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Waiting Line Theory on QUEUING MODEL [C23]

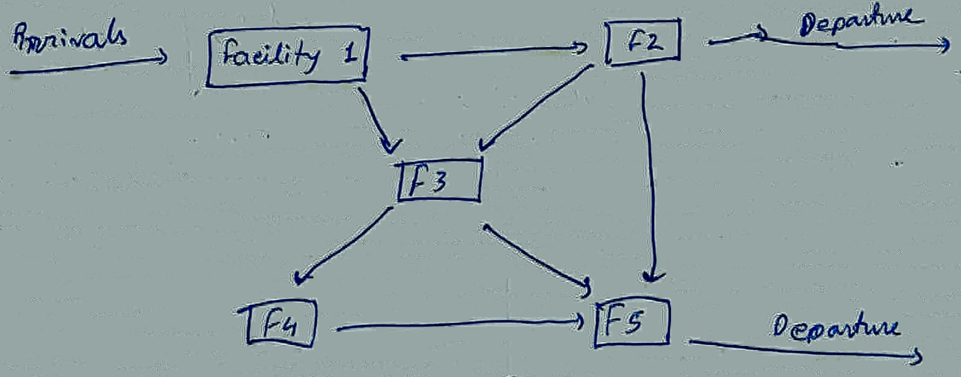
① (M/M/1) : (∞/FIFO)

arrival pattern	Service pattern	Service Channel	Infinite No. Customers	Service Discipline

~~Ftom of~~

- Lack of capability at a time forms a queue.
- Arriving unit that requires some services is customer.
- Queue stands for no of customers to be serviced.

A flow of customers from finite or infinite population towards the service facility forms a queue (waiting line) on account of a lack of capability to serve them all at a time. In the absence of perfect balance between the service facilities and the customers waiting time is required either for the service facilities or for the customers' arrival.



Nature of customers

Service pattern

Service discipline

Queuing system or process

i) The input (arrival pattern)

ii) The service mechanism or service pattern.

iii) The ~~queq~~ queue discipline.

iv) Customer behaviour.

Arrivals → Customer waiting in line → Unit is being served → Unit leaving the queue after service

Arrival process length of the queue waiting time in queue

Unit being served

Output or element leaving the queue

System length waiting time in system.

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[02 4]

Input Process

- Arrivals
  - Nature of the arrivals
    - Size of arrivals
      - Finite popn
      - Infinite popn
    - Arrival time distribution
      - Constant
      - Random
  - Capacity of the system
    - Limited
    - Unlimited
  - Customer Behaviour
    - Balking
    - Collusion
    - Reneging
    - Jockeying

Size of arrivals: The size of arrivals to the service system is greatly

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(25)

NOTATIONS

X : Inter arrival time between two successive customer

Y : The service time required by any customer.

W : The waiting time for any customer before it is taken into service.

V : Time spent by the customer in the system.

n : The number of customer in the system. that is customer in the waiting line at any time including the number of customer being served.

$P_n(t)$: Probability that n customers arrive in the system in time t.

$\phi_n(t)$: Probability that n customers are served in time t.

$u(t)$: Probability distribution of inter arrival time. $P(t \le T)$

$v(t)$: Probability distribution of service time. $P(t \le T)$

$F(n)$: Probability distribution of queue length at any time t.

λ_n : Average number of customers arriving per unit of time when there are already n customer in the system

λ : Average number of customer arriving per unit of time.

μ_n : Average number of customer being served per unit of time when there are already n customer in the system.

μ : Average number of customer being served per unit of time

$1/\lambda$: Inter arrival time between two arrivals.

$1/\mu$: Service time [illegible] customer.

$\rho = \lambda/\mu$: system utility or traffic intensity

Eg: Given arrival time is 8 hours and $\rho = 3/8$ [illegible]

that that out of 8 hours the system is busy for 3 hours

& 5 hours the system is idle.

case I: $(M/M/1):(\infty/FIFO)$

(Poisson arrival / poisson service / No of channel / infinity capacity / FIFO Model)

(system rule)

$\lambda, \mu, \rho = \lambda/\mu \quad (\lambda < \mu)$

i) Probability that the system is empty. $= P_0 = (1-\rho)$

ii) Probability that there are n customer in the system.

