Intuitionistic Logic

Akim Demaille akim@lrde.epita.fr

EPITA — École Pour l'Informatique et les Techniques Avancées

March 22, 2009

Intuitionistic Logic

Constructivity

Constructivity

- Classical logic does not build truth
- it discovers a preexisting truth
- Classical logic assumes facts are either true or false
- $\vdash A \vee \neg A$ Excluded middle, *tertium non datur*

Constructivity

- Classical logic does not build truth
- it discovers a preexisting truth
- Classical logic assumes facts are either true or false
- $\vdash A \vee \neg A$ Excluded middle, *tertium non datur*

Constructivity
Intuitionistic Logic

Constructivity

- Classical logic does not build truth
- it discovers a preexisting truth
- Classical logic assumes facts are either true or false
- $\vdash A \vee \neg A$ Excluded middle, *tertium non datur*

A. Demaille Intuitionistic Logic 4 / 30

Constructivity
Intuitionistic Logic

Constructivity

- Classical logic does not build truth
- it discovers a preexisting truth
- Classical logic assumes facts are either true or false
- $\vdash A \vee \neg A$ Excluded middle, *tertium non datur*

A. Demaille Intuitionistic Logic 4 / 30

Constructivity
Intuitionistic Logic

Excluded Middle

$$\dfrac{}{A \vee \neg A}\,\mathrm{XM}$$

$$\dfrac{\begin{array}{c}\vdots \\ \neg\neg A\end{array}}{A}\,\neg\neg$$

$$\dfrac{\begin{array}{c}[\neg A] \\ \vdots \\ B\end{array} \qquad \begin{array}{c}[\neg A] \\ \vdots \\ \neg B\end{array}}{A}\,\mathrm{Contradiction}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash \neg A, A}\vdash \neg}{\vdash A \vee \neg A, A}\vdash r\vee}{\vdash A \vee \neg A, A \vee \neg A}\vdash l\vee}{\vdash A \vee \neg A}\vdash \mathrm{C}$$

Reductio ad Absurdum

In Real Life

- A You should respect C's belief, for all beliefs are of equal validity and cannot be denied.
- B What about D's belief? (Where D believes something that is considered to be wrong by most people, such as nazism or the world being flat)
- A I agree it is right to deny D's belief.
- B If it is right to deny D's belief, it is not true that no belief can be denied. Therefore, I can deny C's belief if I can give reasons that suggest it too is incorrect.

Reductio ad Absurdum

In Real Life

- A You should respect C's belief, for all beliefs are of equal validity and cannot be denied.
- B
  1. I deny that belief of yours and believe it to be invalid.
  2. According to your statement, this belief of mine (1) is valid, like all other beliefs.
  3. However, your statement also contradicts and invalidates mine, being the exact opposite of it.
  4. The conclusions of 2 and 3 are incompatible and contradictory, so your statement is logically absurd.

Reductio ad Absurdum

Mathematics: The Smallest Positive Rational

There is no smallest positive rational.

1. Suppose there exists one such rational r
2. $r/2$ is rational and positive
3. $r/2 < r$
4. Contradiction □

Reductio ad Absurdum

Mathematics: The Smallest Positive Rational

There is no smallest positive rational.

1. Suppose there exists one such rational r
2. $r/2$ is rational and positive
3. $r/2 < r$
4. Contradiction □

Reductio ad Absurdum

Mathematics: The Smallest Positive Rational

There is no smallest positive rational.

1. Suppose there exists one such rational r
2. $r/2$ is rational and positive
3. $r/2 < r$
4. Contradiction

Reductio ad Absurdum

Mathematics: The Smallest Positive Rational

There is no smallest positive rational.

1. Suppose there exists one such rational r
2. $r/2$ is rational and positive
3. $r/2 < r$
4. Contradiction

Reductio ad Absurdum

Mathematics: The Smallest Positive Rational

There is no smallest positive rational.

