Table of Contents

Question 1:

(a)
(b)
(c)
(d)
(e)
(f)

Question 2:

(a)
(b)

Question 3:

(a)
(b)

Question 4:

Question 1.

(a)

$$\begin{aligned} p(-1) &= 4 \\ p(0) &= 6 \\ p(1) &= 12 \end{aligned} \iff \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix}$$

Eliminate 1st column:

$$P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_1 A = A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

Eliminate 2nd column:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_2 L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \qquad L_2 P_2 L_1 P_1 A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} L_2 P_2 L_1 P_1 A = U &\iff L_2 P_2 L_1 P_2 P_2 P_1 A = U \\ &\iff L_2 (P_2 L_1 P_2) P_2 P_1 A = U \\ &\iff L_2 \widetilde{L_1} P_2 P_1 A = U \\ &\iff P_2 P_1 A = \widetilde{L_1}^{-1} L_2^{-1} U \\ &\iff PA = LU \end{aligned}$$

$$\widetilde{L_1} = P_2 L_1 P_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L = \widetilde{L_1}^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix}$$

$$P = P_2 P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned} A\vec{x} = \vec{b} &\iff PA\vec{x} = P\vec{b} \\ &\iff LU\vec{x} = P\vec{b} \\ &\iff L(U\vec{x}) = P\vec{b} \\ &\iff L\vec{d} = P\vec{b} \end{aligned}$$

$$P\vec{b} = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$$

Forward solve $L\vec{d} = \vec{b}$ for $\vec{d}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 6 \end{bmatrix}$$

$d_1 = 4$

$d_1 + d_2 = 12$

$d_1 + \dfrac{1}{2}d_2 + d_3 = 6$

$d_1 = 4$

$d_2 = 8$

$d_3 = -2$

Backward solve $U\vec{x} = \vec{d}$ for $\vec{x}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -2 \end{bmatrix}$$

$x_3 = 2$

$x_2 = 4$

$x_1 = 6$

$\therefore p(x) = 6 + 4x + 2x^2$

(b)

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \qquad i = 0,\ 1,\ 2$$

$$\begin{aligned} x_0 &= -1 \qquad & y_0 &= 4 \\ x_1 &= 0 & y_1 &= 6 \\ x_2 &= 1 & y_2 &= 12 \end{aligned}$$

$i = 0$

$j = 1,\ 2$

$$\begin{aligned} l_0(x) &= \left(\frac{x-0}{-1-0}\right)\left(\frac{x-1}{-1-1}\right) \\ &= (-x)\left(\frac{x-1}{-2}\right) \\ &= \frac{1}{2}x(x-1) \end{aligned}$$

$i = 1$

$j = 0,\ 2$

$$\begin{aligned} l_1(x) &= \left(\frac{x-(-1)}{0-(-1)}\right)\left(\frac{x-1}{0-1}\right) \\ &= (x+1)(1-x) \end{aligned}$$

$i = 2$

$j = 0,\ 1$

$$\begin{aligned} l_2(x) &= \left(\frac{x-(-1)}{1-(-1)}\right)\left(\frac{x-0}{1-0}\right) \\ &= \left(\frac{x+1}{2}\right)(x) \\ &= \frac{1}{2}x(x+1) \end{aligned}$$

$$\begin{aligned} p(x) &= \sum_{i=0}^{n} l_i(x) y_i \\ &= l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2 \\ &= \frac{1}{2}x(x-1)(4) + (x+1)(1-x)(6) + \frac{1}{2}x(x+1)(12) \\ &= 2x^2 - 2x + (-6x^2) + 6 + 6x^2 + 6x \\ &= 2x^2 + 4x + 6 \end{aligned}$$

(c)

x_i	$y[x_i]$	$y[x_{i+1},\ x_i]$	$y[x_{i+2},\ x_{i+1},\ x_i]$
-1	4		
		$\dfrac{6-4}{1-(-1)} = 2$	
0	6		$\dfrac{6-2}{1-(-1)} = 2$
		$\dfrac{12-6}{1-0} = 6$	
1	12		

$$\begin{aligned} p(x) &= y[x_0] + (x - x_0)y[x_1,\ x_0] + (x - x_0)(x - x_1)y[x_2,\ x_1,\ x_0] \\ &= 4 + \big(x - (-1)\big)(2) + \big(x - (-1)\big)(x - 0)(2) \\ &= 4 + 2x + 2 + 2x^2 + 2x \\ &= 6 + 4x + 2x^2 \end{aligned}$$

(d)

As all the equations are simplified from (a), (b), and (c), we can see that all polynomials are identical.

