Recommended reading:

- Lectures 24, 25, 27 in [4]
- Sections I.6 in [3]
- Sections 6.1, 6.2, 6.4 in [2]
- Kapitel 7 in [1]

Literature:

[1] R. Rannacher.
Numerik 0 - Einführung in die Numerische Mathematik.
Heidelberg University Publishing, 2017.

[2] G. Strang.
Introduction to Linear Algebra.
Wellesley-Cambridge Press, 2003.

[3] G. Strang.
Linear Algebra and Learning from Data.
Wellesley-Cambridge Press, 2019.

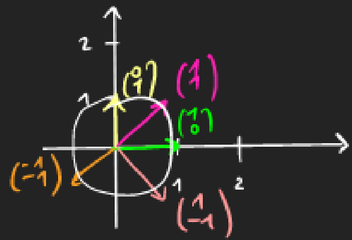
[4] L.N. Trefethen and D. Bau.
Numerical linear algebra.
SIAM, Soc. for Industrial and Applied Math., Philadelphia, 1997.

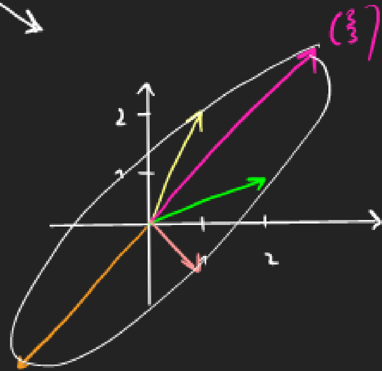
3 Eigenvalues: Theory and Algorithms

3.1 Introduction

Example 3.1 (*Illustration in 2d: Part 1*)

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$





$$A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

eigenvector to the eigenvalue 3

$$A \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvector to the eigenvalue 1

3.2 Eigenvalues and Eigendecompositon

Definition 3.2 (*Eigenvalues and -vectors*) *Let $A \in \mathbb{F}^{n\times n}$ be a matrix. A number $\lambda \in \mathbb{C}$ is called an* ***eigenvalue*** *of A, if*

$$\exists v \in \mathbb{F}^n, v \neq 0: \;\; Av = \lambda v.$$

In that case, v is called an ***eigenvector*** *and (λ, v) an* ***eigenpair***. *The set of all eigenvalues is denoted by*

$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda \text{ is eigenvalue of } A\}$$

and called the ***spectrum of*** *A.*

1) Assume we knew an eigenvalue λ:
   Then we find a corresponding eigenvector by solving the linear equation

   $$(A - \lambda I_n)v = 0$$

   Observation:

   $$v \text{ eigenvector} \quad \Rightarrow \quad \alpha v \text{ eigenvector } \forall \alpha \in \mathbb{F}$$

   We often normalize the eigenvector by $\frac{v}{\|v\|_2}$.

2) Assume we had an eigenvector v:
   Then the corresponding eigenvalue is uniquely determined by the so-called *Rayleigh-Quotient*

   $$\lambda = \frac{v^T A v}{v^T v}$$

The determinant and eigenvalues

Let $A \in \mathbb{F}^{n\times n}$. Then:

1) **Relation between the determinant and eigenvalues:**

$$\begin{aligned}\lambda \in \mathbb{C} \text{ eigenvalue of } A \quad &\Leftrightarrow \quad \exists v \neq 0\colon Av = \lambda v \quad \Leftrightarrow \quad \exists v \neq 0\colon \ (A - \lambda I_n)v = 0 \\ &\Leftrightarrow \quad \exists v \neq 0\colon \ v \in \ker(A - \lambda I_n) \quad \Leftrightarrow \quad (A - \lambda I_n) \text{ not injective} \\ &\Leftrightarrow \quad (A - \lambda I_n) \notin \mathrm{GL}(n, \mathbb{F}) \quad \Leftrightarrow \quad \det(A - \lambda I_n) = 0\end{aligned}$$

2) **Implication:**
By invoking the Laplace formula (see Def.1.37) for the determinant we can show that the function

$$\lambda \mapsto \chi_A(\lambda) := \det(A - \lambda I_n)$$

is a **polynomial of degree** $\leq n$. Thus, we can state:

The eigenvalues of A are the roots of the polynomial $\chi_A(\lambda)$.

