MST125 TMA01

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1 Question 1

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1.1 a

MST125 TMA 01 Question 1

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LaTeX

1.2 b

1. The distance between A$(x_1,\ y_1)$ and B$(x_2,\ y_2)$ is given by

$$\text{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Hence the distance between A$(-3,\ 1)$ and B$(2,\ -2)$ is

$$\begin{aligned}\text{AB} &= \sqrt{(2-(-3))^2 + (-1-1)^2} \\ &= \sqrt{5^2 + (-3)^2} \\ &= \sqrt{34}.\end{aligned}$$

2. The gradient m of the line through $(x_1,\ y_1)$ and $(x_2,\ y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence the gradient of the line through $(-3,\ 1)$ and $(2,\ -2)$ is

$$m = \frac{-2-1}{2-(-3)} = -\frac{3}{5}.$$

3. The gradient of the line is $-\frac{3}{5}$. Hence $\tan\alpha = -\frac{3}{5}$. Let ϕ be the acute angle that the line makes with the nagative direction of the x-axis. Then

$$\tan\phi = \frac{3}{5},$$

so

$$\phi = \tan^{-1}\left(\frac{3}{5}\right) = 0.540\ldots.$$

Hence

$$\alpha = \pi - 0.540\cdots = 2.601\ldots.$$

Therefore the angle α is 2.60 radiants (to 2 d.p.).

1.3 c

I intend to typeset my TMA through the fact that I believe the programatic power of LaTeXis a great advantage. Through MST125 I have even started using LaTeXover Markdown (with mathjax) for school assignemts now.

Using LaTeXin my daily school work leads me to a much DRYer (don't repeat yourself) documents where all my associate code can be refrenced with great accuracy without suffering from complecated compilation scripts.

I have also configured a Vim integration so I can use my primary editor for TMAs.

2 Question 2

(a) Find the least residue of 64^7 modulo 15.

$$\begin{aligned} 64^7 &\equiv x \mod 15 \\ 4^7 &\equiv x \mod 15 \end{aligned}$$

We observe $4^2 \equiv 1 \mod 15$.

$$\begin{aligned} 4^1 &\equiv x \mod 15 \\ 64^7 &\equiv 4 \mod 15 \end{aligned}$$

3 Question 3

3.1 a

3.1.1 i

We find the gcd using Euler's algorithm. These variable names come from CAS tool I wrote in Rust for Unit3.

$$\begin{array}{ccccccccc} a & & q & & b & & r \\ 90 & = & 5 & \times & 17 & + & 5 \\ 17 & = & 3 & \times & 5 & + & 2 \\ 5 & = & 2 & \times & 2 & + & 1 \\ 2 & = & 2 & \times & 1 & + & 0 \end{array}$$

We reshape for substitution

$$\begin{array}{ccccccc} 5 & = & 90 & - & 5 & \times & 17 \\ 2 & = & 17 & - & 3 & \times & 5 \\ 1 & = & 5 & - & 2 & \times & 2 \end{array}$$

And then conduct said substitution

$$\begin{aligned}
a \times 17 + b \times 90 &= 1 \\
&= 5 - 2 \times 2 \\
&= (90 - 5 \times 17) - 2 \times 2 \\
&= (90 - 5 \times 17) - 2 \times (17 - 3 \times 5) \\
&= (90 - 5 \times 17) - 2 \times (17 - 3 \times (90 - 5 \times 17)) \\
&= 7 \times 90 - 37 \times 17
\end{aligned}$$

Then we find the least residual of -37 and deduce that the multiplicative inverse of 17 mod 90 is 53.

This means the solution to $17x \equiv 9 \mod 90$ is $x \equiv 53 \times 9 \equiv 477 \equiv 27 \mod 90$.

3.1.2 ii

We begin by simplifying the congruence.

$$\begin{array}{cccccc}
a & x & \equiv & b & \text{mod} & n \\
9 & x & \equiv & 12 & \text{mod} & 90 \\
3 & x & \equiv & 4 & \text{mod} & 30
\end{array}$$

As we can see here $\gcd(3, 30) \neq 1$ and $3 \nmid 4$ so the congruence is unsolvable.