$P_n = \rho^n \cdot P_0$

iii) Average number of customer in the system.

$$E(n) = \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

iv) Average number of customer in the waiting line.

$$\frac{\rho^2}{1-\rho} = \frac{\lambda^2/\mu^2}{1-\lambda/\mu} = \frac{\lambda^2}{(\mu-\lambda)\mu}$$

v) Average waiting length (mean time in the system)

$$\frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}$$

vi) Average length of waiting time with the condition that it is always greater than zero.

$$W(w) = \frac{\rho}{(1-\rho)^2} = \frac{\lambda/\mu}{\frac{(\mu-\lambda)^2}{\mu^2}} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

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vii) Average time an arrival spends in the system:

$$E(v) = \frac{1}{\mu-\lambda} = E(w/w>0) = \frac{1}{\mu(1-\rho)}$$

viii) System is busy $= P(w>0) = \rho$

ix) Idle time $= 1-\rho$

x) Prob that a customer has to wait on arrival:

$P(w>0) = \rho$

Eg: A TV mechanic finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they come in. If the arrival of sets is approximately poisson with an average rate of 10 per eight hour day. What is the mechanics expected idle time each day? How many jobs are ahead of the average set just brought in?

$$\mu = \frac{1}{30} \qquad \lambda = \frac{10}{8 \times 60} = \frac{1}{48}$$

$$\text{Expected no of jobs} = \frac{\rho}{1-\rho}$$

$$= \frac{\lambda}{\mu - \lambda}$$

$$= \frac{1/48}{1/30 - 1/48} = \frac{1}{\frac{48-30}{30}}$$

$$= \frac{30}{18} = \frac{5}{3} = 1\frac{2}{3} \text{ jobs}$$

~~Expected idle time~~

Since the faction of the time the mechanic is busy equal to λ/μ, the no of hours for which the repairmen remains busy in an eight hour day.

$$8\frac{\lambda}{\mu} = 8 \cdot \frac{30}{48} = \frac{30}{6} = 5 \text{ hours}$$

∴ The time for which the mechanic remains idle in an 8 hours day = 8 − 5 = 3 hours

Eg 2: Arrivals at a telephone booth are considered to be poisson with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 3 minutes.

i) What is the prob that the person arriving at the booth will have to wait?

ii) What is the average length of the queue that forms from time to time?

iii) The telephone dept will install a second booth when convinced that arrival will have to wait atleast 3 minutes for the phone. By how much time must the flow of arrivals be increased in order to justify a second booth.

$\lambda = \frac{1}{10}$ $\qquad \mu = \frac{1}{3}$

i) $P(w>0) = \rho = \frac{\lambda}{\mu} = \frac{1/10}{1/3} = \frac{3}{10} = 0.3$

ii) $L(L>0) = \frac{\mu}{\mu-\lambda} = \frac{1/3}{1/3 - 1/10} = \frac{1}{7/10} = \frac{10}{7} = 1\frac{3}{7}$

$= 1.43$ (approx)

iii) $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$ $\qquad W_q = 3$ $\qquad \mu = \frac{1}{3}$ $\qquad \lambda = \lambda'$ for second booth

$\Rightarrow 3 = \frac{\lambda'}{\frac{1}{3}\left(\frac{1}{3} - \lambda'\right)}$

$\Rightarrow \lambda' = \frac{1}{3} - \lambda'$

$\Rightarrow \lambda' = \frac{1}{6} = 0.16$

Hence, increase in the arrival length $= 0.16 - 0.1 = 0.06$ arrival per min.

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Eg 3| Customer arrive at a one window drive in bank according to Poisson dist with mean 10 per hour. Service time per customer is exponential with mean five minutes. The space in front of the window including that for the service car can be accommodate a maximum of three cars. Others can wait outside the space.

