CALIFORNIA STATE UNIVERSITY SAN MARCOS
DR. DE LEONE, PHYSICS 323

H.W. 5

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PROBLEM 1.13

Consider a quantum system with an observable A that has three possible measurement results: a1, a2, and a3.
a) Write down the three kets $|a_1\rangle, |a_2\rangle, |a_3\rangle$, corresponding to these possible results using matrix notation.

We can write the state vectors in matrix notation by choosing orthogonal directions for each observable, giving them a magnitude of one and writing them in column form.

$$|a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |a_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |a_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

b) The system is prepared in the state

$$|\psi\rangle = 1|a_1\rangle - 2|a_2\rangle + 5|a_3\rangle$$

Write this state in matrix notation and calculate the probabilities of all possible measurement results of the observable A. Plot a histogram of the predicted measurement results.

The magnitude of each observable is multiplied by its unit vector.

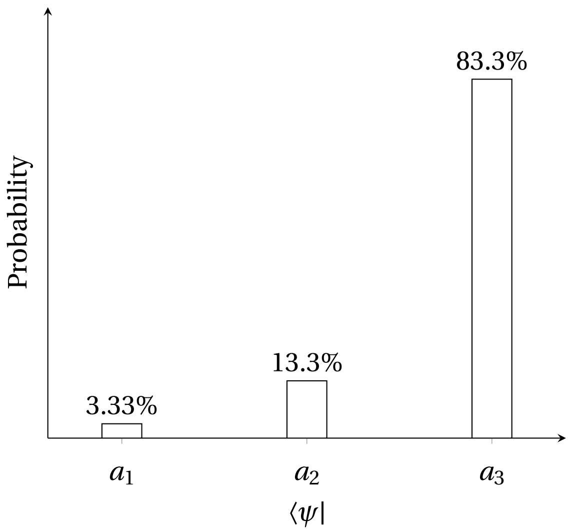
$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

We need to normalize the wave function by dividing the ket by its magnitude,

$$
\begin{aligned}
|\psi\rangle &= C\begin{pmatrix}1\\-2\\5\end{pmatrix}\\
\langle\psi|\psi\rangle &= C^2\begin{pmatrix}1 & -2 & 5\end{pmatrix}\begin{pmatrix}1\\-2\\5\end{pmatrix}\\
&= C^2(1+4+25)\\
C &= \frac{1}{\sqrt{30}}\\
|\psi\rangle &= \frac{1}{\sqrt{30}}\begin{pmatrix}1\\-2\\5\end{pmatrix}
\end{aligned}
$$

We can find the probabilities of each result by multiplying possible bra measurement direction with the normalized ket,

$$
\begin{aligned}
\mathcal{P}_{a_1} &= \left|\langle a_1|\psi\rangle\right|^2 & \mathcal{P}_{a_2} &= \left|\langle a_2|\psi\rangle\right|^2 & \mathcal{P}_{a_3} &= \left|\langle a_3|\psi\rangle\right|^2\\
&= \left|\begin{pmatrix}1 & 0 & 0\end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix}1\\-2\\5\end{pmatrix}\right|^2 & &= \left|\begin{pmatrix}0 & 1 & 0\end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix}1\\-2\\5\end{pmatrix}\right|^2 & &= \left|\begin{pmatrix}0 & 0 & 1\end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix}1\\-2\\5\end{pmatrix}\right|^2\\
&= \left|\frac{1}{\sqrt{30}}\right|^2 & &= \left|\frac{-2}{\sqrt{30}}\right|^2 & &= \left|\frac{5}{\sqrt{30}}\right|^2\\
\mathcal{P}_{a_1} &= \frac{1}{30} & \mathcal{P}_{a_2} &= \frac{4}{30} & \mathcal{P}_{a_3} &= \frac{25}{30} = \frac{5}{6}
\end{aligned}
$$



c) In a different experiment, the system is prepared in the state

$$|\psi\rangle = 2\,|a_1\rangle + 3i\,|a_2\rangle$$

Write this state in matrix notation and calculate the probabilities of all possible measurement results of the observable A. Plot a histogram of the predicted measurement results.