1. Suppose there exists one such rational r
2. $r/2$ is rational and positive
3. $r/2 < r$
4. Contradiction $\square$

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even ($a^2 = 2b^2$)
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even ($a^2 = 2b^2$)
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

Constructivity
Intuitionistic Logic

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even ($a^2 = 2b^2$)
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

A. Demaille Intuitionistic Logic 9 / 30

Constructivity
Intuitionistic Logic

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even ($a^2 = 2b^2$)
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

A. Demaille Intuitionistic Logic 9 / 30

Constructivity
Intuitionistic Logic

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even ($a^2 = 2b^2$)
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

A. Demaille Intuitionistic Logic 9 / 30

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even $(a^2 = 2b^2)$
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

Constructivity
Intuitionistic Logic

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even $(a^2 = 2b^2)$
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

A. Demaille Intuitionistic Logic 9 / 30

Constructivity
Intuitionistic Logic

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even $(a^2 = 2b^2)$
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

A. Demaille Intuitionistic Logic 9 / 30

Constructivity
Intuitionistic Logic

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even $(a^2 = 2b^2)$
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

A. Demaille Intuitionistic Logic 9 / 30

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even $(a^2 = 2b^2)$
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

Reductio ad Absurdum

Mathematics: $\sqrt{2}$ is irrational

$\sqrt{2}$ is irrational.

1. Assume $\sqrt{2}$ is rational: $\exists a, b$ integers st. $a/b = \sqrt{2}$
2. a, b can be taken coprime
3. $\therefore a^2/b^2 = 2$ and $a^2 = 2b^2$
4. $\therefore a^2$ is even $(a^2 = 2b^2)$
5. $\therefore a$ is even
6. Because a is even, $\exists k$ st. $a = 2k$.
7. We insert the last equation of (3) in (6): $2b^2 = (2k)^2$ is equivalent to $2b^2 = 4k^2$ is equivalent to $b^2 = 2k^2$.
8. Since $2k^2$ is even, b^2 is even, hence, b is even
9. By (5) and (8) a, b are even
10. Contradicts 2

Constructivity

Mathematics: Rationality and Power

There are irrational positive numbers a, b such that a^b is rational.

1. $\sqrt{2}$ is known to be irrational
2. Consider $\sqrt{2}^{\sqrt{2}}$:

Constructivity

Mathematics: Rationality and Power

There are irrational positive numbers a, b such that a^b is rational.

1. $\sqrt{2}$ is known to be irrational
2. Consider $\sqrt{2}^{\sqrt{2}}$:
   1. If it is rational, take $a = b = \sqrt{2}$

Constructivity

Mathematics: Rationality and Power

There are irrational positive numbers a, b such that a^b is rational.

1. $\sqrt{2}$ is known to be irrational
2. Consider $\sqrt{2}^{\sqrt{2}}$:
   1. If it is rational, take $a = b = \sqrt{2}$
   2. Otherwise, take $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$, $a^b = 2$ □

Constructivity
Intuitionistic Logic

Constructivity

Mathematics: Rationality and Power

There are irrational positive numbers a, b such that a^b is rational.

1. $\sqrt{2}$ is known to be irrational
2. Consider $\sqrt{2}^{\sqrt{2}}$:
   1. If it is rational, take $a = b = \sqrt{2}$
   2. Otherwise, take $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$, $a^b = 2$ □

A. Demaille Intuitionistic Logic 10 / 30

Constructivity
Intuitionistic Logic

Constructivity

Mathematics: Rationality and Power

There are irrational positive numbers a, b such that a^b is rational.

1. $\sqrt{2}$ is known to be irrational
2. Consider $\sqrt{2}^{\sqrt{2}}$:
   1. If it is rational, take $a = b = \sqrt{2}$
   2. Otherwise, take $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$, $a^b = 2$ □

A. Demaille Intuitionistic Logic 10 / 30

Constructivity
Intuitionistic Logic

Constructivity

Mathematics: Rationality and Power

There are irrational positive numbers a, b such that a^b is rational.

1. $\sqrt{2}$ is known to be irrational
2. Consider $\sqrt{2}^{\sqrt{2}}$:
   1. If it is rational, take $a = b = \sqrt{2}$
   2. Otherwise, take $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$, $a^b = 2$ □

But it is not known which numbers.

Constructivity
Mathematics: Rationality and Power

There are irrational positive numbers a, b such that a^b is rational.

1. $\sqrt{2}$ is known to be irrational
2. Consider $\sqrt{2}^{\sqrt{2}}$:
   1. If it is rational, take $a = b = \sqrt{2}$
   2. Otherwise, take $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}, a^b = 2$ $\square$

But it is not known which numbers.
We proved $A \vee B$, but neither A nor B.

