FORMULAIRE RELATIF AUX OPÉRATEURS

Soient U et V deux champs scalaires et $\vec{a}$ et $\vec{b}$ deux champs vectoriels.

1. Formules portant sur un seul champ:

1. $\vec{\nabla}.(\vec{\nabla} U) = \vec{\nabla}^2 U$ soit $\text{div}(\overrightarrow{\text{grad}}\, U) = \Delta U$

2. $\vec{\nabla} \wedge (\vec{\nabla} U) = \vec{0}$ soit $\overrightarrow{\text{rot}}(\overrightarrow{\text{grad}}\, U) = \vec{0}$

3. $\vec{\nabla}.(\vec{\nabla} \wedge \vec{a}) = 0$ soit $\text{div}(\overrightarrow{\text{rot}}\, \vec{a}) = 0$

4. $\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{a}) = \vec{\nabla}(\vec{\nabla}.\vec{a}) - \vec{\nabla}^2 \vec{a}$ soit $\overrightarrow{\text{rot}}(\overrightarrow{\text{rot}}\, \vec{a}) = \overrightarrow{\text{grad}}(\text{div}\, \vec{a}) - \Delta \vec{a}$

2. Formules portant sur deux champs:

5. $\vec{\nabla}(UV) = V\vec{\nabla}(U) + U\vec{\nabla}(V)$ soit $\overrightarrow{\text{grad}}(UV) = V\, \overrightarrow{\text{grad}}\, U + U\, \overrightarrow{\text{grad}}\, V$

6. $\vec{\nabla}.(U\vec{a}) = \vec{a}(\vec{\nabla} U) + U(\vec{\nabla}.\vec{a})$ soit $\text{div}(U\vec{a}) = \overrightarrow{\text{grad}}\, U.\vec{a} + U\, \text{div}\, \vec{a}$

7. $\vec{\nabla} \wedge (U\vec{a}) = (\vec{\nabla} U) \wedge \vec{a} + U(\vec{\nabla} \wedge \vec{a})$ soit $\overrightarrow{\text{rot}}(U\vec{a}) = \overrightarrow{\text{grad}}\, U \wedge \vec{a} + U\, \overrightarrow{\text{rot}}\, \vec{a}$

8. $\vec{\nabla}.(\vec{a} \wedge \vec{b}) = \vec{b}.(\vec{\nabla} \wedge \vec{a}) - \vec{a}.(\vec{\nabla} \wedge \vec{b})$ soit $\text{div}(\vec{a} \wedge \vec{b}) = \vec{b}.\overrightarrow{\text{rot}}\, \vec{a} - \vec{a}.\overrightarrow{\text{rot}}\, \vec{b}$

9. $\vec{\nabla} \wedge (\vec{a} \wedge \vec{b}) = (\vec{\nabla}.\vec{b})\vec{a} - (\vec{\nabla}.\vec{a})\vec{b} + (\vec{b}.\vec{\nabla})\vec{a} - (\vec{a}.\vec{\nabla})\vec{b}$

soit $\overrightarrow{\text{rot}}(\vec{a} \wedge \vec{b}) = (\text{div}\, \vec{b})\vec{a} - (\text{div}\, \vec{a})\vec{b} + (\vec{b}\, \overrightarrow{\text{grad}})\vec{a} - (\vec{a}.\overrightarrow{\text{grad}})\vec{b}$

10. $\vec{\nabla}(\vec{a}.\vec{b}) = \vec{a} \wedge (\vec{\nabla} \wedge \vec{b}) + \vec{b} \wedge (\vec{\nabla} \wedge \vec{a}) + (\vec{b}.\vec{\nabla})\vec{a} + (\vec{a}.\vec{\nabla})\vec{b}$

soit $\overrightarrow{\text{grad}}(\vec{a}.\vec{b}) = \vec{a} \wedge (\overrightarrow{\text{rot}}\, \vec{b}) + \vec{b} \wedge (\overrightarrow{\text{rot}}\, \vec{a}) + (\vec{b}.\overrightarrow{\text{grad}})\vec{a} + (\vec{a}.\overrightarrow{\text{grad}})\vec{b}$

3. Expressions des opérateurs dans divers systèmes de coordonnées:

a. Gradient:

* cartésiennes: $\overrightarrow{\text{grad}}\, U = \left(\frac{\partial U}{\partial x}\right)\vec{e}_x + \left(\frac{\partial U}{\partial y}\right)\vec{e}_y + \left(\frac{\partial U}{\partial z}\right)\vec{e}_z$

