# 1: GAMES IN STRATEGIC FORM

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The purpose of this topic is to teach you what a game in strategic form is and how economists analyze them.

Define a Game in Strategic Form

## 1. Games in Strategic Form: Description and Examples

1.1. **Defining a Game in Strategic Form.** A game in strategic form consists of several 3 elements. Here we are quite precise about what must be included in the description. There must be the following three things:

Players

(1) A list of players; $i = 1, 2, \ldots, I$. This list can be finite or infinite. Here the names of the players are just numbers but they can be any entity you are interested in studying.

Examples:
(a) In Rock-Paper-Scissors there are two players $i$ =Fred, Daisy.
(b) In an oligopoly with 5 firms there are $n = 5$ players $i$ =Ford, Honda, Toyota, VW, Fiat, GM.
(c) In bargaining there is a buyer and a seller, so the list of players is $i$ =buyer, seller.

Actions

(2) A description of all the actions each player can take. The set/list of possible actions for player i is usually called their pure strategies by game theorists. We will represent this by the set $S_i$ and we write a typical action as $s_i \in S_i$.

Examples:
(a) In Rock-Paper-Scissors, a player has three pure strategies: so $s_i \in \{\text{Rock}, \text{Paper}, \text{Scissors}\} = S_i$, for $i = 1, 2$.
(b) In an oligopoly, each firm, $i$, will choose a non-negative quantity of output to produce, $0 \leq q_i$ the set $S_i$ is the set of all positive output quantities.
(c) In bargaining, the buyer proposes a price $0 \leq p \leq 1$ and the seller makes a proposal too so the strategy sets are just intervals of prices: $S_{\text{buyer}} = \{0 \leq p \leq 1\}$, $S_{\text{seller}} = \{0 \leq p \leq 1\}$.

When you want to write all the actions that were taken by all of the players you will write a list or vector $s = (s_1, \ldots, s_I)$. This is called an action or strategy *profile*. The set of all possible strategy profiles (all possible plays of the game) is written as $S$ , where $s = (s_1, \ldots, s_I) \in S$.

Examples:



3 Elements $\begin{cases} \text{Players} \\ \text{Actions} \\ \text{Payoffs} \end{cases}$

(a) In Rock-Paper-Scissors: an example of a profile is $s = (\text{Scissors}, \text{Paper})$ here player 1 does $S$ and 2 does $P$. The set of all possible profiles $S$ has 9 elements.
(b) In oligopoly: a profile is a list of quantities for every firm each firm, $i$. That is a list $(q_1, q_2, \ldots, q_n)$. The set of all possible profiles is $\mathbb{R}^n_+$.
(c) Bargaining: a profile is two prices $(p_b, p_s)$ and the set of all possible profiles is $[0, 1]^2$.

Payoffs

(3) The final element is the payoffs (or utility or profit) the players get from their actions or pure strategies. We can write this as a utility/payoff function $u_i(s) = u_i(s_1, \ldots, s_I)$, that determines player i?s payoff at every possible play of the game. The payoffs are usually determined by the rules of the game or economic phenomenon you are studying.

Examples:
(a) (Rock Paper Scissors): Recall there are two ① players $i = 1, 2$ and each has ② three actions $S_i = \{R, P, S\}$ and the payoffs can be represented in a table:

③

| | R | P | S |
|---|---|---|---|
| R | 0,0 | -1,1 | 1,-1 |
| P | 1,-1 | 0,0 | -1,1 |
| S | -1,1 | 1,-1 | 0,0 |

$\to S_1$ $\quad u_1(S_1, S_2)\ ,\ u_2(S_1, S_2)$

$S_2$

Usually the first number rep. the payoff of the row player.

(b) (Oligopoly where firms choose quantities): Recall that the ① firms' names are $i = 1, 2, \ldots, n$. ② The firms? actions are outputs $q_i > 0$. The firms' payoffs are their profits, which depends on the demand. Price will depend on total output so we write $P(q_1 + q_2 + .. + q_n)$. (For example $P = 50 - q_1 - q_2 ... - q_n$.) Profit also depends on a firm's costs, we assume these only depend on their output $c(q_i)$. Hence

③
$$\begin{aligned}\text{Profit of Firm } i &= \text{Revenue} - \text{Costs}\\ &= \text{Output of } i \times \text{Price} - \text{Costs of } i\\ &= q_i P(q_1 + q_2 + .. + q_n) - c(q_i)\end{aligned}$$

This completes our initial description of a game in strategic form. We now want to allow for the possibility that players act randomly. We don't necessarily think that players are genuinely randomising. But it may be a very good description of what the players think the others are doing. You may know exactly how you're going to play Paper-Scissors-Rock, but I don't. From my point of view your action looks random. We call random actions *Mixed Strategies.* On mixed action for a player is one probability distribution, so the set of all the player's mixed strategies is the set of all probability distributions on their pure actions. We write the mixed strategy of player $i$ as $\sigma_i$. We will write the profile of mixed strategies for all players as $\sigma = (\sigma_1, \ldots, \sigma_I)$. It means that players don't know the opponents' exact decision.

Mixed Strategies

Example 1: In paper, scissors rock $S_i = \{R, P, S\}$ and $\sigma_i = (p, q, 1 - p - q)$ where $p$ is the probability R is played, $q$ is the probability P is played and $1 - p - q$ is the probability S is played.

▷ The perfect competition market is NOT a game: No way to explain how the price is determined.

Example 2: When firms choose quantities $S_i = [0, \infty)$ is the set of possible quantities. A random choice of a quantity can be represented by a probability distribution over the set of positive numbers. One way of describing such a distribution is to write down its cumulative distribution function (cdf) $F(x) := Pr$(Firm's outputs less than or equal to x).

Payoffs from Mixed Actions: A player's payoff when mixed actions are played is an expectation (or average) taken over all the payoffs they may get multiplied by the probability they get them. This expectation is taken assuming the players randomise independently.

Examples of payoffs from mixed actions (Rock Paper Scissors): Suppose in this game you are the column player and you believe the row player will play action R with probability $p$, P with probability $q$ and S with probability $1-p-q$. You are interested in your payoff from action P. Below we have written out the payoffs and emphasised the relevant numbers for the column player.

| | R | **P** | S |
|---|---|---|---|
| $p$ | 0,0 | -1,**1** | 1,-1 |
| $q$ | 1,-1 | 0,**0** | -1,1 |
| $1-p-q$ | -1,1 | 1,**-1** | 0,0 |

If she plays P she expects to get 1 with probability $p$, 0 with probability $q$ and $-1$ with probability $1-p-q$. Thus on average she expects to get

$$p \times 1 + q \times 0 + (1-p-q) \times (-1) = 2p + q - 1.$$

If we repeat these calculations for each column we can work out what each action will give her in expectation

Player1

| Player2 | | R | **P** | S |
|---|---|---|---|---|
| R: | $p$ | 0,0 | -1,**1** | 1,-1 |
| P: | $q$ | 1,-1 | 0,**0** | -1,1 |
| S: | $1-p-q$ | -1,1 | 1,**-1** | 0,0 |
| | | $1-p-2q$ | $2p+q-1$ | $q-p$ |

Strict Dominance

## 2. Dominance

### 2.1. Strict Dominance.

$s_{-i}$ : A list of actions for all players except player i.

eg. $s_{-1} = (s_2, s_3 \cdots s_n)$

**Definition 1.** *A mixed strategy $\sigma_i$ strictly dominates the pure action $s_i'$ for player $i$, if and only if, player $i$'s payoff when she plays $\sigma_i$ and the other players play actions $s_{-i}$ is strictly higher than her payoff from $s_i'$ against $s_{-i}$ for any actions $s_{-i}$ the others may play:*

$$u_i(\sigma_i, s_{-i}) > u_i(s_i', s_{-i}), \qquad \forall s_{-i}.$$

Consider the following game (we only put in the row player's payoffs as that is all that matters for now).

| | L | R |
|---|---|---|
| T | 3 | 0 |
| M | 0 | 3 |
| B | 1 | 1 |

And consider two strategies for the row player: play the first two rows with equal probability $\sigma_i = (1/2, 1/2, 0)$, or play the bottom row $s_i = B$. We can write the expected payoffs to these strategies in the following way.

$\sigma_i$:

| | L | R |
|---|---|---|
| $\frac{1}{2}$ | 3 | 0 |
| $\frac{1}{2}$ | 0 | 3 |
| 0 | 1 | 1 |
| | $\frac{3}{2}$ | $\frac{3}{2}$ |

$s_i$

| | L | R |
|---|---|---|
| 0 | 3 | 0 |
| 0 | 0 | 3 |
| 1 | 1 | 1 |
| | 1 | 1 |

$L: \frac{3}{2} > 1$

$R: \frac{3}{2} > 1$

From this you can see that if the column player plays $L$ the strategy $\sigma_i$ gives the row player the payoff of $\frac{3}{2}$ but the strategy $s_i$ gives the row player the payoff 1. And, if the column player plays $R$ strategy $\sigma_i$ also gives the row player the payoff of $\frac{3}{2}$ but the strategy $s_i$ gives the row player the payoff 1. Thus the strategy $\sigma_i$ always does better than the strategy $s_i$. To describe this we say $s_i$ is *strictly dominated* and we would never expect a rational player to play this.

Be aware of distinguishing "dominated" / "dominates"

However, eliminating strictly dominated actions can allow us to make strong predictions about what actions the players will use. Consider the following game...

| | L | R |
|---|---|---|
| U | (8,10) | (-100,9) |
| D | (7,6) | (6,5) |

L strictly dominates R

First observe that $R$ is strictly dominated by $L$ for the column player. So a rational column player will never play $R$. If the row player knows this then they should play $U$ getting 8 rather than 7. So we predict $(U, L)$ as the outcome of this game. But, some types of players may be very worried about the $-100$. If you had some doubts about column player's rationality would you be willing to play $U$?

Require both to be rational and row knows the col. to be rational

Weak Dominance

2.2. **Weak Dominance.** The notion of weak domination does not require players to strictly prefer one strategy to another, it is enough to weakly prefer one to the other.

**Definition 2.** *A mixed strategy $\sigma_i$ weakly dominates the pure action $s_i \in S_i$ for player $i$, if and only if, playing $\sigma_i$ is at least as good as playing $s_i$ whatever the other players do:*

$$u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i}), \qquad \forall s_{-i}.$$

| | L | R |
|---|---|---|
| T | (**1**,1) | (**0**,0) |
| M | (0,0) | (0,0) |
| B | (0,0) | (0,0) |

T weakly dominates M for the row player, because T is better than M if column plays L and T is no worse than M is column plays R.

T weakly dominates M/B

Iterative Elimination



2.3. **Iterative Elimination of Dominated Strategies.** If we can repeatedly (or iteratively) eliminate weakly or strictly dominated strategies to end up with one prediction on how the game will be played, then this can be used to solve games.

Example:

| | L | M | ~~R~~ |
|---|---|---|---|
| T | (3,3) | (2,2) | (5,2) |
| M | (2,10) | (4,4) | (6,3) |
| B | (2,1) | (2,7) | (1,0) |

3 > 2
10 > 3
1 > 0

- Step 1: R is strictly dominated by L, so a rational column player will not play R

*Row player has to know the Col. player is rational for this step.

| | L | M | R |
|---|---|---|---|
| T | (3,3) | (2,2) | - |
| M | (2,10) | (4,4) | - |
| ~~B~~ | (2,1) | (2,7) | - |

Being strictly dominated by mixed strategy is also considered as being strictly dominated.

- Step 2: In the remaining game $\frac{1}{2}$T $+\frac{1}{2}$ M strictly dominates B. So if the row player knows the column player is rational and is rational themselves, then they will not play B.

*Col. player has to know the Row. player is rational for this step.

| | L | ~~M~~ | R |
|---|---|---|---|
| T | (3,3) | (2,2) | - |
| M | (2,10) | (4,4) | - |
| B | - | - | - |

3 > 2
10 > 4

- Step 3: In the remaining game L strictly dominates M for the column player. So if the column player is rational, knows that the row player is rational and knows that the row player knows the column player is rational, then they will not play M.

(same)

| | L | M | R |
|---|---|---|---|
| T | (3,3) | - | - |
| ~~M~~ | (2,10) | - | - |
| B | - | - | - |

T strictly dominates M

- Step 4: In the remaining game T strictly dominates M for the row player. So if the row player is rational and the row knows that the column player is rational, and the row knows that the column player knows the row player is rational, then they will not play M.

| | L | M | R |
|---|---|---|---|
| T | (3,3) | - | - |
| M | (2,10) | - | - |
| B | - | - | - |

Summary: the set of strategies that survive the iterated elimination of strictly dominated actions are those actions that can be played by rational players with common knowledge of rationality

Nash Equilibrium

## 3. Nash Equilibrium

**Definition 3.** *A action profile $s^* = (s_1^*, \dots, s_I^*)$ is a Nash Equilibrium if $s_i^*$ is a best response to the actions $(s_{-i}^*)$ of the other players, and this is true for all players $i$. A mixed strategy profile $\sigma^*$ is a Nash Equilibrium if for all players $i$, if the same condition holds for mixed strategies.*

Everyone's action is the best response to the others.

Example: Consider the game:

(Fix 1 player's action and see whether another player's action is optimal)

Nash does not mean "the best" outcome of a game.

| | L | R |
|---|---|---|
| U | (8,10) | (-100,9) |
| D | (7,6) | (6,5) |

If row plays U then the best thing for column to play is L. (We say L is a best response to U.) And, if column plays L then the best thing for row to play is U. (U is also a best response to L.) This means that $(U, L)$ satisfies the definition of a Nash equilibrium. Recall that this game appeared before and this "solution" to the game was found using the elimination of strictly dominated actions. So there appears to be a link between these two concepts.

Example: As a second example, consider the game:

| | L | R |
|---|---|---|
| U | (1,1) | (0,0) |
| D | (0,0) | **(0,0)** |

Now observe that the action pair $(D, R)$ satisfies the definition of a Nash equilibrium. If row plays $D$ then $R$ is a best response. And if column plays $R$ then $D$ is a best response. So although we may believe that this is a very unappealing solution to the game, it is certainly a Nash equilibrium of the game, *because it satisfies the definition above.* This leads us to a new notion of Nash equilibrium.

Strict NE

### 3.1. Strict Nash Equilibrium.

**Definition 4.** *A Nash equilibrium is said to be strict, if it is an equilibrium in pure strategies and each player $i$ has a unique best response to the equilibrium action of their opponents.*

Strict equilibria are robust to small perturbations in the players' payoffs, but do not always exist. This is particularly the case in games where the Nash equilibria are all mixed (involve random actions).

Example1: (Rock Paper Scissors) Not a strict NE!

| | R | P | S |
|---|---|---|---|
| R | 0,0 | -1,1 | 1,-1 |
| P | 1,-1 | 0,0 | -1,1 |
| S | -1,1 | 1,-1 | 0,0 |

This game has a mixed strategy Nash equilibrium where each player plays every action with probability $\frac{1}{3}$. If one player plays each action with equal probability, then their opponent gets zero from *any* action. Every action they play gives them zero,, so every action is a best response to this strategy. One best response is to play $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Example 2 : (Matching Pennies)

| | $\frac{1}{2}$ H | $\frac{1}{2}$ T |
|---|---|---|
| H | (-1,1) | (1,-1) |
| T | (1,-1) | (-1,1) |

For row player :
$E(U) = \frac{1}{2}\times(-1) + \frac{1}{2}\times 1 = 0$
$E(U) = \frac{1}{2}\times 1 + \frac{1}{2}\times(-1) = 0$
→ Indifferent of 2 options [Every option is optimal]

If the column player plays $(\frac{1}{2}, \frac{1}{2})$ then the row player expects to get zero from $H$ and from $T$. Thus, any action is optimal. It follows that $(\frac{1}{2}, \frac{1}{2})$ is a best response by the row player. (This is a general property of mixed equilibria—one player's actions at the equilibrium make their opponent indifferent and therefore willing to randomize.)

☆ At a Mixed Strategy NE, players have to get the same payoffs from actions they play with positive probabilities. (eg. play H or T both gets 0 utility)

## 4. How to Find a Nash Equilibrium

Iterative Elimination

4.1. **Method 1 : Iteratively eliminate dominated strategies.** If this works the actions that remain after you have eliminated all others have to be a Nash equilibrium. You already have already seen an example of this above. If you *only* use strict dominance to eliminate actions, then iterative elimination finds all Nash equilibria, But, if you eliminate some actions using weak dominance then you may also eliminate a Nash equilibrium. For example, in the game below the bold outcome is a Nash equilibrium, but plays weakly dominated actions so it would be eliminated.

→ The outcome will still be a NE, but some NEs may be lost.

| | L | R |
|---|---|---|
| U | (1,1) | (0,0) |
| D | (0,0) | **(0,0)** |

You might think that this example is silly because the players ought to be able to avoid this Nash equilibrium. But suppose we increase the number of players so there are three players (player 3 chooses the matrix) then there still is a weakly dominated Nash equilibrium.

| 3 Plays L | L | R |
|---|---|---|
| U | (1,1,1) | (0,0,0) |
| D | (0,0,0) | (0,0,0) |

| 3 Plays R | L | R |
|---|---|---|
| U | (0,0,0) | (0,0,0) |
| D | (0,0,0) | **(0,0,0)** |

This Nash equilibrium seems much harder to avoid as one of the player (say player 3) now has to persuade both of the others to change there actions. There are, therefore, sensible situations where players end up playing weakly dominated actions at a Nash equilibrium.

Underlining Method

4.2. **Method 2 : The underlining method.** This is a process for finding *all* the pure-strategy Nash equilibria of a game. You only use it in finite games. Here is an example of the process applied to the following game

| | L | M | R |
|---|---|---|---|
| T | (3,3) | (2,2) | (5,2) |
| M | (2,10) | (4,4) | (6,3) |
| B | (2,1) | (2,7) | (1,0) |

- Step 1: Underline the best thing for the column player in each row. First consider the column player's payoffs:

| | L | M | R |
|---|---|---|---|
| T | (3,**3**) | (2,**2**) | (5,**2**) |
| M | (2,**10**) | (4,**4**) | (6,**3**) |
| B | (2,**1**) | (2,**7**) | (1,**0**) |

Then underline the biggest number in each row. (If there are many biggest underline them all.) Results in:

| | L | M | R |
|---|---|---|---|
| T | (3,$\underline{\mathbf{3}}$) | (2,**2**) | (5,**2**) |
| M | (2,$\underline{\mathbf{10}}$) | (4,**4**) | (6,**3**) |
| B | (2,**1**) | (2,$\underline{\mathbf{7}}$) | (1,**0**) |

- Step 2: Underline the best thing for the row player in each column. So consider the row players payoffs and find the biggest one in each column:

Finite Players
Finite Actions > → Must be at least 1 NE
↓ Pure Strategy NE
(or) Mixed Strategy NE

| | L | M | R |
|---|---|---|---|
| T | ($\underline{3}$, $\underline{3}$) | (2, 2) | (5,2) |
| M | (2,$\underline{10}$) | ($\underline{4}$,4) | ($\underline{6}$,3) |
| B | (2,1) | (2,$\underline{7}$) | (1,0) |

- Step 3: Any place where two elements are underlined is a NE

Find Mixed Strategy NE

4.3. **Method 3 : Mixed strategy NE's in $2 \times 2$ games.** This is a process that will help you find Nash equilibria in games with 2 players each of which has 2 actions. To find mixed strategy Nash equilibria is difficult and will usually require a computer for $n \times n$ games. To find a mixed we will consider a specific game below, which is often called the Battle of the Sexes

| | L | R |
|---|---|---|
| T | (3,1) | (0,0) |
| B | (0,0) | (1,3) |

Here is the recipe you follow:

- Step 1: Assume that the column player plays a mixed strategy $(q, 1-q)$and evaluate the row player's payoffs to all their actions.

| | $q$ | $1-q$ | |
|---|---|---|---|
| T | (3,1) | (0,0) | $= 3q + 0(1-q)$ |
| B | (0,0) | (1,3) | $= 0q + 1(1-q)$ |

- Step 2: Choose $q$ to make these two payoffs the same.

$$3q + 0(1-q) = 0q + 1(1-q)$$

This solves to give $q = \frac{1}{4}$
- Step 3: Suppose the row player randomizes $(p, 1-p)$ and evaluate the column player's payoffs to every action.

| | L | R |
|---|---|---|
| $p$ | (3,1) | (0,0) |
| $1-p$ | (0,0) | (1,3) |
| | $1p + 0(1-p)$ | $0p + 3(1-p)$ |

- Step 4: Choose $p$ to make these the same.

$$1p + 0(1-p) = 0p + 3(1-p)$$

solves to give $p = \frac{3}{4}$.
- Summary: Then, $p = 3/4$ and $q = 1/4$ is a mixed strategy NE.

Intersection of Best Response

4.4. **Method 4: The intersection of best responses.** This is a direct generalization of the underlining method. We will apply it to economics models of oligopoly later in this course. This approach applies to all games, not just oligopoly. To understand how this method works it is, again, optimal to do an example.

We will consider a game where. Player 1 chooses a number $x$, where $-\infty < x < \infty$ and Player 2 chooses a number $y$, where again $-\infty < y < \infty$. When $(x, y)$ is chosen they experience costs (which they would like to make as small as possible. The costs are

$$\text{Player 1's Costs} = (x-y)^2 + x^2$$
$$\text{Player 2's Costs} = (y-4)^2 + (x-2y)^2$$

So we now have a well-defined game with payoffs and actions. The main difference here is that the action sets are an interval so we can use calculus to minimise costs. Here is the process to find a NE.

- Step 1: Find the optimal $x$ for player 1 if it knows the $y$ chosen by player 2: That is choose $x$ to minimise $C_1 = (x-y)^2 + x^2$ for a given $y$ value. Thus we differentiate

$$\frac{dC_1}{dx} = 2(x-y) + 2x = 0, \qquad \Rightarrow \qquad x = y/2.$$

- Step 2: find the optimal y for player 2 if it knows the $x$ chosen by player 1. Choose $y$ to minimize $C_2 = (y-4)^2 + (x-2y)^2$ for a given $x$ value.

$$\frac{dC_2}{dy} = 2(y-4) - 4(x-2y) = 0, \qquad \Rightarrow \qquad y = 2(2+x)/5.$$

- Step 3: Solve where the two lines: $(y = 2(2+x)/5$ and $x = y/2)$ intersect. This gives $x = 1/2$ and $y = 1$. This is guaranteed to be a place where player 1's $x$ is best for her given $y = 1$ and player 2's $y$ is best for her given $x = 1/2$.

Here are some observations about this process:

To find all NEs of a game: ① find Strict NEs (Method 1, 2, 4)
② find Mixed NEs (Method 3)

(1) This method is a generalization of the underlining method to continuous actions.
(2) If a player has many best responses, they should all be included in the best response function.
(3) It is possible a player does not have a best response (see Bertrand competition).
(4) This generalizes to $n$ players. (There are always as many equations as actions.)
(5) It is possible that there is no intersection. This means that no pure Nash equilibrium exists.

(There may be mixed NEs.)

## 5. E. Nash Equilibrium Multiplicity

Coordination Games: There are two players who want to meet at location A or location B but we don't know which to meet at. This creates a game with the following payoffs

| | A | B |
|---|---|---|
| A | (1,1) | (0,0) |
| B | (0,0) | (1,1) |

[Unable to justify a prediction here]

These games generally have huge numbers of equilibria (it has two pure strategy equilibria and one mixed strategy equilibrium). Sometimes we need a tool for selecting among Nash equilibria in order to make predictions. In this game it seems virtually impossible, any justification for the $(A, A)$ equilibrium seems like it ought to be a justification for $(B, B)$ also. However, in some games we can make choices.

J-J. Rousseau introduced the Stag-Hunt game. He had in mind the following story. Two hunters can either independently hunt a rabbit (this requires no cooperation), or can try to coordinate to catch a stag. But hunting a stag fails unless we both commit to hunting a stag. Here are the usual payoffs given to this game. Another important feature is that hunting a rabbit gets easier if the other player isn't also hunting a rabbit.

| | Stag | Rabbit |
|---|---|---|
| Stag | (9,9) | (0,8) |
| Rabbit | (8,0) | (7,7) |

A public good problem

Pareto / Risk Dominant NE

This game has two Nash equilibria: $(S, S)$ and $(R, R)$. The $(S, S)$ is Pareto efficient, but quite risky because for me to get 9 I need to be quite sure my opponent is also playing $S$. The $(R, R)$ NE is safer, because whatever my opponent does I do OK (I get at least 7), but not Pareto Efficient. We say that: $(S, S)$ is *Pareto dominant NE* and $(R, R)$ is a *Risk Dominant NE.*

Most games have many Nash equilibria. Some people view this a problem and we have to find a way of choosing a unique way of predicting how the game will be played. Sometimes the names or history of the strategy mean that particular action is most prominent and therefore sensible to coordinate on. This is called a Focal Point. For example, at the end of *Slumdog Millionaire* the two characters have no mobile phones and must meet somewhere in Mumbai. They face a game that looks like:

| | x | y | z | u | v | w | ? |
|---|---|---|---|---|---|---|---|
| x | (1,1) | (0,0) | (0,0) | (0,0) | (0,0) | (0,0) | ? |
| y | (0,0) | (1,1) | (0,0) | (0,0) | (0,0) | (0,0) | ? |
| z | (0,0) | (0,0) | (1,1) | (0,0) | (0,0) | (0,0) | ? |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | |

Where $x, y, z,$? are potential places to wait for the other person. As it stands this game has huge number of NE?s. However, only one location, $x =$Chhatrapati Shivaji Terminus, is significant so they both meet each other there.

Sometimes communication before play will allow the players to choose a NE. The set of NE's of a game is the only set of credible non-binding agreements the players can reach.)

△ Exam question from last years

$$\begin{array}{c|cc} & q & 1-q \\ p & 1,1 & 0,0 \\ 1-p & 0,0 & 0,0 \end{array}$$

2 Pure Strategy NEs

Mixed Strategy NEs : $\begin{cases} q = 0 \\ p = 0 \end{cases}$

∴ Cannot find position p, q that make the player indifferent

∴ No mixed strategy NEs

△ Why there's no Mixed S. NE ?

$\begin{cases} \text{Top row weakly dominates bottom row.} \\ \text{Left row weakly dominates the right row.} \end{cases}$ → And those strategies are not the same!

↓ Always prefer T. L

No incentive to mix their strategies.

However, existence of weak domination does not necessarily imply no Mix S. NE.

Counter exp. : $\begin{bmatrix} 0,0 & 0,0 \\ 0,0 & 0,0 \end{bmatrix}$

- Infinite Mixed Strategy NEs

$$\begin{array}{c|cc} & L & R \\ U & \begin{bmatrix} 2,2 & 5,1 \\ 2,2 & 6,0 \end{bmatrix} \\ D & \end{array}$$

L strictly dominates R

$\therefore \infty$ NE: Col $\to$ L

Row $\to (p, 1-p)\ \forall p \in [0,1]$

△ Math notation

$[0,1] \to 0 \le x \le 1$

$(0,1) \to 0 < x < 1$

- Finite Mixed Strategy NEs (2x2)

Coordination Game:

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1,1 & 0,0 & 0,0 \\ 0,0 & 1,1 & 0,0 \\ 0,0 & 0,0 & 1,1 \end{bmatrix} \end{array}$$

Pure S. NEs:

Mixed S. NEs: I Row $\{ (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}),$

Col $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \}$

II $\{ (\frac{1}{2}, \frac{1}{2}, 0),$

$(\frac{1}{2}, \frac{1}{2}, 0) \}$

$$\begin{array}{cc} \begin{array}{ccc} (\frac{1}{2}, & \frac{1}{2}, & 0) \end{array} & \\ \begin{bmatrix} 1,1 & 0,0 & 0,0 \\ 0,0 & 1,1 & 0,0 \\ 0,0 & 0,0 & 1,1 \end{bmatrix} & \begin{array}{l} \left.\begin{array}{l} \to \frac{1}{2} \\ \to \frac{1}{2} \end{array}\right\} \text{Same} \\ \to 0 \end{array} \end{array}$$

# 2: GAMES IN EXTENSIVE FORM

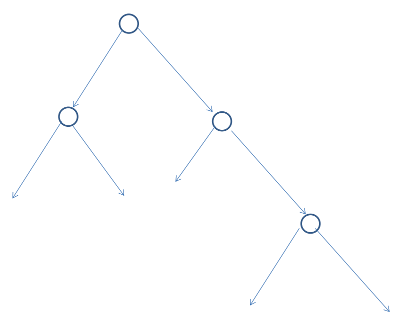
MARTIN CRIPPS

Game Trees

## 1. Perfect Information Games in Extensive Form

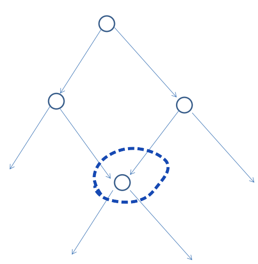
1.1. **Game Trees.** A game in extensive-form captures information about the order of moves, that is ignored in the strategic-form model of a game in the previous section. An extensive form is a diagram that describes the details of the sequence of decisions and actions the players take when they play a game. To do this we write done a mathematical object called a decision tree. This is a collection of nodes (or vertices) linked by arrows (or edges). The tree must have: (i) a unique initial node and (ii) a unique path from the initial node to every successor node. Here is an example of a tree.

Tree

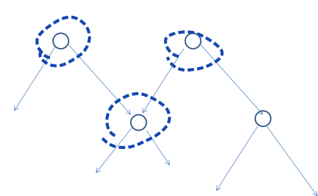


Here are examples of directed graphs that are not trees. the first violates the unique path condition and the second violates the unique initial node condition.

Not a tree



Not a tree

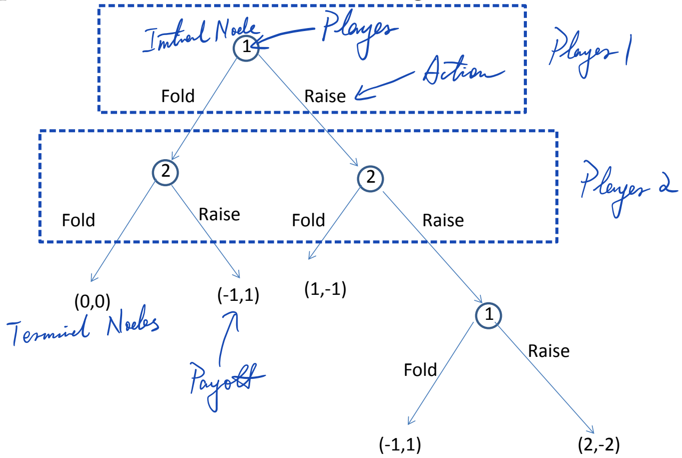


The important feature of these graphs (extensive form games) is the way they are labelled, because it is this that really explains the game. Here are the rules for labelling:

- Each non-terminal node is labelled with a player's name and the edges from that node are actions the player can take at that point in the game.

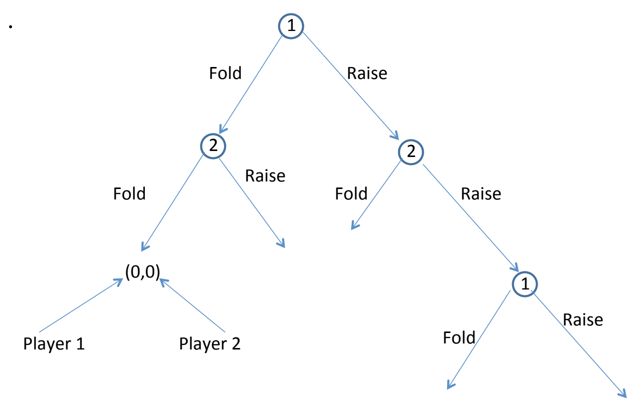
- Each terminal node describes a unique history of play. It is labelled with a payoff to every player in the game for that history.

Here is an example of a tree with such a labelling:

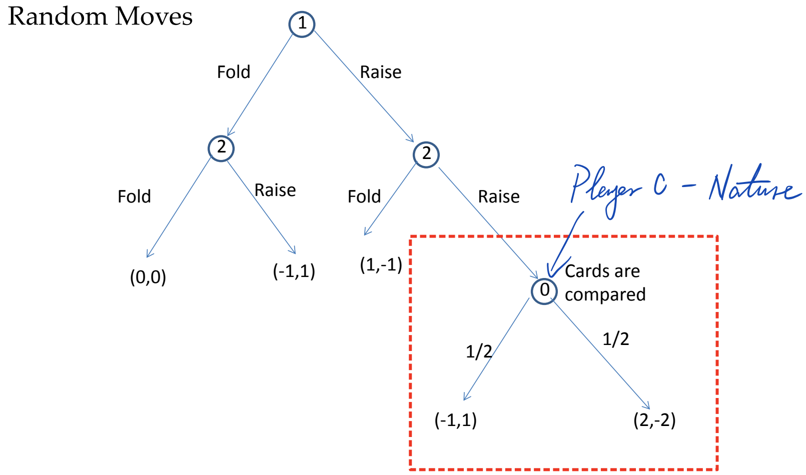



If you wanted to give a verbal description of what goes on in this game. You might proceed as follows... The game begins with Player 1 choosing between two actions Fold or Raise. If Player 1 chooses Fold, Player 2 observes 1's action and then must choose between Fold and Raise and the game ends. The payoffs the players get are either $(0,0)$ or $(-1,1)$. If Player 1 Raises initially, then Player 2 also gets to choose between Fold or Raise. If Player 2 Folds, the game is over and the payoffs are $(1,-1)$. If Player 2 Raises, then Player 1 gets another go and can choose between Fold or Raise a second time after which the game ends. If Player 1 folds, the payoffs are $(-1,1)$ and if Player 1 Raises the payoffs are $(2,-2)$.

I hope you can see that the picture is much clearer than the words. Also notice the convention that we describe the payoffs as a vector giving Player 1's payoff first.




We also want to allow for the possibility that there is some randomness in the game. (Dealing cards, rolling dice, exogenous uncertainty in the model.) To allow for random moves we introduce a player called Nature with a name zero. Each edge from a zero node has a probability attached that describes the probabilities of the random move. For example,




Important: There is a reason these pictures are called *Perfect Information Games* in Extensive Form, this is because there is very little uncertainty in these games. In a perfect information extensive form game each player knows exactly where they are in the game when they take a move/decision. (The only thing they don't know is how future moves will be played.) They do know exactly what has happened at every point they make must make a decision. Examples of perfect information games like this are Chess, Backgammon, Go, Monopoly. However, games such as Poker, Bridge (where the players have private information), or Paper-Scissors-Rock (where the players move at the same time) cannot be described in this way—we will get to them a little later.

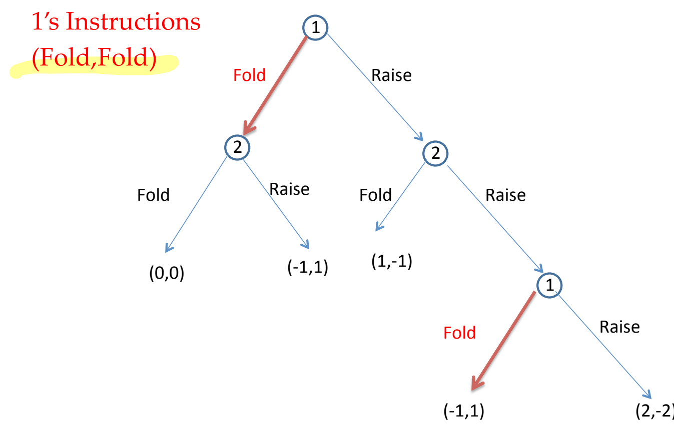
1.2. **Pure Strategies.** Thus far we have been describing a situation the players might find themselves in. Now we want to describe what the players might do.

Pure Strategy in EF G.

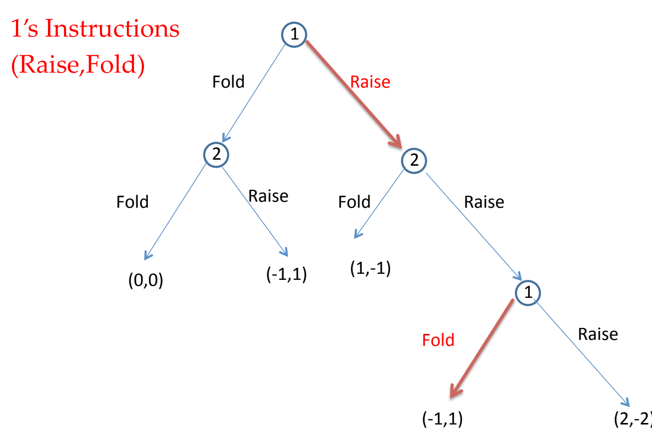
**Definition 1.** *A* Pure Strategy *for a player is an instruction book on how to play the game. This instruction book must be complete and tell the player what to do at every point at which he/she must make a move. Because we want to consider what happens when players make mistakes we must even include apparently redundant instructions.*

(Even if that node will not be reached)

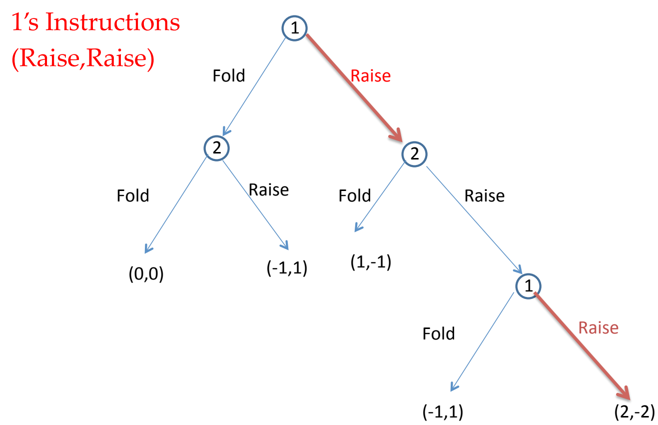
Here is an example of an instruction book (Pure Strategy) for Player 1 in the game we have been studying



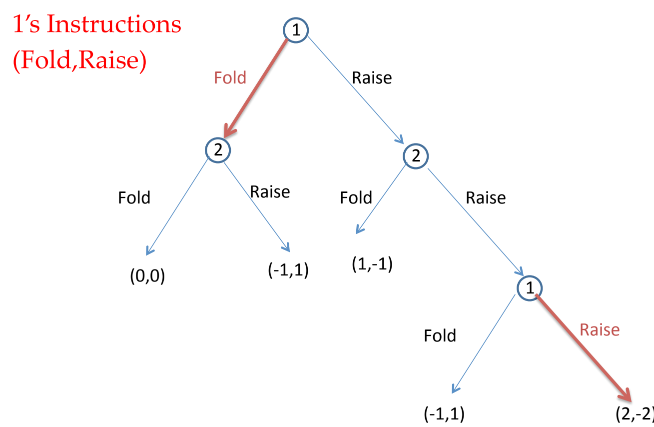

This instruction book has the instruction Fold at the first decision and Fold at the second decision. (Clearly, this second instruction is redundant but we still include it.) Here are some other pure strategies for Player 1 in the game.




