Ox Core Macro - Francesco Zanetti - Growth

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Prelim

Kaldor's Long-Term Facts on Macroeconomy #flashcard

- Roughly constant:capital output ratio, return to capital, capital/labour share of income, consumption/investment to GDP
- Grows: output per worker, capital per worker, real wages

Overlapping Generations (OLG) Model

Baseline OLG: 2-Generation OLG with Cobb-Douglas PF and CRRA Utility

- **Workflow**:
  - Household Optimisation $\rightarrow$ Euler Equation
  - Firm Optimisation $\rightarrow$ FOCs
  - Combine them $\rightarrow$ Capital LOM
  - Combine with Market Clearing $\rightarrow$ Full Equilibrium Results
- **Setup**:
  - Household:

$$\max_{C_t^0, K_{t+1}^1} U(C_t^0) + \beta U(C_{t+1}^1)$$

  - s.t.

$$\begin{cases} BC_0: & K_{t+1}^1 + C_t^0 = w_t \\ BC_1: & C_{t+1}^1 = r_{t+1} K_{t+1}^1 \end{cases}$$

  - and:$U(c) = \ln c$
  - Firm:

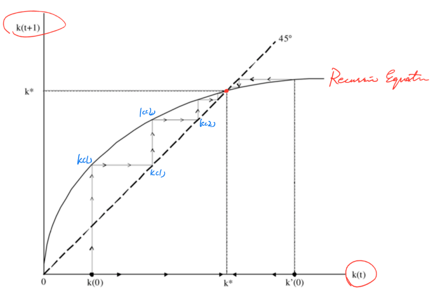
$$\max_{L_t, K_t} F(K_t, L_t) - w_t L_t - r_t K_t$$

  - and $F(K, L) = K^\alpha L^{1-\alpha}$ and normalise $L = 1$ #flashcard
- **Solving the model**:
  - Household optimisation:
    - In this simple setup, we do not need a Lagrangian. Simply combine 2 BCs to eliminate K_{t+1}^1 and take the FOC for C_t
    - $\Longrightarrow$ Consumption Euler Equation:

$$U'(C_t^0) = \beta r_{t+1} U'(C_{t+1}^1)$$

      which can be further calculated using the log-utility assumption
  - Firm's optimisation:
    - Take the FOCs
  - Combining the Euler Equation + Firm's FOC (together with our functional form assumptions), we can already get the Capital LOM:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}(1-\alpha)(K_t^1)^\alpha$$

    - 2 Generations + CDPF $\Longrightarrow$ Convergence to a SS capital level k^*:
    - 

- To solve the full model, we also need the market clearing condition:

$$C_t^0 + C_t^1 + K_{t+1}^1 = F(K_t^1, L_t)$$

- **Comments**:
  - Physical capital forms the basis of intertemporal link in neoclassical models.
  - CDPF ensures convergence towards a steady state.
  - CDPF + 2 Generations with CRRA Utilities $\implies$ Convergence towards the unique globally stable steady state.
  - There's possibility of dynamic inefficiency where capital stock exceeds the Pareto efficient level (equilibrium does not ensures efficiency).
  - In the general case with more generations, there could be multiple equilibria.
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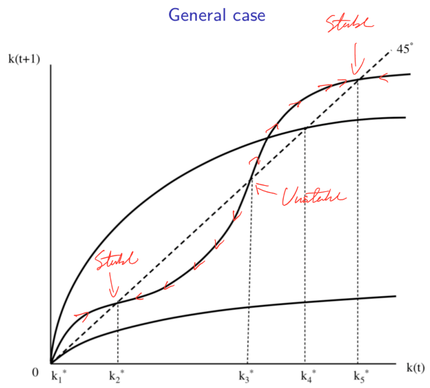


OLG with Taxes / Government Fiscal Policies

- **Workflow**:
  - Household Optimisation $\rightarrow$ Euler Equation
  - Firm Optimisation $\rightarrow$ FOCs
  - Combine them $\rightarrow$ Capital LOM
  - *Combine with specific Government Budget Constraint $\rightarrow$ Different Results*
  - Combine with Market Clearing $\rightarrow$ Full Equilibrium
- **Setup**:
  - Household:

$$\max_{C_t^0, K_{t+1}^1} U(C_t^0 + G_t) + \beta U(C_{t+1}^1)$$

  - s.t.

