OX Y1 Core Micro - Auction

Basics

Independent Private Values (Model) for Auction #flashcard

- There a n bidders competing for a single unit
- Bidder i values the unit v_i, privately known to her
- The distribution of v_i is commonly known:

$$v_i \sim U\left[L, H\right]$$

  with cdf and pdf:

$$\begin{cases} F(v) & = \frac{v-L}{H-L} \\ f(v) & = \frac{1}{H-L} \end{cases}$$

  - This corresponds to linear demand
- All bidders are risk-neutral

Expected k^{th} Highest Value among n Values Independently Drawn from the $U[L, H]$ #flashcard

- The expected k^{th} highest value among n values independently drawn from the $U[L, H]$ is:

$$L + \left(\frac{n+1-k}{n+1}\right)(H-L)$$

  - Intuition: divide the distribution into $n+1$ sections, and the k^{th} highest value means there are k sections above the value $\Longleftrightarrow$ $n+1-k$ sections below

1st Price Auctions

1st Price Sealed Bid Auction in IPV Model #flashcard

- Every player independently chooses a bid without seeing others' bids, and the bidder with the highest bid wins.
- **Symmetric Bayesian Nash Equilibrium**:
  - **Steps**:
  - Setup the expected utility conditioning on $b(\tilde{v})$
  - Optimise by taking FOC wrt $\tilde{v}$
  - Impose $\tilde{b}(.) = b(.), \tilde{v} = v_i$ in a symmetric NE
  - Use cdf, pdf of the uniform distribution
  - Solve the ODE:
  - Specific procedures:
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In this auction every player independently chooses a bid without seeing others' bids, and the object is sold to the bidder who makes the highest bid. The winner pays her bid (i.e. the price is the highest or "first" price bid). We look for a symmetric Nash equilibrium in which a bidder with value v chooses the bid $b(v)$, and assume that b is a continuous strictly increasing function of v, that is, bidders with higher values bid more than bidders with lower values and no range of "types" of bidders all choose the same bid.[6]

continuous + strictly increasing

Consider player i with value v_i. In Nash equilibrium she assumes all other players will bid according to the function $b(v)$, so how should she bid? Imagine she chooses the bid $\tilde{b}$. It is useful to think about the value $\tilde{v}$ of the type of bidder she would just tie with, that is, we let $b(\tilde{v}) = \tilde{b}$.[7] Thus bidding $\tilde{b}$ mimics the way type $\tilde{v}$ would bid, and so beats any other bidder j with the probability that $v_j < \tilde{v}$, which equals $F(\tilde{v})$, in equilibrium. So mimicing $\tilde{v}$ would beat all the other $(n-1)$ bidders with probability $(F(\tilde{v}))^{n-1}$ and yields expected surplus to player

def $\tilde{v}$ s.t. $b(\tilde{v}) = \tilde{b}$

$v_j < \tilde{v} \Rightarrow b(v_j) < \tilde{b} \Rightarrow j$ loses

$P_R(i \text{ wins}) = P_R(j \text{ loses } \forall j)$

$= P_R(b(v_j) < \tilde{b} \ \forall j)$

$= P_R(v_j < \tilde{v} \ \forall j)$

$= (F(\tilde{v}))^{n-1}$

[6] Justifying this is left as a (strictly optional) exercise for the technically minded. They may also look at examples 6.3, 6.4 and (especially) 6.5 in Fudenberg and Tirole's book "Game Theory". Fudenberg and Tirole also show that the symmetric equilibrium is the unique equilibrium.

[7] Why does there exist a type $\tilde{v}$, whose bid our bidder will just tie with? Recall we are looking for an equilibrium in which all types' equilibrium bids are in the range $[b(\underline{V}), b(\overline{V})]$. So if player i bids $\tilde{b} > b(\overline{V})$ she could bid a bit less and still always win. If player i bids $\tilde{b} < b(\underline{V})$ she will never win and so makes zero profit. And if she bids $\tilde{b} \in [b(\underline{V}), b(\overline{V})]$ she will tie with some type because we assumed the bidding function is continuous.

