Akim Demaille akim@lrde.epita.fr

EPITA — École Pour l'Informatique et les Techniques Avancées

March 22, 2009

## Preamble

The following slides are implicitly dedicated to classical logic.

Logical Formalisms Natural Deduction

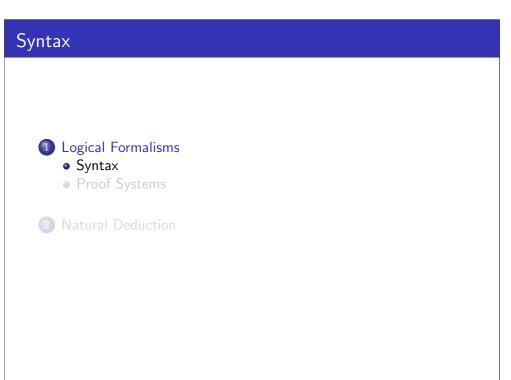
# Logical Formalisms

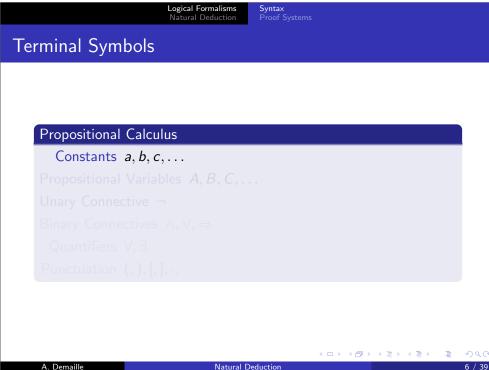
Natural Deduction

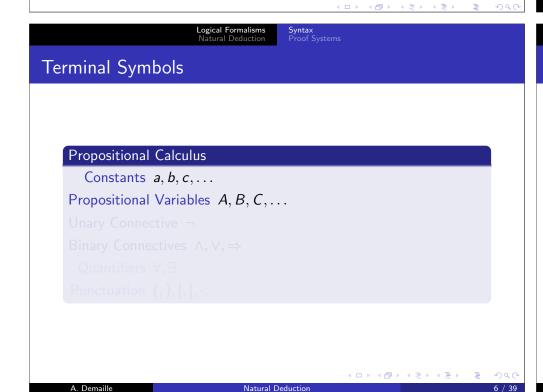
1 Logical Formalisms

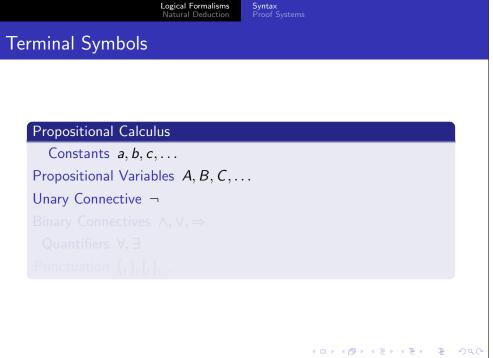
2 Natural Deduction

- Logical Formalisms
  - Syntax
  - Proof Systems
- 2 Natural Deduction

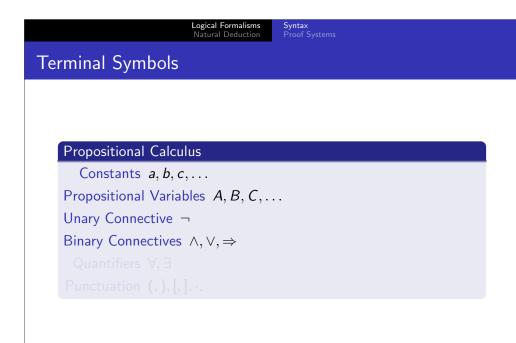








A. Demaille



Logical Formalisms Natural Deduction

Syntax Proof Systems

Terminal Symbols

Propositional Calculus

Constants  $a, b, c, \ldots$ 

Propositional Variables A, B, C, ...

Unary Connective ¬

Binary Connectives  $\land, \lor, \Rightarrow$ 

Quantifiers  $\forall$ ,  $\exists$ 

Punctuation (,),[,],...

Terminal Symbols

Propositional Calculus

Constants  $a, b, c, \ldots$ 

Propositional Variables A, B, C, ...

Unary Connective ¬

Binary Connectives  $\land, \lor, \Rightarrow$ 

Quantifiers  $\forall$ ,  $\exists$ 

**◆□▶ ◆圖▶ ◆園▶ ■ り**90

Natural Deduction

Logical Formalisms Natural Deduction

Syntax Proof Systems

**Terminal Symbols** 

Predicate calculus

Individual Variables x, y, z, ...

