Sequent Calculus
Cut Elimination

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Sequent Calculus Cut Elimination

Preamble

The following slides are implicitly dedicated to classical logic.

Problems of Natural Deduction

Normalization

Proofs hard to find

Some elimination rules used formulas coming out of the blue.

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\,\vee\mathcal{E}$$

Negation is awkward

Sequent Calculus

Syntax

Sequents

Sequent

A sequent is an expression $\Gamma \vdash \Delta$, where Γ, Δ are (finite) sequences of formulas.

Variants:

Sets Γ, Δ can be taken as (finite) sets.

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Single Sided Sequents can be forced to be $\vdash \Gamma$

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- Prevents close examination of… the structural rules

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Reading a Sequent

$\Gamma \vdash \Delta$ If all the formulas of Γ are true, then one of the formulas of Δ is true.

- Commas on the left hand side stand for "and"
- Turnstile, $\vdash$, stands for "implies"
- Commas on the right hand side stand for "or"

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Some Special Sequents

$\Gamma \vdash A$ A is true under the hypotheses Γ

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Some Special Sequents

$\Gamma \vdash A$ A is true under the hypotheses Γ

$\vdash A$ A is true

$\Gamma \vdash$ Γ is in contradiction

Some Special Sequents

$\Gamma \vdash A$	A is true under the hypotheses Γ
$\vdash A$	A is true
$\Gamma \vdash$	Γ is in contradiction
$A \vdash$	$\neg A$

Some Special Sequents

$\Gamma \vdash A$	A is true under the hypotheses Γ
$\vdash A$	A is true
$\Gamma \vdash$	Γ is in contradiction
$A \vdash$	$\neg A$
$\vdash$	contradiction

LK — Classical Sequent Calculus

Identity Group

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

Gentzen's Hauptsatz

The cut rule is redundant, i.e., any sequent provable with cuts is provable without.

Identity Group

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

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Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)}\vdash \mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta}\,\mathrm{X}\vdash$$

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$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{W}\vdash$$

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$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{W}\vdash$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{C} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{C}\vdash$$

Logical Group: Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\vdash\neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\neg\vdash$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\vdash\wedge$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\vdash\wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}l\wedge\vdash$$

Logical Group: Conjunction

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$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}r\wedge\vdash$$

Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge \vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge \vdash$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge \vdash$$

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Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge \vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge \vdash \qquad \&$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge \qquad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge \vdash \qquad \otimes$$

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Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l\vee$$

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$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l\vee$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash r\vee$$

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Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash l\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee\vdash$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash r\vee$$

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$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \vee\vdash$$

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$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash r\vee$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vdash\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \vee\vdash \qquad ⅋$$

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Logical Group: Implication

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow\vdash$$

Logical Group: Implication

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \vdash\Rightarrow$$

Example

Prove $A \wedge B \vdash A \wedge B$

Example

Prove $A \wedge B \vdash A \wedge B$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash \qquad \dfrac{\dfrac{}{B \vdash B}}{A \wedge B \vdash B}\, r\wedge\vdash}{\dfrac{A \wedge B, A \wedge B \vdash A \wedge B}{A \wedge B \vdash A \wedge B}\, \mathrm{C}\vdash}\, \vdash\wedge$$

Example

Prove $A \wedge B \vdash A \vee B$

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$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \wedge B \vdash A}\, l\wedge\vdash}{A \wedge B \vdash A \vee B}\, \vdash\vee$$

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Prove $A \vee B \vdash A \vee B$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash A \vee B}\, \vdash l\vee \qquad \dfrac{\dfrac{}{B \vdash B}}{B \vdash A \vee B}\, \vdash r\vee}{A \vee B \vdash A \vee B}\, \vee\vdash$$

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Example

Prove the equivalence of the two $\wedge$ rules

$$\overset{\text{Multiplicative}}{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\, \vdash\wedge\times} \quad \equiv \quad \overset{\text{Additive}}{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\, \vdash\wedge+}$$

