ADSII3ILV

Algorithms and Data Structures II

Asymptotic Notations

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Agenda

- Recap on Complexity
- Asymptotic Notations

Recap: Slow vs. Fast Fibonacci

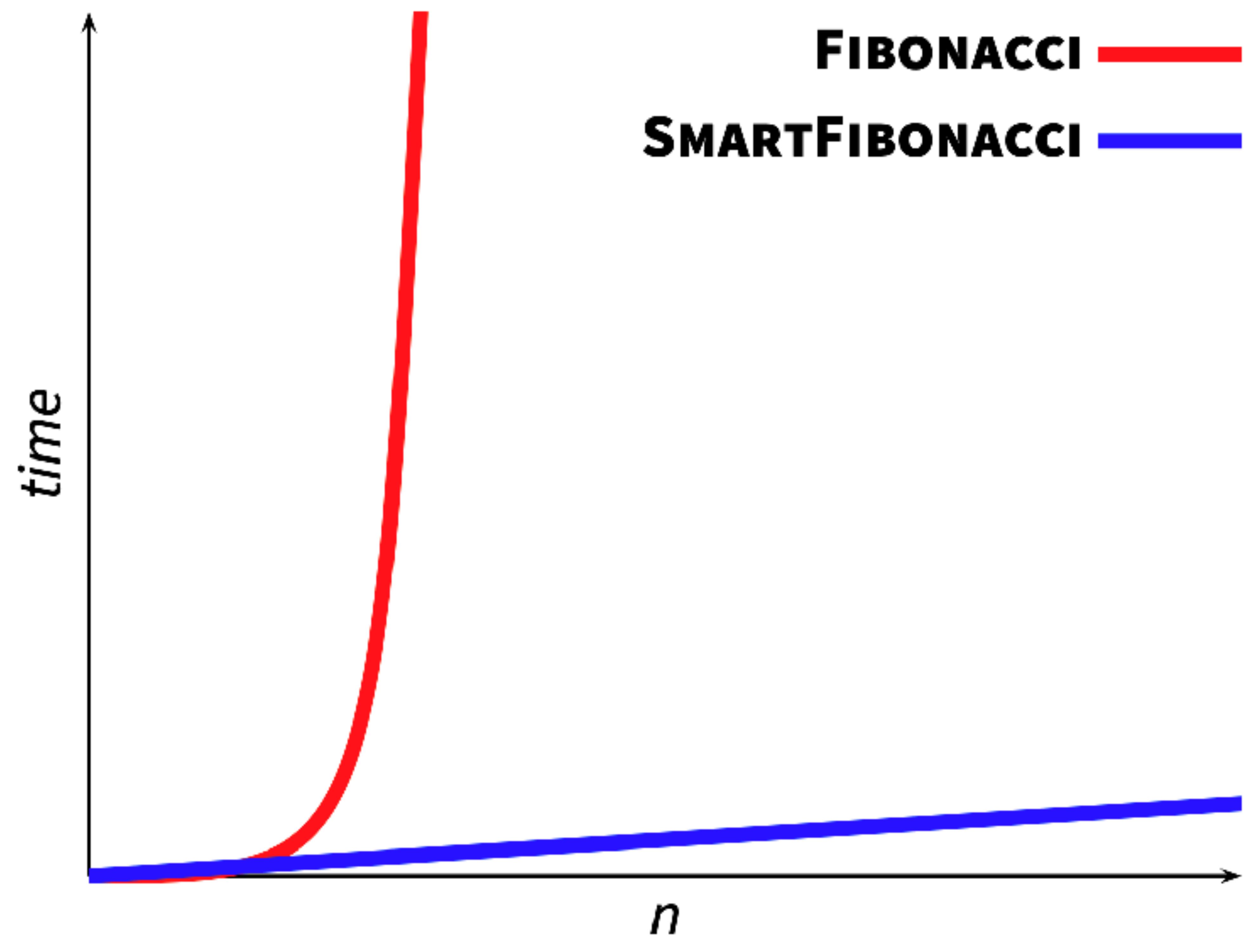
- We informally characterized our two Fibonacci algorithms

```
FIBONACCI(n)
if n == 0
        return 0
elseif n == 1
        return 1
else return FIBONACCI(n − 1) + FIBONACCI(n − 2)
```

```
SMARTFIBONACCI(n)
if n == 0
        return 0
elseif n == 1
        return 1
else pprev = 0
        prev = 1
        for i = 2 to n
                f = prev + pprev
                pprev = prev
                prev = f
return f
```

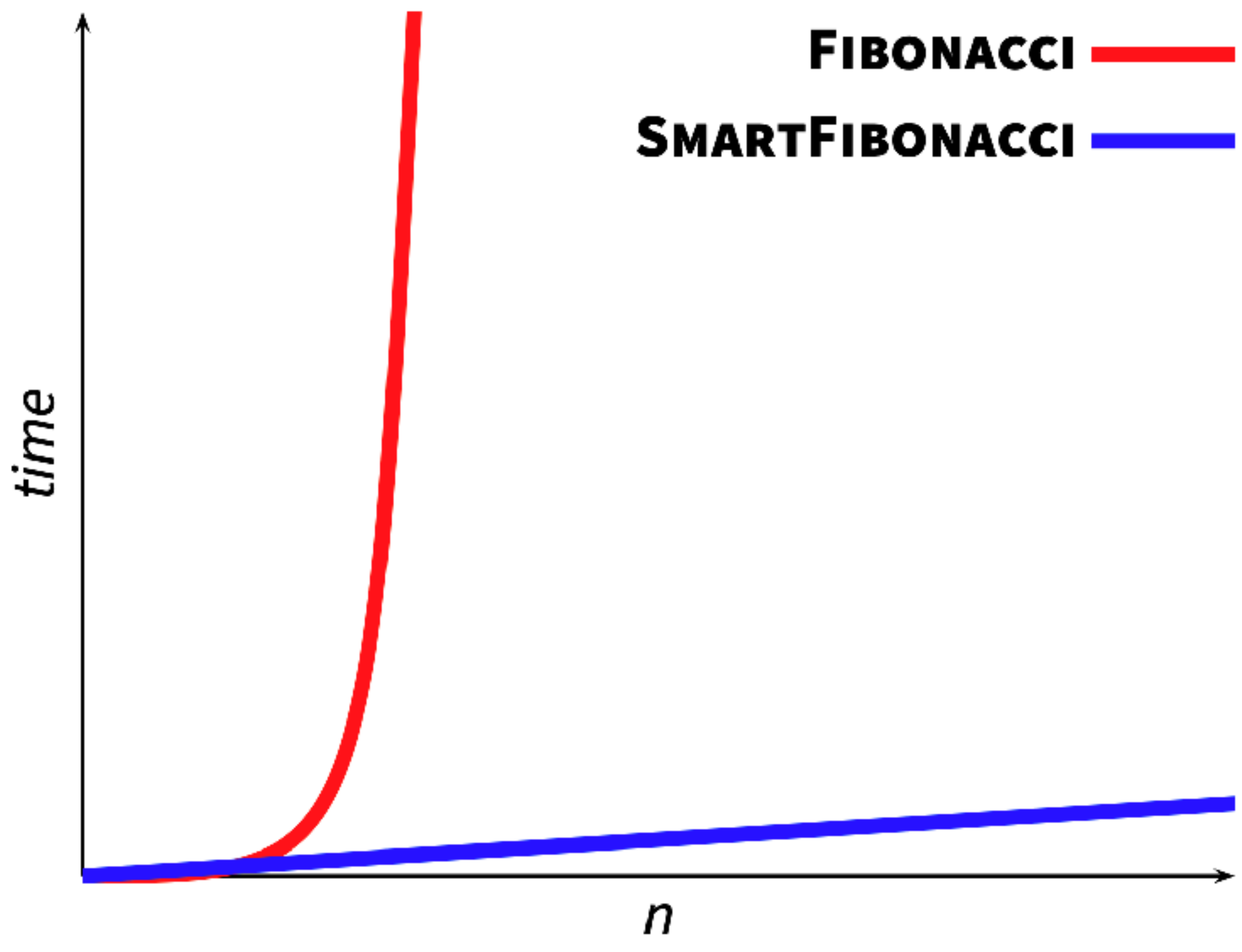
Recap: Slow vs. Fast Fibonacci

- We informally characterized our two Fibonacci algorithms



Recap: Slow vs. Fast Fibonacci

- We informally characterized our two Fibonacci algorithms as
  - **exponential** in n (FIBONACCI)
  - **linear** in n (SMARTFIBONACCI)



Recap: The Random-Access Machine Model

- A model of the computer that **characterizes the complexity of algorithms**
  - in general, and
  - in a way that is specific to the algorithms, but
  - independent of implementation details
- The model has only **basic types** and can perform only **basic steps**
- Each type of basic steps takes **a constant time**


IGNORE EVERY CONSTANT COST FACTOR
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Recap: Order of Growth

- We measure the **complexity** of an algorithm as a function of the **size of the input** (a **problem-dependent** variable or set of variables)
- In general we are interested in the **worst case complexity** to provide (possibly conservative) **guarantees** on how much time it takes for an algorithm to finish while fed with input of increasing size
- Other types of analysis exist:
  - Best case
  - Average case

Recap: Order of Growth

- Higher-order terms **dominate**, i.e., they grow at faster pace, lower-order terms
- Thus, we **ignore** lower-order terms while characterizing the asymptotic worst case behavior of the algorithms
- For instance,
  - given $T(n) = an^2 + bn + c$
  - we consider only the dominant factor $\mathbf{n^2}$
  - we conclude that $T(n)$ is a **quadratic function** in n
  - we write $T(n) = \mathbf{\Theta(n)^2}$

Recap: "Better" Algorithms

- We study the **asymptotic efficiency** of algorithms and consider the size of the input **in the limit**
- We consider one algorithm A_1 to be more efficient than another A_2 if A_1's **worst-case** running time **has a lower order of growth** than A_2's **<u>asymptotically</u>**
- This does NOT imply that A_1 is **always** more efficient than A_2; however, for **large enough inputs**, A_1 will be quicker than A_2 in the worst case

Asymptotic Notations

- We use asymptotic notations **primarily** to describe the running times of algorithms. But asymptotic notation applies to **functions**; thus, we can use them to characterize other aspects of algorithms (e.g., the amount of space they use, aka Space Complexity)
- Standard notations include the Θ-notation, the O- and o-notation, and the Ω- and ω-notation
- Each notation captures an asymptotic notion that characterizes a target function bounds:
  - Is it **above, below** or **between** other functions?
  - Is this bound **tight**?
- Note: the function under analysis might **not be completely known**

$\mathbb{A}$ *absolutely at most c*

- Let $\mathbb{A}(c)$ indicate a quantity that is absolutely at most c
    - e.g., $x = \mathbb{A}(2)$ means that $|x| \leq 2$
- When $x = \mathbb{A}(y)$ we say that "*x is absolutely at most y*"

- **$x = \mathbb{A}(y)$ <u>does not mean</u> that x equals $\mathbb{A}(y)$**
    - $\mathbb{A}(y)$ denotes a set of values
    - $x = \mathbb{A}(y)$ really means $x \in \mathbb{A}(y)$

$\mathbb{A}$ *absolutely at most c*

- $\mathbb{A}(x) + \mathbb{A}(y) = \mathbb{A}(x+y)$
- $\mathtt{k}\mathbb{A}(x) = \mathbb{A}(\mathtt{k}y)$
- $\mathbb{A}(x)\mathbb{A}(y) = \mathbb{A}(xy)$
- $X = \mathbb{A}(3) \Rightarrow X = \mathbb{A}(4)$, but $x = \mathbb{A}(4) \not\Rightarrow x = \mathbb{A}(3)$
- $X = \mathbb{A}(n-1) \Rightarrow X = \mathbb{A}(n^2)$, *for all* n

