# The Curry-Howard Isomorphism

Akim Demaille akim@lrde.epita.fr

EPITA — École Pour l'Informatique et les Techniques Avancées

March 22, 2009



# Heyting's Semantics of Proofs

- Heyting's Semantics of Proofs
- 2 The Curry-Howard Isomorphism

Heyting's Semantics of Proofs
The Curry-Howard Isomorphism

# The Curry-Howard Isomorphism

- Heyting's Semantics of Proofs
- 2 The Curry-Howard Isomorphism

4□ ト 4回 ト 4 直 ト 4 直 ト 9 へ (\*)

A. Demaille

The Curry-Howard Isomorphis

2 / 22

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

# Functional Interpretation

[2, Chap. 5]

- Instead of "when is a sentence A true"
- ask "what is a proof of A"?

#### The Curry-Howard Isomorphis

# Functional Interpretation [1, Sec. 1.2.2]

What is a proof  $\pi$  of A?  $(\pi \vdash A)$ 

$$A \wedge B$$
 A pair  $(\pi_A, \pi_B)$  s.t.  $\pi_A \vdash A$  and  $\pi_B \vdash B$ 

The Curry-Howard Isomorphism

Heyting's Semantics of Proofs
The Curry-Howard Isomorphism

#### Functional Interpretation

[1, Sec. 1.2.2]

What is a proof  $\pi$  of A?  $(\pi \vdash A)$ 

 $A \wedge B$  A pair  $(\pi_A, \pi_B)$  s.t.  $\pi_A \vdash A$  and  $\pi_B \vdash B$ 

 $A \vee B$  A pair  $(i, \pi)$  s.t.

$$i=0$$
  $\pi\vdash A$ 

$$i=1$$
  $\pi \vdash B$ 

 $A \Rightarrow B$  A function f s.t. if  $\pi \vdash A$  then  $f(\pi) \vdash B$ 

Heyting's Semantics of Proofs
The Curry-Howard Isomorphism

#### **Functional Interpretation**

[1, Sec. 1.2.2]

What is a proof  $\pi$  of A?  $(\pi \vdash A)$ 

 $A \wedge B$  A pair  $(\pi_A, \pi_B)$  s.t.  $\pi_A \vdash A$  and  $\pi_B \vdash B$ 

 $A \vee B$  A pair  $(i, \pi)$  s.t.

$$i = 0 \pi \vdash A$$

$$i=1$$
  $\pi \vdash B$ 

A. Demaille

The Curry-Howard Isomorphism

F / 20

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

#### **Functional Interpretation**

[1, Sec. 1.2.2]

What is a proof  $\pi$  of A?  $(\pi \vdash A)$ 

 $A \wedge B$  A pair  $(\pi_A, \pi_B)$  s.t.  $\pi_A \vdash A$  and  $\pi_B \vdash B$ 

 $A \vee B$  A pair  $(i, \pi)$  s.t.

$$i=0 \pi \vdash A$$

$$i=1 \pi \vdash B$$

 $A \Rightarrow B$  A function f s.t. if  $\pi \vdash A$  then  $f(\pi) \vdash B$ 

 $\forall x \cdot A$  A function f s.t.  $f(a) \vdash A[a/x]$ 

ロト 4回 ト 4 重 ト 4 重 ト 1 重 りのの

# **Functional Interpretation**

[1, Sec. 1.2.2]

What is a proof  $\pi$  of A?  $(\pi \vdash A)$ 

$$A \wedge B$$
 A pair  $(\pi_A, \pi_B)$  s.t.  $\pi_A \vdash A$  and  $\pi_B \vdash B$ 

$$A \vee B$$
 A pair  $(i, \pi)$  s.t.

$$i=0$$
  $\pi\vdash A$ 

$$i=1 \pi \vdash B$$

$$A \Rightarrow B$$
 A function f s.t. if  $\pi \vdash A$  then  $f(\pi) \vdash B$ 

$$\forall x \cdot A$$
 A function  $f$  s.t.  $f(a) \vdash A[a/x]$ 

$$\exists x \cdot A$$
 A pair  $(a, \pi)$  s.t.  $\pi \vdash A[a/x]$ 

The Curry-Howard Isomorphism

Heyting's Semantics of Proofs

#### **Functional Interpretation**

[1, Sec. 1.2.2]

