Natural Deduction

Akim Demaille akim@lrde.epita.fr

EPITA — École Pour l'Informatique et les Techniques Avancées

March 22, 2009

Natural Deduction

Preamble

The following slides are implicitly dedicated to classical logic.

Logical Formalisms

Syntax

1. Logical Formalisms
   - Syntax
   - Proof Systems
2. Natural Deduction

Terminal Symbols

Propositional Calculus

Constants $a, b, c, \ldots$

Propositional Variables $A, B, C, \ldots$

Unary Connective $\neg$

Binary Connectives $\wedge, \vee, \Rightarrow$

Quantifiers $\forall, \exists$

Punctuation $(,), [,], \cdot, \cdot$

Logical Formalisms | Syntax
Natural Deduction | Proof Systems

Terminal Symbols

Propositional Calculus

Constants $a, b, c, \ldots$

Propositional Variables $A, B, C, \ldots$

Unary Connective $\neg$

Binary Connectives $\wedge, \vee, \Rightarrow$

Quantifiers $\forall, \exists$

Punctuation $(,), [,], \cdot, \cdot$

Logical Formalisms | Syntax
Natural Deduction | Proof Systems

Terminal Symbols

Propositional Calculus

Constants $a, b, c, \ldots$

Propositional Variables $A, B, C, \ldots$

Unary Connective $\neg$

Binary Connectives $\wedge, \vee, \Rightarrow$

Quantifiers $\forall, \exists$

Punctuation $(,), [,], \cdot, \cdot$

Terminal Symbols

Propositional Calculus

Constants $a, b, c, \ldots$

Propositional Variables $A, B, C, \ldots$

Unary Connective $\neg$

Binary Connectives $\wedge, \vee, \Rightarrow$

Quantifiers $\forall, \exists$

Punctuation $(,), [,], \cdot.$

Terminal Symbols

Propositional Calculus

Constants $a, b, c, \ldots$

Propositional Variables $A, B, C, \ldots$

Unary Connective $\neg$

Binary Connectives $\wedge, \vee, \Rightarrow$

Quantifiers $\forall, \exists$

Punctuation $(,), [,], \cdot.$

Terminal Symbols

Propositional Calculus

Constants $a, b, c, \ldots$

Propositional Variables $A, B, C, \ldots$

Unary Connective $\neg$

Binary Connectives $\wedge, \vee, \Rightarrow$

Quantifiers $\forall, \exists$

Punctuation $(,), [,], \cdot.$

Terminal Symbols

Predicate calculus

Individual Variables $x, y, z, \ldots$

Functions $f, g, h, \ldots$, with a fixed arity

Predicates $P, Q, R, \ldots$, with a fixed arity

Terminal Symbols

Predicate calculus

- Individual Variables $x, y, z, \ldots$
- Functions $f, g, h, \ldots$, with a fixed arity
- Predicates $P, Q, R, \ldots$, with a fixed arity

Terminal Symbols

Predicate calculus

- Individual Variables $x, y, z, \ldots$
- Functions $f, g, h, \ldots$, with a fixed arity
- Predicates $P, Q, R, \ldots$, with a fixed arity

Propositional Formulas

$$
\begin{array}{rcl}
\langle\text{formula}\rangle & ::= & \langle\text{propositional variable}\rangle \\
& | & \neg\langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \wedge \langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \vee \langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \Rightarrow \langle\text{formula}\rangle
\end{array}
$$

Terms

$$
\begin{array}{rcl}
\langle\text{term}\rangle & ::= & \langle\text{constant}\rangle \\
& | & \langle\text{function}\rangle(\langle\text{term}\rangle, \ldots)
\end{array}
$$

With the proper arity.

First Order Formulas

$$
\begin{array}{rcl}
\langle\text{formula}\rangle & ::= & \langle\text{propositional variable}\rangle \\
& | & \neg\langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \wedge \langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \vee \langle\text{formula}\rangle \\
& | & \langle\text{formula}\rangle \Rightarrow \langle\text{formula}\rangle \\
& | & \langle\text{predicate}\rangle(\langle\text{term}\rangle, \ldots) \\
& | & \forall\langle\text{individual variable}\rangle \cdot \langle\text{formula}\rangle \\
& | & \exists\langle\text{individual variable}\rangle \cdot \langle\text{formula}\rangle
\end{array}
$$

With the proper arity.