(i) What is the prob that an arriving customer can drive directly to the space in front of the window?

(ii) What is the probability that an arriving customer will have to wait outside the indicated space?

(iii) How long is an arriving customer expected to wait before starting service?

$\lambda = \frac{10}{60} = \frac{1}{6}$ $\quad \mu = \frac{1}{5}$

$\rho = \frac{1}{2}$

$\rho = \frac{1/6}{1/5} = \frac{5}{6}$

(i) $P_0 + P_1 + P_2$

$\frac{\lambda}{\mu(\mu - \lambda)}$

$= \frac{1/6}{1/5\,(1/5 - 1/6)}$

$= \frac{5/6}{\frac{6-5}{30}} = \frac{5}{6} \cdot \frac{30}{1}$

$$P_0 = 1 - \rho = 1 - \frac{5}{6} = 1/6$$

$$\rho = \frac{\lambda}{\mu} = \frac{1/6}{1/5} = \frac{5}{6}$$

(i) $P_0 + P_1 + P_2$

$\cancel{= 1 - \rho +}$

$$= \frac{1}{6} + \rho \cdot P_0 + \rho^2 P_0$$

$$= \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}$$

$$= \frac{36 + 30 + 25}{216} = 0.42$$

(ii) $1 - 0.42 = 0.58$

(iii) $$\frac{1}{\mu - \lambda} = \frac{1}{1/5 - \frac{1}{6}} = 30$$

$$\frac{\lambda}{\mu}$$

Example 18.1 A T.V. mechanic finds that the time spent on his jobs has an exponential distribution with mean 30 minutes, if he repairs sets in the order in which they come in. If the arrival of sets is approximately Poisson with an average rate of 10 per eight-hour day, what is the mechanic's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution

Here, $\mu = 1/30, \quad \lambda = \dfrac{10}{8 \times 60} = \dfrac{1}{48}$

Expected number of jobs are,

$$L_S = \frac{\dfrac{\lambda}{\mu}}{1 - \dfrac{\lambda}{\mu}} = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{\dfrac{1}{48}}{1/30 - 1/48} = 1\frac{2}{3} \text{ jobs.}$$

Since the fraction of the time the machanic is busy equals to $\dfrac{\lambda}{\mu}$, the number of hours for which the repairman remains busy in an eight-hour day,

$$= 8\left(\frac{\lambda}{\mu}\right) = 8 \times \frac{30}{48} = 5 \text{ hours.}$$

Therefore, the time for which the machanic remains idle in an eight-hour day = (8 − 5) hours = 3 hours.

Example 18.2 At what average rate must a clerk at a supermarket work, in order to insure a probability of 0.90 that the customers will not have to wait longer than 12 minutes? It is assumed that there is only one counter, to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

Solution

Here, $\lambda = \dfrac{15}{60} = \dfrac{1}{4}$ customer/minute, $\mu = ?$

Prob. (waiting time ≥ 12) $= 1 - 0.9 = 0.10$

$$\int_t^{\infty} \lambda(1-\lambda)e^{-(\mu-\lambda)\omega}\, d\omega$$

$$\therefore \quad \int_{12}^{\infty} \lambda\left(1-\frac{\mu}{\lambda}\right) e^{-(\mu-\lambda)\omega}\, d\omega = 0.1$$

$$\lambda\left(1-\frac{\mu}{\lambda}\right)\left(\frac{e^{-(\mu-\lambda)\omega}}{-(\mu-\lambda)}\right)_{12}^{\infty} = 0.1$$

$$\frac{\lambda}{\mu}(e^{-12(\mu-\lambda}) = 0.10$$

$$e^{(3-12\mu)} = 0.4\mu$$

$$\frac{1}{\mu} = 2.48 \text{ minutes per service.}$$

Example 18.3 Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean three minutes.

(*i*) What is the probability that a person arriving at the booth will have to wait?
(*ii*) What is the average length of the queue that forms from time to time?
(*iii*) The telephone department will install a second booth when convinced that an arrival would have to wait at least three minutes for the phone. By how much time must the flow of arrivals be increased in order to justify a second booth?