What is a proof  $\pi$  of A?  $(\pi \vdash A)$ 

Atomic Values Assume we know what a proof is

$$A \wedge B$$
 A pair  $(\pi_A, \pi_B)$  s.t.  $\pi_A \vdash A$  and  $\pi_B \vdash B$ 

$$A \vee B$$
 A pair  $(i, \pi)$  s.t.

$$i = 0 \pi \vdash A$$

$$i=1 \pi \vdash B$$

$$A \Rightarrow B$$
 A function f s.t. if  $\pi \vdash A$  then  $f(\pi) \vdash B$ 

$$\forall x \cdot A$$
 A function  $f$  s.t.  $f(a) \vdash A[a/x]$ 

$$\exists x \cdot A$$
 A pair  $(a, \pi)$  s.t.  $\pi \vdash A[a/x]$ 

The Curry-Howard Isomorphisn

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

#### Heyting's Semantics of Proofs

An informal interpretation

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

# Heyting's Semantics of Proofs

- An informal interpretation
- ullet The handling of  $\vee$  and  $\exists$  are similar to the disjunctive and existential properties

The Curry-Howard Isomorphism

Heyting's Semantics of Proofs

# Heyting's Semantics of Proofs

- An informal interpretation
- ullet The handling of  $\vee$  and  $\exists$  are similar to the disjunctive and existential properties
- Therefore refers to a cut-free proof

The Curry-Howard Isomorphism

# The Curry-Howard Isomorphism

- Heyting's Semantics of Proofs
- 2 The Curry-Howard Isomorphism
  - The Isomorphism
  - Consequences of the Isomorphism

# Heyting's Semantics of Proofs

- An informal interpretation
- ullet The handling of  $\vee$  and  $\exists$  are similar to the disjunctive and existential properties
- Therefore refers to a cut-free proof
- For instance id is a proof of  $A \Rightarrow A$

Heyting's Semantics of Proofs

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

# A Striking Correspondence

#### Type Derivations

$$\frac{M:\sigma\to\tau\quad N:\sigma}{MN:\tau}$$
 app

$$\begin{bmatrix} x : \sigma \\ \vdots \\ M : \tau \\ \hline \lambda x \cdot M : \sigma \to \tau \end{bmatrix} \text{ abs }$$

#### Natural Deduction

$$\frac{A \Rightarrow B \quad A}{B} \Rightarrow \mathcal{E}$$

$$\begin{array}{c}
[A] \\
\vdots \\
B \\
A \Rightarrow B
\end{array} \Rightarrow \mathcal{I}$$

## A Striking Correspondence

#### Type Derivations

$$\frac{M: \sigma \to \tau \quad N: \sigma}{MN: \tau} \operatorname{app}$$

$$\begin{bmatrix} x : \sigma \\ \vdots \\ M : \tau \\ \hline \lambda x \cdot M : \sigma \to \tau \end{bmatrix} \text{ abs }$$

#### Natural Deduction

$$\frac{A \Rightarrow B}{B} \Rightarrow \mathcal{E}$$

$$\begin{array}{c}
[A] \\
\vdots \\
B \\
A \Rightarrow B
\end{array} \Rightarrow \mathcal{I}$$

The Curry-Howard Isomorphism

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism

#### The Isomorphism [1]

#### Curry-Howard Isomorphism

There is a perfect equivalence between two viewpoints:

**Natural Deduction** 

Formulas A, deductions of A, normalization in natural deduction

Typed  $\lambda$ -calculus

Types A, terms of type A, normalization in  $\lambda$ -calculus.

#### The Isomorphism

- Heyting's Semantics of Proofs
- 2 The Curry-Howard Isomorphism
  - The Isomorphism
  - Consequences of the Isomorphism

