# 2. Measurement of Trend

## Measurement of Trend

To measure trend we need to eliminate the other elements, mainly cyclical and seasonal since irregularity is hard to remove.

### Methods of Measuring trend

1. Method of free-hand curve fitting (pretty much useless)
2. Method of moving averages
3. Method of mathametical curves

   **Note**
   if your data has cyclicality and proper seasonality, method of moving average is the best. If your data doesn't have cyclicality then method of mathametical curves is preferred.

   **Note**
   Method of mathametical curves is not focused on eliminating cyclicicality or seasonality but is the only method that is used for forecasting

### Method of Moving Averages

In this method, a series of arethmatic means, each of $k$ successive ovservations of the given data is computed and these means are referred to as the moving averages of period $k$. To begin with, we take the first $k$. for the next item, we exclude the $1^{\text{st}}$ value and include the $(k+1)^{\text{th}}$ value and so on. We repeat this process until we reach the last set of $k$ values. Each mean is placed agaisnt the mid point of the interval it covers. If $k$ is odd, the moving averages correspond to the tabulated times for which the time series is given. On the other hand, when $k$ is even, each moving average falls midway between two tabulated time values, hence a subsequent two item moving average is computed to make the resultant moving average values correspond to the given times (this is called centering). The moving averages are the trend values for the corresponding items

## Method of Mathametical Curves

1. Straight Line Trend (Linear Trend)

$$T_t = a + b \cdot t, \quad b \neq 0$$

2. Second Degree Polynomial (Parabolic Trend)

$$T_t = a + b \cdot t + c \cdot t^2, \quad c \neq 0$$

3. Exponential Curve

$$T_t = a \cdot b^t, \quad a > 0$$

4. Modified Exponential Curve

$$T_t = k + a \cdot b^t, \quad k > 0$$

5. Gompertz

$$T_t = k \cdot a^{b^t}, \quad k > 0$$

**Linear Trend**

$Y_t$ = observed value of the time series at time $t$
$T_t = a + b \cdot t$ (Estimated from the graphical representation)

**Method of Least Squares**

$$S = \sum_t (Y_t - T_t)^2$$

[ To be minimised w.r.t. $a$ and $b$ ]

$$S = \sum_t (Y_t - a - b \cdot t)^2$$

$$\therefore \frac{\delta S}{\delta a} = 0$$

$$\Rightarrow -2 \sum_t (Y_t - a - b \cdot t) = 0$$

$$\Rightarrow \sum_t Y_t = n \cdot a + b \sum_t t$$

..(1)

$$\therefore \frac{\delta S}{\delta b} = 0$$

$$\Rightarrow -2\sum_t (Y_t - a - b\cdot t)\cdot t = 0$$

$$\Rightarrow \sum_t t\cdot Y_t = a\sum_t t + b\sum_t t^2$$

..(2)

eq 1 anbd 2 are called the **normal equations**

**Second Degree Polynomial (Parabolic Trend)**

$$T_t = a + b\cdot t + c\cdot t^2$$

**Fitting a Second Degree Polynomial**

$$S = \sum_t (Y_t - a - b\cdot t - c\cdot t^2)^2$$

[ To be minimised w.r.t. $a, b$ and $c$ ]

**Exponential Curve**

Here the trend eq us given by

$$Y_t = a\cdot b^t, \quad a, b > 0$$

$\therefore \log Y_t = \log a + t\log b$
i.e. $\log Y_t$ is a linear function of $t$

Note that $\frac{Y_t}{Y_{t-1}} = b$, i.e. the **exponential curve** indicates a constant ratio of change

> **Note :** if $0 < b < 1$, the value of $Y_t$ gradually decays but if $b > 1$, the value of $Y_t$ gradually increases

**Fitting an exponential curve** Let the trend eq be

$$T_t = a\cdot b^t$$
$$\Rightarrow \log T_t = \log a + t\log b$$
$$Y_t' = A + Bt$$

where

$$[Y_t' = \log T_t, A = \log a, B = \log b]$$

The normal equations are,

$$\Rightarrow \sum_t Y_t' = n \cdot A + B \sum_t t$$

..(1)

$$\Rightarrow \sum_t t \cdot Y_t = A \sum_t t + B \sum_t t^2$$

..(2)

Let $\hat{A}$ and $\hat{B}$ be the estimates from (1) and (2),

$$\therefore \ \hat{A} = \log \hat{a}$$
$$\Rightarrow \hat{a} = e^{\hat{A}}$$

and

$$\therefore \ \hat{B} = \log \hat{b}$$
$$\Rightarrow \hat{b} = e^{\hat{B}}$$

Hence the fitted trend equation is given by,

$$T_t = \hat{a} \cdot \hat{b}^t$$

**Modified Exponential Curve**

The **modified exponential curve** indicates an increasing or decreasing ratio of change. The curve is given as

$$Y_t = k + ab^t$$

Note that $\Delta Y_t = Y_{t+1} - Y_t = a(b^{t+1} - b^t)$

$$\therefore \frac{\Delta Y_t}{\Delta Y_{t-1}} = \frac{a(b^{t+1} - b^t)}{a(b^t - b^{t-1})} = \frac{b^t(b-1)}{b^{t-1}(b-1)} = b$$

Hence $\frac{\Delta Y_t}{\Delta Y_{t-1}}$ is constant for a modified exponential curve.

