3. Discrete Time Markov Chain

Discrete Time Markov Chain

Let $\{X_n, n \geq 0\}$ be a stochastic process taking values in a state space S that has N states. such a stochastic process is a Markov processes if it satisfies a following property :

$$P(X_{n+1} = k_{n+1}|X_n = k_n, X_{n-1} = k_{n-1}, ...X_1 = k_1) = P(X_{n+1} = k_{n+1}|X_n = k_n)$$

For a markov process, the *future state* only depends on the *present state* and not on the *past states*.

If the state space of a Markov process is discrete, it's called a **Markov Chain**.

To understand the behaviour of this process, we will need to calculate probabilities like,

$$P\left[X_0 = i_0, X_1 = i_1, ..., X_n = i_n\right]$$

..(1)

$\because P(A, B) = P(A) \cdot P(B|A)$, this can be computed by multiplying conditional probabilities as follows.

$$= P(X_0 = i_0) \cdot P(X_1 = i_1|X_0 = i_0) \cdot P(X_2 = i_2|X_1 = i_1, X_0 = i_0)...$$

$$P(X_n = i_n|X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, ..., X_0 = i_0)$$

..(2)

From the markovian property,

$$= P(X_0 = i_0) \cdot P(X_1 = i_1|X_0 = i_0) \cdot P(X_2 = i_2|X_1 = i_1)...P(X_n = i_n|X_{n-1} = i_{n-1})$$

..(3)

State Transition Probabilities

For a discrete time Markov Chain $\{X_n : n = 1, 2, ...\}$ with discrete state space $S = \{0, 1, 2, ...\}$ where this set may be finite or infinite, if $X_n = i$ then the Markov Chain is said to be in state i at time n(or the n^{th} step)

One Step Transition Probability

A discrete time Markov Chain $\{X_n : n = 1, 2, ...\}$ is characterized by

$$P\left[X_{n+1} = i_{n+1} | X_n = i_n, ..., X_0 = i_0\right] = P\left[X_{n+1} = i_{n+1} | X_n = i_n\right]$$

Where $P\left[X_{n+1} = j | X_n = i\right]$ is called one step transition probability

If $P\left[X_{n+1} = j | X_n = i\right]$ is independent of n then the Markov Chain is said to possess stationary transition probabilities and the process is reffered to as a homogeneous Markov Chain. Otherwise the process is called a non-homogeneous Markov Chain.

Transition Probability Matrix

The matrix called the **state transition matrix (t.p.m)** or **transition probability matrix** is usually denoted by P.

Let $\{X_n : n = 1, 2, ...\}$ be a homogenous Markov Chain with a discrete finite state space $S = \{0, 1, 2, ..., m\}$ then

$$p_{ij} = P\left[X_{n+1} = j | X_n = i\right] \quad i \geq 0, j \geq 0$$

regardless of the value of n.

A t.p.m of $\{X_n\}$ is defined by

$$P = \left[p_{ij}\right] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ p_{31} & p_{32} & \cdots & p_{3m} \\ \vdots & \ddots & & \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

Where

$$p_{ij} \geq 0$$

and

$$\sum_{j=1}^{m} p_{ij} = 1, \quad i = 1, 2, ..., m$$

State Transition Diagram

A Markov Chain is usually shown by a state transition diagram. Consider a Markov Chain with three possible states $S = \{1, 2, 3\}$ and the following transition probabilities

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

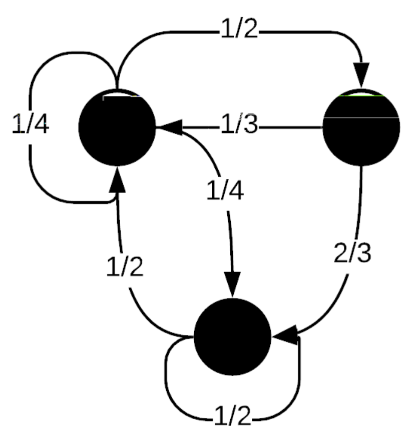
Which satisfies the two criterias, i.e.

$$p_{ij} \geq 0$$

and

$$\sum_{j=1}^{3} p_{ij} = 1, \quad i = 1, 2, 3$$

The figure below shows the state transition diagram for this Markov Chain



n-step Transition Probability

Consider a Markov Chain $\{X_n : n = 0, 1, 2, ...\}$ if $X_0 = i$ then $X_1 = j$ with probability p_{ij} is the probability of going from state i to state j in one step.

Now suppose we're interested in finding the probability of going from state i to state j in two steps, i.e.

$$p_{ij}^{(2)} = P(X_2 = j | X_0 = i)$$

$$p_{ij}^{(2)} = P(X_{n+2} = j | X_n = i)$$

We can find the probability by applying the law of total probability X_1 can take one of the possible values of S

$$p_{ij}^{(2)} = P(X_2 = j | X_0 = i)$$

$$= \sum_{k \in S} P(X_2 = j | X_1 = k, X_0 = i) \cdot P(X_1 = k | X_0 = i)$$

[by law of total probability]

$$= \sum_{k \in S} P(X_2 = j | X_1 = k) \cdot P(X_1 = k | X_0 = i)$$

[by markovian property]

$$= \sum_{k \in S} p_{kj} \cdot p_{ik}$$

$$= \sum_{k \in S} p_{ik} \cdot p_{kj}$$

$$\therefore p_{ij}^{(2)} = P(X_2 = j | X_0 = i) = \sum_{k \in S} p_{ik} \cdot p_{kj}$$

