The Curry-Howard Isomorphism

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The Curry-Howard Isomorphism

Heyting's Semantics of Proofs

Functional Interpretation

[2, Chap. 5]

- Instead of "when is a sentence A true"
- ask "what is a proof of A"?

Functional Interpretation

[1, Sec. 1.2.2]

What is a proof π of A? ($\pi \vdash A$)

$A \wedge B$ A pair (π_A, π_B) s.t. $\pi_A \vdash A$ and $\pi_B \vdash B$

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$A \vee B$ A pair (i, π) s.t.

$i = 0$ $\pi \vdash A$

$i = 1$ $\pi \vdash B$

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$\forall x \cdot A$ A function f s.t. $f(a) \vdash A[a/x]$

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$\forall x \cdot A$ A function f s.t. $f(a) \vdash A[a/x]$

$\exists x \cdot A$ A pair (a, π) s.t. $\pi \vdash A[a/x]$

Functional Interpretation

[1, Sec. 1.2.2]

What is a proof π of A? ($\pi \vdash A$)

Atomic Values Assume we know what a proof is

$A \wedge B$ A pair (π_A, π_B) s.t. $\pi_A \vdash A$ and $\pi_B \vdash B$

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- The handling of $\vee$ and $\exists$ are similar to the disjunctive and existential properties
- Therefore refers to a cut-free proof
- For instance id is a proof of $A \Rightarrow A$

The Curry-Howard Isomorphism

A Striking Correspondence

Type Derivations

$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau}\,\mathrm{app}$$

$$\frac{\begin{array}{c}[x : \sigma] \\ \vdots \\ M : \tau\end{array}}{\lambda x \cdot M : \sigma \to \tau}\,\mathrm{abs}$$

Natural Deduction

$$\frac{A \Rightarrow B \quad A}{B}\Rightarrow\mathcal{E}$$

$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \Rightarrow B}\Rightarrow\mathcal{I}$$

A Striking Correspondence

Type Derivations

$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau}\,\mathrm{app}$$

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Natural Deduction

$$\frac{A \Rightarrow B \quad A}{B} \Rightarrow\mathcal{E}$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \Rightarrow B} \Rightarrow\mathcal{I}$$

The Isomorphism

The Isomorphism

[1]

Curry-Howard Isomorphism

There is a perfect equivalence between two viewpoints:

Natural Deduction
: Formulas A, deductions of A, normalization in natural deduction

Typed λ-calculus
: Types A, terms of type A, normalization in λ-calculus.

Deductions and Terms: Conjuction

$$A^i$$

$$\frac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ B\end{array}}{A \wedge B} \wedge\mathcal{I}$$

$$\frac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{A} \wedge 1\mathcal{E}$$

$$\frac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{B} \wedge 2\mathcal{E}$$

Deductions and Terms: Conjuction

$$x_i^A : A^i$$

$$\dfrac{\begin{matrix}\vdots \\ u : A\end{matrix} \qquad \begin{matrix}\vdots \\ v : B\end{matrix}}{\langle u, v\rangle : A \wedge B} \wedge\mathcal{I}$$

$$\dfrac{\begin{matrix}\vdots \\ u : A \wedge B\end{matrix}}{\pi_1 u : A} \wedge 1\mathcal{E}$$

$$\dfrac{\begin{matrix}\vdots \\ u : A \wedge B\end{matrix}}{\pi_2 u : B} \wedge 2\mathcal{E}$$

Reductions

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B} \wedge\mathcal{I}}{A} \wedge 1\mathcal{E} \\ \vdots\end{matrix} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ A \\ \vdots\end{matrix}$$

Reductions

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\vdots \\ u : A\end{matrix} \qquad \begin{matrix}\vdots \\ v : B\end{matrix}}{\langle u, v\rangle : A \wedge B} \wedge\mathcal{I}}{\pi_1\langle u, v\rangle : A} \wedge 1\mathcal{E} \\ \vdots\end{matrix} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ u : A \\ \vdots\end{matrix}$$