3.1.3 iii

We begin by simplifying the congruence.

$$\begin{array}{cccccc}
a & x & \equiv & b & \text{mod} & n \\
27 & x & \equiv & 72 & \text{mod} & 90 \\
3 & x & \equiv & 8 & \text{mod} & 10
\end{array}$$

This is small enougth for a brute force attack but for clarities sake I will not do this.

We can see they are coprime. Hence we can skip Euler's algo.

$$\begin{aligned}
10a + 3b &= 1 \\
&= 10 - 3 \times 3
\end{aligned}$$

We find least residual and multiply with 8. So $x \equiv -3 \times 8 \equiv 7 \times 8 \equiv 56 \equiv 6 \mod 10$.

3.2 b

3.2.1 i

$$7 \times 15 \equiv 105 \equiv 4 \times 26 + 1 \equiv 1 \mod 26$$

3.2.2 ii

$$\begin{array}{cccccccccc} (& 8 & - & 12 &) & \times & 15 & \equiv & 18 & \mod 26 \\ (& 16 & - & 12 &) & \times & 15 & \equiv & 8 & \mod 26 \\ (& 25 & - & 12 &) & \times & 15 & \equiv & 13 & \mod 26 \\ (& 2 & - & 12 &) & \times & 15 & \equiv & 6 & \mod 26 \end{array}$$

Using the conversion table in Figure 1 we get

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	14	15	16	17	18	19	20	21	22	23	24	25

Figure 1: Linear 0-indexed α-numeric conversion table

$$\text{"IQZC"} \xrightarrow{\alpha-\text{numeric}} \{8,\ 16,\ 25,\ 2\} \xrightarrow{\text{affine}^{-1}} \{18,\ 8,\ 13,\ 6\} \xrightarrow{\alpha-\text{numeric}^{-1}} \text{"SING"}$$

4 Question 4

We rewrite the hyperbola into standard notation for standard possition.

$$\begin{aligned} 9x^2 - 4y^2 - 100 = 0 \\ 9x^2 - 4y^2 = 100 = 2^2 5^2 \\ 9\frac{x^2}{100} - \frac{y^2}{25} = 1 \\ \frac{x^2}{100 \times 9^{-1}} - \frac{y^2}{25} = 1 \\ \frac{x^2}{(10 \times 3^{-1})^2} - \frac{y^2}{5^2} = 1 \end{aligned}$$

This gives ut a few key values

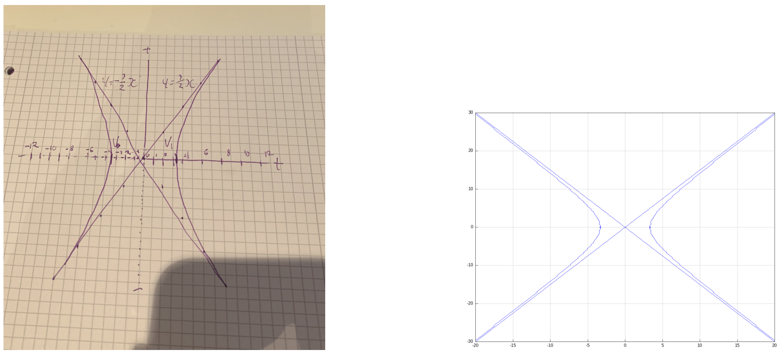
$$a = \frac{10}{3} \qquad b = 5 \qquad e = \sqrt{1 + \frac{25 \times 9}{100}} = \frac{\sqrt{13}}{2}$$

4.0.1 i

The asymtotes are $y = \pm\frac{3}{2}x$. Verticies are $(\pm\frac{10}{3}, 0)$.

4.0.2 ii

I have decided to sketch on paper as the Maxima plot does not show the verticies so well.



