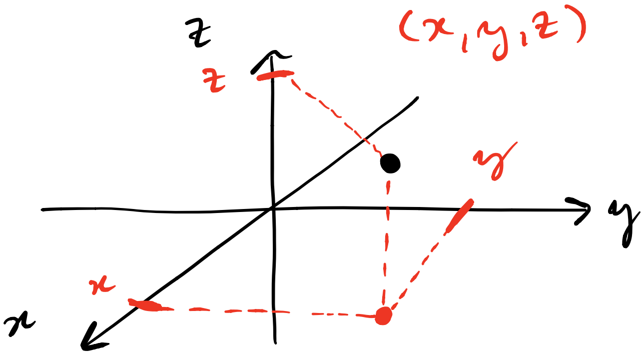
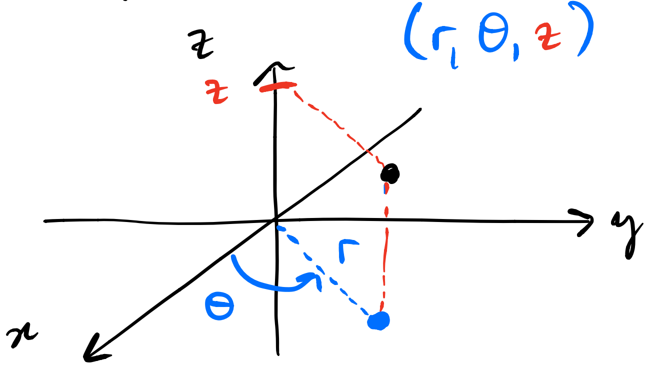
# MATH 267 lecture 35 : Triple Integrals in Cylindrical Coords.

Cartesian coordinates




Cylindrical coordinates




**IDEA:** Change the x and y coordinates to polar coordinates

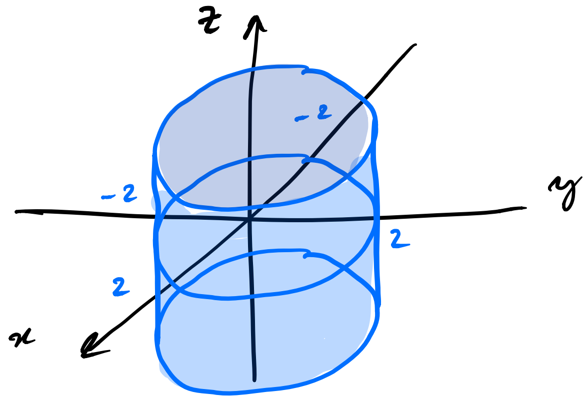
"height" z remains the same

$r^2 = x^2 + y^2$

$\tan\theta = \dfrac{y}{x}$

Why "cylindrical" coordinates?

eg) The circle $r = 2$ of radius 2 in the xy-plane becomes the cylinder of radius 2 in $\mathbb{R}^3$




Homework: Do the examples in the slides about switching points between cylindrical coordinates and surfaces in cylindrical coordinates (they have solutions).

We want to focus on integration.

Let R be a solid and let D be its projection onto the xy-plane

$$\iiint_R f(x,y,z)\, dV = \iint_D \left( \int_{z_0}^{z_1} f(x,y,z)\, dz \right) dA$$

POLAR
$dA = r\, dr\, d\theta$

$\Rightarrow$ In cylindrical coordinates: $\boxed{dV = r\, dz\, dr\, d\theta}$

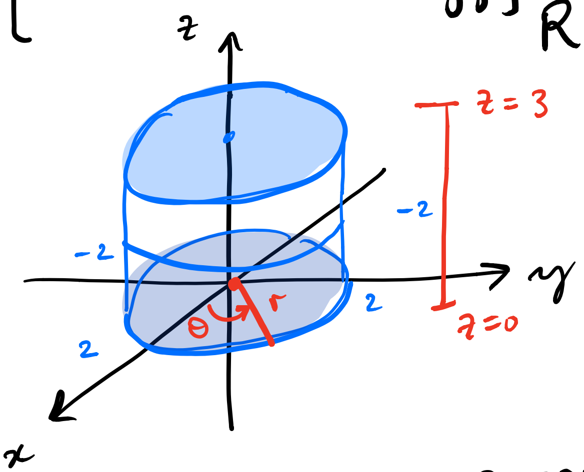
# Fubini's Theorem for Cylindrical Coordinates still works!

SLIDE 22-23

Example:

Suppose $R$ is the region enclosed between the planes $z=0$ and $z=3$ and inside the cylinder $x^2+y^2=4$.