Constructivity
Mathematics: Unknown Numbers

Let σ be the number defined below. Its value is unknown, but it is rational.

For each decimal digit of π, write 3. Stop if the sequence 0123456789 is found.

1. If 0123456789 occurs in π, then $\sigma = 0,3\ldots3 = \frac{10^k-1}{3.10^k}$
2. If it does not, $\sigma = 0,3\ldots = 1/3$

Constructivity
Mathematics: Unknown Numbers

Let σ be the number defined below. Its value is unknown, but it is rational.

For each decimal digit of π, write 3. Stop if the sequence 0123456789 is found.

1. If 0123456789 occurs in π, then $\sigma = 0,3\ldots3 = \frac{10^k-1}{3.10^k}$
2. If it does not, $\sigma = 0,3\ldots = 1/3$

Constructivity
Mathematics: Unknown Numbers

Let σ be the number defined below. Its value is unknown, but it is rational.

For each decimal digit of π, write 3. Stop if the sequence 0123456789 is found.

1. If 0123456789 occurs in π, then $\sigma = 0,3\ldots3 = \frac{10^k-1}{3.10^k}$
2. If it does not, $\sigma = 0,3\ldots = 1/3$

Constructivity

Mathematics: Unknown Numbers

Let σ be the number defined below. Its value is unknown, but it is rational.

For each decimal digit of π, write 3. Stop if the sequence 0123456789 is found.

1. If 0123456789 occurs in π, then $\sigma = 0,3\ldots 3 = \frac{10^k - 1}{3.10^k}$
2. If it does not, $\sigma = 0,3\ldots = 1/3$

We proved $\exists x.P(x)$, but know no $t : P(t)$.

Constructivity

Disjunction Property

If $A \vee B$ is provable, then either A or B is provable, and reading the proof tells which one.

Existence Property

If $\exists x \cdot A(x)$ is provable, then reading the proof allows to exhibit a witness t (i.e., such that $A(t)$).

Constructivity

Disjunction Property

If $A \vee B$ is provable, then either A or B is provable, and reading the proof tells which one.

Existence Property

If $\exists x \cdot A(x)$ is provable, then reading the proof allows to exhibit a witness t (i.e., such that $A(t)$).

Intuitionistic Logic

Intuitionistic Logic

- Classical logic focuses on truth (hence truth values)
- Intuitionistic logic focuses on provability (hence proofs)
- A is true if it is provable
- The excluded middle is. . . excluded

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{\vdash \neg A, A}\vdash\neg}{\vdash A \vee \neg A, A}\vdash r\vee}{\vdash A \vee \neg A, A \vee \neg A}\vdash l\vee}{\vdash A \vee \neg A}\vdash \mathrm{C}$$

Intuitionistic Logic

- Classical logic focuses on truth (hence truth values)
- Intuitionistic logic focuses on provability (hence proofs)
- A is true if it is provable
- The excluded middle is. . . excluded

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{\vdash \neg A, A}\vdash\neg}{\vdash A \vee \neg A, A}\vdash r\vee}{\vdash A \vee \neg A, A \vee \neg A}\vdash l\vee}{\vdash A \vee \neg A}\vdash \mathrm{C}$$

Intuitionistic Logic

- Classical logic focuses on truth (hence truth values)
- Intuitionistic logic focuses on provability (hence proofs)
- A is true if it is provable
- The excluded middle is. . . excluded

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{\vdash \neg A, A}\vdash\neg}{\vdash A \vee \neg A, A}\vdash r\vee}{\vdash A \vee \neg A, A \vee \neg A}\vdash l\vee}{\vdash A \vee \neg A}\vdash \mathrm{C}$$

Intuitionistic Logic

- Classical logic focuses on truth (hence truth values)
- Intuitionistic logic focuses on provability (hence proofs)
- A is true if it is provable
- The excluded middle is. . . excluded

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{\vdash \neg A, A}\vdash\neg}{\vdash A \vee \neg A, A}\vdash r\vee}{\vdash A \vee \neg A, A \vee \neg A}\vdash l\vee}{\vdash A \vee \neg A}\vdash \mathrm{C}$$