(a) Sketch of conic (b) Maxima plot of conic

Figure 2: Representations of conic

4.0.3 iii

$$
\begin{aligned}
&\text{eccentricity} && e = \frac{\sqrt{13}}{2} \\
&\text{foci} && (\pm ae, 0) = (\pm\sqrt{13}\,\frac{5}{3}, 0) \\
&\text{directrices} && y = \pm\frac{a}{e} = \pm\frac{20}{3\sqrt{13}}
\end{aligned}
$$

4.0.4 iv

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \ \sec t \\ 5 \ \tan t \end{pmatrix}$$

To find the bounds we need to observe when we have the signs we want. Since we want a point with a negative x-sign and a possitive y-sign we need to see when the functions satasfy this.

$$\sec x < 0 \text{ if } -3\frac{\pi}{2} < x < -\frac{\pi}{2}$$

and in the same interval we can see that

$$\tan x > 0 \text{ if } -\pi < x < -\frac{\pi}{2}$$

Hence the interval for t is $-\pi < t < -\frac{\pi}{2}$.

4.1 b

$$6x^2 - 8xy + 5y^2 - 3x - 16y - 20 = 0$$

4.1.1 i

$$T = B^2 - 4AC = 64 - 6 \times 5 = 34$$

Since $T > 0$ then the conic must be a Hyperbola.

4.1.2 ii

```
wxdraw2d(
      implicit(
          6 * x * x - 8 * x * y + 5 * y * y - 3 * x - 16 * y - 20
  = 0,
          x,-10,10,
          y,-10,10
      )
)$
```

5 Question 5

5.1 a

$$\text{A} = (4t + 2, t - 1)$$

if $t = 1$ then we get $(6, 0)$. This means the entire line is translated up by 6. If we then plug-in $t = 2$ then we get $(10, 1)$ which if we insert into the two point formula we get that the rise over run of this line is $\frac{10-6}{1-0} = 4$. This means the equation for the curve A traces is $y = 4x + 6$.

5.2 b

$$\begin{aligned}
\mid \vec{\text{BA}} \mid &= \begin{vmatrix} (4t+2) - (2t+5) \\ (t-1) - (3t-1) \end{vmatrix} \\
&= \begin{vmatrix} 2t - 3 \\ -2t \end{vmatrix} \\
&= \sqrt{(2t-3)^2 + (-2t)^2} \\
&= \sqrt{8t^2 - 12t + 9}
\end{aligned}$$

$$d^2 = \mid \vec{\text{BA}} \mid^2 = 8t^2 - 12t + 9$$

6 c

$$\begin{align} d^2 &= 8t^2 - 12t + 9 \tag{1}\\ &= 8\Big(t^2 - \frac{3}{2}t\Big) + 9 \tag{2}\\ &= 8\Big(t - \frac{3}{4}\Big)^2 - \frac{9}{16} + 9 \tag{3}\\ &= 8\Big(t - \frac{3}{4}\Big)^2 + \frac{135}{16} \tag{4} \end{align}$$

the minimum value of d^2 occurs when $t = \frac{3}{4}$.
hence the minimum distance is 8.4m (to 2 s.f)

The expression on (3) ends up evaluating to

$$8(t^2 - \frac{3}{2}t + \frac{9}{16}) - \frac{9}{16} + 9 = 8t^2 - 12t + \frac{9}{2} - \frac{9}{16} + 9$$

which obviously is not equal to (2).

Finally the statement that the distance is $\frac{3}{2}$ is incorrect since this is the square of the distance. The correct answer to the distance (including the compounded errir) is $\frac{\sqrt{135}}{4}$.

Personally I would solve this task with a differentiation and a substutution but employing the same workings as the student we get as follows:

$$\begin{align} d^2 &= 8t^2 - 12t + 9 \tag{5}\\ &= 8\Big(t^2 - \frac{3}{2}t\Big) + 9 \tag{6}\\ &= 8\Big(t - \frac{3}{4}\Big)^2 - \frac{9}{2} + 9 \tag{7}\\ &= 8\Big(t - \frac{3}{4}\Big)^2 + \frac{9}{2} \tag{8} \end{align}$$

The minimum value of d^2 occurs when $t = \frac{3}{4}$.
hence the minimum distance is $\frac{3}{\sqrt{2}}$m = 2.12m
(to 2 s.f)