Player 1 Raises at the first decision node and Folds at the second.




Player 1 Raises at the first decision node and Raises at the second.




Player 1 Folds at the first decision node and Raises at the second.

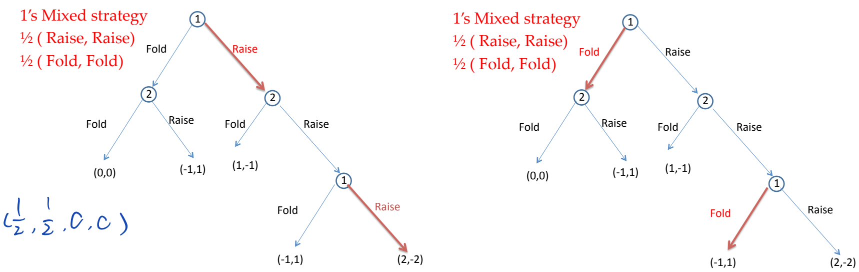
In this game Player 1 has 4 pure strategies: (Raise, Raise), (Raise, Fold), (Fold, Raise), (Fold, Fold). Notice that Player 2 also has 4 pure strategies, but none of her instructions are ever redundant.

1.3. **Mixed and Behaviour Strategies.** From our study of games in strategic form we also want to allow for the players to make random moves. There are two ways of doing this. The first is to imagine a library containing all the possible instruction books for Player 1. (In this case there would be 4 books (Raise, Raise), (Raise, Fold), (Fold, Raise), (Fold, Fold).) Then choosing one of the books at random, for example Player 1 could choose the (Raise,Raise) book with probability $\frac{1}{2}$ and the (Fold, Fold) book with probability $\frac{1}{2}$. This is an example of what is called a *mixed strategy.*

Mixed Strategy in EF G.

**Definition 2.** *A* mixed strategy *for a player is a random choice of an instruction book (i.e. a random choice of a pure strategy).*

This is a picture of what's going on




$(\frac{1}{2}, \frac{1}{2}, 0, 0)$

Notice that the randomness appears to be occurring only once (at the selection stage) but from the perspective of the other players in the game watching what their opponent does it may be that there are many points at which Player 1's move looks random.

If you wanted to describe a general mixed strategies for Player 1 in this game you would write down a probability distribution $(p, q, r, 1-p-q-r)$ where $p$ is the probability $(R, R)$ is played, $q$ is the probability $(R, F)$ is played, $r$ is the probability $(F, R)$ is played, and $1-p-q-r$ is the probability $(F, F)$ is played.

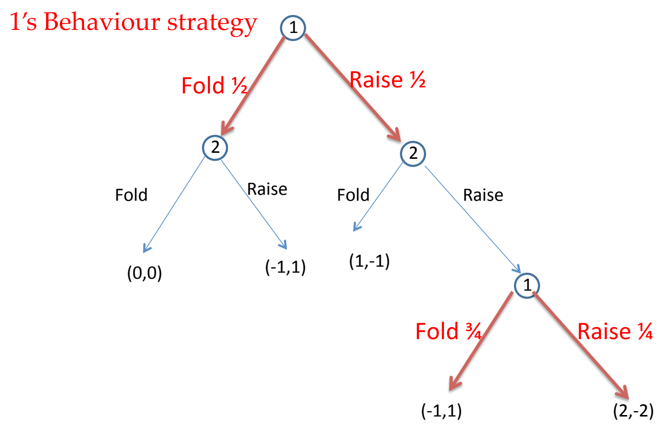
The second way of including randomness is to allow for the instructions themselves to have random elements. That is at each place where the player must move the instruction book tells the player how to randomize at that point. For example such an instruction book for Player 1 might be: At first node with probability $\frac{1}{2}$ play Raise. At second node with probability $\frac{1}{4}$ play Raise. These books with random instructions are called *Behaviour Strategies.*

Behavior Strategy

**Definition 3.** *A Behaviour Strategy for a player is an instruction book that has random instructions.*

No longer random selection of pure strategies

Here's our example of this random instruction book in action.




If you wanted to write down a general behaviour strategy for Player 1 in this game. You would write the pairs $(p, 1-p)$ and $(q, 1-q)$ where $(p, 1-p)$ is the randomization at the first decision Player 1 must take and $(q, 1-q)$ is the randomization at the second decision she must take.

## 2. Incomplete Information Games in Extensive Form

The game trees above have the property that everyone playing the game knows everything about the past events in the game—this rules out the following important features of strategic interaction:

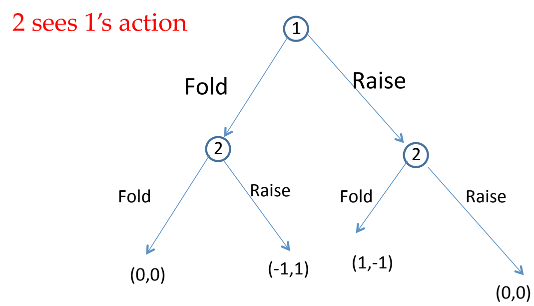
- Players moving simultaneously.
- Or players' past actions being hidden from others.
- Or players knowing something that other players do not.

We need to have a way of representing this in the trees and the device we have to do this is called an *information set*. These are things that we add on to the graphs above to describe what players know or don't know. In particular we the nodes in an information set represent positions in the game that the player is unable to tell apart. Of course, if a player is unable to tell the positions apart she must have the same actions and the same name:

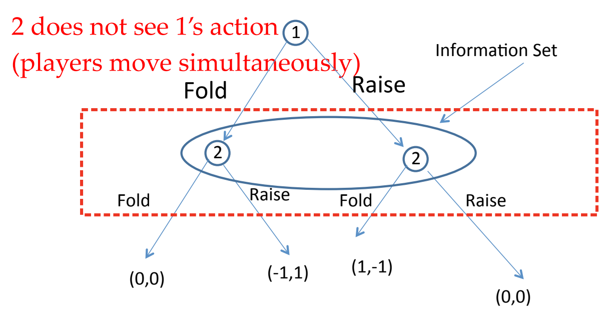
Information Set

**Definition 4.** *An* Information Set *is a collection of nodes with the property that: Each node in the set has the same player's name. Each node in the set has the same actions available.*

Consider the simplest kind of game where two players must each choose two actions simultaneously (such as matching pennies or battle of the sexes or the prisoners' dilemma). The picture below is inadequate, because Player 2 knows what Player 1 has done before she chooses her move.




To get over this we make the following addition to the picture.



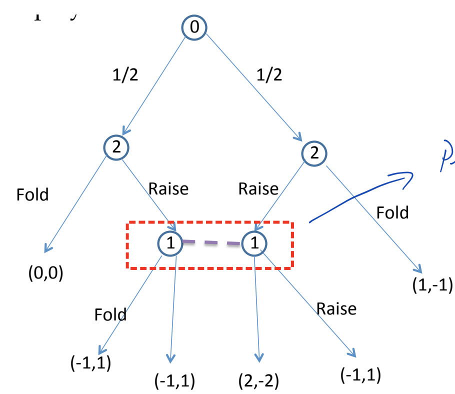

Player 2 cannot distinguish which node is he at in the Information Set

This indicates that 2 does not know what 1 has done when she chooses between Raise and Fold.

Often we will use dashed lines to indicate information sets—this is done in all the following examples.

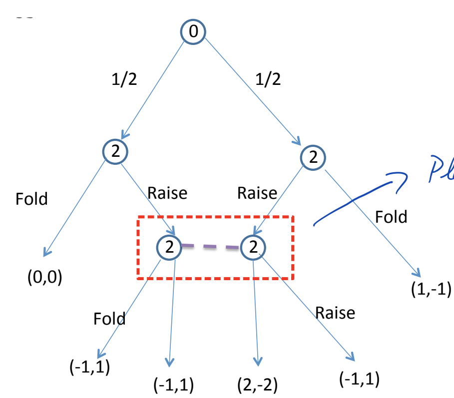
Information sets can be used to describe situations where players move simultaneously. But there are many more subtle things that they can be used to represent

(1) Information Sets can be used to describe situations where a player's past actions are hidden from others (this is like the simultaneous move tree).
(2) Information Sets can be used to describe situations where a player has private information. For example, Player 2 knows the outcome of a coin toss but Player 1 does not.




Player 2 knows the side of the coin and made a decision
However, Player 1 has no info about the side of the coin

(3) Information Sets can be used to describe situations where a player forgets what they knew previously. For example, Player 2 knows the outcome of a coin toss and then forgets it.




Player 2 forgot the side of the coin here.

When we introduce information sets we need to change the definition of what a pure strategy is:

Pure Strategy in Incomplete Info EFG.



**Definition 5.** *A pure strategy for an incomplete information extensive form game is just an instruction for every information set where it must make a move. (NOT instructions for every node.)*

If a player does not know which node at an information set they actually are at they are forced to choose the same action at every node in that set. For example, in the simultaneous move game above Player 2 only has two pure strategies Fold or Raise. But in the game without the information set Player 2 has four pure strategies (Fold,Fold), (Fold,Raise), (Raise,Fold), (Raise,Raise), that is, they can act in a way that depends on what Player 1 did.

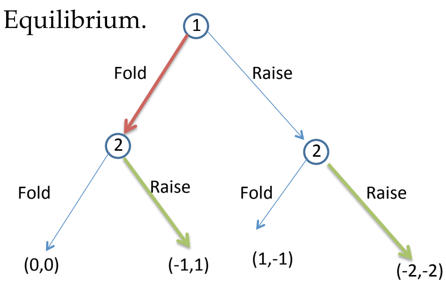
Nash Equilibrium in EFG.

## 3. Nash Equilibrium

We have already defined what a Nash equilibrium is: At a Nash equilibrium each player's strategy is a best response to the other players' strategies. This seems to be a good notion of equilibrium in strategic form games, but (as we will see) it is not such a great notion of equilibrium in extensive form games. Remember the test for a Nash equilibrium is: fix everyone else's strategy then check whether my strategy is optimal for me given that everyone else *is committed* to playing their strategy.

To understand why a Nash equilibrium might not be a good way of describing equilibrium in these games let us look at two Nash equilibria of the same game. Here is a simple two-stage game and one Nash equilibrium of it.

A Nash Equilibrium.




If we fix Player 2's strategy (in green) you can see that changing 1's strategy will reduce her payoff to $-2$ from $-1$, so certainly 1 is doing the best she possibly can given 2's actions. Now let us test 2's strategy. If we change what 2 does at the right decision it has no effect on her payoff (she gets 1), so any action here passes the optimality test of a Nash equilibrium. If we change 2's action at her left decision her payoff goes down to zero from 1, so again she is acting optimally here. I hope this convinces you that this is a Nash equilibrium.

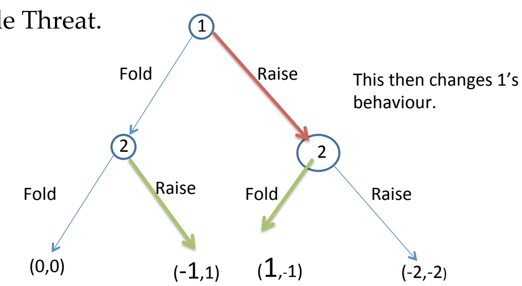
Now we will argue that the above is not a particularly sensible equilibrium. Consider again why Fold is optimal for Player 1. This is because she believes she will get $-2$ if she plays Raise. Is this a sensible belief? The answer to this is yes under the assumption that Player 2 plays the green actions, but is it reasonable to expect that Player 2 will play the green action? The answer to this is no. Player 2 is threatening to play Raise if Player 1

Non-Creditable Threat

Raises. However, this threat is not something Player 2 would actually want to carry out—it is a non-credible threat. Because if 1 actually played Raise, the best thing for Player 2 to do would be to Fold (they would get $-1$ rather than $-2$). Thus the problem with this NE is that Player 2 is threatening to do something that they are not actually going to do and as a result of the NE assumption Player 1 believes this threat.

We have just argued that this is a bad kind of equilibrium, so is there another NE of this game that does not have this problem? Here is another NE of the same game that does not rely on a non-credible threat.

Credible Threat.




This is another Nash equilibrium and at this Nash equilibrium no non-credible threats are made. Can we always find such a Nash equilibrium? The answer is yes and what is more there is a very nice process to find these good Nash equilibria and it is called backwards induction. This always works in games of perfect information (i.e. games without information sets), so in these games we can always find Nash equilibria that only rely on credible threats.

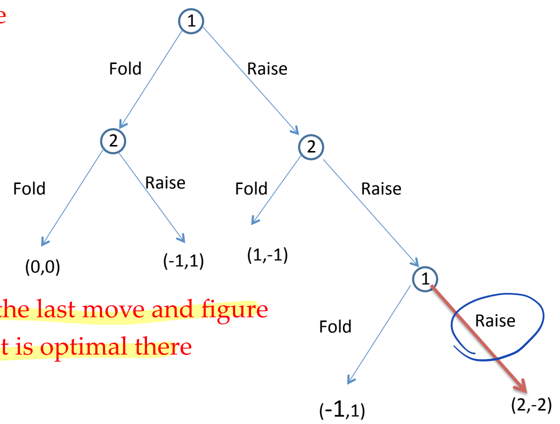
## 4. Backwards Induction in Finite Perfect Information Games

Backward Induction

The process called backwards induction works for all finite, perfect information games and it can be used to find the Nash equilibria that do not rely on non-credible threats.

Here we explain how you do backwards induction by working through an example. First you consider the very last decision that gets taken in the game and the player who takes it. You make the optimal decision for that node.

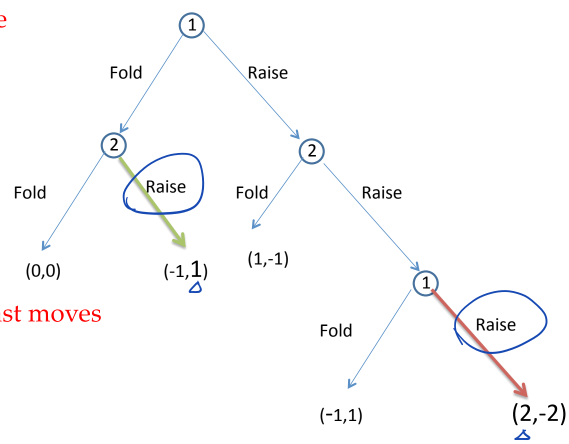
Example




Start at the last move and figure out what is optimal there

(As $-1 < 2$ Player 1 prefers Raise at the last node.) Then you determine the optimal decisions at all the other final nodes.

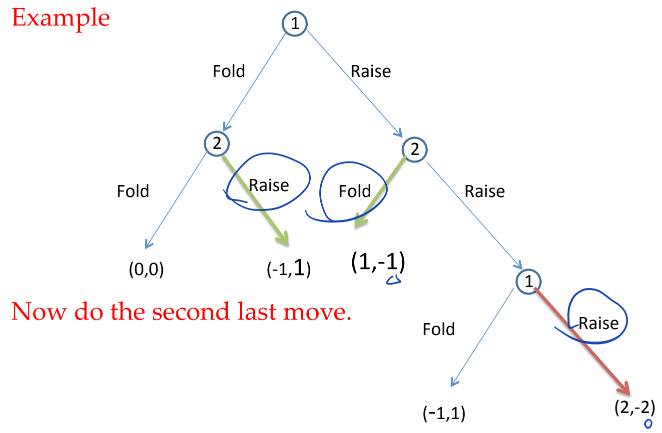
Example




Do all last moves

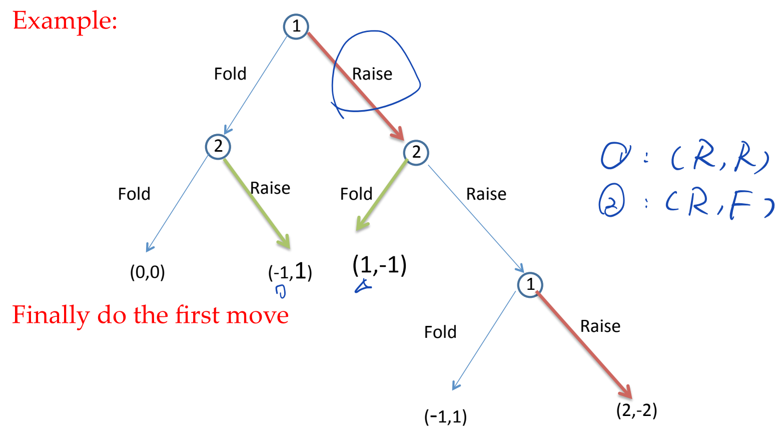
(As $1 > 0$ Player 2 prefers Raise at this node.)

Once the last decisions have been determined it is possible to figure out the second last decision. We assume the two last decisions cannot now be changed. (Notice that we have ensured there are no non-credible threats by making all the last decisions optimal for the Players that have to take them.)




(As $-1 > -2$ Player 2 prefers Fold at the second last decision.)

Now we can finally determine the first decision




(Player 1 prefers Raise as $1 > -1$.)

We have now created an instruction book (or pure strategy) for every player in the game. These pure strategies are a Nash equilibrium, because no change can improve any players' payoff. But also they do not rely on non-credible threats. These equilibria are given a special name, they are called *Subgame Perfect Equilibria*. However, the notion of subgame perfect equilibrium can be applied more broadly to all extensive form games, not just those with perfect information.

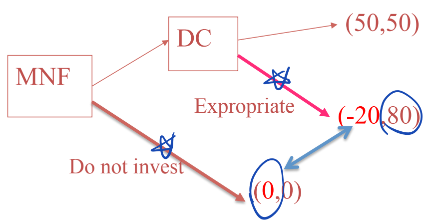
Notes:

- Backwards induction proves the existence of a NE in finite perfect information games.
- This result is also called Zermelo's Theorem and it implies that games such as Chess, Draughts, Backgammon, Go, etc. must have Nash equilibria.
- Backwards induction is a generalization of dynamic programming.

- Backwards induction will not work in games with information sets, so it can only work in perfect information games.
- Reinhart Selten generalized the idea of backwards induction to create the idea of subgame perfection and got the Nobel prize for it (we will study this in the next section).

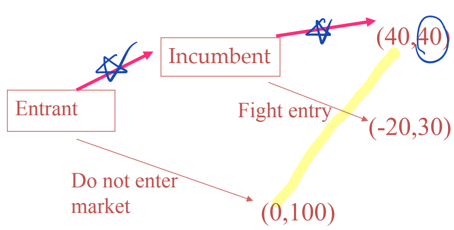
4.1. **Economic Applications of Backwards Induction.** There are many Macroeconomic applications of backwards induction. Here are two microeconomic applications.

4.1.1. *Investment by multinational firms.* In this model we think of a firm first deciding whether to invest overseas or not. Then the government of the destination country decides whether to expropriate the investment or not. The country would like to commit to not expropriating but, in the simple version of this game below, this is not a credible undertaking.




(The first payoff is what the multinational firm (MNF) gets and the second payoff is what the developing country (DC) gets.) The backwards induction solution, illustrated on the figure, has no investment and no expropriation.

4.1.2. *Entry deterrence.* Here we imagine there is a monopolist in an industry and it is trying to deter another firm from entering the industry and competing with it. The entrant moves first and decides whether to enter or not and then if entry has occurred the incumbent gets to decide whether to fight the entrant.




(The first number is the entrant's profit the second is the incumbents.) Again the incumbent would like to commit to fighting the entrant, so entry did not occur. But this is a non-credible threat and the backwards induction solution is for entry to occur. One possible solution would be for the incumbent to build a large plant which is loss making unless it has a large share of the market. This reduces the incumbent's payoff to sharing the market to 20. Then what does the incumbent want to do?

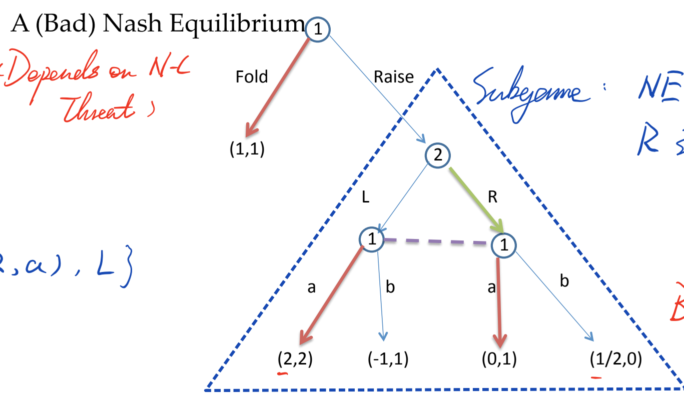
Subgame Perfection

## 5. Subgame Perfect Equilibrium

Subgame perfect equilibrium is a way of generalising backwards induction to games that have information sets. It has the good properties of backwards induction—every threat is credible.

Subgame perfect equilibrium divides the game tree up into subgames (that is, parts of the tree that can be considered separately). It requires that the players' strategies are a Nash equilibrium on every subgame, that is, the players behave in a credible way on all the parts of the game tree that can be considered separately. Again a subgame perfect equilibrium can be found by working backwards, taking the last independent game and finding strategies that are a Nash equilibrium. Then taking the next last independent game and so on. (Backwards induction *always* finds a subgame perfect equilibrium.)

Let us begin with an example of a game and a bad Nash equilibrium.

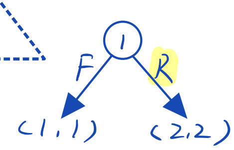



(Depends on N-C Threat)

Subgame: NE: (a, L) [Only NE]
R strictly dominate L

Subgame NE: {(R, a), L}
(Good)

Backward induction doesn't work




The whole game is also a potential subgame!

In this game Player 1 folds because they believe they will get a payoff of 0 if they raise. Player 2's action genuinely has no effect on their payoffs so R is optimal for them. Hence the situation above passes the test for a Nash equilibrium. However, if Player 2 got to actually make a move in this game, then going R gets them 1 while going left gets them 2. It looks like Player 2 is making a threat they wouldn't actually want to carry out. We would like to rule this situation out by doing something like backwards induction. But that's not going to work in this game as we cannot work out what Player 1 would do at the very last move—it might be optimal for 1 to do action $a$ if she thought she was at the left node but it might be optimal for her to do action $b$ if she thought she was at the right node. If there's no dominant strategy in the last node, then this game is not solvable by backward induction

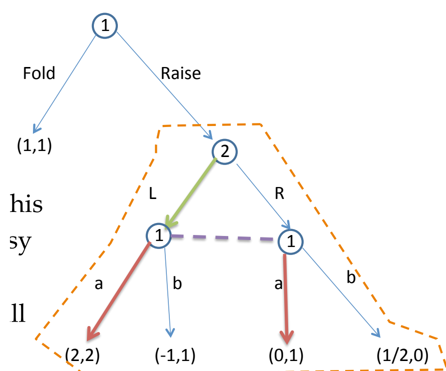
Reinhard Selten suggested that we treat the events after Player 1 moves Raise as a separate independent game, or a subgame, of the original game. He suggested that what players do on this separate independent game must be a Nash equilibrium of that game. In this example, this is a game where Player 2 and Player 1 simultaneously move. The

Non-Creditable Threat Never exist in SgP. NEs.

strategic form for this subgame is given below.

| | L | R |
|---|---|---|
| a | (2,2) | (0,1) |
| b | (-1,1) | ($\frac{1}{2}$,0) |

What the players are doing at the bad NE of this game is $(a, R)$. If you were given just this game to analyse you would predict $(a, L)$ because it is the unique Nash equilibrium and can be found by iteratively eliminating strictly dominated actions. Selten said that we should expect the players to play a NE of this game, so if we replace the $(a, R)$ actions with the Nash equilibrium of the subgame we get the following picture.



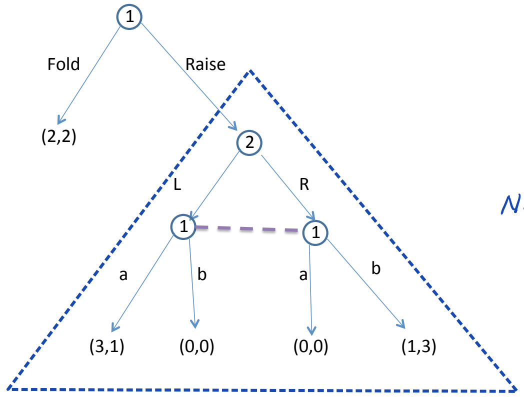

Here there is also a dashed line around the subgame we solved separately (and some text I have been unable to crop). Once we have solved this subgame we can easily figure out what Player 1 will do at the initial move and find a subgame perfect equilibrium of the entire game. If Player 1 plays Fold they will get 1 but if they play Raise and the NE of the subgame is then played, then Player 1 will get 2. Thus this SPE has player 1 Raising.

Summary: The way of finding a subgame perfect equilibrium is to find a Nash equilibrium on the smallest subgames and then work backwards through the treat finding a Nash equilibrium on subgames as you go.

There is a problem with subgame perfect equilibria in that they may not be unique. This occurs because the subgames may have many Nash equilibria. To see this consider a slightly different version of our game.

Backward induction always gives unique NE.

SgP NE may not be unique.



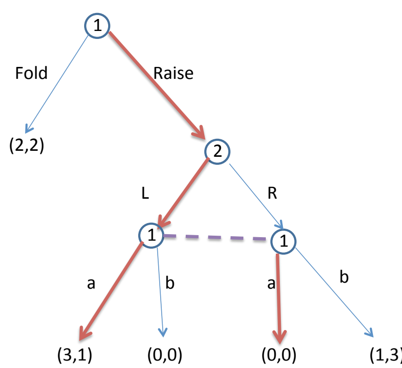

NE (Sub): PS NE: (L, a)
(R, b)
MSNE: ....

The subgame of this extensive form can be represented as the strategic form below.

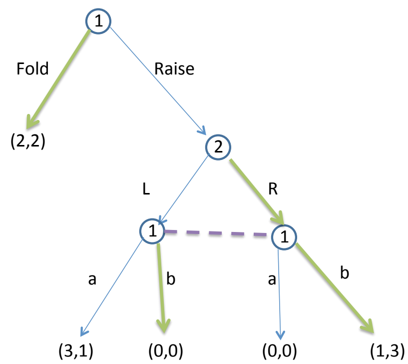
| | L | R |
|---|---|---|
| a | (3,1) | (0,0) |
| b | (0,0) | (1,3) |

This strategic form has 2 pure Nash equilibria at $(3, 1)$ and $(1, 3)$. So which one do we choose to put in the game to find the subgame perfect equilibrium (SPE)? Well we can choose either of these.

If we choose the $(3, 1)$ NE on the subgame we get the SPE below.




If we choose the $(1, 3)$ NE on the subgame we get the SPE below.




It is clear that one can choose the NE to get very different behaviour from Player 1 at the first move. But these are both perfectly fine SPE's.

解多次重复的 Strategic Form Games 的要点：

每个 2×2 Strategic Game 里有一个 Information Set.

所以，2×2 的玩 2 次有 5 个 Information Set，每个人 32 个 Pure Strategies

① Strategic form ⟶ Extensive form

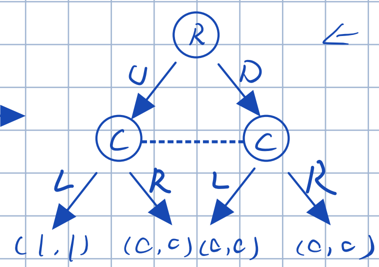
Strategic : Simultaneous + Don't know each other's action

C

2 Player:

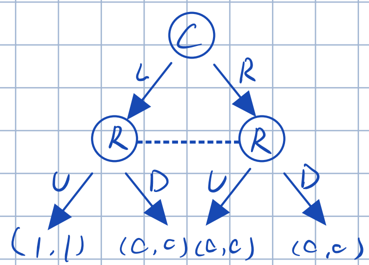
$$R\quad \begin{array}{c|cc} & L & R \\ \hline U & 11 & 00 \\ D & 00 & 00 \end{array}$$

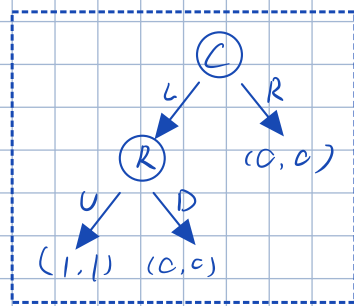
⟶



← Who is the initial player does not matter here!

Or:





3 Players: Player 3

L

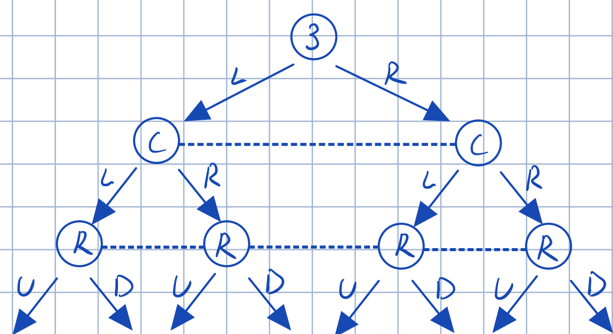
$$C$$

$$R\quad \begin{array}{c|cc} & L & R \\ \hline U & 111 & \cdots \\ D & \cdots & \cdots \end{array}$$

R

$$C$$

$$R\quad \begin{array}{c|cc} & L & R \\ \hline U & \cdots & \cdots \\ D & \cdots & \cdots \end{array}$$

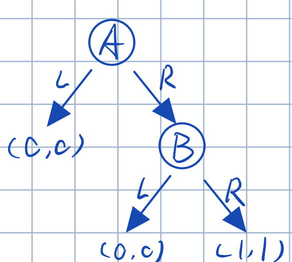


One Strategic form can represent several different Extensive forms.

② Extensive form ⟶ Strategic form

Step 1: Find every player's Pure Strategies

Step 2: Tabulate players' expected payoffs to these

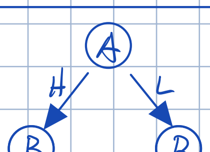


A: {L, R}

B: {L, R}

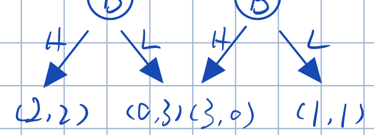
| | | B | |
|---|---|---|---|
| | | L | R |
| A | L | (0,0) | (0,0) |
| | R | (0,0) | (1,1) |

$$\begin{bmatrix} 00 & 00 \\ 00 & 11 \end{bmatrix}$$



Firm A chooses prices, firm B responds

A {H, L}




B { (H,H), (H,L), (L,H), (L,L) }
(AH/L AL/L)
(Instruction book)

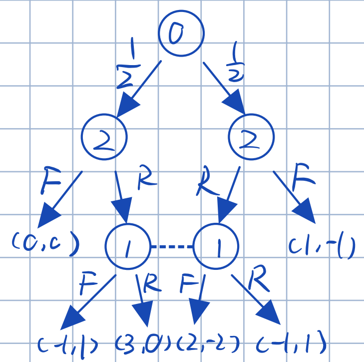
Order is "Implicit" here

| A \ B | (H,H) | (H,L) | (L,H) | (L,L) |
|---|---|---|---|---|
| H | (2,2) | (2,2) | (0,3) | (0,3) |
| L | (3,0) | (1,1) | (3,0) | (1,1) |

No info. on order here

$$\begin{bmatrix} (2,2) & (2,2) & (0,3) & (0,3) \\ (3,0) & (1,1) & (3,0) & \boxed{(1,1)} \end{bmatrix}$$

NE




① : { F, R } because of the info set.

② : { (F,F), (F,R), (R,F), (R,R) }

Dealing with random player 0:

$\frac{1}{2} \times$ (0 ↙)

| ① \ ② | (F,F) | (F,R) | (R,F) | (R,R) |
|---|---|---|---|---|
| F | (0,0) | (0,0) | (-1,1) | (-1,1) |
| R | (0,0) | (0,0) | (3,0) | (3,0) |

$+ \frac{1}{2} \times$ (0 ↘)

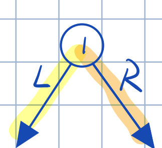
| ① \ ② | (F,F) | (F,R) | (R,F) | (R,R) |
|---|---|---|---|---|
| F | (1,-1) | (2,-2) | (1,-1) | (2,-2) |
| R | (1,-1) | (-1,1) | (1,-1) | (-1,1) |

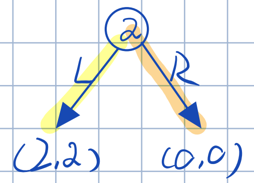
=

| ① \ ② | (F,F) | (F,R) | (R,F) | (R,R) |
|---|---|---|---|---|
| F | $(\frac{1}{2},-\frac{1}{2})$ | (1,-1) | (0,0) | $(\frac{1}{2},-\frac{1}{2})$ |
| R | $(\frac{1}{2},-\frac{1}{2})$ | $(-\frac{1}{2},\frac{1}{2})$ | $(2,-\frac{1}{2})$ | $(1,\frac{1}{2})$ |

How to find a NE that is not a Subgame Perfection:

Find a S-P NE -> Try to make a N-C Threat and force another player to change its behavior





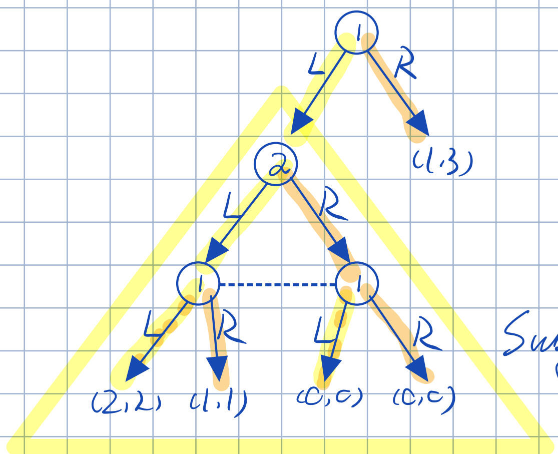
(1,3)

Good NE (SubP) : (L, L)

Bad NE (N-C Threat) : (R, R)

In extensin forms, players do not necessarily play simultaneously, but know each other's choice (no info set.)

Strategic forms do not have time structure



Subgame :

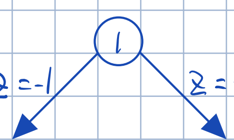
| | | ② L | ② R |
|---|---|---|---|
| ① | L | (2,2) | (0,0) |
| | R | (1,1) | (0,0) |

NE (L, L)

S-P NE : {(L, L) , (L)}

Bad NE : {(R,R) , (R)}

{(R, L) , (R)}



③

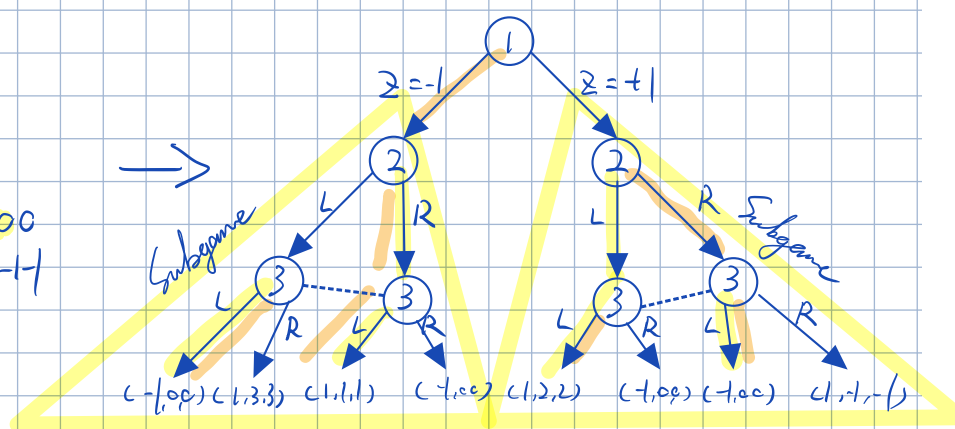
| | | |
|---|---|---|
| ② | -100 | 133 |
| | 111 | -100 |

③

| | | |
|---|---|---|
| ② | 122 | -100 |
| | -100 | 1-1-1 |

and $\{(\frac{1}{4},\frac{3}{4}),(\frac{1}{4},\frac{1}{4})\}$

$\longrightarrow$



Choose one of NE here and proceed

* SPE #1 ② : (L, L)

③ : (R, L)

① : (P, 1-P) $0 \le P \le 1$

# 3: MONOPOLY

MARTIN CRIPPS

In this section we resit some models you probably studied last year so we can warm up for the study of more complex models of firm behaviour.

