Random variables and Distributions

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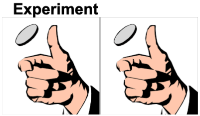
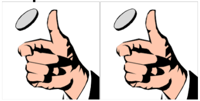
Recap



Recap

Experiment



Outcome

HH

TT

HT

TH

Recap

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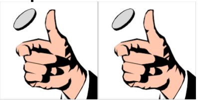
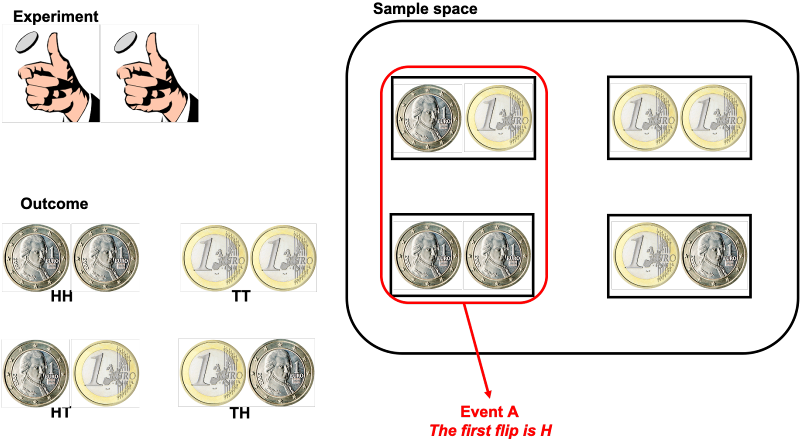


