5.12 THE DESIGN OF FIR FILTERS USING THE 'FREQUENCY SAMPLING' METHOD

The starting point for the previous 'Fourier' or 'windowing' design method (Section 5.6), was the required frequency response. The IFT was then applied to convert the frequency response to the unit impulse response of the filter. The unit impulse response was then shifted, sampled and scaled to give the filter coefficients.

The 'frequency sampling' method is very similar but the big difference here is that the starting point is the *sampled* frequency response. This can then be converted *directly* to the FIR filter unit sample response, and so to the filter coefficients, by using the *IDFT.*

To demonstrate this approach we will design a lowpass FIR filter with a cut-off frequency of 2 kHz – the filter being used with a sampling frequency of 9 kHz. Therefore $f_c = 0.444f_N$.

The required magnitude response of the filter is shown in Fig. 5.18. For the purpose of demonstrating this technique the frequency has been sampled at just

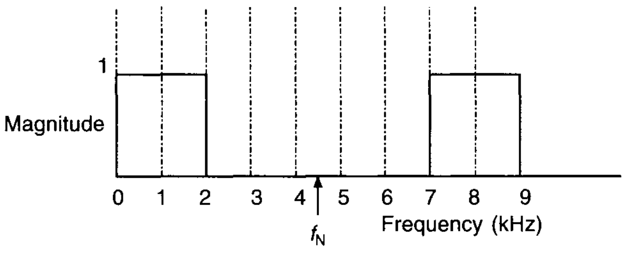

Figure 5.18

nine points, starting at 0 Hz, i.e. every 1 kHz. Note that f_s is *not* one of the sampled frequencies. To understand this, look back to Example 5.1, where a sampled time signal was converted to its corresponding sampled frequency spectrum using the DFT. By doing this, only frequencies up to $(N-1)f_s/N$ were generated, i.e. the sampled frequency spectrum stopped one sample away from f_s. As would be expected, by using the IDFT to process these frequency samples, the original signal was reconstituted. With the frequency sampling method, we start with the sampled frequency response, *stopping one sample short of* f_s, and aim to derive the equivalent sampled time function by applying the IDFT. As explained in Section 5.6, this time function will be the filter unit sample response, i.e. $Y(z) = 1 \times T(z)$, and so the IDFT of $T(z)$ must be the IDFT for $Y(z)$. These sampled signal values must also be the FIR filter coefficients.

The frequency samples for the required filter, derived from Fig. 5.18, are shown in Table 5.4.

Two important points to note. The first is that the response samples shown in Table 5.4 are the *magnitudes* of the sampled frequency response, $|X_k|$, whereas the X_k in the expression for the IDFT, equation (5.6), *also includes the phase angle.*

The second point is that the samples at $n = 2$ and $n = 7$ are shown as 0.5 rather

Table 5.4

n	$\lvert X_k \rvert$
0	1
1	1
2	0.5
3	0
4	0
5	0
6	0
7	0.5
8	1

than 1 or 0. By using the average of the two extreme values, we will design a much better filter, i.e. one that satisfies the specification more closely than if either 1 or 0 had been used.

We will now insert our frequency samples into the IDFT in order to find the sampled values of the unit impulse response and, hence, the filter coefficients. However, before we do this, a few changes will be made to the IDFT, so as to make it more usable.

At the moment our IDFT is expressed in terms of the complex values X_k, rather than the magnitude values, $\lvert X_k \rvert$, of Table 5.4. However, it can be shown that $\lvert X_k \rvert$ and X_k are related by:

$$\boxed{X_k = \lvert X_k \rvert e^{-j\frac{2\pi k}{N}\left(\frac{N-1}{2}\right)}}$$

or $X_k = \lvert X_k \rvert e^{-jk\pi\left(\frac{8}{9}\right)}$ for this particular example.

N.B. This exponential format is just a way of including the phase angle, i.e.

$$X_k = \lvert X_k \rvert e^{-j\frac{2\pi k}{N}\left(\frac{N-1}{2}\right)} = \lvert X_k \rvert \left(\cos \frac{2\pi k}{N}\left(\frac{N-1}{2}\right) - j \sin \frac{2\pi k}{N}\left(\frac{N-1}{2}\right)\right)$$

(from Euler's identity).

$$\therefore \tan \phi = -\frac{\sin \frac{2\pi k}{N}\left(\frac{N-1}{2}\right)}{\cos \frac{2\pi k}{N}\left(\frac{N-1}{2}\right)} = -\tan \frac{2\pi k}{N}\left(\frac{N-1}{2}\right)$$

where ϕ is the phase angle.

