2. Finite Difference Operators

Finite Diferences

Given the function $y = f(x)$, x is called called an **argument** and y is called an **entry**.

Here values of arguments are given at equal intervals
$a, a + h, a + 2h, ..., a + nh$

Corresponding values of y are:
$f(a), f(a + h), f(a + 2h), ..., f(a + nh)$

So we can write:
$f(a + h) - a, f(a + 2h) - f(a + h), ..., f(a + nh) - f(a + (n - 1)h)$

such a representation is called **finite differences**

Finite diference operators

There are five differnce operators, namely:

1. Forward Difference Operator Δ
2. Backward Difference Operator ∇
3. The Shifting Opetator E
4. Central Difference Operator δ
5. The Averaging Operator μ

Forward diference operator Δ

Consider the function $y = f(x)$

Given the values of the function at points
$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, ..., x_n = x_0 + nh$.

Let $y_0 = f(x_0), y_1 = f(x_1), ..., y_n = f(y_n)$.

We define

$$\Delta[f(x)] = f(x + h) - f(x)$$

Thus $\Delta y_0 = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0) = y_1 - y_0 = \Delta f(x_0)$.

$$\therefore \Delta y_0 = y_1 - y_0$$

Further, $x_0, x_1, ..., x_n$ are called arguments. The corresponding $f(x)$ values are called entries and h is the interval of differencing.

Similarly,

$$\begin{gathered} \Delta y_0 = y_1 - y_0 \\ . \\ . \\ . \\ \Delta y_n = y_{n+1} - y_n \end{gathered}$$

Δ is known as the **forward difference operator** and $\Delta y_0, \Delta y_1, ..., \Delta y_{n-1}$ are called the **first forward difference** of the function $y = f(x)$

The second order differences of the function are defined by

$$\begin{gathered} \Delta^2 y_0 = \Delta y_1 - \Delta y_0 \\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1 \\ . \\ . \\ . \\ \Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1} \end{gathered}$$

Similarly, in general the n^{th} order differences are defined as

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i$$

Forward difference table

The differences of the function can be systematically represented in the form of a table called the **forward difference table**. An example of such a table with 6 arguments is given below.

x_0	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y$
$x_1 = x_0 + h$	y_1	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y$
$x_2 = x_0 + 2h$	y_2	$\Delta y_2 = y_3 - y_2$	$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$	$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$	
$x_3 = x_0 + 3h$	y_3	$\Delta y_3 = y_4 - y_3$	$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$		

x_0	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_4 = x_0 + 4h$	y_4	$\Delta y_4 = y_5 - y_4$			
$x_5 = x_0 + 5h$	y_5				

Backward diference operator ∇

The shifting opetator E

Central diference operator δ

The averaging operator μ