Outcome

Sample space

Recap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The first flip is H

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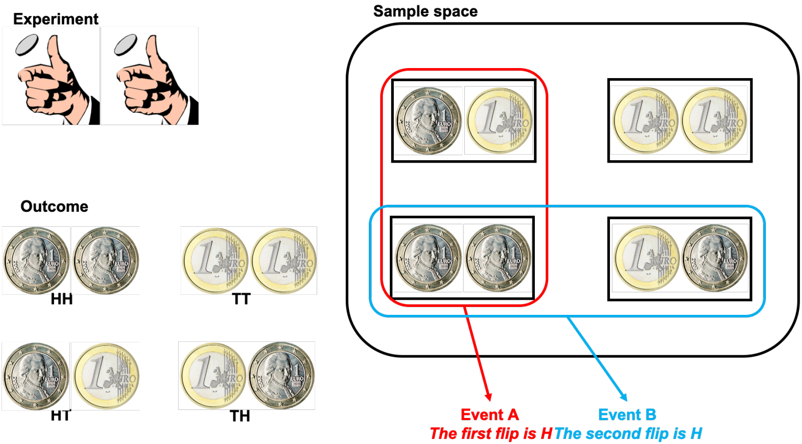
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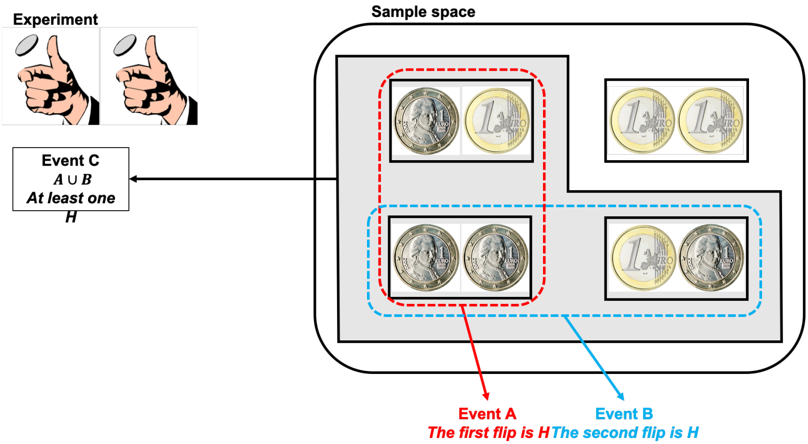
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Sample space
Event C
$A \cup B$
At least one H
Event A
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Event B
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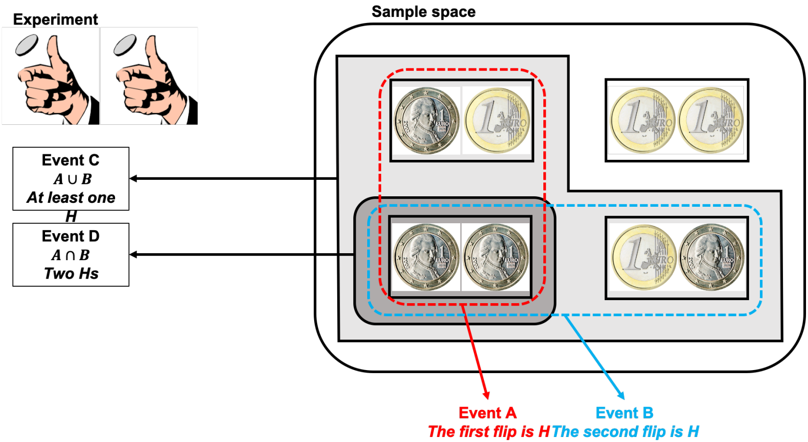
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Two Hs
Event A
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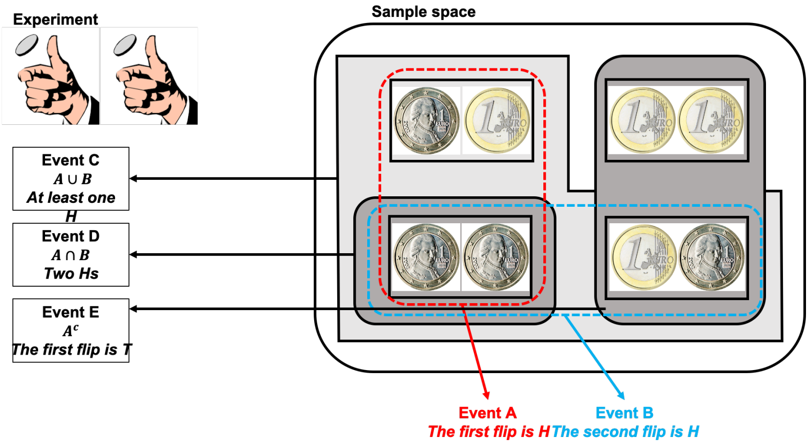
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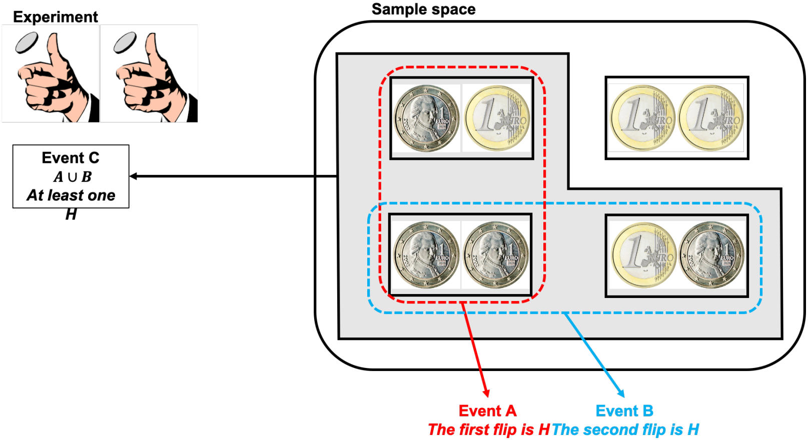
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Recap

Random variables

Given an experiment with sample space S, a random variable (r.v.) is a function from the sample space S to the real numbers R.
r.v. X assigns a value $X(s)$ for each outcome s of experiment.

