

Table of contents

Problem 1:

- (1)
- (2)
- (3)
- (4)

Problem 2:

- (1)
- (2)
- (3)

Problem 3:

- (1)
- (2)
- (3)

Problem 4:

- (1)
- (2)
- (3)

Problem 1.

(1)

Suppose $A \in M_{n \times n}(\mathbb{C})$ is unitarily diagonalizable

$$D = PAP^{-1}$$

D is a diagonal matrix

P is a unitary matrix

WTS: $AA^* = A^*A$

$$A = P^{-1}DP$$

[by given]

$$A^* = (P^{-1}DP)^*$$

[by definition of A]

$$= P^*D^*P^{-1*}$$

[by properties of $*$]

$$= P^{-1}D^*P$$

[by properties of $*$]

$$= P^{-1}\overline{D^T}P$$

[by definition of $*$]

$$= P^{-1}\overline{D}P$$

[since diagonal matrices are symmetrical]

$$AA^* = P^{-1}DPP^{-1}\overline{D}P$$

[by given]

$$= P^{-1}D\overline{D}P$$

[since $PP^{-1} = I$]

$$= P^{-1}\overline{D}DP$$

[since diagonal matrices commute]

$$= P^{-1}\overline{D}PP^{-1}DP$$

[since $PP^{-1} = I$]

$$= A^*A$$

[by definition of A and A^*]

$\therefore A$ is normal

■

(2)

Suppose

$$A = UBU^*$$

$$U \text{ is unitary or } U^* = U^{-1}$$

WTS: A is normal $\iff B$ is normalCase 1: A is normal

$$AA^* = A^*A$$

WTS: $BB^* = B^*B$

$$B = U^*AU$$

[by given]

$$B^* = U^*A^*U$$

[by definition of B]

$$BB^* = U^*AUU^*A^*U$$

[by definition of B]

$$= U^*AA^*U$$

[since $UU^* = UU^{-1} = I$]

$$= U^*A^*AU$$

[since $AA^* = A^*A$]

$$= U^*A^*UU^*AU$$

[since $UU^* = UU^{-1} = I$]

$$= B^*B$$

[by definition of B] $\therefore B$ is normal, as wanted.Case 2: B is normal

$$BB^* = B^*B$$

WTS: $AA^* = A^*A$

$$A = UBU^*$$

[by given]

$$A^* = UB^*U^*$$

[by definition of B]

$$AA^* = UBU^*UB^*U^*$$

[by definition of A]

$$= UBB^*U^*$$

[since $UU^* = UU^{-1} = I$]

$$= UB^*BU^*$$

[since $BB^* = B^*B$]

$$= UB^*U^*UBU^*$$

[since $UU^* = UU^{-1} = I$]

$$= A^*A$$

[by definition of A] $\therefore A$ is normal, as wanted. $\therefore A$ is normal $\iff B$ is normal

(3)

$$\text{Let } B = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ 0 & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \cdots & \lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{nn} \end{bmatrix} \quad B^* = \begin{bmatrix} \overline{\lambda_{11}} & 0 & 0 & \cdots & 0 \\ \overline{\lambda_{12}} & \overline{\lambda_{22}} & 0 & \cdots & 0 \\ \overline{\lambda_{13}} & \overline{\lambda_{23}} & \overline{\lambda_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\lambda_{1n}} & \overline{\lambda_{2n}} & \overline{\lambda_{3n}} & \cdots & \overline{\lambda_{nn}} \end{bmatrix}$$

Given: $BB^* = B^*B$ WTS: $\lambda_{ij} = 0$ if $i \neq j$

$$A = BB^*$$

$$= B^*B$$

$$= \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ 0 & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \cdots & \lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{nn} \end{bmatrix} \begin{bmatrix} \overline{\lambda_{11}} & 0 & 0 & \cdots & 0 \\ \overline{\lambda_{12}} & \overline{\lambda_{22}} & 0 & \cdots & 0 \\ \overline{\lambda_{13}} & \overline{\lambda_{23}} & \overline{\lambda_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\lambda_{1n}} & \overline{\lambda_{2n}} & \overline{\lambda_{3n}} & \cdots & \overline{\lambda_{nn}} \end{bmatrix}$$

$$= \begin{bmatrix} \overline{\lambda_{11}} & 0 & 0 & \cdots & 0 \\ \overline{\lambda_{12}} & \overline{\lambda_{22}} & 0 & \cdots & 0 \\ \overline{\lambda_{13}} & \overline{\lambda_{23}} & \overline{\lambda_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\lambda_{1n}} & \overline{\lambda_{2n}} & \overline{\lambda_{3n}} & \cdots & \overline{\lambda_{nn}} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ 0 & \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \cdots & \lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{nn} \end{bmatrix}$$

$$\begin{aligned}
A_{11} &= \lambda_{11} \overline{\lambda_{11}} && [\text{from } BB^*] && \sum_{i=1}^n |\lambda_{1i}|^2 = |\lambda_{11}|^2 \\
&= |\lambda_{11}|^2 && && \implies \sum_{i=2}^n |\lambda_{1i}|^2 = 0 \\
&= \sum_{i=1}^n \lambda_{1i} \overline{\lambda_{1i}} && [\text{from } B^*B] && \implies |\lambda_{1i}|^2 = 0 \\
&= \sum_{i=1}^n |\lambda_{1i}|^2 && && \implies |\lambda_{1i}| = 0 \\
&&& && \implies \lambda_{1i} = 0
\end{aligned}$$