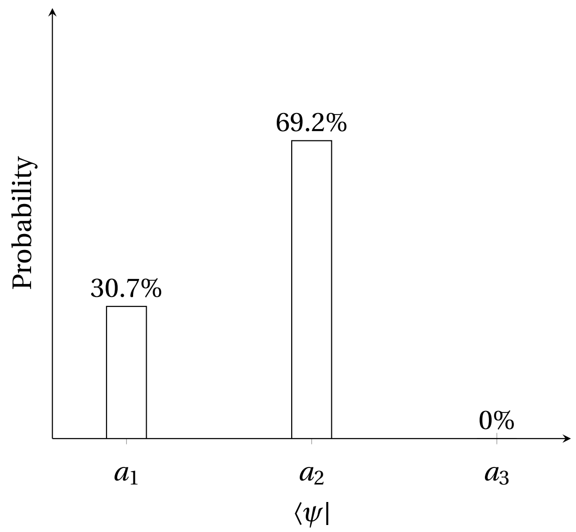
$$|\psi\rangle = \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix} \quad \text{Normalize the function}$$

$$\begin{aligned}
\langle\psi|\psi\rangle &= C^*\begin{pmatrix} 2 & -3i & 0 \end{pmatrix} C \begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix} \\
&= C^2(4+9) \\
C &= \frac{1}{\sqrt{13}} \\
|\psi\rangle &= \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{a_1} &= \Big|\langle a_1|\psi\rangle\Big|^2 \\
&= \left|\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}\right|^2 \\
&= \left|\frac{2}{\sqrt{13}}\right|^2 \\
\mathcal{P}_{a_1} &= \frac{4}{13}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{a_2} &= \Big|\langle a_2|\psi\rangle\Big|^2 \\
&= \left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}\right|^2 \\
&= \left|\frac{3i}{\sqrt{13}}\right|^2 \\
\mathcal{P}_{a_2} &= \frac{9}{13}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{a_3} &= \Big|\langle a_3|\psi\rangle\Big|^2 \\
&= \left|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 3i \\ 0 \end{pmatrix}\right|^2 \\
&= 0 \\
\mathcal{P}_{a_3} &= 0
\end{aligned}$$



PROBLEM 1.15

Consider a quantum system described by a basis $|a_1\rangle, |a_2\rangle$, and $|a_3\rangle$. **The system is initially in a state**

$$|\psi_i\rangle = \frac{i}{\sqrt{3}}|a_1\rangle + \sqrt{\frac{2}{3}}|a_2\rangle.$$

Find the probability that the system is measured to be in the final state

$$|\psi_f\rangle = \frac{1+i}{\sqrt{3}}|a_1\rangle + \frac{1}{\sqrt{6}}|a_2\rangle + \frac{1}{\sqrt{6}}|a_3\rangle.$$

$$\begin{aligned}
\mathcal{P}_{\psi_f} &= \left|\langle\psi_f|\psi_i\rangle\right|^2 \\
&= \left|\begin{pmatrix}\frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\frac{i}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \\ 0\end{pmatrix}\right|^2 \\
&= \left|\frac{1-i}{3} + \sqrt{\frac{2}{9}} + 0\right|^2 \\
&= \left|\frac{1-i}{3} + \frac{1}{3}\right|^2 \\
&= \left(\frac{2}{3} + \frac{1}{3}\right)^2 \\
\mathcal{P}_{\psi_f} &= \frac{5}{9}
\end{aligned}$$

PROBLEM 1.16

The spin components of a beam of atoms prepared in the state $|\psi_{in}\rangle$ **are measured and the following experimental probabilities are obtained:**

$\mathcal{P}_+ = \frac{1}{2}$	$\mathcal{P}_{+x} = \frac{3}{4}$	$\mathcal{P}_{+y} = 0.067$
$\mathcal{P}_- = \frac{1}{2}$	$\mathcal{P}_{-x} = \frac{1}{4}$	$\mathcal{P}_{-y} = 0.933$

From the experimental data, determine the input state.