* cylindriques: $\overrightarrow{\text{grad}}\, U = \left(\frac{\partial U}{\partial r}\right)\vec{e}_r + \frac{1}{r}\left(\frac{\partial U}{\partial \theta}\right)\vec{e}_\theta + \left(\frac{\partial U}{\partial z}\right)\vec{e}_z$

* sphériques: $\overrightarrow{\text{grad}}\, U = \left(\frac{\partial U}{\partial r}\right)\vec{e}_r + \frac{1}{r}\left(\frac{\partial U}{\partial \theta}\right)\vec{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial U}{\partial \varphi}\right)\vec{e}_\varphi$

b. Divergence:

* cartésiennes: $\text{div}\, \vec{a} = \left(\frac{\partial a_x}{\partial x}\right) + \left(\frac{\partial a_y}{\partial y}\right) + \left(\frac{\partial a_z}{\partial z}\right)$

* cylindriques: $\text{div}\, \vec{a} = \frac{1}{r}\left(\frac{\partial\, r a_r}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta}\right) + \left(\frac{\partial a_z}{\partial z}\right)$

* sphériques: $\text{div}\, \vec{a} = \frac{1}{r^2}\left(\frac{\partial\, r^2 a_r}{\partial r}\right) + \frac{1}{r\sin\theta}\left(\frac{\partial\, a_\theta \sin\theta}{\partial \theta}\right) + \frac{1}{r\sin\theta}\left(\frac{\partial a_\varphi}{\partial \varphi}\right)$

c. Rotationnel:

* cartésiennes: $\overrightarrow{\mathrm{rot}}\,\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z$

* cylindriques: $\overrightarrow{\mathrm{rot}}\,\vec{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta}\right)\vec{e}_z$

* sphériques:

$$\overrightarrow{\mathrm{rot}}\,\vec{a} = \frac{1}{r\sin\theta}\left(\frac{\partial (a_\varphi \sin\theta)}{\partial \theta} - \frac{\partial a_\theta}{\partial \varphi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial a_r}{\partial \varphi} - \frac{\partial (r a_\varphi)}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta}\right)\vec{e}_\varphi$$

d. Laplacien:

* cartésiennes: $\Delta U = \left(\frac{\partial^2 U}{\partial x^2}\right) + \left(\frac{\partial^2 U}{\partial y^2}\right) + \left(\frac{\partial^2 U}{\partial z^2}\right)$

* cylindriques: $\Delta U = \left(\frac{\partial^2 U}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 U}{\partial \theta^2}\right) + \left(\frac{\partial^2 U}{\partial z^2}\right)$

* sphériques: $\Delta U = \left(\frac{\partial^2 U}{\partial r^2}\right) + \frac{2}{r}\left(\frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 U}{\partial \varphi^2}\right) + \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U}{\partial \theta}\right)\right)$

4. Action des opérateurs sur le trièdre de base:

a. coordonnées cylindriques:

$\overrightarrow{\mathrm{rot}}\,\vec{e}_r = \vec{0}$; $\overrightarrow{\mathrm{rot}}\,\vec{e}_\theta = \frac{\vec{e}_z}{r}$; $\overrightarrow{\mathrm{rot}}\,\vec{e}_z = \vec{0}$ pour le rotationnel

$\mathrm{div}\,\vec{e}_r = \frac{1}{r}$; $\mathrm{div}\,\vec{e}_\theta = 0$; $\mathrm{div}\,\vec{e}_z = 0$ pour la divergence

b. coordonnées sphériques:

$\overrightarrow{\mathrm{rot}}\,\vec{e}_r = \vec{0}$; $\overrightarrow{\mathrm{rot}}\,\vec{e}_\theta = \frac{\vec{e}_\varphi}{r}$; $\overrightarrow{\mathrm{rot}}\,\vec{e}_\varphi = \frac{\cos\theta}{r\sin\theta}\vec{e}_r - \frac{\vec{e}_\theta}{r}$ pour le rotationnel

$\mathrm{div}\,\vec{e}_r = \frac{2}{r}$; $\mathrm{div}\,\vec{e}_\theta = \frac{1}{r\tan\theta}$; $\mathrm{div}\,\vec{e}_\varphi = 0$ pour la divergence