Random variables

Flip 2 coins.
Sample space: $S = \{SS, NN, SN, NS\}$
Example of r.v.:

- X is the number of heads. r.v with 3 possible values: 0, 1, 2:
  $X(NN) = 2$, $X(NS) = X(SN) = 1 X(SS) = 0$

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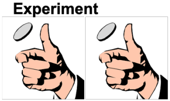
- Y is the number of tails. $Y(s) = 2 - X(s)$
- $I = 1$ if heads first and $I = 0$ otherwise. $I(NN) = I(NS) = 1$, $I(SN) = I(SS) = 0$

Random variables



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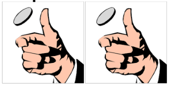

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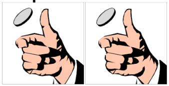

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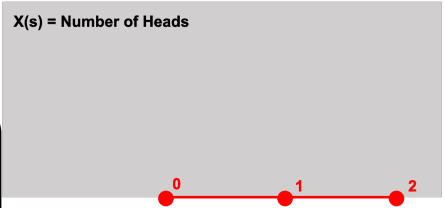

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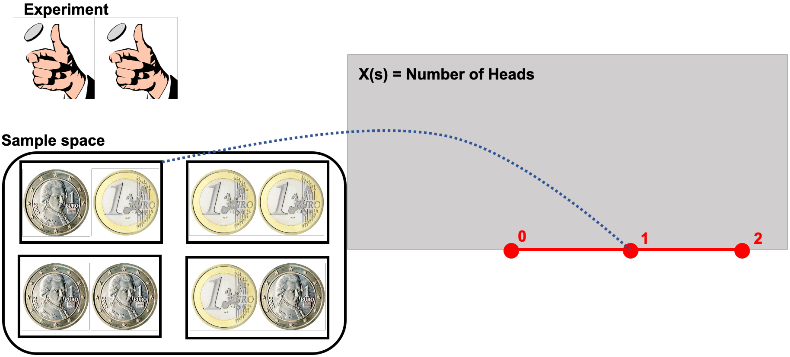

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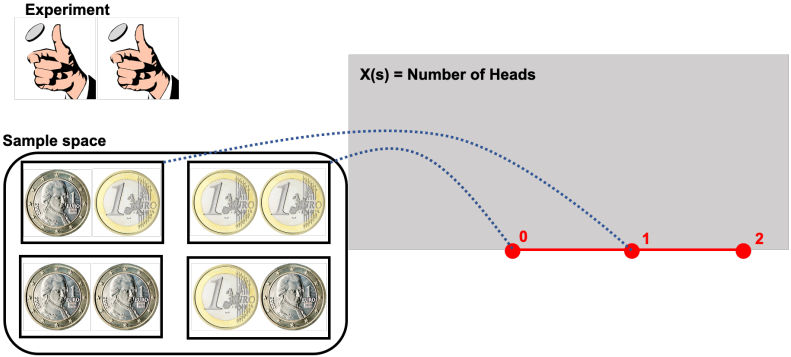

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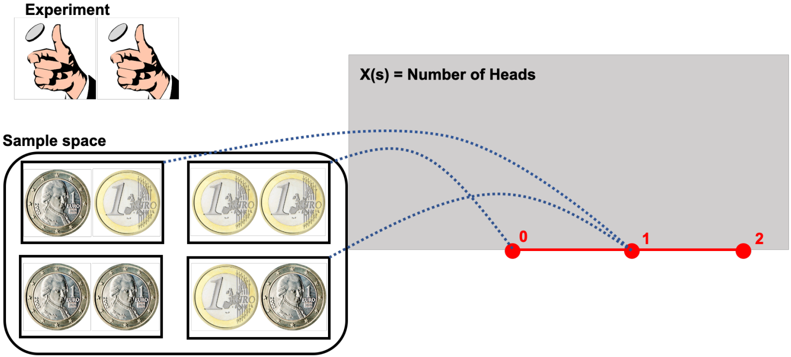