Intuitionistic Logic

- Classical logic focuses on truth (hence truth values)
- Intuitionistic logic focuses on provability (hence proofs)
- A is true if it is provable
- The excluded middle is. . . excluded

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash \neg A, A}\vdash \neg}{\vdash A \vee \neg A, A}\vdash r\vee}{\vdash A \vee \neg A, A \vee \neg A}\vdash l\vee}{\vdash A \vee \neg A}\vdash \mathrm{C}$$

Constructivity
Intuitionistic Logic
NJ: Intuitionistic Natural Deduction
LJ: Intuitionistic Sequent Calculus

Intuitionistic Logic

- Classical logic focuses on truth (hence truth values)
- Intuitionistic logic focuses on provability (hence proofs)
- A is true if it is provable
- The excluded middle is. . . excluded

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\color{red}{\vdash \neg A, A}}\color{red}{\vdash \neg}}{\vdash A \vee \neg A, A}\vdash r\vee}{\vdash A \vee \neg A, A \vee \neg A}\vdash l\vee}{\vdash A \vee \neg A}\vdash \mathrm{C}$$

NJ: Intuitionistic Natural Deduction

1. Constructivity
2. Intuitionistic Logic
   - NJ: Intuitionistic Natural Deduction
   - LJ: Intuitionistic Sequent Calculus

Constructivity
Intuitionistic Logic
NJ: Intuitionistic Natural Deduction
LJ: Intuitionistic Sequent Calculus

Intuitionistic Natural Deduction

- Natural deduction supports very well intuistionistic logic.
- In fact, classical logic does not fit well in natural deduction.

$$\dfrac{}{A \vee \neg A}\mathrm{XM} \qquad \dfrac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\neg\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots \\ A\end{array} \quad \begin{array}{c}\vdots \\ \neg A\end{array}}{\bot}\neg\mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots \\ \neg\neg A\end{array}}{A}\neg\neg \qquad \dfrac{\begin{array}{c}[\neg A]\\ \vdots \\ B\end{array} \quad \begin{array}{c}[\neg A]\\ \vdots \\ \neg B\end{array}}{A}\mathrm{Contradiction}$$

Intuitionistic Natural Deduction

- Natural deduction supports very well intuistionistic logic.
- In fact, classical logic does not fit well in natural deduction.

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A}\neg\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix}\quad\begin{matrix}\vdots\\ \neg A\end{matrix}}{\bot}\neg\mathcal{E}$$

$$\dfrac{}{A \vee \neg A}\mathrm{XM} \qquad \dfrac{\begin{matrix}\vdots\\ \neg\neg A\end{matrix}}{A}\neg\neg \qquad \dfrac{\begin{matrix}[\neg A]\\ \vdots\\ B\end{matrix}\quad\begin{matrix}[\neg A]\\ \vdots\\ \neg B\end{matrix}}{A}\mathrm{Contradiction}$$

Constructivity
Intuitionistic Logic
NJ: Intuitionistic Natural Deduction
LJ: Intuitionistic Sequent Calculus

Intuitionistic Natural Deduction

- Natural deduction supports very well intuistionistic logic.
- In fact, classical logic does not fit well in natural deduction.

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A}\neg\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix}\quad\begin{matrix}\vdots\\ \neg A\end{matrix}}{\bot}\neg\mathcal{E}$$

$$\dfrac{}{A \vee \neg A}\mathrm{XM} \qquad \dfrac{\begin{matrix}\vdots\\ \neg\neg A\end{matrix}}{A}\neg\neg \qquad \dfrac{\begin{matrix}[\neg A]\\ \vdots\\ B\end{matrix}\quad\begin{matrix}[\neg A]\\ \vdots\\ \neg B\end{matrix}}{A}\mathrm{Contradiction}$$

Constructivity
Intuitionistic Logic
NJ: Intuitionistic Natural Deduction
LJ: Intuitionistic Sequent Calculus

Intuitionistic Natural Deduction

- Natural deduction supports very well intuistionistic logic.
- In fact, classical logic does not fit well in natural deduction.