**Fitting of Modified Exponential Curve** The curve has 3 constants, $k, a$ and $b$. Thus we need 3 quations for fitting the curve. To do this, we use the method of partial sums. First, we divide the observed series $Y_t$ into 3 equal sections. Let the number of observations in each section be $n$; i.e. the first section goes from 1 to $n$, te second from $(n+1)$ to $2n$ and the last section goes from $(2n+1)$ to $3n$. The subtotals for each section are denoted as $S_1, S_2$ & $S_3$ respectively.

$$\begin{aligned}
\therefore \quad S_1 &= \sum_{t=1}^{n} k + a \cdot b^t \\
&= nk + a \sum b^t \\
&= nk + a \left[ b^1 + b^2 + ... b^n \right] \\
&= nk + ab \left[ 1 + b^1 + ... b^{n-1} \right] \\
&= nk + ab \tfrac{(b^n - 1)}{b-1}
\end{aligned}$$

..(1)

Similarly,

$$\begin{aligned}
S_2 &= \sum_{t=n+1}^{2} nk + a \cdot b^t \\
&= nk + a \sum b^t \\
&= nk + a \left[ b^{n+1} + b^{n+2} + ... b^{2n} \right] \\
&= nk + ab^{n+1} \left[ 1 + b^1 + ... b^{n-1} \right] \\
&= nk + ab^{n+1} \tfrac{(b^n - 1)}{b-1}
\end{aligned}$$

..(2)

$$
\begin{aligned}
S_3 \quad &= \sum_{t=2n+1}^{3} nk + a \cdot b^t \\
&= nk + a \sum b^t \\
&= nk + a\left[b^{2n+1} + b^{2n+2} + ...b^{3n}\right] \\
&= nk + ab^{2n+1}\left[1 + b^1 + ...b^{n-1}\right] \\
&= nk + ab^{2n+1}\tfrac{(b^n-1)}{b-1}
\end{aligned}
$$

..(3)

$$
\begin{aligned}
\therefore \quad S_2 - S_1 \quad &= nk + ab^{n+1}\left[\tfrac{b^n-1}{b-1}\right] - nk - ab\left[\tfrac{b^n-1}{b-1}\right] \\
&= ab^{n+1}\left[\tfrac{b^n-1}{b-1}\right] - ab\left[\tfrac{b^n-1}{b-1}\right] \\
&= \left[ab^{n+1} - ab\right]\left[\tfrac{b^n-1}{b-1}\right] \\
&= \left[ab \cdot (b^n - 1)\right]\left[\tfrac{b^n-1}{b-1}\right] \\
&= ab\tfrac{(b^n-1)^2}{b-1}
\end{aligned}
$$

..(4)

$$
\begin{aligned}
\therefore \quad S_3 - S_2 \quad &= nk + ab^{2n+1}\left[\tfrac{b^n-1}{b-1}\right] - nk - ab^{n+1}\left[\tfrac{b^n-1}{b-1}\right] \\
&= ab^{2n+1}\left[\tfrac{b^n-1}{b-1}\right] - ab^{n+1}\left[\tfrac{b^n-1}{b-1}\right] \\
&= \left[ab^{2n+1} - abn + 1\right]\left[\tfrac{b^n-1}{b-1}\right] \\
&= \left[ab^{n+1} \cdot (b^n - 1)\right]\left[\tfrac{b^n-1}{b-1}\right] \\
&= ab^{n+1}\tfrac{(b^n-1)^2}{b-1}
\end{aligned}
$$

..(5)

$$
\frac{S_3 - S_2}{S_2 - S_1} = b^n
$$

[From here we get the value of $b$ as $\hat{b}$]

$$\therefore \hat{b} = \left( \frac{S_3 - S_2}{S_2 - S_1} \right)^{1/n}$$

Recall eq (4) which was

$$S_2 - S_1 = ab \frac{(b^n - 1)^2}{b - 1}$$

Substuting $\hat{b}$ into eq (4), we get the value of $a$ as $\hat{a}$.

Recall eq (1) which was

$$S_1 = nk + ab \frac{(b^n - 1)}{b - 1}$$

Substuting $\hat{b}$ & $\hat{a}$ into eq (1), we get the value of $k$ as $\hat{k}$.

> **Note**
> If the number of years is not a multiple of 3, slightly overlapping intervals have to be taken for the sections. For example if there are 10 years, first 4 observations would fall under group 1, (4 to 7)th observation would fall under group 2 and (7 to 10)th observation would fall under group 3. Here the 4th and the 7th observations are getting repeated