Reductions

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\vdots \\ u : A\end{matrix} \qquad \begin{matrix}\vdots \\ v : B\end{matrix}}{\langle u, v\rangle : A \wedge B} \wedge\mathcal{I}}{\pi_1\langle u, v\rangle : A} \wedge 1\mathcal{E} \\ \vdots\end{matrix} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ u : A \\ \vdots\end{matrix}$$

$$\pi_1\langle u, v\rangle \rightsquigarrow u$$
$$\pi_2\langle u, v\rangle \rightsquigarrow v$$

Deductions and Terms: Implication

$$\frac{\begin{matrix}[A]^i \\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I}_i \qquad \frac{\begin{matrix}\vdots \\ A \Rightarrow B\end{matrix} \quad \begin{matrix}\vdots \\ A\end{matrix}}{B} \Rightarrow\mathcal{E}$$

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Deductions and Terms: Implication

$$\frac{\begin{matrix}x_i^A : [A]^i \\ \vdots \\ u : B\end{matrix}}{\lambda x_i^A \cdot u : A \Rightarrow B} \Rightarrow\mathcal{I}_i \qquad \frac{\begin{matrix}\vdots \\ u : A \Rightarrow B\end{matrix} \quad \begin{matrix}\vdots \\ v : A\end{matrix}}{uv : B} \Rightarrow\mathcal{E}$$

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Disjunction

$$\frac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B} \vee l\mathcal{I} \qquad \frac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B} \vee r\mathcal{I} \qquad \frac{\begin{matrix}\vdots \\ A \vee B\end{matrix} \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C} \vee\mathcal{E}$$

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Disjunction

$$\frac{\begin{matrix}\vdots \\ u : A\end{matrix}}{\iota_l u : A \vee B} \vee l\mathcal{I} \qquad \frac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B} \vee r\mathcal{I} \qquad \frac{\begin{matrix}\vdots \\ A \vee B\end{matrix} \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C} \vee\mathcal{E}$$

Disjunction

$$\cfrac{\begin{matrix}\vdots\\ u : A\end{matrix}}{\iota_l u : A \vee B}\,\vee l\mathcal{I} \qquad \cfrac{A \vee B \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathcal{E}$$

$$\cfrac{\begin{matrix}\vdots\\ v : B\end{matrix}}{\iota_r v : A \vee B}\,\vee r\mathcal{I}$$

Disjunction

$$\cfrac{\begin{matrix}\vdots\\ u : A\end{matrix}}{\iota_l u : A \vee B}\,\vee l\mathcal{I} \qquad \cfrac{\begin{matrix}\vdots\\ r : A \vee B\end{matrix} \qquad \begin{matrix}[x : A]\\ \vdots\\ u : C\end{matrix} \qquad \begin{matrix}[y : B]\\ \vdots\\ v : C\end{matrix}}{C}\,\vee\mathcal{E}$$

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Disjunction

$$\cfrac{\begin{matrix}\vdots\\ u : A\end{matrix}}{\iota_l u : A \vee B}\,\vee l\mathcal{I} \qquad \cfrac{\begin{matrix}\vdots\\ r : A \vee B\end{matrix} \qquad \begin{matrix}[x : A]\\ \vdots\\ u : C\end{matrix} \qquad \begin{matrix}[y : B]\\ \vdots\\ v : C\end{matrix}}{\delta\ x \cdot u\ y \cdot v\ r : C}\,\vee\mathcal{E}$$

$$\cfrac{\begin{matrix}\vdots\\ v : B\end{matrix}}{\iota_r v : A \vee B}\,\vee r\mathcal{I}$$

Reductions

$$\begin{matrix}\cfrac{\cfrac{\begin{matrix}\vdots\\ r : A\end{matrix}}{A \vee B}\,\vee l\mathcal{I} \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathcal{E}\\ \vdots\end{matrix} \quad \rightsquigarrow \quad \begin{matrix}\vdots\\ A\\ \vdots\\ C\\ \vdots\end{matrix}$$

