- Mathematical Basics
  - Statements
  - Sets
  - Functions
  - Numbers
  - Sequences

0 Mathematical Basics

In this first section we will provide mathematical notations and concepts which are fundamental and crucial for the further presentation.

0.1 Statements

By a **statement** we mean a linguistic or mental construction that is either true or false.

Example 0.1

- "4 is an even number." is a true statement.
- "Bananas have conic shape." is a false statement.
- "In the night, it is colder than outside." is not a statement.
- "There are infinitely many stars." is a statement, which can be true or false.

Relations and Operations

$\neg A$: A is false (**negation**)

$A \Rightarrow B$: from A follows B; if A is true, then also B is true (**implication**)

we say: A is **sufficient** for B, B is **necessary** for A

$A \Leftrightarrow B$: A is true, if and only if B is true. (**equivalence**)

Note that the following two statements are equivalent

$$A \Rightarrow B$$
$$\neg B \Rightarrow \neg A$$

For example A: "it rains", B: "the street is we", then $\neg A$: ... and $\neg B$: ...

0.2 Sets

Definition 0.2 (*Set*) *According to* Cantor *a* ***set*** *is a well-defined collection of distinct objects, considered as an object in its own right. The objects that make up a set (also known as the set's* ***elements****) can be anything: numbers, people, letters of the alphabet, other sets, and so on.*

Notation: curly brackets $\{\}$

Example 0.3

- $M := \{1, 2, 3\}$
- $N := \{\underbrace{x}_{\text{element}} \mid \underbrace{x \text{ is multiple of } 7}_{\text{element property}}\} = \{x : x \text{ is multiple of } 7\} = \{7, 14, 21, \dots\}$

$\rightarrow$ Note: We use ":=" for definitions

Definition 0.4 (*Cardinality*) *If a set M is* ***finite*** *(i.e., it only contains finitely many elements), then we denote by $|M|$ the number of elements contained in M and call it* ***cardinality of*** *M.*

Set relations and further definitions

$a \in M$ (or $M \ni a$):	a is element of M; M contains a
$a \notin M$ (or $M \not\ni a$):	a is not element of M; M does not contain a

$M = \{1,2,3\}$, $1 \in M$, $\{1\} \notin M$

$M = N$:	M contains the same elements as N
$M \neq N$:	M does not contain the same elements as N

$N := \{1,2\}$, $M = M$, $M \neq N$

$M \subset N$ (or $M \subseteq N$):	M is subset of N, i.e., each element of M is also an element of N; equality of sets is permitted.
$N \supset M$ (or $N \supseteq M$):	N is superset of M; analogously
$M \subsetneqq N$:	M is strict subset of N; $M \neq N$
$\emptyset = \{\}$:	empty set

$N \subset M$ and even $N \subsetneqq M$; what about the relation between $N_2 := \{\{1\},2\}$ and M or $\emptyset$ and M?

Remark: Very useful in practice to show that two sets are equal:

$$M = N \iff M \subset N \ \textbf{ and } \ N \subset M$$

$M \times N$: **Cartesian product** defined by $M \times N := \{(m, n) \colon m \in M, n \in N\}$

$M^n := M \times \ldots \times M$ (n times)

Let us consider

$$M := \{x : x \text{ is multiple of } 2\} = \{2, 4, 6, \ldots\}$$

and

$$N := \{x : x \text{ is multiple of } 4\} = \{4, 8, 12, \ldots\},$$

then we have

$$M \times N = \{(a, b) : a \in M,\ b \in N\} = \{(2,4), (2,8), \ldots, (4,4), (4,8), \ldots\}$$

[image:Cartesian grid]

$\mathcal{P}(M)$ **power set of** M defined by
$\mathcal{P}(M) := 2^M := \{N : N \subset M\}$ (set of all subsets of M)
We find $|\mathcal{P}(M)| = 2^{|M|}$

Let us consider $M = \{1, 2\}$, then $|M| = 2$ and the power set of M is given by

$$\mathcal{P}(M) = 2^M = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

We also find

$$|\mathcal{P}(M)| = 4 = 2^{|M|} = 2^2.$$

Remark: **Binomial theorem**

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

where

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

is the so-called binomial coefficient, which give us the number of subsets with k elements that we can draw from a set with n elements.