(e)

We can use the method in (c) since all we have to do is to add to the table

x_i	$y[x_i]$	$y[x_{i+1},\ x_i]$	$y[x_{i+2},\ x_{i+1},\ x_i]$	$y[x_{i+3},\ x_{i+2},\ x_{i+1},\ x_i]$
-1	4			
		$\frac{6-4}{1-(-1)} = 2$		
0	6		$\frac{6-2}{1-(-1)} = 2$	
		$\frac{12-6}{1-0} = 6$		$\frac{-1-2}{2-(-1)} = -1$
1	12		$\frac{4-6}{2-0} = -1$	
		$\frac{16-12}{2-1} = 4$		
2	16			

$$
\begin{aligned}
p(x) &= y[x_0] + (x - x_0)y[x_1,\ x_0] \\
&\qquad + (x - x_0)(x - x_1)y[x_2,\ x_1,\ x_0] \\
&\qquad + (x - x_0)(x - x_1)(x - x_2)y[x_3,\ x_2,\ x_1,\ x_0] \\
&= 4 + \Big(x - (-1)\Big)(2) + \Big(x - (-1)\Big)(x - 0)(2) + \Big(x - (-1)\Big)(x - 0)(x - 1)(-1) \\
&= 4 + 2x + 2 + 2x^2 + 2x - x^3 + x \\
&= 6 + 5x + 2x^2 - x^3
\end{aligned}
$$

(f)

Equation of line from $(-1,\ 4)$ to $(0,\ 6)$:

$$\begin{aligned}\text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{6-4}{0-(-1)} \\ &= 2\end{aligned}$$

$$\begin{aligned}y - y_1 = \text{slope}_1(x - x_1) &\iff y - 4 = 2\Big(x - (-1)\Big) \\ &\iff y = 2(x+1) + 4 \\ &\iff y = 2x + 6\end{aligned}$$

Equation of line from $(0,\ 6)$ to $(1,\ 12)$:

$$\begin{aligned}\text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{12-6}{1-0} \\ &= 6\end{aligned}$$

$$\begin{aligned}y - y_1 = \text{slope}_1(x - x_1) &\iff y - 6 = 6(x - 0) \\ &\iff y = 6x + 6\end{aligned}$$

Equation of line from $(1,\ 12)$ to $(2,\ 16)$:

$$\begin{aligned}\text{slope}_1 &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{16-12}{2-1} \\ &= 4\end{aligned}$$

$$\begin{aligned}y - y_1 = \text{slope}_1(x - x_1) &\iff y - 12 = 4(x - 1) \\ &\iff y = 4x + 8\end{aligned}$$

$\therefore$ the equation of the linear spline is:

$$\begin{cases} y = 2x + 6 & \text{if } -1 \le x < 0 \\ y = 6x + 6 & \text{if } 0 \le x < 1 \\ y = 4x + 8 & \text{if } 1 \le x \le 2 \end{cases}$$

Question 2.

(a)
When we solve using the Vandermonde method, we get a polynomial of the form:

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{n-1}x^{n-1} + a_nx^n$$

We can factor this polynomial into the form:

$$p(x) = a_0 + x\Bigg(a_1 + x\bigg(a_2 + x\Big(a_3 + \ldots + x(a_{n-1} + xa_n)\cdots\Big)\bigg)\Bigg)$$

There are $n - 1$ additions, and $n - 1$ multiplications, so we have
$2n + \mathcal{O}(1)$ flops.

(b)

For divided-difference, we get a polynomial of the form:

$$
\begin{aligned}
p(x) = y[x_0] &+ (x-x_0)y[x_1,\ x_0] \\
&+ (x-x_0)(x-x_1)y[x_2,\ x_1,\ x_0] \\
&+ (x-x_0)(x-x_1)(x-x_2)y[x_3,\ x_2,\ x_1,\ x_0] \\
&\vdots \\
&+ (x-x_0)(x-x_1)\cdots(x-x_{n-3})(x-x_{n-2})y[x_{n-1},\ x_{n-2},\ \cdots,\ x_1,\ x_0] \\
&+ (x-x_0)(x-x_1)\cdots(x-x_{n-2})(x-x_{n-1})y[x_n,\ x_{n-1},\ \cdots,\ x_1,\ x_0]
\end{aligned}
$$

We can also factor this to be of the form:

$$
\begin{aligned}
p(x) = y[x_0] + (x-x_0)\Bigg(& y[x_1,\ x_0] \\
&+ (x-x_1)\Bigg(y[x_2,\ x_1,\ x_0] \\
&+ (x-x_2)\Bigg(y[x_3,\ x_2,\ x_1,\ x_0] \\
&\vdots \\
&+ (x-x_{n-2})\Big(y[x_{n-1},\ x_{n-2},\ \cdots,\ x_1,\ x_0] \\
&+ (x-x_{n-1})(y[x_n,\ x_{n-1},\ \cdots,\ x_1,\ x_0])\Big)\cdots\Bigg)\Bigg)\Bigg)
\end{aligned}
$$

Assuming $y[]$ has already been computed, then we only have $n-1$ additions, $n-1$ subtractions, and $n-1$ multiplications.