Profit Maximization (Monopoly)

## 1. The Basics of Profit Maximization

We now revise the simplest possible case where there is one firm making a pricing/output decision. A monopolist is faced by a collection of customers who have a demand curve $p = P(q)$. It also has a cost function (that describes the costs of making each output level) that depends on input prices and output $C(q, w)$. With these elements we can write down its profit

$$\begin{aligned} \text{Profit} &= \text{Revenue} - \text{Costs} \\ &= qP(q) - C(q, w) \end{aligned}$$

To maximize profit (and choose the optimal price-output combination) we differentiate and set the slope of profit equal to zero.

$$\begin{aligned} P &= qP(q) - C(q, w) \\ \frac{dP}{dq} &= \left[q\left(\frac{dp}{dq}\right) + P(q)\right] - \frac{dC}{dq} && = 0 \\ &\text{Marginal Revenue} - \text{Marginal Costs} && = 0 \end{aligned}$$

Notes: the formula for marginal revenue does not equal the price. This is because to sell more a monopolist has to cut the price on all units it sells not just the additional unit. Thus

① $MR^* = MC^* < P^*$

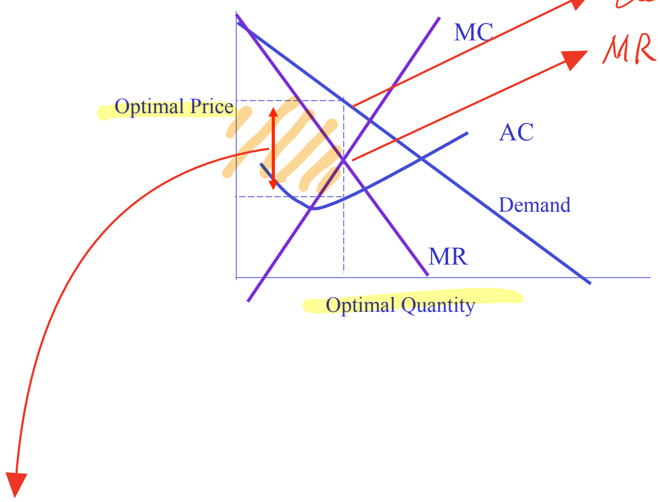
$$\text{MR} = q\frac{dP}{dq} + P(q) < P(q)$$

Negative

(The inequality is true as the first term is negative.)

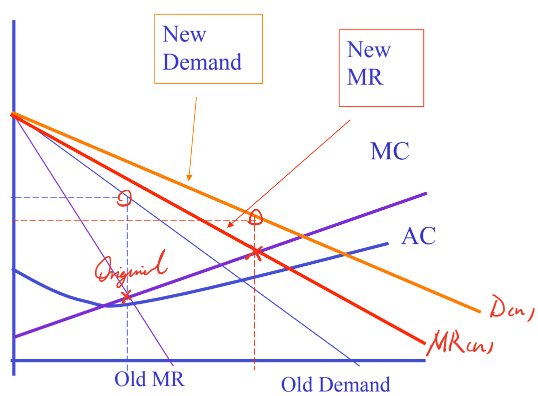
Thus we have derived the first rule for profit maximisation $MR = MC$. This is informative about the firm's optimal quantity (its output choice) but not the optimal price. Here is the usual kind of picture we draw and it has pretty good information about quantities on it, but how price is determined is unclear.

Optimum Quantity




The demand curve then tells the price
MR / MC determines the optimum quantity

(The arrow indicates the profit per unit.) As the demand moves around it is not clear how the price will change. It is possible that an increase in demand reduces the price.




Note: Here the new demand is much flatter and quantity has gone up.

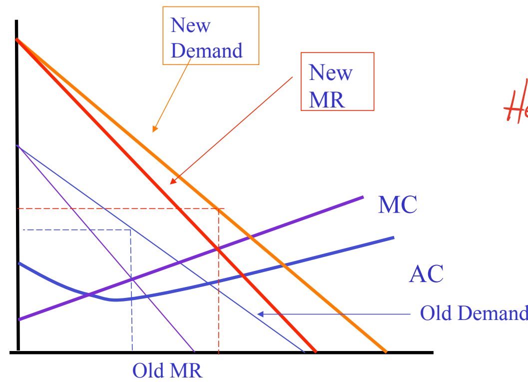
Demand ↑

Price may not be higher

(Here, D↑ P↓)

You see this in DVD's where low demand DVD's often are quite expensive but very popular family films are much cheaper.

It is also possible that an increase in demand reduces the price.




Here, D↑ P↑

Note: Here the new demand is steeper.

You see this in fashion when some new fad becomes popular and simultaneously expensive.

To understand what's going on here we need to take the condition for profit maximisation and reinterpret it in terms of pricing:

MR MC

$$q\frac{dp}{dq} + P(q) = \frac{dC}{dq}$$

$$P(q)\left[\frac{q}{p}\frac{dp}{dq} + 1\right] = \frac{dC}{dq}$$

$$P(q)\left[\left(\frac{p}{q}(\frac{dq}{dp})\right)^{-1} + 1\right] = = \frac{dC}{dq}$$

② $P^* = \frac{MC}{1 + PED^{-1}}$

Hence we get

Rule 2:

$$\boxed{P(q) = \frac{\frac{dC}{dq}}{\left(\frac{p}{q}(\frac{dq}{dp})\right)^{-1} + 1} = \frac{\text{MC}}{1 + \frac{1}{\text{Price Elasticity}}}}$$

Optimum Price = $\frac{MC}{1 + PED^{-1}}$

It is this rule that tells us about pricing:

- As marginal cost goes up so do prices. MC↑ P↑
- If demand becomes more elastic (the elasticity moves from −3 to −4 for example) so prices go down. |PED|↑ P↓

This results in the following typical properties: Steeper demand curves have higher prices and flatter demand curves have lower prices. This is what is going on the in the previous examples.

1.1. **Monopoly Power.** Pure monopoly is rare but firms do face downward sloping demand curves. As one firm varies its price there will be a change in the demand the firm experiences.

The individual firm's demand is more responsive to price than the market demand, but as long as demand does not entirely vanish when it raises its price the firm enjoys some monopoly power.

One way of measuring monopoly power is to consider the share of the price that goes in profit.

Lerner's Measure of Monopoly

$= \frac{P - MC}{P}$

$= - \frac{1}{PED}$

$$\text{Lerner's Measure of Monopoly} = \frac{\text{Price} - \text{MC}}{\text{Price}}$$

$= \frac{-1}{PED}$ (when maximizing profit using monopoly power.

If the firm is maximising profits and we substitute in for the profit maximising price from above into this measure we get that Lerner's measure is equal to $= -1/$Elasticity of demand. So the question of what determines a firm's monopoly power amounts to the question: What determines the firm's elasticity of demand?

- The Elasticity of the market demand
  - You may just be in an industry producing a product with many substitutes, so consumers are pretty sensitive to overall price increases in the industry. For example, the oil industry was successful in forming a cartel, but coffee producers were not because there are substitutes to coffee.
  - A firm may be in an industry with many others, but geography gives each of these firms a strong local monopoly. This is particularly true if goods are difficult to transport.
- The number of competitors a firm faces obviously determines how price-sensitive its customers are. The determinants of the number of firms in an industry are the barriers to entry these are: 1. Legal, 2. Scale, 3. Technological, 4. Strategic. 5 Strategic Behaviour of other firms.

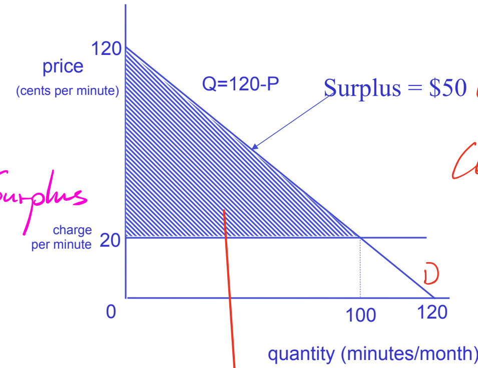
Two Part Tariffs

1.2. **Two Part Tariffs.** This is a topic we will revisit later in this course but also should appear in the section on Monopoly.

A two-part tariff is a common form of pricing where the price paid by the consumer for buying $Q$ units is $P = A + pQ$, where $A$ is a fixed fee (connection charge, membership fee, admission charge) and $p$ is the price for units. Clearly, the average price $= (A + pQ)/Q$ declines in $Q$. Effectively this is a quantity discount, so high-usage customers pay less per unit than low-usage consumers. On the other hand it seems to be a deterrent to consuming small numbers of units. You see two-part pricing in: Theme Parks, Utilities, all you can eat restaurants, razors...

We have talked about how one might choose a price optimally when there is no fixed fee, but how does this decision change in the presence of the fee. To understand the best way of setting the fee $A$ it is easiest to think of the case where there is only one consumer buying units of your good. In this case the area below the demand curve of the individual and above the price represents the value the individual is getting from the good but is not paying for. Here is the example of a demand function and a per unit price.

TR { Max $\pi$ : $MR = MC$
Charge a fixed fee
$F = \Sigma$ Consumer Surplus }




$\left(\frac{1}{2} \times 100 \times 100 \text{ cents}\right)$

Consumer Surplus = Max possible fixed cost

Total Revenue $= P \times Q +$ Consumer Surplus

The area above the price but below the demand is a triangle base 100 height 100 so it has an area 5000 or \$50. This represents the value the consumer is getting at this per unit price that they are not paying for. This is the maximum fixed fee the seller could charge. In all they would make $20 \times 100$ on per-unit sales and then 5000 in a fixed fee. The revenue the firm gets is the entire area under the demand curve between $q = 0$ and $q = 100$. Thus we can now address the optimal choice of a two part tariff by appreciating that in this case the firm's revenue is the area under the demand not just price times quantity.

If the firm charges consumer surplus as a constant fee:

$$\text{Total Revenue} = \int_0^q P(v)dv$$ (All area under Demand curve)

$$\text{Marginal Revenue} = \frac{d}{dq}\left(\int_0^q P(v)dv\right) = P(q)$$ $MR = P$

So to maximise profits you set $MR = MC$, which implies $P(q) = MC$ or that price is set to equal marginal cost—just like a perfectly competitive firm would.

Price Discrimination Problem of one price: If a firm sets one price, ~~then one~~ always customers willing to pay more.

**1.3. Price Discrimination.** In general setting only one price for a product is suboptimal because there are usually customers who are willing to pay more. The practice of setting several prices for the same product is called price discrimination.

Price Discrimination occurs when a firm sells the same good at different prices to different customers. Firms do this to try to capture part of the consumer surplus; this is not a bad thing as generally it increases supply. Here are some examples of it

- Airlines(Business/Vacation)
- Hotels(Business/Vacation)
- Drugs (Generic/Branded)
- Telephone Companies (Fixed Fee and Call charges)
- Service provides Fixed fee and per unit cost (IBM/Xerox)
- Manufacturers(Offer high initial prices and lower ones later)
- Businesses(offer student discounts)
- Manufactures (Retail/Wholesale)

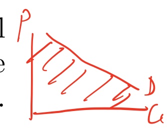
For price discrimination to be possible it is necessary that it is possible to divide the market up in some way and to get the two groups to pay different prices. Thus the necessary conditions for price discrimination are:

Prerequisits

- It is possible to identify at least two market segments with different elasticities of demand. This can be done using personal characteristics such as age or gender as well as geography and time. (Dry Cleaning/Haircuts)
- It is also necessary to think about how price discrimination will be enforced. In particular what stops higher price buyers buying at the lower price? Must also prevent low price types reselling to higher price types. They become the firm's competitors for the high price customers.Resale is easier to prevent in services rather than goods so price discrimination is more common in the service sector.

1st PD

1.3.1. *First-Degree Price Discrimination.* Here a customised price is set for each individual customer and it may be observed in car dealers, dotcoms, telephone companies, college tuition. In practice it is very unlikely, because firms cannot find out customers true value. This theoretical extreme would and it maximise profit if it were feasible.




(If a firm is able to do this, then the revenue is the area under the demand curve. The revenue obtained from selling one more unit = price. Thus, it is optimal for the firm to produce the quantity at which $MC = p$.)

2nd PD

1.3.2. *Second-Degree Price Discrimination.* This describes a situation where a firm is able to charge a price that depends on the quantity purchased (e.g. quantity discounts). Examples: Block Pricing by Power Generators, minutes on your phone, two-part tariffs.


3rd PD

1.3.3. *Third-Degree Price Discrimination.* This describes everything else. For example where the firm divides a market into 2 broad segments and charges these segments different prices. If a firm supplies two markets (for more) A and B with two independent plants, then to maximise profits it wants to set the prices in the segments in the way that will maximize its profit. That is

$P_A^* = \frac{MC_A}{1+PED_A^{-1}}$ $\quad P_B^* = \frac{MC_B}{1+PED_B^{-1}}$

$$\boxed{P_A = \frac{\mathrm{MC}_A}{1+\frac{1}{\text{Price Elasticity}_A}}, \qquad P_B = \frac{\mathrm{MC}_B}{1+\frac{1}{\text{Price Elasticity}_B}}}$$

$|PED| \uparrow \; P \downarrow$

PED是负的.

To see where this result comes from, suppose that the firm supplies output from one plant to two separate markets, then there is a trade-off between supplying one market and another because it increases costs.

$$\text{Profits} = \Pi = q_A p(q_A) + p(q_B) q_B ? C(w, q_A + q_B)$$

Maximizing:

$$0 = \frac{d\Pi}{dq_A}, \qquad 0 = \frac{d\Pi}{dq_B}$$

Or

$$\frac{dC}{dq} = p(q_A) + q_A p'(q_A) \qquad \frac{dC}{dq} = p(q_B) + q_B p'(q_B)$$

Rearranging

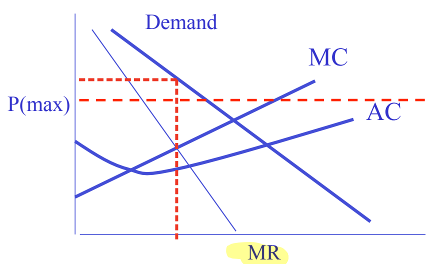
$$\frac{dC}{dq} = p_A[1 + (q_A/p_A)p'(q_A)] \qquad \frac{dC}{dq} = p_B[1 + (q_B/p_B)p'(q_B)]$$

and so...

$$\frac{MC}{1 + (\text{Elasticity}_A)^{-1}} = p_A \qquad \frac{MC}{1 + (\text{Elasticity}_B)^{-1}} = p_B.$$

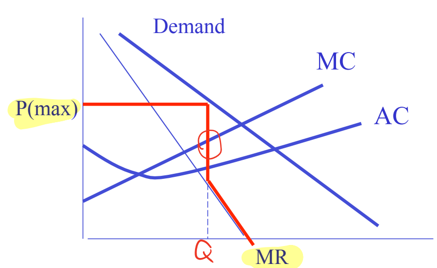
Regulation

1.4. **Regulation of a Monopoly.** The most common form of regulation a firm faces is a maximum price, or a price cap. We will see that this will tend to *increase* the output of a monopoly and decrease its price. Here is a picture of a monopoly and a potential regulated price ($P(max)$). Note that this regulated price is less than the price it would charge if it set $MR = MC$.




Right Regulated Pri intersection of MC, D
(MC = P = AR)

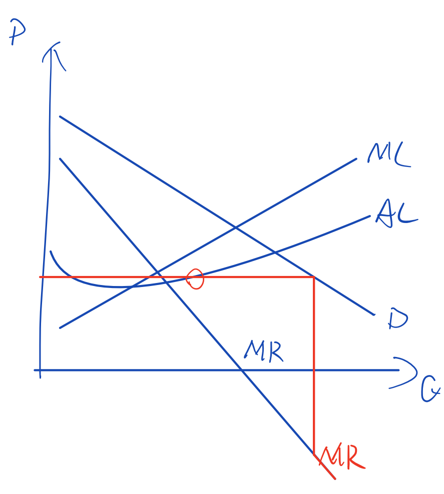
The regulated price means that the firm's $MR$ curve changes. If the firm is selling less than the maximum it can sell at the regulated price, then it can always sell one more unit and get the regulated price for it. Thus the firms $MR$ is flat and is equal to the regulated price, until the regulated price hits the demand curve. At this point the $MR$ is exactly what it was before. This is shown in the figure below.




It should be clear from this figure that $MC = MR$ at the vertical section of the red $MR$ curve. Thus the amount of output the firm has produced has increased and prices have fallen as a result of this regulation.

when it goes wrong:

Shortages

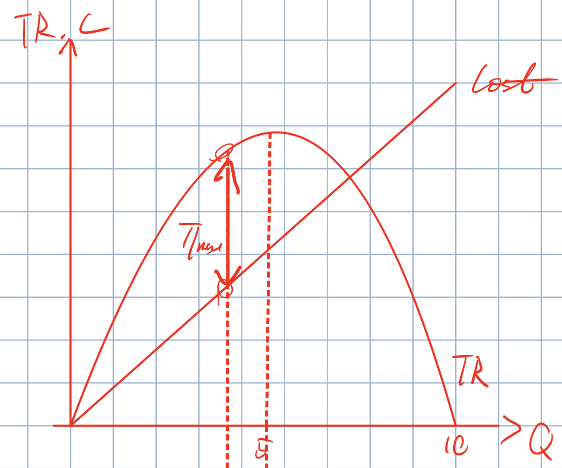


# Regulated Monopoly

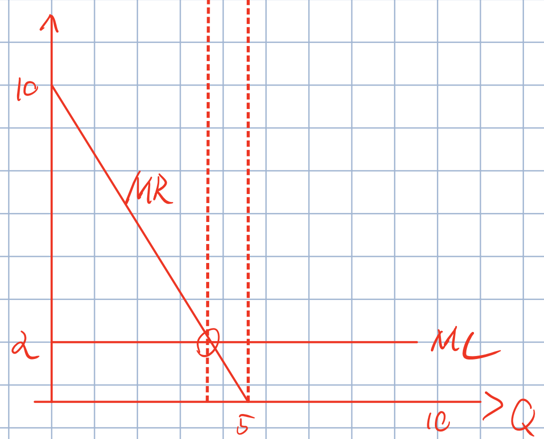
$P = 10 - Q$

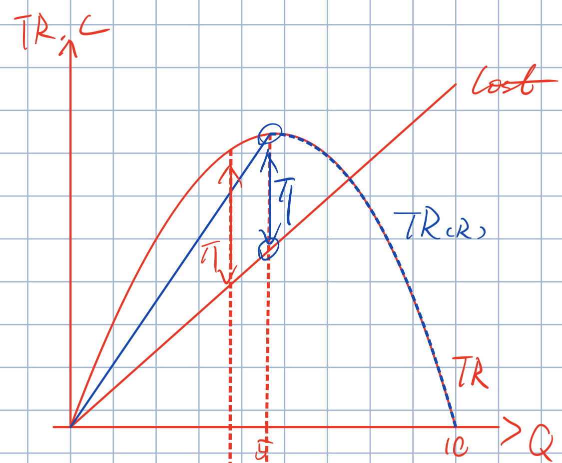
$Cost = 2Q$

$\pi = (10 - Q)Q - 2Q$



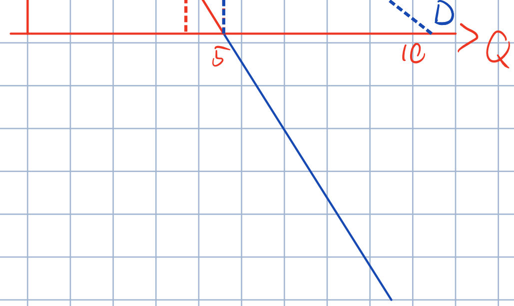

No Regulation

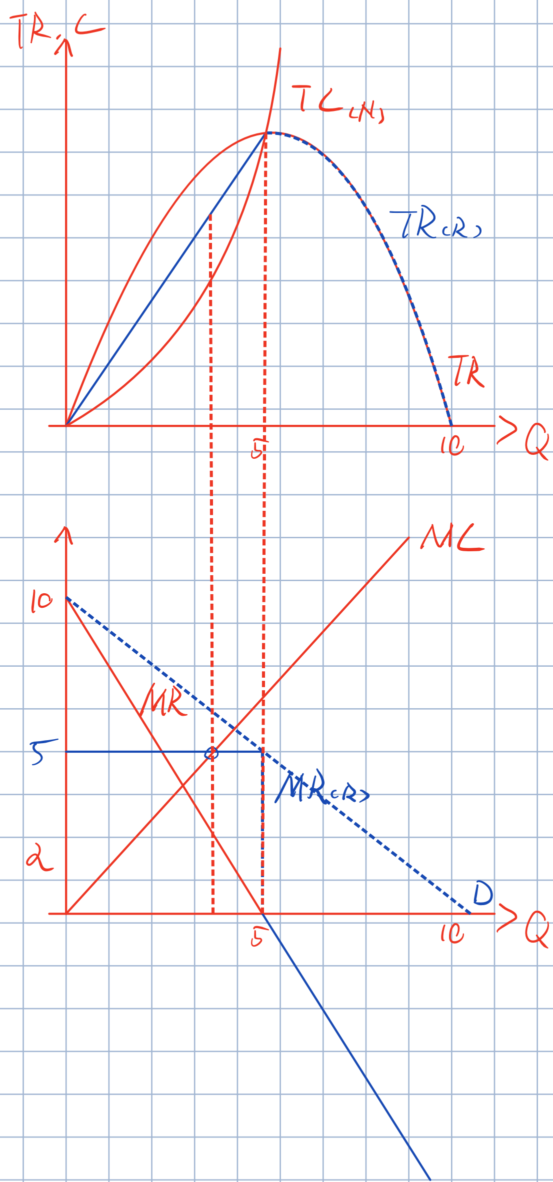



Regulation : $P \leq 5$

$$\begin{cases} P = 5 & Q < 5 \\ P = 10 - Q & Q \geq 5 \end{cases}$$




If a firm has an increasing MC

Regulation : $P \leq 5$

$MC = 2Q$

$TC = Q^2$

# 4: STATIC OLIGOPOLY

MARTIN CRIPPS

In this topic we take the tools of strategic form games (in particular Nash equilibrium and dominance) to analyse models of industries where there are many firms who are in competition with each other. There is no single "correct" model of these kinds of industry. Here we will look at four models of Oligopoly, but there are others you should be aware of.

Cournot's Model of Oligopoly

## 1. COURNOT's Model of Oligopoly

To understand how Cournot's model of oligopoly works it is best to work through an example. Before doing this we imagine a world where firms are producing exactly the same product (homogenous product) and the decision they must take is how much output to produce. By producing more output they drive down the market price for themselves and all other firms in the industry. (We say this negative effect of one firm on others in the industry is a "negative externality".) One interpretation of this model is it describes competition in the long run. That is when firms are deciding what factories to build. It is then, presumably, that they make their quantity decisions.

1.1. **Example of Two-Firm Cournot Oligopoly.** In the example we imagine there are only two firms in the industry each producing exactly the same good (water or oil for example). The demand curve for their product is $P = 30 - Q$, where $Q$ is the total output produced by all the firms and $P$ is the market price. Thus we could write $P = 30-(Q_1+Q_2)$, where $Q_1$ and $Q_2$ are the individual output of firm 1 and firm 2 respectively. Each of the two firms has the same costs of production with a constant Marginal Cost of 6: $C(Q_1) = 6Q_1$ and $C(Q_2) = 6Q_2$. (A quicker way of writing this is $C(Q_i) = 6Q_i$ for $i = 1, 2$.)

With this information it is possible to write down the firms' profits (or payoffs) in this game:

$$\pi_1(Q_1, Q_2) = PQ_1 - C(Q_1) = (30 - (Q_1 + Q_2))Q_1 - 6Q_1,$$
$$\pi_2(Q_1, Q_2) = PQ_2 - C(Q_2) = (30 - (Q_1 + Q_2))Q_2 - 6Q_2.$$

What is different from Monopoly is that each firm's profit depends on the output of another firm.

To summarise the game: Each firm chooses and output level $Q_i \geq 0$. These choices are made simultaneously. The payoffs these choices are given by the firms' profits $\pi_1$ and $\pi_2$ in the above equation.

Our tool to analyse this situation will be to find a Nash equilibrium. And because of the large number of actions the firms have the method of finding the Nash equilibrium will be to find the firm's best responses. We called this "Method 4 The Intersection of Best Responses".

Intersection of Best Responses

1.1.1. *Finding the Best Responses.* Here are two ways of calculating Firm 1's best response to a given output of Firm 2. First write down firm 1's profit.

$$\pi_1(Q_1, Q_2) = (30 - (Q_1 + Q_2))Q_1 - 6Q_1$$

Observe that this is a quadratic function of $Q_1$ os its maximum can be found by differentiating and setting equal to zero.

$$\frac{\partial \pi_1}{\partial Q_1} = 30 - (2Q_1 + Q_2) - 6 = 24 - Q_2 - 2Q_1 = 0.$$

Solving this for $Q_1$ gives

Reaction Function of Firm 1

$$24 - Q_2 = 2Q_1, \qquad \text{or} \qquad Q_1 = 12 - \frac{1}{2}Q_2.$$

Thus firm 1's optimal choice of output is a linear and decreasing function of firm 2's output choice.

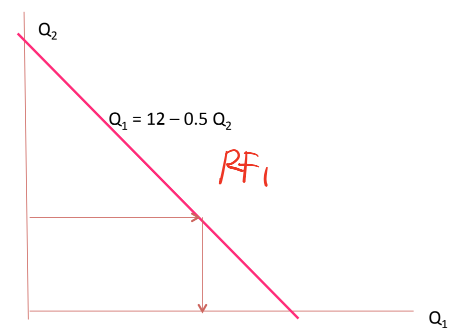
A second and more Economist-y approach to this would be to say that Firm 1's Revenue $= PQ_1 = [30 - Q_1 - Q_2]Q_1$. So we differentiate this to get its Marginal Revenue

$$\text{MR} = \frac{dR}{dQ_1} = 30 - 2Q_1 - Q_2.$$

We also know that the firm's Marginal Cost = 6 from above. So to find where the firm maximises profit we would set MC equal to MR, that is,

$$\text{MC} = 6 = 30 - 2Q_1 - Q_2 = \text{MR} \qquad \text{or} \qquad Q_1 = 12 - \frac{1}{2}Q_2.$$

This tells us firm 1's best output as a function of firm 2's output. This relationship is important and is often plotted by economists. Here's a picture




There are three points worth emphasizing about this picture

(1) It usually slopes down, because as my competitor produces more, there is less demand less for me and as a result of the reduction in demand it is optimal to produce less.

(2) The intercepts are informative. When $Q_2 = 0$ and firm 1 produces its optimal response it is alone in the market. Thus Firm 1's optimal response to $Q_2 = 0$ is the monopoly level of output. In this example 12 is the output of a monopolist.

(3) The intercepts are informative (again). When $Q_2$ is very high, Firm 1 wants to produce zero output, because it makes zero profit. (In fact when $Q = 24$ this is true.) Zero profit is actually a characteristic of Perfect Competition. Thus, in this example $Q = 24$ is the perfectly competitive output.

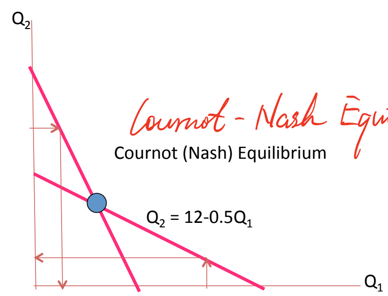
Notice that everything is about Firm 1 is true for Firm 2. So when we try to calculate Firm 2's Best Response we will get a very similar kind of equation:

$$Q_2 = 12 - \frac{1}{2}Q_1.$$

Just the names of the outputs have been swapped.

Cournot / Nash Equilibrium

1.1.2. *Finding the Cournot/Nash Equilibrium.* We now have two reaction functions of the firms: $Q_1 = 12 - \frac{1}{2}Q_2$ and $Q_2 = 12 - \frac{1}{2}Q_1$. If we are going to find a Nash equilibrium of this game we are going to find where they intersect. This is usually called a Cournot, or Cournot-Nash equilibrium, because Cournot found this solution to the model many many years before Nash was even born. Here's the picture of the two reaction functions:




Cournot - Nash Equilibrium

Decisions will converge to C-N E

To find the Cournot equilibrium we must substitute one reaction function into the other

$$\begin{aligned}
Q_1 &= 12 - \frac{1}{2}Q_2 \\
Q_1 &= 12 - \frac{1}{2}[12 - \frac{1}{2}Q_1] \\
\frac{3}{4}Q_1 &= 6 \\
Q_1^* &= 8
\end{aligned}$$

Performing a similar calculation for Firm 2 we get $Q_1^* = Q_2^* = 8$.

There is a short cut that we can use in this particular case because the firms have the same costs. In this case we know the equilibrium will be symmetric $Q_1^* = Q_2^*$. So instead of substituting firm 2's reaction function into firm 1's reaction function we can just substitute

$Q_2 = Q_1$ and get the right answer:

$$Q_1^* = 12 - \frac{1}{2}Q_2^*$$
$$Q_1^* = 12 - \frac{1}{2}Q_1^*$$
$$\frac{3}{2}Q_1^* = 12$$
$$Q_1^* = 8$$

This method of find the Cournot equilibrium can make the algebra *a lot* easier, but you must remember this only works when firms have the same costs and demand!

**1.2. Generalisation of this Approach to Many Firms.** There is nothing particularly difficult about extending the Cournot model to many firms. (Although drawing the pictures gets quite hard.) Suppose now that we have an industry with $N$ identical firms—we will give them the names $i = 1, 2, \dots, N$. Firm $i$ has to choose an output $q_i \geq 0$ for $i = 1, 2, \dots, N$. These choices are made simultaneously.

Now we must write down the firms' profits. Each firm has the marginal cost $c$, or total costs $cq_i$. The final ingredient we need is the demand function and we will assume that $Q$ units can be sold if the price satisfies $Q = D - p$ (so the total amount demanded is always less than $D$). Of course there are many firms so the price is a function of the total output

$$p = D - (q_1 + q_2 + \cdots + q_N) = D - \sum_{i=1}^{N} q_i = D - q_i - \sum_{j \neq i} q_j.$$

This writes the relationship between prices and output in three identical way. For our purposes the last is the most useful. It says price equals $D$ minus firm $i$'s output and the output of all the other firms. It will be useful to write $Q_i = \sum_{j \neq i} q_j$ as the output of all firms apart from firm $i$.

Now we are in a position to write down firm $i$'s profit.

$$\pi_i(q_i, Q_i) = pq_i - cq_i = (D - Q_i - q_i)q_i - cq_i.$$

To find Firm $i$'s best response we want to maximise this by choosing the best $q_i$. So, as above, we differentiate and set equal to zero.

$$\boxed{\frac{\partial \pi_i}{\partial q_i} = D - Q_i - 2q_i - c = 0.}$$

Setting this equal to zero gives $q_i^* = (N - Q_i - c)/2$. If we do this for each firm we will get $N$ equations in $N$ unknowns:

$$q_1^* = (D - Q_1 - c)/2, \qquad q_2^* = (D - Q_2 - c)/2, \qquad \dots \qquad q_N^* = (D - Q_N - c)/2$$

Other firm's output

This can be solved directly, but it is much easier to use the trick we talked about in the earlier section. Here all the firms are the same and have the same costs. We would expect that the Nash equilibrium had all the firms producing the same output $q_1^* = q_2^* = \cdots = q_N^*$.

Symmetry

So let us substitute this into Firm 1's reaction function. With this assumption, the output of all firms apart from firm 1 can $Q_1 = \sum_{i=2}^{N} q_i^* = (N-1)q_1^*$ can be calculated. Hence we get

$$q_1^* = (D - Q_1 - c)/2 = (D - (N-1)q_1^* - c)/2.$$

This solves for $q_1^*$ to give

$$\boxed{q_i^* = \frac{D-c}{N+1}.}$$

Hence as $N$ increases each firm's output tends to 0, there are more and more firms and each becomes vanishingly small relative to the size of the market. We would hope this would look something like perfect competition, and we can check this by finding out how the price behaves as $N$ increases. The price will satisfy

$$p^* = D - \sum q_i^* = D - N\frac{D-c}{N+1} = \frac{Nc + D}{N+1} \to c \quad \text{as} \quad N \to \infty.$$

Here the price tends to Marginal Cost as the number of firms grows. Thus Cournot competition starts to look a lot like perfect competition when there are a lot of firms.

Bertrand Competition
(Price)

## 2. Bertrand Competition

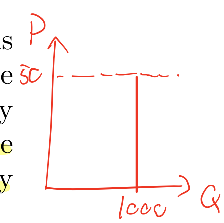
This model of competition is similar to Cournot's in that it assumes that firms are producing exactly the same product. However, it assumes that firms choose the price of their product (not how much to produce). This model of price competition is a good description of what goes on in financial markets or of the price wars that many industries experience from time to time. One interpretation of this is that it describes what happens in the short run, rather than the long run.

To be precise there are three assumptions in Bertrand competition:

(1) The lower-priced firm always claims the entire market. (Firms produce identical products.)
(2) All competition is in prices.
(3) If the firms set equal prices they will share the market.

Again it is easiest to understand Bertrand competition by thinking about an example.

**2.1. An Example of Bertrand Competition.** There are two ferry companies (it is more usual to consider airlines but I am quite find of boats). They serve the same route offering identical service, the only difference between the two companies is the price they charge. The cost per customer is 30 for both companies. There are 1000 customers who are willing to pay up to 50 to make the crossing. Company 1 charges a price $P_1$ and company 2 charges $P_2$.




First let us write down the profits of Firm 1. If it is the low price firm it gets all 1000 customers and has profit

$$\pi_1 = 1000(P_1 - 30), \qquad P_1 < P_2.$$

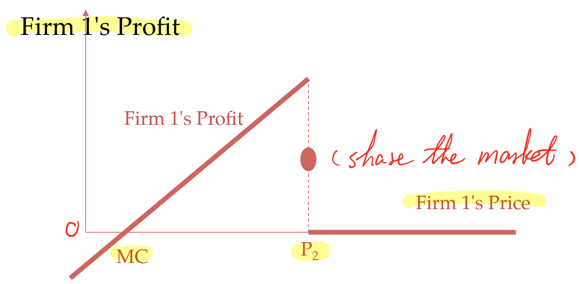
If it has the same price as Firm 2 it gets half of 1000 customers and has profit

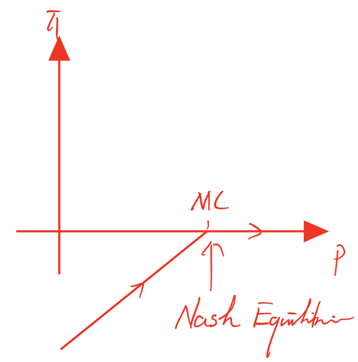
$$\pi_1 = 500(P_1 - 30), \qquad P_1 = P_2.$$

If it has a higher price than Firm 2 it gets no customers and has profit

$$\pi_1 = 0(P_1 - 30) = 0, \qquad P_1 > P_2.$$

Here is a picture of this function (remember $MC = 30$).




The most important thing we notice from this is a problem. Firm 1's profit is highest if it just undercuts firm 2's price. But it also wants to charge as high a price as possible. Thus what Firm 1 would like to do is to charge the highest price just below firm 1's price, *but no such price exists.* Thus, in this game (with continuous prices) the best response function of firm 1 does not exist! No BRF

The question then is: can we find a Nash equilibrium if we cannot find where the best responses intersect? Well the answer is yes, by telling a story. Suppose, the firms set equal prices of 50 and share the market (which is clearly a good situation). The profit of each firm is $500(50 - 30) = 10,000$. If firm 1 cut its price to 49, then it can attract all the customers and make profit $19,000 = 1000(49 - 30)$. In response firm 2 can undercut firm 1 by asking the price 48 and attracting the whole market. This process continues until both firms are charging the price 30 and making zero profit. At this point no firm benefits by undercutting their rival and the firms do actually have a best response. *Summary: The Nash equilibrium is at* $(P_1, P_2) = \underbrace{(30, 30)}_{MC,\, MC}$, *although there is no best response function.*

A cycle of firms undercutting each other

2.2. **General Properties of Bertrand Competition.** Here is a list of points that should be noted:

Continuous Prices

- Pure price competition drives oligopoly to look very much like perfect competition. The Nash equilibrium of the game has the firms setting prices equal to marginal costs. This is why we believe markets with price competition such as financial markets may be quite efficient.
- Price competition does not expand the market in the above example—the demand was always 1000 for low prices. As firms cut prices, one would usually expect more customers to want to buy the good. This expansion of the size of the market will increase the temptation of the firms to undercut their rival's price.
- When prices are not allowed to vary continuously the problem of the non-existence of a best response goes away (see the next section).

2.3. **Extensions to the Basic Model of Bertrand Competition.**

Price competition : SR

Quantity competition : LR

Discrete Prices

2.3.1. *Integer Prices.* A lot of the problems with Bertrand competition came from the fact that firms were allowed to continuously vary their prices. If we assume firms are only able to charge whole number prices then there is no problem with finding the biggest price less than some number. It also increases the number of Nash equilibria of the game. Here are the firms profits for the prices 30, 31, 32, 33, in the model of ferry competition above.

| | $P_2 = 30$ | $P_2 = 31$ | $P_2 = 32$ | $P_2 = 33$ | |
|---|---|---|---|---|---|
| $P_1 = 30$ | (0,0) | (0,0) | (0,0) | (0,0) | ... |
| $P_1 = 31$ | (0,0) | (500,500) | (1000,0) | (1000,0) | ... |
| $P_1 = 32$ | (0,0) | (0,1000) | (1000,1000) | (2000,0) | ... |
| $P_1 = 33$ | (0,0) | (0,1000) | (0,2000) | (1500,1500) | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |

If you apply the underlining method to this game you will find that it has 3 Nash equilibria: $(P_1, P_2) = (30, 30)$, $(P_1, P_2) = (31, 31)$, $(P_1, P_2) = (32, 32)$. This increase in the number of Nash equilibria arises because now you really have to make a big change in your price if you are going to undercut your rival. This big change in the price might hurt your profits more than the increase in customers you experience from being the low-price firm.