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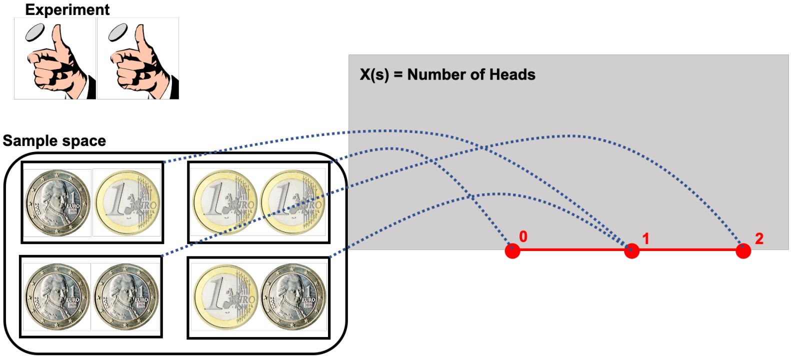

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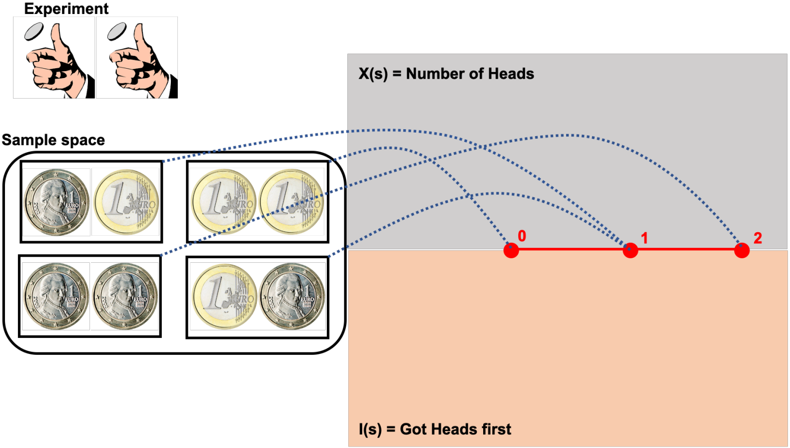

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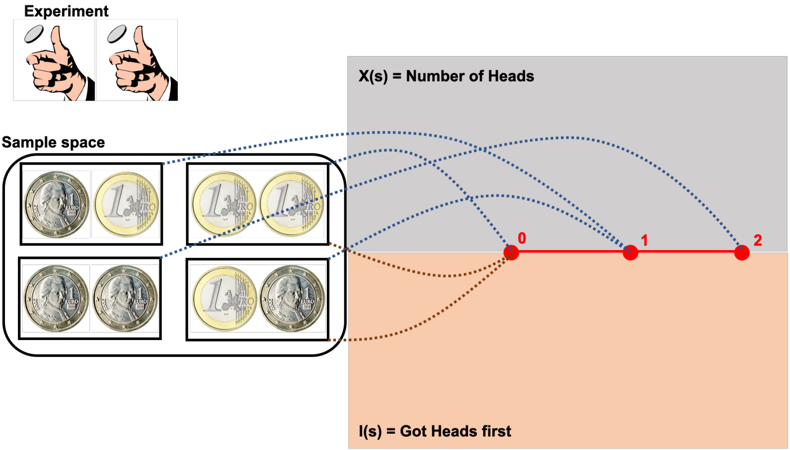