Heyting's Semantics of Proofs
The Curry-Howard Isomorphism

#### **Deductions and Terms: Conjuction**

$$A^{i}$$

$$\vdots \qquad \vdots$$

$$\frac{A \qquad B}{A \wedge B} \wedge \mathcal{I}$$

$$\vdots$$

$$\frac{A \wedge B}{A} \wedge 1\mathcal{E}$$

$$\vdots$$

$$\frac{A \wedge B}{B} \wedge 2\mathcal{E}$$

#### Deductions and Terms: Conjuction

$$x_{i}^{A}: A^{i}$$

$$\vdots \qquad \vdots$$

$$u: A \qquad v: B$$

$$\langle u, v \rangle : A \wedge B$$

$$\vdots$$

$$u: A \wedge B$$

$$\pi_{1}u: A$$

$$\vdots$$

$$u: A \wedge B$$

$$\vdots$$

$$u: A \wedge B$$

$$\pi_{2}u: B$$

The Curry-Howard Isomorphism

Reductions

The Curry-Howard Isomorphism

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism

#### Reductions

$$\frac{u:A \qquad v:B}{\frac{\langle u,v\rangle:A\wedge B}{\pi_1\langle u,v\rangle:A}\wedge 1\mathcal{E}} \wedge \mathcal{I} \qquad \qquad \vdots \\
\frac{u:A \qquad v:B}{\pi_1\langle u,v\rangle:A} \wedge \mathcal{I}\mathcal{E} \qquad \qquad \qquad \vdots \\
\vdots \qquad \qquad \qquad \qquad \qquad \qquad \vdots$$

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism

#### Reductions

$$\frac{u:A \qquad v:B}{\frac{\langle u,v\rangle:A\wedge B}{\pi_1\langle u,v\rangle:A}\wedge 1\mathcal{E}} \wedge \mathcal{I} \qquad \Rightarrow \qquad u:A$$

$$\vdots$$

$$\pi_1\langle u, v \rangle \rightsquigarrow u$$
  
 $\pi_2\langle u, v \rangle \rightsquigarrow v$ 

The Isomorphism

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

Deductions and Terms: Implication

The Isomorphism

# Deductions and Terms: Implication

# $\begin{array}{ccc} [A]^{i} & & \vdots & \vdots \\ \frac{\dot{B}}{A \to B} \Rightarrow \mathcal{I}_{i} & & \frac{A \Rightarrow B}{B} & & A \\ & & & & & & & & & & & & \\ \end{array}$

= 4) d (4

A. Demaille

The Curry-Howard Isomorphism

13 / 22

Heyting's Semantics of Proofs The Curry-Howard Isomorphism The Isomorphism

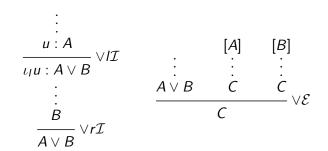
The Isomorphism

Heyting's Semantics of Proofs The Curry-Howard Isomorphism The Isomorphism Consequences of the Isomorphism

#### Disjunction

$$\frac{A}{A \vee B} \vee I\mathcal{I} \qquad \qquad [A] \qquad [B] \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
A \vee B \qquad C \qquad C \\
C \qquad \qquad C$$

# Disjunction

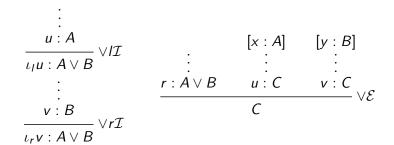


#### Disjunction

$$\frac{u : A}{\iota_{I}u : A \vee B} \vee I\mathcal{I} \qquad [A] \qquad [B] \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
\frac{V : B}{\iota_{r}v : A \vee B} \vee r\mathcal{I}$$

The Curry-Howard Isomorphism

Disjunction

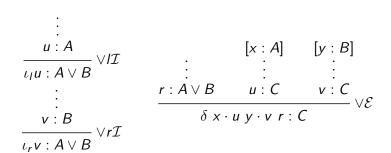


Heyting's Semantics of Proofs
The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

#### Disjunction

A. Demaille



Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

#### Reductions

◆ロ > ◆昼 > ◆差 > を差 > を め へ ○

The Curry-Howard Isomorphism

A. Demaille

The Curry-Howard Isomorphism

◆ロ > ◆昼 > ◆ き > ◆ き \* り へ の



#### The Isomorphism

#### Reductions

$$\frac{r:A}{\iota_{I}r:A\vee B}\vee I\mathcal{I} \qquad \vdots \qquad \vdots \qquad \vdots \qquad A \\
 u:C \qquad v:C \qquad \vee :C \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

4□ > 4□ > 4 = > 4 = > = 90

The Isomorphism

Heyting's Semantics of Proofs
The Curry-Howard Isomorphism

#### Reductions

$$\frac{r:A}{\iota_{I}r:A\vee B}\vee I\mathcal{I} \qquad \vdots \qquad \vdots \qquad r:A \\
\underline{\iota_{I}r:A\vee B} \qquad u:C \qquad v:C \\
\delta \times u \text{ } y\cdot v \text{ } (\iota_{I}r):C \qquad \vdots \qquad u[r/x]:C$$