$$\begin{aligned}
A_{22} &= \lambda_{22} \overline{\lambda_{22}} + \lambda_{12} \overline{\lambda_{12}} && [\text{from } BB^*] && \sum_{i=2}^n |\lambda_{2i}|^2 = |\lambda_{22}|^2 \\
&= |\lambda_{22}|^2 + 0 && && \implies \sum_{i=3}^n |\lambda_{2i}|^2 = 0 \\
&= |\lambda_{22}|^2 && && \implies |\lambda_{2i}|^2 = 0 \\
&= \sum_{i=2}^n \lambda_{2i} \overline{\lambda_{2i}} && [\text{from } B^*B] && \implies |\lambda_{2i}| = 0 \\
&= \sum_{i=2}^n |\lambda_{2i}|^2 && && \implies \lambda_{2i} = 0
\end{aligned}$$

we can continue this pattern for all $i \in \{1, \dots, n\}$

This means that for every $i \neq j$, $A_{ij} = 0$

$\therefore A$ is a diagonal matrix.

(4)

Suppose $A \in M_{n \times n}(\mathbb{C})$ is normal

WTS: $\exists U, D \in M_{n \times n}(\mathbb{C})$ such that $D = UAU^{-1}$, U is unitary, D is diagonal.

By Schur's diagonalizable lemma, $\exists B, U \in M_{n \times n}(\mathbb{C})$ such that $B = U^{-1}AU$, U is unitary, B is an upper triangular matrix.

By (2), A is normal $\iff B$ is normal.

We know A is normal since it is given in the question, so this means B is normal.

by (3), we know that a normal upper triangular matrix is diagonal.

Since B satisfies these conditions, this means B is diagonal.

In the end, we have $B = U^{-1}AU$, U is unitary, B is diagonal.

\therefore Any normal matrix is unitarily diagonalizable ■

Problem 2.

(1)

Suppose T is unitarily diagonalizable with real eigenvalues. $\implies [T]_{\mathcal{U}}$ is diagonal.

WTS: $[T]_{\mathcal{U}} = [T]_{\mathcal{U}}^*$

Since $[T]_{\mathcal{U}}$ is diagonal, let

$$[T]_{\mathcal{U}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \lambda_i \in \mathbb{C}$$

$$[T]_{\mathcal{U}}^* = \begin{bmatrix} \overline{\lambda_1} & 0 & \cdots & 0 \\ 0 & \overline{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\lambda_n} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Since the eigenvalues are real.

Since T is hermitian, T is self-adjoint. ■

(2)

Suppose T is an isometry. $\implies \langle v, w \rangle = \langle T(v), T(w) \rangle$

WTS: $\forall v \neq 0$ such that $T(v) = \lambda v$, $\|\lambda\| = 1$

Let $v \in V$ be an arbitrary eigenvector.

$$\begin{aligned} \langle v, v \rangle &= \langle T(v), T(v) \rangle && \text{[since } T \text{ is an isometry]} \\ &= \langle \lambda v, \lambda v \rangle && \text{[since } v \text{ is an eigenvector]} \\ &= \lambda \overline{\lambda} \langle v, v \rangle && \text{[by taking scalars out of inner product]} \\ &= |\lambda| \langle v, v \rangle && \text{[by definition of magnitude of a complex number]} \end{aligned}$$

We know that $\langle v, v \rangle > 0$ by the property of positive-definite, and that $v \neq 0$ since it is an eigenvector.

This means in order for $\langle v, v \rangle = |\lambda| \langle v, v \rangle$, $|\lambda|$ must equal 1. ■

(3)

Suppose T is unitarily diagonalizable, and all eigenvalues of T has absolute value 1.

WTS: T is an isometry.

Let $\mathcal{U} = \{u_1, \dots, u_n\}$ be an orthonormal basis of V .

So, for all $i \in \{1, \dots, n\}$, u_i is an eigenvector.

Let $T(u_i) = \lambda_i u_i$

$$[T]_{\mathcal{U}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad |\lambda_i| = 1$$

Since for every column of $[T]_{\mathcal{U}}$, we have

$$\text{i}^{\text{th}} \text{ column of } T = \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i}^{\text{th}} \text{ row}$$

So $(\text{i}^{\text{th}} \text{ column of } T) \cdot (\text{j}^{\text{th}} \text{ column of } T)$

$$= \begin{cases} |\lambda_i|^2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \begin{array}{l} \text{since } \lambda_i \cdot \lambda_i = |\lambda_i|^2 \\ \text{since all other entries are 0} \end{array}$$

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

\therefore columns of $[T]_{\mathcal{U}}$ are orthonormal

$\implies [T]_{\mathcal{U}}$ is unitary

$\implies T$ is an isometry. ■

Problem 3.

(1)

(2)

(3)

Problem 4.

(1)

(2)

(3)