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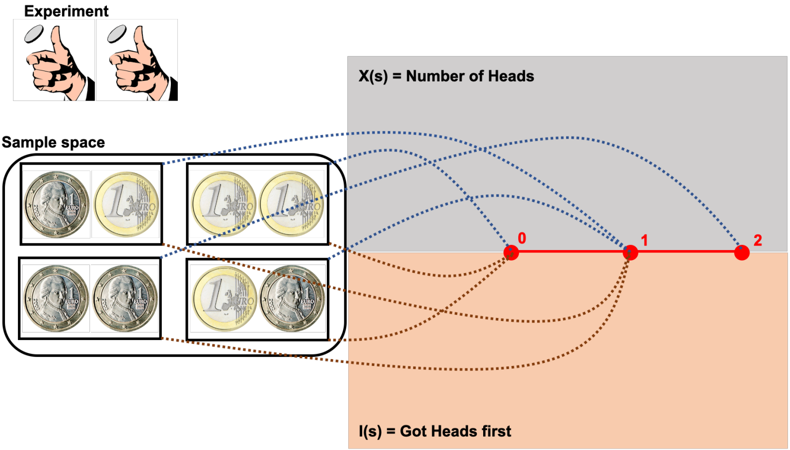

Types of random variables

- Discrete random variables: possible values of X is finite or countable infinite
- Continuous random variables: can take on any real value in an interval

Probability distribution

Distribution of a r.v. specifies the probabilities of all events associated with that r.v.

Probability mass function (PMF)

PMF of a discrete r.v. X is the function p_X given by $p_X(x) = P(X = x)$. This function is positive if x is in the support of X, and 0 otherwise.

Properties of PMF p_X:

- Non-negative: $p_X(x) > 0$ for some x and $p_X(x) = 0$ otherwise
- Sum to 1: $\Sigma_{j=1}^{\infty} p_X(x_j) = 1$

Example 1

Find PMF of the following r.v.:

- X is the number of heads
- Y is the number of tails
- $I = 1$ if heads first and $I = 0$ otherwise

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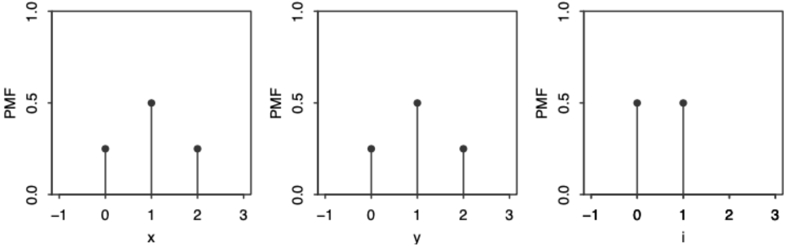
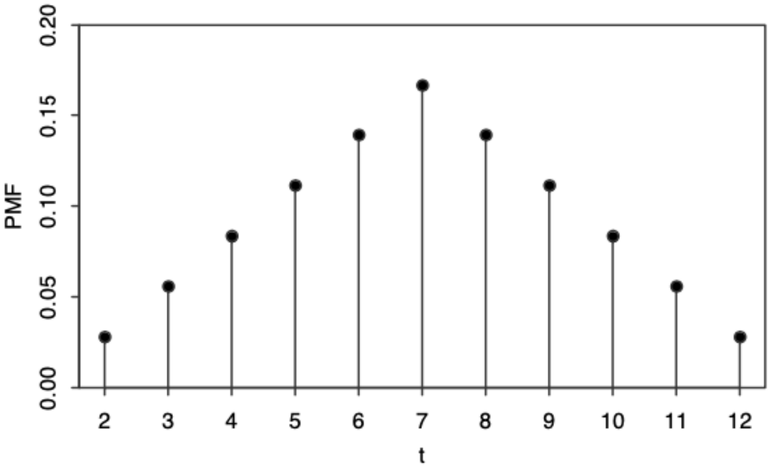


Example 2

Roll 2 fair dices A and B. Calculate PMF of the following r.v.:

- X is the score of dice A.
- Y is the score of dice B
- $Z = 7 - X$
- $T = X + Y$

Example 2



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$P(1 \leq T \leq 4) = P(T = 2) + P(T = 3) + P(T = 4) = 6/36$

Bernoulli distribution

A r.v. X has Bernoulli distribution with parameter p if $P(X = 1) = p$ and $P(X = 0) = 1 - p$, with $0 < p < 1$. Denote as $X \sim Bern(p)$

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Any r.v. whose possible values are 0 and 1 has a $Bern(p)$, with p is the probability of the r.v. equalling 1.