From 𝔸 to O

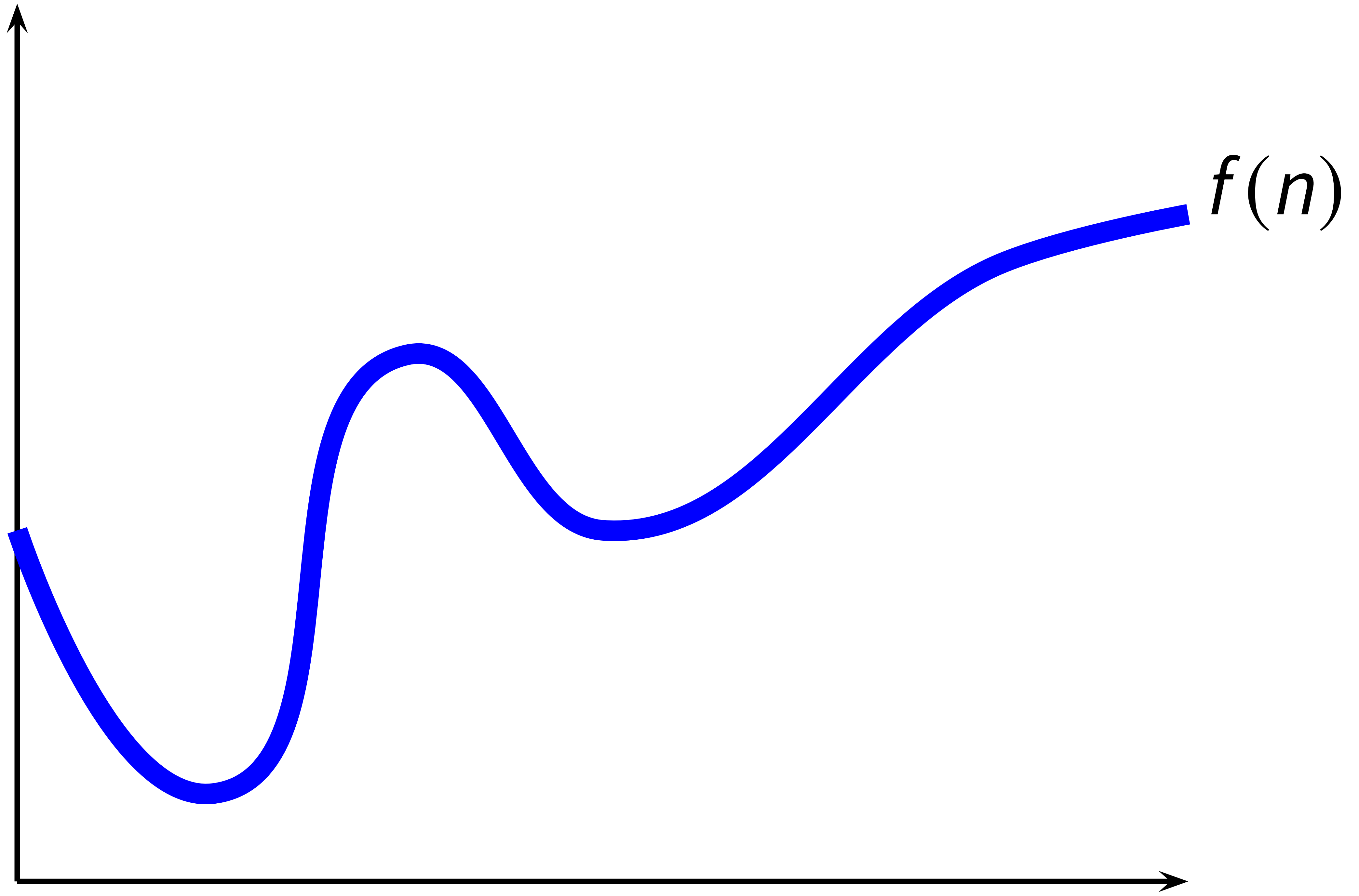
- If f(n) is such that **f(n) = c𝔸(g(n))** for all n **sufficiently large** and for some **positive constant** c (c>0), then we say that **f(n) = O(g(n))**
    - *read "f(n) is big-oh of g(n)" or simply "f(n) is oh of g(n)"*

- O(g(n)) denotes a **family of functions**

- **Warning: f(n) = O(g(n)) <u>does not mean</u> that f(n) equals O(g(n))**

O-Notation

- Given a function g(n), we define the **family of functions** O(g(n))



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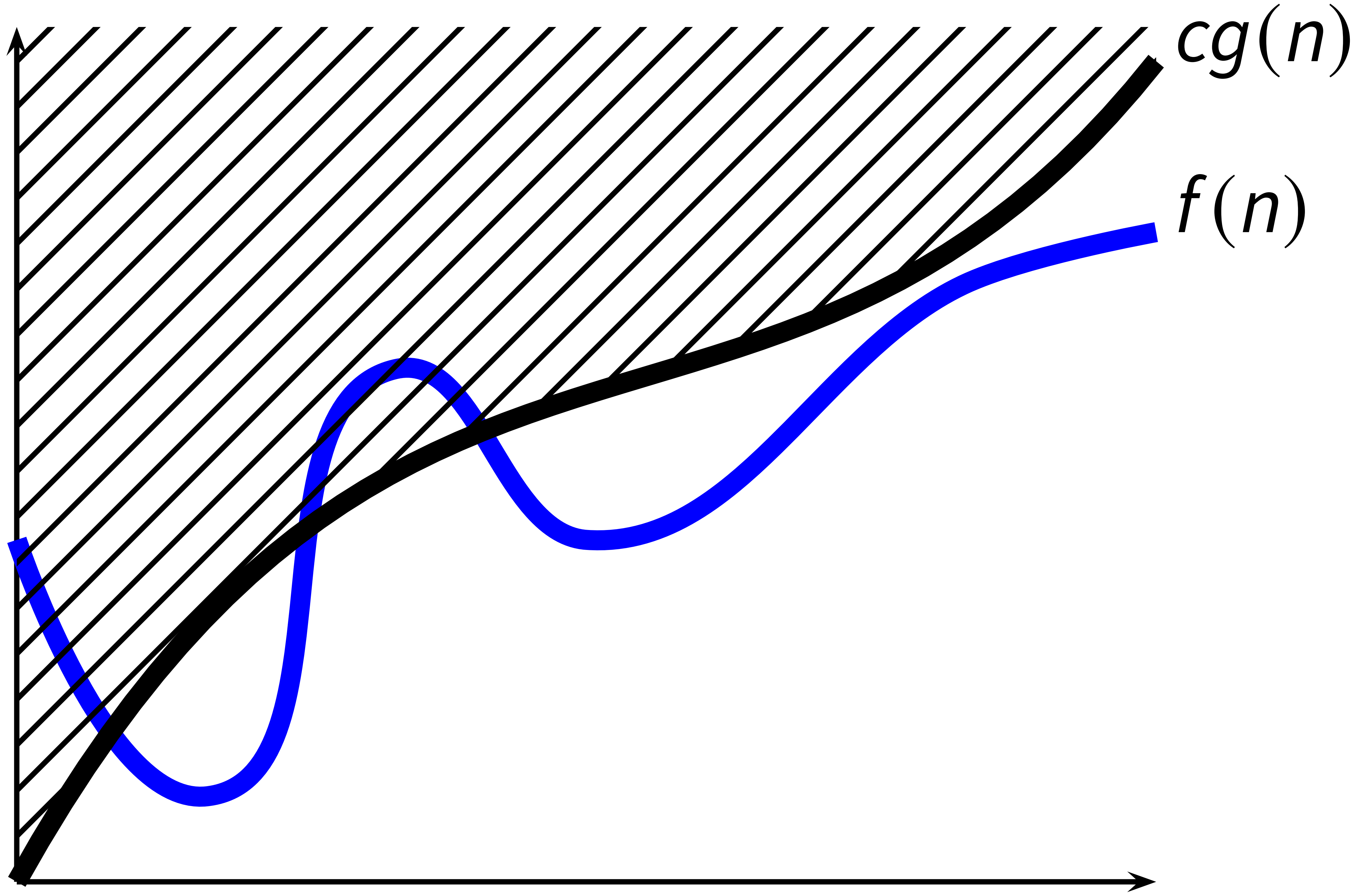

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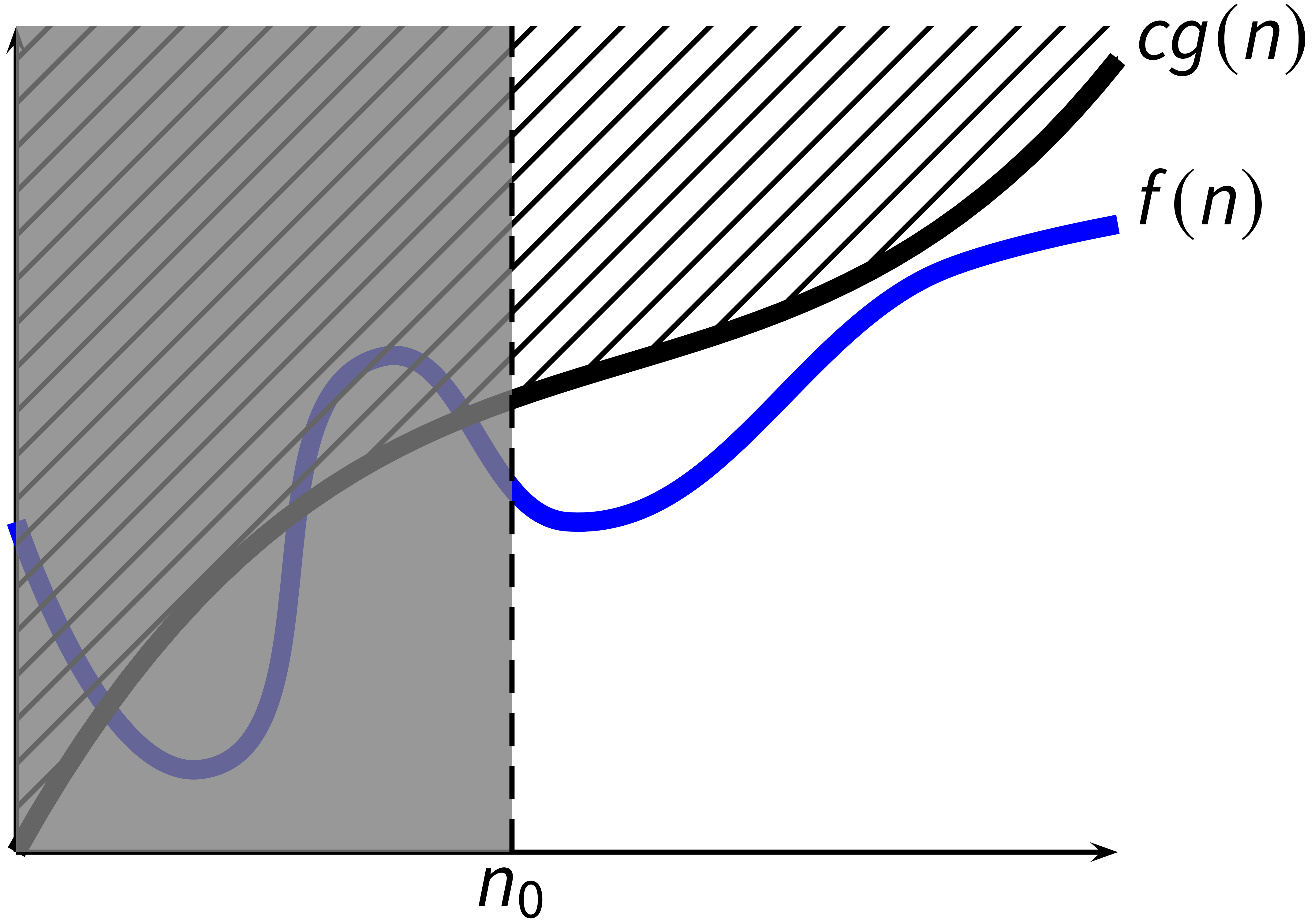
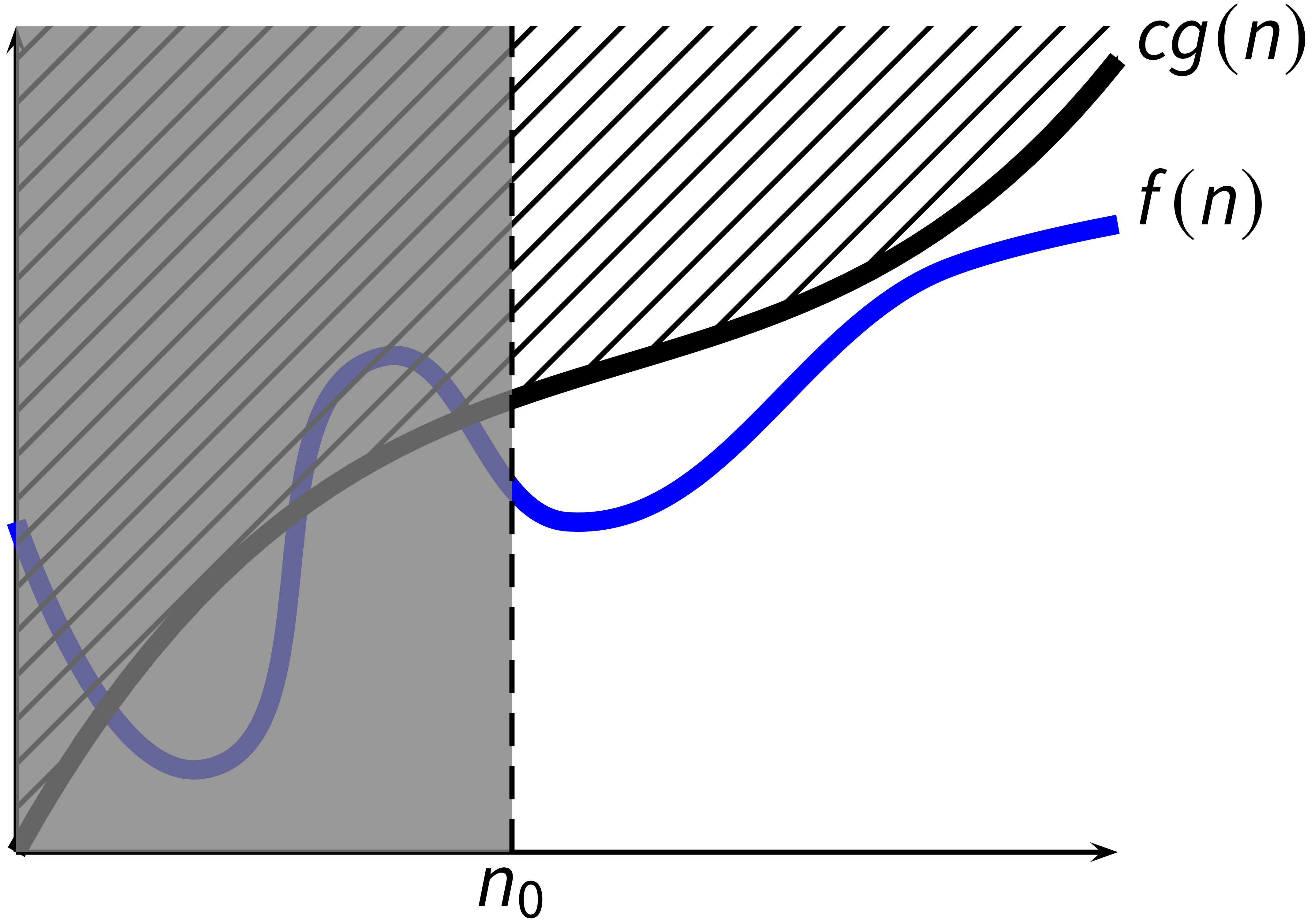