Evaluate: $$\iiint_R x^2 z \, dV$$




change to cylindrical coords.

height: $0 \le z \le 3$

radius: $0 \le r \le 2$

angle: $0 \le \theta \le 2\pi$

$z \longmapsto z$

$x \longmapsto r\cos\theta$

$y \longmapsto r\sin\theta$

$dV = r\,dz\,dr\,d\theta$

$$\iiint_R x^2 z\,dV = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{z=0}^{z=3} (r\cos\theta)^2 z\, r\; dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{3} r^3\cos^2\theta\, z\; dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r^3\cos^2\theta\, \frac{z^2}{2}\Big|_0^3\; dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \frac{9}{2} r^3\cos^2\theta\; dr\,d\theta$$

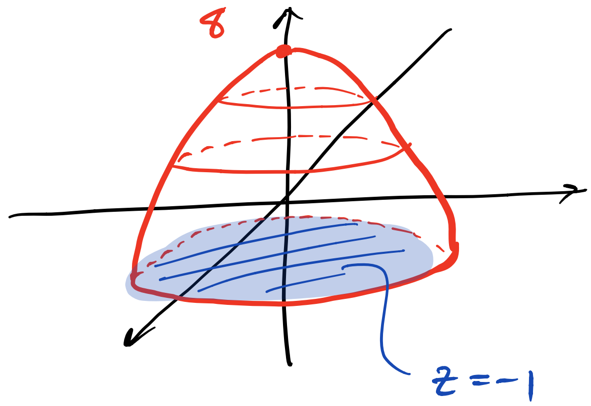
$$= \int_0^{2\pi} \frac{9}{2}\frac{r^4}{4}\cos^2\theta\Big|_0^2 = \int_0^{2\pi} 18\cdot\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 9\theta + \frac{9}{2}\sin 2\theta\Big|_0^{2\pi} = 18\pi.$$

**Example:**

Suppose that $R$ is the solid enclosed in the paraboloid $z = 8 - x^2 - y^2$ and the plane $z = -1$.

Evaluate $\iiint_R \sqrt{z+1}\, dV$




**Boundaries:**

$$-1 \le z \le 8 - x^2 - y^2 = 8 - r^2$$

$$0 \le r \le 3$$

$$0 \le \theta \le 2\pi$$

$$dV = r\, dz\, dr\, d\theta$$

$\Rightarrow$ Bottom has boundary given by

$$-1 = 8 - x^2 - y^2$$

$$\Rightarrow x^2 + y^2 = 9$$

$$\Rightarrow r^2 = 9 \iff r = 3$$

$$\iiint_R \sqrt{z+1}\, dV = \int_0^{2\pi} \int_0^3 \int_{-1}^{8-r^2} \sqrt{z+1}\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 r\, \frac{2}{3}(z+1)^{3/2} \Big|_{-1}^{8-r^2} dr\, d\theta$$

$(z+1)^{1/2}$

substitution

$$= \int_0^{2\pi} \int_0^3 \frac{2}{3} r (9 - r^2)^{3/2}\, dr\, d\theta$$

$u = 9 - r^2$

$-du = 2r\, dr$

| $r$ | $u = 9 - r^2$ |
|---|---|
| 0 | 9 |
| 3 | 0 |

$$= \int_0^{2\pi} \int_9^0 -\frac{1}{3} u^{3/2}\, du\, d\theta = \int_0^{2\pi} \int_0^9 \frac{1}{3} u^{3/2}\, du\, d\theta$$

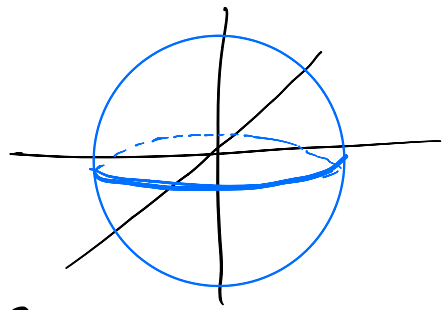
$$= \int_0^{2\pi} \frac{1}{3} \cdot \frac{2}{5} u^{5/2} \Big|_0^9 d\theta = \frac{4}{5 \cdot 3} 9^{5/2} \cdot \theta \Big|_0^{2\pi}$$

$$= \frac{2}{5 \cdot 3} (3)^5 \cdot 2\pi = \frac{4}{5} \cdot 81 \cdot \pi = \frac{324\pi}{5}$$