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Symmetric NE: everyone bids using $b: v_i \Rightarrow$ bid

for individual i: bid $\hat{b} = b(\hat{v}) \Leftrightarrow$ win with $\Pr = \Pr(V_j < \hat{v})\ \forall j = [F(\hat{v})]^{n-1}$

the utility of winning is $v_i - \hat{b} = v_i - b(\hat{v})$ since U_i is linear $\Leftrightarrow$ no risk aversion

$\Rightarrow \mathbb{E}[U_i(\hat{v}); v_i] = S(v_i, \hat{v}) = \underbrace{[v_i - b(\hat{v})]}_{\mathbb{E}[U_i \mid Win]}\underbrace{[F(\hat{v})]^{n-1}}_{\Pr[Win]}$

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$$S(v_i,\widetilde{v}) = \underbrace{(v_i - b(\widetilde{v}))}_{\text{Surplus}|\text{Win}}\underbrace{(F(\widetilde{v}))^{n-1}}_{\Pr(\text{Win})}.$$

Note the tradeoff the bidder faces. Choosing a low $\widetilde{v}$ to mimic, that is, choosing a low bid $b(\widetilde{v})$, increases the value of winning, $(v_i - b(\widetilde{v}))$, but reduces the probability of doing so, $(F(\widetilde{v}))^{n-1}$.

Since everyone uses $b(\cdot)$ in the symmetric NE, opting $\hat{b} \Leftrightarrow$ opting $\hat{v}$

FOC wrt $\hat{v}$:

$$\frac{\partial S(v_i,\hat{v})}{\partial \hat{v}} = -b'(\hat{v})[F(\hat{v})]^{n-1} + [v_i - b(\hat{v})](n-1)[F(\hat{v})]^{n-2}f(\hat{v}) = 0$$

$$b'(\hat{v})[F(\hat{v})]^{n-1} = [v_i - b(\hat{v})](n-1)[F(\hat{v})]^{n-2}f(\hat{v})$$

Choosing the best bid to make is equivalent to choosing the best $\widetilde{v}$ to mimic, which we can do by looking at the first-order condition

$$\frac{\partial S(v_i,\widetilde{v})}{\partial \widetilde{v}} = -b'(\widetilde{v})(F(\widetilde{v}))^{n-1} + (v_i - b(\widetilde{v}))(n-1)(F(\widetilde{v}))^{n-2}f(\widetilde{v}).$$

For the bidding function $b(v)$ to be a (symmetric) equilibrium, i's best-response to all others bidding according to this function must be to do likewise, i.e. her optimal choice of $\widetilde{b}$ is $b(v_i)$ and of $\widetilde{v}$ is v_i.

Since the NE is symmetric, the optimising bidder must also use $b(\cdot)$ at her valuation v_i: $\hat{b} = b(v_i)$, $\hat{v} = v_i$ at Optimum / FOC

$$\Rightarrow b'(v_i)[F(v_i)]^{n-1} = [v_i - b(v_i)](n-1)[F(v_i)]^{n-2}f(v_i)$$

$$b'(v_i) = \frac{[v_i - b(v_i)](n-1)[F(v_i)]^{n-2}f(v_i)}{[F(v_i)]^{n-1}}$$

So

$$\frac{\partial S}{\partial \widetilde{v}}(v_i,\widetilde{v}) = 0 \text{ at } \widetilde{v} = v_i$$

$$\Rightarrow \quad b'(v_i) = (v_i - b(v_i))(n-1)\frac{f(v_i)}{F(v_i)}. \qquad (1)$$

Using the boundary condition

$$b(\underline{V}) = \underline{V}$$

(it is obvious type $\underline{V}$ will not bid more than $\underline{V}$, and we assume the auctioneer will not accept lower bids than $\underline{V}$), we can solve the differential equation (1) for the equilibrium.[8]

Boundary Condition

$b(L) = L$

The agent with the lowest possible valuation will always bid at that value.