Price Guarantees

2.3.2. *Price Guarantees.* If firms advertise deals like "if you find this good cheaper anywhere else we will refund twice the difference" then there is a strange effect. Suppose the firms set prices $(40, 41)$ where would the customers go? If they go to the firm with the sticker price of 40, that is what they will pay. If they go to the firm with the sticker price of 41 they can claim a refund of twice the difference in prices and in fact pay only 39. Thus all the customers would prefer to go to the higher price firm and claim a refund. The sensible response of the firm setting the low price (40) is to *raise* its sticker price, so the customers come to it and claim the refund. If such guarantees are in place we would expect prices to increase rather than decrease. These guarantees are collusive, although they look like they are good for the consumer.

Capacity Constraints / Edgeworth Cycles

2.3.3. *Capacity Constraints.* If firms are unable to fit all 1000 customers on their ferry it never makes sense to cut prices down to Marginal Cost. Instead prices seem to go around in circles. These circles are called Edgeworth cycles.

Consider the following slight change in our example. Suppose that only 800 customers will fit on a ferry. So that when you are the low-price firm your profits are $800(P - 30)$ and when you are the high-price firm your profits are $200(P - 30)$. The high price firm always gets to serve those who cannot get on the low-price ferry. Suppose now that firm 1 sets the price of 35. Firm 2 can undercut this and set the price of (say) 34 making the profit $800(34 - 30) = 3200$. Or firm 2 can embrace being the high-price firm and charge the price of 50 and make the profit $200(50 - 30) = 4000$. Clearly the best response is to set the price $P_2 = 50$. Now (of course) firm 1 will respond by raising prices to $P_1 = 49$ and competition will drive prices back down again to 35 where the cycle begins again.

## 3. Differentiated Product Duopoly

If firms are not producing identical products, then we say their products are differentiated. When products are differentiated it will not be the case that the low-price firm gets all the market, because some consumers just prefer the product of the high-price firm. However, if products are substitutes we would expect their prices to have an effect of the demand for each other. Thus the firms will be playing a game in the choice of price they make. This is what we study here. Again we will do this by working through an example.

Price Competition with Differentiated Products

↓

BRF Method

3.1. **Example of Price Competition with Differentiated Products.** There are two firms producing different goods. $P_1$ is the price chosen by Firm 1 and $P_2$ is the price chosen by Firm 2. We will first describe the demand functions of each firm.

$$\text{Firm 1}: \quad Q_1 = 12 - 2P_1 + P_2$$
$$\text{Firm 2}: \quad Q_2 = 12 + P_1 - 2P_2$$

Notice that each firm's demand is decreasing in its own price but increasing in its rival's price—they are producing substitute goods and are in competition with each other. We will ignore costs here and suppose that all the costs are fixed costs. Fixed Costs = 20.

The first step in defining this game is to write down the Firms' actions. These are the prices they choose $(P_1, P_2)$. The next step is to write down the Firms' payoffs or profits.

$$\text{Firm 1's Profit} := P_1Q_1 - \text{Cost} = P_1(12 - 2P_1 + P_2) - 20$$
$$\text{Firm 2's Profit} := P_2Q_2 - \text{Cost} = P_2(12 - 2P_2 + P_1) - 20$$

This completes the formal description of the game that is being played here.

Now we must find a Nash equilibrium of this game and again we will use the Reaction Function method. We begin by finding the best $P_1$ for firm 1 given it knows the price of firm 2. To do this we maximize firm 1's profit by differentiating and setting equal to zero.

$$\frac{d\pi_1}{dP_1} = 12?4P_1 + P2 = 0$$

This solves to give

$$P_1 = 3 + \frac{1}{4}P_2.$$

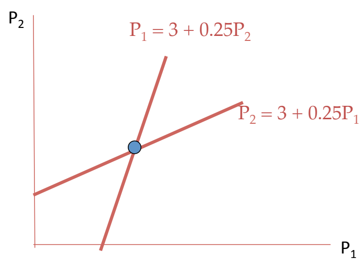
A similar process for firm 2 will also find its optimal choice of $P_2$ as a function of $P_1$:

$$P_2 = 3 + \frac{1}{4}P_1.$$

Note here these reaction functions are upward sloping. As my rivals price goes up it is optimal for me to increase my price too. (I can still undercut by raising my price and an increase in my rivals price has a positive effect on my demand.) To find the Nash equilibrium $(P_1^*, P_2^*)$ we must substitute one reaction function into the other.

$$P_1 = 3 + \frac{1}{4}(3 + \frac{1}{4}P_1).$$

This solves to give $(P_1^*, P_2^*) = (4, 4)$. Here is a picture of what we have just done.




Single Unit Auction

## 4. A Model of a Single-Unit Auction

All the previous models were games where a few sellers compete to sell to a set of buyers. Now we will look at the reverse position where a few buyers compete to acquire a good from a single seller.

First we will describe the buyers. The buyers? values for the good are written as $(v_1, v_2, \ldots, v_N)$ where buyer $I$'s value, $v_i$ is in the interval $0 \leq v_i \leq 1$. We will suppose these values are random and that the buyers only know their value but not the value of the others. The seller does not know the values $(v_1, v_2, \ldots, v_N)$ and so has a random or unknown demand curve.

How the $(v_1, v_2, \ldots, v_N)$ are determined has a big effect on the nature of competition that among the buyers. Here are some different assumptions that might be made:

- **Independent Symmetric Private Values:** $v_i$ is drawn independently from the density $f(v)$ on $[0, 1]$. All buyers have the same distribution of values for the good.
- **Independent Private Values:** $v_i$ is drawn independently from the density $f_i(v)$ on $[0, 1]$. Buyers have distinct views about the good.

Here is the order of events in the auction

(1) The players observe their own values and no-one else?s.
(2) Then they submit a bid.
(3) The rules of the auction determines payoffs.

Here a strategy for a buyer is to describe how they should bid for each different value they observe. Thus players' strategies are bidding functions that takes the player's value and maps it to a bid.

Infinite Pure Strategies (Functions not Bids)

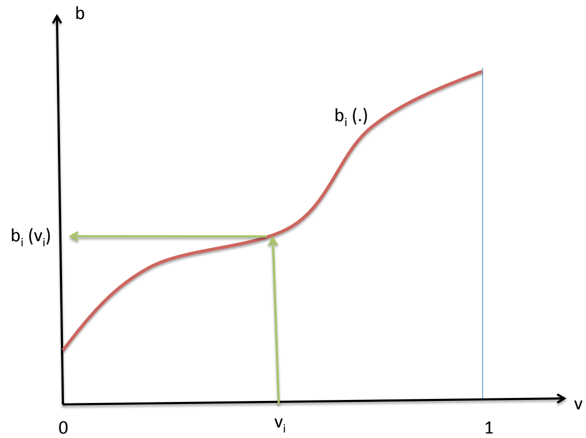
$$b_i : \text{Observed Value} \to \text{Bid}$$
$$b_i : [0, 1] \to [0, \infty)$$
$$v_i \mapsto b_i(v_i)$$

Here is a picture of the bidding function

"Bayes-Nash Equilibrium"

"Don't know other's action, but have beliefs"

Bid




(Just 1 example)

Value

We have to find a whole function to describe a player's equilibrium strategy. To make this easier we will only look for certain kinds of equilibria. That is, equilibria where bids are strictly increasing functions of values.

Second-Price Auctions

**4.1. Equilibrium in Second-Price Auctions.** The rules of the auction determine who wins and who pays what. The easiest and simplest set of auction rules to analyse are second-price auctions. These are auctions where the person submitting the highest bid gets the object, but they pay a price equal to the second highest bid (not their own highest bid). In an auction where the price paid is the second highest bid, the strategy $b_i(v_i) = v_i$ (that is submit a bid equal to your value), weakly dominates all other strategies. This is a famous result due to Vickrey and is much used in Economics.

We now explain why this is true.

(1) We first consider the possibility of bidding above your value. That is, overstating how you feel about the good. Suppose you have a value $v_i$ and consider your payoffs for a bid $b' > v_i$. Your payoff only depends on the highest bid from the other players call this $B$. We will deal separately with the cases where: (1) the highest bid from everyone else is above your new contemplated bid $B > b'$. (In which case you always lose the auction whether you bid truthfully or exaggerate to $b'$). (2) The highest bid from everyone else falls between your value and your increased bid $b' > B > v_i$. (In which case bidding truthfully causes you to lose the auction and get zero while bidding $b'$ causes you to win the auction and pay $B > v_i$ so you get a negative payoff.) (3) The highest bid from everyone else is below your value $v_i > B$. (In which case you win the object whether you bid $v_i$ or $b'$ and the price you pay $B$ is independent of your bid. This gives the matrix of payoffs below. You can see that the top row is weakly dominated by the bottom row.

$v_i \leq B \leq b'$

| | $B > b'$ | $B \in (v_i, b')$ | $B < v_i$ |
|---|---|---|---|
| Bid $b'$ | Lose $= 0$ | Win $= v_i - B < 0$ | Win $= v_i - B > 0$ |
| Bid $v_i$ | Lose $= 0$ | Lose $= 0$ | Win $= v_i - B > 0$ |

( $b' > v_i$ )

(2) Now we consider the possibility of bidding below your value. That is, understating how you feel about the good. Suppose you have a value $v_i$ and consider your payoffs for a bid

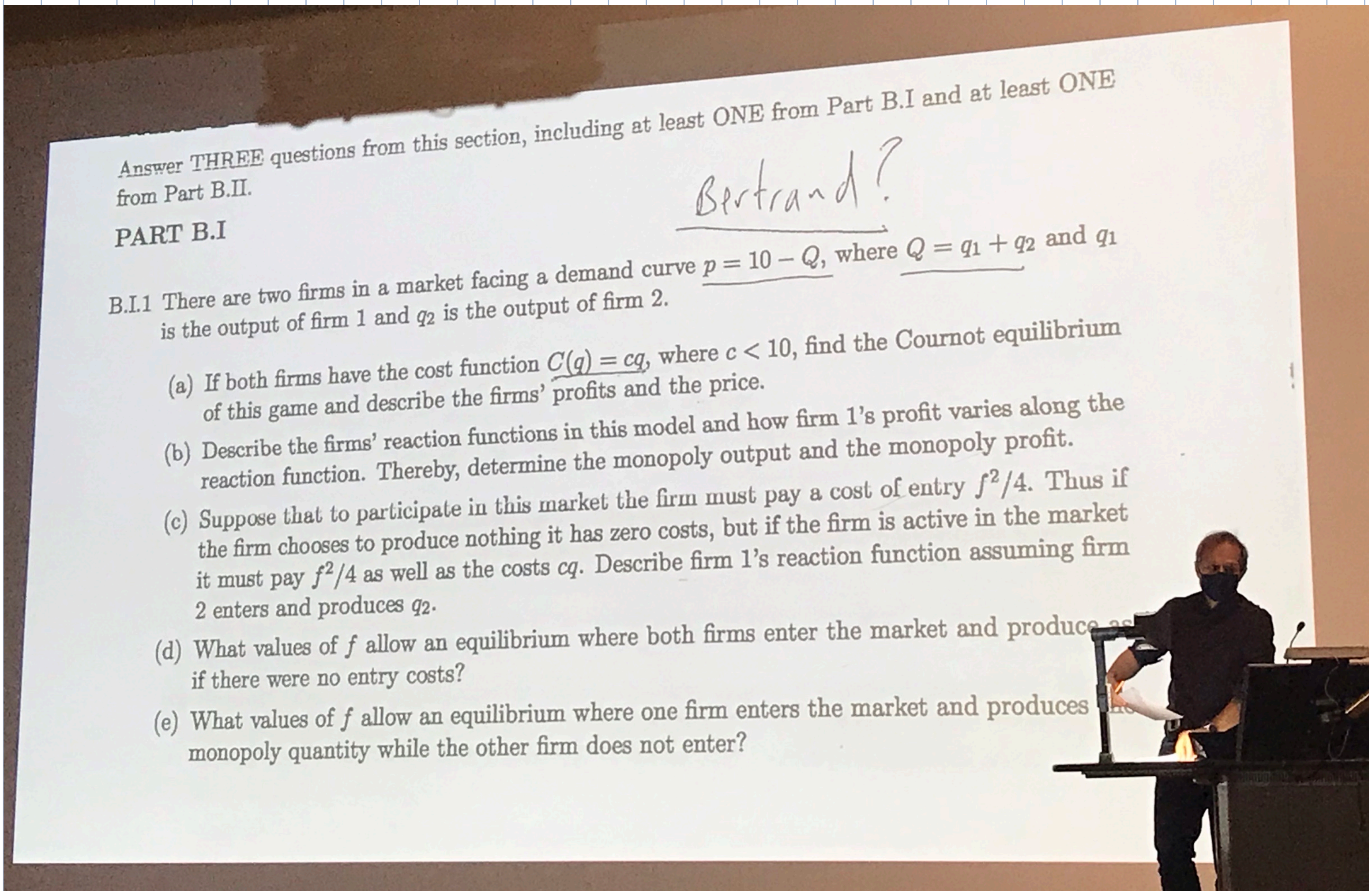
Still negative payoff

$b'' < v_i$. We will deal separately with the cases where: (1) the highest bid from everyone else is above your value $B > v_i$. (In which case you always lose the auction whether you bid truthfully or understate your bid). (2) The highest bid from everyone else falls between your value and your understated bid $b'' < B < v_i$. (In which case bidding truthfully causes you to win the auction and pay a price $B < v_i$ giving positive profit while bidding $b''$ causes you to lose the auction and get nothing.) (3) The highest bid from everyone else is below your understated bid $b'' > B$. (In which case you win the object whether you bid $v_i$ or $b''$ and the price you pay $B$ is independent of your bid. This gives the matrix of payoffs below. Again you can see that the top row is weakly dominated by the bottom row.

| | | $B > v_i$ | $B \in (b'', v_i,)$ | $B < b''$ |
|---|---|---|---|---|
| $(b'' < v_i)$ | Bid $b''$ | Lose $= 0$ | Lose $= 0$ | Win $= v_i - B > 0$ |
| | Bid $v_i$ | Lose $= 0$ | Win $= v_i - B > 0$ | Win $= v_i - B > 0$ |

Hence we can conclude: *it is a weakly dominating strategy to bid truthfully in a second price auction*.

After the auction, the seller knows values of all bidders.



Answer THREE questions from this section, including at least ONE from Part B.I and at least ONE from Part B.II.

Bertrand?

**PART B.I**

B.I.1 There are two firms in a market facing a demand curve $p = 10 - Q$, where $Q = q_1 + q_2$ and $q_1$ is the output of firm 1 and $q_2$ is the output of firm 2.

(a) If both firms have the cost function $C(q) = cq$, where $c < 10$, find the Cournot equilibrium of this game and describe the firms' profits and the price.

(b) Describe the firms' reaction functions in this model and how firm 1's profit varies along the reaction function. Thereby, determine the monopoly output and the monopoly profit.

(c) Suppose that to participate in this market the firm must pay a cost of entry $f^2/4$. Thus if the firm chooses to produce nothing it has zero costs, but if the firm is active in the market it must pay $f^2/4$ as well as the costs $cq$. Describe firm 1's reaction function assuming firm 2 enters and produces $q_2$.

(d) What values of $f$ allow an equilibrium where both firms enter the market and produce as if there were no entry costs?

(e) What values of $f$ allow an equilibrium where one firm enters the market and produces the monopoly quantity while the other firm does not enter?

a) $\pi_1 = pq_1 - cq_1 = (10 - q_1 - q_2)q_1 - cq_1$

$\pi_2 = pq_2 - cq_2 = (10 - q_1 - q_2)q_2 - cq_2$

$\frac{d\pi_1}{dq_1} = 10 - 2q_1 - q_2 - c = 0$

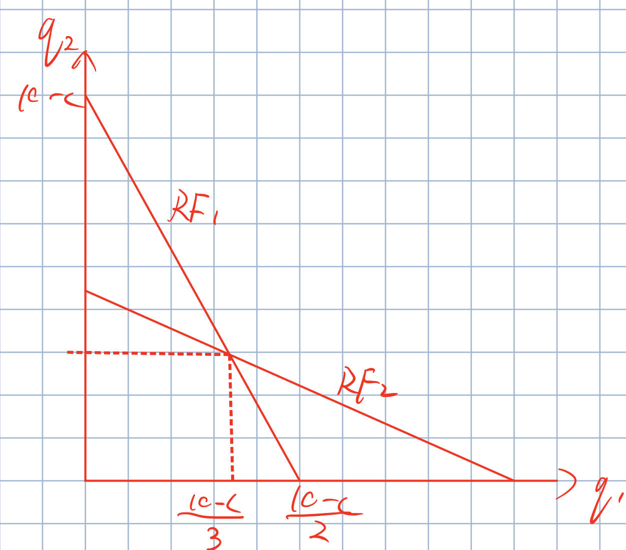
$RF_1 \quad q_1^* = (10 - c - q_2)/2$

$RF_2 \quad q_2^* = (10 - c - q_1)/2$

Symmetry: $q_1^* = q_2^*$ at Cournot Equilibrium

$(10 - c - q^*)/2 = q^*$

$q^* = (10 - c)/3$




$\pi_1 = q_1(10 - q_1 - q_2 - c)$

$\pi_1^* = \frac{10-c}{3} \times (10 - \frac{10-c}{3} \times 2 - c)$

$$= \frac{(10-c)^2}{3} \times \left(1 - \frac{2}{3}\right)$$

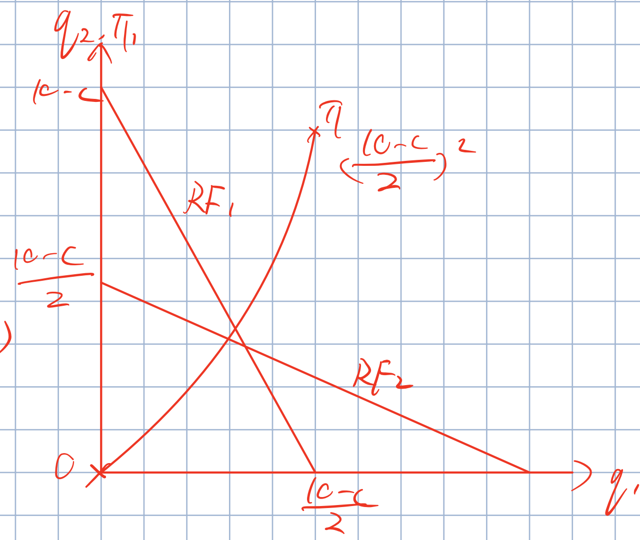
$$= \left(\frac{10-c}{3}\right)^2$$

b) $\pi_1 = q_1(10 - c - q_1 - q_2)$

RF: $q_1^* = (10 - c - q_2)/2$

$$\hat{\pi}_1 = \left(\frac{10-c-q_2}{2}\right)\left(10 - c - q_2 - \frac{10-c-q_2}{2}\right)$$

$$= \left(\frac{10-c-q_2}{2}\right)^2$$

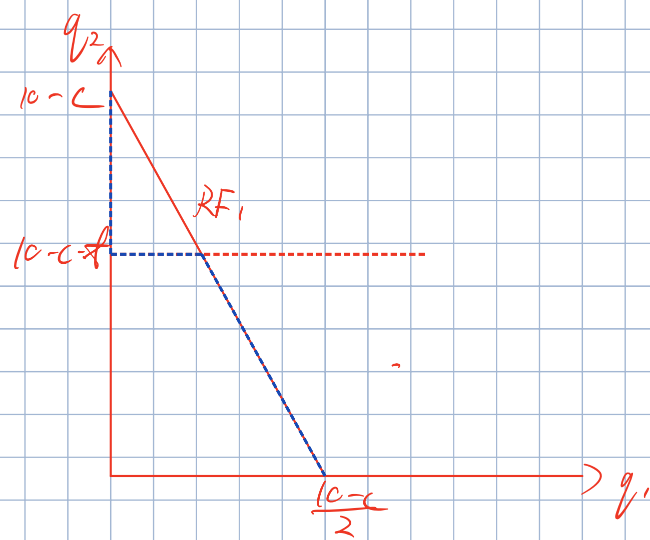



c) Entry cost: $\frac{f^2}{4}$

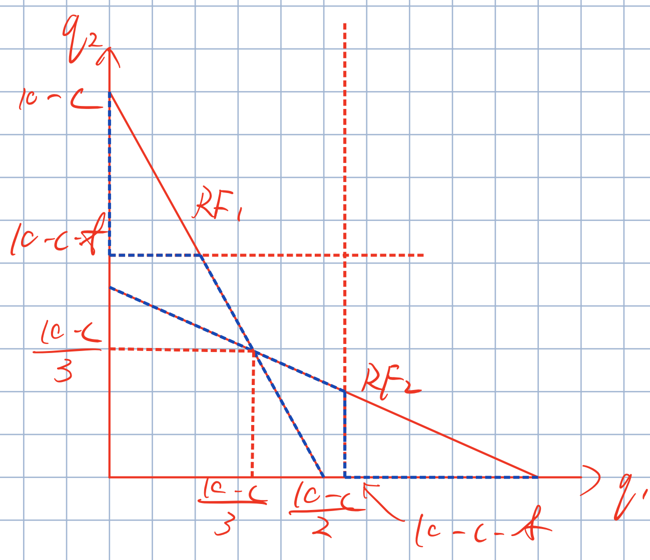
$$\pi_1 = \left(\frac{10-c-q_2}{2}\right)^2 - \frac{f^2}{4}$$

Entry point: $\left(\frac{10-c-q_2}{2}\right)^2 = \frac{f^2}{4}$

$$10 - c - f = q_2$$




d)




$$10 - c - f > \frac{10-c}{3}$$

# 5. DYNAMIC OLIGOPOLY

MARTIN CRIPPS

## 1. Introduction

In the models of oligopoly we studied in the previous section we imagined that the firms were moving simultaneously, either choosing quantities (Cournot) or prices (Bertrand) at exactly the same time. Thus the firms were playing a game in *strategic form*. The right way to study games in strategic form is to use the tools of Nash equilibrium and dominance. So this is how we modelled competition in these games.

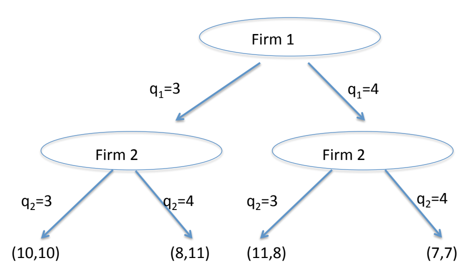
Now we are going to study games where there is an order to the moves. In the first two sections we will study competition in the case where one firm moves first and then the other firm moves. In the final section the firms will move at the same time but there are many periods where they can make different moves. Thus we are considering games in extensive form. In games like these we know Nash equilibrium is inadequate, as it may have non-credible threats. So the way we model competition now will be to use either backwards induction or subgame perfection.

Stackelberg Competition

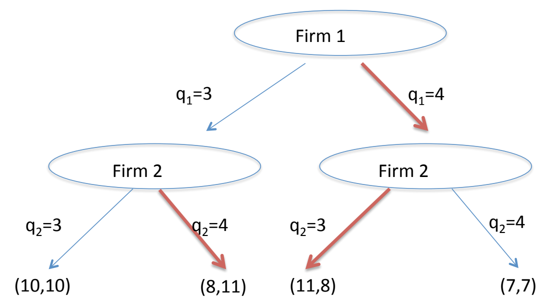
## 2. Stackelberg Competition

Stackelberg criticised Cournot's way of modelling quantity competition, because he believed industries always had one firm that got to dominate the competition in a particular way. In particular he thought there was a specific order in which the firms got to make their quantity choices, there was a leader and then other firms followed, and that this order affected the quantity choices the firms in the industry made.

This can obviously be quite a complicated situation to describe as a game tree, if there are many firms in an industry. But in its simplest form, with only two firms, we could imagine that first firm 1 (the leader) chooses its quantity, then firm 2 (the follower) chooses its quantity, then the price is determined, and finally the firms' profits are realised. Here is the simplest possible version of this in game-tree form.



In this game Firm 1 chooses a quantity $q_1 = 3$ or $q_1 = 4$, then firm 2 sees the quantity chosen by the leader and chooses a quantity $q_2 = 3$ or $q_2 = 4$ themselves. How does this change the outcome of competition? Well it depends on what kind of equilibrium we look for. Stackelberg said we should solve this game using backwards induction (he introduced this idea long before game theorists took it up). Here is the backwards-induction solution to the game.




First look at what firm 2 (the follower) does. It takes Firm 1's output as given and chooses the profit-maximizing output given Firm 1's choice. *Firm 2 is choosing its best response to every output choice of Firm 1.* Thus, in the terminology of reaction functions, we would say that whatever Firm 1 does Firm 2 is always making a choice on its reaction function.

The consequence of this structure is the leader has an advantage, because it can produce a lot and force the follower to cut back its own output. Thus, in contrast to the Cournot equilibrium the firms now behave in different ways although they have the same costs and demand.

Now we will generalise these ideas to the firms we studied in Cournot competition.

2.1. **Stackelberg Competition with Continuous Quantity Choices.** We will now revisit our previous model of oligopoly where demand is $P = 30 - Q_1 - Q_2$ and firms have the profits:

$$\pi_1 = PQ_1 - 6Q_1 = (30 - Q_1 - Q_2)Q_1 - 6Q_1;$$
$$\pi_2 = PQ_2 - 6Q_2 = (30 - Q_1 - Q_2)Q_2 - 6Q_2.$$

Also recall the reaction functions of the two firms which are:

$$Q_1 = 12 - \frac{1}{2}Q_2;$$
$$Q_2 = 12 - \frac{1}{2}Q_1.$$

To find the Stackelberg equilibrium we would like to solve this game by backwards induction, first solving for the follower's (Firm 2's) optimal output and then finding the leader's (Firm 1's) optimal output. But drawing out the game tree will be impossible as there are so many $Q_1$'s to consider. How can we do backwards induction without the picture? Here's how...

2.1.1. *Follower's Behaviour.* If we think about the previous example, what the follower did was to choose the best $Q_2$ given the observed $Q_1$. This is exactly what the reaction function of player 2 does. It describes the best possible $Q_2$ for each $Q_1$. So, the last stage of the backwards induction is precisely described by the function.

$$Q_2 = 12 - \frac{1}{2}Q_1 \qquad \text{Final Stage of Backwards Induction}$$

2.1.2. *Leader's Behaviour.* The leader knows how the follower will respond to any $Q_1$ they choose. If they choose $Q_1 = 3$ the follower will choose $Q_2 = 12 - 3/2 = 10.5$, because this is their optimal response as determined by their reaction function. How should the leader choose $Q_1$ then? Well it should choose $Q_1$ to maximise its profit taking into account the fact that $Q_2$ is constrained to equal $12 - Q_1/2$. That is, the leader solves the problem:

$$\max_{Q_1}(30 - Q_1 - Q_2)Q_1 - 6Q_1, \qquad \text{subject to } Q_2 = 12 - \frac{1}{2}Q_1.$$

This constrained optimization can be solved by substitution:

$$\max_{Q_1}\left(30 - Q_1 - \left[12 - \frac{1}{2}Q_1\right]\right)Q_1 - 6Q_1.$$

Simplifying this gives:

$$\max_{Q_1}\left(30 - Q_1 - \left[12 - \frac{1}{2}Q_1\right]\right)Q_1 - 6Q_1 = \max_{Q_1} 12Q_1 - \frac{1}{2}Q_1^2.$$

Differentiating and setting equal to zero gives:

$$Q_1 = 12.$$

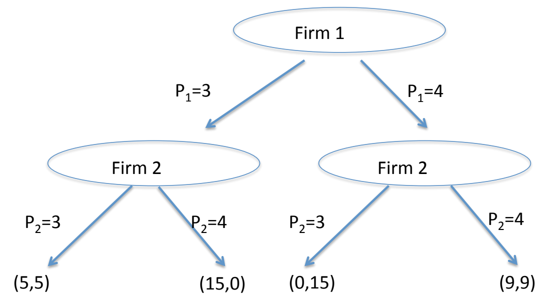
We can substitute this back into the reaction function of Firm 2 to get $Q_2 = 6$.

Let us compare this situation with that when the firms move simultaneously (under Cournot competition). If there is Cournot competition the firms will both produce output 8 ($Q_1 = Q_2 = 8$), so the total supply is 16 and the price is 14. If there is Stackelberg competition the outputs are $Q_1 = 12$ and $Q_2 = 6$, so the total supply is 18 and the price is 12. This is quite generally true. Under Stackelberg competition the leader produces more the follower produces less and the leader makes more profit than under Cournot competition. Thus being a quantity leader confers considerable benefits on the leader and harms the followers.

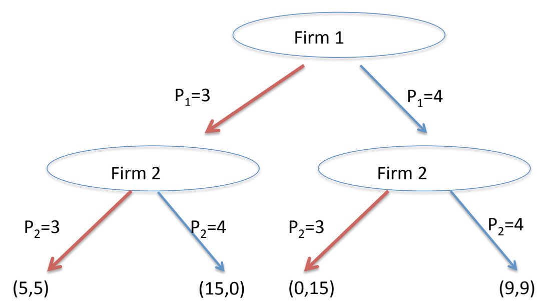
Price Leadership

## 3. Price Leadership

Just as we can think of situations where the firms set quantities in a definite order so we can also imagine situations where firms set prices in an order too. Let us also imagine that Firm 1 is the price leader and Firm 2 is the price follower. Here the effects of the order are entirely different—being a leader in pricing harms the leader and benefits the follower. In its starkest form suppose the firms are producing identical products, then if Firm 1 goes first and sets price, then Firm 2 can undercut this price and take all the customers. Here's a simple example of this.




Here the second mover has and advantage and this is transparent when we consider the backwards-induction solution.




Now let us do price leadership in the case when the firms are *not* producing identical products. Recall our model of price competition with differentiated products where as the goods are not identical, demand does not jump to zero for the highest price firm

$$\text{Firm1 Demand} \qquad Q_1 = 12 - 2P_1 + P_2,$$
$$\text{Firm2 Demand} \qquad Q_2 = 12 + P_1 - 2P_2.$$

This gave rise to the (Price) reaction functions

$$P_1 = 3 + \frac{1}{4}P_2,$$
$$P_2 = 3 + \frac{1}{4}P_1.$$

and a Nash equilibrium in the simultaneous move game where $P_1 = P_2 = 4$.

3.0.3. *Follower's Behaviour.* Just as in the quantity-setting game the price-follower (Firm 2) will choose their price to be an optimal response to what the leader does. Thus Firm 2 will see $P_1$ and then choose

$$P_2 = 3 + \frac{1}{4}P_1.$$

3.0.4. *Leader's Behaviour.* The leader wishes to maximise their profit by choosing the right $P_1$. In the previous section we noted that the leader's profit was equal to

$$\pi_1 = P_1 Q_1 - 20 = 12P_1 - 2P_1^2 + P_1 P_2 - 20.$$

However,the leader is constrained because whatever $P_1$ they chose will lead to the choice of a $P_2 = 3 + 0.25P_1$ by the follower. Thus the leader solves the problem:

$$\max_{P_1} 12P_1 - 2P_1^2 + P_1 P_2 - 20, \qquad \text{subject to:} \qquad P_2 = 3 + \frac{1}{4}P_1.$$

This can be solved by substitution:

$$\max_{P_1} 12P_1 - 2P_1^2 + P_1\left(3 + \frac{1}{4}P_1\right) - 20 = 15P_1 - 1.75P_1^2 - 20.$$

Differentiating this and setting this equal to zero to find the maximum gives

$$0 = 15 - 3.5P_1 \qquad \text{or} \qquad P_1 = 30/7 = 4 + \frac{2}{7}.$$

Thus the price leader chooses to set a higher price than 4 (the equilibrium in the simultaneous move game). The price follower sets the price

$$P_2 = 3 + \frac{1}{4}\frac{30}{7} = 4 + \frac{1}{14}$$

Better to be a quantity leader rather then a price leader.

and undercuts the leader. They do well at this lower price.Their demand is higher than at 4 because the leader has raised their price and they the follower are undercutting them.

Cooperation Through Repetition

## 4. Cooperation through Repetition

We have seen that competition on prices will typically drive an industry to low profits. This is reflected in the strategic form of a (static/one-shot) pricing game below. In this game it is strictly dominant for each firm to choose a low price although they would prefer to be able to coordinate on high prices.

| | High Price | Low Price |
|---|---|---|
| High Price | (10,10) | (5,12) |
| Low Price | (12,5) | (7,7) |

Now we use this one shot game as a building block in a dynamic or repeated pricing game that has no end at all. We suppose the pricing game above is played again and again over infinitely many periods t=0,1,2,.... In each period the firms can choose a price and the price they choose may depend upon what has happened in the past.

This repeated game structure presents a number of technical problems. First, it is difficult to draw the game tree even if the players only have a couple of actions in each period, because there are so many periods. Second, the players have so many strategies it is difficult to find and check that we have a Nash Equilibrium or a Subgame Perfect Equilibrium. Finally, we will also need to find a way of adding up the profits the firms make over infinitely many periods of play to determine their profits for the entire game.

To add up the profits over many periods we will use the rate of interest $r$ to calculate the firm's *discounted present value* of the profit it earns every period. For example, if the firm earns 7 units of profit every period of the game we will say its payoff from playing the entire repeated game is

$$= 7 + \frac{7}{(1+r)} + \frac{7}{(1+r)^2} + \frac{7}{(1+r)^3} + \dots$$

(Recall that $r > 0$ is an interest rate.) We can use what we know about geometric series to work out this sum.

$$\begin{aligned}
\text{Payoff} &= 7 + \frac{7}{(1+r)} + \frac{7}{(1+r)^2} + \frac{7}{(1+r)^3} + \dots \\
&= 7\left(1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \dots\right) \\
&= 7\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \\
&= 7\frac{1}{1 - \frac{1}{(1+r)}} \\
&= 7\frac{1+r}{r}
\end{aligned}$$

Tis repeated game has many equilibria. Now we will give an example of one equilibrium which is particularly beneficial to the firms. When the play out the equilibrium the firms coordinate on the high price and only threaten to play a low price if one player cheats on this. We will assume they both play the strategy:

> *"Set a high price if my opponent has set the high price in every past period, but play the low price in all other cases."*

Such a strategy promotes cooperation by making something very bad happen if a player cheats on the cooperation. (The very bad thing is permanently low prices.) These strategies are usually called "Grim Trigger" strategies by game theorists.

To see why it is an equilibrium first notice that if I stick to playing the high price (and my opponent does) the previous calculation tells me I will have profits

$$\text{Payoff of Not Cheating} = \frac{10(1+r)}{r}.$$

If I cheat on high prices I get 12 for one period and then 7 forever, so in total I get

$$\begin{aligned}
\text{Payoff of Cheating} &= 12 + \frac{7}{(1+r)} + \frac{7}{(1+r)^2} + \frac{7}{(1+r)^3} + \dots \\
&= 12 + \frac{7}{(1+r)}\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \\
&= 12 + \frac{7}{r}
\end{aligned}$$

For it not to be optimal to cheat we need

$$\text{Payoff of Not Cheating} > \text{Payoff of Cheating}$$
$$\frac{10(1+r)}{r} > 12 + \frac{7}{r}$$
$$10(1+r) > 12r + 7$$
$$3 > 2r$$

This tells us that if $r < 1.5$ it is not optimal to cheat. That is, it is a Nash equilibrium to play out the strategy. To check that this strategy is also a subgame perfect equilibrium we also need to check that the players want to carry out the threat of always playing the low price. This is easy to check because playing the low price is the best thing to do if you believe your opponent will also play the low price today.