The fundamental theorem of algebra then assures the existence of eigenvalues (at most n distinct ones).

Definition: The polynomial $\chi_A(\lambda)$ is called **characteristic polynomial of A**.

Example 3.3 (*Illustration in 2d: Part 2*)
Let us consider the (2×2) matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

from Example 3.1 above.

- We compute its eigenvalues by solving the following root finding problem:

$$0 = \chi_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}\right) = (2-\lambda)^2 - 1$$
$$\Leftrightarrow \quad \lambda \in \{3, 1\} =: \{\lambda_1, \lambda_2\} = \sigma(A)$$

- Now that we have the eigenvalues we can find corresponding eigenvectors by solving the following homogeneous linear systems:
  - For $\lambda_1 = 3$:

$$(A - \lambda_1 I)v^1 = 0 \quad \Leftrightarrow \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} v^1 = 0 \quad \Rightarrow \quad v^1_1 - v^1_2 = 0$$

Thus, the set of all eigenvectors corresponding to the eigenvalue λ_1 is given by

$$E(\lambda_1) := \{v \in \mathbb{R}^2 : \ Av = \lambda_1 v\} \quad = \quad \{v \in \mathbb{R}^2 : \ v^1_1 = v^1_2\} = \quad \left\{\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \in \mathbb{R}^2 : \ \alpha \in \mathbb{R}\right\} = \quad \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

Sometimes it is reasonable to choose eigenvectors of length 1, so that we normalize: $v^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- For $\lambda_2 = 1$:

$$(A - \lambda_2 I)v^2 = 0 \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v^2 = 0 \quad \Leftrightarrow \quad v_1^2 + v_2^2 = 0$$

  Thus, the set of all eigenvectors corresponding to the eigenvalue λ_2 is given by

$$E(\lambda_2) := \{v \in \mathbb{R}^2 : \; Av = \lambda_2 v\} = \{v \in \mathbb{R}^2 : \; v_1^2 = -v_2^2\} = \{\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} \in \mathbb{R}^2 : \; \alpha \in \mathbb{R}\} = \text{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

  Normalization: Choose $v^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Remark:
The set of all eigenvectors corresponding to the eigenvalue $\lambda \in \sigma(A)$, i.e.,

$$E(\lambda) = \ker(A - \lambda I) \subset \mathbb{F}^n$$

is called **eigenspace to the eigenvalue λ of A**.

Lemma 3.4 (*Matrix and Eigenvalue Properties*)

i) *Power of a matrix:* $A \in \mathbb{F}^{n\times n}$, $\lambda \in \sigma(A) \quad \Rightarrow \quad \lambda^k \in \sigma(A^k)$ *for any* $k \in \mathbb{N}$

ii) *Inverse matrix:* $A \in GL_n(\mathbb{F})$, $\lambda \in \sigma(A) \quad \Rightarrow \quad \lambda \neq 0$, $\frac{1}{\lambda} \in \sigma(A^{-1})$

iii) *Scaling:* $A \in \mathbb{F}^{n\times n}$, $\lambda \in \sigma(A) \quad \Rightarrow \quad \alpha\lambda \in \sigma(\alpha A)$ *for any* $\alpha \in \mathbb{F}$

iv) $A \in \mathbb{F}^{n\times n}$*hermitian* $(A = A^H) \quad \Rightarrow \quad \sigma(A) \subset \mathbb{R}$.