Summary: Set relations and further definitions

$a \in M$ (or $M \ni a$):	a is element of M; M contains a
$a \notin M$ (or $M \not\ni a$):	a is not element of M; M does not contain a
$M = N$:	M contains the same elements as N
$M \neq N$:	M does not contain the same elements as N
$M \subset N$ (or $M \subseteq N$):	M is subset of N, i.e., each element of M is also an element of N; equality of sets is permitted.
$N \supset M$ (or $N \supseteq M$):	N is superset of M; analogously
$M \subsetneqq N$:	M is strict subset of N; $M \neq N$
$\emptyset = \{\}$:	empty set
$M \times N$:	**Cartesian product** defined by $M \times N := \{(m,n) \colon m \in M, n \in N\}$ $M^n := M \times \ldots \times M$ (n times)
$\mathcal{P}(M)$	**power set of** M defined by $\mathcal{P}(M) := 2^M := \{N : N \subset M\}$ (set of all subsets of M) We find $\lvert\mathcal{P}(M)\rvert = 2^{\lvert M\rvert}$

Set operations

$$
\begin{array}{lcll}
M \cup N & := & \{a : a \in M \text{ or } a \in N\} & (\textbf{union}) \\
M \cap N & := & \{a : a \in M \text{ and } a \in N\} & (\textbf{intersection}) \\
M \setminus N & := & \{a : a \in M \text{ and } a \notin N\} & (\textbf{difference}) \\
\text{If } N \subset M & & & \\
N^c & := & \overline{N} := M \setminus N & (\textbf{complement of N with respect to M})
\end{array}
$$

M, N are called **disjoint**, if $M \cap N = \emptyset$.

Example 0.5
Let $M = \{1, 2, 3, 4\}$ and $N = \{1, 3\}$, then

$M \cup N = \{1, 2, 3, 4\}$, we always have $M \subset M \cup N$ and $N \subset M \cup N$

$M \cap N = \{1, 3\}$

$\overline{N} = M \setminus N = \{2, 4\}$

For combinations of those set operations we have the following result:

Lemma 0.6 (*De Morgan's laws*) *Let Ω be a set and $M, N \subset \Omega$. Then we find*

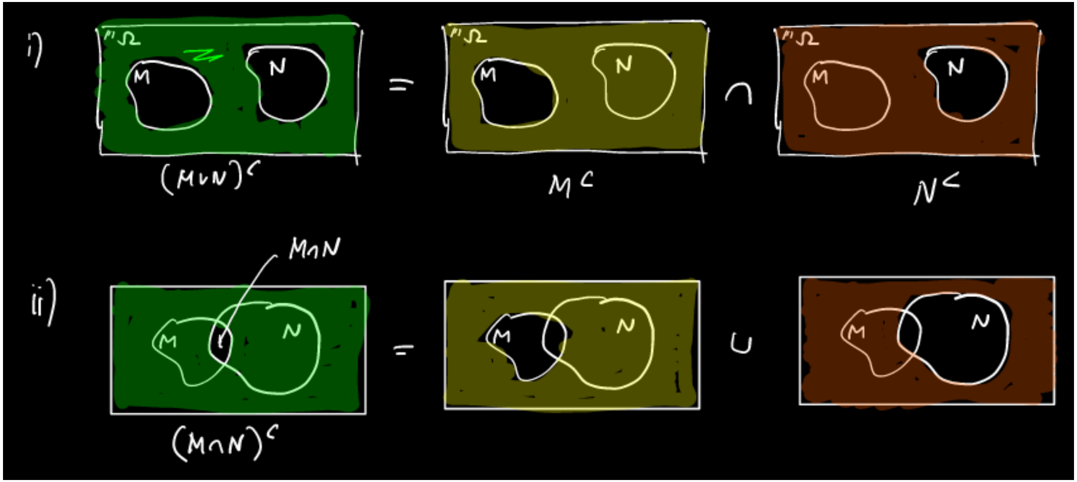
i) $(M \cup N)^c = M^c \cap N^c$,

ii) $(M \cap N)^c = M^c \cup N^c$.