$$O(g(n)) = \{f(n) : \exists c>0, \exists n_0 >0 : 0 \leq f(n) \leq cg(n) \text{ for } n \geq n_0\}$$

O-Notation

- Given a function g(n), we define the **family of functions** O(g(n))



f(n) = O(g(n)), i.e., f(n) ∈ O(g(n)) which reads "*f(n) is big-oh of g(n)*"

Exercise

- Given T(n) find a suitable O(n)

$n^2 + 10n + 100$	
$n+10\log(n)$	
$n \log(n) + n \operatorname{sqrt}(n)$	
$2^{n/6} + n^7$	
$(10+n)/(n^2)$	

Exercise

- Given T(n) find a suitable O(n)

$n^2 + 10n + 100$	$O(n^2)$, $O(n^3)$
$n+10\log(n)$	$O(2^n)$
$n \log(n) + n\ \mathrm{sqrt}(n)$	$O(n^2)$
$2^{n/6} + n^7$	$O((1.5)^n)$
$(10+n)/(n^2)$	$O(1)$

From O to Ω

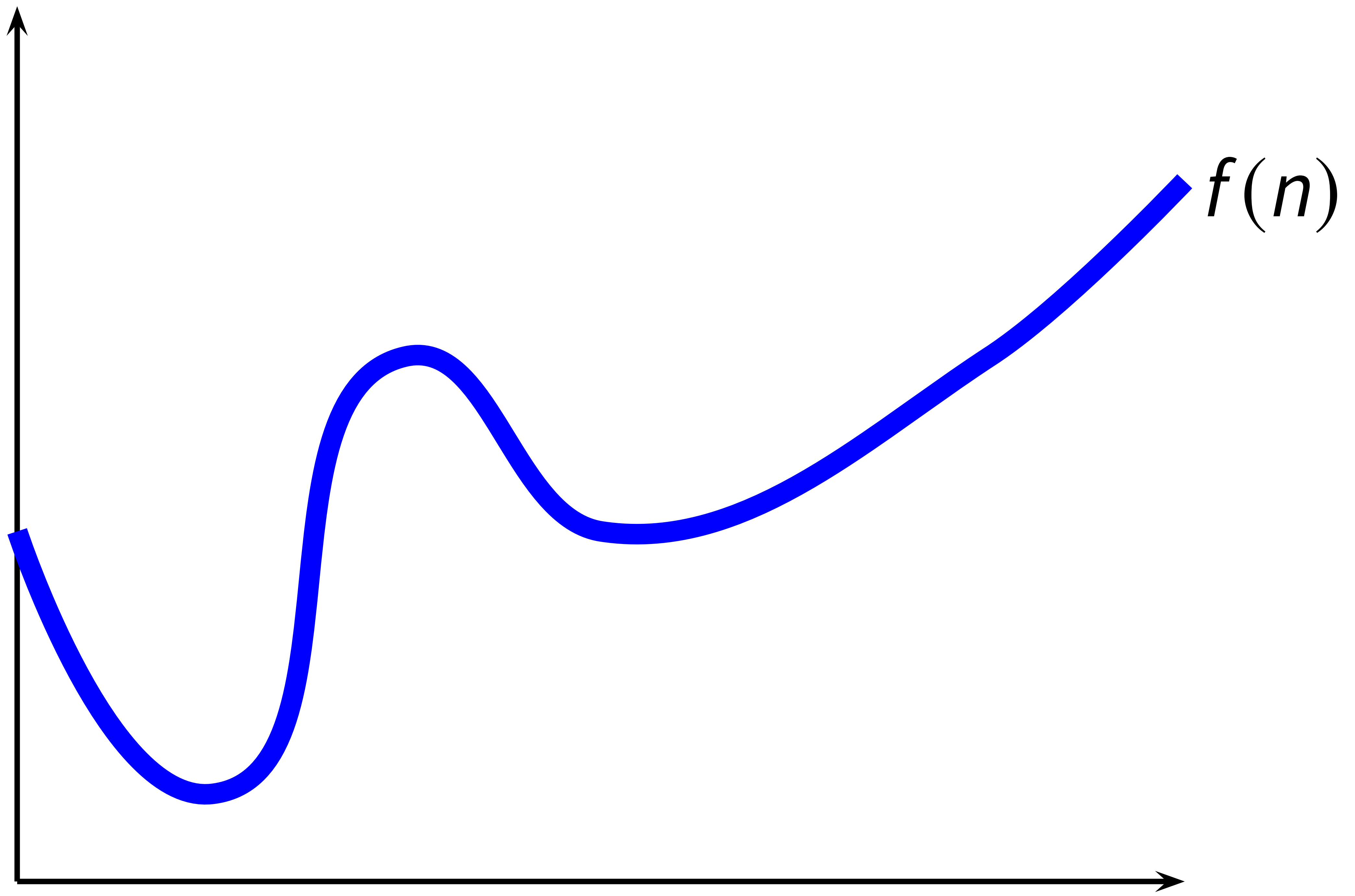
- If g(n) = O(f(n)) then we can **also** say that g(n) **asymptotically dominates** f(n), which we can also write as **f(n) = Ω(g(n))**
    - *read "f(n) is big-omega of g(n)" or simply "f(n) is omega of g(n)"*

- Ω(g(n)) denotes a **family of functions**

- **Warning: f(n) = Ω(g(n)) does not mean that f(n) equals Ω(g(n))**

Ω-Notation

- Given a function g(n), we define the **family of functions** Ω(g(n))



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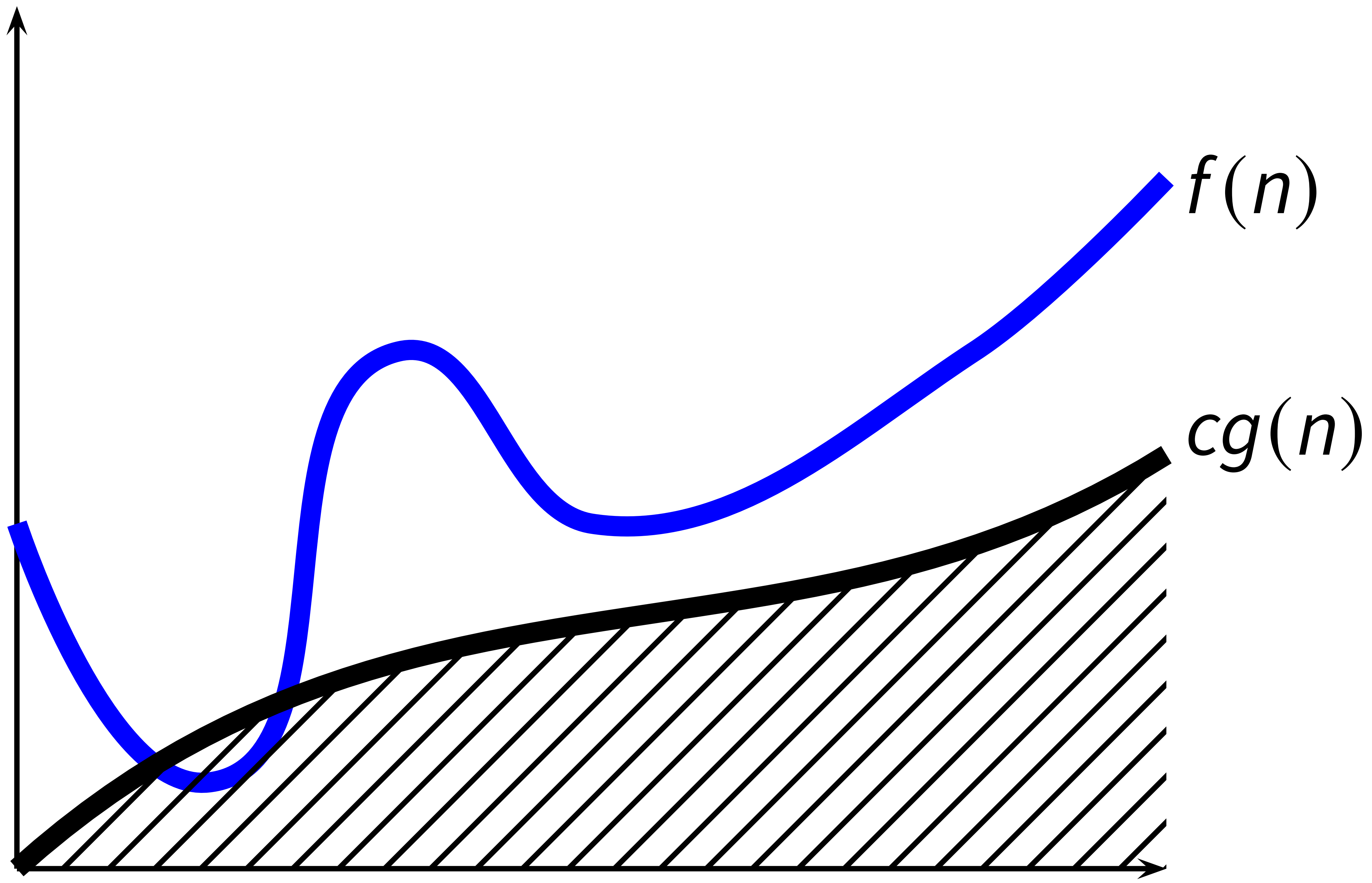