$$\begin{cases} BC_0: & K_{t+1}^1 + B_{t+1}^1 + C_t^0 = w_t - T_t^0 \\ BC_1: & C_{t+1}^1 = r_{t+1}(K_{t+1}^1 + B_{t+1}^1) - \cancelto{0 \text{ (assumption)}}{T_{t+1}^1} \end{cases}$$

  - and:$U(c) = \ln c$
  - Firm:

$$\max_{L_t, K_t} F(K_t, L_t) - w_t L_t - r_t K_t$$

  - and $F(K, L) = K^\alpha L^{1-\alpha}$ and normalise $L = 1$ #flashcard
- **Solving the model**:
  - Household optimisation:
  - In this simple setup, we do not need a Lagrangian. Simply combine 2 BCs to eliminate K_{t+1}^1 and take the FOC for C_t
  - $\implies$ Consumption Euler Equation:

$$U'(C_t^0 + G_t) = \beta r_{t+1} U'(C_{t+1}^1)$$

  which can be further calculated using the log-utility assumption
  - Firm's optimisation
  - Take the FOCs
  - Combining the Euler Equation + Firm's FOC (together with our functional form assumptions), we can already get the Capital LOM:

$$K_{t+1} = \frac{\beta}{1+\beta}\left(\frac{\partial F(1, K_t^1)}{\partial L} - T_t^0 + G_t\right) - B_{t+1}^1$$

- Combine with specific Government Budget Constraint to get the actual Capital Dynamics:
- Consider 1-period government spending at t:

$$\begin{cases} G_t & = \overline{G} \quad \text{for } t \\ G_{t+j} & = 0 \quad \text{for } j > 0 \end{cases}$$

- **Balanced Budget $\implies$ Ricardian Equivalence**
- The young generation is taxed at t to finance $t \iff$ Government keeps a balanced budget per period; no gov debt:

$$\begin{cases} T_t^0 & = G_t \\ B_{t+1}^1 & = 0 \end{cases}$$

- This implies the Capital LoM:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}\left(\frac{\partial F(1, K_t^1)}{\partial L}\right)$$

which is exactly the same as the baseline OLG model $\implies$ no fiscal distortion (neutral fiscal policy).
- **Deficit Financed Government Spending $\implies$ Distortion**
- The government issues bond B_{t+1}^1 to finance its spending at t and tax the young at $t+1$ to repay debt:

$$\begin{cases} T_t^0 & = 0 \\ B_{t+1}^1 & = G_t \\ T_{t+1}^0 & = (1+r_{t+1})G_t \end{cases}$$

- Substituting into the Capital LoM, we will see a distortion in $t, t+1, t+2$ but no distortion from $t+3$
- On capital:
- In period $t+1$, government debt decreases capital through the Euler Equation $t \to t+1$
- In period $t+2$, interest on debt decreases capital through the Euler Equation $t+1 \to t+2$
- From $t+3$ onward, no effect.

- **Insights**:
  - Balanced government budget per period $\implies$ Ricardian Equivalence / No Distortion
  - Deficit (Bond) Financed $\implies$ Distortion

OLG with Pension System

- **Workflow**:
  - (same, just different setup)
  - Household Optimisation $\to$ Euler Equation
  - Firm Optimisation $\to$ FOCs
  - Combine them $\to$ Capital LOM
  - Combine with Market Clearing $\to$ Full Equilibrium Results
- **Setup**:
  - Household:

$$\max_{C_t^0, K_{t+1}^1} U(C_t^0) + \beta U(C_{t+1}^1)$$

  - s.t.

$$\begin{cases} BC_0: & C_t^0 + S_t + D_t = w_t \\ BC_1: & C_{t+1}^1 = (1+r_{t+1})S_t + B_{t+1} \end{cases}$$

  where S_t is saving, D_t is the contribution to the pension system, and B_{t+1} is the benefit got from the pension system at $t+1$.
  - and:$U(c) = \ln c$
  - Firm:

$$\max_{L_t, K_t} F(K_t, L_t) - w_t L_t - r_t K_t$$

  - and $F(K, L) = K^\alpha L^{1-\alpha}$ and normalise $L = 1$
  - Capital:

$$K_{t+1} = S_t + D_t$$

- **Solving the model**: #flashcard
  - **Fully-Funded (Self-Founded) Pension System $\implies$ No Distortion**
  - Benefits when old is financed by the contribution when young:

$$B_{t+1} = (1+r_t)D_t$$

- Essentially, D_t play the same role as S_t, and we will have exactly the same Capital LoM:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}\left(\frac{\partial F(1, K_t^1)}{\partial L}\right)$$