Here the complements are taken with respect to Ω.

Proof. *Exercise.* □

Illustration:



0.3 Functions

Definition 0.7 (*function*) *Let M, N be two sets. A **function** or **mapping** f from M to N (notation: $f : M \to N$) is determined by*

*its **domain** M,*

*its **codomain** N,*

*and a **rule**,*

that uniquely assigns to each element $a \in M$ an $b := f(a) \in N$ (notation: $a \mapsto f(a)$).

Two functions $f_1 : M_1 \to N_1$ and $f_2 : M_2 \to N_2$ are called **equal** (abbr. $f_1 \equiv f_2$) (identical), if $M_1 = M_2$, $N_1 = N_2$ and $f_1(a) = f_2(a)$ for all $a \in M_1$ (i.e., equal, domain, codomain and rule).

Example 0.8
Let $M := N := \{1, 2, 3, 4, 5 \ldots\}$ and consider $f : M \to N$, $a \mapsto 2a$. How could we "visualize" this function?

i) a **table** with two columns (one for domain and codomain each)

follow where points go from $A = \{1, 2\} \subset M$

find points which would produce $B = \{4, 6\} \subset N$

ii) draw the **graph** into a coordinate system

We introduce function related sets:

Definition 0.9 (*Image, preimage, graph*) *Let $A \subset M$ and $B \subset N$, then*

i) *the set $f(A) = \{f(a) : a \in A\} \subset N$ is called **image set** of A (under f),*

ii) *the set $f^{-1}(B) = \{a \in M : f(a) \in B\} \subset M$ is called **preimage** of B (under f),*

iii) *the set graph$(f) := \{(a, f(a)) : a \in M\} \subset M \times N$ is called the **graph** of f.*

$\rightarrow$ **Attention:** Here, f^{-1} is not the inverse function (see below).

– [abstract picture with with image, preimage and graph]

– With the definitions from above we have

for $A = \{1, 2\}$ we find $f(A) = \{2, 4\}$

for $B = \{4, 6\}\}$ we find $f^{-1}(B) = \{2, 3\}$

graph$(f) = \{(1, 2), (2, 4), (3, 6), \ldots\}$

Important properties of functions:

Definition 0.10 (*Injective, surjective, bijective*) *A function $f : M \to N$ is called*

i) ***injective*** *(one-to-one), if $f(a) \neq f(\tilde{a})$ for all $a, \tilde{a} \in M$ with $a \neq \tilde{a}$;*

ii) ***surjective*** *(onto), if for all $b \in N$ there exists an $a \in M$ with $f(a) = b$ (or equivalently $f(M) = N$);*

iii) ***bijective****, if f is injective as well as surjective relation.*

We can invert bijective functions:

Definition 0.11 (*Inverse function*) *Let $f\colon M \to N$ be a bijective (invertible) function. Then there exists a (unique) function $f^{-1}\colon N \to M$, the so-called **inverse of** f, such that*

$$f(a) = b \iff f^{-1}(b) = a.$$

Example 0.12 Consider again example from above and the visualization with the help of the table.

We can concatenated two or more functions:

Definition 0.13 (*Composition*) *Let $f\colon M \to N$ and $g\colon N \to P$ be functions, then we call the function*

$$g \circ f\colon M \to P,\ a \mapsto g(f(a))$$

***composition** of f and g.*

– [abstract picture for composition of the functions

– Concrete example:

$$M = \{1, 2, 3, 4, \dots\},\ N := \{x :\ x \text{ is even}\},\ P := \{x :\ x \text{ is odd}\},$$
$$f : M \to N, a \mapsto 2a,\ g : N \to P, b \mapsto b - 1$$

$$(g \circ f)(6) = g(\underbrace{f(6)}_{=12}) = g(12) = 12 - 1 = 11$$

A function with "no effect":

Definition 0.14 (*Identity function*) *Let M be a set. Then the function*

$$id := id_M\colon M \to M,\ a \mapsto a$$

*is called the **identity function on** M.*

0.4 Numbers

The notion of **number** has been extended over the centuries, here we do not go in detail through the axiomatic construction but just point out some properties that are useful in the remaining.