v) $Q \in \mathbb{F}^{n\times n}$ *unitary* $(Q^H Q = I)$, $\lambda \in \sigma(Q) \quad \Rightarrow \quad |\lambda| = 1$

vi) $A \in \mathbb{F}^{n\times n}$ *positive definite (semi-definite)* $(x^H Ax > 0\ (\geq 0)) \quad \Leftrightarrow \quad \forall \lambda \in \sigma(A)\colon\ \lambda > 0\ (\lambda \geq 0)$

vii) *The eigenvalues of an upper (lower) triangular matrix are sitting on the diagonal.*

viii) *Similarity transformation:* $A \in \mathbb{F}^{n\times n}$, $T \in GL_n(\mathbb{F}) \quad \Rightarrow \quad \sigma(A) = \sigma(T^{-1}AT)$

ix) *Shifts:* $A \in \mathbb{F}^{n\times n}$, (λ, v) *eigenpair of* $A \quad \Rightarrow \quad \forall s \in \mathbb{F}\colon\ (\lambda + s, v)$ *eigenpair of* $A + sI$

Attention: The following rules do not hold in general:

- $\lambda \in \sigma(A), \mu \in \sigma(B) \quad \not\Rightarrow \quad (\lambda + \mu) \in \sigma(A + B)$
- $\lambda \in \sigma(A), \mu \in \sigma(B) \quad \not\Rightarrow \quad (\lambda \cdot \mu) \in \sigma(A \cdot B)$

Proof. Exercise. Here, we exemplary prove viii): □

Diagonalizing a matrix

Let us consider a matrix $A \in \mathbb{F}^{n\times n}$ with eigenpairs $(\lambda_i, v_i) \in \mathbb{F} \times \mathbb{F}^n$, so that

$$Av_i = \lambda_i v_i, \quad \text{for } 1 \leq i \leq n.$$

Using matrix notation this can be written as

$$A \cdot \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}}_{=:V\in\mathbb{F}^{n\times n}} = \begin{pmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{=:\Lambda\in\mathbb{F}^{n\times n}}$$

which is equivalent to

$$AV = V\Lambda.$$

If V is invertible (note that this is not necessarily the case!), then we can rearrange this into the following decomposition

$$V^{-1}AV = \Lambda \quad \Leftrightarrow \quad A = V\Lambda V^{-1}.$$

One central question arises: When is V invertible?

Let us first revisit the example from above (see Examples 3.1 and 3.3)

Example 3.5 (*Illustration in 2d: Part 3)*

Let us again consider the *real symmetric* matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

with eigenpairs

$$\lambda_1 = 3,\ v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 1,\ v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Assembling the normalized eigenvectors into the matrix V yields

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since for the columns we have

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 - 1 = 0$$

and by construction

$$\|v_1\|_2 = \frac{1}{\sqrt{2}} \underbrace{\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|}_{=\sqrt{2}} = 1, \quad \text{and similarly} \quad \|v_2\|_2 = 1,$$

we find that V is orthogonal and thus in particular invertible.

In the previous Example 3.5 the matrix V of eigenvectors turned out to be orthogonal. The next theorem states, that this is true for any real symmetric matrix.

Theorem 3.6 (*Eigendecomposition of real symmetric matrices*) *For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, there is an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ (i.e., $Q^\top Q = I$) such that*

$$Q^\top A Q = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} =: \mathit{diag}(\lambda_1, \dots, \lambda_n) \quad \text{(= diagonal matrix)}$$

and $\lambda_i \in \mathbb{R}, i \in \{1, \dots, n\}$, are the eigenvalues of A. The columns of Q are the normalized eigenvectors.