- **Pay-As-You-Go Pension System** $\implies$ **Distortion**
- Benefits of the old is paid by the current-period young:

$$B_t = D_t$$

- Budget Constraint:

$$\begin{cases} C_t^0 + S_t + D_t & = w_t \\ C_{t+1}^1 & = (1+r_{t+1})S_t + D_{t+1} \end{cases}$$

- Combine BC:

$$C_t^1 = (1+r_{t+1})(w_t - D_t - C_t^0) + D_{t+1}$$

- Same Euler Equation, but different expressions for C $\implies$ Distorted Capital LoM:

$$K_{t+1} = \frac{\beta}{1+\beta}\left(\frac{\partial F(1, K_t^1)}{\partial L} - D_t + \frac{1}{\beta(1+r_{t+1})}D_{t+1}\right)$$

- The initial generation enjoys, but the latter generations suffer from lower capital accumulation.

OLG with Technological Progress #flashcard

- **Hicks Neutral TP**:

$$Y_t = \theta_t F(K_t, L_t)$$

- **Capital-Augmenting TP**:

$$Y_t = F(\theta_t K_t, L_t)$$

- **Labour-Augmenting TP**:

$$Y_t = F(K_t, \theta_t L_t)$$

- This leads to a BGP that is consistent with the Kaldor Facts.
- Key: write the capital LoM in terns of $\frac{K_{t+1}}{\theta_{t+1}}$. Everything else remains the same.

Ramsey Model

Stock Variables and Flow Variables #flashcard

- **Stock Variables** are not affected by the length of time Δ
    - e.g. Capital
- **Flow Variables** have to be adjusted for the length of time Δ
  - e.g. Investment, Depreciation

Baseline Ramsey Model

- **Setup**
    - Household:

$$\max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)\, dt \;\text{ s.t. }\; \dot{k}_t = w_t l + r_t k_t + \pi_t - c_t - \delta k_t$$

    - Firms:

$$\max_{k_t, l_t} y_t - w_t l_t - r_t k_t \;\text{ s.t. }\; y_t = f(k_t, l_t)$$

- **Solving the Ramsey Model** #flashcard
  - Household Optimisation
  - CV Hamiltonian:

$$H_{cv} = u(c_t) + \lambda(t)[w_t l + r_t k_t + \pi_t - c_t - \delta k_t]$$

  - CV Maximum Principle

- Hamiltonian Maximisation:

$$\frac{\partial H_{cv}}{\partial c_t} = 0$$

- Co-state Equation:

$$-\frac{\partial H}{\partial k_t} = \dot{\lambda}_t - \rho\lambda_t$$

- State Equation:

$$\dot{k}_t = w_t l + r_t k_t + \pi_t - c_t - \delta k_t$$

- Transversality Condition:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0$$

- Differentiate the Hamiltonian Maximisation equation wrt t and combine with the Co-state Equation. Then, substitute the definition of RRA (don't forget the negative sign!!!) to get Consumption Euler Equation:

$$\frac{\dot{c}_t}{c_t} = \frac{\epsilon_t - \delta - \rho}{\sigma(c_t)}$$

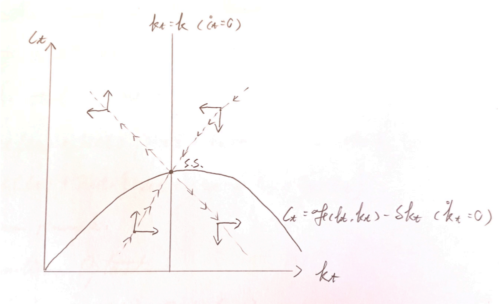
- Firm Optimisation
- FOCs:

$$\begin{cases} w_t & = \frac{\partial f}{\partial l_t} \\ r_t & = \frac{\partial f}{\partial k_t} \end{cases}$$

- Equilibrium Dynamics
- Combine with the Capital Accounting Equation, Firm FOCs, and the TVC, we have the equilibrium dynamics:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{\frac{\partial f}{\partial k_t} - \delta - \rho}{\sigma(c_t)} \\ \dot{k}_t & = f(k_t, l) - c_t - \delta k_t \\ \lim_{t\to\infty} & e^{-\rho t}\lambda_t k_t = 0 \end{cases}$$