Ω-Notation

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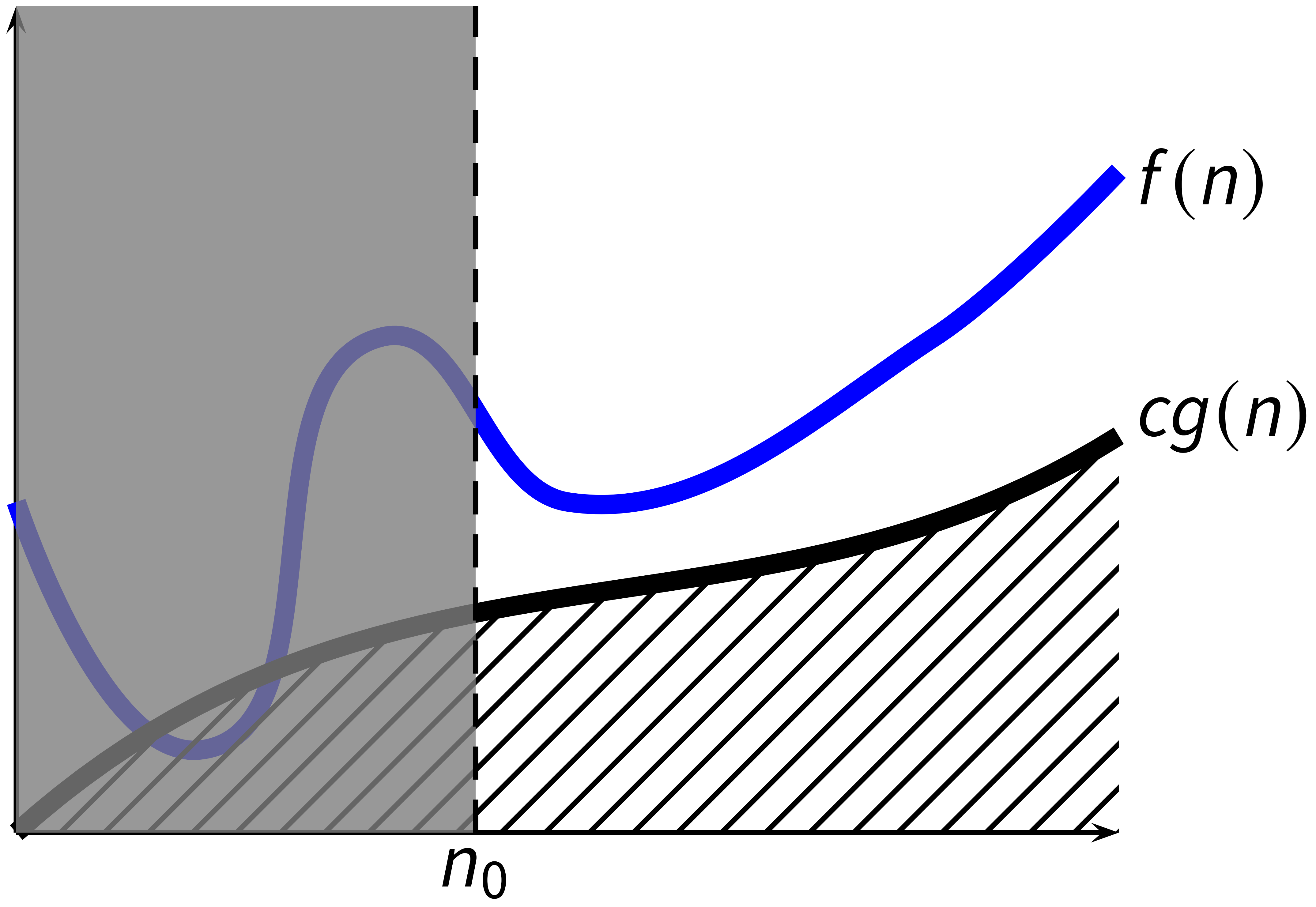
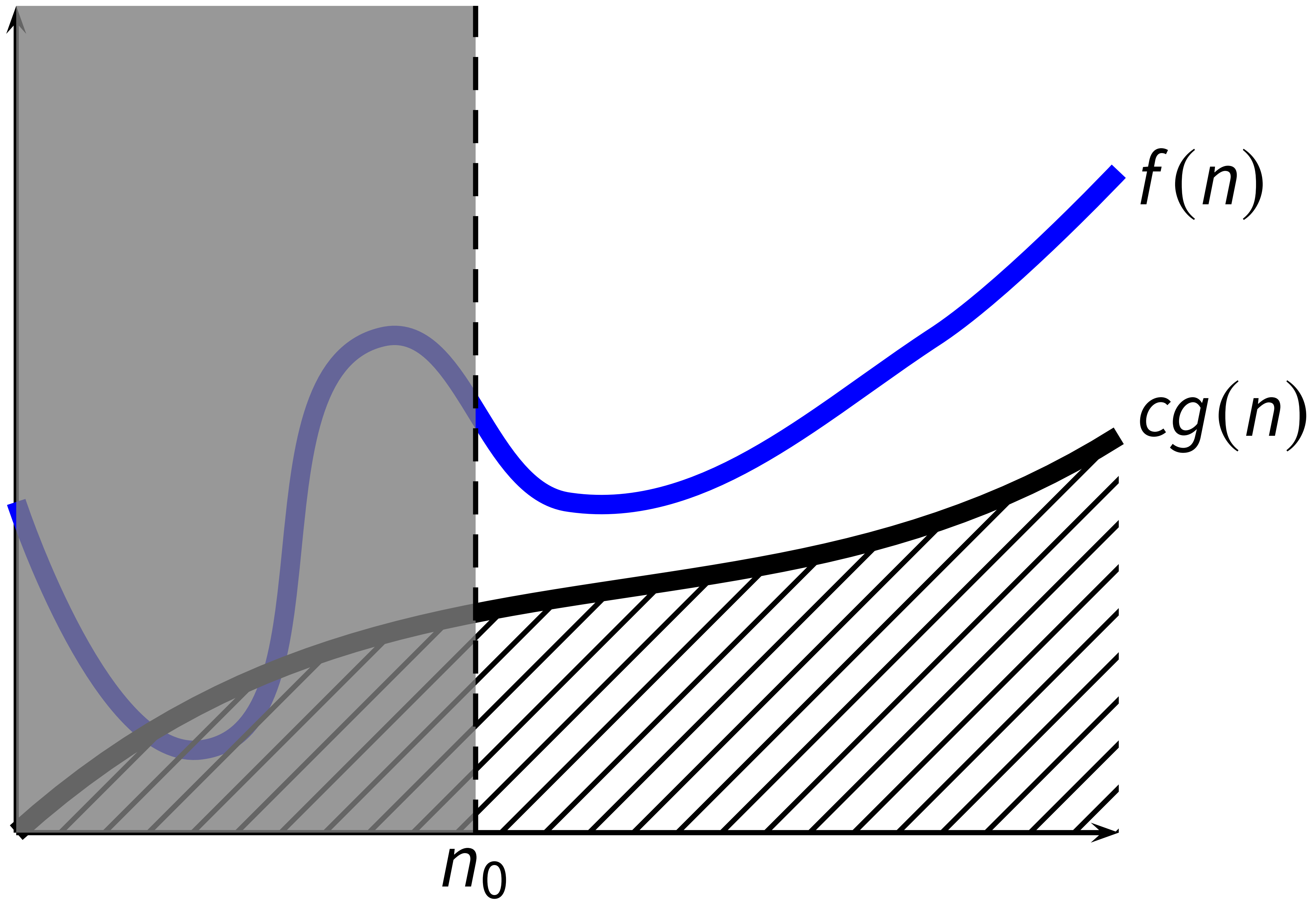


$\Omega(g(n))=\{f(n) :\ \exists\, c>0, \exists\ \exists\, n_0>0 : 0 \leq cg(n) \leq f(n) \text{ for } n \geq n_0\}$

Ω-Notation

- Given a function g(n), we define the **family of functions** Ω(g(n))



f(n) = Ω(g(n)), i.e., f(n) ∈ Ω(g(n)) "*f(n) is omega of g(n)*"

Exercise

- Given T(n) find a suitable Ω(n)

$n^2 + 10n + 100$	
$n+10\log(n)$	
$n \log(n) + n\ \mathrm{sqrt}(n)$	
$2^{n/6} + n^7$	
$(10+n)/(n^2)$	

Exercise

- Given T(n) find a suitable Ω(n)

$n^2 + 10n + 100$	$\Omega(n^2)$, $\Omega(n)$
$n+10\log(n)$	$\Omega(n)$, $\Omega(\log(n))$
$n \log(n) + n\,\mathrm{sqrt}(n)$	$\Omega(n)$
$2^{n/6} + n^7$	$\Omega(2^{n/6})$, $\Omega(n^7)$
$(10+n)/(n^2)$	$\Omega(1/n)$

From O and Ω to Θ

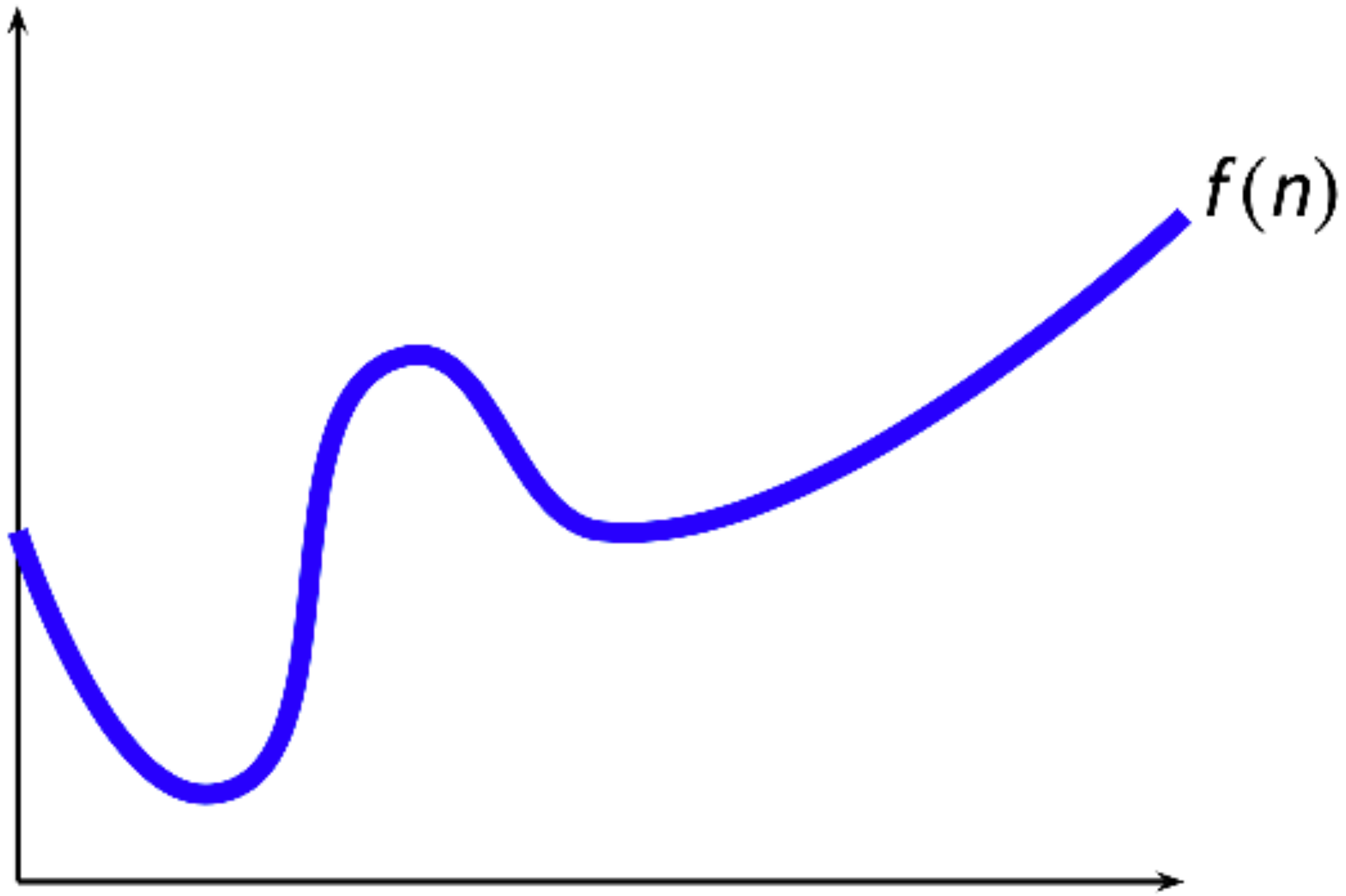
- When $f(n) = O(g(n))$ **and** $f(n) = \Omega(g(n))$, then $\mathbf{f(n) = \Theta(g(n))}$
    - *read "f(n) is theta of g(n)"*

- $\Theta(g(n))$ denotes a **family of functions**

- **Warning: $f(n) = \Theta(g(n))$ does not mean that $f(n)$ equals $\Theta(g(n))$**

Θ-Notation

- Given a function g(n), we define the **family of functions** Θ(g(n))