Therefore the *magnitude* of the phase angle is

$$\frac{2\pi k}{N}\left(\frac{N-1}{2}\right)$$

Although it's not obvious, this particular phase angle indicates a filter delay of $(N-1)/2$ sample periods. When we designed FIR filters using the 'Fourier' method in Section 5.6, we assumed zero phase before shifting, and then shifted our time response by $(N-1)/2$ samples to the right, i.e. we effectively introduced a filter delay of $(N-1)/2$ samples. This time-shifting results in a filter which has a phase response which is no longer zero for all frequencies *but the new phase response is linear*. In other words, by using this particular expression for the phase angle, we are imposing an appropriate constant filter delay for all signal frequencies. The exponential expression of $e^{-j\frac{2\pi k}{N}\left(\frac{N-1}{2}\right)}$ is the same for all linear-phase FIRs that have a delay of $(N-1)/2$ samples – which is the norm. For a fuller explanation, see Damper (1995).

Therefore, replacing X_k with $|X_k|e^{-j\frac{2\pi k\alpha}{N}}$, in the IDFT (equation (5.6)), where $\alpha = (N-1)/2$:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} |X_k| e^{-j\frac{2\pi k\alpha}{N}} e^{j\frac{2\pi kn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} |X_k| e^{j\frac{2\pi k(n-\alpha)}{N}}$$

Therefore, from Euler's identity:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} |X_k| (\cos[2\pi k(n-\alpha)/N] + j\sin[2\pi k(n-\alpha)/N])$$

Further, as $x[n]$ values must be real, i.e. they are actual signal samples, then we can ignore the imaginary 'sin' terms.

$$\therefore x[n] = \frac{1}{N}\sum_{k=0}^{N-1} |X_k| \cos[2\pi k(n-\alpha)/N]$$

Finally, as all X terms, apart from X_0, are partnered by an equivalent symmetric sample, i.e. $|X_1| = |X_8|$, $|X_2| = |X_7|$ etc. (Fig. 5.18), then we can further simplify to:

$$x[n] = \frac{1}{N}\left[\sum_{k=1}^{(N-1)/2} 2|X_k| \cos[2\pi k(n-\alpha)/N] + X_0\right] \tag{5.7}$$

We now only have to sum between $k = 1$ and $k = (N-1)/2$ rather than 0 and $(N-1)$ and so avoid some work! In our example this means summing between 1 and 4 rather than 0 and 8.

To recap, we can use equation (5.7) to find the values, $x[n]$, of the filter unit sample response. *These values are also the FIR filter coefficients.*

O.K – so let's use equation (5.7) to find the filter coefficients. Substituting $n = 0$ and $\alpha = (9-1)/2 = 4$ into equation (5.7), and carrying out the summation:

$$x[0] = \frac{1}{9}[2(\cos[2\pi(0-4)/9] + 0.5\cos[4\pi(0-4)/9]) + 1] = -0.0126$$

As $|X_3|$ and $|X_4| = 0$ we only have two terms to sum.

- $n = 1$:

$$x[1] = \frac{1}{9}[2(\cos[2\pi(1-4)/9] + 0.5\cos[4\pi(1-4)/9]) + 1] = -0.0556$$

- $n = 2$:

$$x[2] = \frac{1}{9}[2(\cos[2\pi(2-4)/9] + 0.5\cos[4\pi(2-4)/9]) + 1] = 0.0453$$

Check for yourself that $x[3] = 0.3006$ and $x[4] = 0.4444$.

N.B. As we are designing a *linear-phase* filter, then the signal samples must be symmetric about the central value, $x[4]$, i.e. $x[0] = x[8]$, $x[1] = x[7]$, etc., and so we only need to find the five $x[n]$ values, $x[0]$ to $x[4]$.