Syntactic Conventions

Associativity

- $\wedge, \vee$ are left-associative (unimportant)
- $\Rightarrow$ is right-associative (very important)

Precedence (increasing)

1. $\forall, \exists$
2. $\Rightarrow$
3. $\vee$
4. $\wedge$
5. $\neg$

Syntactic Conventions

Associativity

- $\wedge, \vee$ are left-associative (unimportant)
- $\Rightarrow$ is right-associative (very important)

Precedence (increasing)

1. $\forall, \exists$
2. $\Rightarrow$
3. $\vee$
4. $\wedge$
5. $\neg$

Free Variables

$$
\begin{array}{rcl}
\mathrm{FV}(X) & = & \emptyset \\
\mathrm{FV}(P(x_1, x_2, \cdots, x_n)) & = & \{x_1, x_2, \cdots, x_m\} \\
\mathrm{FV}(\neg A) & = & \mathrm{FV}(A) \\
\mathrm{FV}(A \vee B) & = & \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(A \wedge B) & = & \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(A \Rightarrow B) & = & \mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}(\forall x \cdot A) & = & \mathrm{FV}(A) - \{x\} \\
\mathrm{FV}(\exists x \cdot A) & = & \mathrm{FV}(A) - \{x\}
\end{array}
$$

Proof Systems

1. Logical Formalisms
   - Syntax
   - Proof Systems
2. Natural Deduction

Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

Logical Formalisms | Syntax
Natural Deduction | Proof Systems

Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

A. Demaille | Natural Deduction | 14 / 39

Logical Formalisms | Syntax
Natural Deduction | Proof Systems

Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

A. Demaille | Natural Deduction | 14 / 39

Proof Systems

- Hilbertian Systems
- Natural Deduction
- Sequent Calculus
- Natural Deduction in Sequent Calculus

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables that range over a specific domain

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables that range over a specific domain

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables that range over a specific domain

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Axioms

- Axioms are formulas that are considered true a priori

$$\forall x \cdot x + 0 = x$$

- Axiom schemes use meta-variables that range over a specific domain

$$X + Y = Y + X$$

- Axiom schemes are used when quantifiers are not welcome

$$\begin{aligned} SXYZ &\rightarrow XZ(YZ) \\ KXY &\rightarrow X \end{aligned}$$

- Axiom schemes are used when quantifiers do not apply

$$A \vee \neg A$$

Inference Rules

$$\frac{H_1 \quad H_2 \quad \cdots \quad H_n}{C}\,\text{Rule name}$$

$$\frac{}{A}\,\text{Axiom name}$$

Hilbertian System

- A single inference rule: the modus ponens

$$\frac{A \qquad A \Rightarrow B}{B}\,\textit{modus ponens}$$

- Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B$$

$$A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$$
$$A \vee B \Rightarrow \ (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \qquad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$A \vee \neg A \qquad A \Rightarrow \neg A \Rightarrow B$$

Hilbertian System

- A single inference rule: the modus ponens

$$\frac{A \qquad A \Rightarrow B}{B}\,\textit{modus ponens}$$

- Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B$$

$$A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$$
$$A \vee B \Rightarrow \ (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \qquad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$A \vee \neg A \qquad A \Rightarrow \neg A \Rightarrow B$$

Hilbertian System

- A single inference rule: the modus ponens

$$\frac{A \qquad A \Rightarrow B}{B}\ \textit{modus ponens}$$

- Many axioms to define the connectives

$$A \Rightarrow B \Rightarrow A \wedge B \qquad A \wedge B \Rightarrow A \qquad A \wedge B \Rightarrow B$$

$$A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$$
$$A \vee B \Rightarrow\ (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$$

$$A \Rightarrow B \Rightarrow A \qquad (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$$

$$\Rightarrow A \vee \neg A \qquad A \Rightarrow \neg A \Rightarrow B$$

Hilbertian System: Prove $A \Rightarrow A$

Hilbertian System: Prove $A \Rightarrow A$

$$\dfrac{\dfrac{\dfrac{}{(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow (A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A} \qquad \dfrac{}{A \Rightarrow (A \Rightarrow A) \Rightarrow A}}{(A \Rightarrow A \Rightarrow A) \Rightarrow A \Rightarrow A} \qquad \dfrac{}{A \Rightarrow A \Rightarrow A}}{A \Rightarrow A}$$

Natural Deduction

Syntax

Deduction

Deduction

A deduction is a tree whose root (A) is the conclusion and whose leafs (Γ) is the set of hypotheses.

$$\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}$$

Any formula A is a valid hypothesis.

Proof (Demonstration)

A proof is a deduction without hypotheses.

Deduction

Deduction

A deduction is a tree whose root (A) is the conclusion and whose active leafs (Γ) is the set of hypotheses.

$$\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}$$

Any formula A is a valid hypothesis.

Proof (Demonstration)

A proof is a deduction without hypotheses.

Deductions

What's this?

$$A$$

Deductions

What's this?

$$A$$

A deduction of A under the hypothesis A.