Here is an overview:

$\mathbb{N}$	**Natural**	$\{1,2,3,4,5,\ldots\}$	counting objects **order relation**: $m \leq n$ $m+n$, $m \cdot n$ proof concept of **induction**
$\mathbb{Z}$	**Integer**	$\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$	adding zero and negative numbers (borrowing money,...) $(\mathbb{Z},+)$ **ordered, commutative group**
$\mathbb{Q}$	**Rational**	$\left\{\frac{p}{q} : \; p,q \in \mathbb{Z}, q \neq 0\right\}$	adding fractions of objects (one half of a cake) $(\mathbb{Q},+,\cdot)$ **ordered field**
$\mathbb{R}$	**Real**	$\mathbb{Q} \cup \{$limits of sequences in $\mathbb{Q}\}$	adding square roots ($\sqrt{2}$,$\sqrt{5}$,...), π,... $(\mathbb{R},+,\cdot)$ **ordered and complete field**
$\mathbb{C}$	**Complex**	$\{a+ib : \; a,b \in \mathbb{R}\}$, $i := \sqrt{-1}$	adding e.g. square root of negative numbers $\mathbb{R} \times \mathbb{R}$ with a special multiplication $(\mathbb{C},+,\cdot)$ **complete field** (not ordered)

We have

$$\mathbb{N} \subsetneqq \mathbb{Z} \subsetneqq \mathbb{Q} \subsetneqq \mathbb{R} \subsetneqq \mathbb{C}$$

0.4.1 Complex Numbers $\mathbb{C}$

$\mathbb{C} =$ "$\mathbb{R} \times \mathbb{R}$ with a special multiplication"

- extension: e.g., imaginary unit $i := \sqrt{-1}, \sqrt{-3}, \ldots$
- in real life: electricity and roots of polynomials (e.g., which do not touch the x-axis)

Definition 0.15 (*Complex numbers* $\mathbb{C}$) *We define the field of complex numbers* $(\mathbb{C}, +, \cdot)$ *by* $\mathbb{C} := \mathbb{R} \times \mathbb{R}$ *with the binary operations*

$$\begin{aligned} + :\ & (a_1, b_1) + (a_2, b_2) := (a_1 + a_2,\ b_1 + b_2), \\ \cdot :\ & (a_1, b_1)\ \cdot\ (a_2, b_2) := (a_1 a_2 - b_1 b_2,\ a_1 b_2 + a_2 b_1). \end{aligned}$$

Note that $\mathbb{R}$ *itself is identified with* $\mathbb{R} \times \{0\} \subset \mathbb{C}$.

Remarks

- the product in $\mathbb{C}$ is all the magic
- $(\mathbb{C}, +, \cdot)$ is a (*complete*) **field**
- as a 2-dimensional object, $\mathbb{C}$ does **not** possess an order relation

In order to alleviate the memorizing of the product definition, it is customary to use the so-called **imaginary unit** $i := \sqrt{-1}$ and perform computations as if it would be a real number:

For $z = (a_1, b_1),\ w = (a_2, b_2) \in \mathbb{C}$ we write

$$z = a_1 + ib_1 \text{ and } w = a_2 + ib_2.$$

Then the product naturally computes as

$$z \cdot w = (a_1 + ib_1) \cdot (a_2 + ib_2) = a_1a_2 + ib_1a_2 + ia_1b_2 + \underbrace{i^2}_{-1} b_1b_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

Example 0.16

$$z \in \mathbb{C} : z = a + ib$$

$$(1 + 2i) \cdot (3 + 4i) = 3 + 4i + 6i + 8i^2 = 3 + 10i - 8 = -5 + 10i$$

$$\frac{1 + 2i}{3 + 4i} = \frac{(1 + 2i)(3 - 4i)}{(3 + 41)(3 - 4i)} = \frac{3 - 4i + 6i - 8i^2}{9 + 16} = \frac{11 + 2i}{25}$$

[note on real and imaginary part, complex conjugate]