Index:

- Production
  - Basics
    - Marginal Rate of Transformation [MRT] Output $i$ -> Output $j$ | $F(y)=0$ Technical Effect

      $$MRT_{i,j} = -\frac{\partial F/\partial y_i}{\partial F/\partial y_j}$$

    - Marginal Product [MP] Input -> Output

      $$MP_i = \frac{\partial f}{\partial z_i}$$

    - Marginal Rate of Technical Substitution [MRTS] Input $i$ -> Input $j$ | $f(z)=q$

      $$MRTS_{i,j} = \frac{\partial f/\partial z_i}{\partial f/\partial z_j} = \frac{MP_i}{MP_j}$$

    - Elasticity of Substitution Rate of change of the slope of the isoquant

      $$\sigma_{i,j} = -\frac{\partial \ln\left(\frac{z_i}{z_j}\right)}{\partial \ln (MRTS_{i,j})}$$

    - Convexity of Input Requirement Set: Decreasing MRTS
    - Homotheticity: $f(z^A) = f(z^B) \rightarrow f(\lambda z^A) = f(\lambda z^B)$

      Homogenous: $f(\lambda z) = \lambda^{\alpha} f(z)$

    - Returns to Scale:

      DRS $\lambda f(z) > f(\lambda z)$; $C(\lambda q, w) > \lambda C(q,w)$ $\forall \lambda > 1$

      CRS $\lambda f(z) = f(\lambda z)$; $C(q,w) = q \times K(w)$ $\forall \lambda > 1$

      IRS $\lambda f(z) < f(\lambda z)$; $C(\lambda q, w) < \lambda C(q,w)$ $\forall \lambda > 1$

  - Profit Maximization
    - First Order Condition

      $$p\frac{\partial f}{\partial z_i} - w_i = 0 \rightarrow MP_i = w_i/p$$

    - Second Order Condition

      $$\frac{\partial^2 f}{\partial z_i^2} \leq 0 \quad \text{(Concave)}$$

    - With profit maximization: $MRTS_{i,j} = \frac{\partial f/\partial z_i}{\partial f/\partial z_j} = w_i/w_j$
    - Unconditional Demand Functions $D(p,w)$

      由解 1st Order Condition 得到

- Supply Function $S(p,w)$

  由$z^*$代入$f(z)$得到

- Characteristics:

  - Homogenous of degree 0 : $D(\lambda p, \lambda w) = D(p,w)$ $\quad S(\lambda p, \lambda w) = S(p,w)$
  - $$\begin{cases} p^A q^A - w^A z^A \geq p^A q^B - w^A z^B \\ p^B q^B - w^B z^B \geq p^B q^A - w^B z^A \end{cases}$$

    $$\begin{cases} p^A \Delta q - w^A \Delta z \geq 0 \\ p^B \Delta q - w^B \Delta z \leq 0 \end{cases} \qquad \Delta = {}^A - {}^B$$

    $$\Delta p \Delta q - \Delta w \Delta z \geq 0$$

    $$\Delta p \Delta q \geq 0 \geq \Delta w \Delta z$$

- Profit Function $\pi(p,w)$

  Homogenous of Degree 1 : $\pi(\lambda p, \lambda w) = \lambda \pi(p,w)$

- Hotelling's Lemma

  - $\frac{\partial \pi}{\partial p} = S(p,w)$
  - $\frac{\partial \pi}{\partial w_i} = -D_i(p,w)$

- Symmetry

  - $\frac{\partial D_i}{\partial w_j} = \frac{\partial D_j}{\partial w_i}$
  - $\frac{\partial D_i}{\partial p} = -\frac{\partial S}{\partial w_i}$

- Cost Minimization

  - Conditional Input Demands $H(q,w)$

    Min cost subject to $f(z) \geq q$

    - Homogenous of Degree 0 in $w$ : $H(q, \lambda w) = H(q,w)$
    - $w^A z^A \leq w^A z^B \quad w^B z^B \leq w^B z^A$

      $\Delta w \Delta z \leq 0$

  - Cost Function

$C(q, w) = w H(q, w)$ Homogenous of Degree 1 in $w$

- Shephard's Lemma
  - $\frac{\partial C}{\partial w_i} = H_i(q, w)$
  - $\frac{\partial H_i}{\partial w_j} = \frac{\partial H_j}{\partial w_i}$
- Relationship between Conditional / Unconditional Input Demands

$$D(p, w) = H(S(p, w), w)$$

$$\frac{\partial D_i(p, w)}{\partial w_i} = \frac{\partial H_i(S(p, w), w)}{\partial w_i} + \frac{\partial H_i(S(p, w), w)}{\partial q} \times \frac{\partial S(p, w)}{\partial w_i}$$

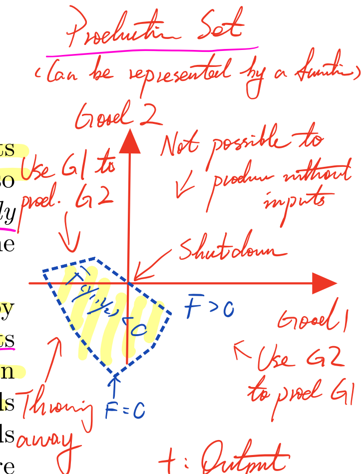
# ECON2001 Microeconomics Lecture Notes

### *Term 1: Production*

### Ian Preston

Production

## Technology




Suppose a firm produces a single output $q$ with inputs $z$. The constraints imposed by technology can be represented using a production function f(z) so that it is feasible to produce $q$ using $z$ if $q \leq f(z)$. Production is *technically efficient* if $q = f(z)$ so the greatest possible output is being produced given the inputs.

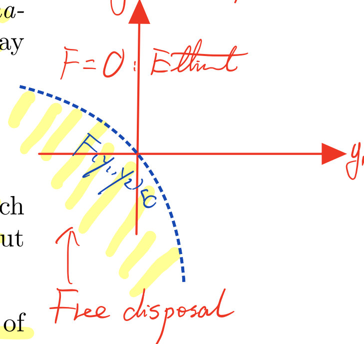
$F(\vec{q}, -\vec{z}) \leq 0$

Transformation Frontier

$F(\vec{q}, -\vec{z}) = 0$

More general than this is a formulation representing a firm's activities by a *production plan*, *production vector* or *net output vector*, $y$, in which outputs and inputs are not distinguished. Negative quantities in a particular production plan correspond to goods consumed (*inputs*) and positive quantities to goods produced (*outputs*). This allows for multiple outputs and also for certain goods to be either inputs or outputs depending on the production plan. In this more general setting we can represent technological feasibility with what is called a transformation function, $F(y)$, with feasible plans satisfying $F(y) \leq 0$. For the single output case, $y = (q, -z)$ and $F(q, -z) = q - f(z)$. The *transformation frontier* is the set of technically efficient production plans, which is to say satisfying $F(y) = 0$.

+: Output
−: Input




## Properties of production

- **Possibility of shutdown**: Shutdown is usually assumed possible which means that $0 \leq f(0)$ or, in the more general case, $F(0) \leq 0$. If no output can be produced without using some inputs then these are equalities.

- **Monotonicity**: If production is technically efficient then net output of one good can be increased only by decreasing net output of another. Output can only be increased by increasing some input. To produce the same output after decreasing one input requires increasing another. If there is only one output then the production functions must be increasing $\partial f/\partial z_i \geq 0$. In the more general formulation $\partial F/\partial y_i \geq 0$.

## Input requirement sets

Suppose there is only one output. Pick a particular level $q$. The input requirement set $Z(q)$ is the set of all input vectors capable of producing $q$. In other words, all $z$ such that $f(z) \geq q$ (or equivalently, $F(q, -z) \leq 0$). The boundary of the input requirement set is known as an *isoquant*. [$q = f(z)$]

Possibility of shutdown: $0 \leq f(0)$

# Marginal rates of transformation

The slopes of outputs with respect to inputs, of inputs with respect to inputs holding output constant and of outputs with respect to other outputs holding inputs constant can all be thought of in terms of rates at which one net output can be transformed into another while the firm remains on its transformation frontier. These are known as *marginal rates of transformation* or MRT. If we hold $F(y) = 0$ and differentiate with respect to any one output $y_i$ letting another $y_j$ change to maintain technical efficiency then

$F(y) = 0$

$\frac{dF}{dy_i}$ $$\boxed{\frac{\partial F}{\partial y_i} + \frac{\partial F}{\partial y_j} \left.\frac{\mathrm{d}y_j}{\mathrm{d}y_i}\right|_{F(y)=0} = 0}$$ $\frac{d0}{dy_i}$

Rate of transforming one net OUTPUT into another OUTPUT while remains on the transformation frontier.

Marginal Rates of Transformation [MRT]

Output$_i$ → Output$_j$

$$\boxed{\Rightarrow MRT_{ij} = \left.\frac{\mathrm{d}y_j}{\mathrm{d}y_i}\right|_{F(y)=0} = -\frac{\partial F/\partial y_i}{\partial F/\partial y_j}}$$

For example, if we assume only one output:

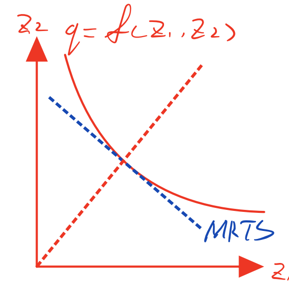
- The rate at which any input can be transformed into output is known as its *marginal product* and is equal to the slope of the production function

Marginal Product [MP]

Input → Output

$$\boxed{MP_i = \left.\frac{\mathrm{d}q}{\mathrm{d}z_i}\right|_{q=f(z)} = \frac{\partial f}{\partial z_i}}$$

(It may seem pedantic to note it but it may help to avoid confusion in later uses of the idea to note that since inputs are minus net outputs this is minus the marginal rate of transformation between net outputs in this case.)




- The rate at which any input has to be increased as we decrease another holding output constant is known as the *marginal rate of technical substitution* between the two and is equal to the slope of the isoquant

Marginal Rate of Technical Substitution [MRTS]

Input$_i$ → Input$_j$

$$\boxed{MRTS_{ij} = -\left.\frac{\mathrm{d}z_j}{\mathrm{d}z_i}\right|_{q=f(z)} = \frac{\partial f/\partial z_i}{\partial f/\partial z_j} = \frac{MP_i}{MP_j}}$$

(No negative sign)

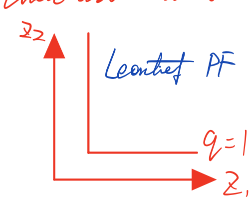
## Convexity of input requirement sets

MRTS changes slowly ⇒ Flexible

MRTS changes dramatically ⇒ Inflexible

(Think about this)

*Convexity* of the input requirement set $Z(q)$ means that convex combinations of input vectors $z^A$ and $z^B$ which produce the same output $q$ will also produce at least $q$. In terms of the slope of isoquants, convexity implies a decreasing MRTS.




The rate at which the MRTS changes as you move around an isoquant is an indication of its curvature and will prove to be important for the responsiveness of input choices to economic incentives. This is usually measured, for any two inputs, by the *elasticity of substitution*, defined as the elasticity of the input ratio $z_i/z_j$ to the MRTS between the two

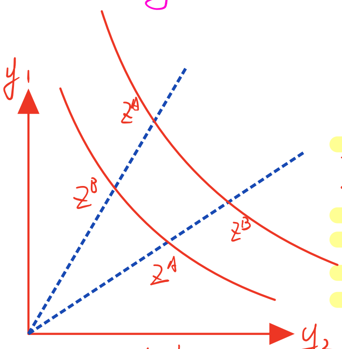
Elasticity of Substitution

$$\boxed{\sigma_{ij} = -\frac{\partial \ln (z_i/z_j)}{\partial \ln (MRTS_{ij})}}$$

= Rate of changes of Slope of Isoquants

Calculation: ① Calculate MRTS ② Expand ln MRTS ③ Rearrange ④ Differentiated by MRTS

## Homotheticity

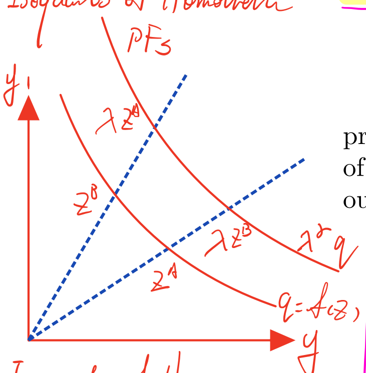



Isoquants of Homothetic PFs

Production is *homothetic* if, whenever input vectors $z^A$ and $z^B$ produce the same output, $f(z^A) = f(z^B)$, then scaling those input vectors up or down by the same factor will also give input vectors which produce the same output. In other words, for any $\lambda > 0$, $f(\lambda z^A) = f(\lambda z^B)$. This means that isoquants corresponding to higher levels of output are just magnified versions of isoquants producing lower levels of output. It also means that marginal rates of technical substitution are constant along rays through the origin. No necessarily constant

Production is said to be homogeneous of degree $\alpha$ if $f(\lambda z) = \lambda^\alpha f(z)$. Homogeneous production functions always have the property of homotheticity.

## Returns to scale




Isoquants of Homogeneous PFs

*Returns to scale* are concerned with the feasibility of scaling up and down production plans. (Remember that a production plan includes a specification of outputs as well as inputs so we are now talking about scaling up inputs *and* output.)

Returns to Scale

- There are *decreasing returns to scale* (DRS) if scaling up the input vector results in a less than proportionate increase in output. Thus $\lambda f(z) > f(\lambda z)$ for $\lambda > 1$. It is therefore possible to scale production plans down but not up. DRS <-> Increasing AC

  If production is homogeneous then there are decreasing returns to scale if $\alpha < 1$.

  If there is only one output and one input then concavity of the production function, $f'' < 0$, implies decreasing returns to scale given $f(0) = 0$. ∝ DRS

- There are *increasing returns to scale* (IRS) if scaling up the input vector results in a more than proportionate increase in output. Thus $\lambda f(z) < f(\lambda z)$ for $\lambda > 1$. It is therefore possible to scale production plans up but not down.

  If production is homogeneous then there are decreasing returns to scale if $\alpha > 1$.

  If there is only one output and one input then convexity of the production function, $f'' > 0$, implies increasing returns to scale given $f(0) = 0$. ∝ IRS

- There are *constant returns to scale* (CRS) if scaling up the input vector results in exactly the same scaling up of production. Thus $\lambda f(z) = f(\lambda z)$ for any $\lambda$. This is equivalent to homogeneity of degree one of the production function. If there is also only one input then production is linear, $q = az$ for some $a$.

# Some examples

- **Perfect substitutes, CRS** $q = f(z_1, z_2) = az_1 + bz_2$: MP of $z_1$ is $a$ and MP of $z_2$ is $b$ and both are constant. The MRTS is the ratio of the marginal products $a/b$ and is constant. Isoquants are therefore parallel straight lines. The elasticity of substitution is infinite. Production is homothetic and there are constant returns to scale.

- **Perfect substitutes, general** $q = f(z_1, z_2) = G(az_1 + bz_2)$: The MRTS is still constant at $a/b$, isoquants are therefore still parallel straight lines and the elasticity of substitution is still infinite. Production is also homothetic as the MRTS is constant along rays (because it is constant everywhere). Returns to scale depend however on the function $G$.

- **Perfect complements, CRS** $q = f(z_1, z_2) = \min[az_1, bz_2]$: MP of $z_1$ is $a$ if $az_1 < bz_2$ and zero otherwise; MP of $z_2$ is $b$ if $az_1 > bz_2$ and zero otherwise. Isoquants are L-shaped with the kinks lying on a ray through the origin of slope $a/b$. Production is homothetic and there are constant returns to scale.

- **Cobb-Douglas**: $q = f(z_1, z_2) = Az_1^a z_2^b$: MP of $z_1$ is $aq/z_1$ and MP of $z_2$ is $bq/z_2$. The MRTS is therefore $az_2/bz_1$ which is diminishing so input requirement sets are convex. Since $\ln(z_1/z_2) = \ln(a/b) - \ln MRTS$ the elasticity of substitution is constant at 1. The MRTS depends only on the input ratio $z_1/z_2$ so production is homothetic. Since $f(\lambda z_1, \lambda z_2) = A(\lambda z_1)^\alpha (\lambda z_2)^\beta = \lambda^{a+b} f(z_1, z_2)$, the production function is homogeneous of degree $a + b$ and shows IRS if $a + b > 1$, DRS if $a + b < 1$ and CRS if $a + b = 1$.

- **Constant elasticity of substitution**: $q = f(z_1, z_2) = [Az_1^a + Bz_2^a]^b$ with $a < 1$: MP of $z_1$ is $abAq^{1-1/b} z_1^{a-1}$ and MP of $z_2$ is $abBq^{1-1/b} z_2^{a-1}$. The MRTS is therefore $(A/B)(z_1/z_2)^{a-1}$ which is diminishing so input requirement sets are convex. Since $\ln(z_1/z_2) = [\ln(a/b) + \ln MRTS]/(a-1)$ the elasticity of substitution is constant at $1/(1 - a)$. The MRTS depends only on the input ratio $z_1/z_2$ so production is homothetic. Since $f(\lambda z_1, \lambda z_2) == \lambda^{ab} f(z_1, z_2)$, the production function is homogeneous of degree $ab$ and shows IRS if $ab > 1$, DRS if $ab < 1$ and CRS if $ab = 1$.

- **A non-homothetic example:** $q = f(z_1, z_2) = \sqrt{z}_1 + z_2$ The MP of $z_1$ is $1/2\sqrt{z}_1$ and the MP of $z_2$ is 1. MRTS is therefore $1/2\sqrt{z}_1$ which is not constant along rays so production is not homothetic. Input requirement sets are convex but the elasticity of substitution is not constant. There is DRS if $z_1 > 0$since $f(\lambda z_1, \lambda z_2) = \sqrt{\lambda}\sqrt{z}_1 + \lambda z_2 < \lambda f(z_1, z_2)$ if $\lambda > 1$ (and CRS if $z_1 = 0$).

- **A multiple product example:** Suppose a firm uses a single input $z$ which it can allocate to production of either of two goods. If it uses $\zeta$ in producing good 1 then $q_1 = \sqrt{\zeta}$ and $q_2 = \sqrt{z - \zeta}$. Thus $q_1^2 + q_2^2 = z$.

The transformation function is $F(q_1, q_2, z) = q_1^2 + q_2^2 - z$. There is DRS as $F(\lambda y) > 0$ if $F(y) = 0$ and $\lambda > 1$.

Profit Maximization

## Profit maximisation

Suppose there is a single output priced at $p$, input prices are $w$ and the firm regards these prices as given. This would be sensible for a small firm under competitive conditions. Firm profits are $pf(z) - w'z$ and we assume that the firm makes production decisions so as to maximise its profits. $\pi = pq - wz$

First order conditions, assuming all inputs are used in positive but finite quantities at the optimum, require

$$p\frac{\partial f}{\partial z_i} - w_i \;=\; 0$$ for each $w_i$

for each input, or equivalently

$$MP_i \;=\; w_i/p. \quad = -MRT_{\text{output. input}}$$

The firm uses the input until its marginal product equals its input price divided by the price of the firm's output. In other words, the price ratio between input and output is equated to (minus) the marginal rate of transformation between output and input. price

Second order conditions include the requirement that $\partial^2 f/\partial z_i^2 \leq 0$ so that output is a concave function of each input.

If we take the conditions relating to any two inputs then we can divide through to eliminate the product price

$$\boxed{\frac{\partial f/\partial z_i}{\partial f/\partial z_j} \;=\; w_i/w_j}$$

which is another condition relating a marginal rate of transformation to a price ratio

$$MRTS_{ij} \;=\; w_i/w_j.$$

## Supply functions and unconditional input demand functions

Unconditional Demand Function $D(p,w)$ (From 1st Order Conditions)

The input demands which solve the profit maximisation problem can be written as functions of output prices $p$ and input prices $w$ known as *unconditional input demand functions*, which we denote by the vector-valued function $D(p, w)$. $z_i^* = D_i(p,w)$

Putting these demands into the production function $f(z)$ will give output as a function of output and input prices which we call the firm's *supply function* and denote $S(p, w)$.

Supply Function $S(p,w)$ (Sub $D(p,w)$ into $f(z)$)

Some basic properties of these functions follow by very simple arguments

• $D_i(p,w)$ decreases in $w_i$

- The profit maximising input choice cannot be altered by scaling both $p$ and $w$ by a common factor so both functions are homogeneous of degree zero
$$D(\lambda p, \lambda w) = D(p,w) \qquad S(\lambda p, \lambda w) = S(p,w).$$

- Consider a comparison of two situations with different prices, say $(p^A, w^A)$ and $(p^B, w^B)$. Call the output and input decisions in the two cases $(q^A, z^A)$ and $(q^B, z^B)$. The fact that the firm is maximising profit in each case shows that

  we know $\pi_A > \pi_B$

$$p^A q^A - w^A z^A \geq p^A q^B - w^A z^B$$
$$p^B q^A - w^B z^A \leq p^B q^B - w^B z^B$$

  $\pi_A^* > \pi$[A's prices, B's quantities]

  $\pi$[B's prices, A's quantities] $< \pi_B^*$

  By simple subtraction, letting $\Delta q = q^A - q^B$, $\Delta p = p^A - p^B$, $\Delta z = z^A - z^B$ and $\Delta w = w^A - w^B$
$$p^A \Delta q - w^A \Delta z \geq 0 \qquad p^B \Delta q - w^B \Delta z \leq 0$$

  $\Delta$Price × $\Delta$Quantity $\geq \Delta$P of inputs × $\Delta$D for inputs

  and therefore
$$\Delta p \Delta q - \Delta w \Delta z \geq 0.$$

  $(p^A - p^B)(q^A - q^B) \geq (w^A - w^B)(z^A - z^B)$

  $\Delta p \Delta q \geq 0 \geq \Delta w \Delta z$

  Thus if output price goes up ($\Delta p > 0$) holding input prices constant ($\Delta w = 0$) then output cannot fall ($\Delta q \geq 0$). Likewise if one input price rises ($\Delta w_i > 0$) while output price stays the same ($\Delta p = 0$) as do all other input prices ($\Delta w_j = 0,\ j \neq i$) then demand for that input cannot rise ($\Delta z_i \leq 0$). $\Delta p \Delta q \geq 0$

  $0 \geq \Delta w \Delta z$

  Hence $S(p,w)$ is increasing in $p$ and $D_i(p,w)$ is decreasing in $w_i$.

# The profit function

If we evaluate the profit earned by the profit maximising choices then we can also write maximum profit as a function of output and input prices
$$\pi(p,w) \;=\; \max_z pf(z) - w'z \;=\; pS(p,w) - w'D(p,w).$$
This function is known as the *profit function*.

It also has some obvious properties.

- Since scaling up output and input prices by a common factor $\lambda$ makes no difference to the profit maximising choices, profit must also be scaled up by $\lambda$. The profit function is therefore homogeneous of degree one in $p$ and $w$
$$\pi(\lambda p, \lambda w) = \lambda \pi(p,w).$$

- Increasing the price of output cannot decrease profit and increasing the price of an input cannot increase profit.

## Hotelling's Lemma

Hotelling's Lemma

If we take $\pi(p.w)$ and differentiate with respect to output price $p$ then $\pi(p,w) = \max_z \{p f(z) - w^T z\}$

$$\begin{aligned}\boxed{\frac{\partial\pi}{\partial p}} &= f(D(p,w)) + \sum_j \left[p\frac{\partial f}{\partial z_j} - w_j\right]\frac{\partial D_j(p,w)}{\partial p} \\ &= f(D(p,w)) = \boxed{S(p,w)}\end{aligned}$$

$= \max_z \{p S(p,w) - w^T D(p,w)\}$

Homogeneous of 0

Homogeneous of 1

$\pi(p,w)$ is homogeneous of degree 1

$\pi(\lambda p, \lambda w) = \lambda\pi(p,w)$

$\frac{\partial \pi}{\partial p} = S(p,w)$

$\frac{\partial \pi}{\partial w_i} = -D_i(p,w)$

using the first order condition that says $p\partial f/\partial z_j - w_j = 0$. Hence the derivative *is* the supply function.

Similarly, if we differentiate with respect to an input price $w_i$ then

$$\begin{aligned}\boxed{\frac{\partial\pi}{\partial w_i}} &= -D_i(p,w) + \sum_j \left[p\frac{\partial f}{\partial z_j} - w_j\right]\frac{\partial D_j(p,w)}{\partial w_i} \\ &= \boxed{-D_i(p,w)}\end{aligned}$$

so that this derivative is minus the conditional input demand function.

These are both examples of what is known as *Hotelling's Lemma.*

(Ignoring D, S in [illegible])

## Symmetry

$\frac{\partial \pi}{\partial w_i} = -D_i(p,w)$

$\frac{\partial^2 \pi}{\partial w_i \partial w_j} = -\frac{\partial D_i}{\partial w_j}$

$\frac{\partial \pi}{\partial w_j} = -D_j(p,w)$

$\frac{\partial^2 \pi}{\partial w_j \partial w_i} = -\frac{\partial D_j}{\partial w_i}$

Given these relationships

$$\boxed{\frac{\partial D_i}{\partial w_j}} = -\frac{\partial^2\pi}{\partial w_i \partial w_j} = -\frac{\partial^2\pi}{\partial w_j \partial w_i} = \boxed{\frac{\partial D_j}{\partial w_i}}.$$

Thus the unconditional effect of a change in one input price $w_j$ on demand for another input $z_i$ must be *identical* to the unconditional effect of a change in $w_i$ on demand for input $z_j$. If this is not true of the firm's input decisions then the firm cannot be maximising profit.

Also, by a similar argument,

$$\boxed{\frac{\partial D_i}{\partial p}} = -\frac{\partial^2\pi}{\partial w_i \partial p} = -\frac{\partial^2\pi}{\partial p \partial w_i} = \boxed{-\frac{\partial S}{\partial w_i}}$$

so effects of output price on input demands must be of the same magnitude as (but of opposite sign to) effects of input price on output supply.

## Profit maximisation in the general case

In the more general formulation we can let $r$ be the price vector corresponding to the net output vector $y$, profits are simply $r'y$ and the firm's problem is to solve $\max_y r'y$ subject to $F(y) \leq 0$. The profit function $\pi(r)$ defined by the solution to this problem is homogeneous of degree one in $p$ and $\partial\pi/\partial r_i = y_i$ (this being the general form of Hotelling's lemma). Hence $\pi(r)$ is increasing in the prices of goods for which net outputs are positive and decreasing in those for which net outputs are negative.

Cost Minimization

# Cost minimisation

For firms producing a single output, it is useful to model firm decision making as a two-stage process with inputs chosen to minimise costs given output $q$ and output $q$ then chosen to maximise profit given what that implies for how costs vary with $q$. An implication of profit maximisation is that the chosen output level is always produced at least cost.

This is a useful perspective because firms may not be price takers in output markets but nonetheless still act as input-price-taking cost minimisers in input markets.

Consider, then, a firm choosing inputs $z$ to solve

$$\min_z w'z \; s.t. \; q \leq f(z).$$

$\min_z w^T z$ Subject to $q \leq f(z)$

We can solve by setting up a Lagrangean

$$w'z \; + \; \mu [q - f(z)]$$

$L = w^T z + \lambda (q - f(z))$

where $\mu$ is a Lagrange multiplier associated with the constraint requiring production of the necessary output $q$.

First order conditions require

$\frac{dL}{dz_i} = 0$

$$\mu \partial f / \partial z_i - w_i \; = \; 0$$

$w_i = \mu \frac{\partial f}{\partial z_i}$ (Then Substitute back $\mu$)

and therefore imply equality between the marginal rate of technical substitution and input price ratio, $MRTS_{ij} = w_i/w_j$ exactly as derived in the profit maximisation problem above.

Solve these using $f(z) = q$

$z_i^* = H_i(q, w)$

We can visualise the optimum choice as occuring at a tangency between an isoquant and an isocost, at a point where the slope of the isocost equals the ratio of input prices.

# Conditional input demands

Conditional Input Demands $H(q,w)$ (Cost-Minimizing GD)

The cost minimising input quantities are known as the *conditional input demands*, $H(q, w)$, written as a function of output $q$ and $w$ rather than price $p$ and $w$ since we are now taking $q$ as given rather than chosen jointly with inputs given $p$.

The properties of these functions follow from arguments somewhat similar to those deployed above

- Since scaling up input prices by a common factor $\lambda$ makes no difference to the cost minimising choices, conditional input demands are homogeneous of degree zero in $w$, $H(q, \lambda w) = H(q, w)$.
- Suppose that $z^A$ minimises the cost of producing $q$ at prices $w^A$ and $z^B$ minimises costs of producing the same output at prices $w^B$. Then

$$w^A z^A \leq w^A z^B \qquad w^B z^A \geq w^B z^B$$

so that, by repeated subtraction, letting $\Delta z = z^A - z^B$ and $\Delta w = w^A - w^B$,

$$\Delta w \Delta z \leq 0.$$

$\Delta w^T \Delta z \leq 0$

If only one input price changes so that $\Delta w_i > 0$ and $\Delta w_j = 0$ for $i \neq j$, then $\Delta z_i \leq 0$. Conditional input demands must therefore be decreasing in own price.

## Cost function

Cost Function $C(q,w)$

The function giving minimum cost given $q$ and $z$ is known as the *cost function*, $C(q, w) = w' H(q, w)$. It is

$\sum_{i=1}^{n} w_i z_i^*$

- increasing in the price of every input which is used $q\uparrow / w\uparrow,\ C\uparrow$
- homogeneous of degree one in $w$. $C(q, \lambda w) = \lambda(q,w)$

## o Satisfies → Shephard's Lemma

Shephard's Lemma

If we differentiate the cost function with respect to one of the input prices then we get a relationship linking the derivative to the conditional input demand.

$\frac{\partial C}{\partial w_i} = z_i^*$

$\frac{\partial C}{\partial w_i} = H_i(q,w)$

$$\begin{aligned}
\boxed{\frac{\partial C}{\partial w_i}} &= H_i(q,w) + \sum_j w_j \frac{\partial H_j(p,w)}{\partial w_i} \\
&= H_i(q,w) + \mu \sum_j \frac{\partial f}{\partial z_j} \frac{\partial H_j(p,w)}{\partial w_i} \\
&\boxed{= H_i(q,w)}
\end{aligned}$$

(Conditional input demand)

where substitution has been made from the first order condition $\mu \partial f / \partial z_j = w_j$ and note has been taken that $\sum_j \partial f / \partial z_j \cdot \partial H_j(p,w) / \partial w_i = \partial q / \partial w_i = 0$ since $q$ is being held fixed.

(Differentiating the constraint)

This result is known as *Shephard's lemma*.

Because of this link between derivatives and demands, conditional input demands have symmetry properties similar to unconditional input demands

$\frac{\partial H_i}{\partial w_j} = \frac{\partial H_j}{\partial w_i}$

$$\boxed{\frac{\partial H_i}{\partial w_j}} = \frac{\partial^2 C}{\partial w_i \partial w_j} = \frac{\partial^2 C}{\partial w_j \partial w_i} = \boxed{\frac{\partial H_j}{\partial w_i}}.$$

$\frac{\partial C}{\partial w_i}$

## Relationship between conditional and unconditional input demands

$D_i(p,w) \neq H_i(q,w)$

Relationship between C/UC Input Demands

$D(p,w) = H(S(p,w),w)$

The two types of input demand are related by the identity

$$D(p,w) \equiv H(S(p,w),w)$$

Supply function $(q^*) = f(z^*) = f(D(p,w))$

Conditional Input Demand Function

Unconditional Input Demand Function

where the unconditional demands are recognised as the conditional input demands at profit maximising supply. $\frac{\partial H_i(S(p,w),w)}{\partial w_i}$

Differentiating, we see that

?

$$\boxed{\frac{\partial D_i(p,w)}{\partial w_i} = \frac{\partial H_i(S(p,w),w)}{\partial w_i} + \frac{\partial H_i(S(p,w),w)}{\partial q} \cdot \frac{\partial S(p,w)}{\partial w_i}}$$

so the unconditional effect of increased input price includes both an effect at fixed output $\partial H_i/\partial w_i$ and an effect coming through adjustment to chosen output level $\partial H_i/\partial q \cdot \partial S/\partial w_i$.

## Price responsiveness of input demands

Since cost minimisation requires $MRTS_{ij} = w_i/w_j$, responsiveness of $z_i/z_j$ to $MRTS_{ij}$ around an isoquant is immediately informative about responsiveness of $z_i/z_j$ to the input price ratio $w_i/w_j$ holding output constant. But this is exactly what the elasticity of substitution measures, introduced earlier as a measure of curvature of isoquants. Responsiveness of input choices to economic incentives is therefore directly related to the curvature of input requirement sets.

## Homotheticity and input demands

What about responsiveness of cost-minimising input demands to changes in output $q$? This is not about what happens as one moves around an isoquant (holding $q$ constant and changing $w$) but about what happens as one moves *between* isoquants (changing $q$ and holding $w$ constant).

Homotheticity was a special property of technology under which MRTS remains constant as inputs are scaled up or down in the same proportions. Hence, under homotheticity, as we move between isoquants holding MRTS constant (because equated to unchanging input price ratios) then we keep input proportions unchanged, expanding or contracting all input choices equiproportionately as we move out or in along rays through the origin. Cost shares of different inputs are accordingly independent of the scale of production.

## Returns to scale and cost functions

Returns to Scale & Cost Functions

If there are constant returns to scale (CRS) then the production function is homogeneous, technology is homothetic and scaling up output by a factor $\lambda$ will be achieved by scaling up all inputs by the same factor $\lambda$, also therefore scaling up costs by $\lambda$. The cost function is therefore proportional to output

CRS implies : $$C(q,w) = q\kappa(w).$$

If there are decreasing returns to scale (DRS) then it is impossible to scale production plans up and multiplying output by $\lambda > 1$ will multiply costs by more than $\lambda$

DRS implies : $$C(\lambda q, w) > \lambda C(q,w) \qquad \text{for } \lambda > 1.$$

If there are increasing returns to scale (IRS) on the other hand then it is possible to scale production up but not down and multiplying output by $\lambda > 1$ will multiply costs by less than $\lambda$

IRS implies : $$C(\lambda q, w) < \lambda C(q, w) \qquad \text{for } \lambda > 1.$$

## Marginal and average costs

Average cost is defined simply by $C(q, w)/q$. Straightforward calculus shows that

$$\frac{\partial(C/q)}{\partial q} = \frac{1}{q}\left(\frac{\partial C}{\partial q} - \frac{C}{q}\right)$$

so that average cost is falling if it exceeds marginal cost $\partial C/\partial q$ and rising if it is below marginal cost. Any output at which marginal and average cost coincide is a stationary point for average cost. It is common to draw a U-shaped average cost curve cut from below by marginal cost at its minimum, though that is only one of many possible configurations.

The lowest quantity $q$ at which average cost reaches a minimum is known as the *minimum efficient scale*.

## Profit maximisation, again

Having defined the cost function we can use it to rewrite the problem of output choice as

$$\max_q pq - C(q, w).$$

The first order condition for this problem requires that price be equated to marginal cost, $p = \partial C/\partial q$. The second order condition requires $\partial^2 C/\partial q^2 \geq 0$.

Because shutdown has been assumed possible, firms cannot make negative profits. The condition that price exceed average cost $p \geq C/q$ can therefore be imposed on firm decisions. Given that $p = \partial C/\partial q$, this implies $\partial C/\partial q \geq C/q$ so that it is only output ranges over which average costs are rising that will be chosen. The competitive firm's supply curve is therefore that segment of the marginal cost curve lying above the average cost curve.

## Some examples, again

- **Perfect substitutes, DRS** $q = f(z_1, z_2) = \sqrt{az_1 + bz_2}$: The firm produces using only $z_1$ if $a/w_1 > b/w_2$ and using only $z_2$ if $a/w_1 < b/w_2$. Conditional input demands are therefore

$$H_1(q, w) = \begin{cases} q^2/a & \text{if } a/w_1 > b/w_2 \\ 0 & \text{if } a/w_1 < b/w_2 \end{cases}$$
$$H_2(q, w) = \begin{cases} 0 & \text{if } a/w_1 > b/w_2 \\ q^2/b & \text{if } a/w_1 < b/w_2 \end{cases}$$

The cost function is

$$C(q, w) \ = \ q^2 \min[w_1/a, w_2/b].$$

Profits are therefore

$$\pi(p, w) \ = \ \min_q pq - C(q, w) \ = \ p^2/4 \min[w_1/a, w_2/b],$$

the supply function is

$$S(p, w) = \frac{p}{2 \min[w_1/a, w_2/b]}$$

and unconditional demands are

$$D_1(q, w) \ = \ \begin{cases} p^2 a/4w_1^2 & \text{if } a/w_1 > b/w_2 \\ 0 & \text{if } a/w_1 < b/w_2 \end{cases}$$

$$D_2(q, w) \ = \ \begin{cases} 0 & \text{if } a/w_1 > b/w_2 \\ p^2 b/4w_2^2 & \text{if } a/w_1 < b/w_2 \end{cases}$$

- **Perfect complements, DRS** $q = f(z_1, z_2) = \sqrt{\min[az_1, bz_2]}$: The firm uses $z_1$ and $z_2$ in the same ratio whatever input prices.

$$H_1(q, w) \ = \ q^2/a \qquad H_2(q, w) \ = \ q^2/b$$

The cost function is

$$C(q, w) \ = \ q^2[w_1/a + w_2/b].$$

Profits are therefore

$$\pi(p, w) \ = \ \min_q pq - C(q, w) \ = \ p^2/4[w_1/a + w_2/b],$$

the supply function is

$$S(p, w) = \frac{p}{2[w_1/a + w_2/b]}$$

and unconditional demands are

$$D_1(q, w) \ = p^2/4a[w_1/a + w_2/b]^2 \qquad D_2(q, w) \ = \ p^2/4b[w_1/a + w_2/b]^2$$

- **Cobb-Douglas**: $q = f(z_1, z_2) = Az_1^a z_2^b$: MRTS $az_2/bz_1$ is equated to $w_1/w_2$. Thus, by substituting into the production function

$$H_1(q, w) \ = \ \left(\frac{q}{A}\right)^{1/(a+b)} \left(\frac{bw_1}{aw_2}\right)^{-b/(a+b)} \qquad H_2(q, w) \ = \ \left(\frac{q}{A}\right)^{1/(a+b)} \left(\frac{bw_1}{aw_2}\right)^{a/(a+b)}$$

The cost function is

$$C(q,w) \;=\; \left(\frac{q}{A}\right)^{1/(a+b)} \left(\frac{w_1}{a}\right)^{a/(a+b)} \left(\frac{w_2}{b}\right)^{b/(a+b)}.$$

Assume $a+b<1$ so that there is DRS. Then the profit function is

$$\pi(p,w) \;=\; \left(\frac{Ap}{(a+b)(w_1/a)^a(w_2/b)^b}\right)^{1/(1-a-b)}$$

- **A non-homothetic example:** $q = f(z_1,z_2) = \sqrt{z}_1 + z_2$: Equating the MRTS to the input price ratio gives $1/2\sqrt{z_1} = w_1/w_2$ at interior solutions. Cost-minimising input demands are therefore

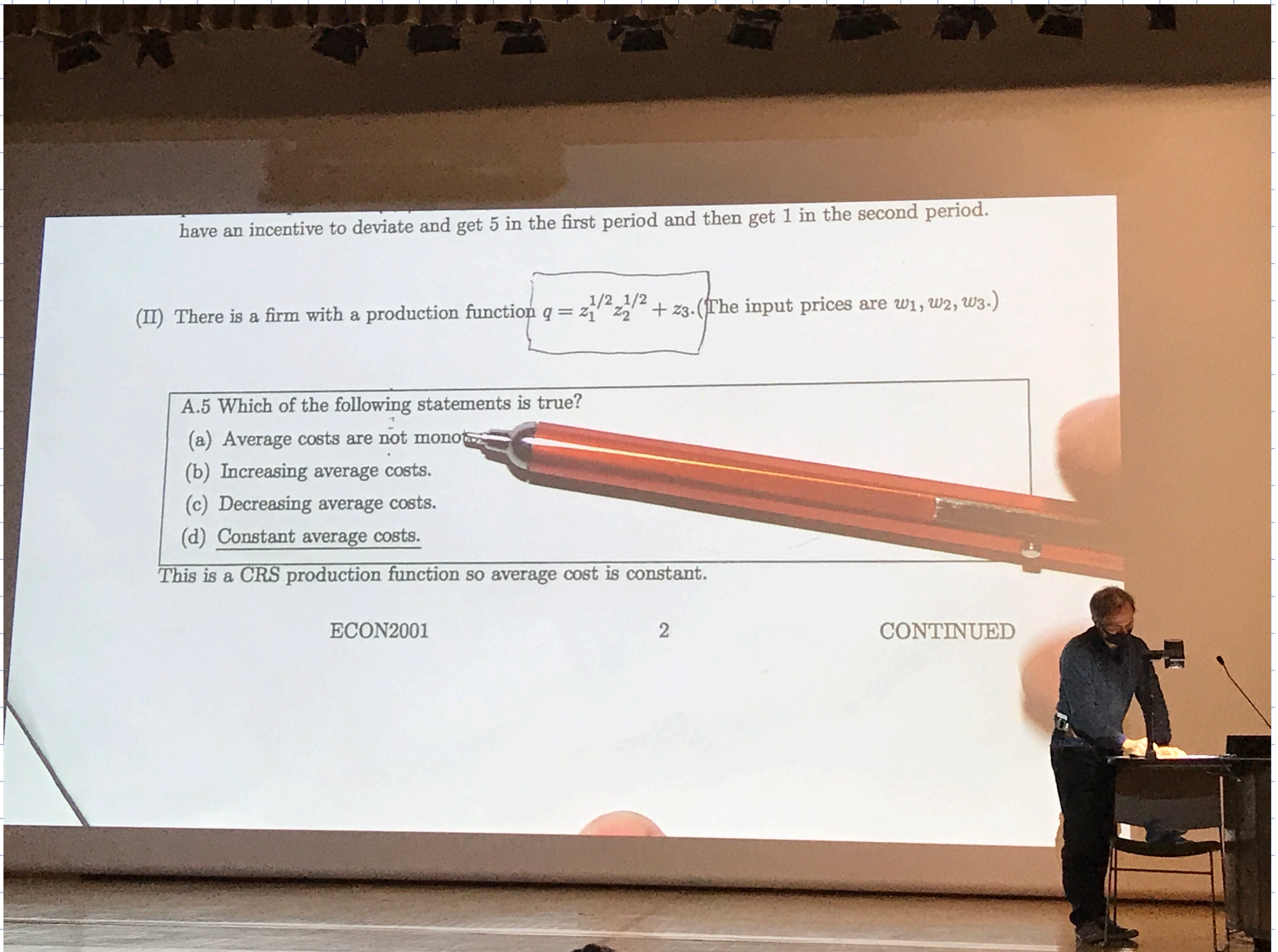
  $$H_1(q,w) \;=\; \begin{cases} (w_2/2w_1)^2 & \text{if } q \geq w_2/2w_1 \\ q^2 & \text{if } q \leq w_2/2w_1 \end{cases}$$
  $$H_2(q,w) \;=\; \begin{cases} q - w_2/2w_1 & \text{if } q \geq w_2/2w_1 \\ 0 & \text{if } q \leq w_2/2w_1 \end{cases}$$

  The cost function is

  $$C(q,w) \;=\; \begin{cases} w_2 q - w_2^2/4w_1 & \text{if } q \geq w_2/2w_1 \\ w_1 q^2 & \text{if } q \leq w_2/2w_1 \end{cases}$$

  Profits are $(p-w_2)\,q + w_2^2/4w_1$ if $q \geq w_2/2w_1$ and $pq - w_1q^2$ otherwise. If $p > w_2$ then profits are everywhere increasing in $q$; only if $p < w_2$ is there a well-defined maximum at finite $q$ with $z_2 = 0$ in which case $\pi(p,w) = p^2/4w_1$. Unconditional input demands in that case are $D_1(p,w) = (p/2w_1)^2$ and $D_2(p,w) = 0$ and the supply function is $S(p,w) = p/2w_1$.

