

MST125 TMA01

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1 Question 1

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1.1 a

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1.2 b

1. The distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Hence the distance between $A(-3, 1)$ and $B(2, -2)$ is

$$\begin{aligned} AB &= \sqrt{(2 - (-3))^2 + (-1 - 1)^2} \\ &= \sqrt{5^2 + (-3)^2} \\ &= \sqrt{34}. \end{aligned}$$

2. The gradient m of the line through (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence the gradient of the line through $(-3, 1)$ and $(2, -2)$ is

$$m = \frac{-2 - 1}{2 - (-3)} = -\frac{3}{5}.$$

3. The gradient of the line is $-\frac{3}{5}$. Hence $\tan \alpha = -\frac{3}{5}$. Let ϕ be the acute angle that the line makes with the negative direction of the x -axis. Then

$$\tan \phi = \frac{3}{5},$$

so

$$\phi = \tan^{-1} \left(\frac{3}{5} \right) = 0.540 \dots$$

Hence

$$\alpha = \pi - 0.540 \dots = 2.601 \dots$$

Therefore the angle α is 2.60 radians (to 2 d.p.).

1.3 c

I intend to typeset my TMA through the fact that I believe the programatic power of \LaTeX is a great advantage. Through MST125 I have even started using \LaTeX over Markdown (with mathjax) for school assignments now.

Using \LaTeX in my daily school work leads me to a much DRYer (don't repeat yourself) documents where all my associated code can be referenced with great accuracy without suffering from complicated compilation scripts.

I have also configured a Vim integration so I can use my primary editor for TMAs.

2 Question 2

(a) Find the least residue of 64^7 modulo 15.

$$64^7 \equiv x \pmod{15}$$

$$4^7 \equiv x \pmod{15}$$

We observe $4^2 \equiv 1 \pmod{15}$.

$$4^1 \equiv 4 \pmod{15}$$

$$64^7 \equiv 4 \pmod{15}$$

3 Question 3

3.1 a

3.1.1 i

We find the gcd using Euler's algorithm. These variable names come from CAS tool I wrote in Rust for Unit3.

$$\begin{array}{rclcl} a & & q & & b & & r \\ 90 & = & 5 & \times & 17 & + & 5 \\ 17 & = & 3 & \times & 5 & + & 2 \\ 5 & = & 2 & \times & 2 & + & 1 \\ 2 & = & 2 & \times & 1 & + & 0 \end{array}$$

We reshape for substitution

$$\begin{array}{rclcl} \textcircled{5} & = & \textcircled{90} & - & 5 & \times & \textcircled{17} \\ \textcircled{2} & = & \textcircled{17} & - & 3 & \times & \textcircled{5} \\ \textcircled{1} & = & \textcircled{5} & - & 2 & \times & \textcircled{2} \end{array}$$

And then conduct said substitution

$$\begin{aligned}
a \times \textcircled{17} + b \times \textcircled{90} &= \textcircled{1} \\
&= \textcircled{5} - 2 \times \textcircled{2} \\
&= (\textcircled{90} - 5 \times \textcircled{17}) - 2 \times \textcircled{2} \\
&= (\textcircled{90} - 5 \times \textcircled{17}) - 2 \times (\textcircled{17} - 3 \times \textcircled{5}) \\
&= (\textcircled{90} - 5 \times \textcircled{17}) - 2 \times (\textcircled{17} - 3 \times (\textcircled{90} - 5 \times \textcircled{17})) \\
&= 7 \times \textcircled{90} - 37 \times \textcircled{17}
\end{aligned}$$

Then we find the least residual of -37 and deduce that the multiplicative inverse of $17 \pmod{90}$ is 53 .

This means the solution to $17x \equiv 9 \pmod{90}$ is $x \equiv 53 \times 9 \equiv 477 \equiv 27 \pmod{90}$.