$$\frac{1}{2} = |\langle +|\psi\rangle|^2 = \left|\langle +|\left\{a|+\rangle + b|-\rangle\right\}\right|^2 = \left|a\langle +|+\rangle\right|^2 = \left|a\right|^2 \qquad a = \frac{1}{\sqrt{2}}$$

$$\frac{1}{2} = |\langle -|\psi\rangle|^2 = \left|\langle -|\left\{a|+\rangle + b|-\rangle\right\}\right|^2 = \left|b\langle -|-\rangle\right|^2 = \left|b\right|^2 \qquad b = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}
\tfrac{3}{4} &= |\langle +_x|\psi\rangle|^2 \\
\tfrac{3}{4} &= \left|\tfrac{1}{\sqrt{2}}\Big(\langle +| + \langle -|\Big)\tfrac{1}{\sqrt{2}}\Big(|+\rangle + e^{i\phi}|-\rangle\Big)\right|^2 \\
&= \left|\tfrac{1}{2}\big(1+e^{i\phi}\big)\right|^2 \\
&= \tfrac{1}{4}\big(1+e^{i\phi}\big)\big(1+e^{-i\phi}\big) \\
&= \tfrac{1}{4}\big(1^2 + 2\cos(\phi) + e^0\big) \\
&= \tfrac{1}{4}\big(2 + 2\cos(\phi)\big) \\
&+ \tfrac{1}{2}\big(1 + \cos(\phi)\big) \\
\tfrac{3}{2} - 1 &= \cos(\phi) \\
\cos^{-1}(\tfrac{1}{2}) &= \pm\tfrac{\pi}{3}
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{4} &= |\langle -_x|\psi\rangle|^2 \\
\tfrac{1}{4} &= \left|\tfrac{1}{\sqrt{2}}\Big(\langle +| - \langle -|\Big)\tfrac{1}{\sqrt{2}}\Big(|+\rangle + e^{i\phi}|-\rangle\Big)\right|^2 \\
&= \left|\tfrac{1}{2}\big(1-e^{i\phi}\big)\right|^2 \\
&= \tfrac{1}{4}\big(1-e^{i\phi}\big)\big(1-e^{-i\phi}\big) \\
&= \tfrac{1}{4}\big(1^2 - 2\cos(\phi) + e^0\big) \\
&= \tfrac{1}{4}\big(2 - 2\cos(\phi)\big) \\
&= \tfrac{1}{2}\big(1 - \cos(\phi)\big) \\
\cos(\phi) &= 1 - \tfrac{1}{2} \\
\cos^{-1}(\tfrac{1}{2}) &= \pm\tfrac{\pi}{3}
\end{aligned}$$

$$\begin{aligned}
0.067 &= |\langle +_y|\psi\rangle|^2 \\
&= \left|\tfrac{1}{\sqrt{2}}\Big(\langle +| - i\langle -|\Big)\tfrac{1}{\sqrt{2}}\Big(|+\rangle + e^{i\phi}|-\rangle\Big)\right|^2 \\
&= \left|\tfrac{1}{2}\big(1-ie^{i\phi}\big)\right|^2 \\
&= \tfrac{1}{4}\big(1-ie^{i\phi}\big)\big(1+ie^{-i\phi}\big) \\
&= \tfrac{1}{4}\big(1^2 + 2\sin(\phi) + e^0\big) \\
&= \tfrac{1}{4}\big(1 + 2\sin(\phi)\big) \\
&= \tfrac{1}{2}\big(1 + \sin(\phi)\big) \\
\sin(\phi) &= 2(0.067) - 1 \\
\sin^{-1}(-0.866) &= -\tfrac{\pi}{3} \; or \; -\tfrac{2\pi}{3}
\end{aligned}$$

$$\begin{aligned}
0.933 &= |\langle +_y|\psi\rangle|^2 \\
&= |\tfrac{1}{\sqrt{2}}\big(\langle +| + i\langle -|\big)\tfrac{1}{\sqrt{2}}\big(|+\rangle + e^{i\phi}|-\rangle\big)\Big|^2 \\
&= \left|\tfrac{1}{2}\big(1+ie^{i\phi}\big)\right|^2 \\
&= \tfrac{1}{4}\big(1+ie^{i\phi}\big)\big(1-ie^{-i\phi}\big) \\
&= \tfrac{1}{4}\big(1^2 + 2\sin(\phi) + e^0\big) \\
&= \tfrac{1}{4}\big(1 + 2\sin(\phi)\big) \\
&= \tfrac{1}{2}\big(1 + \sin(\phi)\big) \\
\sin(\phi) &= 2(0.933) - 1 \\
\sin^{-1}(0.866) &= \pm\tfrac{\pi}{3}
\end{aligned}$$

$$|\psi_{in}\rangle = \tfrac{1}{\sqrt{2}}\big(|+\rangle + e^{-i\frac{\pi}{3}}|-\rangle\big)$$