- **A multiple product example:** $q_1^2+q_2^2 = z$: The firm maximises $p_1q_1 + p_2q_2 - w[q_1^2+q_2^2]$. Supply functions are $S_1(p,w) = p_1/2w$ and $S_2(p,w) = p_2/2w$ and the unconditional input demand function is $D(p,w) = \left(p_1^2+p_2^2\right)/4w^2$. The profit function is $\pi(p,w) = \left(p_1^2+p_2^2\right)/4w$.



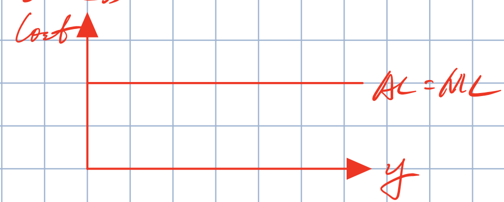

PF: $q_1 = z_1^{\frac{1}{2}} z_2^{\frac{1}{2}} + z_3$

Use returns to scale to analyse AC

$$(\alpha z_1)^{\frac{1}{2}} (\alpha z_2)^{\frac{1}{2}} + \alpha z_3 = \alpha z_1^{\frac{1}{2}} z_2^{\frac{1}{2}} + \alpha z_3$$

$$= \alpha (z_1^{\frac{1}{2}} z_2^{\frac{1}{2}} + z_3)$$

$$= \alpha q_1$$




$\therefore$ Constant RTS $\rightarrow$ Constant AC

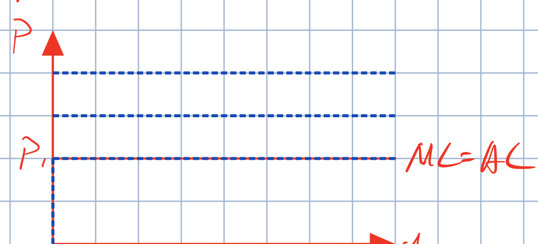
$\therefore$ The PF is Homogenous degree 1. $\rightarrow$ Homothetic

Further investigation of this PF:

Maximizing Profit:

$P < P_1$ : 0 output

$P \geq P_1$ : $\infty$

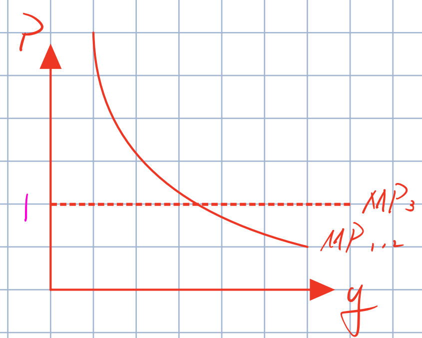



Supply Curve of this Firm


Marginal Output: $MP_1 = \frac{dy}{dz_1} = \frac{1}{2} z_1^{-\frac{1}{2}} z_2^{\frac{1}{2}}$

$MP_2 = \frac{dy}{dz_2} = \frac{1}{2} z_1^{\frac{1}{2}} z_2^{-\frac{1}{2}}$

$MP_3 = \frac{dy}{dz_3} = 1$




Minimizing Cost: Min $w_1 z_1 + w_2 z_2 + w_3 z_3 \mid y = \sqrt{z_1 z_2} + z_3$

Method: Assumption: $z_3$ is fixed

$$\text{Min } w_1 z_1 + w_2 z_2 \mid Y = y - z_3 = \sqrt{z_1 z_2}$$

$$L = w_1 z_1 + w_2 z_2 + \lambda (Y - \sqrt{z_1 z_2})$$

$$\begin{cases} \frac{\partial L}{\partial z_1} = w_1 - \frac{1}{2}\lambda z_1^{-\frac{1}{2}} z_2^{\frac{1}{2}} = 0 \\ \frac{\partial L}{\partial z_2} = w_2 - \frac{1}{2}\lambda z_1^{\frac{1}{2}} z_2^{-\frac{1}{2}} = 0 \end{cases}$$

$$\frac{w_1}{w_2} = \frac{z_2^*}{z_1^*} \qquad w_1 z_1^* = w_2 z_2^*$$

$$Y = \sqrt{z_1^* z_2^*} \rightarrow Y = \sqrt{z_1^* \frac{w_1}{w_2} z_1^*}$$

$$\begin{cases} Y\sqrt{\frac{w_2}{w_1}} = z_1^* \\ Y\sqrt{\frac{w_1}{w_2}} = z_2^* \end{cases}$$

$$w_1 z_1^* + w_2 z_2^* = w_1 Y \sqrt{\frac{w_2}{w_1}} + w_2 Y \sqrt{\frac{w_1}{w_2}}$$

$$= 2Y\sqrt{w_1 w_2}$$

$$\text{SR Cost} = w_1 z_1^* + w_2 z_2^* + w_3 z_3 = 2(y - z_3)\sqrt{w_1 w_2} + w_3 z_3$$

$$= 2\sqrt{w_1 w_2}\, Y + z_3 (w_3 - 2\sqrt{w_1 w_2})$$

$$\min_{z_3 \in \{z_3 : 0 \le z_3 \le y\}} \text{SR Cost} = \underbrace{2\sqrt{w_1 w_2}}_{\text{Constant}}\, y + z_3 \underbrace{(w_3 - 2\sqrt{w_1 w_2})}_{\text{Constant}}$$

It becomes a linear function

① If $w_3 > \underbrace{2\sqrt{w_1 w_2}}_{=AC=MC}$ : $z_3^* = 0$, $z_1^* = y\sqrt{\frac{w_1}{w_2}}$

$z_2^* = y\sqrt{\frac{w_1}{w_2}}$

② If $w_3 < 2\sqrt{w_1 w_2}$ : $z_3^* = y$, $z_1^* = z_2^* = 0$

$= AC = MC$

Input Demand:

$$z_1^* = \begin{cases} 0 & \text{if } w_3 < 2\sqrt{w_1 w_2} \\ y\sqrt{\frac{w_2}{w_1}} & \text{if } w_3 > 2\sqrt{w_1 w_2} \end{cases}$$

$\downarrow$

$$z_1^* = \begin{cases} 0 & \text{if } w_1 > \frac{w_3^2}{4w_2} \\ y\sqrt{\frac{w_2}{w_1}} & \text{if } w_1 < \frac{w_3^2}{4w_2} \end{cases}$$

A different case: $y = z_1^2 z_2^2 + z_3$

RTS: $(\alpha z_1)^2 (\alpha z_2)^2 + \alpha z_3$

$= \alpha^4 z_1^2 z_2^2 + \alpha z_3 > \alpha(z_1^2 z_2^2 + z_3^2)$

Increasing RTS if $\alpha > 1$, $z_1, z_2 > 0$

Homothetic? : Given $\begin{cases} y = z_1^2 z_2^2 + z_3 \\ y = x_1^2 x_2^2 + x_3 \end{cases}$ 2 Pts on the isotope $\rightarrow z_1^2 z_2^2 + z_3 = x_1^2 x_2^2 + x_3$

Whether $(\alpha z_1)^2 (\alpha z_2)^2 + (\alpha z_3) = (\alpha x_1)^2 (\alpha x_2)^2 + (\alpha x_3)$ ?

$\alpha^4 z_1^2 z_2^2 + \alpha z_3 = \alpha^4 x_1^2 x_2^2 + \alpha x_3$

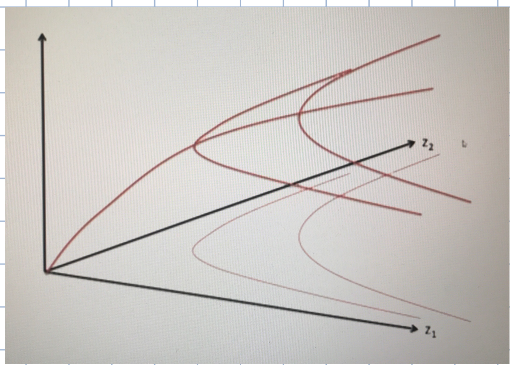
$\alpha^3 z_1^2 z_2^2 + z_3 = \alpha^3 x_1^2 x_2^2 + x_3$

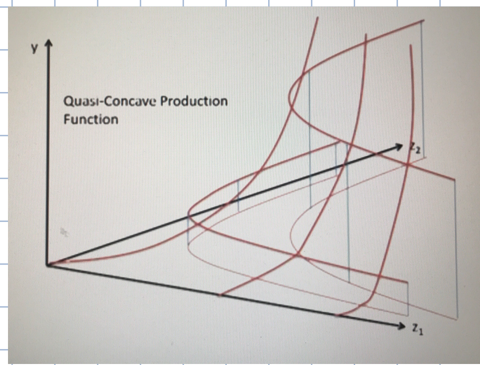
Both to be true?

Cost Min?

Concave PF → Decreasing RS

Quasi-Concave PF → Uncertain RS

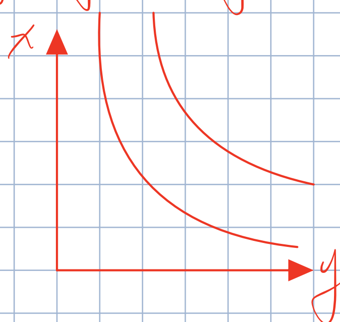



Concave    Quasi-Concave

$f(x,y) = \log x + \log y \qquad xy = e^q$

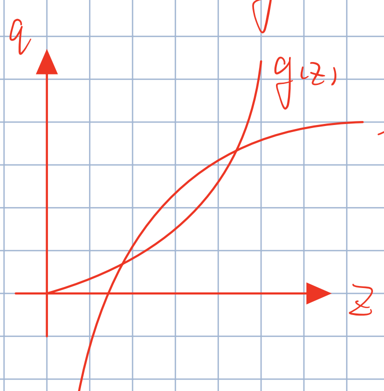
$g(x,y) = (xy)^a \qquad xy = q^{\frac{1}{a}}$




Both have isoquants like this

In z direction: $f(z) = \log z + \log z$

$g(z) = (zz)^a = z^{2a}$ (Assume $a > \frac{1}{2}$)



Example: Supply func of Cobb-Douglas PF

$$q = f(z_1, z_2) = z_1^a z_2^a, \quad 2a < 1$$

$$\pi = p f(z_1, z_2) - w_1 z_1 - w_2 z_2$$

$$= p z_1^a z_2^a - w_1 z_1 - w_2 z_2$$

First Order Conditions: $w_1 = p \frac{\partial f}{\partial z_1} = p a z_1^{a-1} z_2^a$

$$w_2 = p \frac{\partial f}{\partial z_2} = p a z_1^a z_2^{a-1}$$

$$\frac{w_1}{w_2} = \frac{z_2}{z_1} = MRTS_{1,2}$$

$$w_1 z_1 = w_2 z_2$$

$$z_2 = z_1 \times \frac{w_1}{w_2}$$

$$w_1 = p a z_1^{a-1} \left(z_1 \times \frac{w_1}{w_2}\right)^a$$

$$= p a z_1^{2a-1} w_1^a w_2^{-a}$$

$D_1(p,w)$ : $z_1^* = [w_1^{a-1} w_2^{-a} a p]^{\frac{1}{1-2a}}$

$D_2$ ....

$S(p,w) = f(z_1^*, z_2^*) = [w_1 w_2 (ap)^{-2}]^{\frac{a}{2a-1}}$

Adverse Selection

# 6. ADVERSE SELECTION

MARTIN CRIPPS

## 1. Introduction

We now study models of markets where there is uncertainty about the type of the buyers or the type of the sellers. If a buyer knows what type she is but the seller doesn't we are in a situation of asymmetric information. (One person in the game/market knows more than the other.) In models of trading that economists study, asymmetric information gives rise to a phenomenon termed "adverse selection". We will see that uncertainty (or asymmetric information) has profound effects on markets. In particular uncertainty makes optimization difficult, harms efficiency and introduces a role for signalling.

## 2. The Market for Lemons

We begin by, informally, studying a classic model of adverse selection due to George Akerloff. Here we will just draw demand and supply curves and chat. In the later sections of this topic we will make these ideas more rigourous. The message of this model is that it may be impossible to trade some kinds of goods even though there are people who want to buy the good and others who want to sell the good because of lack of information. In this case the good that cannot be traded is good-quality cars. These are squeezed out of the market place by the presence of bad cars or "lemons".

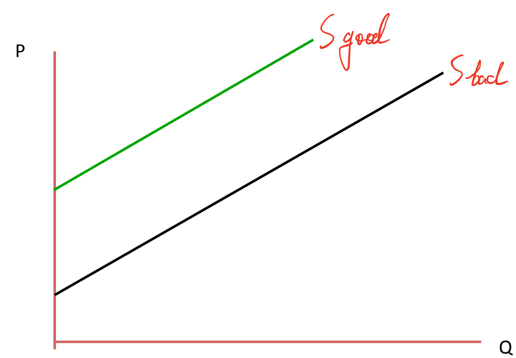
The model assumes that there is a large number of small sellers of used cars and a large number of small buyers (so the market is perfectly competitive). There are two types of used cars for sale *Good used cars* or *Bad used cars* "Lemons". If it was clear whether a used car was good or bad, then there would be two separate markets. There would be a market for good used cars with an equilibrium price and quantity. And there would be a market for bad used cars with a different equilibrium price and quantity. If there were lot of buyers and sellers in each market they would be perfectly competitive and efficient.

Now we change this simple world, so there is asymmetric information. To be precise, the sellers know whether their car is good or bad but the buyers don't know the quality of a particular car they are offered. *This means there can be only one market price for used cars.* Sellers of good used cars might like to ask for a higher price for their car and claim that their car was good. But there is no particular reason for buyers to believe this claim, because sellers of bad cars could also make such a claim. Thus the sellers of cars (good or bad) are forced to offer their cars for sale at the same price.

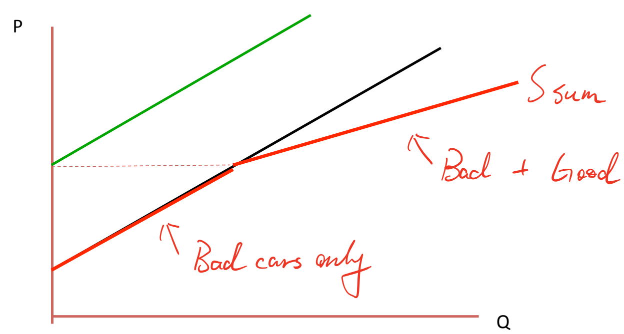
Prices are not creditable signals

Buyers have to decide whether to buy a given car or not at the market price. This decision is obviously dependent on what they think about the quality of cars being offered. One can make very extreme assumptions: the buyers are very pessimistic and believe all cars are bad, the buyers are very optimistic and believe all cars are good, or some intermediate belief. We will assume that buyers are not completely ignorant and are not dogmatic, but that their beliefs reflect the aggregate mix of good and bad cars being traded. That is, in an equilibrium the buyers know the general shares of good and bad cars for sale at the current price.

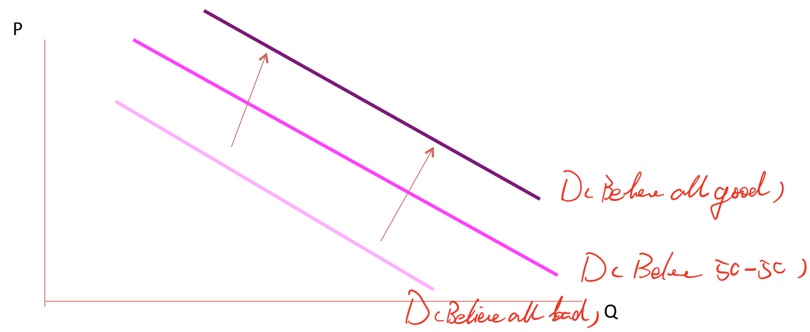
Now let us start analysing the model. First we plot the two supply curves for good and bad used cars. The supply of good cars (green) is above the supply of bad cars (black), because sellers of a good car require a higher price to be willing to part with it.



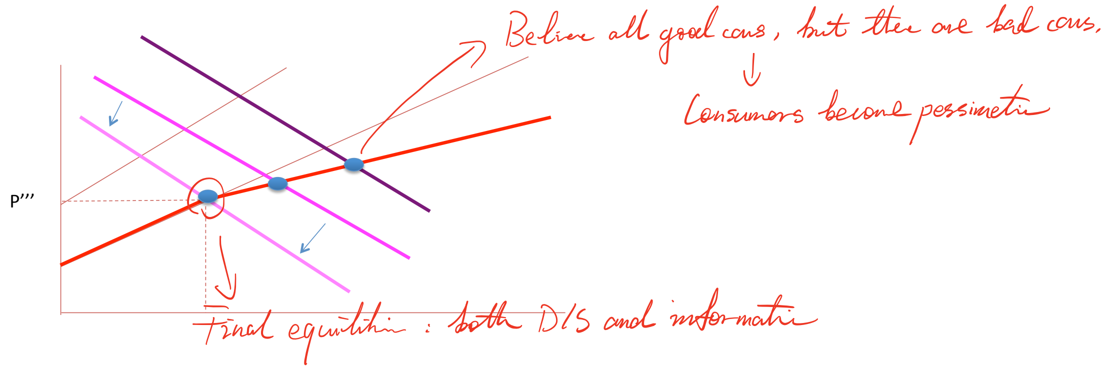

If these two supply curves are summed, we get the total supply of cars at each price (red). Observe that at low prices only the bad used cars are offered and as the price rises the share of good used cars increases.




Now we draw the demand for used cars from the buyers. This will obviously depend upon the buyers' beliefs about the quality of cars being offered for sale. This figure has 3 demand curves. The lowest (light pink) is the demand curve if the buyers believe all used cars are bad. The highest demand curve is the demand if the buyers think all used cars are good. And the intermediate demand is when the buyers think the mix is 50:50. We cannot draw the demands for all possible beliefs, but we assume that they will smoothly adjust downwards as the buyers become more pessimistic about quality.




Now we combine the demands and supply. Suppose first that the buyers were optimistic and believed that all the cars sold were good. Then demand equals supply at the rightmost blue dot in the figure below. At this point it is clear that there are not all good cars being sold but a mix of good and bad. Thus the buyers should become more pessimistic and the demand should shift down. As the demand shifts down, less and less good cars are sold at equilibrium. In some cases demand will continue to shift down until only bad cars a sold and buyers believe only bad cars are sold.




Notice that sellers of good cars would like to be able to convince the buyers that they are selling a good car, but they cannot because anything they say could also be said by low-quality sellers. That is, there is no credible way of distinguishing your good car from someone else's bad car. As a result low quality squeezes out high quality. This is called *Adverse Selection*. One interpretation of what is going on here is that buyers lower prices to protect themselves against the risk of buying a bad used car. But by lowering prices they encourage fewer good cars to be sold.

No market for the good car

Adverse selection is important in financial markets, because sellers may be selling a stock because the stock is bad and buyers may be buying because the stock is good. This leads to Bid-Ask spreads: Sellers get offered lower prices to protect the market from those people who are selling because the news is bad. Buyers get offered higher prices to protect the market from those people who are buying because the stock is good.

?

2.1. **Cursed Equilibrium.** In the Lemons model, the buyers face a problem of adverse selection. Another word for this problem is the winner's curse. That is, when a buyer succeeds in getting a car they may come to regret their success because the car is worse than they thought it would be. This is a feature of many markets where there is less than perfect information and competition among buyers—the winners curse is particularly problematic in auctions.

In the above model the buyers are very smart and they make the correct guess about the proportion of good and bad cars in the market, so they will not be disappointed on average. One extension to the lemons model would be to allow some of the buyers to have too optimistic beliefs about the quality of cars. That is, a share of the buyers are irrational and believe the cars are good while the rest of the buyers correctly perceive the share of good and bad cars. This allows us to incorporate irrationality into this world pretty easily. This is studied in papers on cursed equilibrium

## 3. The Insurance Market and Adverse Selection

Insurance Market

In this section we consider a second example of a perfectly competitive market with asymmetric information, but in a much more rigourous model. In this model one side of the market knows more about the product than the other: buyers know their riskiness and sellers don't. We will study an insurance market— not used cars—but the ideas are the same. In this case the uniformed side of the market (sellers) raise prices to protect themselves against the presence of bad types (high-risk buyers). But by raising prices they reduce the share of low-risk buyers in the market and increase further the share of high-risk buyers. The result we will get is that, although the market is competitive, the market is inefficient, because not all trade that ought to occur does occur. This type of outcome is called *market failure.*

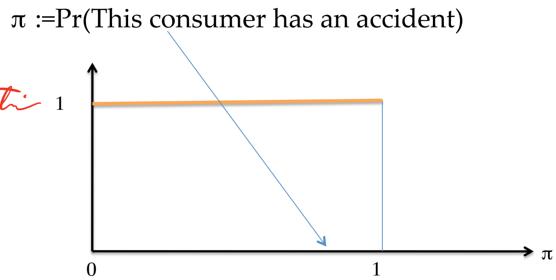
Later we will talk about two kinds of remedies for this inefficiency: *Signalling* (where the informed players are able to send credible signals about what they know). *Screening* (where players' choices indicate their type).

3.1. **The Model.** The model will derive a demand for insurance at each price. The price will also determine the type of customers who demand insurance. To find an equilibrium in this market we will then use the condition that perfectly competitive firms must make zero profit. It is this zero-profit condition that will ultimately determine the equilibrium in this market. The first step is to describe the buyers

Buy Side

3.1.1. *The Buyers of Insurance.* There is an infinity of buyers of insurance and the buyers come in many different types. Each buyer's type is described by $\pi$, which is their probability of having an accident. We suppose their is a continuum of buyers. And the aggregate distribution of buyers $0 \leq \pi \leq 1$ is a uniform distribution on the interval $[0, 1]$.

Represents a normalized population




So for example if we found that buyers with riskiness above $\pi = 3/4$ brought insurance, we would have a demand for insurance that was equal to $1/4$.

Now we describe the buyers' utility for insurance. We will assume that each buyer has a wealth $w > 0$, but if they have an accident they will incur some costs $L$. So as a result of their accident their wealth will fall to $w - L$. We will also assume each buyer has a utility of wealth $U(w) = w^{1/2}$.

Now let us consider the expected utility of a buyer that has a risk off accident $\pi$. With probability $\pi$ they will have an accident and end up with utility $(w-L)^{1/2}$. With probability $1-\pi$ they will not have an accident and end up with utility $w^{1/2}$. Thus the buyer has expected utility

$$\text{Utility of No Insurance} = \pi(w-L)^{1/2} + (1-\pi)(w)^{1/2}$$

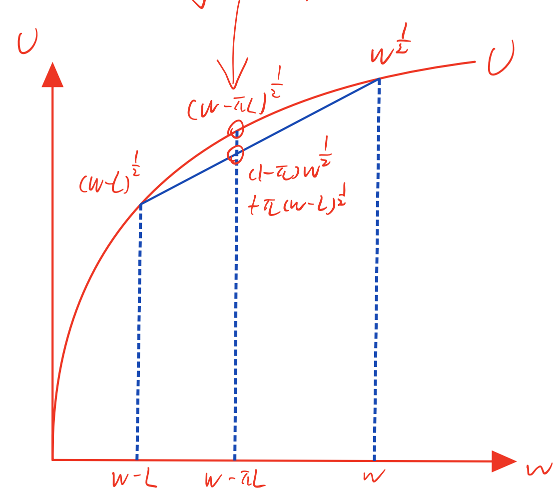
Now suppose the buyer buys insurance. That is, she pays a price $P > 0$ but when an accident occurs her loss gets compensated. This means when she has an accident she has utility $(w-P)^{1/2}$, because she has to pay the price of the insurance. When she doesn't have an accident she also has utility $(w-P)^{1/2}$, because she brought the insurance before she knew whether she was going to have an accident. Thus

$$\text{Utility of Insurance} = \pi(w-P)^{1/2} + (1-\pi)(w-P)^{1/2} = (w-P)^{1/2}$$

We can compare these utilities to figure out who is going to buy insurance. A buyer will buy insurance if

$$\text{Utility of Insurance} > \text{Utility of No Insurance}$$
$$(w-P)^{1/2} > \pi(w-L)^{1/2} + (1-\pi)(w)^{1/2}$$
$$\left(1-\frac{P}{w}\right)^{1/2} > \pi\left(1-\frac{L}{w}\right)^{1/2} + (1-\pi)$$
$$\pi\left[1-\left(1-\frac{L}{w}\right)^{1/2}\right] > 1-\left(1-\frac{P}{w}\right)^{1/2}$$
$$\boxed{\pi > \frac{1-(1-\frac{P}{w})^{1/2}}{1-(1-\frac{L}{w})^{1/2}}}$$

Utility from expected income




$$\frac{1-\sqrt{1-\frac{P}{w}}}{1-\sqrt{1-\frac{L}{w}}}$$

This inequality is very important to the subsequent analysis. First, notice that we do have adverse selection, because this inequality says that only types with high riskiness will buy insurance. That is, all the types with $\pi$ satisfying:

$$\boxed{h(P) = \frac{1 - \left(1 - \frac{P}{w}\right)^{1/2}}{1 - \left(1 - \frac{L}{w}\right)^{1/2}} < \pi \leq 1}$$

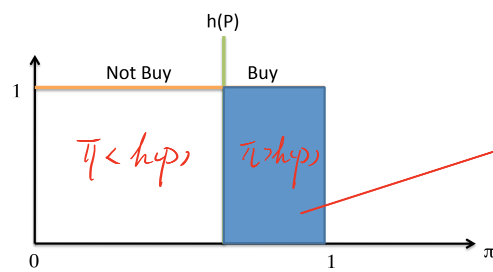
strictly prefer to buy insurance. We will write the function on the left here as $h(P)$ so all those people with $h(P) \leq \pi \leq 1$ will buy insurance. How does the number of people change who buy insurance when the price adjusts. Well it is not to hard to see that as $P$ increases so does $h(P)$ so fewer people buy insurance. It is also easy to see

p = 0 (free) : $$h(0) = \frac{1 - (1 - 0)^{1/2}}{1 - \left(1 - \frac{L}{w}\right)^{1/2}} = 0,$$ All people buy insurance

p = L (not worthy) $$h(L) = \frac{1 - \left(1 - \frac{L}{w}\right)^{1/2}}{1 - \left(1 - \frac{L}{w}\right)^{1/2}} = 1.$$ No people buys

So if insurance were free everyone would buy it and if insurance cost as much as the loss you were going to make no-one would buy it.

Once we know who is going to buy insurance we can work out the demand for insurance at each price. Below we have a picture of the mass of consumers who choose to buy insurance at the price $P$.




Area = Demand

The total area in this figure is the size of the demand for insurance at the price $P$. This area is equal to $1 \times (1 - h(P))$. So we now can write down the demand for insurance at the price $P$

$$\boxed{\text{Demand}(P) = 1 - h(P) = \frac{\left(1 - \frac{P}{w}\right)^{1/2} - \left(1 - \frac{L}{w}\right)^{1/2}}{1 - \left(1 - \frac{L}{w}\right)^{1/2}}}.$$

Sell Side

3.1.2. *The Sellers of Insurance.* There is one aspect of the buyers' behaviour that will particularly matter to the sellers and that is how risky are the customers when the current price is $P$.

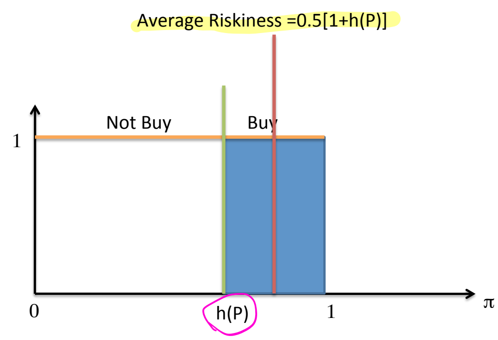
First let us investigate how sellers behave if they know the riskiness of their customer. Consider a customer with known risk $\pi$. A seller will have a cost equal to zero for the customer if they don't have an accident and a cost $L$ for the customer if they do. This makes the expected cost of the customer

$$\text{Cost of Customer}(\pi) = \pi L + (1-\pi)0 = \pi L.$$

Exp Profit : $P - \pi L$

As the seller charges the price $P$, the profit from a customer of known riskiness is $P - \pi L$. Thus, as long as $P \geq \pi L$ the firm will make a profit from selling to this customer if it knows their riskiness. $\rightarrow P = \pi L$

Now suppose the price is $P$ and the firm does ~~know~~ not know the riskiness of their customers. Of course, at an equilibrium they do know that only customers with riskiness above $h(P)$ will buy insurance. So what is the average riskiness of the customers that do buy insurance at the price $P$? In our particular example this can be easily seen from the following picture.




The halfway point between $h(P)$ and 1 is $(1 + h(P))/2$, so this is the average of the customers in the blue region.[1]

Now we have the average riskiness of the customers that buy when the price is $P$, we can work out the firm's costs from selling insurance in this case. It is

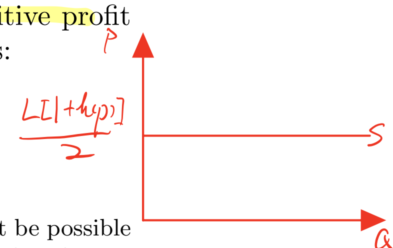
$$\text{Expected Costs} = L(1 + h(P))/2.$$

$\pi = P - L(1+h(P))/2$ per person

So now we can work out the firms expected profit from selling insurance as $P - L(1 + h(P))/2$.

We know that firms will enter the industry as long as the make strictly positive profit from selling insurance so we could now write the supply curve in this market as:

$$\text{Supply} = \begin{cases} \infty & : \quad P \geq L(1 + h(P))/2 \\ 0 & : \quad P < L(1 + h(P))/2 \end{cases}$$




[1]BUT, this is the average riskiness here because the distribution is flat. In general it will not be possible to just draw a line like this. The correct way to calculate the average riskiness for an arbitrary distribution $f(\pi)$ would be the following conditional expectation

$$\text{Average Riskiness of Buyers in}[h(P), 1] = \frac{\int_{h(P)}^{1} \pi f(\pi) d\pi}{\int_{h(P)}^{1} f(\pi) d\pi}$$

where $f(\pi) = 1$ is the density of the distribution of $\pi$. You can check this here by substituting in $f(\pi) = 1$ and doing the calculation.

This gives the supply curve as a horizontal line where $P = L(1 + h(P))/2$. Where this crosses the demand curve above gives the equilibrium in this market.

Equilibrium

3.1.3. *Equilibrium in a Perfectly Competitive Economy.* To find the equilibrium price we must find where supply equals demand. As supply will be zero or infinite if firms are not making zero profits, to find equilibrium we find the place where firms make exactly zero profit. That is, we must solve the equation

Perfect Competition : Zero profit : P = L

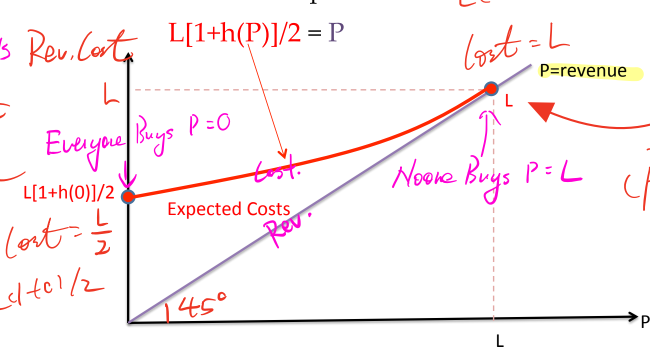
$$P = L(1 + h(P))/2.$$

Here is a picture of the two sides of this equation

Only equilibrium

Perfect Competition Firm's Rev. Cost

Revenue / Cost of the insurance company




(Possibility 1)

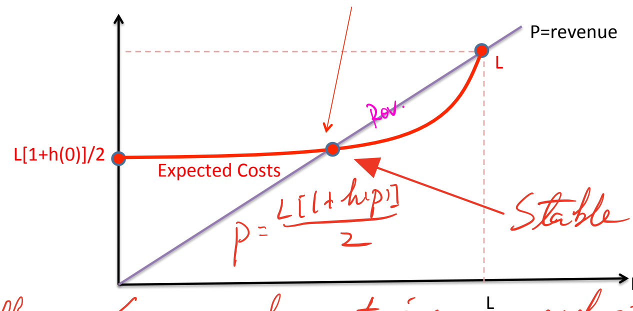
(People with risk > h(p) buy)

From this alone we can see that there must be an equilibrium where $P = L$ and no-one buys insurance. But maybe there are more equilibria and the picture looks like

2 Solutions:
(1) Signals
(2) Menu of distinct contracts (Screening)

Only if Slope of $\left(\frac{1+h(p)}{2}\right) > 1$ when $P = L$

(Possibility 2)




Stable equilibrium

Inefficiency: Some people want insurance and willing to pay the cost, but they cannot get it.

It is quite hard to determine this. One way is to look at the slope of the function $L(1 + h(P))/2$ where $P = L$. this is because if the slope of this function is more than 1 when $P = L$ it means the picture looks like the above. (This has been set as an exercise.)

3.1.4. *Equilibrium with Monopoly.* We could also consider the problem of a monopolist selling insurance. Then, if the monopolist set the price $P$ they would sell to $1 - h(P)$ customers. Each would pay the price $P$ and on average the customers would cost the firm $L(1 + h(P))/2$. Thus the monopolist would make the profit

$$(1 - h(P))(P - L(1 + h(P))/2).$$

The monopolist would then choose $P$ to maximise this expression.

## 4. Two-Part Tariffs, Screening and Adverse Selection

Signaling

We have now shown in two examples that adverse selection can result in extreme market failure where only the highest possible insurance type and only the worst possible customer gets insurance. This is a massive inefficiency, because almost nobody gets insurance (buys a good car) although everybody wants one. There are two kinds of solution to this problem. The first is to use *signalling*—this allows customers to tell insurance suppliers (credibly) their type by showing they have good health if they are buying life insurance or a good driving record if they are buying car insurance. The second is *screening*—here insurance companies offer products that only low-risk types want to buy and get the different types of consumer to choose different types of product. (This occurs when insurance companies offer copays, excesses or deductibles because only low risk customers are willing to accept this kind of policy.) Notice the difference in the order of moves. In screening, first the firms select a list of a list of different products to offer for sale and then the different types of customers choose one. In signalling, first the customers choose a signal then the firms decide whether to sell to them.