For example, for the uniform distribution, (1) yields

$$b'(v_i) = (v_i - b(v_i))(n-1)\left(\frac{1}{v_i - \underline{V}}\right)$$

which is solved by[9]

$$b(v_i) = \underline{V} + \left(\frac{n-1}{n}\right)(v_i - \underline{V}).$$

$$v_i \sim U[L, H] \Rightarrow \begin{cases} F(v_i) = \dfrac{v_i - L}{H - L} \\ f(v_i) = \dfrac{1}{H-L} \end{cases}$$

$$\Rightarrow b'(v_i) = \frac{[v_i - b(v_i)](n-1)\left(\frac{v_i - L}{H-L}\right)^{n-2}\frac{1}{H-L}}{\left(\frac{v_i-L}{H-L}\right)^{n-1}}$$

$$b'(v_i) = \frac{[v_i - b(v_i)](n-1)\frac{(v_i-L)^{n-2}}{\cancel{(H-L)^{n-1}}}}{\frac{(v_i-L)^{n-1}}{\cancel{(H-L)^{n-1}}}}$$

$$b'(v_i) = \frac{[v_i - b(v_i)](n-1)}{v_i - L}$$

Solution to this ODE:

$$b(v_i) = L + \frac{n-1}{n}(v_i - L)$$

[8] We will give the general solution in Section 2.

[9] As I'll emphasise in the lectures, I am teaching economics not mathematics, so I won't require you to know how to solve differential equations (though I would now expect you to recognise that $b(v) = C + (\frac{K}{K+1})(v - C)$ solves $b'(v) = (v - b(v))\frac{K}{(v-C)}$, for any constants C and K!).

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Since $b(\cdot)$ is strictly monotonic, Seller's expected revenue is the expected bid from the bidder with the highest valuation:

$$v_{top} = L + \frac{n}{n+1}(H - L)$$

$$b(v_{top}) = L + \frac{n-1}{n}\left[\cancel{L} + \frac{n}{n+1}(H-L) - \cancel{L}\right]$$

$$= L + \frac{n-1}{\cancel{n}}\frac{\cancel{n}}{n+1}(H-L)$$

$$= L + \frac{n-1}{n+1}(H-L)$$

For this example, the seller's expected revenue is

$$E\left\{\max_{i=1,..,n} b(v_i)\right\} = E\left\{\underline{V} + \left(\frac{n-1}{n}\right)\left(\max_{i=1,..,n} v_i - \underline{V}\right)\right\}.$$

But using the result from Section 1.2,

$$E\left\{\max_{i=1,..,n} v_i\right\} = \underline{V} + \left(\frac{n}{n+1}\right)(\overline{V} - \underline{V}),$$

so the expected revenue is $\underline{V} + \left(\frac{n-1}{n+1}\right)(\overline{V} - \underline{V})$.

- Bidding function and seller's expected revenue

$$\begin{cases} b(v_i) & = L + \left(\frac{n-1}{n}\right)(v_i - L) \\ \mathbb{E}\left[\text{Revenue}\right] & = L + \left(\frac{n-1}{n+1}\right)(H - L) \end{cases}$$

Dutch Auction / Descending-Bid Auction / Open 1st Price Aunction in IPV Model #flashcard

- The auctioneer starts at a very high price and lowers the price continuously. The 1st bidder who accepts the current price wins the object at the current price.
- Results:
  - Each bidder's problem is static: she must choose a price at which she will call out conditional on no other bidder having called out yet, and the bidder who chooses the highest callout price wins.
  - $\Longrightarrow$ strategically equivalent to the 1st Price Sealed Bid Auction
  - $\Longrightarrow$ resulting to the same bidding function $b(.)$ and the same expected payoff of the seller in a **Symmetric Bayesian Nash Equilibrium**:

$$\begin{cases} b(v_i) & = L + \left(\frac{n-1}{n}\right)(v_i - L) \\ \mathbb{E}\left[\text{Revenue}\right] & = L + \left(\frac{n-1}{n+1}\right)(H - L) \end{cases}$$

2nd Price Auctions

2nd Price Sealed Bid Auction / Vickrey Auction in IPV Model #flashcard

- Each bidder independently chooses a bid without seeing the others' bids, and the object is sold to the bidder who makes the highest bid but *at the price of the second-highest bid*.
- **Dominant Strategy Nash Equilibrium**:
  - Bidding Strategy