Reductions

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ r : A\end{matrix}}{\iota_l r : A \vee B}\vee l\mathcal{I} \qquad \begin{matrix}[x : A]\\ \vdots \\ u : C\end{matrix} \qquad \begin{matrix}[y : B]\\ \vdots \\ v : C\end{matrix}}{\begin{matrix}C\\ \vdots\end{matrix}}\vee\mathcal{E} \quad \leadsto \quad \begin{matrix}\vdots\\ A\\ \vdots\\ C\\ \vdots\end{matrix}$$

Reductions

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ r : A\end{matrix}}{\iota_l r : A \vee B}\vee l\mathcal{I} \qquad \begin{matrix}[x : A]\\ \vdots \\ u : C\end{matrix} \qquad \begin{matrix}[y : B]\\ \vdots \\ v : C\end{matrix}}{\begin{matrix}\delta\ x \cdot u\ y \cdot v\ (\iota_l r) : C\\ \vdots\end{matrix}}\vee\mathcal{E} \quad \leadsto \quad \begin{matrix}\vdots\\ A\\ \vdots\\ C\\ \vdots\end{matrix}$$

Reductions

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ r : A\end{matrix}}{\iota_l r : A \vee B}\vee l\mathcal{I} \qquad \begin{matrix}[x : A]\\ \vdots \\ u : C\end{matrix} \qquad \begin{matrix}[y : B]\\ \vdots \\ v : C\end{matrix}}{\begin{matrix}\delta\ x \cdot u\ y \cdot v\ (\iota_l r) : C\\ \vdots\end{matrix}}\vee\mathcal{E} \quad \leadsto \quad \begin{matrix}\vdots\\ r : A\\ \vdots\\ u[r/x] : C\\ \vdots\end{matrix}$$

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$$\delta\ x \cdot u\ y \cdot v\ (\iota_l r) \leadsto u[r/x]$$
$$\delta\ x \cdot u\ y \cdot v\ (\iota_r s) \leadsto v[s/y]$$

Commutative Reductions

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix}\quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\vee\mathcal{E}\quad \vdots}{\begin{matrix}D\\ \vdots\end{matrix}}R\mathcal{E}$$

$$\rightsquigarrow\quad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}\quad \vdots}{D}R\mathcal{E}\quad \dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix}\quad \vdots}{D}R\mathcal{E}}{\begin{matrix}D\\ \vdots\end{matrix}}\vee\mathcal{E}$$

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$$\rightsquigarrow\quad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}\quad \vdots}{D}R\mathcal{E}\quad \dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix}\quad \vdots}{D}R\mathcal{E}}{\begin{matrix}D\\ \vdots\end{matrix}}\vee\mathcal{E}$$

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$$\dfrac{\dfrac{\begin{matrix}\vdots \\ r : A \vee B\end{matrix} \quad \begin{matrix}[x:A] \\ \vdots \\ u : C\end{matrix} \quad \begin{matrix}[y:B]\\ \vdots \\ v : C\end{matrix}}{\delta\ x \cdot u\ y \cdot v\ r : C}\vee\mathcal{E} \quad \vdots}{\cdots(\delta\ x\cdot u\ y\cdot v\ r)\cdots : D} R\mathcal{E}$$

$$\vdots$$

$$\rightsquigarrow \quad \dfrac{\begin{matrix}\vdots\\ r:A\vee B\end{matrix} \quad \dfrac{\begin{matrix}[x:A]\\ \vdots\\ u:C\end{matrix} \quad \vdots}{D} R\mathcal{E} \quad \dfrac{\begin{matrix}[B]\\ \vdots \\ C\end{matrix}\quad \vdots}{D}R\mathcal{E}}{D}\vee\mathcal{E}$$