Implication

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ A\Rightarrow B\end{matrix}}{B} \Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.
Note the connection with (left) contraction: any number of A (including 0) is discharged.

Implication

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ A\Rightarrow B\end{matrix}}{B} \Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.
Note the connection with (left) contraction: any number of A (including 0) is discharged.

Implication

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ A\Rightarrow B\end{matrix}}{B} \Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.
Note the connection with (left) contraction: any number of A (including 0) is discharged.

Implication

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \Rightarrow B}\Rightarrow\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A \Rightarrow B\end{matrix}}{B}\Rightarrow\mathcal{E}$$

Deduction theorem, and Modus Ponens.
Note the connection with (left) contraction: any number of A (including 0) is discharged.

Proving $A \Rightarrow A$ in Natural Deduction

Proving $A \Rightarrow A$ in Natural Deduction

$$\dfrac{[A]}{A \Rightarrow A}\Rightarrow\mathcal{I}$$

Conjunction

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}\wedge\mathcal{I} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\wedge l\mathcal{E} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B}\wedge r\mathcal{E}$$

Conjunction

$$\dfrac{\overset{\vdots}{A} \qquad \overset{\vdots}{B}}{A \wedge B}\,\wedge\mathcal{I} \qquad \dfrac{\overset{\vdots}{A \wedge B}}{A}\,\wedge l\mathcal{E} \qquad \dfrac{\overset{\vdots}{A \wedge B}}{B}\,\wedge r\mathcal{E}$$

Universal Quantification

$$\dfrac{\overset{\vdots}{A[y/x]}}{\forall x \cdot A}\,\forall\mathcal{I} \quad y \notin \mathrm{FV}(hyp(A)) \qquad \dfrac{\overset{\vdots}{\forall x \cdot A}}{A[t/x]}\,\forall\mathcal{E}$$

$$\dfrac{\dfrac{[A]}{\forall x \cdot A}\,\forall\mathcal{I}}{A \Rightarrow \forall x \cdot A}\,\Rightarrow\mathcal{I} \qquad \dfrac{\dfrac{[A]}{A \Rightarrow A}\,\Rightarrow\mathcal{I}}{\forall x \cdot (A \Rightarrow A)}\,\forall\mathcal{I}$$

Universal Quantification

$$\dfrac{\overset{\vdots}{A[y/x]}}{\forall x \cdot A}\,\forall\mathcal{I} \quad y \notin \mathrm{FV}(hyp(A)) \qquad \dfrac{\overset{\vdots}{\forall x \cdot A}}{A[t/x]}\,\forall\mathcal{E}$$

$$\dfrac{\dfrac{[A]}{\forall x \cdot A}\,\forall\mathcal{I}}{A \Rightarrow \forall x \cdot A}\,\Rightarrow\mathcal{I} \qquad \dfrac{\dfrac{[A]}{A \Rightarrow A}\,\Rightarrow\mathcal{I}}{\forall x \cdot (A \Rightarrow A)}\,\forall\mathcal{I}$$

Universal Quantification

$$\dfrac{\overset{\vdots}{A[y/x]}}{\forall x \cdot A}\,\forall\mathcal{I} \quad y \notin \mathrm{FV}(hyp(A)) \qquad \dfrac{\overset{\vdots}{\forall x \cdot A}}{A[t/x]}\,\forall\mathcal{E}$$

$$\dfrac{\dfrac{[A]}{\forall x \cdot A}\,\forall\mathcal{I}}{A \Rightarrow \forall x \cdot A}\,\Rightarrow\mathcal{I} \qquad \dfrac{\dfrac{[A]}{A \Rightarrow A}\,\Rightarrow\mathcal{I}}{\forall x \cdot (A \Rightarrow A)}\,\forall\mathcal{I}$$

Absurd

$$\frac{\begin{matrix}\vdots\\ \bot\end{matrix}}{A}\,\bot\mathcal{E}$$

Disjunction

$$\frac{\begin{matrix}\vdots\\ A\end{matrix}}{A \vee B}\,\vee l\mathcal{I} \qquad \frac{\begin{matrix}\vdots\\ B\end{matrix}}{A \vee B}\,\vee r\mathcal{I} \qquad \frac{\begin{matrix}\vdots\\ A \vee B\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathcal{E}$$

Existential Quantification

$$\frac{\begin{matrix}\vdots\\ A[t/x]\end{matrix}}{\exists x \cdot A}\,\exists\mathcal{I} \qquad \frac{\begin{matrix}\vdots\\ \exists x \cdot A\end{matrix} \quad \begin{matrix}[A[y/x]]\\ \vdots\\ B\end{matrix}}{B}\,\exists\mathcal{E} \quad y \notin \mathrm{FV}(B, hyp(B))$$

For elimination, $y \notin hyp(B)$, i.e., not in the hypotheses other than the discharged A.