Solution

Given, $\lambda = 1/10$, $\mu = 1/3$

(*i*) Probability $(w > 0) = 1 - P_0 = \frac{\lambda}{\mu} = \frac{1}{10} \times \frac{3}{1} = 3/10 = 0.3$

(*ii*) $(L/L > 0) = \frac{\mu}{\mu - \lambda} = \frac{1/3}{1/3 - 1/10} = 1.43$ persons

(*iii*) $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

Since, $W_q = 3$, $\mu = \frac{1}{3}$, $\lambda = \lambda'$ for second booth,

$$3 = \frac{\lambda'}{\frac{1}{3}\left(\frac{1}{3} - \lambda'\right)} \Rightarrow \lambda' = 0.16$$

Hence, increase in the arrival rate $= 0.16 - 0.10 = 0.06$ arrival per minute.

Example 18.4 As in example 18.3, in a telephone booth with Poisson arrivals spaced 10 minutes apart on the average and exponential call length averaging three minutes.

(*i*) What is the probability that an arrival will have to wait for more than 10 minutes before the phone becomes free?
(*ii*) What is the probability that it will take him more than 10 minutes in total to wait for the phone and complete his call?
(*iii*) Estimate the fraction of a day that the phone will be in use.
(*iv*) Find the average number of units in the system.

Solution Given,

$$n\lambda = 0.1 \text{ arrival/minute}$$
$$\mu = 0.33 \text{ service/minute}$$

(*i*) Probability (waiting time ≥ 10) $= \int_{10}^{\infty}\left(1-\frac{\lambda}{\mu}\right)\lambda e^{-(\mu-\lambda)W}\,dW$

$$= -\frac{\lambda}{\mu}\left(e^{-(\mu-\lambda)\omega}\right)_{10}^{\infty}$$
$$= 0.3\,e^{-2.3} = 0.03$$

(*ii*) Probability (waiting time in the system ≥ 10)

$$= \int_{10}^{\infty}(\mu-\lambda)e^{-(\mu-\lambda)W}\,dW$$
$$= e^{-10(\mu-\lambda)} = e^{-2.3} = 10$$

(*iii*) The fraction of a day that the phone will be busy = traffic intensity

$$\rho = \frac{\lambda}{\mu} = 0.3.$$

(*iv*) Average number of units in the system,

$$L_S = \frac{\lambda}{\mu-\lambda} = \frac{\frac{1}{10}}{\frac{1}{3}-\frac{1}{10}} = 3/7 = 0.43 \text{ customer.}$$

Example 18.5 Customers arrive at a one-window drive-in bank according to Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean five minutes. The space in front of the window including that for the serviced car can accommodate a maximum of three cars. Others can wait outside this space.

(*i*) What is the probability that an arriving customer can drive directly to the space in front of the window?

(*ii*) What is the probability that an arriving customer will have to wait outside the indicated space?

(*iii*) How long is an arriving customer expected to wait before starting service?

Solution Given,

$$\lambda = 10 \text{ per hour}$$
$$\mu = \frac{1}{5}\times 60 = 12 \text{ per hour}$$
$$\rho = \frac{\lambda}{\mu} = \frac{10}{12}$$

(*i*) The probability that an arriving customer can drive directly to the space in front of the window,

$$P_0 + P_1 + P_2 = P_0 + \frac{\lambda}{\mu}P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0$$
$$= P_0\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2\right) \quad \because \quad P_0 = 1-\frac{\lambda}{\mu}$$

$$= \left(1-\frac{10}{12}\right)\left(1+\frac{10}{12}+\frac{100}{144}\right) = 0.42$$

(ii) Probability that an arriving customer will have to wait outside the indicated space,

$$S = 1-0.42 = 0.58$$

(iii) Average waiting time of a customer in a queue,

$$= \frac{\lambda}{\mu}\frac{1}{\mu-\lambda} = \frac{10}{12}\left(\frac{1}{12-10}\right) = \frac{5}{12}$$

$$= 0.417 \text{ hours.}$$

Example 18.6 In a supermarket, the average arrival rate of customers is 10 every 30 minutes, following Poisson process. The average time taken by a cashier to list and calculate the customer's purchase is two and a half minutes following exponential distribution. What is the probability that the queue length exceeds six? What is the expected time spent by a customer in the system?