Which means that in order to get to state j from i, we need to pass through some intermediate state k.

$$i \longrightarrow k \longrightarrow j$$

n-step Transition Probability Matrix

We can define the **two-step transition matrix** as

$$P^{(2)} = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} & \cdots & p_{1m}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} & \cdots & p_{2m}^{(2)} \\ p_{31}^{(2)} & p_{32}^{(2)} & \cdots & p_{3m}^{(2)} \\ \vdots & \ddots & & \\ p_{m1}^{(2)} & p_{m2}^{(2)} & \cdots & p_{mm}^{(2)} \end{bmatrix}$$

We conclude that two-step transition matrix can be obtained by squaring the state transition matrix.

$$P^{(2)} = P \cdot P = P^2$$

Similarly,

$$P^{(3)} = P \cdot P^2 = P \cdot P^{(2)}$$

Generally we can define the n-step transition probability $p_{ij}^{(n)}$ as

$$p_{ij}^{(n)} = P(X_n = j | X_0 = i), \qquad i = 0, 1, 2, ...$$

In order to get from state i to state j, we need to pass through $n-1$ intermediate states $k_1, k_2, ..., k_{n-1}$

$$i \longrightarrow k_1 \longrightarrow k_2 \longrightarrow ... \longrightarrow k_{n-1} \longrightarrow j$$

The n**-step transition matrix** is defined as follows

$$P^{(n)} = \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} & ... & p_{1r}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & ... & p_{2r}^{(n)} \\ p_{31}^{(n)} & p_{32}^{(n)} & ... & p_{3r}^{(n)} \\ \vdots & \ddots & & \\ p_{r1}^{(n)} & p_{r2}^{(n)} & ... & p_{rr}^{(n)} \end{bmatrix}$$

$$P^{(n)} = P^n$$

Let m and n be two positive integers and assume $X_0 = i$. In order to get to state j in $(m+n)$ steps, the chain will be at some intermediate state k after m steps.

To obtain

$$\begin{aligned} p_{ij}^{(m+n)} &= P\left[X_{n+m} = j | X_0 = i\right] \\ &= \sum_{k \in S} p_{ik}^{(n)} \cdot p_{kj}^{(m)} \end{aligned}$$

This equation is called the **Chapman–Kolmogorov Equation**

Probability distribution of $X_n, n \geq 0$

Consider a Markov Chain $\{X_n : n = 0, 1, 2, ...\}$. Suppose we know the probability distribution of X_0.

Define the row vector $\pi^{(0)}$ as

$$\pi^{(0)} = \begin{bmatrix} P(X_0 = 1) & P(X_0 = 2) & ... & P(X_0 = r) \end{bmatrix}$$

Now, we can obtain the probability distribution of $X_1, X_2, ...$

Using the law of total probability, for anu $j \in S$, we can write

$$\begin{aligned} P(X_1 = j) &= \sum_{k=1}^{r} P(X_1 = j | X_0 = k) \cdot P(X_0 = k) \\ &= \sum_{k=1}^{r} p_{kj} \cdot P(X_0 = k) \end{aligned}$$

$$\pi^{(n)} = \begin{bmatrix} P(X_n = 1) & P(X_n = 2) & ... & P(X_n = r) \end{bmatrix}$$

Given the state transition matrix P, we can rewrite the above results in the form of matrix multiplication

$$\begin{aligned} \pi^{(1)} &= \pi^{(0)} \cdot P \\ \pi^{(2)} &= \pi^{(1)} \cdot P \\ &\vdots \\ \pi^{(n)} &= \pi^{(n-1)} \cdot P \end{aligned}$$

or

$$\pi^{(n)} = \pi^{(0)} \cdot P^n$$

Stationary Distribution of a Markov Chain

Given a t.p.m P of a markov chain $\{X_n : n = 0, 1, 2, ...\}$, if there exists a probability vector $\hat{p}$ which satisfies

$$\hat{p} \cdot P = \hat{p}$$

..(1)

Then $\hat{p}$ is called a stationary distribution for the given markov chain.

The stationary distribution vector represents the distribution of all states over an infinitely long run.

Types of States

States can be categorized into the following types

1. Accessible States

A state j is said to be **accessible** from a state i if

$$p_{ij}^{(n)} > 0 \quad \text{for some } n$$

We denote it as $i \longrightarrow j$.

2. Communicative States

Two states i and j are said to be **communicative** if i is accessible from j and vice versa. We denote such states as $i \longleftrightarrow j$.

In other words,

$$i \longrightarrow j \;\; \& \;\; j \longrightarrow i \Rightarrow i \longleftrightarrow j$$

Irreducible Markov Chain

A Markov Chain is said to be irreducible if all states communicate with each other.

Properties of an irreducible Markov Chain

1. Every state communicates with each other.

$$i \longleftrightarrow j \quad \forall i, j$$

2. $i \longleftrightarrow j$ implies $j \longleftrightarrow i$.
3. $i \longleftrightarrow j$ and $j \longleftrightarrow k$ together implies $i \longleftrightarrow k$.

Regular Transition Matrix A transition matrix P is said to be **regular** if there is some n for which $P^{(n)}$ contains all positive non-zero elements.

- If the transition matrix is not irreducible then it is *not regular*.
- If the transition matrix is irreducible and at least one entry of the main diagonal is non-zero, then it is *regular*
- If all entries of the main diagonal are zero, but there exists some n for which $P^{(n)}$ contains all positive entries, then it is *regular*