In $\mathbb{C}$ every non-constant polynomial has at least one root in $\mathbb{C}$ (we say $\mathbb{C}$ is *algebraically closed*):

Theorem 0.17 (*Fundamental theorem of algebra*) *Let $\alpha_0, \alpha_1 \ldots, \alpha_n \in \mathbb{C}$ with $n \geq 1$, $\alpha_n \neq 0$ (i.e., nonzero leading coefficient) and consider the nonconstant polynomial $p \colon \mathbb{C} \to \mathbb{C}$,*

$$p(z) := \sum_{i=0}^{n} \alpha_i z^i.$$

Then, there are numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$p(z) = \alpha_n \prod_{i=1}^{n} (z - \lambda_i) = \alpha_n \cdot (z - \lambda_1) \cdot \ldots \cdot (z - \lambda_n), \quad \forall z \in \mathbb{C}.$$

In particular, the λ_i are precisely the roots of p, i.e., $p(\lambda_i) = 0$ for $i = 1, \ldots, n$.

Example 0.18
Consider the polynomial $p \colon \mathbb{C} \to \mathbb{C},\ p(z) := z^2 + 1$.
What are the roots of p?

$$0 = p(z) = z^2 + 1 \Leftrightarrow z^2 = -1 \Leftrightarrow z \in \{i, -i\}$$

We find

$$p(z) = (z - i)(z + i) \qquad (= z^2 + 1)$$

The polynomial p has no roots considered as a function $\mathbb{R} \to \mathbb{R}$, but exactly two roots as a function $\mathbb{C} \to \mathbb{C}$

0.4.2 Summary

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

$\mathbb{N}$	**Natural**	$\{(0), 1, 2, 3, \dots\}$	order relation
$\mathbb{Z}$	**Integer**	$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$	$(\mathbb{Z}, +)$ ordered, commutative group
$\mathbb{Q}$	**Rational**	$\{\frac{p}{q} : \ p, q \in \mathbb{Z}, q \neq 0\}$	$(\mathbb{Q}, +, \cdot)$ ordered field
$\mathbb{R}$	**Real**	$\mathbb{Q} \cup \{$limits of sequences in $\mathbb{Q}\}$	$(\mathbb{R}, +, \cdot)$ ordered and complete field
$\mathbb{C}$	**Complex**	$\{a + ib : \ a, b \in \mathbb{R}\}$, $i := \sqrt{-1}$	$(\mathbb{C}, +, \cdot)$ algebraically closed field

Most **theoretical** investigations deal with real numbers $r \in \mathbb{R}$.

Numerical computations can only be performed with *floating point numbers* (short: *floats*) with a relative error (typically 10^{-16}) in each operation.

Many of the following results hold for general fields, say $(\mathbb{F}, +, \cdot)$. However the only fields we will know about are the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$; thus we always think of $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

0.5 Sequences

Numerical methods often produce sequences which (in the best case) *converge* to a desired solution. Also besides this, the concept of a limiting process to *infinity* is the basis for many other notions in mathematics (differentiation/integration/...).

For simplicity, in the following we only consider sequences in $\mathbb{R}$. In order to have a notion of "distance" we will consider the metric

$$d\colon \mathbb{R}\times\mathbb{R} \to [0,+\infty), \quad d(x,y) := |x-y|,$$

where

$$|x| := \mathrm{abs}(x) := \begin{cases} x\,, & \text{if } x \geq 0 \\ -x\,, & \text{else} \end{cases} \qquad \textbf{(absolute value} \text{ of } x\text{).}$$

In the following, $\mathbb{R}$ can also be replaced by any set X which can be equipped with a so-called **metric** d (in math we call (X,d) a metric space).

Example 0.19 Consider the set

$$\left\{\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots\right\} = \left\{\left(\frac{1}{2}\right)^k : k \in \mathbb{N}\right\} = \{x(k) : k \in \mathbb{N}\}$$

What happens for large k? Also have a look at the graph.

