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Question 1.

(a)

Let $\mathcal{L}_k$ be a Gauss Transform. Then it has the form

$$\mathcal{L}_k = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_n & 0 & \cdots & 1 \end{bmatrix}$$

Then let $m_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -a_{k+1} \\ \vdots \\ -a_n \end{bmatrix}$

Now $-m_k {e_k}^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_{k+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_n & 0 & \cdots & 0 \end{bmatrix}$

So $\mathcal{L}_k = I - m_k {e_k}^T$

(b)

By definition, the inverse of a Gauss Transform is simply flipping the signs of the non-identity values. So if we invert the signs of $-m_k$, we have

$$m_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix}$$

Then

$$I + m_k {e_k}^T = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -a_{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -a_n & 0 & \cdots & 1 \end{bmatrix} = {\mathcal{L}_k}^{-1}$$

(c)

WTS: $(L_k L_j)^{-1} = (I + m_k {e_k}^T) + (m_j {e_j}^T) - I$

$$
\begin{aligned}
LHS &= (L_k L_j)^{-1} && \text{[given]} \\
&= {L_j}^{-1} {L_k}^{-1} && \text{[by inverse laws]} \\
&= (I + m_j {e_j}^T)(I + m_k {e_k}^T) && \text{[by (b)]} \\
&= I + m_j {e_j}^T + m_k {e_k}^T + m_j {e_j}^T m_k {e_k}^T && \text{[expanding]} \\
&= I + m_j {e_j}^T + m_k {e_k}^T + 0 && \text{[since } j < k \text{ and } {e_j}^T m_k = 0\text{]} \\
&= I + m_j {e_j}^T + m_k {e_k}^T + I - I && [I - I = 0] \\
&= (I + m_k {e_k}^T) + (I + m_j {e_j}^T) - I && \text{[rearranging]} \\
&= RHS && \text{[given]}
\end{aligned}
$$

as wanted

(d)

WTS: $\widetilde{\mathcal{L}_k} = \mathcal{L}_k$ with multipliers i and j swapped

$$\begin{aligned}
LHS &= \widetilde{\mathcal{L}_k} && \text{[given]} \\
&= P_i \mathcal{L}_k P_i && \text{[given]} \\
&= P_i(I - m_k {e_k}^T) P_i && \text{[by (a)]} \\
&= (P_i - P_i m_k {e_k}^T) P_i && \text{[expanding]} \\
&= P_i P_i - P_i m_k {e_k}^T P_i && \text{[expanding]} \\
&= I - P_i m_k {e_k}^T P_i && \text{[since } P_i P_i = I\text{]} \\
&= \mathcal{L}_k \text{ with multipliers } i \text{ and } j \text{ swapped} && \text{[since } m_k {e_k}^T \text{ multiplers got swapped]} \\
&= RHS && \text{[given]}
\end{aligned}$$

as wanted

Question 2.

$$
\begin{aligned}
PA = LU \quad &\iff \quad \det(PA) = \det(LU) && \text{[det both sides]} \\
&\iff \quad \det(P)\det(A) = \det(L)\det(U) && \text{[by det laws]} \\
&\iff \quad \det(A) = \det(L)\det(U) && \text{[since } \det(P) = 1\text{]} \\
&\iff \quad \det(A) = 1 \times \det(U) && \text{[since } L \text{ is a unit lower triangle]} \\
&\iff \quad \det(A) = \prod_{i=1}^{n} a_{ii} && \text{[since } U \text{ is an upper triangle]}
\end{aligned}
$$

This is much more efficient than just finding the determinant of A since the determinant of a triangular matrix is just the product of the diagonal.

Question 3.

(a)

$$A = \begin{bmatrix} 3 & 3 & 9 & 6 \\ 4 & 4 & 4 & 4 \\ 1 & 1 & 5 & 5 \\ 2 & 2 & 4 & 6 \end{bmatrix}, \qquad b = \begin{bmatrix} 21 \\ 24 \\ 10 \\ 16 \end{bmatrix}$$

Eliminate 1st column:

$$P_1 = P_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P_1 A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 3 & 3 & 9 & 6 \\ 1 & 1 & 5 & 5 \\ 2 & 2 & 4 & 6 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3/4 & 1 & 0 & 0 \\ -1/4 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 1 \end{bmatrix} \qquad L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

Eliminate 3rd column:

$$P_3 = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P_3 L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & 1 \end{bmatrix} \qquad L_3 P_3 1 L_1 P_1 A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b)

$\det(A) = \prod_{i=1}^{n} a_{ii} = 0$

So there's either infinite, or no solutions to $A\vec{x} = \vec{b}$

(c)

(d)

It's much more efficient to factor first if we have multiple $A\vec{x} = \vec{b_i}$ equations to solve

Question 4.