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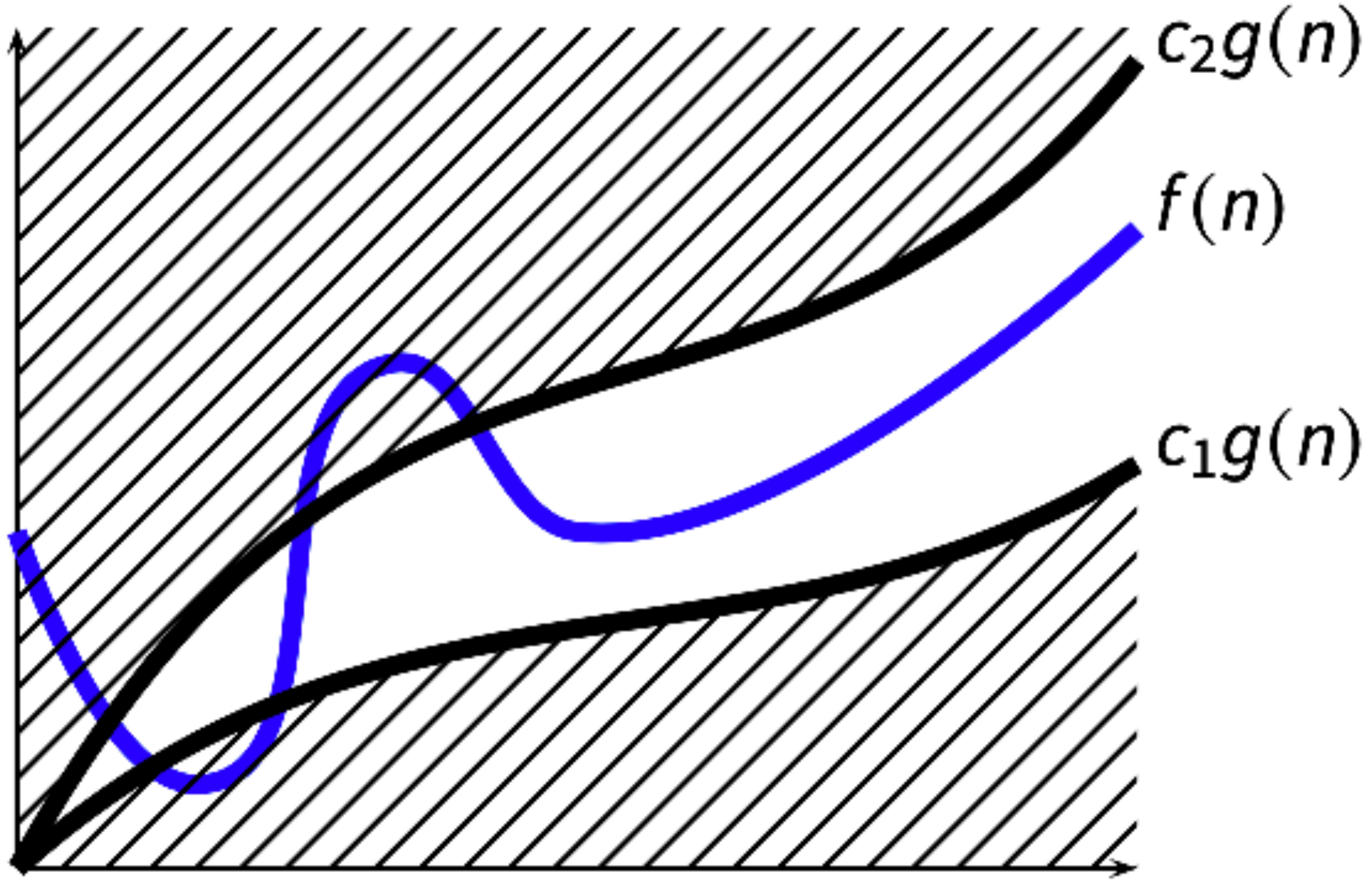

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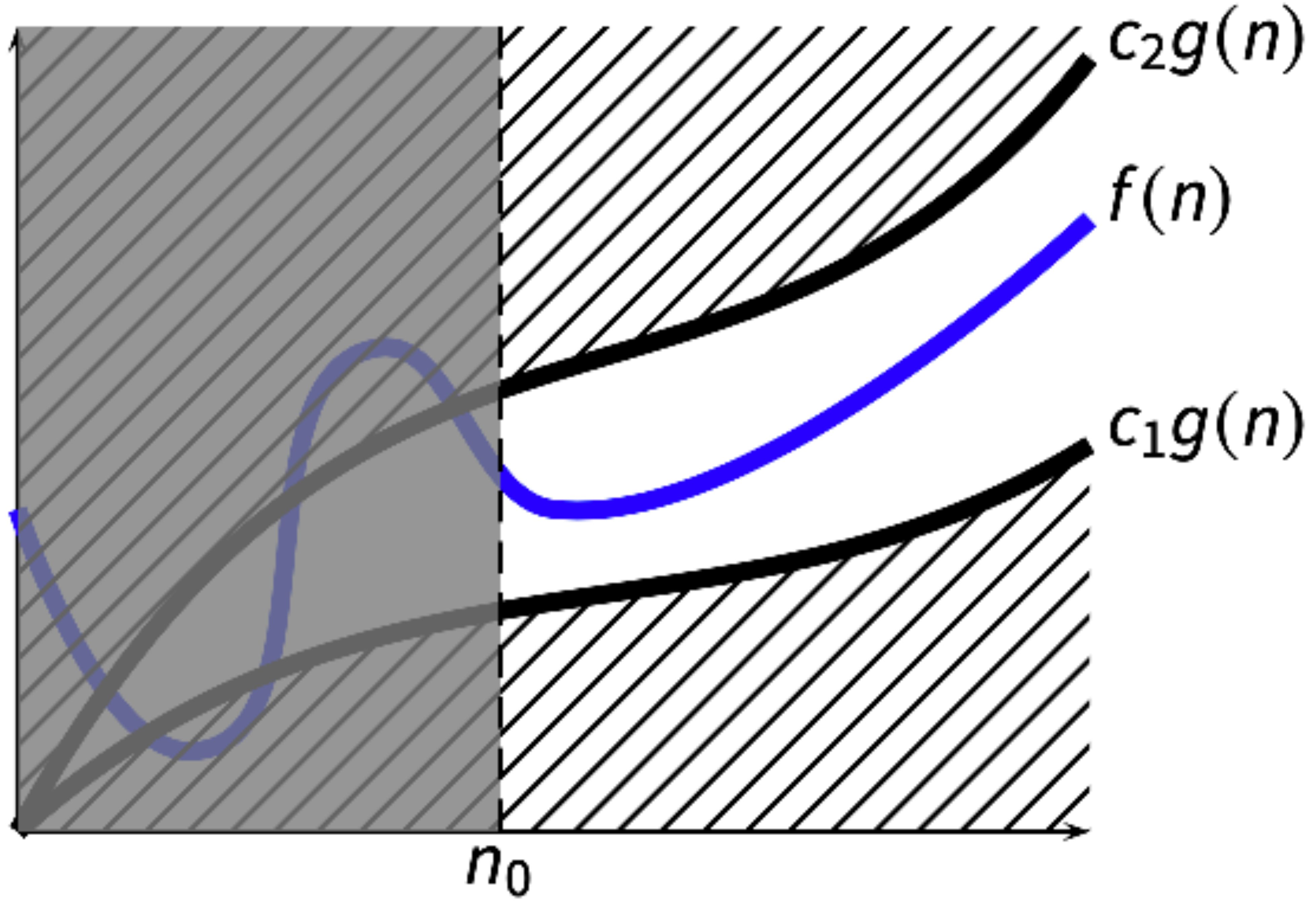
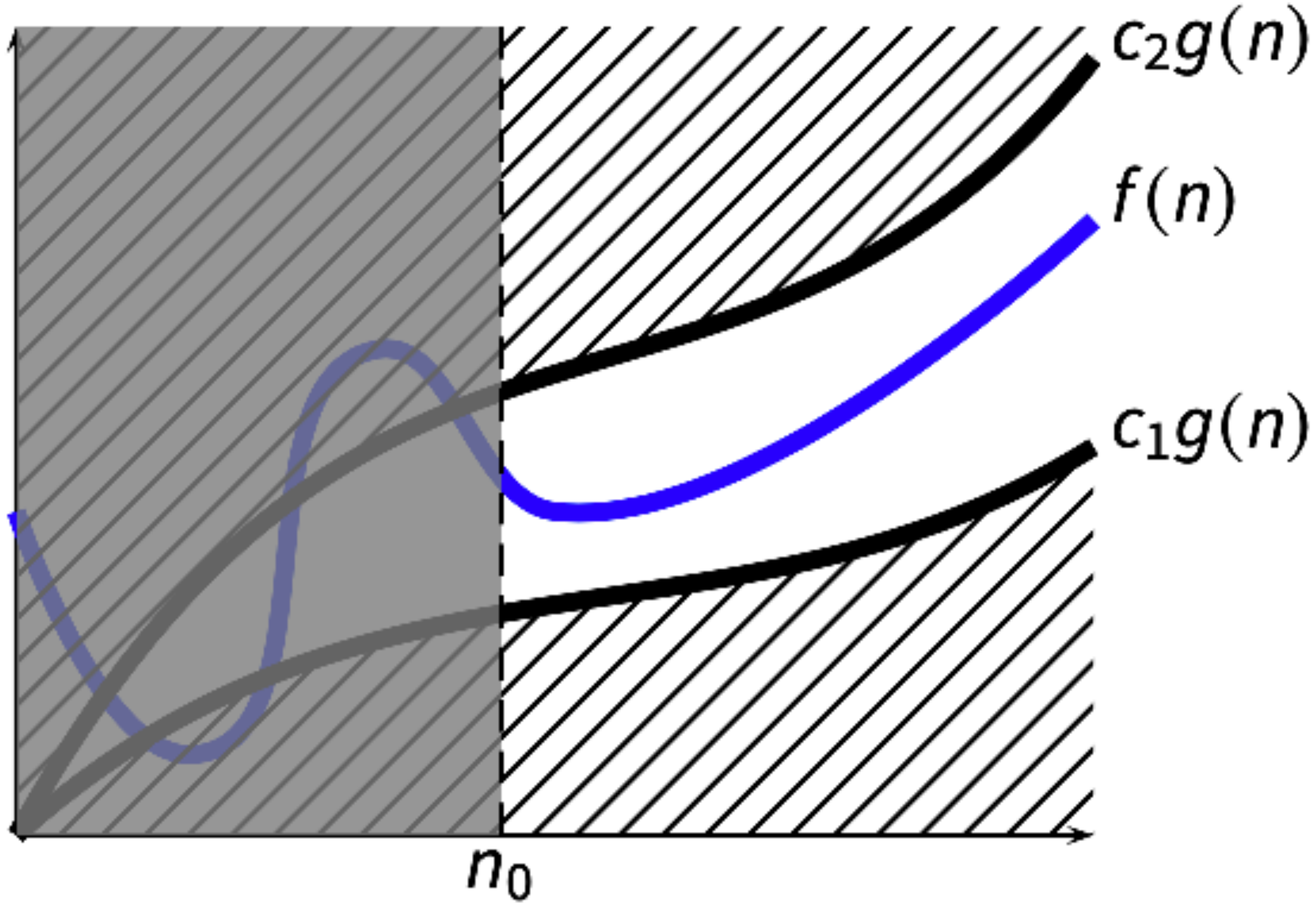


Θ-Notation

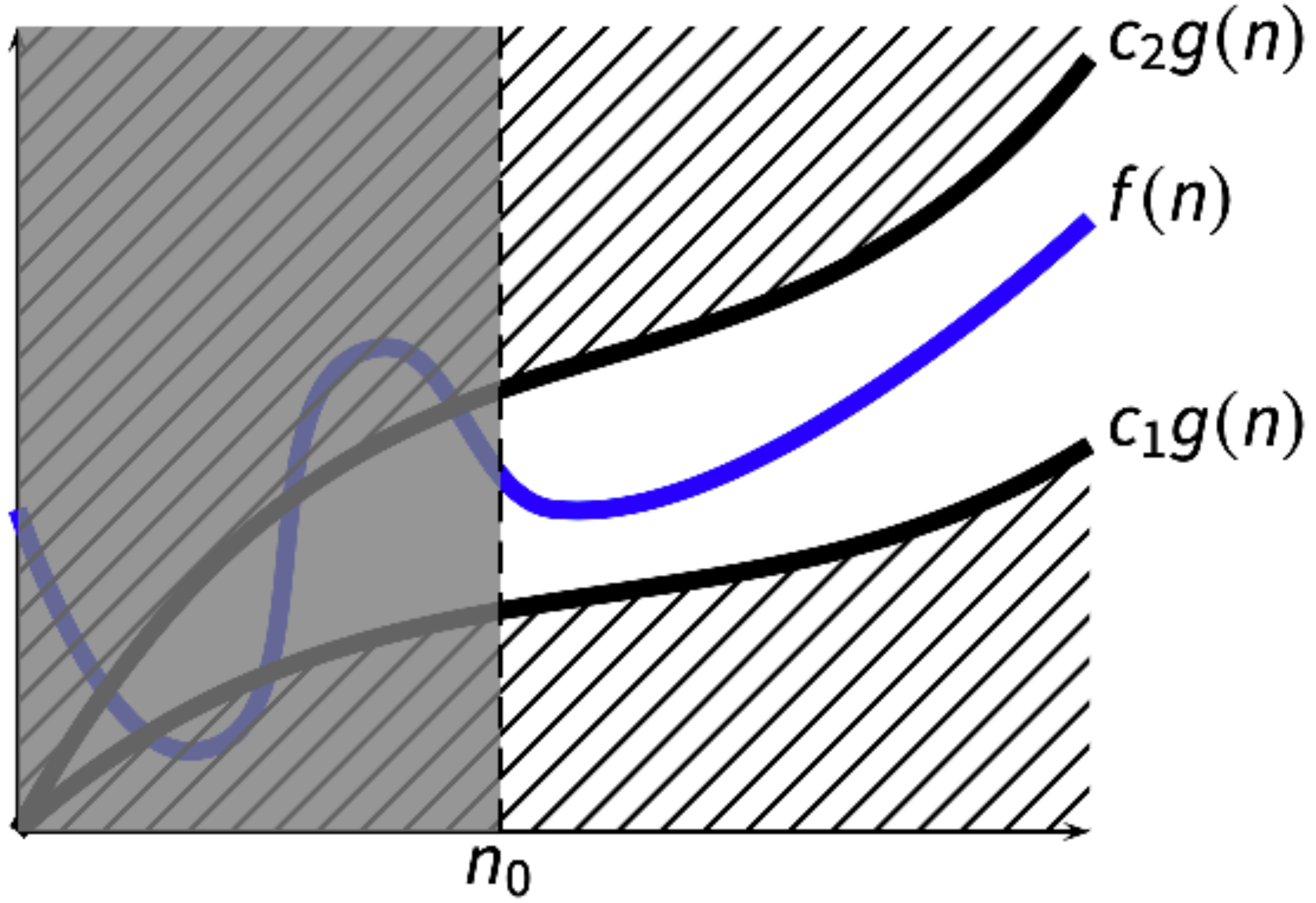
- Given a function g(n), we define the **family of functions** Θ(g(n))