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A}\neg\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix}\quad\begin{matrix}\vdots\\ \neg A\end{matrix}}{\bot}\neg\mathcal{E}$$

$$\dfrac{}{A \vee \neg A}\mathrm{XM} \qquad \dfrac{\begin{matrix}\vdots\\ \neg\neg A\end{matrix}}{A}\neg\neg \qquad \dfrac{\begin{matrix}[\neg A]\\ \vdots\\ B\end{matrix}\quad\begin{matrix}[\neg A]\\ \vdots\\ \neg B\end{matrix}}{A}\mathrm{Contradiction}$$

Constructivity
Intuitionistic Logic
NJ: Intuitionistic Natural Deduction
LJ: Intuitionistic Sequent Calculus

Intuitionistic Natural Deduction

- Natural deduction supports very well intuistionistic logic.
- In fact, classical logic does not fit well in natural deduction.

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ \bot\end{matrix}}{\neg A}\neg\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix}\quad\begin{matrix}\vdots\\ \neg A\end{matrix}}{\bot}\neg\mathcal{E}$$

Intuitionistic Negation

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A}\,\neg\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ \neg A\end{array}}{\bot}\,\neg\mathcal{E}$$

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \Rightarrow B}\,\Rightarrow\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ A \Rightarrow B\end{array}}{B}\,\Rightarrow\mathcal{E}$$

Intuitionistic Negation

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ \bot\end{array}}{\neg A}\,\neg\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ \neg A\end{array}}{\bot}\,\neg\mathcal{E}$$

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \Rightarrow B}\,\Rightarrow\mathcal{I} \qquad \dfrac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ A \Rightarrow B\end{array}}{B}\,\Rightarrow\mathcal{E}$$

So define $\neg A := A \Rightarrow \bot$.

Prove $A \vdash \neg\neg A$

Prove $A \vdash \neg\neg A$

$$\dfrac{\dfrac{A \quad [A \Rightarrow \bot]^1}{\bot}\,\Rightarrow\mathcal{E}}{(A \Rightarrow \bot) \Rightarrow \bot}\,\Rightarrow\mathcal{I}_1$$

Prove $\neg\neg\neg A \vdash \neg A$

Prove $\neg\neg\neg A \vdash \neg A$

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^2 \quad [A \Rightarrow \bot]^1}{\bot}\,{\Rightarrow\mathcal{E}}}{(A \Rightarrow \bot) \Rightarrow \bot}\,{\Rightarrow\mathcal{I}_1} \qquad ((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow \bot}{\bot}\,{\Rightarrow\mathcal{E}}}{A \Rightarrow \bot}\,{\Rightarrow\mathcal{I}_2}$$

Intuitionistic Natural Deduction

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B}\,{\Rightarrow\mathcal{I}} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A \Rightarrow B\end{matrix}}{B}\,{\Rightarrow\mathcal{E}} \qquad \dfrac{\begin{matrix}\vdots \\ \bot\end{matrix}}{A}\,{\bot\mathcal{E}}$$

$$\dfrac{A \quad B}{A \wedge B}\,{\wedge\mathcal{I}} \qquad \dfrac{A \wedge B}{A}\,{\wedge l\mathcal{E}} \qquad \dfrac{A \wedge B}{B}\,{\wedge r\mathcal{E}}$$

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B}\,{\vee l\mathcal{I}} \qquad \dfrac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B}\,{\vee r\mathcal{I}} \qquad \dfrac{\begin{matrix}\vdots \\ A \vee B\end{matrix} \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\,{\vee\mathcal{E}}$$

LJ: Intuitionistic Sequent Calculus

LK: Identity Group

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

LK: Identity Group

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{Cut}$$

LJ: Identity Group

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A \quad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\,\mathrm{Cut}$$

LK: Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)}\,\vdash \mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta}\,\mathrm{X} \vdash$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\,\vdash \mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{W} \vdash$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\,\vdash \mathrm{C} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,\mathrm{C} \vdash$$

LK: Structural Group

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)} \vdash \mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta} \mathrm{X} \vdash$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \vdash \mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{W} \vdash$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \vdash \mathrm{C} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \mathrm{C} \vdash$$

LJ: Structural Group

$$\frac{\Gamma \vdash B}{\sigma(\Gamma) \vdash B} \mathrm{X} \vdash$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \mathrm{W} \vdash$$