Proof. In the exercises we will prove this statement for the special case that the matrix has n distinct eigenvalues. The general proof is rather technical and can be found in any standard textbook. $\square$

$\rightarrow$ **Thus:** "knowing the eigenpairs = knowing the matrix"

An immediate consequence of Theorem 3.6 is this:

Corollary 3.7 *A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is invertible, if and only if all its eigenvalues are nonzero.*

Let us again continue our example:

Example 3.8 (*Illustration in 2d: Part 4*)
For the *symmetric* matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

with eigenpairs

$$\lambda_1 = 3,\ v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 1,\ v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

let us set

$$Q := V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \Lambda := \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

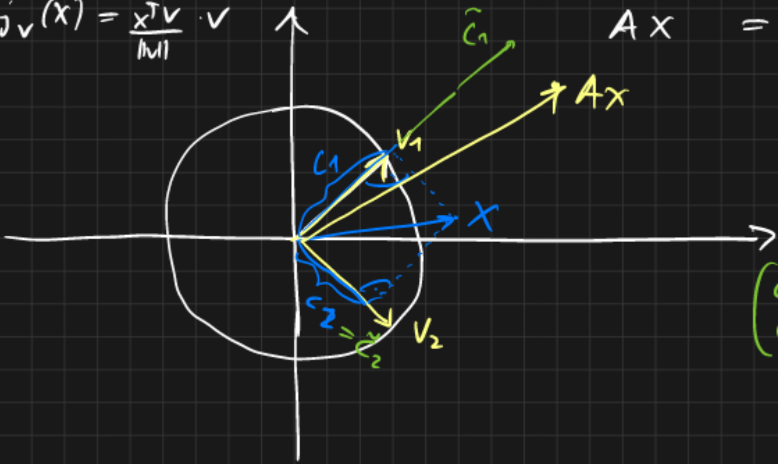
Indeed, we can verify

$$A = Q\Lambda Q^T = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{=\begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix}}}_{=\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Geometry of the eigendecomposition:

$$\mathrm{proj}_v(x) = \frac{x^T v}{\|v\|} \cdot v$$



$$Ax = Q \Lambda \underbrace{Q^T x}$$

$$\begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix} x = \begin{pmatrix} v_1^T x \\ v_2^T x \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix} = \begin{pmatrix} 3c_1 \\ c_2 \end{pmatrix}$$

$$Ax = \underbrace{Q \underbrace{\Lambda \underbrace{Q^T x}_{\text{projection}}}_{\text{scaling}}}_{\text{embedding}}$$

$$\begin{array}{ccc} c & \xrightarrow{\Lambda} & \tilde{c} \\ {\scriptstyle Q^T}\uparrow & & \downarrow{\scriptstyle Q} \\ x & \longmapsto & Ax \end{array}$$

3.3 Eigenvalue Algorithms: Solving the eigenvalue problem

Aim: Solving the ***eigenvalue problem*** defined by

$$\textit{Given } A \in \mathbb{F}^{n\times n}, \textit{ find eigenpairs } (\lambda_i, v_i) \textit{ so that, for all } i = 1, \dots, n,$$

$$v_i \neq 0 \quad \textit{and} \quad Av_i = \lambda_i v_i.$$

Sometimes we are only interested in a few eigenpairs (λ_i, v_i) (for example the one with largest eigenvalue in magnitude). In this case we call it a *partial* eigenvalue problem.

Overview

1. **A first naive approach: Direct method**
   $\rightarrow$ only feasible for very small matrices: $n \in \{2, 3, 4\}$

2. **Partial eigenvalue problem: Simple iterative methods** (here: The Power Method)
   $\rightarrow$ determine a *single* eigenpair

3. **A second advanced approach: General iterative method** (here: The QR algorithm)
   $\rightarrow$ determine *all* eigenpairs

3.3.1 A first naive approach: Direct method

Recipe:

a) Eigenvalues:
Solving **root finding problem** for the characteristic polynomial

$$\chi_A(\lambda) := \det(A - \lambda I) = 0$$

yields the eigenvalues λ_i.

b) Eigenvectors:
Solving the homogeneous **linear system**

$$(A - \lambda_i I)v_i = 0$$

for each distinct λ_i, gives the corresponding eigenvectors v_i (or more precisely, eigenspaces).

