3. Discrete Time Markov Chain

Discrete Time Markov Chain

Let $\{X_n, n \geq 0\}$ be a stochastic process taking values in a state space S that has N states. such a stochastic process is a Markov processes if it satisfies a following property :

$$P(X_{n+1} = k_{n+1}|X_n = k_n, X_{n-1} = k_{n-1}, ...X_1 = k_1) = P(X_{n+1} = k_{n+1}|X_n = k_n)$$

For a markov process, the *future state* only depends on the *present state* and not on the *past states*.

If the state space of a Markov process is discrete, it's called a **Markov Chain**.

To understand the behaviour of this process, we will need to calculate probabilities like,

$$P\left[X_0 = i_0, X_1 = i_1, ..., X_n = i_n\right]$$

..(1)

$\because P(A, B) = P(A) \cdot P(B|A)$, this can be computed by multiplying conditional probabilities as follows.

$$= P(X_0 = i_0) \cdot P(X_1 = i_1|X_0 = i_0) \cdot P(X_2 = i_2|X_1 = i_1, X_0 = i_0)...$$

$$P(X_n = i_n|X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, ..., X_0 = i_0)$$

..(2)

From the markovian property,

$$= P(X_0 = i_0) \cdot P(X_1 = i_1|X_0 = i_0) \cdot P(X_2 = i_2|X_1 = i_1)...P(X_n = i_n|X_{n-1} = i_{n-1})$$

..(3)

State Transition Probabilities

For a discrete time Markov Chain $\{X_n : n = 1, 2, ...\}$ with discrete state space $S = \{0, 1, 2, ...\}$ where this set may be finite or infinite, if $X_n = i$ then the Markov Chain is said to be in state i at time n(or the n^{th} step)

One Step Transition Probability

A discrete time Markov Chain $\{X_n : n = 1, 2, ...\}$ is characterized by

$$P\left[X_{n+1} = i_{n+1} | X_n = i_n, ..., X_0 = i_0\right] = P\left[X_{n+1} = i_{n+1} | X_n = i_n\right]$$

Where $P\left[X_{n+1} = j | X_n = i\right]$ is called one step transition probability

If $P\left[X_{n+1} = j | X_n = i\right]$ is independent of n then the Markov Chain is said to possess stationary transition probabilities and the process is reffered to as a homogeneous Markov Chain. Otherwise the process is called a non-homogeneous Markov Chain.

Transition Probability Matrix

The matrix called the **state transition matrix (t.p.m)** or **transition probability matrix** is usually denoted by P.

Let $\{X_n : n = 1, 2, ...\}$ be a homogenous Markov Chain with a discrete finite state space $S = \{0, 1, 2, ..., m\}$ then

$$p_{ij} = P\left[X_{n+1} = j | X_n = i\right] \quad i \geq 0, j \geq 0$$

regardless of the value of n.

A t.p.m of $\{X_n\}$ is defined by

$$P = \left[p_{ij}\right] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ p_{31} & p_{32} & \cdots & p_{3m} \\ \vdots & \ddots & & \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

Where

$$p_{ij} \geq 0$$

and

$$\sum_{j=1}^{m} p_{ij} = 1, \quad i = 1, 2, ..., m$$

State Transition Diagram

A Markov Chain is usually shown by a state transition diagram. Consider a Markov Chain with three possible states $S = \{1, 2, 3\}$ and the following transition probabilities

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

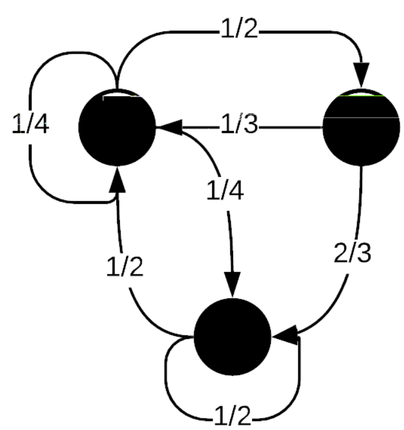
Which satisfies the two criterias, i.e.

$$p_{ij} \geq 0$$

and

$$\sum_{j=1}^{3} p_{ij} = 1, \quad i = 1, 2, 3$$

The figure below shows the state transition diagram for this Markov Chain



n-step Transition Probability Matrix

Consider a Markov Chain $\{X_n : n = 0, 1, 2, ...\}$ if $X_0 = i$ then $X_1 = j$ with probability p_{ij} is the probability of going from state i to state j in one step.

Now suppose we're interested in finding the probability of going from state i to state j in two steps, i.e.