Exercises on λ-calculus and Deduction Systems

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Exercises on λ-calculus and Deduction Systems

λ-calculus

Untyped λ-calculus

Substitutions

$$[\lambda z \cdot zz/x]\lambda y \cdot xy \equiv$$

λ-calculus | Untyped λ-calculus
Deduction Systems | Simply Typed λ-calculus

Substitutions

$$\begin{aligned}[]
[\lambda z \cdot zz/x]\lambda y \cdot xy &\equiv \lambda y \cdot \lambda z \cdot zzy \\
[yy/z](\lambda xy \cdot zy) &\equiv
\end{aligned}$$

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Substitutions

$$\begin{aligned}[]
[\lambda z \cdot zz/x]\lambda y \cdot xy &\equiv \lambda y \cdot \lambda z \cdot zzy \\
[yy/z](\lambda xy \cdot zy) &\equiv \lambda xu \cdot yyu
\end{aligned}$$

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β-Reductions

$$(\lambda x \cdot xyx)\lambda z \cdot z \rightarrow$$

β-Reductions

$$\begin{array}{rcl}(\lambda x \cdot xyx)\lambda z \cdot z & \to & (\lambda z \cdot z)y(\lambda z \cdot z) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to & \end{array}$$

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β-Reductions

$$\begin{array}{rcl}(\lambda x \cdot xyx)\lambda z \cdot z & \to & (\lambda z \cdot z)y(\lambda z \cdot z) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to & (\lambda x \cdot x)(x) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to & \end{array}$$

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$$\begin{array}{rcl}(\lambda x \cdot xyx)\lambda z \cdot z & \to & (\lambda z \cdot z)y(\lambda z \cdot z) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to & (\lambda x \cdot x)(x) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to & ((\lambda y \cdot y)x) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to^* & \end{array}$$

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β-Reductions

$$\begin{array}{rcl}(\lambda x \cdot xyx)\lambda z \cdot z & \to & (\lambda z \cdot z)y(\lambda z \cdot z) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to & (\lambda x \cdot x)(x) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to & ((\lambda y \cdot y)x) \\ (\lambda x \cdot x)((\lambda y \cdot y)x) & \to^* & x \\ (\lambda x \cdot xx)((\lambda x \cdot xx)y) & \to^* & \end{array}$$

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β-Reductions

$$
\begin{array}{rcl}
(\lambda x \cdot xyx)\lambda z \cdot z & \rightarrow & (\lambda z \cdot z)y(\lambda z \cdot z) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow & (\lambda x \cdot x)(x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow & ((\lambda y \cdot y)x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow^* & x \\
(\lambda x \cdot xx)((\lambda x \cdot xx)y) & \rightarrow^* & yy(yy) \\
(\lambda x \cdot xx)((\lambda x \cdot x)y) & \rightarrow^* &
\end{array}
$$

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β-Reductions

$$
\begin{array}{rcl}
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(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow^* & x \\
(\lambda x \cdot xx)((\lambda x \cdot xx)y) & \rightarrow^* & yy(yy) \\
(\lambda x \cdot xx)((\lambda x \cdot x)y) & \rightarrow^* & yy \\
(\lambda x \cdot x)((\lambda x \cdot xx)y) & \rightarrow^* &
\end{array}
$$

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β-Reductions

$$
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(\lambda x \cdot xyx)\lambda z \cdot z & \rightarrow & (\lambda z \cdot z)y(\lambda z \cdot z) \\
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(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow & ((\lambda y \cdot y)x) \\
(\lambda x \cdot x)((\lambda y \cdot y)x) & \rightarrow^* & x \\
(\lambda x \cdot xx)((\lambda x \cdot xx)y) & \rightarrow^* & yy(yy) \\
(\lambda x \cdot xx)((\lambda x \cdot x)y) & \rightarrow^* & yy \\
(\lambda x \cdot x)((\lambda x \cdot xx)y) & \rightarrow^* & yy
\end{array}
$$

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Simply Typed λ-calculus

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Type derivations are trees built from the following nodes.

$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau}$$

Simply Typed λ-calculus

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$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau} \qquad \frac{\begin{array}{c} x : \sigma \\ \vdots \\ M : \tau \end{array}}{}$$

Simply Typed λ-calculus

Type derivations are trees built from the following nodes.

$$\frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau} \qquad \frac{\begin{array}{c} [x : \sigma] \\ \vdots \\ M : \tau \end{array}}{\lambda x \cdot M : \sigma \to \tau}$$

Type Statements

Type $\lambda fx \cdot f(fx)$

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$$\vdash \lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma$$

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Type $\lambda fx \cdot f(fx)$

$$\vdash \lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma$$

$$\dfrac{\dfrac{\dfrac{[f : \sigma \to \sigma]^{(2)} \qquad \dfrac{[f : \sigma \to \sigma]^{(2)} \quad [x : \sigma]^{(1)}}{fx : \sigma}}{f(fx) : \sigma}}{\lambda x \cdot f(fx) : \sigma \to \sigma}\,(1)}{\lambda fx \cdot f(fx) : (\sigma \to \sigma) \to \sigma \to \sigma}\,(2)$$

