4
 \end{aligned}
 \qquad
 \begin{aligned}
 y - y_1 &= \text{slope}_1(x - x_1) \iff y - 12 = 4(x - 1) \\
 &\iff y = 4x + 8
 \end{aligned}$$

 $\therefore$  the equation of the linear spline is:

$$\begin{cases} y = 2x + 6 & \text{if } -1 \leq x < 0 \\ y = 6x + 6 & \text{if } 0 \leq x < 1 \\ y = 4x + 8 & \text{if } 1 \leq x \leq 2 \end{cases}$$



## Question 2.

(a)

When we solve using the Vandermonde method, we get a polynomial of the form:

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_{n-1}x^{n-1} + a_nx^n$$

We can factor this polynomial into the form:

$$p(x) = a_0 + x \left( a_1 + x \left( a_2 + x \left( a_3 + \dots + x(a_{n-1} + xa_n) \dots \right) \right) \right)$$

There are  $n - 1$  additions, and  $n - 1$  multiplications, so we have

$2n + \mathcal{O}(1)$  flops.

(b)

For divided-difference, we get a polynomial of the form:

$$\begin{aligned}
 p(x) = & y[x_0] + (x - x_0)y[x_1, x_0] \\
 & + (x - x_0)(x - x_1)y[x_2, x_1, x_0] \\
 & + (x - x_0)(x - x_1)(x - x_2)y[x_3, x_2, x_1, x_0] \\
 & \vdots \\
 & + (x - x_0)(x - x_1) \cdots (x - x_{n-3})(x - x_{n-2})y[x_{n-1}, x_{n-2}, \dots, x_1, x_0] \\
 & + (x - x_0)(x - x_1) \cdots (x - x_{n-2})(x - x_{n-1})y[x_n, x_{n-1}, \dots, x_1, x_0]
 \end{aligned}$$

We can also factor this to be of the form:

$$\begin{aligned}
 p(x) = & y[x_0] + (x - x_0) \left( y[x_1, x_0] \right. \\
 & + (x - x_1) \left( y[x_2, x_1, x_0] \right. \\
 & + (x - x_2) \left( y[x_3, x_2, x_1, x_0] \right. \\
 & \vdots \\
 & + (x - x_{n-2}) \left( y[x_{n-1}, x_{n-2}, \dots, x_1, x_0] \right. \\
 & \left. \left. \left. + (x - x_{n-1}) (y[x_n, x_{n-1}, \dots, x_1, x_0]) \right) \cdots \right) \right) \left. \right)
 \end{aligned}$$

Assuming  $y[\ ]$  has already been computed, then we only have  $n - 1$  additions,  $n - 1$  subtractions, and  $n - 1$  multiplications.

So in total, we have  $3n + \mathcal{O}(1)$  flops.

## Question 3.

(a)

Suppose we are given  $p(x)$  of the form  $p(x) = \sum_{i=0}^n b_i(x-c)^i$

By the binomial theorem,  $(x-y)^i = \sum_{k=0}^i \binom{i}{k} x^{i-k}(-y)^k = \sum_{k=0}^i \binom{i}{k} x^{i-k}(-1)^k y^k$