3.1.2 ii

We begin by simplifying the congruence.

$$\begin{array}{rclcl}
a & x & \equiv & b & \pmod{n} \\
9 & x & \equiv & 12 & \pmod{90} \\
3 & x & \equiv & 4 & \pmod{30}
\end{array}$$

As we can see here $\gcd(3, 30) \neq 1$ and $3 \nmid 4$ so the congruence is unsolvable.

3.1.3 iii

We begin by simplifying the congruence.

$$\begin{array}{rclcl}
a & x & \equiv & b & \pmod{n} \\
27 & x & \equiv & 72 & \pmod{90} \\
3 & x & \equiv & 8 & \pmod{10}
\end{array}$$

This is small enough for a brute force attack but for clarities sake I will not do this.

We can see they are coprime. Hence we can skip Euler's algo.

$$\begin{aligned}
\textcircled{10}a + \textcircled{3}b &= 1 \\
&= \textcircled{10} - 3 \times \textcircled{3}
\end{aligned}$$

We find least residual and multiply with 8 . So $x \equiv -3 \times 8 \equiv 7 \times 8 \equiv 56 \equiv 6 \pmod{10}$.

3.2 b

3.2.1 i

$$7 \times 15 \equiv 105 \equiv 4 \times 26 + 1 \equiv 1 \pmod{26}$$

3.2.2 ii

$$\begin{aligned} \begin{pmatrix} 8 & - & 12 \end{pmatrix} \times 15 &\equiv 18 \pmod{26} \\ \begin{pmatrix} 16 & - & 12 \end{pmatrix} \times 15 &\equiv 8 \pmod{26} \\ \begin{pmatrix} 25 & - & 12 \end{pmatrix} \times 15 &\equiv 13 \pmod{26} \\ \begin{pmatrix} 2 & - & 12 \end{pmatrix} \times 15 &\equiv 6 \pmod{26} \end{aligned}$$

Using the conversion table in Figure 1 we get

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	14	15	16	17	18	19	20	21	22	23	24	25

Figure 1: Linear 0-indexed α -numeric conversion table

$$\text{"IQZC"} \xrightarrow{\alpha\text{-numeric}} \{8, 16, 25, 2\} \xrightarrow{\text{affine}^{-1}} \{18, 8, 13, 6\} \xrightarrow{\alpha\text{-numeric}^{-1}} \text{"SING"}$$

4 Question 4

We rewrite the hyperbola into standard notation for standard possition.

$$\begin{aligned} 9x^2 - 4y^2 - 100 &= 0 \\ 9x^2 - 4y^2 &= 100 = 2^2 5^2 \\ 9 \frac{x^2}{100} - \frac{y^2}{25} &= 1 \\ \frac{x^2}{100 \times 9^{-1}} - \frac{y^2}{25} &= 1 \\ \frac{x^2}{(10 \times 3^{-1})^2} - \frac{y^2}{5^2} &= 1 \end{aligned}$$

This gives ut a few key values

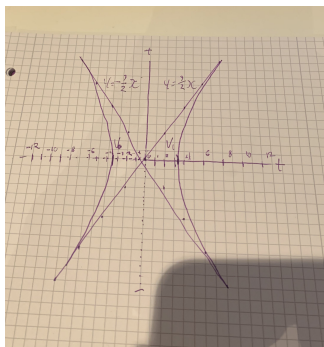
$$a = \frac{10}{3} \quad b = 5 \quad e = \sqrt{1 + \frac{25 \times 9}{100}} = \frac{\sqrt{13}}{2}$$

4.0.1 i

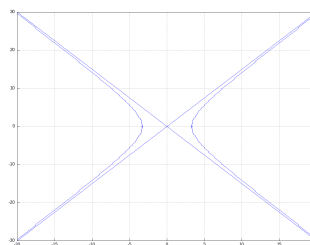
The asytmotes are $y = \pm \frac{3}{2}x$. Verticies are $(\pm \frac{10}{3}, 0)$.

4.0.2 ii

I have decided to sketch on paper as the Maxima plot does not show the vertices so well.