We introduce the notion of sequence and some important related properties:

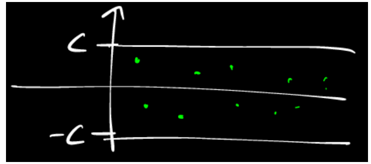
Definition 0.20 (*Sequence*) *Let M be a set. Then a function $x\colon \mathbb{N} \to M$ is called a* ***sequence****.*
Notation: $(x^k)_{k\in\mathbb{N}}$, $\{x^k\}_{k\in\mathbb{N}}$ or $\{x^k\}_{k=1}^{\infty}$.

Example 0.21 Let $a \in \mathbb{R}$.
i) The **constant sequence** $\{a\}_{k\in\mathbb{N}}$, i.e., $x^k := a$ for $k \in \mathbb{N}$ trivially converges to $\bar{x} := a$.
ii) The sequence $\{\frac{1}{k}\}_{k\in\mathbb{N}}$, i.e., $x^k := \frac{1}{k}$ for $k \in \mathbb{N}$ is a null sequence, since it converges to $\bar{x} := 0$.
iii) The sequence $\{a + \frac{1}{k}\}_{k\in\mathbb{N}}$, i.e., $x^k := a + \frac{1}{k}$ converges to $\bar{x} := a$.
iv) The sequence $\{a \cdot \frac{1}{k}\}_{k\in\mathbb{N}}$, i.e., $x^k := a \cdot \frac{1}{k}$ converges to $\bar{x} := 0$.
v) The **alternating sequence** $\{a(-1)^k\}_{k\in\mathbb{N}}$, i.e., $x^k := a(-1)^k$ does not converge.

Definition 0.22 *Let $(x^k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then $(x^k)_{k\in\mathbb{N}}$ is called*

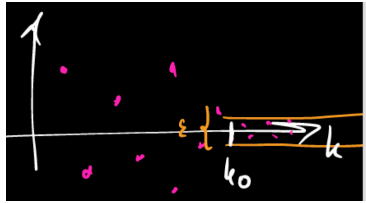
i) ***bounded**, if there exists a <u>uniform bound</u> $C > 0$ such that for all $k \in \mathbb{N}$*

$$d(x^k, 0) = |x^k| \leq C.$$



ii) ***Cauchy**, if for any $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that for all $m, n \geq k_0$*

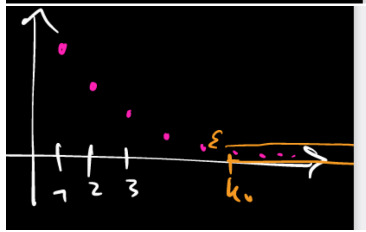
$$d(x^m, x^n) = |x^m - x^n| < \varepsilon.$$



iii) ***null sequence**, if for any $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$*

$$d(x^k, 0) = |x^k| < \varepsilon.$$

We write $\lim\limits_{k\to\infty} x^k = 0$.



iv) ***convergent**, if there exists a $\bar{x} \in \mathbb{R}$ such that $(x^k - \bar{x})_{k\in\mathbb{N}}$ is a null sequence, i.e., $\lim\limits_{k\to\infty}(x^k - \bar{x}) = 0$.*
We write $\lim\limits_{k\to\infty} x^k = \bar{x}$.

v) ***divergent**, if it does not converge.*

Example 0.23 Let $a \in \mathbb{R}$. The following sequences are bounded:
i) The constant sequence $\{a\}_{k \in \mathbb{N}}$ (choose $C = a$),
ii) The sequence $\{\frac{1}{k}\}_{k \in \mathbb{N}}$ (choose $C = 1$),
iii) The **alternating sequence** $\{a(-1)^k\}_{k \in \mathbb{N}}$ (choose $C = a$). This sequence is not convergent but bounded.

An example of an unbounded sequence is given by
iv) the sequence $\{k^2\}_{k \in \mathbb{N}}$.

One can show for a sequence $(x^k)_{k \in \mathbb{N}}$ in $\mathbb{R}$:

$$(x^k)_{k \in \mathbb{N}} \text{ convergent } \Rightarrow (x^k)_{k \in \mathbb{N}} \text{ Cauchy } \Rightarrow (x^k)_{k \in \mathbb{N}} \text{ bounded.}$$