It follows that the nine samples of the filter unit sample response are:

–0.0126, –0.0556, 0.0453, 0.3006, 0.444, 0.3006, 0.0453, –0.0556, –0.0126

and, as these must also be our filter coefficients:

$$T(z) = -0.0126 - 0.0556z^{-1} + 0.0453z^{-2} + 0.3006z^{-3} + 0.444z^{-4} + 0.3006z^{-5} + 0.0453z^{-6} - 0.0556z^{-7} - 0.0126z^{-8}$$

or

$$T(z) = \frac{-0.0126z^8 - 0.0556z^7 + \ldots + 0.444z^4 + \ldots - 0.0556z - 0.0126}{z^8}$$

Figure 5.19 shows the frequency response of our filter and it looks pretty good. The cut-off frequency is not exactly the 2 kHz ($0.444f_N$) required, but it's not too far away. To be fair, to save too much mathematical processing getting in the way of the solution, far too few frequency samples were used. If we had taken a more realistic number of frequency samples than nine then the response would obviously have been much better.

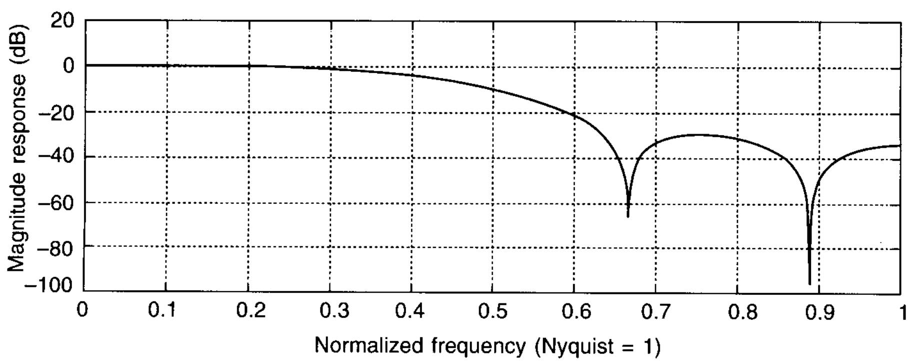

Figure 5.19

Here we used an odd number of frequency samples. The expression for the IDFT for an *even* number of samples is given in equation (5.8a). For completeness, the expression for an odd number is repeated as equation (5.8b).

$$x[n] = \frac{1}{N}\left[\sum_{k=1}^{(N/2)-1} 2\,|X_k|\cos[2\pi k(n-\alpha)/N] + X_0\right] \quad \text{(for even } N\text{)} \qquad (5.8a)$$

$$x[n] = \frac{1}{N}\left[\sum_{k=1}^{(N-1)/2} 2\,|X_k|\cos[2\pi k(n-\alpha)/N] + X_0\right] \quad \text{(for odd } N\text{)} \qquad (5.8b)$$

The 'even' expression is the same as for an odd number of samples apart from the k range being from 1 to $(N/2) - 1$ rather than 1 to $(N - 1)/2$. If you inspect equation (5.8a) you will notice that the $N/2$ sample is not included in the IDFT. For example, if we used eight samples, then X_0, X_1, X_2 and X_3 only would be used, and not X_4. I won't go into this in any detail but using an even number of samples does cause a slight problem in terms of the $N/2$ frequency term. One solution, which gives acceptable results, is to treat $X_{N/2}$ as zero – this is effectively what has been done in equation (5.8a). See Damper (1995) for an alternative approach.

5.13 SELF-ASSESSMENT TEST

A *highpass* FIR filter is to be designed, using the 'frequency sampling' technique. The filter is to have a cut-off frequency of 3 kHz and is to be used with a sampling frequency of 9 kHz. The frequency response is to be sampled every 1 kHz.

5.14 RECAP

- The discrete Fourier transform (DFT), is used to convert a sampled signal to its sampled frequency spectrum, while the inverse discrete Fourier transform (IDFT) achieves the reverse process.

- The sampled frequency spectrum obtained using the DFT consists of frequency components at frequencies of kf_s/N, for $k = 0$ to $N - 1$, where f_s is the sampling frequency and N the number of samples taken.

- By beginning with the desired sampled frequency response of a filter, it is possible to find the coefficients of a suitable linear-phase FIR filter by applying the IDFT.