Negation

$$\frac{\begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{\neg A}\,\neg\mathcal{I} \qquad \frac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ \neg A\end{matrix}}{\bot}\,\neg\mathcal{E}$$

Negation

$$\frac{\begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{\neg A}\ \neg\mathcal{I} \qquad \frac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ \neg A\end{matrix}}{\bot}\ \neg\mathcal{E}$$

Plus one of these equivalent formulation of the fact that classical negation is involutive.

$$\frac{}{A \vee \neg A}\ \mathrm{XM} \qquad \frac{\begin{matrix}\vdots\\ \neg\neg A\end{matrix}}{A}\ \neg\neg \qquad \frac{\begin{matrix}[\neg A]\\ \vdots\\ B\end{matrix} \quad \begin{matrix}[\neg A]\\ \vdots\\ \neg B\end{matrix}}{A}\ \mathrm{Contradiction}$$

Normalization

Cut

Cut: Introduction of a connective followed by its elimination.

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ B\end{matrix}}{A \wedge B}\ \wedge\mathcal{I}}{\begin{matrix}A\\ \vdots\end{matrix}}\ \wedge l\mathcal{E}$$

Normalization

The normalization process eliminates the cuts.

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \begin{matrix}\vdots\\ B\end{matrix}}{A \wedge B}\ \wedge\mathcal{I}}{A}\ \wedge l\mathcal{E} \quad \rightsquigarrow \quad \begin{matrix}\vdots\\ A\\ \vdots\end{matrix}$$

Normalizing Conjunctions

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\A\end{array}\quad\begin{array}{c}\vdots\\B\end{array}}{A\wedge B}\wedge\mathcal{I}}{\begin{array}{c}A\\\vdots\end{array}}\wedge l\mathcal{E}\quad\rightsquigarrow\quad\begin{array}{c}\vdots\\A\\\vdots\end{array}$$

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\A\end{array}\quad\begin{array}{c}\vdots\\B\end{array}}{A\wedge B}\wedge\mathcal{I}}{\begin{array}{c}B\\\vdots\end{array}}\wedge r\mathcal{E}\quad\rightsquigarrow\quad\begin{array}{c}\vdots\\B\\\vdots\end{array}$$

Normalizing Implications

$$\cfrac{\begin{array}{c}\vdots\\A\end{array}\quad\cfrac{\begin{array}{c}[A]\\\vdots\\B\end{array}}{A\Rightarrow B}\Rightarrow\mathcal{I}}{\begin{array}{c}B\\\vdots\end{array}}\Rightarrow\mathcal{E}\quad\rightsquigarrow\quad\begin{array}{c}\vdots\\A\\\vdots\\B\\\vdots\end{array}$$

Normalizing Universal Quantifiers

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\A\end{array}}{\forall x\cdot A}\forall\mathcal{I}}{\begin{array}{c}A[t/x]\\\vdots\end{array}}\forall\mathcal{E}\quad\rightsquigarrow\quad\begin{array}{c}\vdots\\A[t/x]\\\vdots\end{array}$$

x must not be free in the hypotheses, otherwise the reduction would change them.

Normalizing Disjunction

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\A\end{array}}{A\vee B}\vee l\mathcal{I}\quad\begin{array}{c}[A]\\\vdots\\C\end{array}\quad\begin{array}{c}[B]\\\vdots\\C\end{array}}{\begin{array}{c}C\\\vdots\end{array}}\vee\mathcal{E}\quad\rightsquigarrow\quad\begin{array}{c}\vdots\\A\\\vdots\\C\\\vdots\end{array}$$

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\B\end{array}}{A\vee B}\vee r\mathcal{I}\quad\begin{array}{c}[A]\\\vdots\\C\end{array}\quad\begin{array}{c}[B]\\\vdots\\C\end{array}}{\begin{array}{c}C\\\vdots\end{array}}\vee\mathcal{E}\quad\rightsquigarrow\quad\begin{array}{c}\vdots\\B\\\vdots\\C\\\vdots\end{array}$$

Bibliography Notes

[1] A short (160p.) book addressing all the concerns of this course, and more (especially Linear Logic). Easy and pleasant to read. Now available for free.

[2] A much more comprehensive book focusing on logic and its connections with computer science. In French.

Bibliography I

J.-Y. Girard, Y. Lafont, and P. Taylor.
Proofs and Types.
Cambridge University Press, 1989.
http://www.cs.man.ac.uk/~pt/stable/Proofs+Types.html.

Jean-Yves Girard.
Cours de Logique, Rome, Automne 2004.
http://logica.uniroma3.it/uif/corso/, 2004.