$$\vdots$$

Commutative Reductions

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ r : A \vee B\end{matrix} \quad \begin{matrix}[x:A] \\ \vdots \\ u : C\end{matrix} \quad \begin{matrix}[y:B]\\ \vdots \\ v : C\end{matrix}}{\delta\ x \cdot u\ y \cdot v\ r : C}\vee\mathcal{E} \quad \vdots}{\cdots(\delta\ x\cdot u\ y\cdot v\ r)\cdots : D} R\mathcal{E}$$

$$\vdots$$

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$$\vdots$$

Commutative Reductions

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ r : A \vee B\end{matrix} \quad \begin{matrix}[x:A] \\ \vdots \\ u : C\end{matrix} \quad \begin{matrix}[y:B]\\ \vdots \\ v : C\end{matrix}}{\delta\ x \cdot u\ y \cdot v\ r : C}\vee\mathcal{E} \quad \vdots}{\cdots(\delta\ x\cdot u\ y\cdot v\ r)\cdots : D} R\mathcal{E}$$

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$$\vdots$$

Commutative Reductions

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ r : A \vee B\end{matrix} \quad \begin{matrix}[x:A] \\ \vdots \\ u : C\end{matrix} \quad \begin{matrix}[y:B]\\ \vdots \\ v : C\end{matrix}}{\delta\ x \cdot u\ y \cdot v\ r : C}\vee\mathcal{E} \quad \vdots}{\cdots(\delta\ x\cdot u\ y\cdot v\ r)\cdots : D} R\mathcal{E}$$

$$\vdots$$

$$\rightsquigarrow \quad \dfrac{\begin{matrix}\vdots\\ r:A\vee B\end{matrix} \quad \dfrac{\begin{matrix}[x:A]\\ \vdots\\ u:C\end{matrix} \quad \vdots}{u' : D} R\mathcal{E} \quad \dfrac{\begin{matrix}[y:B]\\ \vdots \\ v:C\end{matrix}\quad \vdots}{v' : D}R\mathcal{E}}{\delta\ x\cdot u'\ y\cdot v'\ r : D}\vee\mathcal{E}$$

$$\vdots$$

Commutative Conversions: Disjunction vs. Disjunction

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\vee D\end{array}}{C\vee D}\vee\mathcal{E} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{\begin{array}{c}E\\ \vdots\end{array}}\vee\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \quad \dfrac{\begin{array}{c}[A]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\vee\mathcal{E} \quad \dfrac{\begin{array}{c}[B]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\vee\mathcal{E}}{\begin{array}{c}E\\ \vdots\end{array}}\vee\mathcal{E}$$

Commutative Conversions: Disjunction vs. Disjunction

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ t : A\vee B\end{array} \quad \begin{array}{c}[x : A]\\ \vdots\\ u : C\vee D\end{array} \quad \begin{array}{c}[y : B]\\ \vdots\\ v : C\vee D\end{array}}{C\vee D}\vee\mathcal{E} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{\begin{array}{c}E\\ \vdots\end{array}}\vee\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \quad \dfrac{\begin{array}{c}[A]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\vee\mathcal{E} \quad \dfrac{\begin{array}{c}[B]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\vee\mathcal{E}}{\begin{array}{c}E\\ \vdots\end{array}}\vee\mathcal{E}$$

Commutative Conversions: Disjunction vs. Disjunction

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ t : A\vee B\end{array} \quad \begin{array}{c}[x : A]\\ \vdots\\ u : C\vee D\end{array} \quad \begin{array}{c}[y : B]\\ \vdots\\ v : C\vee D\end{array}}{\delta\ x\cdot u\ y\cdot v\ t : C\vee D}\vee\mathcal{E} \quad \begin{array}{c}[x' : C]\\ \vdots\\ u' : E\end{array} \quad \begin{array}{c}[y' : D]\\ \vdots\\ v' : E\end{array}}{\begin{array}{c}E\\ \vdots\end{array}}\vee\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \quad \dfrac{\begin{array}{c}[A]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\vee\mathcal{E} \quad \dfrac{\begin{array}{c}[B]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\vee\mathcal{E}}{\begin{array}{c}E\\ \vdots\end{array}}\vee\mathcal{E}$$