(a) Sketch of conic



(b) Maxima plot of conic

Figure 2: Representations of conic

4.0.3 iii

eccentricity	$e = \frac{\sqrt{13}}{2}$
foci	$(\pm ae, 0) = (\pm\sqrt{13} \frac{5}{3}, 0)$
directrices	$y = \pm \frac{a}{e} = \pm \frac{20}{3\sqrt{13}}$

4.0.4 iv

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \sec t \\ 5 \tan t \end{pmatrix}$$

To find the bounds we need to observe when we have the signs we want. Since we want a point with a negative x -sign and a positive y -sign we need to see when the functions satisfy this.

$$\sec x < 0 \text{ if } -3\frac{\pi}{2} < x < -\frac{\pi}{2}$$

and in the same interval we can see that

$$\tan x > 0 \text{ if } -\pi < x < -\frac{\pi}{2}$$

Hence the interval for t is $-\pi < t < -\frac{\pi}{2}$.

4.1 b

$$6x^2 - 8xy + 5y^2 - 3x - 16y - 20 = 0$$

4.1.1 i

$$T = B^2 - 4AC = 64 - 6 \times 5 = 34$$

Since $T > 0$ then the conic must be a Hyperbola.

4.1.2 ii

```
1 wxdraw2d(  
2     implicit(  
3         6 * x * x - 8 * x * y + 5 * y * y - 3 * x - 16 * y - 20  
4     = 0,  
5         x, -10, 10,  
6         y, -10, 10  
7     )  
8 )$
```

5 Question 5

5.1 a

$$A = (4t + 2, t - 1)$$

if $t = 1$ then we get $(6, 0)$. This means the entire line is translated up by 6. If we then plug-in $t = 2$ then we get $(10, 1)$ which if we insert into the two point formula we get that the rise over run of this line is $\frac{10-6}{1-0} = 4$. This means the equation for the curve A traces is $y = 4x + 6$.

5.2 b

$$\begin{aligned} |\vec{BA}| &= \left| \begin{pmatrix} 4t+2 \\ t-1 \end{pmatrix} - \begin{pmatrix} 2t+5 \\ 3t-1 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} 2t-3 \\ -2t \end{pmatrix} \right| \\ &= \sqrt{(2t-3)^2 + (-2t)^2} \\ &= \sqrt{8t^2 - 12t + 9} \end{aligned}$$

$$d^2 = |\vec{BA}|^2 = 8t^2 - 12t + 9$$

6 c

$$d^2 = 8t^2 - 12t + 9 \quad (1)$$

$$= 8\left(t^2 - \frac{3}{2}t\right) + 9 \quad (2)$$

$$= 8\left(t - \frac{3}{4}\right)^2 - \frac{9}{16} + 9 \quad (3)$$

$$= 8\left(t - \frac{3}{4}\right)^2 + \frac{135}{16} \quad (4)$$

the minimum value of d^2 occurs when $t = \frac{3}{4}$.
hence the minimum distance is 8.4m (to 2 s.f)

The expression on (3) ends up evaluating to

$$8\left(t^2 - \frac{3}{2}t + \frac{9}{16}\right) - \frac{9}{16} + 9 = 8t^2 - 12t + \frac{9}{2} - \frac{9}{16} + 9$$

which obviously is not equal to (2).

Finally the statement that the distance is $\frac{3}{2}$ is incorrect since this is the square of the distance. The correct answer to the distance (including the compounded error) is $\frac{\sqrt{135}}{4}$.

Personally I would solve this task with a differentiation and a substitution but employing the same workings as the student we get as follows:

$$d^2 = 8t^2 - 12t + 9 \quad (5)$$

$$= 8\left(t^2 - \frac{3}{2}t\right) + 9 \quad (6)$$

$$= 8\left(t - \frac{3}{4}\right)^2 - \frac{9}{2} + 9 \quad (7)$$

$$= 8\left(t - \frac{3}{4}\right)^2 + \frac{9}{2} \quad (8)$$

The minimum value of d^2 occurs when $t = \frac{3}{4}$.
hence the minimum distance is $\frac{3}{\sqrt{2}}\text{m} = 2.12\text{m}$
(to 2 s.f)

7 Question 6

7.1 a

We can see the force diagram as Figure 3. This represents a simplification of the forces; angles nor magnitudes are correct.

