3. Discrete Time Markov Chain

Discrete Time Markov Chain

Let $\{X_n, n \geq 0\}$ be a stochastic process taking values in a state space S that has N states. such a stochastic process is a Markov processes if it satisfies a following property :

$$P(X_{n+1} = k_{n+1}|X_n = k_n, X_{n-1} = k_{n-1}, ...X_1 = k_1) = P(X_{n+1} = k_{n+1}|X_n = k_n)$$

For a markov process, the *future state* only depends on the *present state* and not on the *past states*.

If the state space of a Markov process is discrete, it's called a **Markov Chain**.

To understand the behaviour of this process, we will need to calculate probabilities like,

$$P\left[X_0 = i_0, X_1 = i_1, ..., X_n = i_n\right]$$

..(1)

$\because P(A, B) = P(A) \cdot P(B|A)$, this can be computed by multiplying conditional probabilities as follows.

$$= P(X_0 = i_0) \cdot P(X_1 = i_1|X_0 = i_0) \cdot P(X_2 = i_2|X_1 = i_1, X_0 = i_0)...$$

$$P(X_n = i_n|X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, ..., X_0 = i_0)$$

..(2)

From the markovian property,

$$= P(X_0 = i_0) \cdot P(X_1 = i_1|X_0 = i_0) \cdot P(X_2 = i_2|X_1 = i_1)...P(X_n = i_n|X_{n-1} = i_{n-1})$$

..(3)

State Transition Probabilities

For a discrete time Markov Chain $\{X_n : n = 1, 2, ...\}$ with discrete state space $S = \{0, 1, 2, ...\}$ where this set may be finite or infinite, if $X_n = i$ then the Markov Chain is said to be in state i at time n(or the n^{th} step)

One Step Transition Probability

A discrete time Markov Chain $\{X_n : n = 1, 2, ...\}$ is characterized by

$$P\left[X_{n+1} = i_{n+1} | X_n = i_n, ..., X_0 = i_0\right] = P\left[X_{n+1} = i_{n+1} | X_n = i_n\right]$$

Where $P\left[X_{n+1} = j | X_n = i\right]$ is called one step transition probability

If $P\left[X_{n+1} = j | X_n = i\right]$ is independent of n then the Markov Chain is said to possess stationary transition probabilities and the process is reffered to as a homogeneous Markov Chain. Otherwise the process is called a non-homogeneous Markov Chain.

Transition Probability Matrix

The matrix called the **state transition matrix (t.p.m)** or **transition probability matrix** is usually denoted by P.

Let $\{X_n : n = 1, 2, ...\}$ be a homogenous Markov Chain with a discrete finite state space $S = \{0, 1, 2, ..., m\}$ then

$$p_{ij} = P\left[X_{n+1} = j | X_n = i\right] \quad i \geq 0, j \geq 0$$

regardless of the value of n.

A t.p.m of $\{X_n\}$ is defined by

$$P = \left[p_{ij}\right] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ p_{31} & p_{32} & \cdots & p_{3m} \\ \vdots & \ddots & & \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

Where

$$p_{ij} \geq 0$$

and

$$\sum_{j=1}^{m} p_{ij} = 1, \quad i = 1, 2, ..., m$$

State Transition Diagram

A Markov Chain is usually shown by a state transition diagram. Consider a Markov Chain with three possible states $S = \{1, 2, 3\}$ and the following transition probabilities

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

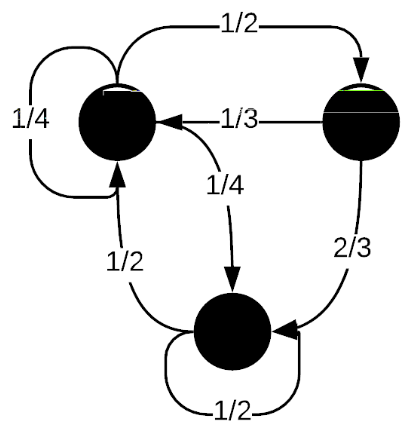
Which satisfies the two criterias, i.e.

$$p_{ij} \geq 0$$

and

$$\sum_{j=1}^{3} p_{ij} = 1, \quad i = 1, 2, 3$$

The figure below shows the state transition diagram for this Markov Chain



n-step Transition Probability

Consider a Markov Chain $\{X_n : n = 0, 1, 2, ...\}$ if $X_0 = i$ then $X_1 = j$ with probability p_{ij} is the probability of going from state i to state j in one step.