Commutative Conversions: Disjunction vs. Disjunction

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ t : A\vee B\end{array} \quad \begin{array}{c}[x : A]\\ \vdots\\ u : C\vee D\end{array} \quad \begin{array}{c}[y : B]\\ \vdots\\ v : C\vee D\end{array}}{\delta\ x\cdot u\ y\cdot v\ t : C\vee D}\vee\mathcal{E} \quad \begin{array}{c}[x' : C]\\ \vdots\\ u' : E\end{array} \quad \begin{array}{c}[y' : D]\\ \vdots\\ v' : E\end{array}}{\begin{array}{c}\delta\ x'\cdot u'\ y'\cdot v'\ (\delta\ x\cdot u\ y\cdot v\ t) : E\\ \vdots\end{array}}\vee\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \quad \dfrac{\begin{array}{c}[A]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\vee\mathcal{E} \quad \dfrac{\begin{array}{c}[B]\\ \vdots\\ C\vee D\end{array} \quad \begin{array}{c}[C]\\ \vdots\\ E\end{array} \quad \begin{array}{c}[D]\\ \vdots\\ E\end{array}}{E}\vee\mathcal{E}}{\begin{array}{c}E\\ \vdots\end{array}}\vee\mathcal{E}$$

Commutative Conversions: Disjunction vs. Disjunction

$$\dfrac{\dfrac{\overset{\vdots}{t : A \vee B} \quad \overset{[x : A]}{\overset{\vdots}{u : C \vee D}} \quad \overset{[y : B]}{\overset{\vdots}{v : C \vee D}}}{\delta\ x \cdot u\ y \cdot v\ t : C \vee D}\vee\mathcal{E} \quad \overset{[x' : C]}{\overset{\vdots}{u' : E}} \quad \overset{[y' : D]}{\overset{\vdots}{v' : E}}}{\delta\ x' \cdot u'\ y' \cdot v'\ (\delta\ x \cdot u\ y \cdot v\ t) : E}\vee\mathcal{E}$$

$$\vdots$$

$$\dfrac{\overset{\vdots}{t : A \vee B} \quad \dfrac{\overset{[x : A]}{\overset{\vdots}{u : C \vee D}} \quad \overset{[x' : C]}{\overset{\vdots}{u' : E}} \quad \overset{[y' : D]}{\overset{\vdots}{v' : E}}}{\delta\ x' \cdot u'\ y' \cdot v'\ u : E}\vee\mathcal{E} \quad \dfrac{\overset{[y : B]}{\overset{\vdots}{v : C \vee D}} \quad \overset{[x' : C]}{\overset{\vdots}{u' : E}} \quad \overset{[y' : D]}{\overset{\vdots}{v' : E}}}{\delta\ x' \cdot u'\ y' \cdot v'\ v : E}\vee\mathcal{E}}{E}\vee\mathcal{E}$$

$$\vdots$$

Commutative Conversions: Disjunction vs. Disjunction

$$\dfrac{\dfrac{\overset{\vdots}{t : A \vee B} \quad \overset{[x : A]}{\overset{\vdots}{u : C \vee D}} \quad \overset{[y : B]}{\overset{\vdots}{v : C \vee D}}}{\delta\ x \cdot u\ y \cdot v\ t : C \vee D}\vee\mathcal{E} \quad \overset{[x' : C]}{\overset{\vdots}{u' : E}} \quad \overset{[y' : D]}{\overset{\vdots}{v' : E}}}{\delta\ x' \cdot u'\ y' \cdot v'\ (\delta\ x \cdot u\ y \cdot v\ t) : E}\vee\mathcal{E}$$