A. Demaille

# Terminal Symbols

#### Predicate calculus

Individual Variables x, y, z, ...

Functions  $f, g, h, \ldots$ , with a fixed arity

#### Predicate calculus

Terminal Symbols

Individual Variables x, y, z, ...

Functions  $f, g, h, \ldots$ , with a fixed arity

Predicates  $P, Q, R, \ldots$ , with a fixed arity

Syntax Proof Systems

## Propositional Formulas

(propositional variable)  $\langle formula \rangle ::=$ 

> $\neg \langle formula \rangle$  $\langle formula \rangle \wedge \langle formula \rangle$

 $\langle \text{formula} \rangle \vee \langle \text{formula} \rangle$ 

 $\langle \text{formula} \rangle \Rightarrow \langle \text{formula} \rangle$ 

Syntax Proof Systems

## **Terms**

 $\langle constant \rangle$  $\langle \text{term} \rangle ::=$  $\langle \text{function} \rangle (\langle \text{term} \rangle, \ldots)$ 

With the proper arity.

## First Order Formulas

 $\langle \text{formula} \rangle ::= \langle \text{propositional variable} \rangle$ 

 $\neg \langle formula \rangle$ 

 $\langle \text{formula} \rangle \wedge \langle \text{formula} \rangle$ 

 $\langle \text{formula} \rangle \vee \langle \text{formula} \rangle$ 

 $\langle \text{formula} \rangle \Rightarrow \langle \text{formula} \rangle$ 

 $\langle \text{predicate} \rangle (\langle \text{term} \rangle, \ldots)$ 

 $\forall \langle individual \ variable \rangle \cdot \langle formula \rangle$ 

 $\exists \langle individual \ variable \rangle \cdot \langle formula \rangle$ 

With the proper arity.

Natural Deduction

Natural Deduction

Logical Formalisms

Syntax

Logical Formalisms Natural Deduction

Syntax

## Syntactic Conventions

Associativity

- ∧, ∨ are left-associative (unimportant)
- ⇒ is right-associative (very important)

Precedence (increasing)

- **●**,∀,∃
- $2 \Rightarrow$
- **3** \
- 4

## Syntactic Conventions

Associativity

- ∧, ∨ are left-associative (unimportant)
- ⇒ is right-associative (very important)

Free Variables

$$FV(X) = \emptyset$$

$$FV(P(x_1, x_2, \dots, x_n)) = \{x_1, x_2, \dots, x_m\}$$

$$FV(\neg A) = FV(A)$$

$$FV(A \lor B) = FV(A) \cup FV(B)$$

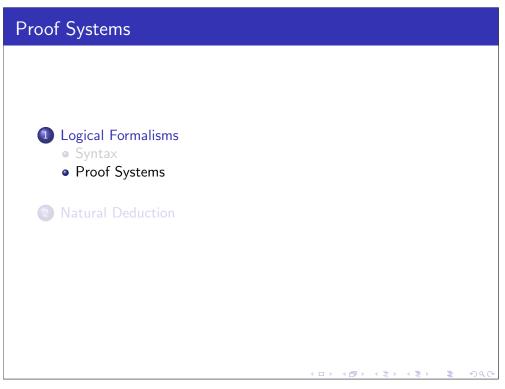
$$FV(A \wedge B) = FV(A) \cup FV(B)$$

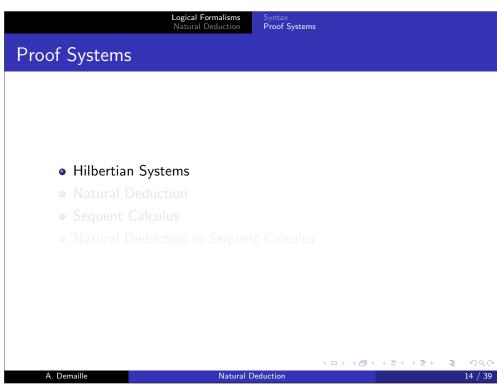
$$FV(A \Rightarrow B) = FV(A) \cup FV(B)$$

