Feuille d'exercices n°12

Fractions rationnelles

(du lundi 20 mai 2013 au vendredi 24 mai 2013)

Exercice 1

Décomposer en éléments simples dans $\mathbb{R}(X)$ les fractions rationnelles suivantes :

1. $\dfrac{X^5+1}{X(X-1)^2}$

2. $\dfrac{X^3-1}{(X-1)(X-2)(X-3)}$

3. $\dfrac{X^4+X^2+2}{(X+1)(X+2)^2}$

4. $\dfrac{X^2-4}{(X-1)^2(X+1)^2}$

5. $\dfrac{X^6+1}{(X-1)(X^2+1)^2}$

6. $\dfrac{2X^4+1}{(X-1)^3(X^2+1)}$

Exercice 2

Simplifier les sommes suivantes $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$ et $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$

Feuille exercices 12
Fraction rationelles

Ex1:

① 1) $\dfrac{X^5+1}{X(X-1)^2} = E + \dfrac{a}{X} + \dfrac{b}{(X-1)} + \dfrac{c}{(X-1)^2}$

Valeur interdite : 0 / 1 : singularité

Valeur singularité 0 = 1 / 1 = 2

Si d°(numérateur) ≥ degré (dénominateur) alors
⇒ ∃ une partie entière E

$X(X-1)^2 = X^3 - 2X^2 + X$

$$\begin{array}{l|l} X^5 + 1 & X(X-1)^2 \\ -(X^5 - 2X^4 + X^3) & \underbrace{X^2 + 2X + 3}_{E} \\ \quad 2X^4 - X^3 + 1 & \\ -(2X^4 - 4X^3 + 2X^2) & \\ \quad 3X^3 - 2X^2 + 1 & \\ -(3X^3 - 6X^2 + 3X) & \\ \quad 4X^2 - 3X + 1 & \end{array}$$

s'arrête quand son degré est inférieur au quotient

2) Calcul de a :

$F(X) \cdot X$ et je remplace X par 0

$F(X) \cdot X = E \cdot X$

$\Leftrightarrow \dfrac{X^5+1}{(X-1)^2} = E \cdot X + a + \dfrac{bX}{(X-1)} + \dfrac{cX}{(X-1)^2}$

pour $X=0$ (0 car il annule $\frac{a}{X}$)

$\Rightarrow 1 = a$

3) Calcul de C :

toujours commencer par la + grande singularité.

$$F(X) = \frac{X^5+1}{X(X-1)^2} = E + \frac{a}{X} + \frac{b}{X-1} + \frac{c}{(X-1)^2}$$

$$F(X)\cdot(X-1)^2 = E\cdot(X-1)^2$$

$$\frac{X^5+1}{\cancel{X(X-1)^2}} = E(X-1)^2 + \frac{a(X-1)^2}{X} + b(X-1) + c$$

On remplace X par 1 :

$2 = c$

4) Calcul de B :

$$F(X) = \frac{X^5+1}{X(X-1)^2} = (4X^2-3X+1) + \frac{1}{X} + \frac{b}{X-1} + \frac{2}{(X-1)^2}$$

$$\frac{X^5+1}{X(X-1)^2} - (X^2+2X+3) - \frac{1}{X} - \frac{2}{(X-1)^2} = \frac{b}{(X-1)}$$

X^3-2X^2+X

$$\frac{X^5+1-(X^2+2X+3)X(X-1)^2-(X-1)^2-2X}{X(X-1)^2} = \frac{b}{(X-1)}$$

$$\frac{\cancel{X^5}+1-(\cancel{X^5}-\cancel{2X^4}+\cancel{X^3}+\cancel{2X^4}-\cancel{4X^3}-2X^2+\cancel{3X^3}+6X^2+3X)-X^2+2X-1-2X}{X(X-1)^2}$$

$$\Leftrightarrow \quad \frac{3x^2-3x}{X(x-1)^2} = \frac{b}{(x-1)}$$

$$\Leftrightarrow \quad \frac{3x(x-1)}{x(x-1)^2} = \frac{b}{(x-1)}$$

$$\frac{3}{(x-1)} = \frac{b}{(x-1)}$$

$$b = 3$$

$$\frac{X^5+1}{X(x-1)^2} = (X^2+2X+3) + \frac{1}{X} + \frac{3}{(X-1)} + \frac{2}{(X-1)^2}$$