$$\vdots$$

$$\dfrac{\overset{\vdots}{t : A \vee B} \quad \dfrac{\overset{[x : A]}{\overset{\vdots}{u : C \vee D}} \quad \overset{[x' : C]}{\overset{\vdots}{u' : E}} \quad \overset{[y' : D]}{\overset{\vdots}{v' : E}}}{\delta\ x' \cdot u'\ y' \cdot v'\ u : E}\vee\mathcal{E} \quad \dfrac{\overset{[y : B]}{\overset{\vdots}{v : C \vee D}} \quad \overset{[x' : C]}{\overset{\vdots}{u' : E}} \quad \overset{[y' : D]}{\overset{\vdots}{v' : E}}}{\delta\ x' \cdot u'\ y' \cdot v'\ v : E}\vee\mathcal{E}}{\delta\ x \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ u)\ y \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ v)\ t : E}\vee\mathcal{E}$$

$$\vdots$$

Commutative Reductions

$$\delta\ x' \cdot u'\ y' \cdot v'\ (\delta\ x \cdot u\ y \cdot v\ t)$$
$$\rightsquigarrow \quad \delta\ x \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ u)\ y \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ v)\ t$$

$$\pi_1(\delta\ x \cdot u\ y \cdot v\ t)$$
$$\rightsquigarrow \quad \delta\ x \cdot (\pi_1 u)\ y \cdot (\pi_1 v)\ t$$

$$\pi_2(\delta\ x \cdot u\ y \cdot v\ t)$$
$$\rightsquigarrow \quad \delta\ x \cdot (\pi_2 u)\ y \cdot (\pi_2 v)\ t$$

$$(\delta\ x \cdot u\ y \cdot v\ t) w$$
$$\rightsquigarrow \quad \delta\ x \cdot (uw)\ y \cdot (vw)\ t$$

Commutative Reductions

$$\delta\ x' \cdot u'\ y' \cdot v'\ (\delta\ x \cdot u\ y \cdot v\ t)$$
$$\rightsquigarrow \quad \delta\ x \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ u)\ y \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ v)\ t$$

$$\pi_1(\delta\ x \cdot u\ y \cdot v\ t)$$
$$\rightsquigarrow \quad \delta\ x \cdot (\pi_1 u)\ y \cdot (\pi_1 v)\ t$$

$$\pi_2(\delta\ x \cdot u\ y \cdot v\ t)$$
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Commutative Reductions

$$\delta\ x' \cdot u'\ y' \cdot v'\ (\delta\ x \cdot u\ y \cdot v\ t) \quad \rightsquigarrow \quad \delta\ x \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ u)\ y \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ v)\ t$$

$$\pi_1(\delta\ x \cdot u\ y \cdot v\ t) \quad \rightsquigarrow \quad \delta\ x \cdot (\pi_1 u)\ y \cdot (\pi_1 v)\ t$$

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$$\delta\ x' \cdot u'\ y' \cdot v'\ (\delta\ x \cdot u\ y \cdot v\ t) \quad \rightsquigarrow \quad \delta\ x \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ u)\ y \cdot (\delta\ x' \cdot u'\ y' \cdot v'\ v)\ t$$

$$\pi_1(\delta\ x \cdot u\ y \cdot v\ t) \quad \rightsquigarrow \quad \delta\ x \cdot (\pi_1 u)\ y \cdot (\pi_1 v)\ t$$

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Consequences of the Isomorphism

1. Heyting's Semantics of Proofs
2. The Curry-Howard Isomorphism
   - The Isomorphism
   - Consequences of the Isomorphism

Correspondences

Logic	λ-calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
formula	type	interface
conjunction	Cartesian product	record
disjunction	direct sum	variant
implication	functional type	function type
universal q.	dependent products	
existential q.	dependent sums	
contradiction	empty type	

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The system **F**

[1, Chap. 11],[2, Chap. 6]

- Discovered by Jean-Yves Girard.
- A typed λ-calculus
- Known as the second-order or polymorphic λ-calculus
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- $\vdash \Lambda\alpha \cdot \lambda x^\alpha \cdot x : \forall\alpha \cdot \alpha \to \alpha$

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