Example: $n = 2$

Consider a general (2×2)-matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

a) Root finding problem:
Above, we have derived a closed formula for the determinant of a (2×2)-matrix, which applied to $A - \lambda I$ gives

$$0 = \chi_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\right) = (a-\lambda)(d-\lambda) - cb = \lambda^2 - (a+d)\lambda + (ad - cb)$$

$$\rightarrow \lambda_{1/2} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - (ad - cb)}.$$

b) Linear system:
We then have to solve

$$\begin{pmatrix} a-\lambda_i & b \\ c & d-\lambda_i \end{pmatrix} \begin{pmatrix} v_1^i \\ v_2^i \end{pmatrix} \quad \text{for} \quad i = 1, 2.$$

$$\rightarrow v^1, v^2$$

Note: For $n = 3$ we can proceed similarly by applying the rule of Sarrus in step a).

Problem:

In practice, for general, potentially very large, matrices the root finding problem is infeasible, because:

$$A \text{ with large dimension } n \quad \Rightarrow \quad \chi_A \text{ high polynomial degree} \quad \Rightarrow \quad \text{high risk of rounding errors}$$

See for example:
https://en.wikipedia.org/wiki/Root-finding_algorithms#Roots_of_polynomials

Abel–Ruffini theorem (see related discussion in [4, Theorem 25.1]):
There are no "closed formulas" for the roots of general polynomials with degree higher than 4.

As a consequence:

We cannot solve the eigenvalue problem in finitely many steps.
Instead, any eigenvalue algorithm has to be iterative!

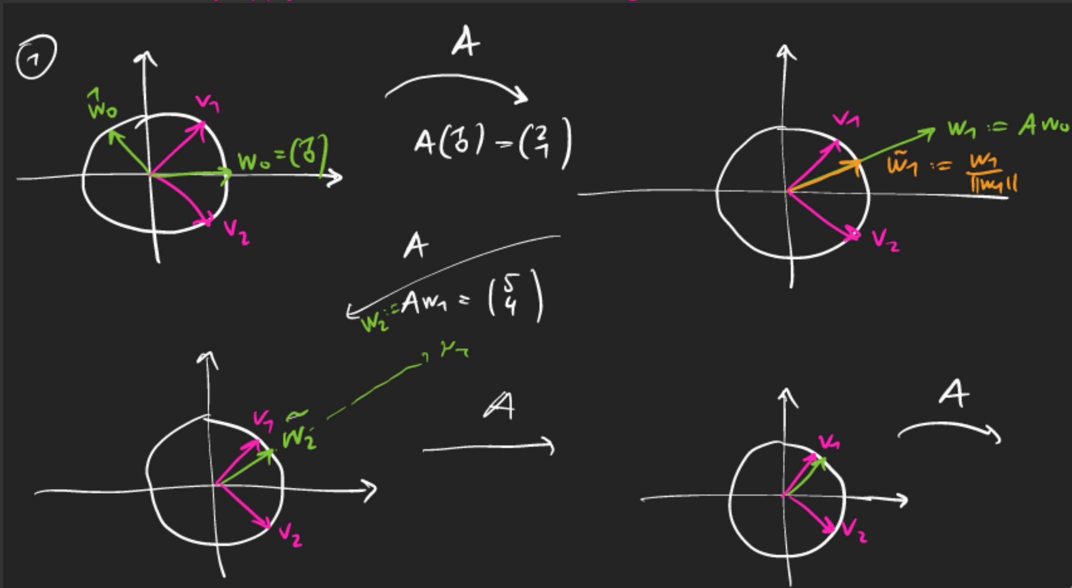
3.3.2 Simple Iterative Method: The Power Iteration

$\rightarrow$ basis for PageRank algorithm from Google and the WTF algorithm from Twitter

Example 3.9 (*Illustration in 2d: Part 5*)

Again, let us consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\lambda_1 = 3$, $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = 1$, $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Let us successively apply the matrix A to an initial guess $w^0 \in \mathbb{R}^n$:



Note: The normalization step can be performed with respect to any norm.