#### Reductions

$$\frac{r:A}{\iota_{l}r:A\vee B}\vee l\mathcal{I} \qquad \vdots \qquad \vdots \qquad \vdots \qquad A \\
\underline{\iota_{l}r:A\vee B} \vee l\mathcal{I} \qquad \vdots \qquad \vdots \qquad A \\
\underline{\delta \times \iota u \ y \cdot v \ (\iota_{l}r):C} \qquad \vee \mathcal{E} \qquad \sim \qquad \vdots \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
C$$

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism

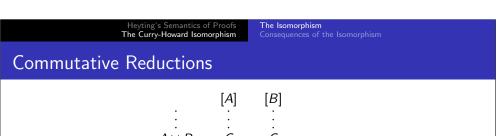
Heyting's Semantics of Proofs The Curry-Howard Isomorphism

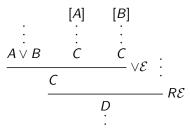
The Isomorphism

#### Reductions

A. Demaille

$$\delta x \cdot u y \cdot v (\iota_l r) \rightsquigarrow u[r/x] 
\delta x \cdot u y \cdot v (\iota_r s) \rightsquigarrow v[s/y]$$



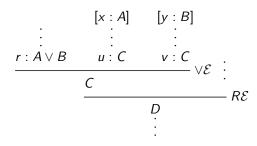


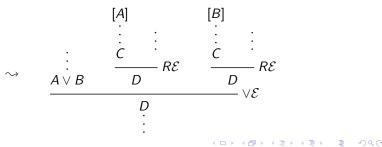
The Isomorphism
Consequences of the Isomorphism

#### Commutative Reductions

Heyting's Semantics of Proofs The Curry-Howard Isomorphism The Isomorphism

#### Commutative Reductions





A. Demaille

The Curry-Howard Isomorph

Heyting's Semantics of Proofs The Curry-Howard Isomorphism The Isomorphism

#### Commutative Reductions

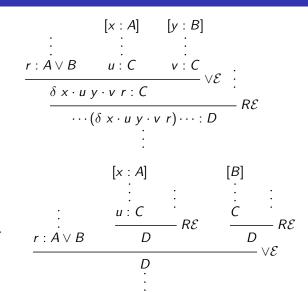
A. Demaille

The Curry-Howard Isomorphism

16 / 22

<ロ> <回> <回> <=> <=> <=> の<</p>

#### Commutative Reductions



7 ti Demaine

The Curry-Howard Isomorphis

16 / 22

#### Commutative Reductions

Heyting's Semantics of Proofs The Curry-Howard Isomorphism The Isomorphism
Consequences of the Isomorphism

#### Commutative Reductions

$$[x : A] \quad [y : B]$$

$$\vdots \quad \vdots \quad \vdots$$

$$r : A \lor B \quad u : C \quad v : C$$

$$\frac{\delta x \cdot u \ y \cdot v \ r : C}{\cdots (\delta x \cdot u \ y \cdot v \ r) \cdots : D} R \mathcal{E}$$

$$\vdots \quad \vdots \quad \vdots$$

$$[x : A] \quad [y : B]$$

$$\vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots$$

$$u : C \quad \vdots$$

$$u : C \quad v : C$$

$$r : A \lor B \quad u' : D \quad R \mathcal{E} \quad v : C$$

$$D \quad \vdots$$

$$D \quad \vdots$$

$$\vdots$$

$$The Curry-Howard Isomorphism$$

Heyting's Semantics of Proofs
The Curry-Howard Isomorphism

The Isomorphism

#### Commutative Reductions

#### Commutative Conversions: Disjunction vs. Disjunction

#### Commutative Conversions: Disjunction vs. Disjunction

4 D > 4 A > 4 B > 4 B > B = 40 Q Q

## Commutative Conversions: Disjunction vs. Disjunction

## Commutative Conversions: Disjunction vs. Disjunction

#### Commutative Conversions: Disjunction vs. Disjunction

#### Commutative Conversions: Disjunction vs. Disjunction

Heyting's Semantics of Proofs
The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