Then

$$\begin{aligned}
 p(x) &= \sum_{i=0}^n b_i(x-c)^i \\
 &= \sum_{i=0}^n b_i \sum_{k=0}^i \binom{i}{k} x^{i-k} (-c)^k \\
 &= \sum_{i=0}^n \sum_{k=0}^i b_i \binom{i}{k} x^{i-k} (-c)^k \\
 &= \sum_{k=0}^0 b_0 \binom{0}{k} x^{0-k} (-c)^k \\
 &\quad + \sum_{k=0}^1 b_1 \binom{1}{k} x^{1-k} (-c)^k \\
 &\quad + \sum_{k=0}^2 b_2 \binom{2}{k} x^{2-k} (-c)^k \\
 &\quad \vdots \\
 &\quad + \sum_{k=0}^{n-1} b_{n-1} \binom{n-1}{k} x^{(n-1)-k} (-c)^k \\
 &\quad + \sum_{k=0}^n b_n \binom{n}{k} x^{n-k} (-c)^k \\
 &= b_0 \binom{0}{0} x^{0-0} (-c)^0 \\
 &\quad + b_1 \binom{1}{0} x^{1-0} (-c)^0 + b_1 \binom{1}{1} x^{1-1} (-c)^1 \\
 &\quad + b_2 \binom{2}{0} x^{2-0} (-c)^0 + b_2 \binom{2}{1} x^{2-1} (-c)^1 + b_2 \binom{2}{2} x^{2-2} (-c)^2 \\
 &\quad \vdots \\
 &\quad + b_{n-1} \binom{n-1}{0} x^{(n-1)-0} (-c)^0 + b_{n-1} \binom{n-1}{1} x^{(n-1)-1} (-c)^1 + \dots + b_{n-1} \binom{n-1}{n-2} x^{(n-1)-(n-2)} (-c)^{n-2} + b_{n-1} \binom{n-1}{n-1} x^{(n-1)-(n-1)} (-c)^{n-1} \\
 &\quad + b_n \binom{n}{0} x^{n-0} (-c)^0 + b_n \binom{n}{1} x^{n-1} (-c)^1 + \dots + b_n \binom{n}{n-1} x^{n-(n-1)} (-c)^{n-1} + b_n \binom{n}{n} x^{n-n} (-c)^n
 \end{aligned}$$

We can notice the last element of all sums are  $x^0$ , 2<sup>nd</sup> last element (if it has one) are  $x^1$ , and so on.  
We can factor out all the  $x$ 's with the same power.

$$\begin{aligned}
p(x) = & x^0 \left( b_0 \binom{0}{0} (-c)^0 + b_1 \binom{1}{1} (-c)^1 + b_2 \binom{2}{2} (-c)^2 + \dots + b_{n-1} \binom{n-1}{n-1} (-c)^{n-1} + b_n \binom{n}{n} (-c)^n \right) \\
& + x^1 \left( b_1 \binom{1}{0} (-c)^0 + b_2 \binom{2}{1} (-c)^1 + b_3 \binom{3}{2} (-c)^2 + \dots + b_{n-1} \binom{n-1}{n-2} (-c)^{n-2} + b_n \binom{n}{n-1} (-c)^{n-1} \right) \\
& + x^2 \left( b_2 \binom{2}{0} (-c)^0 + b_3 \binom{3}{1} (-c)^1 + b_4 \binom{4}{2} (-c)^2 + \dots + b_{n-1} \binom{n-1}{n-3} (-c)^{n-3} + b_n \binom{n}{n-2} (-c)^{n-2} \right) \\
& \vdots \\
& + x^{n-1} \left( b_{n-1} \binom{n-1}{0} (-c)^0 + b_n \binom{n}{1} (-c)^1 \right) \\
& + x^n \left( b_n \binom{n}{0} (-c)^0 \right)
\end{aligned}$$

so

$$\begin{aligned}
p(x) &= \sum_{i=0}^n \left( \sum_{k=i}^n b_k \binom{k}{k-i} (-c)^{k-i} \right) x^i \\
\implies p(x) &= \sum_{i=0}^n a_i x^i \text{ where } a_i = \sum_{k=i}^n b_k \binom{k}{k-i} (-c)^{k-i}
\end{aligned}$$

(b)

When calculating the reciprocal condition of the Vandermonde matrix for values of  $c$ , we get the following table:

c	Reciprocal condition
0	4.2535e-07
0.5	1.9436e-06
1	7.5962e-06
1.5	2.6885e-05
2	5.3226e-05
2.5	0.0001131
<b>3</b>	<b>0.00030227</b>
3.5	0.00016034
4	0.00014415
4.5	3.5049e-05
5	4.8742e-05
5.5	1.9436e-06
6	4.2535e-07

We can see that for  $c = 3$ , we get the biggest reciprocal condition  $\implies$  it minimizes the condition of the Vandermonde matrix.