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \mathrm{C} \vdash$$

LK: Logical Group: Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \vdash \neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg \vdash$$

LK: Logical Group: Negation

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \vdash \neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg \vdash$$

LJ: Logical Group: Negation

LK: Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge \vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge \vdash$$

LK: Logical Group: Conjunction

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge \vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge \vdash$$

LJ: Logical Group: Conjunction

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \vdash \wedge \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} l\wedge \vdash \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C} r\wedge \vdash$$

LK: Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash l\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\vee\vdash$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash r\vee$$

LK: Logical Group: Disjunction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash l\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\vee\vdash$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash r\vee$$

LJ: Logical Group: Disjunction

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vdash l\vee \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\vee\vdash$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vdash r\vee$$

LK: Logical Group: Implication

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'}\Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\vdash\Rightarrow$$

LK: Logical Group: Implication

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \vdash\Rightarrow$$

LJ: Logical Group: Implication

$$\frac{\Gamma \vdash A \qquad \Gamma', B \vdash B'}{\Gamma, \Gamma', A \Rightarrow B \vdash B'} \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \vdash\Rightarrow$$

LJ

$$\frac{}{A \vdash A}\,\mathrm{Id} \qquad \frac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\,\mathrm{Cut}$$

$$\frac{\Gamma \vdash B}{\sigma(\Gamma) \vdash B}\,\mathrm{X}\vdash \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathrm{W}\vdash \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathrm{C}\vdash$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \vdash\wedge \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\, l\wedge\vdash \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\, r\wedge\vdash$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vdash l\vee \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vdash r\vee \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \vee\vdash$$

$$\frac{\Gamma \vdash A \qquad \Gamma', B \vdash B'}{\Gamma, \Gamma', A \Rightarrow B \vdash B'} \Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \vdash\Rightarrow$$

Prove $A \vdash \neg\neg A$

Prove $A \vdash \neg\neg A$

$$\dfrac{\dfrac{\overline{A \vdash A} \qquad \overline{\bot \vdash \bot}}{A, A \Rightarrow \bot \vdash \bot}\Rightarrow\vdash}{A \vdash (A \Rightarrow \bot) \Rightarrow \bot}\vdash\Rightarrow$$

Prove $A \vdash \neg\neg A$

$$\dfrac{\dfrac{\overline{A_- \vdash A_+} \qquad \overline{\bot_- \vdash \bot_+}}{A_-, A_+ \Rightarrow \bot_- \vdash \bot_+}\Rightarrow\vdash}{A_- \vdash (A_+ \Rightarrow \bot_-) \Rightarrow \bot_+}\vdash\Rightarrow$$

Prove $\neg\neg\neg A \vdash \neg A$

Prove $\neg\neg\neg A \vdash \neg A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A} \qquad \overline{\bot \vdash \bot}}{A, A \Rightarrow \bot \vdash \bot}\Rightarrow\vdash}{A \vdash (A \Rightarrow \bot) \Rightarrow \bot}\vdash\Rightarrow \qquad \overline{\bot' \vdash \bot'}}{A, ((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow \bot' \vdash \bot'}\Rightarrow\vdash}{((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow \bot' \vdash A \Rightarrow \bot'}\vdash\Rightarrow$$

Prove $\neg\neg\neg A \vdash \neg A$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A_- \vdash A_+} \quad \dfrac{}{\bot_- \vdash \bot_+}}{A_-, A_+ \Rightarrow \bot_- \vdash \bot_+} \Rightarrow\vdash}{A_- \vdash (A_+ \Rightarrow \bot_-) \Rightarrow \bot_+} \vdash\Rightarrow \quad \dfrac{}{\bot'_- \vdash \bot'_+}}{\dfrac{A_-, ((A_+ \Rightarrow \bot_-) \Rightarrow \bot_+) \Rightarrow \bot'_- \vdash \bot'_+}{((A_+ \Rightarrow \bot_-) \Rightarrow \bot_+) \Rightarrow \bot'_- \vdash A_- \Rightarrow \bot'_+} \vdash\Rightarrow} \Rightarrow\vdash
$$

Therefore, in intuitionistic logic $\neg\neg\neg A \equiv \neg A$, but $\neg\neg A \not\equiv A$.