Now suppose we're interested in finding the probability of going from state i to state j in two steps, i.e.

$$p_{ij}^{(2)} = P(X_2 = j | X_0 = i)$$

$$p_{ij}^{(2)} = P(X_{n+2} = j | X_n = i)$$

We can find the probability by applying the law of total probability X_1 can take one of the possible values of S

$$p_{ij}^{(2)} = P(X_2 = j | X_0 = i)$$

$$= \sum_{k \in S} P(X_2 = j | X_1 = k, X_0 = i) \cdot P(X_1 = k | X_0 = i)$$

[by law of total probability]

$$= \sum_{k \in S} P(X_2 = j | X_1 = k) \cdot P(X_1 = k | X_0 = i)$$

[by markovian property]

$$= \sum_{k \in S} p_{kj} \cdot p_{ik}$$

$$= \sum_{k \in S} p_{ik} \cdot p_{kj}$$

$$\therefore p_{ij}^{(2)} = P(X_2 = j | X_0 = i) = \sum_{k \in S} p_{ik} \cdot p_{kj}$$

Which means that in order to get to state j from i, we need to pass through some intermediate state k.

$$i \longrightarrow k \longrightarrow j$$

n-step Transition Probability Matrix

We can define the **two-step transition matrix** as

$$P^{(2)} = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} & \cdots & p_{1m}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} & \cdots & p_{2m}^{(2)} \\ p_{31}^{(2)} & p_{32}^{(2)} & \cdots & p_{3m}^{(2)} \\ \vdots & \ddots & & \\ p_{m1}^{(2)} & p_{m2}^{(2)} & \cdots & p_{mm}^{(2)} \end{bmatrix}$$

We conclude that two-step transition matrix can be obtained by squaring the state transition matrix.

$$P^{(2)} = P \cdot P = P^2$$

Similarly,

$$P^{(3)} = P \cdot P^2 = P \cdot P^{(2)}$$

Generally we can define the n-step transition probability $p_{ij}^{(n)}$ as

$$p_{ij}^{(n)} = P(X_n = j | X_0 = i), \qquad i = 0, 1, 2, ...$$

In order to get from state i to state j, we need to pass through $n-1$ intermediate states $k_1, k_2, ..., k_{n-1}$

$$i \longrightarrow k_1 \longrightarrow k_2 \longrightarrow ... \longrightarrow k_{n-1} \longrightarrow j$$

The n**-step transition matrix** is defined as follows

$$P^{(n)} = \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} & ... & p_{1r}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & ... & p_{2r}^{(n)} \\ p_{31}^{(n)} & p_{32}^{(n)} & ... & p_{3r}^{(n)} \\ \vdots & \ddots & & \\ p_{r1}^{(n)} & p_{r2}^{(n)} & ... & p_{rr}^{(n)} \end{bmatrix}$$

$$P^{(n)} = P^n$$

Let m and n be two positive integers and assume $X_0 = i$. In order to get to state j in $(m+n)$ steps, the chain will be at some intermediate state k after m steps.

To obtain

$$\begin{aligned} p_{ij}^{(m+n)} &= P\left[X_{n+m} = j | X_0 = i\right] \\ &= \sum_{k \in S} p_{ik}^{(n)} \cdot p_{kj}^{(m)} \end{aligned}$$

This equation is called the **Chapman–Kolmogorov Equation**

Probability distribution of $X_n, n \geq 0$

Consider a Markov Chain $\{X_n : n = 0, 1, 2, ...\}$. Suppose we know the probability distribution of X_0.

Define the row vector $\pi^{(0)}$ as

$$\pi^{(0)} = \begin{bmatrix} P(X_0 = 1) & P(X_0 = 2) & ... & P(X_0 = r) \end{bmatrix}$$

Now, we can obtain the probability distribution of $X_1, X_2, ...$

Using the law of total probability, for anu $j \in S$, we can write

$$\begin{aligned} P(X_1 = j) &= \sum_{k=1}^{r} P(X_1 = j | X_0 = k) \cdot P(X_0 = k) \\ &= \sum_{k=1}^{r} p_{kj} \cdot P(X_0 = k) \end{aligned}$$

$$\pi^{(n)} = \begin{bmatrix} P(X_n = 1) & P(X_n = 2) & ... & P(X_n = r) \end{bmatrix}$$

Given the state transition matrix P, we can rewrite the above results in the form of matrix multiplication

$$\begin{aligned} \pi^{(1)} &= \pi^{(0)} \cdot P \\ \pi^{(2)} &= \pi^{(1)} \cdot P \\ &\vdots \\ \pi^{(n)} &= \pi^{(n-1)} \cdot P \end{aligned}$$

or

$$\pi^{(n)} = \pi^{(0)} \cdot P^n$$