Type Statements

Type $\lambda xy \cdot x$

Type Statements

Type $\lambda xy \cdot x$

$$\vdash \lambda xy \cdot x : \sigma \to \tau \to \sigma$$

Type Statements

Type $\lambda xy \cdot x$

$$\vdash \lambda xy \cdot x : \sigma \to \tau \to \sigma$$

$$\dfrac{\dfrac{[x : \sigma]^{(1)}}{\lambda y \cdot x : \tau \to \sigma}}{\lambda xy \cdot x : \sigma \to \tau \to \sigma}\,(1)$$

Deduction Systems

Natural Deduction

Intuitionistic Natural Deduction

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{A \quad A \Rightarrow B}{B} \Rightarrow\mathcal{E} \qquad \dfrac{\bot}{A} \bot\mathcal{E}$$

$$\dfrac{A \quad B}{A \wedge B} \wedge\mathcal{I} \qquad \dfrac{A \wedge B}{A} \wedge l\mathcal{E} \qquad \dfrac{A \wedge B}{B} \wedge r\mathcal{E}$$

$$\dfrac{A}{A \vee B} \vee l\mathcal{I} \qquad \dfrac{B}{A \vee B} \vee r\mathcal{I} \qquad \dfrac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C} \vee\mathcal{E}$$

Prove $A \wedge B \Rightarrow B \wedge A$

Prove $A \wedge B \Rightarrow B \wedge A$

$$
\dfrac{\dfrac{\dfrac{[A \wedge B]^1}{B}\,\wedge r\mathcal{E} \qquad \dfrac{[A \wedge B]^1}{A}\,\wedge l\mathcal{E}}{B \wedge A}\,\wedge\mathcal{I}}{A \wedge B \Rightarrow B \wedge A}\,\Rightarrow\mathcal{I}_1
$$

Prove $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

Prove $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

$$
\dfrac{\dfrac{A \wedge (B \vee C)}{B \vee C}\,\wedge r\mathcal{E} \qquad \dfrac{\dfrac{\dfrac{A \wedge (B \vee C)}{A}\,\wedge l\mathcal{E} \quad [B]^1}{A \wedge B}\,\wedge\mathcal{I}}{(A \wedge B) \vee (A \wedge C)}\,\vee l\mathcal{I} \qquad \dfrac{\dfrac{\dfrac{A \wedge (B \vee C)}{A}\,\wedge l\mathcal{E} \quad [C]^1}{A \wedge C}\,\wedge\mathcal{I}}{(A \wedge B) \vee (A \wedge C)}\,\vee r\mathcal{I}}{(A \wedge B) \vee (A \wedge C)}\,\vee\mathcal{E}_1
$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

$$\begin{prooftree}
\AxiomC{$[A]^2$}
\AxiomC{$[A \Rightarrow A]^1$}
\RightLabel{$\Rightarrow\mathcal{E}$}
\BinaryInfC{A}
\RightLabel{$\Rightarrow\mathcal{I}_2$}
\UnaryInfC{$A \Rightarrow A$}
\RightLabel{$\Rightarrow\mathcal{I}_1$}
\UnaryInfC{$(A \Rightarrow A) \Rightarrow A \Rightarrow A$}
\end{prooftree}$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

$$\begin{prooftree}
\AxiomC{$[A]^2$}
\AxiomC{$[A \Rightarrow A]^1$}
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\BinaryInfC{A}
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\UnaryInfC{$A \Rightarrow A$}
\RightLabel{$\Rightarrow\mathcal{I}_1$}
\UnaryInfC{$(A \Rightarrow A) \Rightarrow A \Rightarrow A$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$[A]^1$}
\RightLabel{$\Rightarrow\mathcal{I}_1$}
\UnaryInfC{$A \Rightarrow A$}
\RightLabel{$\Rightarrow\mathcal{I}$}
\UnaryInfC{$(A \Rightarrow A) \Rightarrow A \Rightarrow A$}
\end{prooftree}$$

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

$$\dfrac{\dfrac{\dfrac{[A]^1 \quad A \Rightarrow B}{B}\Rightarrow\mathcal{E} \qquad \dfrac{\dfrac{[A]^1 \quad A \Rightarrow B}{B}\Rightarrow\mathcal{E} \quad B \Rightarrow C}{C}\Rightarrow\mathcal{E}}{B \wedge C}\wedge\mathcal{I}}{A \Rightarrow (B \wedge C)}\Rightarrow\mathcal{I}_1$$

Prove $A \vee B, \neg B \vdash A$ (Intuitionistic) (LOFO-2005)

Prove $A \vee B, \neg B \vdash A$ (Intuitionistic) (LOFO-2005)

Recall that $\neg B := B \Rightarrow \bot$.