$$\begin{array}{r} 81 \\ \times\ 4 \\ \hline 324 \end{array}$$

**Example:** let $D$ be the solid sphere of radius $R$

Compute $\iiint_R 1 \, dV$



Sphere $R^2 = x^2+y^2+z^2$

TOP $\frac{1}{2}$ $\quad z = \sqrt{R^2-x^2-y^2} =$

BOTTOM $\frac{1}{2}$ $\quad z = -\sqrt{R^2-x^2-y^2}$

$0 \le r \le R$

$0 \le \theta \le 2\pi$

$$\iiint_R 1 \, dV = \int_0^{2\pi}\int_0^R \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^R 2r\sqrt{R^2-r^2} \, dr \, d\theta = \int_0^{2\pi} -\frac{2}{3}(R^2-r^2)^{3/2} \Big|_0^R \, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3}R^3 \, d\theta = \frac{2}{3}R^3\theta \Big|_{\theta=0}^{\theta=2\pi} = \frac{4\pi}{3} R^3.$$

**Example:**

Let $D$ be the solid cylinder given by $2 \le z \le 5$ and $x^2+y^2 \le 4$. Calculate the integral

$r \le 2$

$$\iiint_D \frac{z}{x^2+y^2+1} \, dV$$

$$= \int_0^{2\pi}\int_0^2\int_2^5 \frac{z}{r^2+1} r \, dz \, dr \, d\theta$$

I skipped several steps because all of the limits of integration are numbers.

$$= 2\pi \left(\frac{z^2}{2}\Big|_2^5\right) \cdot \left(\frac{1}{2}\ln(r^2+1)\Big|_0^2\right) = \pi(25-4) \cdot \frac{1}{2}(\ln 5)$$

$$= \frac{21}{2}\pi \cdot \ln(5)$$

**Example:** Let $D$ be the solid bounded above by the paraboloid $z = 4 - x^2 - y^2$ and bounded below by the plane $z = 0$. Find the triple integral

$$\iiint_D \sqrt{4-x^2-y^2}\, dV$$

Projection to $z=0$ (xy-plane)

$0 = 4 - x^2 - y^2$

$x^2 + y^2 = r^2 = 4$

$\Rightarrow \quad r = 2$

$$= \int_0^{2\pi} \int_0^{2} \int_0^{4-r^2} \sqrt{4-r^2}\; r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{2} r\sqrt{4-r^2}\,(4-r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{2} r\,(4-r^2)^{3/2}\, dr\, d\theta$$

$u = 4 - r^2$

$-\frac{1}{2}\, du = r\, dr$

| $r$ | $u = 4 - r^2$ |
|---|---|
| 0 | 4 |
| 2 | 0 |

$$= \int_0^{2\pi} \int_4^{0} -\frac{1}{2}\, u^{3/2}\, du\, d\theta$$

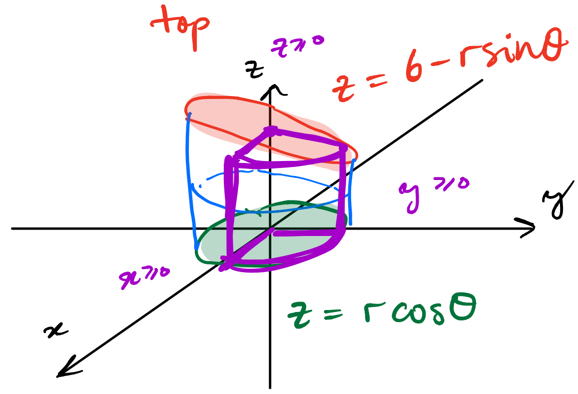
$$= \frac{1}{2}\int_0^{2\pi} \int_0^{4} u^{3/2}\, du\, d\theta = \frac{1}{2} \cdot 2\pi\, \frac{2u^{5/2}}{5} \Big|_0^4$$

$$= \frac{2\pi}{5} \cdot 2^5 = \frac{64\pi}{5}$$

**Example:** Let $D$ be the solid in the first octant (i.e. $x \geq 0,\ y \geq 0,\ z \geq 0$) bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = x$ and $z = 6 - y$

$r^2 = 3 \Rightarrow r = 3$     bottom     top

Find the integral $\iiint_D z\, dV$.