Solution

$$\lambda = \frac{10}{30} \text{ per minute}$$

$$\mu = \frac{1}{2.5} \text{ per minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{3}}{1/2.5} = 0.8333$$

(i) The probability of queue size $> 6 = \rho^6$

Expected waiting time $W_S = \dfrac{1}{\mu-\lambda} = (0.8333)^6 = 0.3348.$

(ii) $$W_S = \frac{L_S}{\lambda} = \frac{\frac{\rho}{1-\rho}}{\lambda} = \frac{0.833}{1-0.8333}\times 3$$

$$= 14.96 \text{ minutes.}$$

Example 18.7 On an average, 96 patients per 24-hour day require the service of an emergency clinic. Also, on an average a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic ₹ 100 per patient treated, to obtain an average servicing time of 10 minutes and thus, each minute of decrease in this average time would cost ₹ 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ patients?

Solution

Given, $$\lambda = \frac{96}{24\times 60} = \frac{1}{15} \text{ patient/minute}$$

$$\mu = \frac{1}{10} \text{ patient/minute}$$

Average number of patients in the queue,

$$L_q = \frac{\lambda}{\mu} - \frac{\lambda}{\lambda - \mu} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{1 - \frac{\lambda}{\mu}}$$

$$= \frac{\left(\frac{1}{15}\right)^2}{\left(\frac{1}{10} - \frac{1}{15}\right)\frac{1}{10}} = 1\frac{1}{3} \text{ patients}$$

But, $\quad L_q = 1\frac{1}{3}$ is reduced to $L_q' = 1/2$

$\therefore$ Substituting $L_q' = 1/2, \lambda' = \lambda = \frac{1}{15}$ in the formula

$$L_q' = \frac{\lambda^{12}}{\mu'(\mu' - \lambda')}$$

$$\frac{1}{2} = \frac{\left(\frac{1}{15}\right)^2}{\mu'(\mu' - 1/15)} \Rightarrow \mu' = 2/15 \text{ patients/minute}$$

Hence, the average rate of treatment required is, $\frac{1}{\mu'} = 7.5$ minutes. Decrease in time required by each patient

$$= 10 - \frac{15}{2} = \frac{5}{2} \text{ minutes}$$

$\therefore$ The budget required for each patient

$$= 100 + \frac{5}{2} \times 10 = ₹\,125$$

So, in order to get the required size of the queue, the budget should be increased from ₹ 100 to ₹ 125 per patient.

Example 18.8 In a public telephone booth, the arrivals on an average are 15 per hour. A call on an average takes three minutes. If there is just one phone, find (*i*) the expected number of callers in the booth at any time (*ii*) the proportion of the time, the booth is expected to be idle?

Solution

Given, $\quad \lambda = 15$ per hour

$$\mu = \frac{1}{3} \times 60 \text{ per hour}$$

(*i*) Expected length of the non-empty queue

$$= \frac{\mu}{\mu - \lambda} = \frac{20}{20 - 15} = 4$$

(ii) The service is busy $= \dfrac{\lambda}{\mu} = \dfrac{15}{20} = \dfrac{3}{4}$

$\therefore$ the booth is expected to be idle for $1 - \dfrac{3}{4} = \dfrac{1}{4}$ hours = 15 minutes.

Example 18.9 In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that inter-arrival time and service time distribution follows an exponential distribution with an average of 30 minutes, calculate the following:

(i) The mean queue size.
(ii) The probability that queue size exceeds 10.
(iii) If the input of the train increases to an average of 33 per day, what will be the changes in (i) and (ii)?

Solution:

Given,
$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ trains/minute}$$

$$\mu = \frac{1}{30} \text{ trains/minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{30}{48} = \frac{5}{8}$$