$$\Theta(g(n)) = \{f(n) : \exists\, c_1 > 0, \exists\, c_2 > 0, \exists\, n_0 > 0 \mid 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for } n \ge n_0\}$$

Θ-Notation

- Given a function g(n), we define the **family of functions** Θ(g(n))



f(n) = Θ(g(n)), i.e., f(n) ∈ Θ(g(n)), which reads as "*f(n) is theta of g(n)*"

Exercise

- Given T(n) find a suitable Θ(n)

$n^2 + 10n + 100$	
$n+10\log(n)$	
$n \log(n) + n\, \mathrm{sqrt}(n)$	
$2^{n/6} + n^7$	
$(10+n)/(n^2)$	

Exercise

- Given T(n) find a suitable Θ(n)

$n^2 + 10n + 100$	$\Theta(n^2)$
$n+10\log(n)$	$\Theta(n)$
$n \log(n) + n\ \mathrm{sqrt}(n)$	$\Theta(n\ \mathrm{sqrt}(n))$
$2^{n/6} + n^7$	$\Theta(2^{n/6})$
$(10+n)/(n^2)$	$\Theta(1/n)$

Θ, O and Ω as Relations

- The Θ-, Ω-, and O-notation can be viewed as the **asymptotic version** of the "=", "≥", and "≤" **relations for functions**
- For two functions f(n) and g(n), $f(n) = \Omega(g(n))$ *and* $f(n) = O(g(n))$ *iff* $f(n) = \Theta(g(n))$ can be interpreted as saying $f \geq g \wedge f \leq g \Leftrightarrow f = g$
- When $f(n) = O(g(n))$ we say that g(n) is an **upper bound** for f(n)
  - g(n) **dominates** f (n)
- When $f(n) = \Omega(g(n))$ we say that g(n) is a **lower bound** for f(n)

Θ, O and Ω as Anonymous Functions

- We can use the Θ-, Ω-, and O-notation to represent **unknown or unspecified functions**

- For instance, $f(n) = 10n^2 + O(n)$ means that $f(n)$ is equals to $10n^2$ plus a(ny) function that we do not know (or care) which is **at most linear** in n ... **asymptotically** speaking

Exercise

- Are the following equalities true or false? And why?
  - $n^2 + 4n - 1 = n^2 + \Theta(n)$?
  - $n^2 + \Omega(n) - 1 = O(n^2)$?
  - $n^2 + O(n) - 1 = O(n^2)$?
  - $n \log(n) + \Theta(\mathrm{sqrt}(n)) = O(n\ \mathrm{sqrt}(n))$?

Exercise

- Are the following equalities true or false? And why?
  - $n^2 + 4n - 1 = n^2 + \Theta(n)$? YES
  - $n^2 + \Omega(n) - 1 = O(n^2)$? NO
  - $n^2 + O(n) - 1 = O(n^2)$? YES
  - $n \log(n) + \Theta(\text{sqrt}(n)) = O(n\ \text{sqrt}(n))$? YES

Exercise

Are the following formulae true or false?

- $f(n) = \Theta(g(n)) \Rightarrow f(n) = O(g(n))$?
- $f(n) = \Theta(g(n)) \wedge g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$?
- $f(n) = O(g(n)) \wedge g(n) = \Theta(h(n)) \Rightarrow f(n) = O(h(n))$?
- $n \log(n) + \Theta(\mathrm{sqrt}(n)) = O(n\ \mathrm{sqrt}(n))$?
- $n^2 + 10n + 100 = O(n \log(n))$?
- $n^2 + (1.5)^n = O(2^{n/2})$?

Exercise

Are the following formulae true or false?

- $f(n) = \Theta(g(n)) \Rightarrow f(n) = O(g(n))$? Yes
- $f(n) = \Theta(g(n)) \wedge g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$? Yes
- $f(n) = O(g(n)) \wedge g(n) = \Theta(h(n)) \Rightarrow f(n) = O(h(n))$? Yes
- $n \log(n) + \Theta(\mathrm{sqrt}(n)) = O(n\ \mathrm{sqrt}(n))$? Yes
- $n^2 + 10n + 100 = O(n \log(n))$? No
- $n^2 + (1.5)^n = O(2^{n/2})$? No

Exercise @ Home

Are the following formulae true or false?

- $f(n) = O(2^n) \Rightarrow f(n) = O(n^2)$?
- $f(n) = O(2^n) \Rightarrow f(n) = \Theta(n^2)$?
- $f(n) = \Theta(2^n) \Rightarrow f(n) = O(n^2 2^n)$?
- $f(n) = \Theta(n^2 2^n) \Rightarrow f(n) = O(2^{n+2 \log 2(n)})$?
- $\mathrm{sqrt}(n) = O(\log^2 n)$?

o-Notation

- The O-notation defines an upper bound that **might not** be asymptotically tight
  - $n \log(n) = O(n^2)$ is not asymptotically tight
  - $n^2 - n + 10 = O(n^2)$ is asymptotically tight
- The o-notation denotes upper bounds that **are not** asymptotically tight
- Given a function g(n), we define the families of functions o(g(n))as:

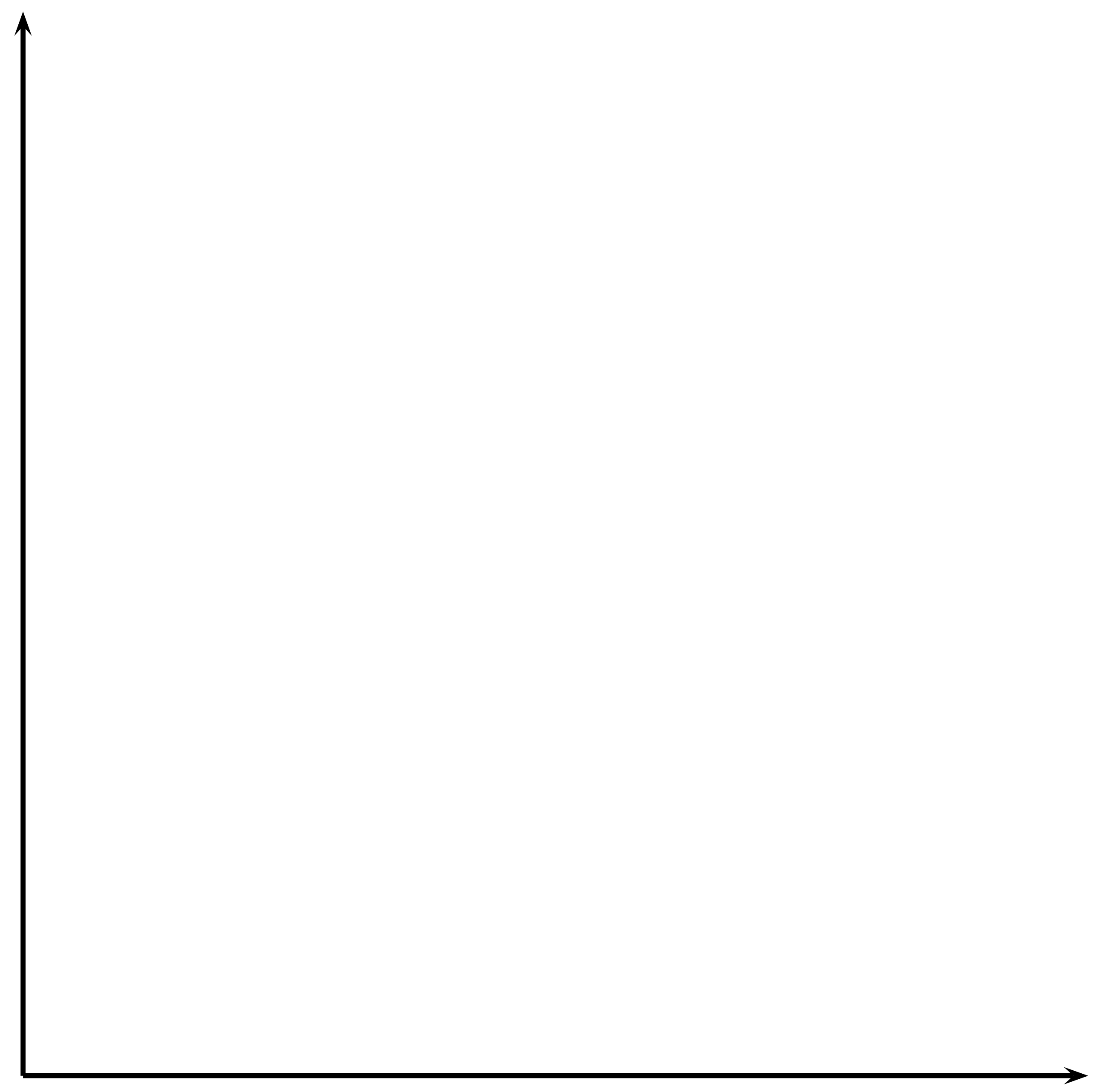
$$o(g(n)) = \{f(n): \forall c>0, \exists n_0 >0 : 0 \leq f(n) < c\, g(n) \text{ for } n \geq n_0\}$$

ω-Notation

- The Ω-notation defines a lower bound that **might not** be asymptotically tight
  - $2^n = \Omega(n \log(n))$ is not asymptotically tight
  - $n + 4n \log(n) = \Omega(n \log(n))$ is asymptotically tight
- The ω-notation define lower bounds that **are not** asymptotically tight
- Given a function g(n), we define the families of functions ω(g(n)) as:

$$\omega(g(n)) = \{f(n): \forall\, c>0, \exists\, n_0 > 0 : 0 \leq cg(n) < f(n) \text{ for } n \geq n_0\}$$