$$= \int_0^{\pi/2} \int_0^{3} \int_{r\cos\theta}^{6 - r\sin\theta} z \cdot r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 \frac{r}{2}\, z^2 \Big|_{r\cos\theta}^{6-r\sin\theta} \, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 \frac{r}{2} \left[ (6-r\sin\theta)^2 - r^2\cos^2\theta \right] dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 \frac{r}{2} \left[ 36 - 12r\sin\theta + r^2\sin^2\theta - r^2\cos^2\theta \right] dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^3 18r - 6r^2\sin\theta + \frac{1}{2} r^3 \left(\sin^2\theta - \cos^2\theta\right) dr\, d\theta.$$

$$= \int_0^{\pi/2} 9r^2 - 2r^3\sin\theta + \frac{1}{8} r^4 \left(\underline{\sin^2\theta - \cos^2\theta}\right) \Big|_0^3 \, d\theta$$

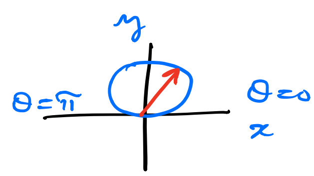
$$= \int_0^{\pi/2} 81 - 54\sin\theta + \frac{81}{8}\left(\underline{-\cos 2\theta}\right) d\theta$$

$\cos^2\theta - \sin^2\theta$
$= \cos 2\theta$.

$$= \frac{81\pi}{2} + 54\cos\theta \Big|_0^{\pi/2} - \frac{81}{16} \sin(2\theta) \Big|_0^{\pi/2}$$

$$= \frac{81\pi}{2} - 54$$

<u>Example</u>: Let $D$ be the solid enclosed inside the cylinder $r = 2\sin\theta$ bounded above by the paraboloid $z = x^2 + y^2$ and below by the $xy$ plane. Calculate $\iiint_D x \, dV$ and $\iiint_D y \, dV$.




<u>Solution</u>:

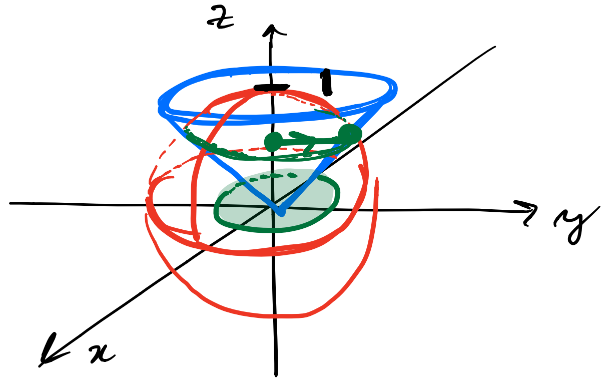
$r = 2\sin\theta$ is the cylinder $x^2 + (y-1)^2 = 1$ with axis $(0,1,z)$. We get the entire cylinder with $0 \le \theta \le \pi$.
The paraboloid is $z = r^2 = x^2 + y^2$

$$\iiint_D x\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} r\cos\theta\, r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} r^2\cos\theta\,dz\,dr\,d\theta = 0$$

$$\iiint_D y\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} r^2\sin\theta\,dz\,dr\,d\theta = 2\pi$$

**Example:** Find the volume of the solid where the cone $z=r$ (bottom), where $r\geq 0$ and the sphere (top) $x^2+y^2+z^2=1$ intersect




$$V = \iiint_D 1\,dV$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_r^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} r\left(\sqrt{1-r^2}-r\right)dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}-r^2\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\cdot\frac{2}{3}(1-r^2)^{3/2} - \frac{1}{3}r^3\Big|_0^{\frac{1}{\sqrt{2}}}\,d\theta$$

$$= 2\pi\left(-\frac{1}{3}\left(1-\frac{1}{2}\right)^{3/2}+\frac{1}{3}(1)^{3/2}-\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3\right)$$

$$= 2\pi\left(\underbrace{-\frac{1}{3}\cdot\frac{1}{2\sqrt{2}}+\frac{1}{3}-\frac{1}{3}\frac{1}{2\sqrt{2}}}_{-2\cdot\frac{1}{3}\cdot\frac{1}{2\sqrt{2}}=-\frac{1}{3\sqrt{2}}}\right) = \frac{2\pi}{3}-\frac{2\pi}{3\sqrt{2}} = \frac{1}{3}\pi(2-\sqrt{2})$$

$z^2 = 1-x^2-y^2 = 1-r^2$

$z = \sqrt{1-r^2}$ Top

shadow: $\boxed{0\leq r\leq\ ?}$ Tricky part

intersection of $z=r$ and $z^2=1-r^2$

$z^2=r^2$

$r^2=1-r^2 \Rightarrow 2r^2=1$

$\Rightarrow r^2=\frac{1}{2} \Rightarrow r=1/\sqrt{2}$