Example

Prove the equivalence of the two $\wedge$ rules

$$\begin{array}{ccc} \text{Multiplicative} & & \text{Additive} \\ \dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge\times & \equiv & \dfrac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge+ \end{array}$$

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$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \Gamma' \vdash A, \Delta} \mathrm{W} \vdash}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'} \vdash \mathrm{W} \quad \dfrac{\dfrac{\Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta'} \mathrm{W} \vdash}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'} \vdash \mathrm{W}}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash \wedge\times$$

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Logical Group: Quantifiers

$$\dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x \cdot A, \Delta} \vdash \forall \qquad \dfrac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x \cdot A \vdash \Delta} \forall \vdash$$

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Logical Group: Quantifiers

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$$\dfrac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x \cdot A, \Delta} \vdash \exists \qquad \dfrac{\Gamma, A \vdash \Delta}{\Gamma, \exists x \cdot A \vdash \Delta} \exists \vdash$$

In $\vdash \forall$ and $\exists \vdash$, $x \notin \mathrm{FV}(\Gamma, \Delta)$.

Single Sided

Defining the Negation

- Alternatively, one can define the negation as a notation instead of defining it by inference rules.

$$
\begin{array}{rcl}
\neg(\neg p) & := & p \\
\neg(A \wedge B) & := & \neg A \vee \neg B \\
\neg(A \vee B) & := & \neg A \wedge \neg B \\
\neg(\forall x \cdot A) & := & \exists x \cdot \neg A \\
\neg(\exists x \cdot A) & := & \forall x \cdot \neg A
\end{array}
$$

- Then define the sequents as $\vdash \Gamma$
- I.e., $\Gamma \vdash \Delta \rightsquigarrow \vdash \neg\Gamma, \Delta$

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\end{array}
$$

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Single Sided

The Full Sequent Calculus

$$
\frac{}{\vdash \neg A, A}\,\mathrm{Id} \qquad \frac{\vdash \Gamma, A \quad \vdash \neg A, \Delta}{\vdash \Gamma, \Delta}\,\mathrm{Cut}
$$

$$
\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\,\mathrm{X} \qquad \frac{\vdash \Gamma}{\vdash A, \Gamma}\,\mathrm{W} \qquad \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}\,\mathrm{C}
$$

$$
\frac{\vdash A, \Delta}{\vdash A \vee B, \Delta}\,l\vee \qquad \frac{\vdash B, \Delta}{\vdash A \vee B, \Delta}\,r\vee \qquad \frac{\vdash A, \Delta \quad \vdash B, \Delta}{\vdash A \wedge B, \Delta}\,\wedge
$$

$$
\frac{\vdash A, \Delta}{\vdash \forall x \cdot A, \Delta}\,\forall \qquad \frac{\vdash A[t/x], \Delta}{\vdash \exists x \cdot A, \Delta}\,\exists
$$

Cut Elimination

Subformula Property

What can be told from the last rule of a proof?

- If the last rule is a cut,

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

- Otherwise...

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge \vdash \qquad \dots$$

Subformula Property

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  nothing can be said!

- Otherwise...

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Subformula Property

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Subformula Property

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  nothing can be said!
- Otherwise...

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\, l\wedge\vdash \qquad \ldots$$

  premisses can only use subformulas of the conclusion!

Cut Elimination

- replace “complex” cuts by simpler cuts (smaller formulas)
- until the cut is on the simplest form, the identity
- where it is not longer needed!

Cut Elimination

Logical Rules

$$\frac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \dfrac{\Gamma', A \vdash \Delta'}{\Gamma', A \wedge B \vdash \Delta'}\, l\wedge\vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

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Logical Rules

$$\frac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \dfrac{\Gamma', A \vdash \Delta'}{\Gamma', A \wedge B \vdash \Delta'}\, l\wedge\vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

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Logical Rules

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\vdash\wedge \qquad \dfrac{\Gamma', A \vdash \Delta'}{\Gamma', A \wedge B \vdash \Delta'}\, l\wedge\vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

For all the connectives.