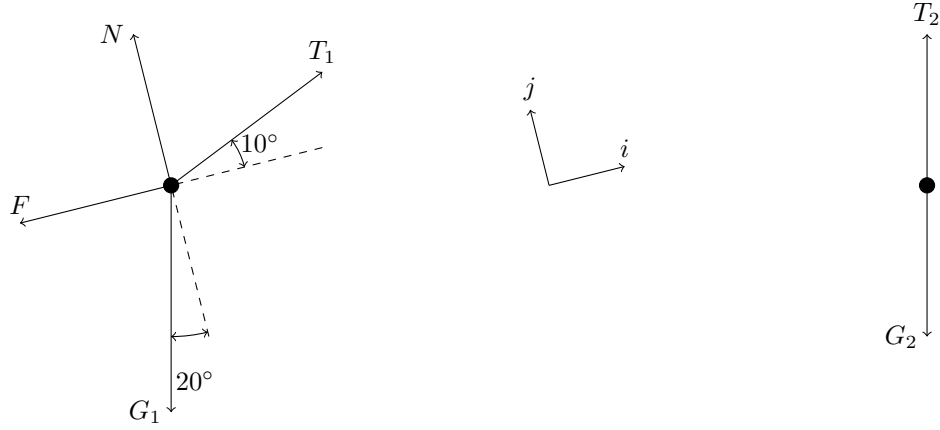


Figure 3: Force diagram

7.2 b

$$\mathbf{F} + \mathbf{N} + \mathbf{T} + \mathbf{G} = 0$$

$$-i F + j N + (i \cos 10^\circ, j \sin 10^\circ) T + (-i \sin 20^\circ - j \cos 20^\circ) G = 0$$

$$-i \mu N + j N + (i \cos 10^\circ, j \sin 10^\circ) T + (-i \sin 20^\circ - j \cos 20^\circ) G = 0$$

NOTE: $|\mathbf{X}| = X$ and in this case for simplicity $T = T_1$ and $G = G_1$.
So if we split the equation into equations of its components we get:

$$-\mu N + \cos 10^\circ T - \sin 20^\circ G = 0 \quad N + T \sin 10^\circ - G \cos 20^\circ = 0 \quad (9)$$

$$-T \sin 10^\circ + G \cos 20^\circ = N \quad (10)$$

We can now substitute N into the first equation of (9) getting

$$-\mu N + \cos 10^\circ T - \sin 20^\circ G = 0$$

$$-\mu(-T \sin 10^\circ + G \cos 20^\circ) + \cos 10^\circ T - \sin 20^\circ G = 0$$

$$T(\mu \sin 10^\circ + \cos 10^\circ) - G(\mu \cos 20^\circ + \sin 20^\circ) = 0$$

Finally we can now get an expression for T_1 .

$$\begin{aligned}
T_1 &= \frac{G(\mu \cos 20^\circ + \sin 20^\circ)}{\mu \sin 10^\circ + \cos 10^\circ} \\
&= \frac{17 \times 9.8(\mu \cos 20^\circ + \sin 20^\circ)}{\mu \sin 10^\circ + \cos 10^\circ} \\
&\approx 17 \times 9.8 \times 0.5281 \\
&\approx 88\text{N}
\end{aligned}$$

NOTE this answer is to two sigfigs.

7.3 c

Stemming for the fact that

$$T_2 - G_2 = 0 \iff T_2 = G_2$$

we know that

$$T_1 = T_2 = G_2 = mg$$

then we can use the answer from 7.2 to state that the mass of ball is

$$\frac{88}{9.8} \approx 9.0\text{kg}$$