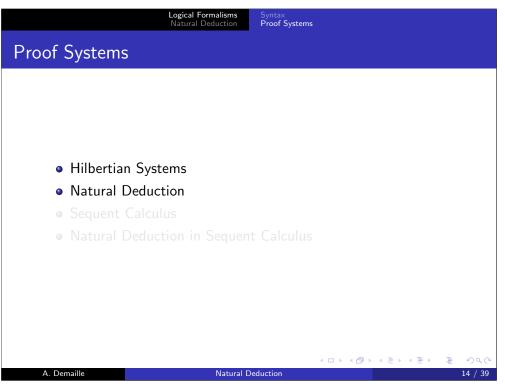
$$FV(\forall x \cdot A) = FV(A) - \{x\}$$

$$FV(\exists x \cdot A) = FV(A) - \{x\}$$

Natural Deduction







Proof Systems

Hilbertian Systems

Natural Deduction

Natural Deduction

Sequent Calculus

Natural Deduction in Sequent Calculus

## Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

イロン イ部 とくきと くきと 一度 。

Natural Deduction

Logical Formalisms

Logical Formalisms Natural Deduction

Proof Systems

#### **Axioms**

• Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

• Axiom schemes use meta-variables that range over a specific domain

$$X + Y = Y + X$$

$$A \vee \neg A$$

• Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

$$X + Y = Y + X$$

$$\begin{array}{ccc} SXYZ & \to & XZ(YZ) \\ KXY & \to & X \end{array}$$

Natural Deduction

Proof Systems

#### **Axioms**

• Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

• Axiom schemes use meta-variables that range over a specific domain

$$X + Y = Y + X$$

• Axiom schemes are used when quantifiers are not welcome

$$\begin{array}{ccc} SXYZ & \to & XZ(YZ) \\ KXY & \to & X \end{array}$$

$$A \vee \neg A$$

### **Axioms**

• Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

 Axiom schemes use meta-variables that range over a specific domain

$$X + Y = Y + X$$

• Axiom schemes are used when quantifiers are not welcome

$$\begin{array}{ccc} SXYZ & \to & XZ(YZ) \\ KXY & \to & X \end{array}$$

Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Proof Systems

4□▶ 4₫▶ 4½▶ 4½▶ ½ 900

A. Demaille

Natural Deduction

Logical Formalisms Natural Deduction "

39

Natural Deduction

 $\frac{H_1 \quad H_2 \quad \cdots \quad H_n}{G}$  Rule name

- Axiom name

Logical Formalisms Natural Deduction

Syntax Proof Systems

## Hilbertian System

• A single inference rule: the modus ponens

$$\frac{A \longrightarrow B}{B}$$
 modus ponens

Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \land B$$
  $A \land B \Rightarrow A$   $A \land B \Rightarrow B$ 

$$A \Rightarrow A \lor B$$
  $B \Rightarrow A \lor B$   
 $A \lor B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$ 

$$A \Rightarrow B \Rightarrow A$$
  $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$ 

ロト (日) (日) (日) (日)

Inference Rules

• A single inference rule: the modus ponens

$$\frac{A \longrightarrow B}{B}$$
 modus ponens

Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \land B$$
  $A \land B \Rightarrow A$   $A \land B \Rightarrow B$ 

$$A \Rightarrow A \lor B$$
  $B \Rightarrow A \lor B$   
 $A \lor B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$ 

$$A \Rightarrow B \Rightarrow A$$
  $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$ 

$$A \lor \neg A$$
  $A \Rightarrow \neg A \Rightarrow B$ 

40.49.41.41. 1 000

Matural Da

Natural Deduction

17 / 3

A. Demaille

Natural Deduction

17 / 3

Proof Systems

Hilbertian System: Prove  $A \Rightarrow A$ 

Proof Systems

## Hilbertian System

• A single inference rule: the modus ponens

$$\frac{A \qquad A \Rightarrow B}{B} \text{ modus ponens}$$

Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \land B \qquad A \land B \Rightarrow A \qquad A \land B \Rightarrow B$$

$$A \Rightarrow A \lor B \qquad B \Rightarrow A \lor B$$

$$A \lor B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \qquad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$\Rightarrow A \lor \neg A \qquad A \Rightarrow \neg A \Rightarrow B$$

<ロ > < @ > < き > くき > き の < @ へ

Natural Deduction

Natural Deduction

Proof Systems

## Hilbertian System: Prove $A \Rightarrow A$

$$(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow (A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A \qquad A \Rightarrow (A \Rightarrow A) \Rightarrow A$$

 $(A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A$ 

 $A \Rightarrow A \Rightarrow A$ 

 $A \Rightarrow A$ 

- Logical Formalisms
- Natural Deduction
  - Syntax
  - Normalization



- Logical Formalisms
- 2 Natural Deduction
  - Syntax
  - Normalization



Syntax Normalization Logical Formalisms Natural Deduction

## Deduction

#### Deduction

A deduction is a tree whose root (A) is the conclusion and whose active leafs  $(\Gamma)$  is the set of hypotheses.

Natural Deduction

Any formula A is a valid hypothesis.