To check more accurately, we can use finer values of  $c$ . So we have:

c	Reciprocal condition
2.5	0.0001131
2.55	0.00011994
2.6	0.00012769
2.65	0.00013636
2.7	0.00014597
2.75	0.00015654
2.8	0.00016809
2.85	0.00018069
2.9	0.00019448
2.95	0.00020968
<b>3</b>	<b>0.00030227</b>
3.05	0.00028431
3.1	0.0002672
3.15	0.00025084
3.2	0.00023527
3.25	0.00022053
3.3	0.00020671
3.35	0.00019392
3.4	0.00018223
3.45	0.00017171
3.5	0.00016034

## Question 4.

$x_i$	$y[x_i]$	$y[x_{i+1}, x_i]$	$y[x_{i+2}, x_{i+1}, x_i]$	$y[x_{i+3}, \dots, x_i]$	$y[x_{i+4}, \dots, x_i]$	$y[x_{i+5}, \dots, x_i]$	$y[x_{i+6}, \dots, x_i]$
-1	4	$\frac{7-4}{0-(-1)} = 3$	$\frac{6-3}{0-(-1)} = 3$	$\frac{15-3}{1-(-1)} = 6$	$\frac{20-6}{1-(-1)} = 7$	$\frac{15-7}{1-(-1)} = 4$	$\frac{7-4}{2-(-1)} = 1$
0	7	$\frac{y'(0)}{1!} = 6$	$\frac{21-6}{1-0} = 15$	$\frac{35-15}{1-0} = 20$	$\frac{35-20}{1-0} = 15$	$\frac{29-15}{2-0} = 7$	
0	7	$\frac{28-7}{1-0} = 21$	$\frac{56-21}{1-0} = 35$	$\frac{70-35}{1-0} = 35$	$\frac{93-35}{2-0} = 29$		
1	28	$\frac{y'(1)}{1!} = 56$	$\frac{y''(1)}{2!} = 70$	$\frac{163-70}{2-1} = 93$			
1	28	$\frac{y'(1)}{1!} = 56$	$\frac{219-56}{2-1} = 163$				
1	28	$\frac{247-28}{2-1} = 219$					
2	247						

$$\begin{aligned}
p(x) &= y[x_0] + (x - x_0)y[x_1, x_0] \\
&\quad + (x - x_0)(x - x_1)y[x_2, x_1, x_0] \\
&\quad + (x - x_0)(x - x_1)(x - x_2)y[x_3, x_2, x_1, x_0] \\
&\quad + (x - x_0)(x - x_1)(x - x_2)(x - x_3)y[x_4, x_3, x_2, x_1, x_0] \\
&\quad + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)y[x_5, x_4, x_3, x_2, x_1, x_0] \\
&\quad + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)y[x_6, x_5, x_4, x_3, x_2, x_1, x_0] \\
&= 4 + (x - (-1))(3) \\
&\quad + (x - (-1))(x)(3) \\
&\quad + (x - (-1))(x)(x)(6) \\
&\quad + (x - (-1))(x)(x)(x - 1)(7) \\
&\quad + (x - (-1))(x)(x)(x - 1)(x - 1)(4) \\
&\quad + (x - (-1))(x)(x)(x - 1)(x - 1)(x - 1)(1) \\
&= 4 + 3(x + 1) + 3x(x + 1) + 6x^2(x + 1) + 7x^2(x + 1)(x - 1) + 4x^2(x + 1)(x - 1)^2 + x^2(x + 1)(x - 1)^3 \\
&= x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 7
\end{aligned}$$

$$p'(x) = 6x^5 + 10x^4 + 12x^3 + 12x^2 + 10x + 6$$

$$p''(x) = 30x^4 + 40x^3 + 36x^2 + 24x + 10$$

$$\begin{aligned}
p(-1) &= 4 & p(0) &= 7 & p(1) &= 28 & p(2) &= 247 \\
p'(-1) &= 6 & p'(0) &= 6 & p'(1) &= 56 & & \\
p''(-1) &= 10 & p''(0) &= 10 & p''(1) &= 140 & & 
\end{aligned}$$