Visualizing the Relation between Functions



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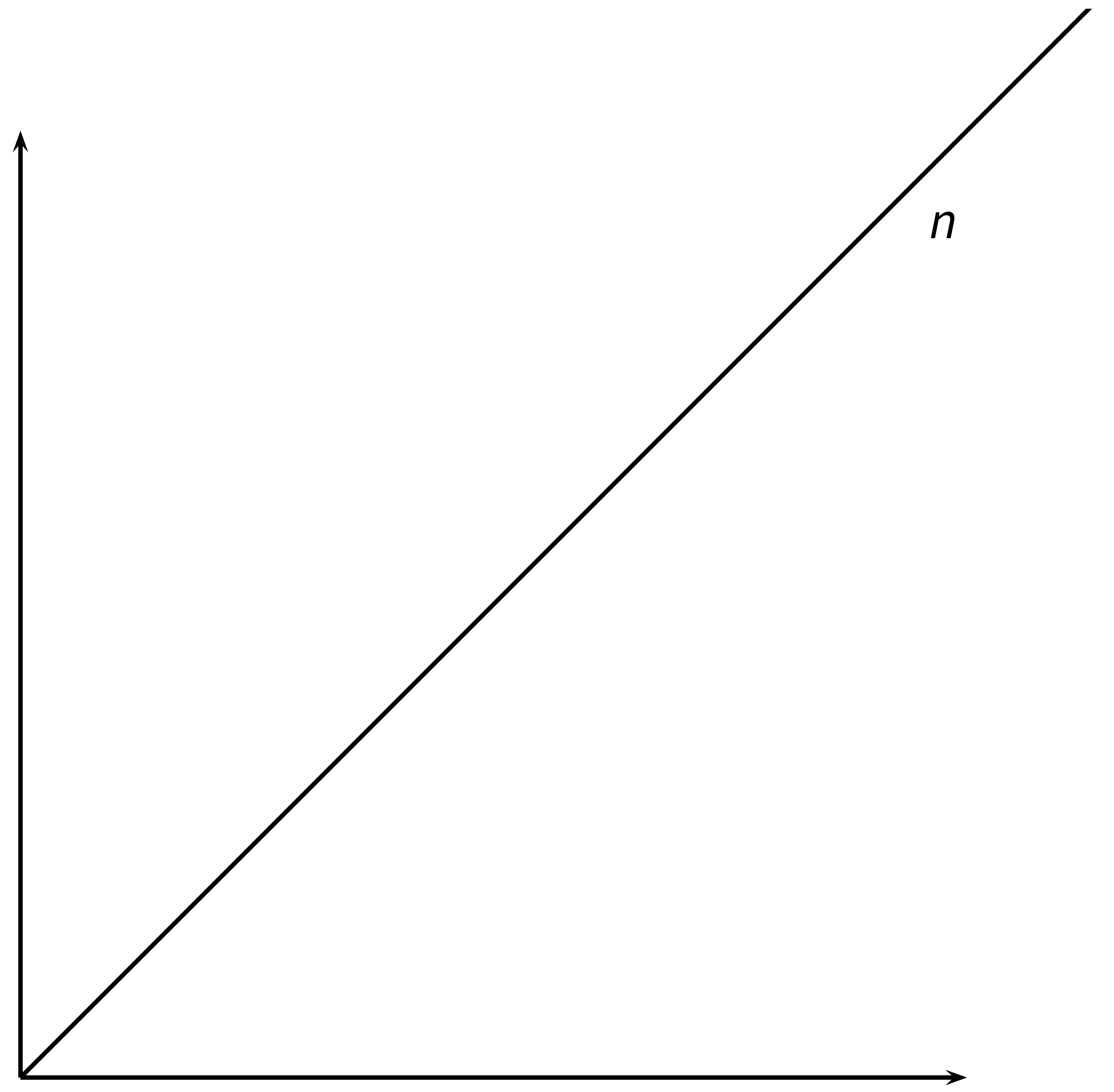

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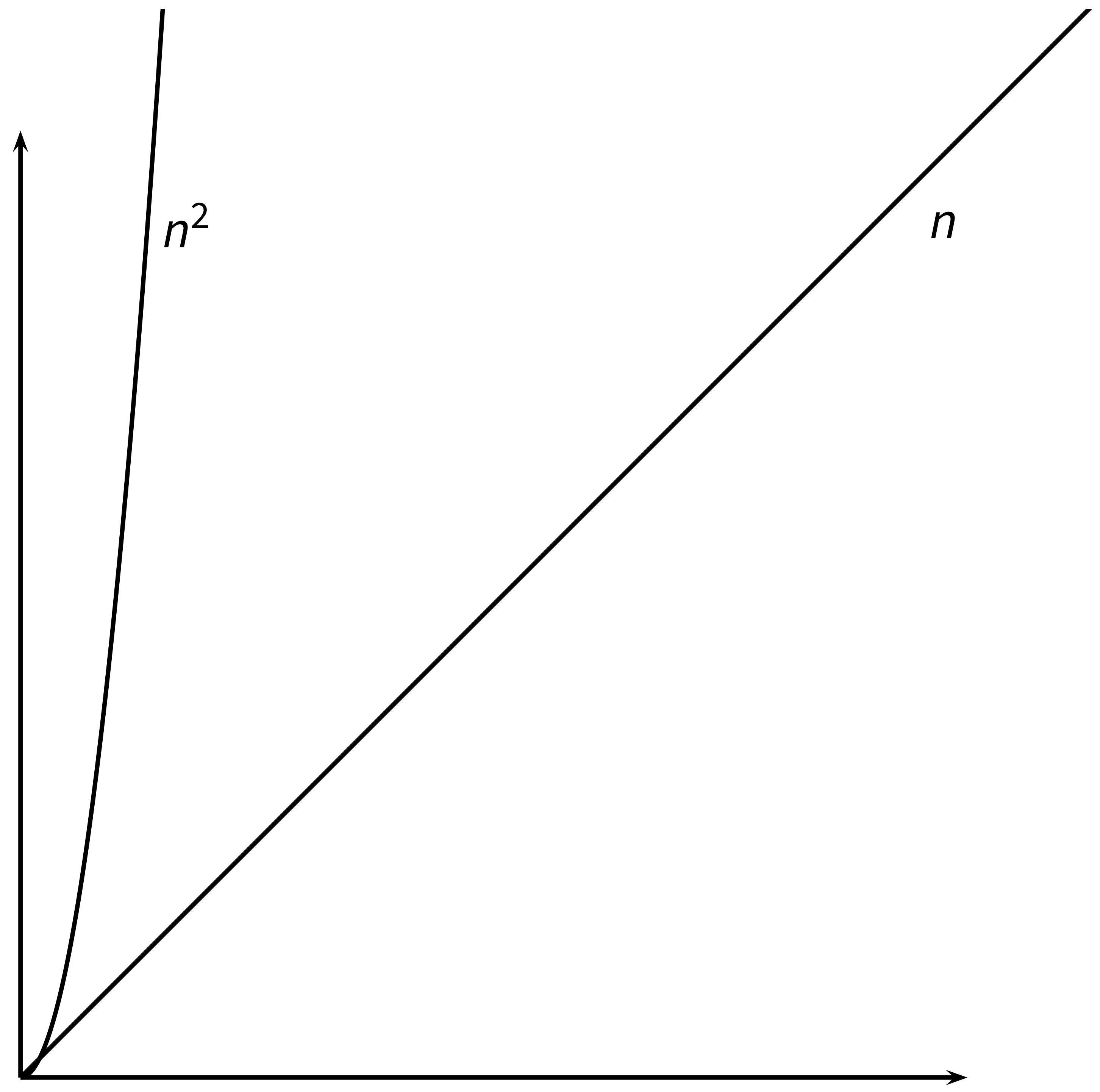

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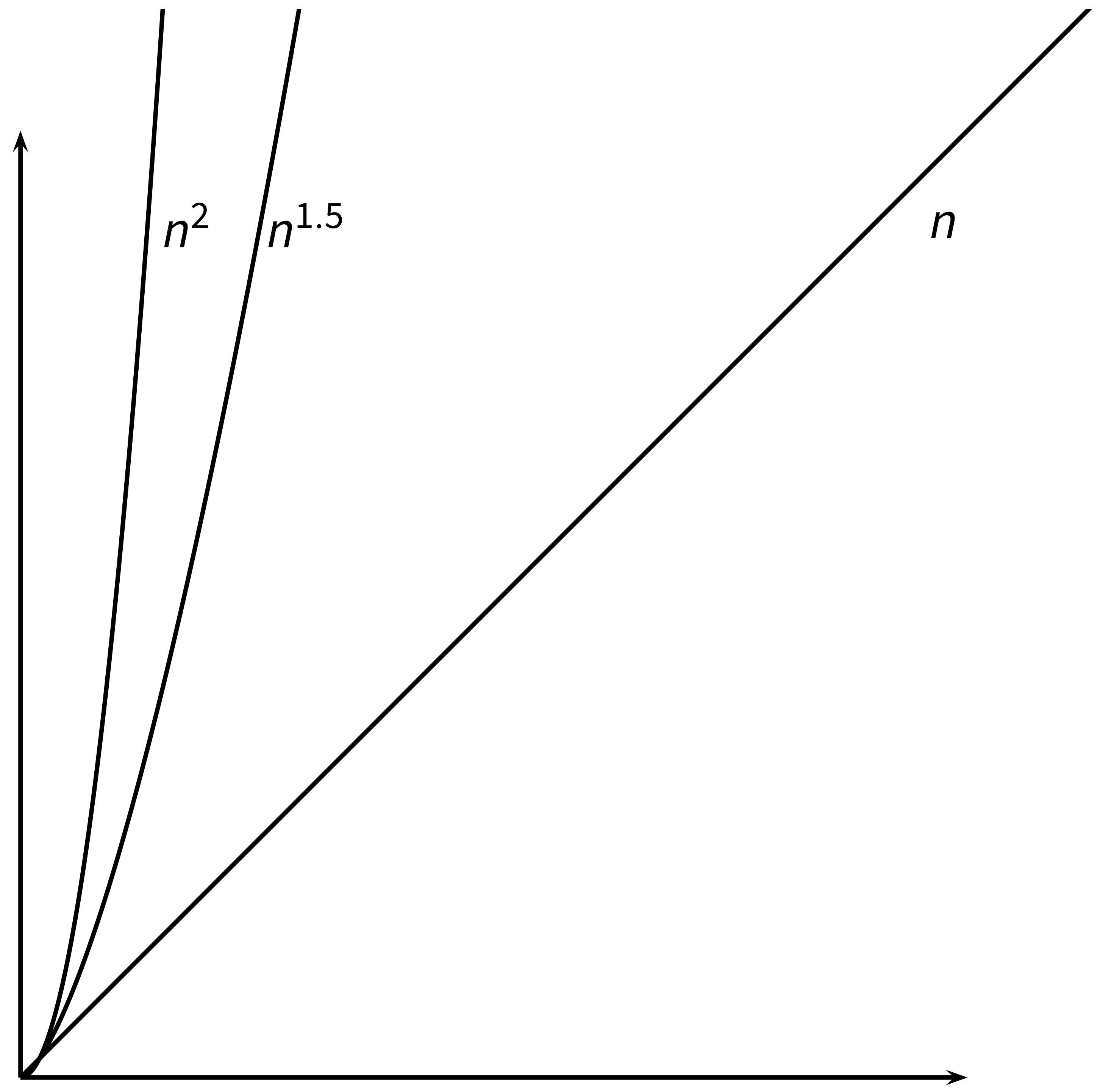

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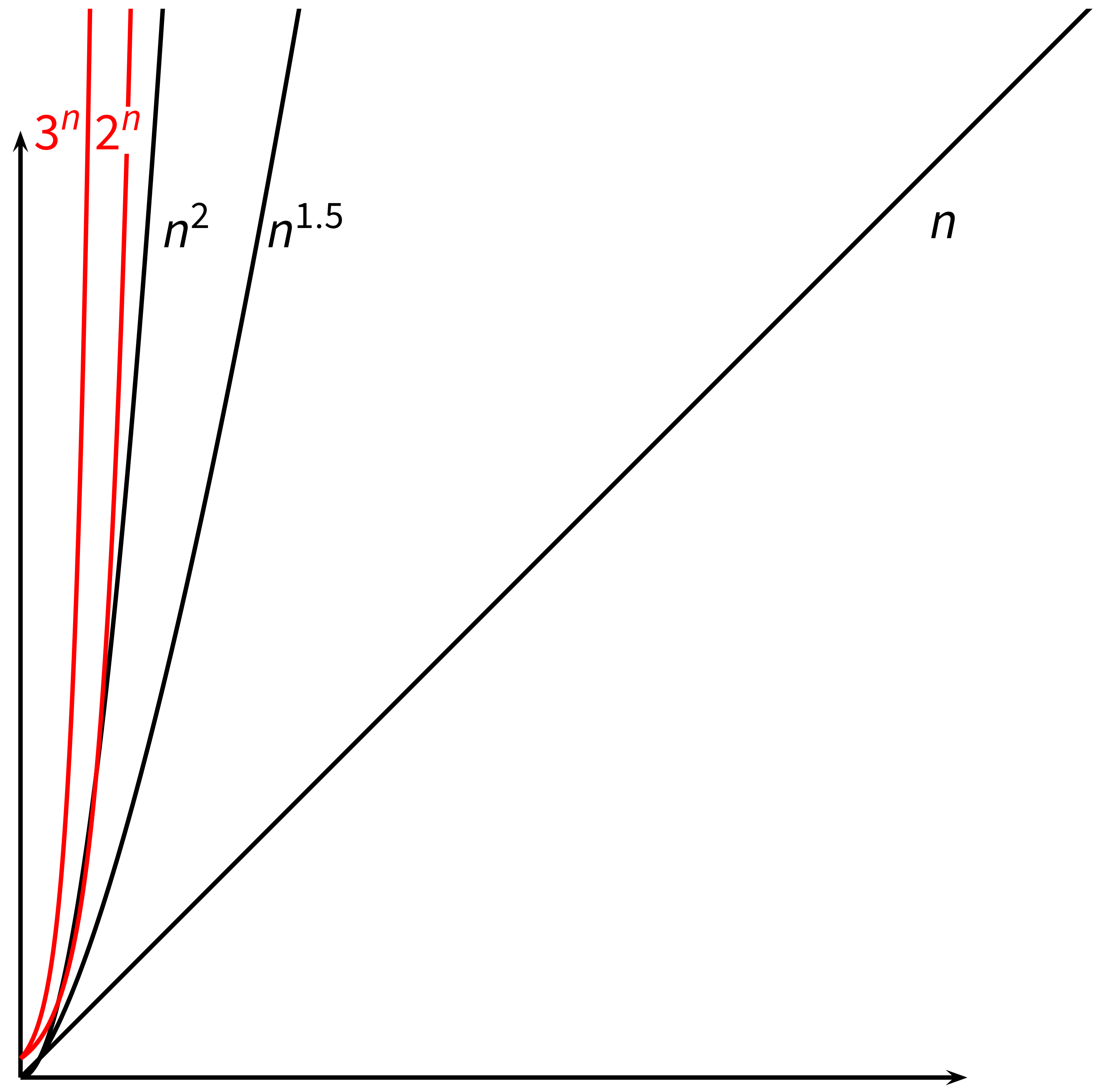