Cut Elimination

Removal of a Cut

$$\dfrac{\dfrac{}{A \vdash A}\,\mathrm{Identity} \qquad \begin{array}{c}\vdots\\ \Gamma, A \vdash \Delta\end{array}}{\Gamma, A \vdash \Delta}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

Cut Elimination

Removal of a Cut

$$\dfrac{\dfrac{}{A \vdash A}\,\mathrm{Identity} \qquad \begin{array}{c}\vdots\\ \Gamma, A \vdash \Delta\end{array}}{\Gamma, A \vdash \Delta}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

$$\begin{array}{c}\vdots\\ \Gamma, A \vdash \Delta\end{array}$$

Cut Elimination

Commutations

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma, C \vdash A, \Delta}{\Gamma, B \vee C \vdash A, \Delta}\vee\vdash \qquad \Gamma', A \vdash \Delta'}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\rightsquigarrow$$

Cut Elimination

Commutations

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma, C \vdash A, \Delta}{\Gamma, B \vee C \vdash A, \Delta}\vee\vdash \qquad \Gamma', A \vdash \Delta'}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, B, \Gamma' \vdash \Delta, \Delta'}\text{Cut} \qquad \dfrac{\Gamma, C \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, C, \Gamma' \vdash \Delta, \Delta'}\text{Cut}}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\vee\vdash$$

Problems of Natural Deduction | Syntax
Sequent Calculus | LK — Classical Sequent Calculus
Natural Deduction in Sequent Calculus | Cut Elimination

Cut Elimination

Commutations

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma, C \vdash A, \Delta}{\Gamma, B \vee C \vdash A, \Delta}\vee\vdash \qquad \Gamma', A \vdash \Delta'}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, B \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, B, \Gamma' \vdash \Delta, \Delta'}\text{Cut} \qquad \dfrac{\Gamma, C \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, C, \Gamma' \vdash \Delta, \Delta'}\text{Cut}}{\Gamma, B \vee C, \Gamma' \vdash \Delta, \Delta'}\vee\vdash$$

Beware of the duplication!

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Cut Elimination

Structural Rules: Weakening

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\vdash\text{W} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

$$\rightsquigarrow$$

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Cut Elimination

Structural Rules: Weakening

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\vdash\text{W} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}\text{W}\vdash}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\vdash\text{W}$$

Cut Elimination

Structural Rules: Contraction

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

$$\leadsto$$

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Cut Elimination

Structural Rules: Contraction

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

$$\leadsto$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\mathrm{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{C}$$

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Structural Rules: Contraction

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

Nice!

$$\leadsto$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\mathrm{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{C}$$

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Structural Rules: Contraction

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

Nice!

but wrong

$$\leadsto$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\mathrm{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{C}$$

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Structural Rules: Contraction

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,{\vdash}\mathrm{C} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\leadsto$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,\mathrm{Cut} \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{C}$$

Nice!

but wrong

might loop for ever

Cut Elimination

Structural Rules: Complications with Contractions

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,{\vdash}\mathrm{C} \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}{\vdash}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\leadsto$$

Cut Elimination

Structural Rules: Complications with Contractions

$$\dfrac{\dfrac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,{\vdash}\mathrm{C} \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}{\vdash}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

$$\leadsto$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A, A, \Delta \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}{\vdash}}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,\mathrm{Cut} \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\mathrm{C}{\vdash}}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,\mathrm{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{C}$$

Natural Deduction in Sequent Calculus

Recommended readings

[1], Chapters 5 & 13 A short (160p.) book addressing all the concerns of this course, and more (especially Linear Logic). Easy and pleasant to read. Now available for free.

Bibliography I

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