Exo 3 APIN 2014

1) pour $f(x) = 1$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} dx = \left[x\right]_{-1}^{1} = 2$$

$$\frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) = \frac{1}{3} + \frac{4}{3} + \frac{1}{3} = \frac{6}{3} = 2$$

$$\Rightarrow E(x \to 1) = 0$$

pour $f(x) = x$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

$$\frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) = -\frac{1}{3} + \frac{1}{3} = 0$$

$$\Rightarrow E(x \to x) = 0$$

pour $f(x) = x^2$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

$$\frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow E(x \to x^2) = 0$$

$f(x) = x^3$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$\|$$

$$\frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) = -\frac{1}{3} + \frac{1}{3} = 0$$

$$E(x \to x^3) = 0$$

$f(x) = x^4$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^4 dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$\frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) = \frac{2}{3} \quad \neq$$

$$E(x \to x^4) = \frac{2}{5} - \frac{2}{3} = -\frac{4}{15} \neq 0$$

$$\Rightarrow N = 3$$

2) $K_3(t) = E\left(x \to (x-t)_+^3\right)$

$$\forall t \in [-1, 1],\ K_3(t) = \int_{-1}^{1} (x-t)_+^3 dx - \left(\frac{1}{3}(-1-t)_+^3 + \frac{4}{3}(0-t)_+^3 - \frac{1}{3}(1-t)_+^3\right)$$

$$K_3(t) = \int_{t}^{1} (x-t)^3 dx - \left(\underbrace{\frac{1}{3}(-1-t)_+^3}_{0} + \frac{4}{3}(-t)_+^3 - \frac{1}{3}(1-t)_+^3\right)$$

$$K_3(t) = \left[\frac{(x-t)^4}{4}\right]_{t}^{1} - \frac{4}{3}(-t)_+^3 + \frac{1}{3}(1-t)_+^3$$

si $0 \geq t \geq -1$ alors

$$K_3(t) = \frac{(1-t)^4}{4} - \left(0 + \frac{4}{3}(-t)^3 - \frac{1}{3}(1-t)^3 \right)$$

$$K_3(t) = \frac{(1-t)^4}{4} - \frac{4}{3}t^3 + \frac{1}{3}(1-t)^3 = \underbrace{(1-t)^3}_{\geq 0}\left(\frac{1-t}{4} + \frac{1}{3}\right) \underbrace{- \frac{4}{3}t^3}_{\geq 0}$$

si $1 \geq t \geq 0$

$$K_3(t) = \frac{(1-t)^4}{4} - \left(0 + 0 - \frac{1}{3}(1-t)^3 \right)$$

$$= \frac{(1-t)^4}{4} + \frac{1}{3}(1-t)^3$$

$$= \underbrace{(1-t)^3}_{\geq 0} \left(\frac{(1-t)}{4} + \frac{1}{3} \right) \geq 0$$

3) $K_3(t) \geq 0$ sur $[-1, 1]$

4) D'après le corollaire de Peano

$\exists t \in [-1, 1]$, $E(f) = \frac{f^{(4)}(t)}{4!} E(X \to X^4)$

$$E(f) = \frac{f^{(4)}(t)}{24} \cdot \left(-\frac{4}{15}\right) = -4\frac{f^{(4)}(t)}{360} = -\frac{f^{(4)}(t)}{90}$$

$$|E(f)| \leq C \cdot \left(-\frac{1}{90}\right) \qquad C = \mathrm{Max}\left|f^{(4)}(x)\right|$$