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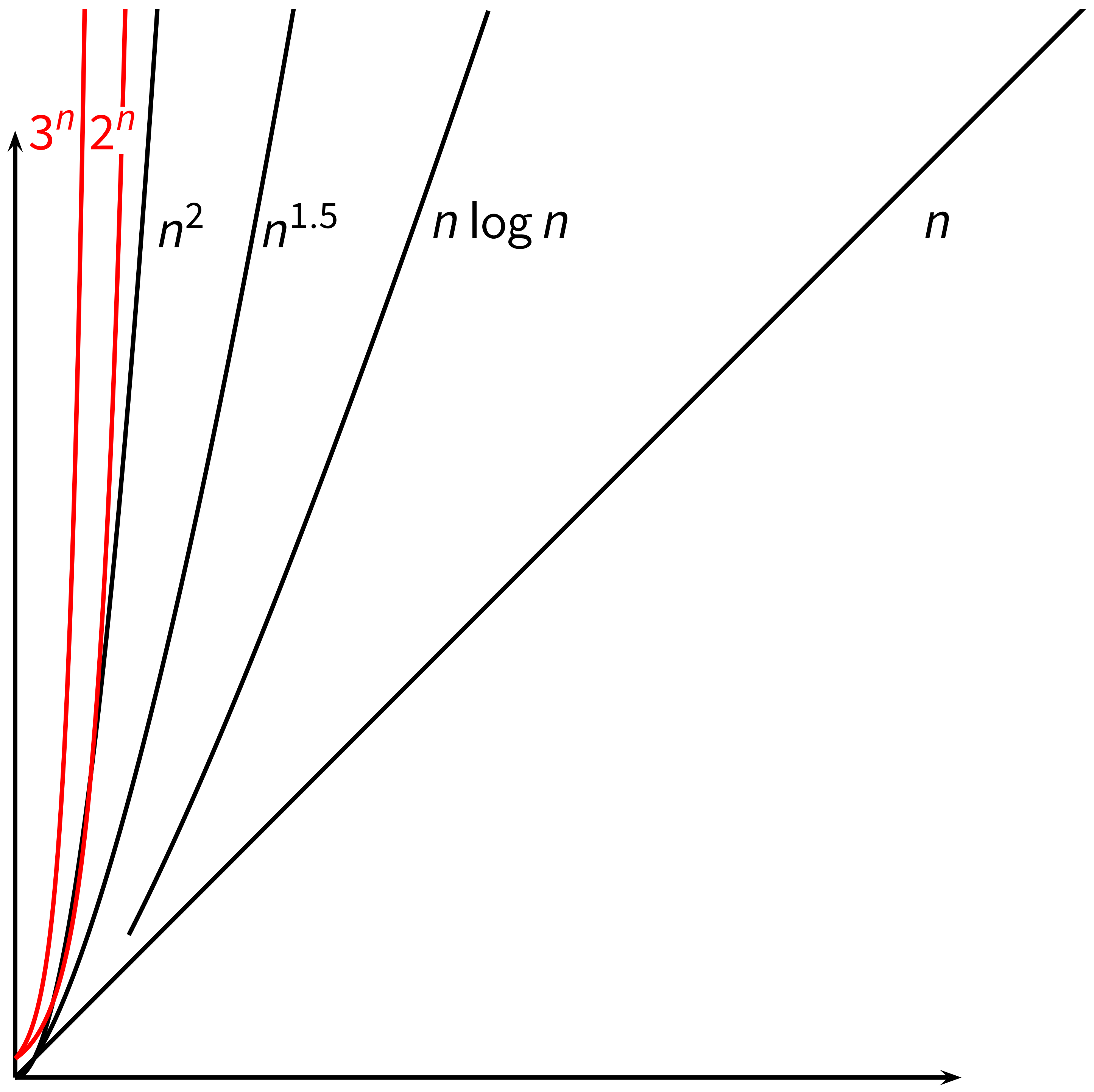

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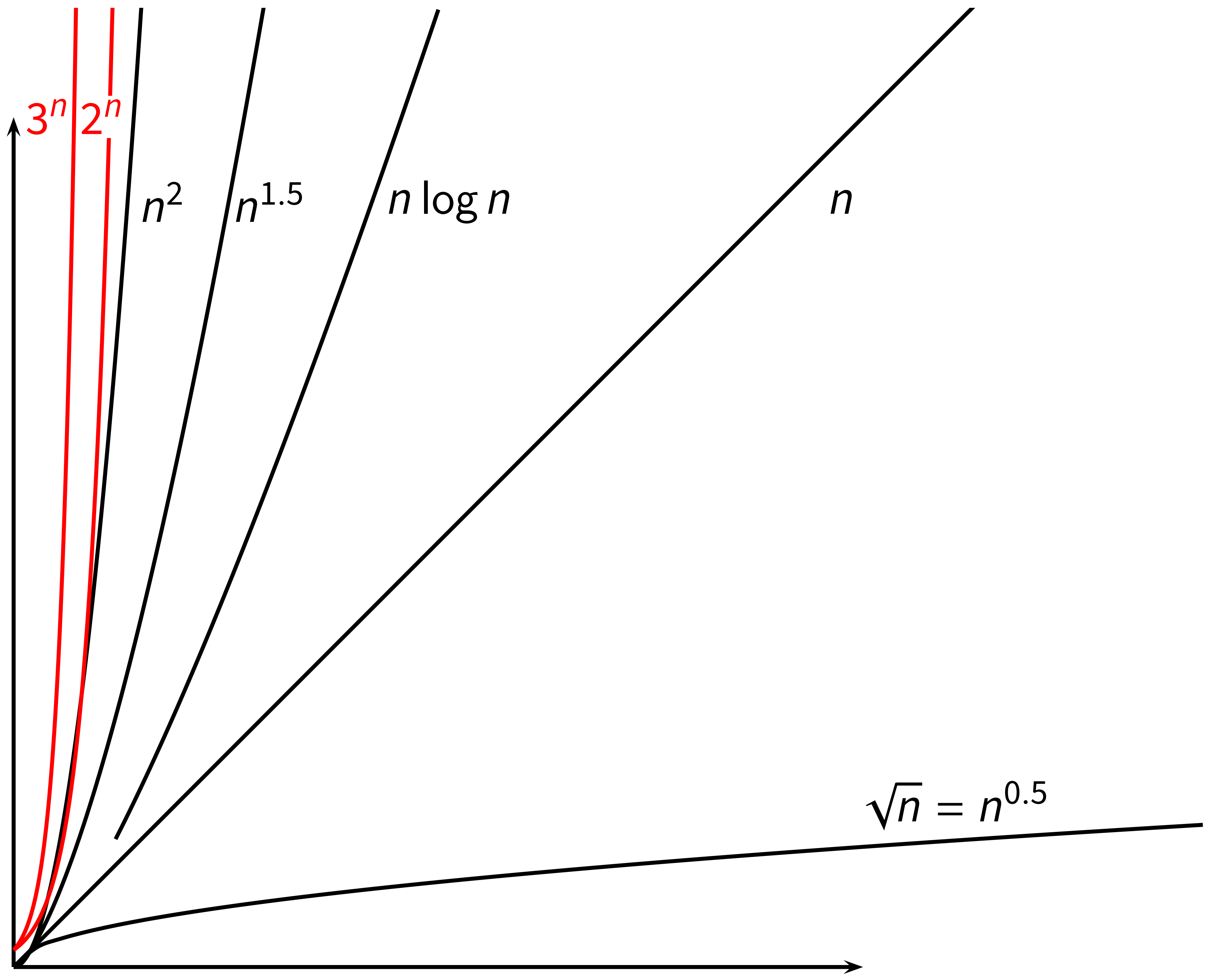

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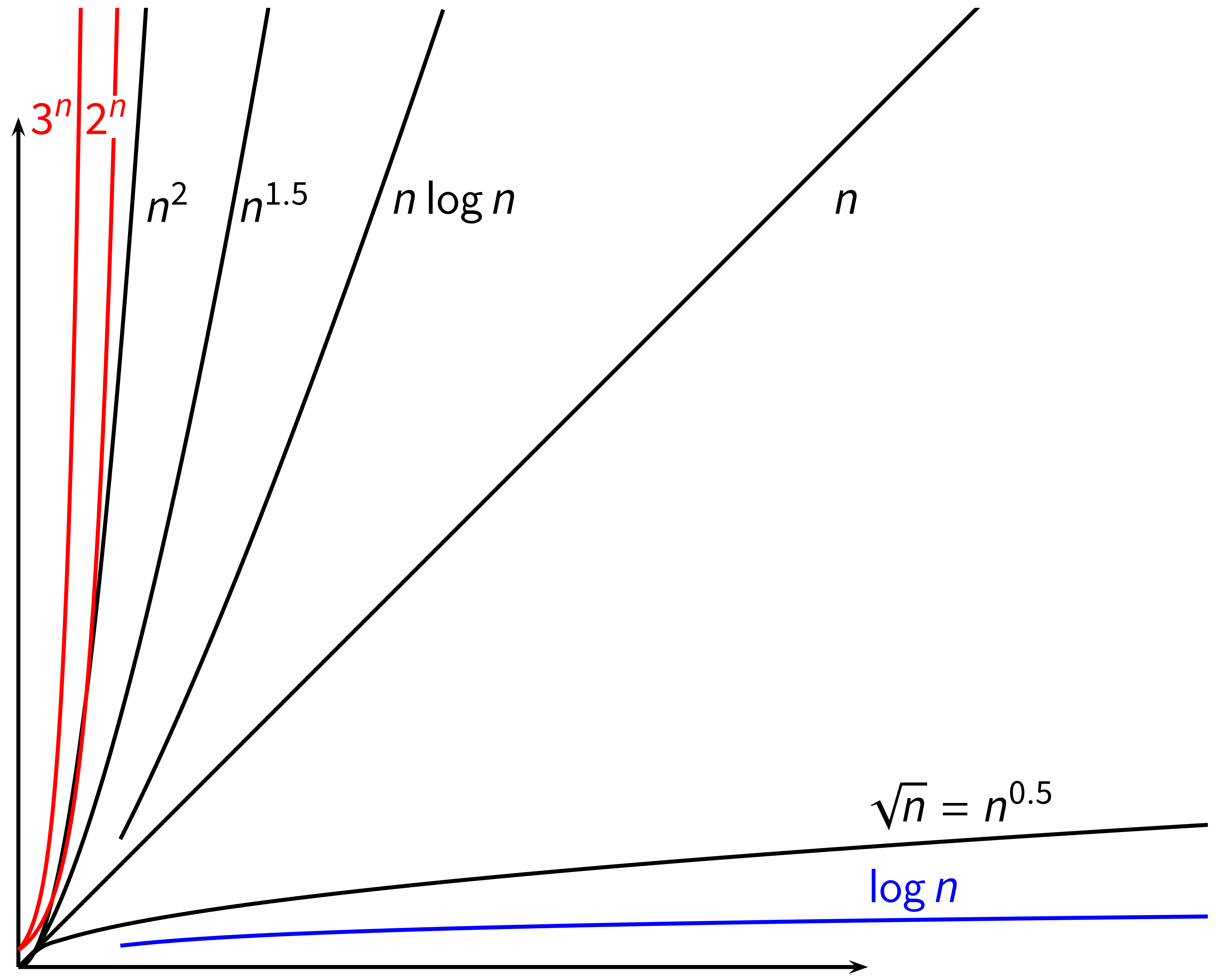

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