We will now build a model of screening where a monopolist offers two deals call them deal A and deal B. The deals are carefully constructed so the A-type customers prefer to buy the A deal and the B-type customers prefer to buy the B deal. Thus the firm can get the customers to tell it what type they are and, hopefully, increase their profits as a result.

**4.1. No Adverse Selection.** We will begin by describing a simple model of monopoly. The demand the monopolist faces comes from (for now) identical customers. Each customer has an income $y$. Their value from buying $x$ units of a good is described by a function $\psi(x)$. We will assume $\psi(.)$ has the following properties

Utility from purchasing :
$$\boxed{\psi(0) = 0, \qquad \frac{d\psi}{dx} > 0, \qquad \frac{d^2\psi}{dx^2} < 0.}$$

So $\psi(.)$ is an increasing function (the consumer likes to have more $x$) but has a decreasing slope (so the Marginal benefit of additional units of $x$ decreases). When our consumer buys $x$ units of the good and pays $F$ in total for these units they get a total value equal to

$$\psi(x) - F + y,$$

Utility - Cost + Income (Here, $F = px$ is variable cost)

if they buy nothing they have value $y$.

If our monopolist chooses to set the price $p$ for the good, each consumer will choose a quantity to buy that maximizes their value. That is they solve the problem

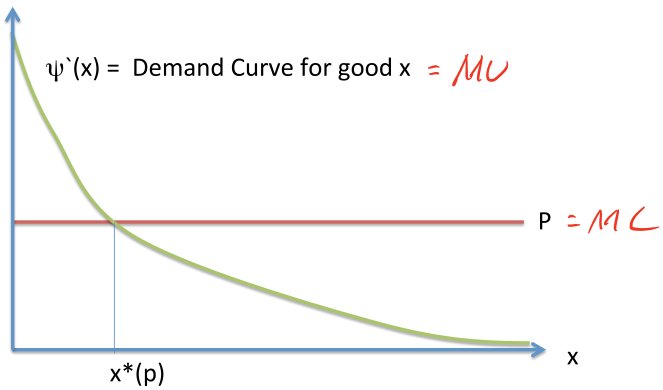
$$\text{Maximise}_{x\geq 0}\ \psi(x) - px + y.$$

To solve this we differentiate with respect to $x$, set equal to zero and get the first order condition

$\psi'(x) - p = 0$

$$p = \psi'(x), \qquad \text{or} \qquad p = \psi'(x^*(p)).$$

This can be interpreted as setting the marginal cost of extra units $p$ equal to the marginal benefit of extra units $\psi'(x)$. As we assumed the function $\psi''$ was decreasing this is the picture of what's going on here.




We can also interpret $\psi'(.)$ as this individual's demand curve.

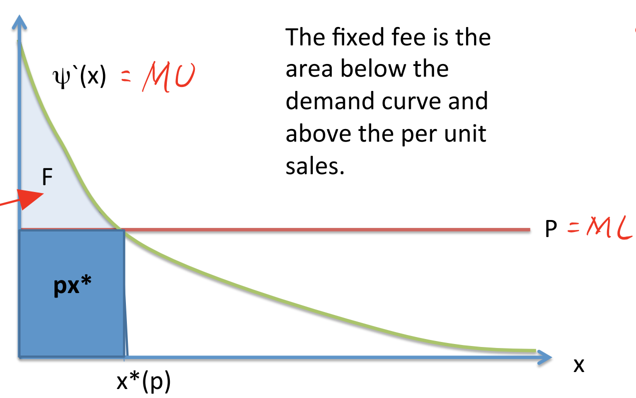
We can see from this picture that there is value that the customer is not paying for. The monopolist could charge the customer a fixed fee $F$ as well as a price per unit $p$ and the customer would still be willing to buy $x^*$ units. What is the biggest fixed fee the customer is willing to pay? The customer is willing to pay any fee $F$ that gives her greater utility than not buying anything, that is: Max $\psi(x) - px - F + y$ (Here, $F$ is a fixed fee)

2 Part Tariff

$$\psi(x^*) - px^* - F + y \geq \psi(0) + y \quad \Leftrightarrow \quad \psi(x^*) - px^* \geq F.$$

So the customer will buy the deal $x^*$ at the price $p$ and fee $F$ if $\psi(x^*) - px^* \geq F$ and will not buy if $\psi(x^*) - px^* < F$. That is, the customer either buys the quantity where marginal benefit equals marginal cost (when the value it receives is greater than the fixed cost) or nothing. Another way of saying this is that if the fixed fee is less than the total value the customer obtains then she is prepared to buy the good. The picture shows the optimal fixed fee where $\psi(x^*) - px^* = F$.

$$\begin{cases} \text{buy } x^*(p), \text{ if } \psi(x^*) - px^* \geq F \\ \text{buy } 0 \text{ if } \psi(x^*) - px^* \leq F \end{cases}$$




Max $F$ possible for the consumer to start buying $= \psi(x^*) - px^*$

Now we can think about what is the price-fee combination that maximises the firm's profit. We suppose that the monopolist has a cost per unit of $c > 0$. As usual one of the

Monopoly's Revenue = Total blue area

most important things is writing down profit correctly:

$$\begin{aligned}\text{Profit} &= \text{Revenue} - \text{Cost}\\ &= px^* + F - cx^*\end{aligned}$$

Recall that the optimal fixed cost satisfies the condition $\psi(x^*) - px^* = F$ so if this is substituted in we get

$$\begin{aligned}\text{Profit} &= px^* + \psi(x^*) - px^* - cx^*\\ &= \psi(x^*) - cx^*\end{aligned}$$

Only $x^*$ matters

Thus to maximise profit our monopolist wants to choose $x^*$ to make the above expression as large as possible, that is, differentiating

$$0 = \psi'(x^*) - c$$

$p = MC$

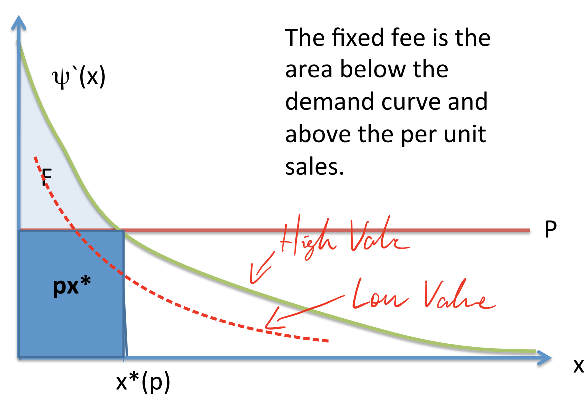
so the optimal thing for the monopolist to do is to set $p = c$ (price equals marginal costs) and all the profit comes from the fixed fee.

Summary: In this simple model the optimal thing for a monopolist to do is to sell the quantity $x^*$ where Marginal Benefit equals Marginal Cost, $\psi'(x^*) = c$. Then it should make the customer pay all their utility to them as a fixed fee and a price per unit $\psi(x^*) = F + px^*$.

4.2. **Introducing adverse Selection.** Now we introduce adverse selection by introducing two types of customers. Those who get high value from the product and those who get a lower value from it. The way we do this is by allowing the two types to have different value functions.

$$\text{High Value} = \alpha\psi(x) + y \qquad \text{Population Share} = \phi;$$
$$\text{Low Value} = \beta\psi(x) + y \qquad \text{Population Share} = 1 - \phi.$$

We assume $\alpha > \beta$ so the utility and marginal utility of the high value customers is above that of the low value customers.




Adverse selection is now present in this model, because the monopolist would like to offer to sell the good at marginal cost and take all the value. (But if it did this it would have to offer a lower fixed fee to the low-value customers and a higher fixed fee to the high-value customers.) This is impossible as all the high-value customers would then pretend to be low-value customers.

The screening solution to the adverse selection problem. Now we will describe the menu of deals the monopolist. The first deal $(x_a, F_a)$ is a quantity and a payment (this payment covers both the fixed fee and the per unit price). The monopolist would like this deal to be bought by the high-value customers. The second deal $(x_b, F_b)$ is a second quantity and payment, which the monopolist would like to be bought by the low-value customers. In total the monopolist will offer a menu of deals $((x_a, F_a); (x_b, F_b))$. Suppose the monopolist were successful in getting the two types of customers to buy the appropriate deal, then it would make the profit.

$$\text{Profits} = \phi(F_a - cx_a) + (1 - \phi)(F_b - cx_b).$$

Screening

4.2.1. *Getting the Customers to Buy: Individual Rationality Constraints.* The first thing to notice is that the high-value customers are not willing to buy any deal. If the price was very high they would rather not buy anything. If they are going to buy the deal $(x_a, F_a)$ then they need to get positive from it. That is,

$$\text{High-Value's Utility from A-Deal} \geq \text{Utility from not buying}$$
$$\alpha\psi(x_a) - F_a + y \geq y.$$

Or $\alpha\psi(x_a) \geq F_a$. Similarly if the monopolist is going to get the low value types to buy the B-Deal, then

$$\text{Low-Value's Utility from B-Deal} \geq \text{Utility from not buying}$$
$$\beta\psi(x_b) - F_b + y \geq y.$$

Or $\beta\psi(x_b) \geq F_b$. The result of these calculations are two constraints on the kinds of deals the monopolist can choose to maximise its profit

$$\alpha\psi(x_a) \geq F_a \qquad \beta\psi(x_b) \geq F_b.$$

These are called *Individual Rationality Constraints* which kind of means you cannot force customers to buy things they don't want to buy. They are sometimes also called participation constraints.

4.2.2. *Getting the Customers to Buy the Right Deal: Incentive Compatibility Constraints.* The second thing the monopolist needs to worry about is that the high-value customers actually choose to buy the low-value customers' deal.$(x_b, F_b)$ was offering the good at a very low price the high-value types would rather buy this than $(x_a, F_a)$. If they are going to buy the deal $(x_a, F_a)$ then they need to get higher utility from it than from $(x_b, F_b)$. That is,

$$\text{High-Value's Utility from A-Deal} \geq \text{High-Value's Utility from B-Deal}$$
$$\alpha\psi(x_a) - F_a + y \geq \alpha\psi(x_b) - F_b + y.$$

Incentive for consumers to buy the right deal

Or $\alpha\psi(x_a) - F_a \geq \alpha\psi(x_b) - F_b$. Similarly if the monopolist is going to get the low value types to buy the B-Deal and not the A-Deal

$$\text{Low-Value's Utility from B-Deal} \geq \text{Low-Value's Utility from A-Deal}$$
$$\beta\psi(x_b) - F_b + y \geq \beta\psi(x_a) - F_a + y.$$

Or $\beta\psi(x_b) - F_b \geq \beta\psi(x_a) - F_a$. The result of these calculations are two constraints on the kinds of deals the monopolist can choose to maximise its profit

$$\alpha\psi(x_a) - F_a \geq \alpha\psi(x_b) - F_b \qquad \beta\psi(x_b) - F_b \geq \beta\psi(x_a) - F_a.$$

These are called *Incentive Compatibility Constraints* which kind of means you are giving the customers the incentive to buy the things you want them to buy.

4.2.3. *Reducing the Number of Constraints.* Now we have found four different constraints on the monopolist's choice of a menu $((x_a, F_a); (x_b, F_b))$. Here is the list

IRCs & ICCs

Individual Rationality Constraints : IRCs

$$\alpha\psi(x_a) - F_a \geq 0 \tag{1}$$
(Guaranteed if the rest 3 hold)
$$\beta\psi(x_b) - F_b \geq 0 \tag{2}$$

Incentive Compatibility Constraints : ICCs

$$\alpha\psi(x_a) - F_a \geq \alpha\psi(x_b) - F_b \tag{3}$$
$$\beta\psi(x_b) - F_b \geq \beta\psi(x_a) - F_a \tag{4}$$

Consider the individual rationality constraint for the high value types, (1), this is ensured because the A's could pretend to be low-value types and they can get positive value when they do this. Here is the argument. First the high values don't want the low-value deal, this is the constraint (3)

$$\alpha\psi(x_a) - F_a \overbrace{\geq}^{(3)} \alpha\psi(x_b) - F_b$$

Then we notice that $\alpha > \beta$, so $\alpha\psi(x_b) \geq \beta\psi(x_b)$ and

$$\alpha\psi(x_a) - F_a \overbrace{\geq}^{(3)} \alpha\psi(x_b) - F_b \overbrace{\geq}^{\alpha\geq\beta} \beta\psi(x_b) - F_b.$$

Finally we use the fact that this last expression is the low type's utility from buying the B-deal, which must be positive.

$$\alpha\psi(x_a) - F_a \overbrace{\geq}^{(3)} \alpha\psi(x_b) - F_b \overbrace{\geq}^{\alpha\geq\beta} \beta\psi(x_b) - F_b \overbrace{\geq}^{(2)} 0.$$

Thus by using (3) and $\alpha \geq \beta$ and (2) we can conclude that $\alpha\psi(x_a) - F_a \geq 0$ which is (1). Thus this condition is implied by the remaining three constraints and we only need them. Here is the list:

$$\beta\psi(x_b) - F_b \geq 0, \tag{5}$$
$$\alpha\psi(x_a) - F_a \geq \alpha\psi(x_b) - F_b, \tag{6}$$
$$\beta\psi(x_b) - F_b \geq \beta\psi(x_a) - F_a. \tag{7}$$

Now think about adding a small amount $f$ to the payments $F_a$ and $F_b$. If you did this the new constraints would be:

$$\begin{aligned}
\beta\psi(x_b) - F_b - f &\geq 0, \\
\alpha\psi(x_a) - F_a - f &\geq \alpha\psi(x_b) - F_b - f, \\
\beta\psi(x_b) - F_b - f &\geq \beta\psi(x_a) - F_a - f.
\end{aligned}$$

This change would increase what the monopolist earned from all its customers. And so would have to be good for the monopolist. The change would not be a problem for the two last constraints, because the $f$ affects both sides of these equally. However, it might be a problem for the first constraint, because as $f$ gets bigger eventually it would become negative. Thus we can deduce that the first constraint has to be an equality when the monopolist is maximising its profit.

① $F_a$, $F_b$ can ↑ by the same amount until

② $F_a$↑ until this becomes =

$$\begin{aligned}
\beta\psi(x_b) - F_b &= 0 \\
\alpha\psi(x_a) - \overbrace{F_a}^{+} &\geq \alpha\psi(x_b) - F_b \\
\beta\psi(x_b) - F_b &\geq \beta\psi(x_a) - \overbrace{F_a}^{+}
\end{aligned}$$

Now the first constraint pins down $F_b$, but the monopolist would still like to make $F_a$ as big as possible. This will be easy in the last constraint, because it just makes the right smaller, but will be harder for the middle constraint as the left gets smaller. We can conclude that the monopolist will increase $F_a$ until the middle constraint is also an equality:

Final Constraints

$$\begin{aligned}
\beta\psi(x_b) - F_b &= 0, \quad \text{IRC} \\
\alpha\psi(x_a) - F_a &= \alpha\psi(x_b) - F_b, \\
\beta\psi(x_b) - F_b &\geq \beta\psi(x_a) - F_a. \quad \text{ICC}
\end{aligned}$$

We have now done with manipulation of the constraints and we will consider the monopolist's optimisation.

4.2.4. *Writing Down the Monopolist's Optimisation.* Now we will consider how the monopolist will choose the menu $((x_a, F_a); (x_b, F_b))$ to maximise its profits. This is the problem

$$\max_{((x_a,F_a);(x_b,F_b))} \phi(F_a - cx_a) + (1-\phi)(F_b - cx_b), \qquad \text{subject to}$$

$$\begin{aligned}
\beta\psi(x_b) - F_b &= 0 \\
\alpha\psi(x_a) - F_a &= \alpha\psi(x_b) - F_b \\
\beta\psi(x_b) - F_b &\geq \beta\psi(x_a) - F_a
\end{aligned}$$

We have already simplified this problem by implementing the reduction in the constraints we talked about in the previous section. Now let us use the 2 equality constraints to

Simplification



substitute out $F_a$ and $F_b$. Substituting out $F_b = \beta\psi(x_b)$ we get

$$\max_{((x_a,F_a);(x_b,F_b))} \phi(F_a - cx_a) + (1-\phi)(\beta\psi(x_b) - cx_b), \qquad \text{subject to}$$

$$\alpha\psi(x_a) - F_a = \alpha\psi(x_b) - \beta\psi(x_b)$$
$$0 \geq \beta\psi(x_a) - F_a$$

Now if we take the remaining equality constraint we have $F_a = \alpha\psi(x_a) + (\beta-\alpha)\psi(x_b)$. If this is substituted in to our optimisation we now have the problem

$$\max_{((x_a,F_a);(x_b,F_b))} \phi(\alpha\psi(x_a) + (\beta-\alpha)\psi(x_b) - cx_a) + (1-\phi)(\beta\psi(x_b) - cx_b),$$
$$\text{subject to } \ 0 \geq \beta\psi(x_a) - \alpha\psi(x_a) - (\beta-\alpha)\psi(x_b)$$

Or tidying up this becomes,

$$\max_{((x_a,F_a);(x_b,F_b))} \phi\alpha\psi(x_a) + (\beta-\phi\alpha)\psi(x_b) - \phi cx_a - (1-\phi)cx_b,$$
$$\text{subject to } \ (\alpha-\beta)\psi(x_a) \geq (\alpha-\beta)\psi(x_b)$$

The constraint can be made even simpler, because dividing by $(\alpha-\beta)$ says $\psi(x_a) \geq \psi(x_b)$. Then we know that $\psi(.)$ is an increasing function, so the only way this constraint can hold is if $x_a \geq x_b$. So the problem we need to solve is

$$\max_{(x_a;x_b)} \phi\alpha\psi(x_a) + (\beta-\phi\alpha)\psi(x_b) - \phi cx_a - (1-\phi)cx_b, \qquad \text{s.t. } \ x_a \geq x_b.$$

We have got rid of all the constraints and we now have one very simple condition.

4.2.5. *Solving the Monopolist's Optimisation.* We now have a simple inequality constrained optimisation that tells us what the monopolist wants to do. Let us write down the Lagrangean for this problem

$$L(x_a, x_b) = \phi\alpha\psi(x_a) + (\beta-\phi\alpha)\psi(x_b) - \phi cx_a - (1-\phi)cx_b + \lambda(x_a - x_b).$$

Finding the first order conditions.

$$\boxed{\begin{aligned} \frac{\partial L}{\partial x_a} &= \phi\alpha\psi'(x_a) - \phi c + \lambda &&= 0, \\ \frac{\partial L}{\partial x_b} &= (\beta-\phi\alpha)\psi'(x_b) - (1-\phi)c - \lambda &&= 0. \end{aligned}}$$

① Binding

Now we need to worry about whether the constraint binds or not. Suppose the constraint was binding ($\lambda > 0$) and $x_a = x_b = x$ then if these two equations are added together we get $\beta\psi'(x_a) = c$. Then substitute this into the first equation to get

(Not Possible)

$$\phi(\alpha-\beta)\psi'(x_a) + \lambda = 0,$$

which is impossible, because everything on the left is positive. This contradiction tells us the constraint cannot be binding and $\lambda = 0$.

Answer :



② Not Binding

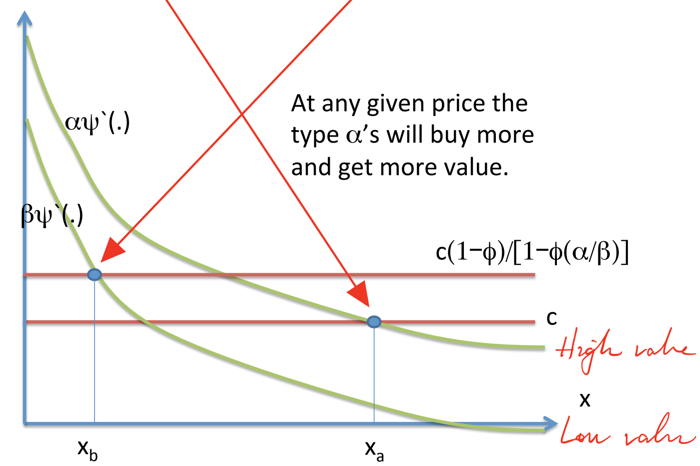
$\lambda = 0$

Substituting the fact that the constraint is not binding into the first order conditions we get:

$P = MC(\text{High}) = \text{Cost of Production}$ $\qquad$ $P$ for low value $>$ Cost of production

$$\alpha\psi'(x_a) = c, \qquad \beta\psi'(x_b) = \frac{\beta(1-\phi)}{\beta - \phi\alpha}c > c.$$

So we learn that the high value types get sold their optimal quantity and the low value types get sold less than their optimal quantity.



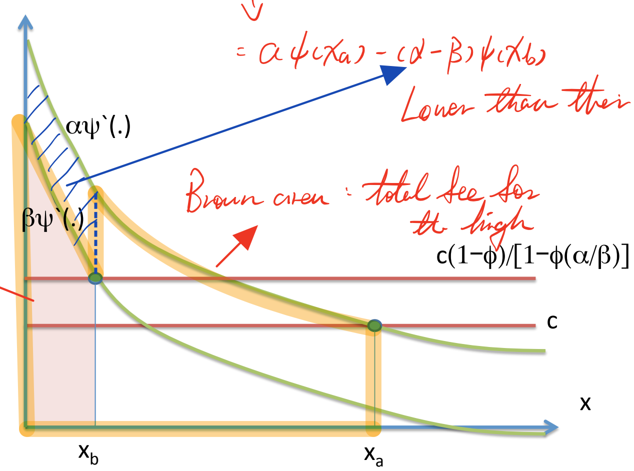

What fees do they pay? We substituted these out of the problem with the substitutions $F_b = \beta\psi(x_b)$ and $F_a = \alpha\psi(x_a) + (\beta - \alpha)\psi(x_b)$. The first of these, $F_b = \beta\psi(x_b)$, says the low-value types get not extra value. The fee they pays takes all their utility.

$= \alpha\psi(x_a) - (\alpha - \beta)\psi(x_b)$

Lower than their total utility (to prevent them from pretending to be the low value)

Red Area: Total Fee paid by the low value $= \beta\psi(x_b)$

Brown area: total fee for the high




But notice the high-value types don't pay all their utility to the monopolist. They pay less than their total utility: $F_a = \alpha\psi(x_a) - (\alpha - \beta)\psi(x_b)$. This is because they could pretend to be low-value types. If they make this pretence they get strictly positive utility $\alpha\psi(x_b) - F_b = (\alpha - \beta)\psi(x_b) > 0$. So this is the amount of utility they get to keep.

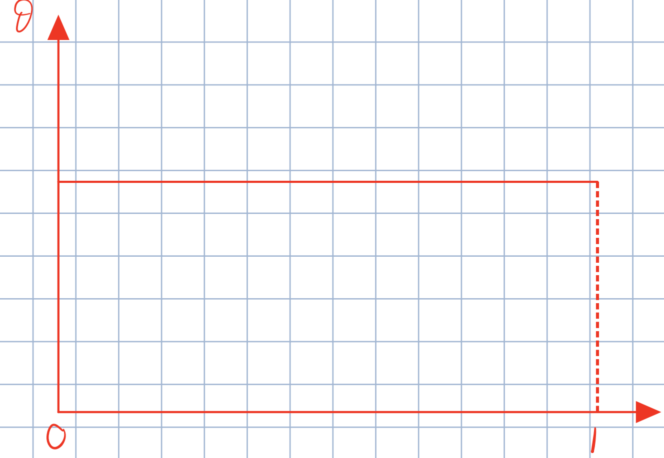
To conclude: The right way to solve the adverse selection problem for the monopolist is to efficiently serve the high-value customers and treat the low-value types suboptimally.

Lecture November 16

B.I.2 Patients have many different levels of illness. The level of illness for a given patient is described by $\theta$. (Assume that $\theta$ has a uniform distribution on the interval $0 \leq \theta \leq 1$.) A patient that does not go to the doctor (is untreated) has utility $-\theta$. A patient that goes to the doctor has a probability $p$ of being cured and getting utility zero but if they are uncured they still have utility $-\theta$.

(a) *If it costs $c > 0$ to travel to the doctor, what is the expected utility of going to the doctors?* [1 marks]

a)




Don't go to the doctor: $U = -\theta$

Go to the doctor: $U = p \times 0 - (1-p)\theta - c$

$= -\theta + p\theta - c$

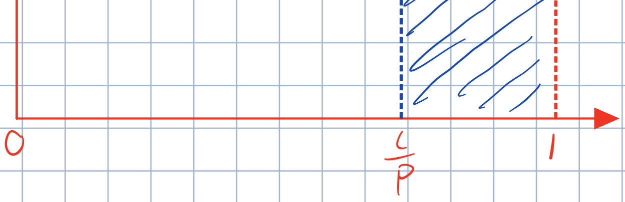
b) Who goes to the doctors?

$-\theta + p\theta - c > -\theta$

$p\theta - c > 0$

$p\theta > c$

$\theta > \frac{c}{p}$




c) Average level of illness of people who go to the doctor?

$$E(\theta \mid \theta > \tfrac{c}{p}) = \frac{1}{2}\left(\frac{c}{p}+1\right) = \frac{c}{2p} + \frac{1}{2}$$

$$E(\theta \mid \theta > \tfrac{c}{p}) = \frac{\int_{\frac{c}{p}}^{1} \theta \times f(\theta)\, d\theta}{\int_{\frac{c}{p}}^{1} f(\theta)\, d\theta} = \frac{\int_{\frac{c}{p}}^{1} \theta \times 1\, d\theta}{\int_{\frac{c}{p}}^{1} 1\, d\theta} = \frac{\left[\frac{1}{2}\theta^2\right]_{\frac{c}{p}}^{1}}{\left[\theta\right]_{\frac{c}{p}}^{1}} = \frac{\frac{1}{2}\left(1 - \frac{c^2}{p^2}\right)}{1 - \frac{c}{p}}$$

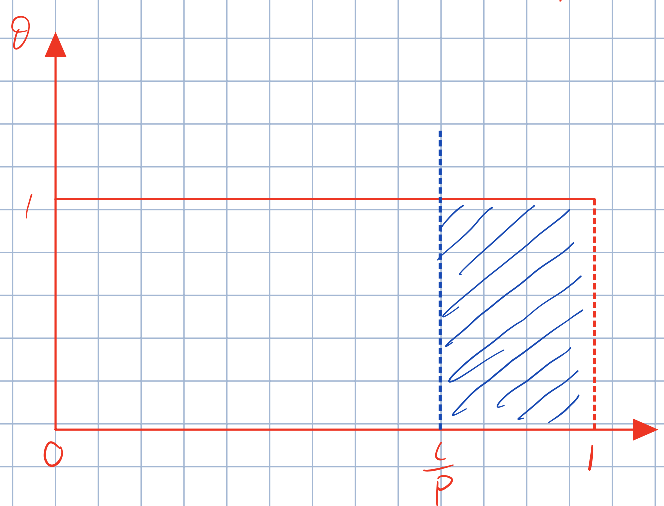
$$= \frac{1}{2}\left(1 + \frac{c}{p}\right)$$

$$= \frac{c}{2p} + \frac{1}{2}$$

d) How many people go to the doctor?

$$1 \times \left(1 - \frac{c}{p}\right) = 1 - \frac{c}{p}$$

e) Doctors' cost assumed to be $\theta^2$.
Calculate the cost in terms of $c$ and $p$.




$$\text{Cost} = \int_{\frac{c}{p}}^{1} \theta^2 f(\theta)\, d\theta = \left[\frac{1}{3}\theta^3\right]_{\frac{c}{p}}^{1}$$

Integrate, don't be too confident!

A distinct question: $\theta \sim [0, 2]$

Staying at home: $U = \log\theta$

Go to the doctors: $E[U] = p(0-c) + (1-p)(\log\theta - c)$

$p(0-c) + (1-p)(\log\theta - c) > \log\theta$ Mind $\log\theta$ can be negative

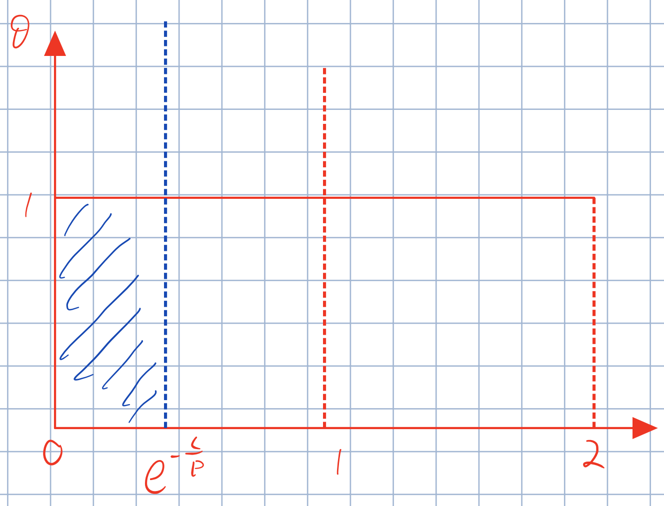
$(1-p)\log\theta - c > \log\theta$

$-c > \log\theta - (1-p)\log\theta$

$-c > p\log\theta$

$-\frac{c}{p} > \log\theta$

$e^{-\frac{c}{p}} > \theta$



Lecture November 23

B.I.2 A firm has two types of customer. High-value customers have utility $3 - \frac{1}{1+x} - p$ if they buy $x$ units at the price $p$. Low-value customers have utility $2 - \frac{1}{1+x} - p$ if they buy $x$ units at the price $p$. There are 10 customers of each type. The firm has the marginal cost $c < 1$ for producing $x$. If customers do not buy anything they get zero utility

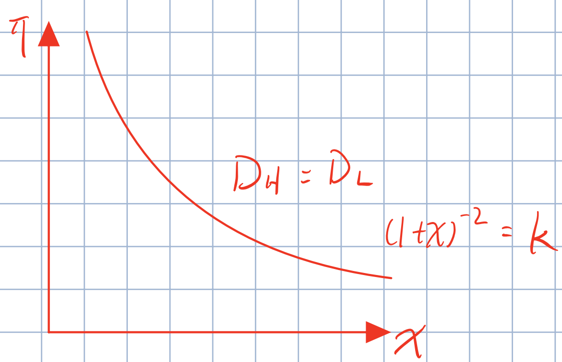
$p$ is the total price

$U_{high} = 3 - \frac{1}{1+x} - p$ 10 customers

$U_{low} = 2 - \frac{1}{1+x} - p$ 10 customers Not buy: $U = 0$

$MC = c < 1$

* (Not in Question) $\frac{dU_H}{dx} = \frac{d(3 - \frac{1}{1+x} - xk)}{dx} = (1+x)^{-2} - k = 0$




Demands are the same $\longrightarrow$ Unable to screen

Qa: $(x, p)$ (I) Only H buy?

$$\begin{cases} 3 - \frac{1}{1+x} - p \geq 0 \\ 2 - \frac{1}{1+x} - p < 0 \end{cases}$$

$$3 - \frac{1}{1+x} - p \geq 0 > 2 - \frac{1}{1+x} - p$$

$$3 \geq \frac{1}{1+x} + p > 2$$

(II) Both buy?

$$\begin{cases} 3 - \frac{1}{1+x} - p \geq 0 \\ 2 - \frac{1}{1+x} - p \geq 0 \end{cases}$$

$$2 > \frac{1}{1+x} + p$$

(III) Only L buy?

$$\begin{cases} 3 - \frac{1}{1+x} - p < 0 \\ 2 - \frac{1}{1+x} - p \geq 0 \end{cases}$$

$$3 - \frac{1}{1+x} - p < 0 \leq 2 - \frac{1}{1+x} - p$$

Not possible.

Q6 Selling only to H. Max $\pi$

$$3 \geq \frac{1}{1+x} + p > 2$$

$$3 = \frac{1}{1+x} + p^*$$

$$p = 3 - \frac{1}{1+x}$$

$$\pi = 10(p - cx)$$

$$= 10(3 - \frac{1}{1+x} - cx)$$

$$\frac{\partial \pi}{\partial x} = 10(1+x)^{-2} - 10c = 0$$

$$(1+x)^{-2} = c$$

$$(1+x)^2 = 1/c$$

$$1+x = \frac{\sqrt{c}}{c}$$

$$x = \frac{\sqrt{c}}{c} - 1$$

$$\pi^* = 10(3 - \sqrt{c} - \sqrt{c} - c)$$

$$= 30 - 20\sqrt{c} - 10c$$

Qc: Selling to H & L. Max $\pi$.

$$2 > \frac{1}{1+x} + p$$

$$2 = \frac{1}{1+x} + p$$

$$p = 2 - \frac{1}{1+x}$$

$$\pi = 20(p - xc)$$

$$= 20(2 - \frac{1}{1+x} - xc)$$

$$\frac{\partial \pi}{\partial x} = 20(1+x)^{-2} - 20c = 0$$

$$(1+x)^{-2} = c$$

$$x = \frac{\sqrt{c}}{c} - 1$$

$$p = 2 - \sqrt{c}$$

$$\pi^* = 20(2 - \sqrt{c} - \sqrt{c} - c)$$

$$= 40 - 40\sqrt{c} - 20c$$

Qb/c: Better off?

$$\Delta \pi^* = 40 - 40\sqrt{c} - 20c - (30 - 20\sqrt{c} - 10c)$$

$$= 10 - 20\sqrt{c} - 10c$$

When: $10 - 20\sqrt{c} - 10c > 0$

$$1 - 2\sqrt{c} - c > 0$$

Qe: $\overset{a}{(x_a, p_a)}$ $\overset{b}{(x_b, p_b)}$ to diff customers.

Find the conditions for H -> a, L -> b.

ICCs: $3 - \frac{1}{1+x_a} - p_a \geq 3 - \frac{1}{1+x_b} - p_b$

$$2 - \frac{1}{1+x_b} - p_b \geq 2 - \frac{1}{1+x_a} - p_a$$

IRCs: $3 - \frac{1}{1+x_a} - p_a \geq 0$ (Redundant)

$$\alpha - \frac{1}{1+x_b} - p_b \geq 0$$

$$\frac{1}{1+x_b} + p_b \leq \frac{1}{1+x_a} + p_a$$

$$\frac{1}{1+x_a} + p_a \leq \frac{1}{1+x_b} + p_b$$

$$\frac{1}{1+x_a} + p_a = \frac{1}{1+x_b} + p_b$$

Unable to screen customers.

# 7. MORAL HAZARD AND INCENTIVES

MARTIN CRIPPS

## 1. Introduction

Economic relationships often have the form of a *Principal* who contracts with an *Agent* to take certain actions that the principal cannot observe directly. In each of these cases the

Principal - Agent

Table 1. Examples of Principal-Agent Pairs

| **Principal** | **Agent** | **Hidden Action** |
|---|---|---|
| Physician | Patient | exercise |
| Employer | Employee | effort |
| Stockholder | Manager | strategy |
| Insurer | insuree | caution |
| Bank | Borrower | prudence |
| Teacher | Student | study time |

principal wants to control what the agent does, because this affects the principal and the agent's payoffs, but has limited ability to directly control the agent's actions. These are all examples of what economists call *Moral Hazard.* We think of moral hazard as arising in situations where individuals have hidden or secret actions they can take.

For there to be an economic problem here it is necessary that three conditions hold: (1) The Principal and Agent have different payoffs/objectives. (2) The Agent has actions it can take. (3) The actions the Agent takes are not directly monitored by the Principal. If any one of the three conditions fail, then there is not problem. If (1) is not true then the Principal can rely on the Agent to take actions they both like. If (2) is not true then there is nothing the Agent can do that will harm the Principal. And if (3) is not true then the Principal can choose to only reward the Agent if they take the Principal's most preferred action.

If an incentive problem/moral hazard problem exists there will be a loss in efficiency. The Principal will try to devise a reward/incentive scheme that will encourage the Agent to take actions the Principal likes. These incentives impose an economic cost and harm efficiency. We are all familiar with examples of bad incentive schemes. Here are two examples:

- 1960's: dentists in the UK health service had a government contract that paid them for every cavity/filling they drilled independently of whether the tooth needed drilling.

- Cost Plus Contracts: A Principal will reimburse a builder for allowable costs and a provision for normal rates of profit (a cost plus contract) on a construction project.

However, even the very best incentive scheme will still impose economic costs and generate inefficiency!

1.1. **Adverse Selection or Moral Hazard?** Under adverse selection there is hidden information and under moral hazard there are hidden actions. Sometimes it is difficult to tell the difference between moral hazard and adverse selection. Here is an example. Suppose it was observed that Volvos go through stop signs more than other makes of car. This has two potential explanations one is about moral hazard: "Volvo drivers believe they?re safer and take less care when they drive as a result". The other is about adverse selection: "Bad drivers tend to buy Volvos." Clearly both might be true but there are very different implications for these two explanations of the same fact.

1.2. **General Features of Incentive Schemes.**

1.2.1. *Proportionality.* This is a common feature of incentive schemes. Managers get a share in a company. Farmers get to keep a share of their crops. And students tend to get grades that increase linearly with their performance on a course. These proportional schemes tend to get used when good performance is easy to measure and closely correlated with the Principal's desired outcome.

1.2.2. *High-Powered Incentives.* This is when a small change in observable outcome has a big change in the Principal's decision. You see these when it is particularly important to avoid a bad outcome: Pass-Fail Grading Schemes, Sentences for Serious Crimes, Certification of Nuclear Power Plants are all examples of high-powered incentives.

1.2.3. *Relative Performance Evaluation.* When good performance by the agent is particularly difficult to measure, incentives will compare the performance of one agent with another agent to judge good performance. Tournaments, Benchmarking and Competition are all examples of relative performance evaluation.

1.2.4. *Target and Penalty Schemes.* These are schemes that award a bonus when a certain level of performance is achieved. The bonus has the effect of discouraging further effort and there are mild penalties for under-perfomance—these schemes provide low-powered incentives.