So in total, we have $3n + \mathcal{O}(1)$ flops.

Question 3.

(a)

Suppose we are given $p(x)$ of the form $p(x) = \sum_{i=0}^{n} b_i(x-c)^i$

By the binomial theorem, $(x-y)^i = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(-y)^k = \sum_{k=0}^{n} \binom{n}{k} x^k (-y)^{n-k}$

Then

$$\begin{aligned}
p(x) &= \sum_{i=0}^{n} b_i(x-c)^i \\
&= \sum_{i=0}^{n} b_i \sum_{k=0}^{i} \binom{i}{k} x^{i-k}(-c)^k \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} b_i \binom{i}{k} x^{i-k}(-c)^k \\
&= \sum_{k=0}^{0} b_0 \binom{0}{k} x^{0-k}(-c)^k \\
&\quad + \sum_{k=0}^{1} b_1 \binom{1}{k} x^{1-k}(-c)^k \\
&\quad + \sum_{k=0}^{2} b_2 \binom{2}{k} x^{2-k}(-c)^k \\
&\quad \vdots \\
&\quad + \sum_{k=0}^{n-1} b_{n-1} \binom{n-1}{k} x^{(n-1)-k}(-c)^k \\
&\quad + \sum_{k=0}^{n} b_n \binom{n}{k} x^{n-k}(-c)^k \\
&= b_0 \binom{0}{0} x^{0-0}(-c)^0 \\
&\quad + b_1 \binom{1}{0} x^{1-0}(-c)^0 + b_1 \binom{1}{1} x^{1-1}(-c)^1 \\
&\quad + b_2 \binom{2}{0} x^{2-0}(-c)^0 + b_2 \binom{2}{1} x^{2-1}(-c)^1 + b_2 \binom{2}{2} x^{2-2}(-c)^2 \\
&\quad \vdots \\
&\quad + b_{n-1} \binom{n-1}{0} x^{(n-1)-0}(-c)^0 + b_{n-1} \binom{n-1}{1} x^{(n-1)-1}(-c)^1 + \ \dots \ + b_{n-1} \binom{n-1}{n-2} x^{(n-1)-(n-2)}(-c)^{n-2} + b_{n-1} \binom{n-1}{n-1} x^{(n-1)-(n-1)}(-c)^{n-1} \\
&\quad + b_n \binom{n}{0} x^{n-0}(-c)^0 + b_n \binom{n}{1} x^{n-1}(-c)^1 + \ \dots \ + b_n \binom{n}{n-1} x^{n-(n-1)}(-c)^{n-1} + b_{n-1} \binom{n}{n} x^{n-n}(-c)^n
\end{aligned}$$

We can notice the last element of all sums are x^0, 2^{nd} last element (if it has one) are x^1, and so on. We can factor out all the x's with the same power.

$$
\begin{aligned}
p(x) = x^0 & \left(b_0\binom{0}{0}(-c)^0 + b_1\binom{1}{1}(-c)^1 + b_2\binom{2}{2}(-c)^2 + \ \dots\ + b_{n-1}\binom{n-1}{n-1}(-c)^{n-1} + b_n\binom{n}{n}(-c)^n\right) \\
& + x^1\left(b_1\binom{1}{0}(-c)^0 + b_2\binom{2}{1}(-c)^1 + b_3\binom{3}{2}(-c)^2 + \ \dots\ + b_{n-1}\binom{n-1}{n-2}(-c)^{n-2} + b_n\binom{n}{n-1}(-c)^{n-1}\right) \\
& + x^2\left(b_2\binom{2}{0}(-c)^0 + b_3\binom{3}{1}(-c)^1 + b_4\binom{4}{2}(-c)^2 + \ \dots\ + b_{n-1}\binom{n-1}{n-3}(-c)^{n-3} + b_n\binom{n}{n-2}(-c)^{n-2}\right) \\
& \vdots \\
& + x^{n-1}\left(b_{n-1}\binom{n-1}{0}(-c)^0 + b_n\binom{n}{1}(-c)^1\right) \\
& + x^n\left(b_n\binom{n}{0}(-c)^0\right)
\end{aligned}
$$

so

$$
p(x) = \sum_{i=0}^{n}\left(\sum_{k=i}^{n} b_k\binom{k}{k-i}(-c)^{k-i}\right)x^i
$$

$$
\implies p(x) = \sum_{i=0}^{n} a_i x^i \text{ where } a_i = \sum_{k=i}^{n} b_k\binom{k}{k-i}(-c)^{k-i}
$$

(b)

When calculating the reciprocal condition of the Vandermode matrix for values of c, we get the following table:

c	Reciprocal condition
0	4.2535e-07
0.5	1.9436e-06
1	7.5962e-06
1.5	2.6885e-05
2	5.3226e-05
2.5	0.0001131
3	0.00030227
3.5	0.00016034
4	0.00014415
4.5	3.5049e-05
5	4.8742e-05
5.5	1.9436e-06
6	4.2535e-07

We can see that for $c = 3$, we get the biggest reciprocal condition $\implies$ it minimizes the condition of the Vandermonde matrix.