#### Commutative Reductions

$$\delta x' \cdot u' \ y' \cdot v' \ (\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\delta x' \cdot u' \ y' \cdot v' \ u) \ y \cdot (\delta x' \cdot u' \ y' \cdot v' \ v) \ t 
\pi_1(\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\pi_1 u) \ y \cdot (\pi_1 v) \ t 
\pi_2(\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\pi_2 u) \ y \cdot (\pi_2 v) \ t 
(\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (uw) \ y \cdot (vw) \ t$$

The Curry-Howard Isomorphism

#### Commutative Reductions

$$\delta x' \cdot u' \ y' \cdot v' \ (\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\delta x' \cdot u' \ y' \cdot v' \ u) \ y \cdot (\delta x' \cdot u' \ y' \cdot v' \ v) \ t 
\pi_1(\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\pi_1 u) \ y \cdot (\pi_1 v) \ t 
\pi_2(\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\pi_2 u) \ y \cdot (\pi_2 v) \ t 
(\delta x \cdot u \ y \cdot v \ t) w 
\sim \delta x \cdot (uw) \ y \cdot (vw) \ t$$

#### Commutative Reductions

$$\delta x' \cdot u' \ y' \cdot v' \ (\delta x \cdot u \ y \cdot v \ t)$$

$$\sim \delta x \cdot (\delta x' \cdot u' \ y' \cdot v' \ u) \ y \cdot (\delta x' \cdot u' \ y' \cdot v' \ v) \ t$$

$$\pi_1(\delta x \cdot u \ y \cdot v \ t)$$

$$\sim \delta x \cdot (\pi_1 u) \ y \cdot (\pi_1 v) \ t$$

$$\pi_2(\delta x \cdot u \ y \cdot v \ t)$$

$$\sim \delta x \cdot (\pi_2 u) \ y \cdot (\pi_2 v) \ t$$

$$(\delta x \cdot u \ y \cdot v \ t)_W$$

$$\sim \delta x \cdot (uw) \ y \cdot (vw) \ t$$

The Curry-Howard Isomorphism

#### Commutative Reductions

$$\delta x' \cdot u' \ y' \cdot v' \ (\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\delta x' \cdot u' \ y' \cdot v' \ u) \ y \cdot (\delta x' \cdot u' \ y' \cdot v' \ v) \ t 
\pi_1(\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\pi_1 u) \ y \cdot (\pi_1 v) \ t 
\pi_2(\delta x \cdot u \ y \cdot v \ t) 
\sim \delta x \cdot (\pi_2 u) \ y \cdot (\pi_2 v) \ t 
(\delta x \cdot u \ y \cdot v \ t) w 
\sim \delta x \cdot (uw) \ y \cdot (vw) \ t$$

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

# Consequences of the Isomorphism

- Heyting's Semantics of Proofs
- 2 The Curry-Howard Isomorphism
  - The Isomorphism
  - Consequences of the Isomorphism

#### Correspondences

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
	redex	function call etc.
	reduction	execution step
	normal form	value
	type	interface
	Cartesian product	record
	direct sum	variant
	functional type	function type
	dependent products	
	dependent sums	
	empty type	

The Isomorphism
Consequences of the Isomorphism

#### Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

## Correspondences

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
	reduction	execution step
	normal form	value
	type	interface
	Cartesian product	record
	direct sum	variant
	functional type	function type
	dependent products	
	dependent sums	
	empty type	

The Curry-Howard Isomorphism

イロト イ団ト イミト イミト 一意

20 / 22

A. Demaille

The Curry-Howard Isomorphism

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism Consequences of the Isomorphism

#### Correspondences

A. Demaille

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
	type	interface
	Cartesian product	record
	direct sum	variant
	functional type	function type
	dependent products	
	dependent sums	
	empty type	

 $\lambda$ -calculus **Programs** Logic strongly normalizable term proof halting program redex function call etc. cut cut elimination reduction execution step normal form value interface type

Cartesian product record direct sum variant functional type function type dependent products dependent sums

empty type

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

#### Correspondences

Correspondences

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
formula	type	interface
	Cartesian product	record
	direct sum	variant
	functional type	function type
	dependent products	
	dependent sums	
	empty type	

A. Demaille The Curry-Howard Isomorphism

20 / 22

A. Demaille

The Isomorphism
Consequences of the Isomorphism

## Heyting's Semantics of Proofs The Curry-Howard Isomorphism

Consequences of the Isomorphism

# Correspondences

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
formula	type	interface
conjunction	Cartesian product	record
	direct sum	variant
	functional type	function type
	dependent products	
	dependent sums	
	empty type	