Prove $A \vee B, \neg B \vdash A$ (Intuitionistic) (LOFO-2005)

Recall that $\neg B := B \Rightarrow \bot$.

$$\dfrac{A \vee B \qquad [A]^1 \qquad \dfrac{\dfrac{[B]^2 \quad B \Rightarrow \bot}{\bot}\Rightarrow\mathcal{E}}{A}\bot\mathcal{E}}{A}\vee\mathcal{E}_1$$

Sequent Calculus

Classical Sequent Calculus

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau(\Delta)}\vdash \mathrm{X} \qquad \frac{\Gamma \vdash \Delta}{\sigma(\Gamma) \vdash \Delta}\mathrm{X}\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{W} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\mathrm{W}\vdash \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\vdash \mathrm{C} \qquad \frac{\Gamma, A, A \vdash}{\Gamma, A \vdash \Delta}$$

$$\frac{}{F \vdash F}\mathrm{Id} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\mathrm{Cut}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\vdash \neg \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\neg\vdash$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\vdash \wedge \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} l\wedge\vdash \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} r\wedge\vdash$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash l\vee \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\vdash r\vee \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\vee\vdash$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma', B \vdash \Delta'}{}\Rightarrow\vdash \qquad \frac{\Gamma, A \vdash B, \Delta}{}\vdash\Rightarrow$$

Prove $A \wedge B \vdash A \wedge B$

Prove $A \wedge B \vdash A \wedge B$

$$\frac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \wedge B \vdash A} l\wedge\vdash \qquad \dfrac{\dfrac{}{B \vdash B}}{A \wedge B \vdash B} r\wedge\vdash}{A \wedge B, A \wedge B \vdash A \wedge B}\vdash\wedge}{A \wedge B \vdash A \wedge B}\mathrm{C}\vdash$$

Prove $A \wedge B \vdash A \vee B$

λ-calculus Natural Deduction
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Prove $A \wedge B \vdash A \vee B$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \wedge B \vdash A}\; l\wedge\vdash}{A \wedge B \vdash A \vee B}\; \vdash\vee$$

Prove $A \vee B \vdash A \vee B$

λ-calculus Natural Deduction
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Prove $A \vee B \vdash A \vee B$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A \vdash A \vee B}\; \vdash l\vee \qquad \dfrac{\dfrac{}{B \vdash B}}{B \vdash A \vee B}\; \vdash r\vee}{A \vee B \vdash A \vee B}\; \vee\vdash$$

Prove $(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H)$

Prove $(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F \vdash F} \quad \dfrac{}{G \vdash G}}{F, G \vdash F \wedge G}\wedge\vdash \qquad \dfrac{}{H \vdash H}}{F, G, (F \wedge G) \Rightarrow H \vdash H}\Rightarrow\vdash}{F, G, (F \wedge G) \Rightarrow H \vdash H, H}\vdash \mathrm{W}}{F, (F \wedge G) \Rightarrow H \vdash H, G \Rightarrow H}\vdash\Rightarrow}{(F \wedge G) \Rightarrow H \vdash F \Rightarrow H, G \Rightarrow H}\vdash\Rightarrow}{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H), G \Rightarrow H}\vdash r\vee}{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H), (F \Rightarrow H) \vee (G \Rightarrow H)}\vdash l\vee}{(F \wedge G) \Rightarrow H \vdash (F \Rightarrow H) \vee (G \Rightarrow H)}\vdash \mathrm{C}$$

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

Prove $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ (LOFO-2005)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{A \vdash A}}{A, A \Rightarrow A \vdash A}\Rightarrow\vdash}{A \Rightarrow A \vdash A \Rightarrow A}\vdash\Rightarrow}{\vdash (A \Rightarrow A) \Rightarrow A \Rightarrow A}\vdash\Rightarrow$$

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

Prove $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow (B \wedge C)$ (LOFO-2005)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A, A \Rightarrow B \vdash B} \Rightarrow\vdash}{A, A \Rightarrow B, B \Rightarrow C \vdash B}\,\mathrm{W}\vdash \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{C \vdash C}}{A, B \Rightarrow C \vdash C} \Rightarrow\vdash}{A, A \Rightarrow B, B \Rightarrow C \vdash C}\,\mathrm{W}\vdash}{A, A \Rightarrow B, A, B \Rightarrow C \vdash B \wedge C}\,\wedge\vdash}{A, A \Rightarrow B, B \Rightarrow C \vdash B \wedge C}\,\mathrm{C}\vdash$$

Prove $A \vee B, \neg B \vdash A$ (Classical) (LOFO-2005)

Prove $A \vee B, \neg B \vdash A$ (Classical) (LOFO-2005)

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, \neg B \vdash A}\,\mathrm{W}\vdash \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}}{B \vdash B, A}\,\vdash\mathrm{W}}{B, \neg B \vdash A}\,\vdash\neg}{A \vee B, \neg B \vdash A}\,\vee\vdash$$