(2) 1) $$\frac{X^3-1}{(X-1)(X-2)(X-3)} = E + \frac{a}{(x-1)} + \frac{b}{(x-2)} + \frac{c}{(x-3)}$$

$E = 1$ car $d^\circ(\text{num}) = d^\circ(\text{denom.})$

2) Calcul de a

$$F(x).(X-1) = E(X-1)$$

$$\frac{X^3-1}{(x-2)(x-3)} = E(x-1) + a + \frac{b(x-1)}{(x-2)} + \frac{c(x-1)}{(x-3)}$$

Pour $x = 1$:

$$0 = a$$

3) Calcul de b :

$$F(x) = C \ldots$$

$$F(x) \cdot (x-2) = C(x-2) \ldots$$

$$\frac{(x^3-1)}{(x-1)(x-3)} = C(x-2) + \frac{a(x-2)}{(x-1)} + b + \frac{c(x-2)}{(x-3)}$$

pour $x = 2$:

$$-7 = b$$

4) Calcul de c :

$$F(x) = C \ldots$$

$$F(x)(x-3) = C(x-3) \ldots$$

$$\frac{(x^3-1)}{(x-1)(x-2)} = C(x-3) \frac{a(x-3)}{(x-1)} + \frac{b(x-3)}{(x-2)} + c .$$

pour $x = 3$:

$$\frac{26}{2} = 13 = c .$$

Suite ② :

$$F(x) = 1 - \frac{7}{(x-2)} + \frac{13}{(x-3)}$$

③ 1) $$\frac{X^4 + X^2 + 2}{(x+1)(x+2)^2} = E + \frac{a}{(x+1)} + \frac{b}{(X+2)} + \frac{c}{(x+2)^2}$$

$$(X+1)(X^2+4X+4) = X^3 + 4X^2 + 4X + X^2 + 4X + 4$$
$$= X^3 + 3X^2 + 8X + 4$$

$$\begin{array}{l|l} X^4 + X^2 + 2 & X^3 + 5X^2 + 8X + 4 \\ \hline -(X^4 + 5X^3 + 6X^2 + 8X) & X - 5 \\ \hline -5X^3 - 7X^2 - 8X + 2 & \\ -(-5X^3 - 25X^2 - 24X - 12) & \\ \hline 18X^2 + 16X + 10 & \end{array}$$

Calcul de a :

$F(X) = E \ldots$

$F(X)(x+1) = E(x+1) \ldots$

$$\frac{X^4 + X^2 + 2}{(x+2)^2} = E(x+1) + a + \frac{b(x+1)}{(x+2)} + \frac{c(x+1)}{(x+2)^2}$$

Pour $X = -1$:

$4 = a$

Calcul de C :

$$F(x) = E \ldots$$

$$F(x)\cdot(x+2)^2 = E(x+2)^2 \ldots$$

$$\frac{x^4+x^2+2}{(x+1)} = E(x+2)^2 + \frac{a(x+2)^2}{(x+1)} + \frac{b(x+2)^2}{(x+2)} + C$$

Pour $x=-2$:

$$\frac{22}{-1} = -22 = C$$

Calcul de B :

$$F(x) = \frac{(x^4+x^2+2)}{(x+1)(x+2)^2} = (x-5) + \frac{4}{(x+1)} + \frac{b}{(x+2)} - \frac{22}{(x+2)^2}$$

$(x-5)($

$$= \frac{x^4+x^2+2}{(x+1)(x+2)^2} = \frac{(x-5)(x+1)(x+2)^2}{(x+1)(x+2)^2} + \frac{4(x+2)^2}{(x+1)(x+2)^2} + \frac{b(x+2)}{(x+1)(x+2)} - \frac{22(x+1)}{(x+1)(x+2)^2}$$

Ou autre méthode :

Soit $x=0$ et $x=1$

$x=0$: $\frac{2}{4} = \frac{1}{2} = -5 + 4 + \frac{b}{2} - \frac{22}{4}$

$$2 = -4 + 2b - 22$$

$$b = 14$$

(1) $$\frac{x^2-4}{(x-1)^2(x+1)^2} = E + \frac{a}{(x-1)} + \frac{b}{(x-1)^2} + \frac{c}{(x+1)} + \frac{d}{(x+1)^2}$$

↳ degré (num) < degré (denom) → pas de $E = 0$.