To check more accurately, we can use finer values of `c`. So we have:

c	Reciprocal condition
2.5	0.0001131
2.55	0.00011994
2.6	0.00012769
2.65	0.00013636
2.7	0.00014597
2.75	0.00015654
2.8	0.00016809
2.85	0.00018069
2.9	0.00019448
2.95	0.00020968
3	0.00030227
3.05	0.00028431
3.1	0.0002672
3.15	0.00025084
3.2	0.00023527
3.25	0.00022053
3.3	0.00020671
3.35	0.00019392
3.4	0.00018223
3.45	0.00017171
3.5	0.00016034

Question 4.

x_i	$y[x_i]$	$y[x_{i+1},\ x_i]$	$y[x_{i+2},\ x_{i+1},\ x_i]$	$y[x_{i+3},\ \cdots,\ x_i]$	$y[x_{i+4},\ \cdots,\ x_i]$	$y[x_{i+5},\ \cdots,\ x_i]$	$y[x_{i+6},\ \cdots,\ x_i]$
-1	4						
		$\frac{7-4}{0-(-1)} = 3$					
0	7		$\frac{6-3}{0-(-1)} = 3$				
		$\frac{y'(0)}{1!} = 6$		$\frac{15-3}{1-(-1)} = 6$			
0	7		$\frac{21-6}{1-0} = 15$		$\frac{20-6}{1-(-1)} = 7$		
		$\frac{28-7}{1-0} = 21$		$\frac{35-15}{1-0} = 20$		$\frac{15-7}{1-(-1)} = 4$	
1	28		$\frac{56-21}{1-0} = 35$		$\frac{35-20}{1-0} = 15$		$\frac{7-4}{2-(-1)} = 1$
		$\frac{y'(1)}{1!} = 56$		$\frac{70-35}{1-0} = 35$		$\frac{29-15}{2-0} = 7$	
1	28		$\frac{y''(1)}{2!} = 70$		$\frac{93-35}{2-0} = 29$		
		$\frac{y'(1)}{1!} = 56$		$\frac{163-70}{2-1} = 93$			
1	28		$\frac{219-56}{2-1} = 163$				
		$\frac{247-28}{2-1} = 219$					
2	247						

$$
\begin{aligned}
p(x) &= y[x_0] + (x-x_0)y[x_1,\ x_0] \\
&\qquad + (x-x_0)(x-x_1)y[x_2,\ x_1,\ x_0] \\
&\qquad + (x-x_0)(x-x_1)(x-x_2)y[x_3,\ x_2,\ x_1,\ x_0] \\
&\qquad + (x-x_0)(x-x_1)(x-x_2)(x-x_3)y[x_4,\ x_3,\ x_2,\ x_1,\ x_0] \\
&\qquad + (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)y[x_5,\ x_3,\ x_2,\ x_1,\ x_0] \\
&\qquad + (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)y[x_6,\ x_5,\ x_3,\ x_2,\ x_1,\ x_0] \\
&= 4 + \big(x-(-1)\big)(3) \\
&\quad + \big(x-(-1)\big)(x)(3) \\
&\quad + \big(x-(-1)\big)(x)(x)(6) \\
&\quad + \big(x-(-1)\big)(x)(x)(x-1)(7) \\
&\quad + \big(x-(-1)\big)(x)(x)(x-1)(x-1)(4) \\
&\quad + \big(x-(-1)\big)(x)(x)(x-1)(x-1)(x-1)(1) \\
&= 4 + 3(x+1) + 3x(x+1) + 6x^2(x+1) + 7x^2(x+1)(x-1) + 4x^2(x+1)(x-1)^2 + x^2(x+1)(x-1)^3 \\
&= x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 7
\end{aligned}
$$

$$
\begin{aligned}
p'(x) &= 6x^5 + 10x^4 + 12x^3 + 12x^2 + 10x + 6 \\
p''(x) &= 30x^4 + 40x^3 + 36x^2 + 24x + 10
\end{aligned}
$$

$$
\begin{array}{cccc}
p(-1) = 4 & p(0) = 7 & p(1) = 28 & p(2) = 247 \\
 & p'(0) = 6 & p'(1) = 56 & \\
 & & p''(1) = 140 &
\end{array}
$$