- $\sigma(c_t)$ is often assumed to be constant $\iff$ CRRA utility
- Use the concavity of production function to get conditions for $\dot{c}_t > 0, \dot{k}_t > 0$and draw the Phase Diagram on c_t to k_t plane



- Key points:
  - Don't forget the NEGATIVE SIGN and c_t in the definition of RRA
  - Use the basic accounting equation for the capital dynamics (no need for the equilibrium equation)
- Comments:
  - This is a purely deterministic model with no stochasticity $\implies$ everything is pinned down at time 0 and no need to model expectations

Ramsey with Capital Taxation and Rebate

- **Modification**:
  - The government imposes a tax τ_t on capital returns and rebate the tax revenue $\tau_t r_t k_t$ to households
  - Household Optimisation becomes

$$\max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)\, dt \text{ s.t. } \dot{k}_t = w_t l + (1-\tau_t) r_t k_t + \pi_t - c_t - \delta k_t + T_t$$

  where $T_t = \tau_t r_t k_t$ #flashcard
- **Results**
  - Solve the model exactly the same as the Baseline Ramsey

- Resulting Dynamics:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{(1-\tau_t)\frac{\partial f}{\partial k_t} - \delta - \rho}{\sigma(c_t)} \\ \dot{k}_t & = f(k_t, l) - c_t - \delta k_t \\ \lim_{t\to\infty} & e^{-\rho t}\lambda_t k_t = 0 \end{cases}$$

- $\sigma(c_t)$ is often assumed to be constant $\iff$ CRRA utility
- Phase Diagram and Transition Dynamics:
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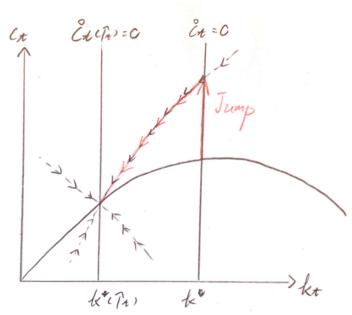

- Transition Dynamics:
- *TVC implies that: in the period where the tax is imposed, the economy immediately jumps upwards onto the saddle path of the new SS and converges to the new SS continuously*.
- In the new SS, *both capital and consumption are lower than before*.

Ramsey with Technological Progress: Exogenous Growth

- **Motivation**: to match Kaldor facts
  - Standard Ramsey model implies convergence towards SS.
    - Over time:
      - Capital converges to a fixed amount k^*
      - Interest rate falls $\rightarrow$ 0
      - Growth rate falls $\rightarrow$ 0
      - Capital-labour ration grows and gradually approaches a fixed quantity
  - However, Kaldor found no change over time in interest rates, output growth, capital share, capital-labour ration, capital-output ratio, etc.
- **Modification**
  - Labour-Augmenting Technological Progress:

$$y_t = f(A_t l, k_t)$$

  where A_t grows exogenously at a constant rate:

$$\frac{\dot{A}_t}{A_t} = \gamma$$

  #flashcard
- **Results**:
  - Solve the model exactly the same as the Baseline Ramsey
  - Equilibrium Dynamics
  - Combine with the Capital Accounting Equation, Firm FOCs, and the TVC, we have the equilibrium dynamics:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{\frac{\partial f(A_t l, k_t)}{\partial k_t} - \delta - \rho}{\sigma(c_t)} \\ \dot{k}_t & = f(k_t, A_t l) - c_t - \delta k_t \\ \lim_{t\to\infty} & e^{-\rho t}\lambda_t k_t = 0 \end{cases}$$

  - $\sigma(c_t)$ is often assumed to be constant $\iff$ CRRA utiltiy
  - Stationarise the variables:

$$\begin{cases} \tilde{k}_t & = \frac{k_t}{A_t} \\ \tilde{c}_t & = \frac{c_t}{A_t} \end{cases}$$

- Differentiate the new variables wrt t to get:

$$\begin{cases} \frac{\dot{k}_t}{A_t} & = \left(\dot{\tilde{k}_t}\right) + \gamma \tilde{k}_t \\ \frac{\dot{c}_t}{A_t} & = (\dot{\tilde{c}_t}) + \gamma \tilde{c}_t \end{cases}$$