4□ > 4圖 > 4 = > 4 = > = 900

A. Demaille

The Curry-Howard Isomorphism

20 / 22

The Isomorphism
Consequences of the Isomorphism

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

# Correspondences

A. Demaille

	· · ·	_
Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
formula	type	interface
conjunction	Cartesian product	record
disjunction	direct sum	variant
implication	functional type	function type
universal q.	dependent products	
	dependent sums	
	empty type	
	dependent sums	

#### ◆□▶ ◆□▶ ◆■▶ ◆■ ● 釣۹@

# Correspondences

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
formula	type	interface
conjunction	Cartesian product	record
disjunction	direct sum	variant
	functional type	function type
	dependent products	
	dependent sums	
	empty type	

The Curry-Howard Isomorphism

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

# Correspondences

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
formula	type	interface
conjunction	Cartesian product	record
disjunction	direct sum	variant
implication	functional type	function type
universal q.	dependent products	
	dependent sums	
	empty type	

**◆□▶◆□▶◆■▶◆■▶ ■ り९◎** 

20 / 22 The Curry-Howard Isomorphism

A. Demaille

The Isomorphism
Consequences of the Isomorphism

#### Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

#### Correspondences

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
formula	type	interface
conjunction	Cartesian product	record
disjunction	direct sum	variant
implication	functional type	function type
universal q.	dependent products	
existential q.	dependent sums	
contradiction	empty type	

The Curry-Howard Isomorphism

20 / 22

Correspondences

Logic	$\lambda$ -calculus	Programs
proof	strongly normalizable term	halting program
cut	redex	function call etc.
cut elimination	reduction	execution step
cut-free proof	normal form	value
formula	type	interface
conjunction	Cartesian product	record
disjunction	direct sum	variant
implication	functional type	function type
universal q.	dependent products	
existential q.	dependent sums	
contradiction	empty type	

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism Consequences of the Isomorphism

# The system **F**

[1, Chap. 11],[2, Chap. 6]

- Discovered by Jean-Yves Girard.

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

# The system **F**

[1, Chap. 11],[2, Chap. 6]

- Discovered by Jean-Yves Girard.
- A typed  $\lambda$ -calculus

A. Demaille

The Isomorphism
Consequences of the Isomorphism

The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

# The system **F**

[1, Chap. 11],[2, Chap. 6]

- Discovered by Jean-Yves Girard.
- A typed  $\lambda$ -calculus
- Known as the second-order or polymorphic  $\lambda$ -calculus

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

# The system **F**

[1, Chap. 11],[2, Chap. 6]

- Discovered by Jean-Yves Girard.
- A typed  $\lambda$ -calculus
- Known as the second-order or polymorphic  $\lambda$ -calculus
- Formalizes the notion of parametric polymorphism in programming languages
- Corresponds to a second-order logic via Curry-Howard

# The system **F**

[1, Chap. 11],[2, Chap. 6]

- Discovered by Jean-Yves Girard.
- A typed  $\lambda$ -calculus
- Known as the second-order or polymorphic  $\lambda$ -calculus
- Formalizes the notion of parametric polymorphism in programming languages

Heyting's Semantics of Proofs The Curry-Howard Isomorphism

The Isomorphism
Consequences of the Isomorphism

The system **F** 

[1, Chap. 11],[2, Chap. 6]

- Discovered by Jean-Yves Girard.
- A typed  $\lambda$ -calculus
- Known as the second-order or polymorphic  $\lambda$ -calculus
- Formalizes the notion of parametric polymorphism in programming languages
- Corresponds to a second-order logic via Curry-Howard
- $\bullet \vdash \Lambda \alpha \cdot \lambda x^{\alpha} \cdot x : \forall \alpha \cdot \alpha \rightarrow \alpha$

The Curry-Howard Isomorphism

A. Demaille

# Bibliography I



J.-Y. Girard, Y. Lafont, and P. Taylor.

Proofs and Types.

Cambridge University Press, 1989.

http:

//www.cs.man.ac.uk/~pt/stable/Proofs+Types.html.



Jean-Yves Girard.

Cours de Logique, Rome, Automne 2004.

http://logica.uniroma3.it/uif/corso/, 2004.