## Proof (Demonstration)

A proof is a deduction without hypotheses.

## Deduction

#### Deduction

A deduction is a tree whose root (A) is the conclusion and whose leafs ( $\Gamma$ ) is the set of hypotheses.

Any formula A is a valid hypothesis.

#### Proof (Demonstration)

A proof is a deduction without hypotheses.

Natural Deduction

Logical Formalisms Natural Deduction

#### **Deductions**

What's this?

Α

Natural Deduction

Syntax

## **Deductions**

What's this?

Α

Natural Deduction

A deduction of A under the hypothesis A.

22 / 39

**Implication** 

Natural Deduction

Logical Formalisms Natural Deduction

Syntax Normalization

Logical Formalisms Natural Deduction

Syntax Normalization

## **Implication**

$$\begin{array}{ccc}
[A] & \vdots & \vdots \\
\frac{B}{A \to B} \Rightarrow \mathcal{I} & \frac{A \to B}{B} \Rightarrow \mathcal{E}
\end{array}$$

**Implication** 

Deduction theorem, and Modus Ponens.

A. Demaille

Syntax

Logical Formalism

Proving  $A \Rightarrow A$  in Natural Deduction

Syntax Normalization

## **Implication**

# $\begin{array}{ccc} [A] & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ A \to B \Rightarrow \mathcal{I} & A \to B \\ \hline A \to B \Rightarrow \mathcal{E}$

Deduction theorem, and Modus Ponens. Note the connection with (left) contraction: any number of *A* (including 0) is discharged.

| ロ ト 4 団 ト 4 里 ト 4 里 ト 9 Q C

A. Demaille

TVacarar E

A. Demaille

Natural Deduction

Logical Formalisms Natural Deduction Syntax Normalization 24 /

Logical Formalisms
Natural Deduction

Syntax Normalization

## Proving $A \Rightarrow A$ in Natural Deduction

$$\frac{[A]}{A \to A} \Rightarrow \mathcal{I}$$

Conjunction

## Conjunction

$$\begin{array}{ccc} \vdots & \vdots & & \vdots \\ \frac{A & B}{A \wedge B} \wedge \mathcal{I} & & \frac{A \wedge B}{A} \wedge l\mathcal{E} & & \frac{A \wedge B}{B} \wedge r\mathcal{E} \end{array}$$

## Universal Quantification

$$\frac{A[y/x]}{\forall x \cdot A} \,\forall \mathcal{I} \quad y \notin \text{FV}(hyp(A)) \qquad \frac{\forall x \cdot A}{A[t/x]} \,\forall \mathcal{E}$$

$$\frac{\frac{[A]}{\forall x \cdot A} \, \forall \mathcal{I}}{A \Rightarrow \forall x \cdot A} \Rightarrow \mathcal{I}$$

$$\frac{A \Rightarrow A}{A \Rightarrow A} \Rightarrow \mathcal{I}$$

Universal Quantification

Logical Formalisms
Natural Deduction

## Universal Quantification

$$\frac{A[y/x]}{\forall x \cdot A} \,\forall \mathcal{I} \quad y \notin FV(hyp(A)) \qquad \frac{\forall x \cdot A}{A[t/x]} \,\forall \mathcal{E}$$

$$\frac{[A]}{\forall x \cdot A} \forall \mathcal{I}$$

$$A \Rightarrow \forall x \cdot A \Rightarrow \mathcal{I}$$

$$\frac{[A]}{A \Rightarrow A} \Rightarrow \mathcal{I}$$

$$\forall x \cdot (A \Rightarrow A) \quad \forall x \in \mathcal{I}$$

 $\frac{[A]}{\forall x \cdot A} \forall \mathcal{I}$   $A \Rightarrow \forall x \cdot A \Rightarrow \mathcal{I}$   $\frac{[A]}{A \Rightarrow A} \Rightarrow \mathcal{I}$   $\forall x \cdot (A \Rightarrow A) \forall \mathcal{I}$ 

Natural Deduction

Natural Deduction

 $\frac{A[y/x]}{\forall x \cdot A} \, \forall \mathcal{I} \quad y \not\in \mathrm{FV}(hyp(A)) \qquad \frac{\forall x \cdot A}{A[t/x]} \, \forall \mathcal{E}$ 

## Absurd

$$\frac{\vdots}{A} \perp \mathcal{E}$$

Natural Deduction

Logical Formalisms Natural Deduction

A. Demaille

Logical Formalisms Natural Deduction

Syntax Normalization

Syntax Normalization

## **Existential Quantification**

$$\frac{A[t/x]}{\exists x \cdot A} \exists \mathcal{I} \qquad \frac{\exists x \cdot A \qquad B}{B} \exists \mathcal{E} \quad y \notin FV(B, hyp(B))$$

For elimination,  $y \notin hyp(B)$ , i.e., not in the hypotheses other than the discharged A.