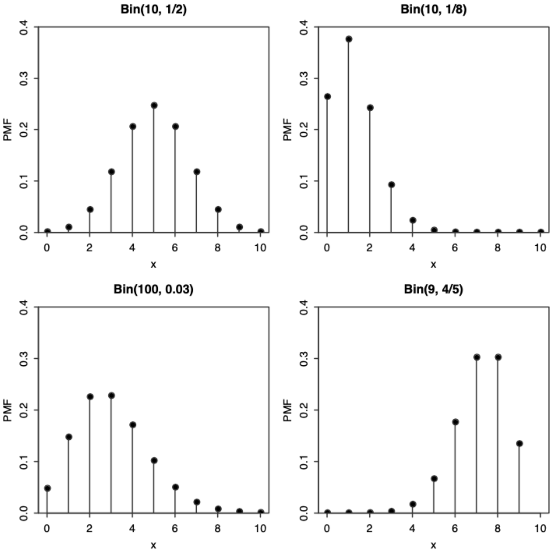
An experiment that can result in either a "success" or a "failure" (but not both) is called a **Bernoulli trial**. A Bernoulli random variable can be thought of as the indicator of success in a Bernoulli trial.

Binomial distribution

Perform n independent Bernoulli trials, each with the same success probability p. Let X be the number of successes. The distribution of X is the Binomial distribution with parameters n and p. We write $X \sim Bin(n, p)$
If $X \sim Bin(n, p)$, PMF of X is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial distribution



Cumulative distribution functions (CDF)

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- Increasing: if $x_1 \leq x_2$ then $F(x_1) \leq F(x_2)$
- Convergence to 0 and 1 at the limits: $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$

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- Variance: describe the spread of the distribution

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Expected value of Binomial distribution: if $X \sim Bin(n, p)$ then $E(X) = np$

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$$Var(X) = E(X - EX)^2 = E(X^2) - (EX)^2$$

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Properties of variance:

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Variance of Binomial distribution: if $X \sim Bin(n, p)$ then
$Var(X) = np(1 - p)$

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X is a r.v. with PDF f. Then CDF của X is:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

và probability that X is from a to b is:

$$P(a < X \leq b) = \int_{a}^{b} f(x)dx$$

Probability density function (PDF)

Properties of PDF

- Non-negative: $f(x) \geq 0$
- Integrates to 1: $\int_{-\infty}^{\infty} f(x)dx = 1$

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Expected value of r.v. X with PDF f is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Standard normal distribution

A continuous r.v. Z is said to have the standard Normal distribution if its PDF φ is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \exp^{-z^2/2}, -\infty < z < \infty$$

Z has mean 0 and variance 1. Denote as $Z \sim N(0, 1)$

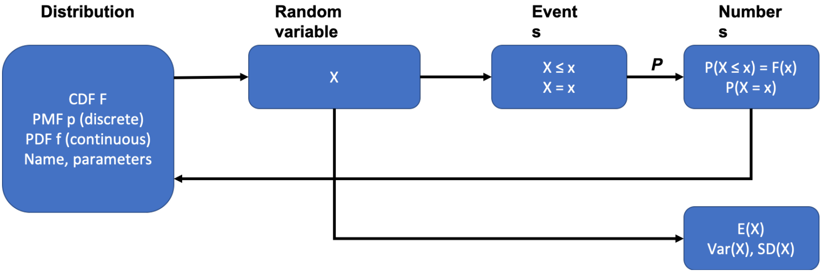
Normal distribution

If $Z \sim N(0,1)$ then

$$X = \mu + \sigma Z$$

has the Normal distribution with mean μ và variance σ^2. Denote as $X \sim N(\mu, \sigma^2)$

Recap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Number s
CDF F
PMF p (discrete)
PDF f (continuous)
Name, parameters
X
X ≤ x
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P(X = x)
E(X)
Var(X), SD(X)