Theorem 3.10 (*Convergence of power iteration*) *Let $A \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues λ_i for $i \in \{1,\dots,n\}$ which satisfy $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$ and whose eigenvectors form a basis of $\mathbb{R}^n$. Also, let the sequence of vectors $\{w^k\}_{k=0}^{\infty}$ be defined by the so-called* ***power iteration***

$$w^{k+1} := \frac{Aw^k}{\|Aw^k\|_p}, \; k \geq 0, p \geq 1, \quad \text{with} \quad w^0 \text{ such that } (v^1, w^0)_2 \neq 0,$$

where v^1 is the normalized (i.e., $\|v^1\|_p = 1$) eigenvector corresponding to λ_1. Then, for $k \to \infty$, we find $w^k \longrightarrow \pm v^1$ and also the so-called Rayleigh quotients

$$\mu_k := \frac{(w^k, Aw^k)_2}{(w^k, w^k)_2} \longrightarrow \lambda_1.$$

Proof. See, e.g., [1, Satz 7.3] or [4, Theorem 27.1]. The idea: Let $v^i \in \mathbb{R}^n$ be the corresponding eigenvectors. Then we can write the initial guess as linear combination $w^0 = \sum_{j=1}^n \mu_j v^j (\mu_1 \neq 0)$, so that with $c_k := \frac{1}{\|Aw^k\|_p}$ we find

$$w^k = c_k A^k w^0 = c_k \sum_{j=1}^n \mu_j A^k v^j = c_k \sum_{j=1}^n \mu_j \lambda_j^k v^j = c_k \lambda_1^k \left(\mu_1 v^1 + \sum_{j=2}^n \mu_j \left(\frac{\lambda_j}{\lambda_1}\right)^k v^j \right).$$

The fractions $\left(\frac{\lambda_j}{\lambda_1}\right)^k$ vanish as $k \to \infty$ and the limit vector is in $\text{span}(v^1)$. Since $\|w^k\|_p = \|v^1\|_p = 1$ the result follows. □

Remark:

- A variant of this approach is given by the so-called **inverse power method**, which can estimate any eigenpair, assumed a good initial guess for the eigenvalue is available.
- The assumption on the eigenvectors is satisfied, e.g., for real symmetric matrices (see Theorem 3.6)
- From the proof idea one finds that the convergence speed is determined by the fraction $\left(\frac{\lambda_2}{\lambda_1}\right)^k$ (potentially very slow).

3.3.3 A second advanced approach: General iterative method

Recall: (Lemma 3.4)

a) **Similar matrices** have the same eigenvalues:

$$\sigma(A) = \sigma(T^{-1}AT) \quad \text{for} \quad T \in GL_n(\mathbb{F}).$$

b) **Simple matrices**: Eigenvalues of an upper triangular matrix U (e.g., a diagonal matrix) are found on its diagonal, i.e.,

$$\sigma(U) = \{u_{11}, \ldots, u_{nn}\}.$$

Recipe:

a) Iteratively applying **similarity transformations** $T_k \in GL_n(\mathbb{F})$ to $A =: A_0$ thereby producing a sequence

$$A_k = T_k^{-1} A_{k-1} T_k.$$

b) Choose T_k so that this sequence converges to a **simple matrix** (triangular or even diagonal)

$$A_\infty := \lim_{k \to \infty} A_k.$$

$\to$ **Question:** Choice of the T_k's?