## Negation

Disjunction

$$\begin{bmatrix} A \\ \vdots \\ \frac{\perp}{\neg A} \neg \mathcal{I} & \vdots & \vdots \\ \frac{A}{\neg A} \neg A \\ \end{bmatrix} \neg \mathcal{E}$$

## Negation

$$\begin{array}{cccc}
 [A] & \vdots & \vdots \\
 \vdots & & A & \neg A \\
 \frac{\bot}{\neg A} \neg \mathcal{I} & & \frac{A}{\bot} & \neg \mathcal{E}
\end{array}$$

Plus one of these equivalent formulation of the fact that classical negation is involutive.

$$\frac{1}{A \vee \neg A} XM \qquad \frac{\vdots}{A} \neg \neg A \qquad \frac{\neg A}{A} \neg \neg \qquad \frac{B}{A} \qquad \neg B \text{ Contradiction}$$

4□ > 4□ > 4 = > 4 = > = 90

A. Demaille

Natural Deduction

30 / 39

## Normalization

- 1 Logical Formalisms
- 2 Natural Deduction
  - Syntax
  - Normalization

←□ → ←□ → ← □ → ← □ → ○ へ ○

Logical Formalisms Natural Deduction

Syntax Normalization

#### Cut

Cut: Introduction of a connective followed by its elimination.

$$\frac{A \quad B}{A \wedge B} \wedge \mathcal{I}$$

$$\frac{A \wedge B}{A} \wedge I\mathcal{E}$$

Logical Formalisms Natural Deduction Syntax Normalization

## Normalization

The normalization process eliminates the cuts.

$$\frac{A \quad B}{A \wedge B} \wedge I \quad \sim \quad A$$

$$\frac{A \wedge B}{A} \wedge I\mathcal{E}$$

## Normalizing Conjunctions

$$\frac{A \qquad B}{A \wedge B} \wedge \mathcal{I} \qquad \Rightarrow \qquad A$$

$$\frac{A \wedge B}{A} \wedge I\mathcal{E} \qquad \Rightarrow \qquad A$$

$$\vdots$$

$$\frac{A \qquad B}{A \wedge B} \wedge \mathcal{I} \qquad \Longrightarrow \qquad B$$

$$\frac{B}{B} \wedge r\mathcal{E} \qquad \Longrightarrow \qquad B$$

A. Demaille

Natural Deduction

34

Normalizing Implications

$$\begin{array}{ccc}
[A] & \vdots & \vdots \\
\vdots & B & \vdots \\
A & \overline{A \Rightarrow B} \Rightarrow \mathcal{I} & A \\
\hline
B & \vdots & B \\
\vdots & \vdots & B
\vdots$$

A Demaille

Vatural Deduction

Syntax Normalization

Logical Formalisms Natural Deduction 35 /

Logical Formalisms
Natural Deduction

Syntax

Normalizing Universal Quantifiers

 $\frac{A}{\frac{A}{\forall x \cdot A}} \forall \mathcal{I} \qquad \qquad \vdots \\
\frac{A[t/x]}{A[t/x]} \forall \mathcal{E} \qquad \qquad \Rightarrow \qquad A[t/x]$ 

*x* must not be free in the hypotheses, otherwise the reduction would change them.

Normalizing Disjunction

$$\frac{A}{A \vee B} \vee I\mathcal{I} \qquad \vdots \qquad \vdots \qquad A \\
C \qquad C \qquad C \\
\vdots \qquad \vdots \qquad \vdots$$

$$\frac{\vdots}{B} \vee r\mathcal{I} \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
\frac{B}{C} \vee \mathcal{E} \qquad C \qquad C \qquad C \qquad C \qquad C \qquad \vdots$$

## Bibliography Notes

- [1] A short (160p.) book addressing all the concerns of this course, and more (especially Linear Logic). Easy and pleasant to read. Now available for free.
- [2] A much more comprehensive book focusing on logic and its connections with computer science. In French.

Bibliography I



J.-Y. Girard, Y. Lafont, and P. Taylor.

Proofs and Types.

Cambridge University Press, 1989.

http:

//www.cs.man.ac.uk/~pt/stable/Proofs+Types.html.



Jean-Yves Girard.

Cours de Logique, Rome, Automne 2004.

http://logica.uniroma3.it/uif/corso/, 2004.



Natural Deduction