Les pôles : 1 double

Calcul de b :

$$F(x)(x-1)^2 = a(x-1) + b + \frac{c(x-1)^2}{(x+1)} + \frac{d(x-1)^2}{(x+1)^2}$$

pour $x = 1$,

$$\frac{x^2-4}{(x+1)^2} = b \Rightarrow -\frac{3}{4} = b.$$

Calcul de d

$$F(x)(x+1)^2 = \frac{a(x+1)^2}{(x-1)} + \frac{b(x+1)^2}{(x-1)^2} + \frac{c(x+1)^2}{(x+1)} + d$$

Pour $x = -1$:

$$\frac{-3}{4} = d$$

Calcul de a/c

Soit $x = 0$

$$F(0) = -4 = \frac{a}{-1} + \frac{\left(-\frac{3}{4}\right)}{1} + \frac{c}{1} + \frac{\left(-\frac{3}{4}\right)}{1}$$

$$-4 = -a + c - \frac{6}{4}$$

$$xF(x) = \frac{x^2-4}{(x-1)^2(x+1)^2}\,x$$

$$= \frac{a}{x-1}\,x + \frac{b}{(x-1)^2}\,x + \frac{c\,x}{x+1} + \frac{d\,x}{(x+1)^2}$$

$\sim 1 \qquad \sim 1/x \qquad \sim 1 \qquad \sim 1/x$

Lorsque $x \to +\infty$

$$\begin{cases} 0 = a + c \\ c - a = -\frac{5}{2} \end{cases} \iff \begin{cases} a = -c \\ 2c = -\frac{5}{2} \end{cases} \quad \begin{cases} a = \frac{5}{4} \\ c = -\frac{5}{4} \end{cases}$$

(5) $$\frac{x^6+1}{(x-1)(x^2+1)^2}$$

$$= \frac{(x^2+1)(x^4-x^2+1)}{(x-1)(x^2+1)^2}$$

$$\boxed{= \frac{(x^2 \cdot x^2+1)}{(x-1)(x^2+1)}}$$

$$F(x) = c + \frac{a}{x-1} + \frac{bx+c}{x^2+1}$$

$$(x-1)(x^2+1) = x^3 + x - x^2 - 1$$
$$= x^3 - x^2 + x - 1$$

$$\begin{array}{r|l} x^4 - x^2 + 1 & x^3 - x^2 + x - 1 \\ \hline -(x^4 - x^3 + x^2 - x) & x + 1 \\ \hline x^3 - 2x^2 + x + 1 & \\ -(x^3 - x^2 + x - 1) & \\ \hline -x^2 + 2 & \end{array}$$

$$F(x) = x+1 + \frac{a}{x-1} + \frac{bx+c}{x^2+1}$$

Calcul de a :

$$F(x)(x-1)\Big|_{x=1} \to a = \frac{1}{2}$$

Calcul de b et c :

$$F(x)(x^2+1)\Big|_{x=i} \to b, c$$

$$F(x)(x^2+1) = (x^2+1)(x+1) + \frac{a(x^2+1)}{(x-1)} + (bx+c)$$

$$= \frac{x^4 - x^2 + 1}{(x-1)}$$

Pour $x = i$:

$$\frac{(i)^4 - (i)^2 + 1}{i-1} = bi + c$$

$$\frac{3}{i-1} = bi + c$$

$$\Leftrightarrow \frac{3(-1-i)}{(-1)^2 - (i)^2} = bi + c$$

$$\frac{-3-3i}{2} = bi + c$$

2 complexes sont égaux si :

$$\begin{cases} -\frac{3}{2} = c \\ -\frac{3}{2} = b \end{cases}$$

②

$$F(x) = \frac{2x^4+1}{(x-1)^3(x^2+1)}$$

$$= \frac{a}{(x-1)} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{dx+e}{(x^2+1)}$$