Requirements on the transformations T_k:

1. easily constructed from A_{k-1}
2. easy to invert (e.g., orthogonal matrix)
3. $(A_k)_k$ converges to something simple

One Implementation:

a) The QR-**Algorithmn** defines such transformations T_k through

$$\begin{aligned}
&A_0 = A \\
&\textbf{for } k = 1, \ldots, \infty : \\
&\qquad Q_k R_k := A_{k-1} \\
&\qquad A_k := R_k Q_k
\end{aligned}$$

Thus, inserting the first equation $R_k = Q_k^T A_{k-1}$ into the second gives

$$A_k = R_k Q_k = Q_k^T A_{k-1} Q_k = Q_k^T Q_{k-1}^T A_{k-2} Q_{k-1} Q_k = \cdots = \overline{Q}_k^T A \overline{Q}_k,$$

where

$$\overline{Q}_k := Q_1 \cdot Q_2 \cdots Q_{k-1} \cdot Q_k$$

Here: $T_k = Q_{k-1}$, where Q_{k-1} is derived from the QR decomposition of A_{k-1}.

b) We find: $A_k = \overline{Q}_k^T A \overline{Q}_k \xrightarrow[k\to\infty]{} U$, where U is (quasi) upper triangular; given as follows:

Theorem 3.11 (*QR **Algorithm***) *Consider a matrix $A \in \mathbb{R}^{n\times n}$ with distinct eigenvalues λ_i for $i = 1, \ldots, n$, i.e., $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$. Then the iterates $A_k \in \mathbb{R}^{n\times n}$ produced by the QR algorithm converge to the diagonal matrix $\Lambda :=$ diag$(\lambda_1, \ldots, \lambda_n)$ which consists of the eigenvalues of A, i.e.,*

$$\lim_{k\to\infty} A_k = \Lambda.$$

Proof. See, e.g., [1, Satz 7.8]. □

Finally: **What about the eigenvectors?**

One can further show that similar to the power iteration, we find that the columns of

$$\overline{Q}_\infty := \lim_{k\to\infty} \overline{Q}_k$$

are normalized eigenvectors of A.

3.3.4 In Practice: Combined Iterative Methods

Problems:

- QR decomposition for general and very large matrices too expensive
- Exact Schur complement is not reached in finitely many steps (= many QR decompositions needed)

However:

- Any matrix can be reduced to a Hessenberg matrix (= simple matrix) in *finitely many* steps
- QR decomposition for this type of matrix is cheap

This leads to:

(3) A third state-of-the-art approach: Combined iterative methods

a) **Similarity transformation via reduction** (e.g., Householder, Wilkinson, Givens) to something simple such as Hessenberg or even tridiagonal
(→ finite steps)

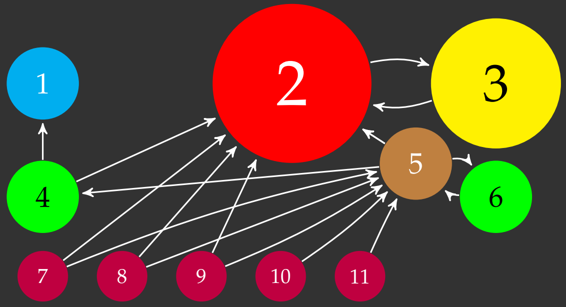
b) **Similarity transformation via iterative method** (e.g., QR or LR algorithm)
(→ potentially infinitely many steps)

Standard: QR Algorithm (with performance optimized modifications (shifts etc...))

c) Determine eigenvalues from the limiting simple matrix (and eigenvectors from the similarity transformations).

Common combination in practice: (a) Householder reflection + (b) QR algorithm
→ Works pretty well for matrices up to 1 mio. columns $n \approx 10^6$
→ for larger matrices one needs to develop problem-tailored structure exploiting methods

3.4 Example: The PageRank Algorithm from Google



Aim: Rank search enginge results according to the *"importance"* of the web pages.

1998: For this purpose, Larry Page and Sergei Brin develop the PageRank algorithm as the basis of the Google empire.

Assumption: *"important"* means more links from other (important) web pages.

→ More details on the sheet and in the video.