$$b(v_i) = v_i$$

  i.e. reporting the true valuation is a *weakly dominant strategy* for each bidder
  - Argument:
  - Bidding lower $b_i = v_i - x$; denote the highest bid other than you as w
  - $w \leq b_i = v_i - x \implies$ you still win and pay the same w
  - $w \geq v_i \implies$ you still lose the auction and get the same 0
  - $v_i - x = b_i < w < v_i \implies$ originally, you could win and get $v_i - w$, but now you lose the auction and get 0 (worse off)
  - Bidding higher $b_i = v_i + x$; denote the highest bid other than you as w
  - $w \geq b_i = v_i + x \implies$ you still lose and get the same 0
  - $w \leq v_i \implies$ you still win and pay the same w
  - $v_i < w < b_i = v_i + x \implies$ originally, you lose and get 0, but new you win and get $v_i - w < 0$ (worse off)
  - $\implies$ Reporting the true valuation is a weakly dominant strategy
  - Seller's Expected Payoff: since everyone is bidding her true value, the seller's expected payoff is the expected 2nd highest value of the n values. With the $U[L, H]$ distribution in the IPV model, this is:

$$\mathbb{E}\left[\text{Revenue}\right] = L + \left(\frac{n-1}{n+1}\right)(H - L)$$

  - Summary:

$$\begin{cases} b(v_i) & = v_i \\ \mathbb{E}\left[\text{Revenue}\right] & = L + \left(\frac{n-1}{n+1}\right)(H - L) \end{cases}$$

English and Japanese Aunction / Ascending-Bid Auction / Open 2nd Price Auction in IPV Model #flashcard

- **Japanese Auction / Open 2nd Price Sealed Auction**: price rises continuously until only one bidder remains, who wins the auction at the current price.
- Dominant Strategy Nash Equilibrium:
  - It's obvious that a bidder should give up at her true valuation.
  - Results:

$$\begin{cases} b(v_i) & = v_i \\ \mathbb{E}\left[\text{Revenue}\right] & = L + \left(\frac{n-1}{n+1}\right)(H - L) \end{cases}$$

- This is strategically equivalent to the 2nd Price Sealed Auction if and only if there are 2 players.
  - *If there are more than 2 bidders, the open/sealed 2nd price auctions are strategically different*.
  - In an open 2nd price auction / Japanese auction with more than 2 bidders, a bidder can see when others quit and condition their behaviour on this information.
  - In the IPV model, this does not alter the bidders' behaviour since each it's weakly dominant to reporting each bidder's true valuation, which is known to herself without uncertainty.
  - However, when a bidder is unsure about her own valuation and other players have private information about her value, her quitting price will depend on others' behaviour. *Information revealing will be important* and open 2nd price auction will generate different results from sealed 2nd price auction.

Discussions

Comparing NEs from 1st/2nd Price Auctions #flashcard

- NEs in Sealed 2nd Price Auctions are Dominant Strategy Nash Equilibria, which are stronger than Bayesian Nash Equilibria in Sealed/Open 1st Price Auctions.
  - In DSNE for Sealed 2nd Price Auctions, each bidder bids her true value regardless of the others' bids --- getting private information about an opponent's value is worthless
  - For Open 2nd Price Auctions, the argument depends on specific set up --- if there's uncertainty in player's own valuation and there are more than 2 players, information on others' value will be important.
  - In BNE for Sealed/Open 1st Price Auctions, perception about the opponents' value'/behaviour will alter a bidder's behaviour.

Revenue Equivalence Theorem for IPV Models #flashcard

- RET for IPV Models
  - Assume:
  - Each of n risk-neutral potential buyers of an object has a privately known value independently drawn from a common distribution $F(v)$ that is strictly increasing and atomless on $[L, H]$.
  - Any auction mechanism with the following characteristics:
  - the object always go to the buyer with the highest value
  - any bidder with value L expects zero surplus
  - will:
  - yield the same expected revenue for the seller
  - result in a buyer with value v making the same expected payment:

$$\begin{aligned}\mathbb{E}\left[\text{Payment}(v)\right] &= Pr(\text{Win}) \times \text{Payment}|\text{Win} \\ &= [F(v)]^{n-1}v - \int_{x=L}^{v} [F(v)]^{n-1}\, dx\end{aligned}$$

- Robustness:
  - RET is robust to:
  - more than 1 objects being sold
  - simple common-value models
  - RET will break down if:
  - bidders are not risk-neutral
  - risk-averseness makes first-price auctions more profitable than ascending(2nd-price) ones (from PS4 Q1)
  - bidders have budget constraint
  - bidders' valuations are not independently
  - bidders' valuations are not drawn from an identical distribution
  - bidders can collude
  - the number of bidders is endogenous to the auction desgin