algébrique
fonctionnelle
linéaire

Calcul de c :

$$F(x)(x-1)^3 \Big|_{x=1}$$

$$\frac{3}{2} = c$$

Calcul de d et e :

$$F(x)(x^2+1) \Big|_{x=i}$$

$$\frac{2x^4+1}{(x-1)^3} = dx+e + \frac{a(x^2+1)}{(x-1)} + \frac{b(x^2+1)}{(x-1)^2} + \frac{c(x^2+1)}{(x-1)^3}$$

$$\frac{3}{(i-1)^3} = di+e$$

$$\frac{2-2i}{2-2i} \times \frac{3}{2+2i} = di+e$$

$$\Leftrightarrow \frac{6-6i}{8} = di+e$$

$\Rightarrow \frac{-3}{4}i + \frac{3}{4} = di + e$

$$\begin{cases} d = -\frac{3}{4} \\ e = \frac{3}{4} \end{cases}$$

Calcul de a et b :

$xF(x) \underset{+\infty}{\sim} \frac{2x^5}{x^5} \sim \frac{ax}{x} + \frac{bx}{x^2} + \frac{cx}{x^2} + \frac{dx^2}{x^2} + \frac{ex}{x^2}$

$\xrightarrow[x \to \infty]{} 2 = a + d$

$$\begin{cases} a + d = 2 \\ -1 = -a + b - c + e \end{cases}$$

$F(0) \rightarrow$

$$\begin{cases} a - \frac{3}{4} = 2 \iff a = 2 + \frac{3}{4} = \frac{11}{4} \\ -1 = -a + b - \frac{3}{2} + \frac{3}{4} \end{cases}$$

$-\frac{4}{4} = -\frac{11}{4} + b - \frac{6}{4} + \frac{3}{4}$

$b = -\frac{4}{4} + \frac{11}{4} + \frac{6}{4} - \frac{3}{4} = \frac{5}{2}$

$F(x) = \frac{11}{4(x-1)} + \frac{5}{2(x-1)^2} + \frac{3}{2(x-1)^3} + \frac{-\frac{3}{4}x + \frac{3}{4}}{(x^2+1)}$

$\Rightarrow$

Exercice 2 :

1/ Simplifier $\sum_{k=1}^{n} \frac{1}{k(k+1)}$

$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}$$

$$a = \frac{1}{k+1} - \frac{bk}{k+1} \quad \Big|_{k=0} \Rightarrow a = 1$$

$$b = \frac{1}{k} - \frac{a(k+1)}{k} \quad \Big|_{k=-1} \Rightarrow b = -1$$

$$\Rightarrow \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

2/ $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} =$

$$\frac{1}{k(k+1)(k+2)} = \frac{a}{k} + \frac{b}{(k+1)} + \frac{c}{(k+2)}$$

$$\Rightarrow \begin{cases} a = \frac{1}{2} \\ b = -1 \\ c = \frac{1}{2} \end{cases}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} \right)$$

$$\frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k+2}$$

$$= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} - \sum_{K=2}^{n+1} \frac{1}{K} + \frac{1}{2}\sum_{K'=3}^{n+2} \frac{1}{K'} \quad k+1 = K \quad k+2 = K'$$

$$= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} + \frac{1}{2}\sum_{k=3}^{n+2} \frac{1}{k}$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{2}\sum_{k=3}^{n} \frac{1}{k} - \left[\left(\frac{1}{2} + \frac{1}{n+1}\right) + \sum_{k=3}^{n} \frac{1}{k}\right]$$

$$+ \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right) + \frac{1}{2}\sum_{k=3}^{n} \frac{1}{k}$$

$$= \frac{3}{4} + \frac{1}{2}\sum_{k=3}^{n} \frac{1}{k} - \frac{1}{2} - \frac{1}{n+1} - \sum_{k=3}^{n} \frac{1}{k}$$

$$+ \frac{1}{2(n+1)} + \frac{1}{2(n+2)} + \frac{1}{2}\sum_{k=3}^{n} \frac{1}{k}$$

$$= \frac{1}{4} + \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right) - \frac{1}{n+1}$$