- Divide both sides of the original dynamics equations by A_t, and use the CRS/HoD1 assumption on PF ($\implies$ $F(A_t l, k_t) = F(l, \tilde{k}_t), \frac{\partial F(A_t l, k_t)}{\partial k_t} = \frac{\partial F(l, \tilde{k}_t)}{\partial \tilde{k}_t}$):

$$\begin{cases} \frac{(\dot{\tilde{c}_t})}{\tilde{c}_t} & = \frac{\frac{\partial F(l, \tilde{k}_t)}{\partial \tilde{k}_t} - \delta - \rho - \sigma\gamma}{\sigma} \\ \left(\dot{\tilde{k}_t}\right) & = F(l, \tilde{k}_t) - \tilde{c}_t - (\delta + \gamma)\tilde{k}_t \end{cases}$$

- *These imply a SS in transformed variables* $\implies$ *BGP where c_t, k_t grow at constant rate γ.*
- *Transition Dynamics: converging to BGP.*

Endogenous Growth: AK Model

Why AK Model Has Endogenous Growth? → Inada Conditions #flashcard

- **Inada Conditions**: essentially diminishing marginal returns
    - $$\lim_{K \to 0} F_K(K, L) = \infty, \quad \lim_{K \to \infty} F_K(K, L) = 0$$
    - $$\lim_{L \to 0} F_L(K, L) = \infty, \quad \lim_{L \to \infty} F_L(K, L) = 0$$
- Inada conditions prevent *endogenous* growth from capital accumulation along: we will suffer from diminishing marginal returns to capital, and the marginal utility of consuming will eventually outweigh saving/investing.
    - Cobb-Douglas PF satisfies Inada conditions $\implies$ Models with CDPF can only have exogenous growth (e.g. from TFP or from population growth)
- Instead, AK PF demonstrates constant marginal return to capital $\implies$ violates Inada conditions $\implies$ allows for long-term endogenous growth through capital accumulation

Baseline AK Model

- **Setup**
    - Household:

$$\max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)\, dt \;\text{ s.t. }\; \dot{k}_t = r_t k_t + \pi_t - c_t - \delta k_t \qquad \text{(no labour)}$$

    - Firms:

$$\max_{k_t} y_t - r_t k_t \;\text{ s.t. }\; y_t = A k_t$$

        - Key: Inada condition is violated due to constant marginal return to capital!:

$$\lim_{K \to 0} F_K(K, L) = A, \quad \lim_{K \to \infty} F_K(K, L) = A$$

- **Solving the AK Model** #flashcard
  - Use the exact same method as Ramsey:
  - Results:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{A - \rho - \delta}{\sigma(c_t)} \\ \frac{\dot{k}_t}{k_t} & = A - \delta - \frac{c_t}{k_t} \\ \lim_{t \to \infty} & e^{-\rho t}\lambda(t) k_t = 0 \end{cases}$$

  - Insights:
  - We *always have positive and exogenously determined rate of consumption growth if $A - \delta > \rho$ (patient households and low depreciation)*
  - *Capital growth rate depends not only on A, δ, but also on the consumption-capital ratio, hence on the relative growth of consumption and capital*.
  - Consumption grows faster than capital $\implies$ capital depletion (implosion in LR)
  - Consumption grows slower than capital $\implies$ inefficiency (violates TVC in LR)

- $\implies$ *only BGP ensures sustainability and efficiency*
- Impose further assumption: CRRA utility $\iff$ $\sigma(c_t)$ is constant
- Dynamics:
- Define the Consumption-Capital Ratio:

$$Q_t \equiv \frac{c_t}{k_t}$$

- Analyse the Consumption-Capital Ratio dynamics:

$$\frac{\dot{Q}_t}{Q_t} = Q_t - \rho$$

- Balanced Growth Path (BGP)
- BGP is defined as constant consumption-capital ratio $\iff$ constant growth rate of consumption and capital:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{\dot{k}_t}{k_t} & = g \\ \dot{Q}_t & = 0 \end{cases}$$

- This implies:

$$\frac{\dot{Q}_t}{Q_t} = Q_t - \rho = 0 \implies Q_t^* = \rho \text{ on the BGP}$$

- Transitional Dynamics:
- *There is no transitional dynamics in AK model: growth rate of consumption, capital, and output are always constant.*
- We are *always on the BGP* hence:

$$\boxed{\frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = A - \delta - \rho}$$

Microfoundations of AK Models

- The key of AK model is the AK production function:

$$y_t = Ak_t$$

  with exogenous A
- There are many microfoundations leading to this production function. #flashcard
- **Leaning by Doing** #notes/tbd
- **1-Sector Human Capital Accumulation**
- **2-Sector Human Capital Accumulation**

Real Business Cycle (RBC) Model