1.2.5. *Efficiency Wages.* Firms sometimes choose to pay their workers more than the market wage. If they paid exactly what a worker could get elsewhere, then the worker does not care if she is fired or not. (She can immediately obtain the same wage elsewhere.) The more a worker is paid in his current job relative to the market wage, the more costly it is for her to get fired (and she doesn't want to shirk). Companies want to raise wages and perpetuate unemployment as this gives workers incentives to try hard in their current job.

**1.3. Ownership: One Solution to the Incentive Problem.** We've seen that incentive problems arise if 3 conditions hold: 1. There is a divergence of interests. 2. There is a need for the individuals to transact. 3. There are observation problems. One solution to this problem is to sell the issue to the agent. Once the worker owns the company, they will of course be willing to put in effort. Thus they should be willing to buy the company from its owners at a price that correctly reflects its full value.

There are 2 problems with this argument. First the agent may not be able to borrow sufficient money to acquire the company (we say the agent is credit-constrained). Second, the company might be risky and the agent may care more about the risks of owning the company than the current owners do.

**1.4. Incentives and Risk.** In a situation of moral hazard, the Principal observes an outcome that is correlated with the action that she wants to encourage and rewards the agent based on this outcome. Usually, this outcome has a random element in it. Sometimes the agent puts in effort but is just unlucky and a bad outcome occurs. This randomness in the outcomes makes the agent's life risky because the rewards they get from the Principal are random and usually individuals don?t like this risk.

Thus incentives introduce an inefficiency because they force Agents (workers, students etc.) to bear risk. Although it is usually better for the Principal (firms, government etc.) to bear the risk not the individual agents. It is actually essential for agents to bear some risk, because if they experience no risk there are no incentives! So as economists we want individuals to bear some risk pros (to give them incentives) but not too much cons (to impose too great costs on them). Also, they might simply refuse to participate in schemes that are too risky. (Of course, if the agent is risk-neutral and doesn't care about risk this is not a problem.)

Perfectly observe agent's performance -> Easier to provide incentives

**1.5. Summary.** In setting up incentives the general message is you must balance several forces. They are (1) The increased benefit from better behaviour from agents. (2) The costs of risk borne by agents (risk aversion). (3) How precisely you can measure performance. (4) How much effort will increase in response to incentives anyway!

Incentive Intensity Principle: Incentives should be most intense when agents are able to take actions to respond to them.

Moral Hazard in Lending

## 2. Moral Hazard in Lending: Loan Contracts and Bankruptcy

In this section we work through a very simple example of moral hazard. The Principal will be a lender deciding whether to make a loan or not. The Agent will be a borrower who can decide whether to put in effort or not into an investment project. We will see that the optimal loan contract for the principal is not to make a loan unless the agent can put in sufficient collateral. There will be no costs to risk here, both the Principal and the Agent are risk neutral. However, the agent will be able to declare bankruptcy and this gives the agent skewed incentives.

2.1. **The Model.** A company has £$x$ in collateral, but needs a sum of money £$1 >$ £$x$ to finance a risky investment project. The project has returns that are risky and also depends on the effort put in by the company. If the company puts in *no effort* the project will pay out

$$\text{£}3 \ \text{ probability} = 1/3; \qquad \text{£}0 \ \text{ probability} = 2/3.$$

If the company puts in *high effort* the project will pay

$$\text{£}3 \ \text{ probability} = 2/3; \qquad \text{£}0 \ \text{ probability} = 1/3.$$

High effort costs the company £0.8 in lost opportunities elsewhere but low effort is free.

2.2. **Optimal Decision When Investment is Self Financed.** If the investment is made from the company's own assets, the company's profits from its two effort levels are:

$$\text{Profit High Effort} = (2/3)3 + (1/3)0 - 1 - 0.8 = \text{£}0.2,$$
$$\text{Profit Low Effort} = (1/3)3 + (2/3)0 - 1 = \text{£}0.$$

It is efficient in this case for the project to go ahead and for the company to put in high effort.

2.3. **Optimal Decision When Investment is Bank Financed.** Now suppose the company borrows all the cost of the investment from the bank and must pay the £1 back from any money it makes on the project. Then the above calculation becomes:

$$\text{Profit High Effort} = (2/3)(3-1) + (1/3)0 - 0.8 = \text{£}0.53,$$
$$\text{Profit Low Effort} = (1/3)(3-1) + (2/3)0 = \text{£}0.666.$$

In this case the company is better off putting in low effort (this save it the costs of effort). Notice that when the project doesn't succeed the loan is not repaid, so this doesn't affect the company's profits. (This is the company declaring bankruptcy.) Essentially, what happens when the investment is borrowed is that the company gets to gamble with the bank's money and would rather not put in any effort of its own. The borrower here bears no real costs of risk.

2.4. **Optimal Decision When Company Puts in Some Collateral.** Suppose the company puts in £$x$ in collateral and borrows $1-x$ from the bank. Then the profits from high and low effort are

也需要 $-x$!

$$\text{Profit High Effort} = (2/3)(3-(1-x)) + (1/3)0 - 0.8, \ -x$$
$$\text{Profit Low Effort} = (1/3)(3-(1-x)) + (2/3)0. \ -x$$

Suppose he bank will only loan the funds if it knows the company will put in high effort, because this maximises the probability of it getting its loan repaid. What value of collateral ensures the company will put in high effort? This requires

$$\begin{aligned}
\text{Profit High Effort} &> \text{Profit Low Effort} \\
2/3[3-(1-x)]-0.8 &> 1/3[3-(1-x)] \\
6-2(1-x) &> 5.4-(1-x) \\
0.6 &> 1-x \\
x &> 0.4
\end{aligned}$$

Thus if the bank insists that the borrower puts in collateral greater than 0.4 it can be certain that the borrower will put in high effort into the project. The incentive for high effort comes from the desire of the borrower not to lose its collateral. Both the bank and the firm are bearing some of the risks.

Labour Contracting Model

## 3. A Labour Contracting Model

The previous section was a warm-up exercise. It didn't really have anything for the Principal to do—the bank either made a loan or it didn't—the point was how the agent responded to different levels of the collateral. Now we will study the optimal choice of an incentive scheme by a principal, who takes into account how the agent responds to different incentives. Here the principal will be a manager and the agent will be a worker.

Assume "Risk Neutral": no cost of risk (ettlement)

3.1. **The Model.** The worker can choose either to put in low effort $e = 0$ or high effort $e = 1$. The amount of effort the agent puts in affects how much output $q$ gets produced. Here is the relationship between effort and output:

Low effort $e = 0, \qquad \Rightarrow q = 0;$

High effort $e = 1, \qquad \Rightarrow \begin{cases} q = 1 & \text{prob} = \pi \\ q = 0 & \text{prob} = 1-\pi \end{cases}$

Thus when output is produced ($q = 1$) the manager knows that the worker put in high effort, but if no output is produced ($q = 0$) the manager does not know whether the worker put in high effort but was unlucky or just put in low effort (shirked).

We will assume that $e = 1$ is costly for the worker and costs them $c > 0$, but $e = 0$ is costless for the worker. We will also assume the worker can go elsewhere and get utility $U > 0$, so it doesn't have to work for the firm and cannot be forced to have arbitrarily bad utility.

The manager is able to provide an incentive scheme for the worker. In this case she can see what output the worker produces but not see the effort they put in. So we allow the manager to choose an employment contract that pays two different wages $(w_0, w_1)$, where $w_0$ is the wage the worker gets paid if they produce nothing and $w_1$ is the wage the worker gets if they produce some output. Thus the worker has a payoff that depends on its expected wage net of its effort costs. There are three different actions he can take and three different expected wages:

Worker's Expected Utility

$$\begin{aligned}
&(1) \qquad e = 1 \qquad U = \pi w_1 + (1-\pi)w_0 - c\\
&(2) \qquad e = 0 \qquad U = w_0\\
&(3) \qquad \text{Outside Option} \qquad U = U
\end{aligned}$$

Now let us think about the profits of the firm (what the manager gets). These will depend on how much output gets produced and how much wages are paid. We will assume that output can be sold at the price of one, so profit equals $1q - w = q - w$. Again the manager's expected profit will depend on the worker's action:

Manager's Expected Profit

$$\begin{aligned}
&(4) \qquad e = 1 \qquad \text{Expected Profit} = \pi 1 + (1-\pi)0 - (\pi w_1 + (1-\pi)w_0)\\
&(5) \qquad e = 0 \qquad \text{Expected Profit} = -w_0
\end{aligned}$$

This completes the description of the model, now we proceed to solve it...

3.2. **The Worker's Optimal Behaviour.** As usual we work backwards. We first understand what the worker will do for various different wage contracts $(w_0, w_1)$ and then use this to understand what the right choice of contract is for the manager. We already have figured out what the worker will get from each of his 3 actions. These are the expressions (1),(2),(3) above. For the worker to be willing to come and work for the firm he must do better at the firm than taking his outside option, that is, one of (1),(2) must be bigger than (3).

max { Utility (e=1) , Utility (e=0) } ≥ U

IRC $$\max\{\pi w_1 + (1-\pi)w_0 - c, w_0\} \geq U \qquad \text{Individual Rationality}$$

So if one of: $\pi w_1 + (1-\pi)w_0 - c \geq U$ or $w_0 \geq U$ then the worker is better off working for the firm than not. This is called the individual rationality constraint, or the participation constraint, and is analogous to the IR constraint in adverse selection models.

If the worker works for the firm and prefers to provide high effort ($e = 1$) then his utility from high effort (1) must be greater than his utility from low effort (2). This gives us a second constraint

Utility (e=1) ≥ Utility (e=0)

ICC $$\pi w_1 + (1-\pi)w_0 - c \geq w_0 \qquad \text{Incentive Compatibility}$$

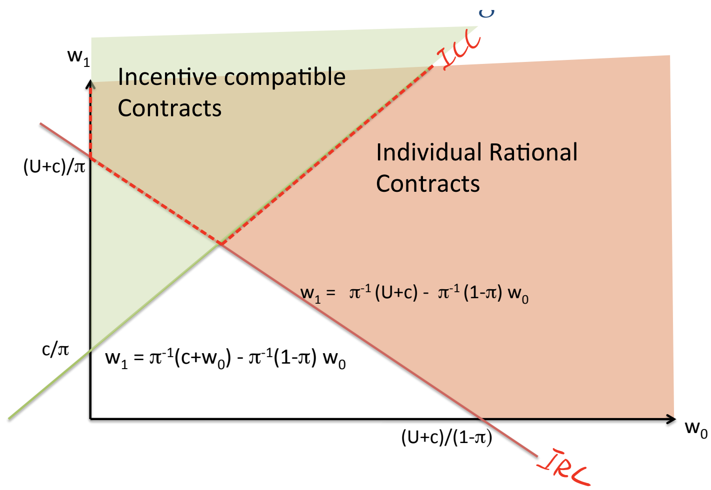
which is called the incentive compatibility constraint. This says the wage contract gives the worker the incentive to put in high effort.

If the manager of the firm wants the worker to work and to provide high effort then two constraints must hold:

Constraints on the Manager

$$\begin{aligned}
&\pi w_1 + (1-\pi)w_0 - c \geq U \qquad \text{IR}\\
&\pi w_1 + (1-\pi)w_0 - c \geq w_0 \qquad \text{IC}
\end{aligned}$$

Here is a picture of these two constraints.




But if the manager wants the worker to provide low effort then the constraints are

$$
\begin{aligned}
& w_0 \geq U && \text{IR} \\
& w_0 \geq \pi w_1 + (1-\pi)w_0 - c && \text{IC}
\end{aligned}
$$

See if you can draw these two constraints.

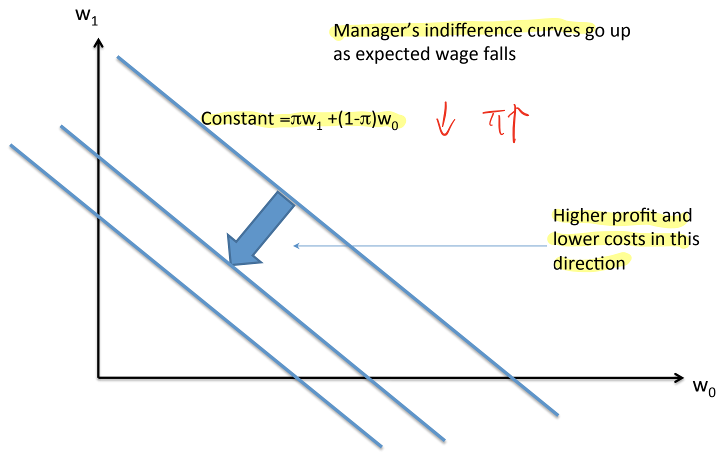
3.3. **The Manager's Problem.** The manager wants to maximise the profits made. We have written profits down in the expressions (4) and (5). Suppose the manager wanted the worker to provide high effort then she would choose a contract $(w_0, w_1)$ to solve the profit maximisation problem

$$
\max_{w_0,w_1} \overbrace{\pi 1 + (1-\pi)0}^{\text{Constant}} - \overbrace{(\pi w_1 + (1-\pi)w_0)}^{\text{Cost}\uparrow\ \pi\downarrow} \qquad (\pi \mid e=1)
$$

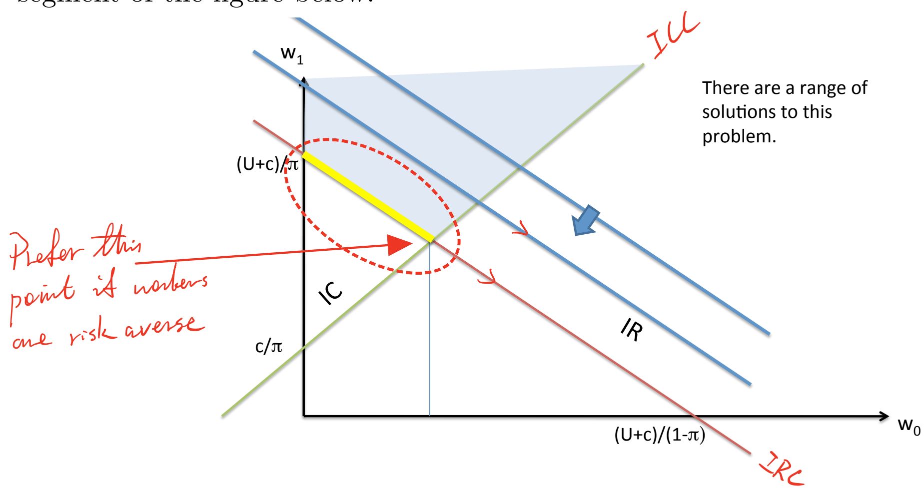
$$
\left.
\begin{aligned}
\text{subject to: } & \pi w_1 + (1-\pi)w_0 - c \geq U && \text{IRC} \\
& \pi w_1 + (1-\pi)w_0 - c \geq w_0 && \text{ICC}
\end{aligned}
\right\} \text{Ensure } e=1
$$

Linear programing cannot be solved by Lagragen

We have a picture of the constraints on this optimisation above. What about the manager's objectives. Well they want to maximise the profits which is equivalent to minimising the expected wage they have to pay. Thus there profits increase as the expected wage decreases.




If we now combine the objectives of the manager with the constraint set above we can see that the manager would maximise her profit by choosing a wage contract on the yellow segment of the figure below.




This line segment has the endpoints $(w_0, w_1) = (0, (U+c)/\pi)$ and $(w_0, w_1) = (U, U+(c/\pi))$. Any convex combination of these two points is an optimal contract. The expected wage that is paid by the manager at the optimal contract is

$$(1-\pi)0 + \pi\frac{U+c}{\pi} = (1-\pi)U + \pi(U + \frac{c}{\pi} = U + c.$$

The manager thus gets a maximised profit equal to $\pi - U - c$.

If $\pi > U + c$ the manager could make positive profit from this type of contract. Notice that this is fully efficient. If the expected output $\pi$ is greater than the social costs of high effort $c + U$ (where $U$ is the opportunity cost of the worker coming to the firm), then it is economically efficient for the worker to provide high effort. Thus there appear to be no efficiency costs of incentives in this model.

The expected wage above barely satisfies the workers IR constraint. (The worker is just indifferent between working for the firm or giving up and working elsewhere.) The situation is different for the IC constraint. At the contract $(w_0, w_1) = (U, U + (c/\pi))$ the worker is just indifferent between supplying high effort and shirking. However, at all other contracts the worker strictly prefers higher effort. For example, at the contract $(w_0, w_1) = (0, (U + c)/\pi)$ the wage for no output is so low that the worker strictly prefers to provide high effort rather than low effort. Notice that this wage contract is very risky. The worker only gets paid if they give high effort. As the worker only cares about his expected wage, this is not a problem here. But if the worker disliked risk, then they would prefer the contract $(w_0, w_1) = (U, U + (c/\pi))$ to the contract $(w_0, w_1) = (0, (U + c)/\pi)$.

Suppose the manager decided to have low effort from the worker, then they would get zero output (from the low effort) and would want to minimise their wage bill. To do this they could always pay zero (and encourage the worker to go elsewhere).

Including Risk Aversion

## 4. Contracts and Risk

The previous section found the full optimal contract for the manager, but missed a crucial aspect of incentives—risk. Because this aspect was missing the optimal contract was actually efficient. Now we include an agent who doesn't like risk. This has the effect of giving only one optimal contract—the one where the wage is least risky—and efficiency costs.

4.1. **The Model.** The model is very similar to the one in the previous section again there will be two effort levels and two output levels. Now the effort level will be called $e^-$ (low effort) and $e^+$ (high effort) and the output levels will be called $y^-$ (low output) and $y^+$ (high output). When the worker puts in low effort, $e^-$, the probability of high output is $\pi(e^-)$. And when the worker puts in high effort, $e^+$, the probability of high output is $\pi(e^+)$. Of course we will assume $\pi(e^+) > \pi(e^-)$. In summary we have:

Low effort $$e = e^- \qquad \Rightarrow \begin{cases} y = y^- & \text{prob} = 1 - \pi(e^-) \\ y = y^+ & \text{prob} = \pi(e^-) \end{cases};$$

High effort $$e = e^+ \qquad \Rightarrow \begin{cases} y = y^- & \text{prob} = 1 - \pi(e^+) \\ y = y^+ & \text{prob} = \pi(e^+) \end{cases}.$$

In contrast to the previous section, it is possible for the worker to produce high output even when they put in low effort. Thus, the manager is never certain what the worker did. The difference $\pi(e^+) - \pi(e^-)$ measures how effectively the manager can monitor the worker. If $\pi(e^+) - \pi(e^-)$ is zero, it is impossible for the manager to ever know anything about the worker's behaviour or to provide incentives. If $\pi(e^+) - \pi(e^-) = 1$ the manager perfectly knows what the worker has done.

As before we will write a wage contract as a pair $(w^-, w^+)$, where $w^-$ is the wage the worker gets after low output is realised and $w^+$ is the worker's wage after high output is achieved. The worker has a utility function $u(w)$ and a cost of effort function $c(e)$, so we can write down the worker's payoffs to the three different actions they can take.

$$\text{Workers' Payoffs}\left\{\begin{aligned} e^+ \quad &= \pi(e^+)u(w^+) + (1-\pi(e^+))u(w^-) - c(e^+) \\ e^- \quad &= \pi(e^-)u(w^+) + (1-\pi(e^-))u(w^-) - c(e^-) \\ \text{Outside Option} \quad &= U \end{aligned}\right.$$

4.2. **The Constraints.** We will focus on contracts where the worker provides high effort. In this case the individual rationality constraint only requires that the worker prefers high effort to quitting

$$\text{IRC: } \pi(e^+)u(w^+) + (1-\pi(e^+))u(w^-) - c(e^+) \geq U. \qquad \text{Individual Rationality}$$

The IR constraint cannot be manipulated in a useful way. That is not true of the IC constraint. If the worker is to provide high effort he must also prefer high effort to low effort that is

$$\text{ICC: } \pi(e^+)u(w^+)+(1-\pi(e^+))u(w^-)-c(e^+) \geq \pi(e^-)u(w^+)+(1-\pi(e^-))u(w^-)-c(e^-). \qquad \text{IC}$$

Now we can do the following manipulations.

$$\begin{aligned} \pi(e^+)u(w^+) + (1-\pi(e^+))u(w^-) - c(e^+) &\geq \pi(e^-)u(w^+) + (1-\pi(e^-))u(w^-) - c(e^-) \\ \pi(e^+)[u(w^+) - u(w^-)] - c(e^+) &\geq \pi(e^-)[u(w^+) - u(w^-)] - c(e^-) \\ \pi(e^+)[u(w^+) - u(w^-)] - c(e^+) &\geq \pi(e^-)[u(w^+) - u(w^-)] - c(e^-) \\ [\pi(e^+) - \pi(e^-)][u(w^+) - u(w^-)] &\geq c(e^+) - c(e^-) \end{aligned}$$

Marginal Gain = ↑(Probability) × ↑(Wage) ≥ Marginal Cost

This says the increase in the probability of a high wage times in the increase in the utility of the high wage must be greater than the increase in the cost of a high wage.

4.3. **The Firm's Objectives.** If the worker puts in high effort, there is probability $\pi(e^+)$ of a high output and the firm then has to pay a high wage and there is probability $1-\pi(e^+)$ of low output and the firm then has to pay a low wage. Thus the firm's expected profits, when the worker puts in high effort is

$$\pi(e^+)(y^+ - w^+) + (1-\pi(e^+))(y^- - w^-)$$

So, if we assume that high effort is worth paying for, then the firm?s profit maximisation problem boils down to maximising the above expression (by choosing wages) subject to the two constraints in the previous section. That is

$$\begin{aligned} &\max_{w^+,w^-} \pi(e^+)(y^+ - w^+) + (1-\pi(e^+))(y^- - w^-) \\ &\text{subject to: } \pi(e^+)u(w^+) + (1-\pi(e^+))u(w^-) \geq U + c(e^+) \\ &\qquad [\pi(e^+) - \pi(e^-)][u(w^+) - u(w^-)] \geq c(e^+) - c(e^-) \end{aligned}$$

4.4. **Solving the Firm's Problem.** We now really need to use our Lagrangean technique to solve the firm's problem. It has two constraints and it wants to choose a wage contract to maximise its expected profits. We are going to have to write down a rally big Lagrangean:

$$\begin{aligned} L(w^+, w^-) = {} & \pi(e^+)(y^+ - w^+) + (1 - \pi(e^+))(y^- - w^-) \\ & + \lambda\left\{\pi(e^+)u(w^+) + (1 - \pi(e^+))u(w^-) - c(e^+) - U\right\} \quad \text{IRC} \\ & + \mu\left\{[\pi(e^+) - \pi(e^-)][u(w^+) - u(w^-)] - c(e^+) + c(e^-)\right\} \quad \text{ICC} \end{aligned}$$

Then differentiate it with respect to $w^+$ and $w^-$. You can see that each of these two variables appear in many places here. There are 3 different places where $w^-$ appears this gives

$$\frac{\partial L}{\partial w^-} = (1 - \pi(e^+))(-1) + \lambda(1 - \pi(e^+))u'(w^-) + \mu[\pi(e^+) - \pi(e^-)][-u'(w^-)] = 0$$

If we divide this equation though by $u'(w^-)(1 - \pi(e^+))$ it becomes

$$\frac{-1}{u'(w^-)} + \lambda - \mu\frac{\pi(e^+) - \pi(e^-)}{1 - \pi(e^+)} = 0$$

Or

$$\lambda - \mu\frac{\pi(e^+) - \pi(e^-)}{1 - \pi(e^+)} = \frac{1}{u'(w^-)}$$

Now let us differentiate the Lagrangean with respect to $w^+$. This gives

$$\frac{\partial L}{\partial w^-} = \pi(e^+)(-1) + \lambda\pi(e^+)u'(w^+) + \mu[\pi(e^+) - \pi(e^-)]u'(w^+) = 0$$

Now we divide this through by $\pi(e^+)u'(w^+)$ to get

$$\frac{-1}{u'(w^+)} + \lambda + \mu\frac{\pi(e^+) - \pi(e^-)}{\pi(e^+)} = 0$$

Or

$$\lambda + \mu\frac{\pi(e^+) - \pi(e^-)}{\pi(e^+)} = \frac{1}{u'(w^+)}$$

We will now start to do some thinking about these conditions and what the tell us. Let us start by writing them in a slightly different way:

$$\boxed{\begin{aligned} u'(w^-) &= \frac{1}{\lambda - \mu\frac{\pi(e^+) - \pi(e^-)}{1 - \pi(e^+)}} \\ u'(w^+) &= \frac{1}{\lambda + \mu\frac{\pi(e^+) - \pi(e^-)}{\pi(e^+)}}. \end{aligned}}$$

Suppose that $\lambda > 0$ and $\mu > 0$ (we will address this issue in a little while). Then $1/(\lambda - \dots)$ is bigger than $1/(\lambda + \dots)$, so we know that the the marginal utility at the wage $w^-$ is greater than the marginal utility at the wage $w^+$. This means that $w^+ > w^-$. The extent to which this is true depends on the shape of the utility function and how quickly marginal utility varies with the wages.

$\triangle$ Probability $\uparrow$ $\quad$ $\triangle$ Wage $\uparrow$

$[\pi(e^+) - \pi(e^-)] \uparrow$ $\quad$ $(w^+ - w^-) \uparrow$



The gap between the two wages determines the strength of the incentives in this model. What makes $w^+ - w^-$ big? Making $\mu$ or $\pi(e^+) - \pi(e^-)$ big will strengthen incentives, because it is this that makes the denominators in these two fractions far apart.

$$\lambda + \mu \frac{\pi(e^+) - \pi(e^-)}{\pi(e^+)} - \left[ \lambda - \mu \frac{\pi(e^+) - \pi(e^-)}{1 - \pi(e^+)} \right] = \mu \frac{[\pi(e^+) - \pi(e^-)][1 - 2\pi(e^+)]}{(1 - \pi(e^+))\pi(e^+)}$$

But this makes perfect sense when $\mu$ is big the constraint on getting high effort is very important so it makes sense for strong incentives to be provided. Also when $\pi(e^+) - \pi(e^-)$ is big the firm gets very accurate information from its output signals so strong incentives are powerful.

Technical Issue: Now we might start to worry about the Lagrange multipliers being zero. Suppose, $\lambda = 0$ then the first order condition for $w^-$ becomes

$$-\mu \frac{\pi(e^+) - \pi(e^-)}{1 - \pi(e^+)} = \frac{1}{u'(w^-)}$$

but this is impossible because it says marginal utility is negative. So we can conclude the $\lambda > 0$. Now suppose $\mu = 0$. If this is substituted into the first order conditions we get Then we have to have $\frac{1}{u'(w^-)} = \frac{1}{u'(w^+)} = \lambda$. This says $w^+ = w^-$, because marginal utility increases in $w$. But it is impossible to satisfy the IC constraint if the same wage is paid for both output levels.

Workers' MC $\downarrow$ $\longrightarrow$ Stronger Incentive

Better Info. of Performance

Small $\triangle$ Effort leads to large $\triangle$ Output

# Adverse Selection

$U^H(p)$ $U^L(p)$ $U^0$ - Outside Option

IRCs: $U^L(p^L) \ge U^0$

$U^H(p^H) \ge U^0$ *

ICCs: $U^L(p^L) \ge U^L(p^H)$

$U^H(p^H) \ge U^H(p^L)$

} All Matter

Firm: $p^L, p^H \to$ Max $\pi$

# Moral Hazard

$U(\dots, \text{action})$ $\begin{cases} U(\dots, e=1) \\ U(\dots, e=0) \end{cases}$

$U^0$ - Outside Option

IRCs: eg: $U(e=1) \ge U^0$ *

$U(e=0) \ge U^0$ †

ICCs: eg.: $U(e=1) \ge U(e=0)$ *

or $U(e=1) \le U(e=0)$ †

} Not All Matter

Firm chooses $e, w$ to max $\pi$.

B.I.3 A worker can either exert effort or shirk. It costs the worker $c > 0$ to provide effort and shirking costs nothing. The utility of earning wage $w$ and providing effort is $\sqrt{w} - c$. The utility of earning wage $w$ and not providing effort is $\sqrt{w}$. When the worker provides effort there is a probability $p$ of an innovation, when the worker shirks there is probability $q < p$ of an innovation.

The manager of the firm decides to offer a wage $w$ for unsuccessful workers but a wage $v$ to workers who generate an innovation. If workers leave the company, they can obtain the wage 0

$e=0$ Shirk $c=0$ $\to$ $U = \sqrt{w}$

$e=1$ Effort $c=c$ $\to$ $U = \sqrt{w} - c$

$U=0$ Not Work

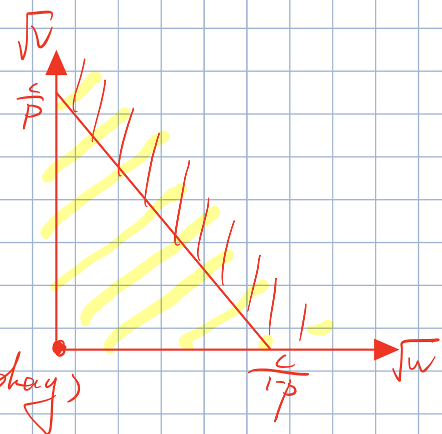
$q$, $p$ } Innov - ($W=V$)

a) Conditions for $v, w$ for the worker to be willing to work.

IRCs: $e=1$: $U = p\sqrt{v} + (1-p)\sqrt{w} - c \geq 0$

$p\sqrt{v} + (1-p)\sqrt{w} \geq c$

$e=0$: $U = q\sqrt{v} + (1-q)\sqrt{w} \geq 0$




As long as $\sqrt{v}, \sqrt{w} > 0$, workers come
(Only satisfying Shirk IRC is okay)

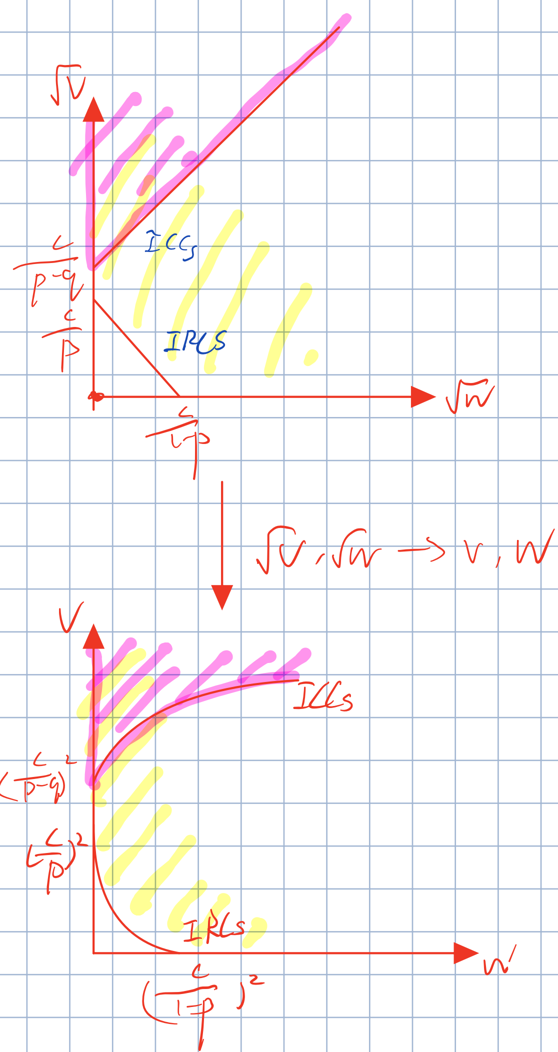
b) ensure $e=1$?

$$\begin{cases} \text{IRCs: } p\sqrt{v} + (1-p)\sqrt{w} \geq c \\ p\sqrt{v} + (1-p)\sqrt{w} - c \geq q\sqrt{v} + (1-q)\sqrt{w} \quad \text{(ICCs)} \end{cases}$$

$(p-q)\sqrt{v} + (q-p)\sqrt{w} \geq c$

$(\sqrt{v} - \sqrt{w})(p-q) \geq c$

$\sqrt{v} - \sqrt{w} \geq c/p-q$



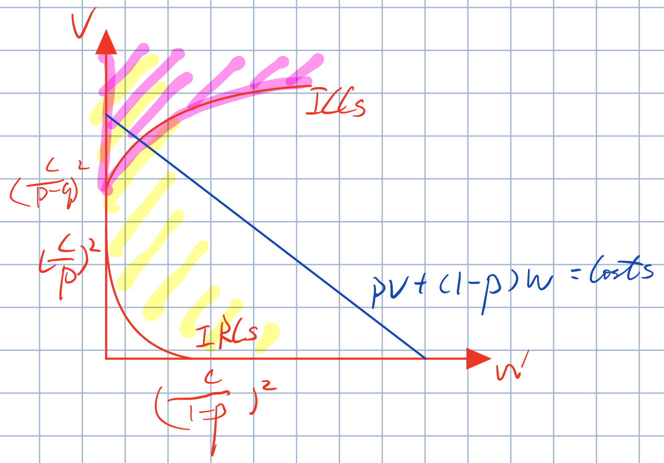

c) Firms Optimization?

Cheapest way for $e=0$?

$\min_{v,w} qv + (1-q)w$ s.t. ...

$v = w = 0$

Cheapest way for $e=1$?



# ECON2001 7. DESIGNING ECONOMIC SYSTEMS

MARTIN CRIPPS

## 1. An Example of a Mechanism Design Problem: Myerson-Satterthwaite Result

We will begin by considering a simple problem. Is it possible for someone to organise a buyer and a seller to trade efficiently? This problem is made difficult, because there are some kinds of buyer and seller combinations that cannot beneficially trade (for example, high-cost sellers and low-value buyers). So it is not always optimal for trade to occur. What also makes it difficult is that the costs of the sellers and the values of the buyers are private information and they may not be willing to reveal this to you—the organiser of the trade. The final constraint on this problem is that agreeing to do what the organiser says is voluntary—the organiser cannot force the buyer or the seller to take part. We will show that it is impossible for the efficient trade to be organised in general. Thus no matter how clever the organiser is, there has to be some gains from trade that cannot be realised in this situation.

Here are the details of the example. There is a project under consideration. A seller can build the project, a buyer can use it. The project gets built if the seller and buyer can agree on terms (i.e. a price paid by the buyer to the seller). The project is either "easy" or "hard" for the seller. If the project is easy the seller has costs $C = 35$ and if the project is hard the seller has costs $C = 60$. These two types of seller are equally likely do the high cost seller and the low cost seller each occur with probability $\frac{1}{2}$. (But of course you as the organiser of trade don't actually know which one has occurred.) The buyer also comes in two types. The low-value and the high-value buyer. The low-value buyer places value $V = 40$ on the project and the high-value buyer places value $V = 65$ on it. Again the low and high-value buyer's types are equally likely—each occurring with probability $\frac{1}{2}$. Neither the buyer nor the seller can be forced to participate in the deal and they can get zero by just walking away.

Suppose we look at the potential gains from trade in this world.

Organize a Trade

Value (Buyer)

| | | V$\stackrel{(\frac{1}{2})}{=}$40 | V$\stackrel{(\frac{1}{2})}{=}$65 | |
|---|---|---|---|---|
| Cost | C$\stackrel{(\frac{1}{2})}{=}$35 | 40-35=5 $(\frac{1}{4})$ | 65-35=30 $(\frac{1}{4})$ | Gains |
| (Seller) | C$\stackrel{(\frac{1}{2})}{=}$60 | 0 [No Trade] $(\frac{1}{4})$ | 65-60=5 $(\frac{1}{4})$ | |

When $C = 60$ and $V = 40$ it is inefficient for the buyer and seller to trade, however, in all other cases trade is efficient. The total expected increase in value that efficient trade can generate is

$$10 = \frac{1}{4}0 + \frac{1}{4}5 + \frac{1}{4}5 + \frac{1}{4}30. \tag{1}$$

Thus if we could organise trade in all the cases where trade should occur, then welfare could be increased by 10, however, we will no establish (in this example) the following result:

Unable to organize trade efficiently in all cases where values and costs are unknown.

Myerson - Satterthwaite

**Result 1** (Myerson Satterthwaite)**.** *There is no way of organising trade in this example under which the project gets built if and only if it is efficient to do so.*

Sometimes efficient to trade, Sometimes efficient not to trade → Generally impossible to organize trade efficiently.

1.1. **Split the Difference Doesn't Work.** Suppose you tried to organise trade by getting the buyer and seller to tell you their values and then choosing a price that would ensure they shared the gains from trade equally. Then the prices you would choose are

Choose $P = \frac{1}{2}(C+V)$

| | V=40 | V=65 |
|---|---|---|
| C=35 | P=37.5 | P=50 |
| C=60 | No Trade | P=62.5 |

(The missing entry in this table would be where no trade would occur.) Would the seller want to tell the truth in this scheme? If the seller was low cost and truthfully announced her type, she could expect to trade at the price $P = 37.5$ half the time and at the price $P = 50$ half the time, making an expected profit of $(37.5 - 35)0.5 + (50 - 35)0.5 = 8.75$. But if she lied and said her costs were high she would trade half the time at the price 62.5 and not trade the rest of the time. This would give her the expected profit $0.5(62.5 - 35) = 13.75$. Thus, she would be better off lying and not telling the truth and the split the difference scheme does not work.

Low Cost Seller: $E[U(\text{Lies})] > E[U(\text{Truth})]$

As an exercise check whether the high cost seller wants to tell the truth and whether the buyer's two types want to tell the truth here.

1.2. **Myerson Satterthwaite in this Example.** Now we will give an argument to show that it is impossible to get the buyer and the seller to truthfully reveal their type in this example. Thus it is impossible to organise trade efficiently. The reason this argument works is that it is necessary to give the buyer and seller incentives to tell the truth, because you the organiser needs to know when trade should occur and when it should not. These incentives are costly. The total amount of value that can be created from trade is insufficient to provide these incentives.

First notice that the total amount of surplus created from trade is 10, by the calculation (1) above. Now suppose we begin by thinking about the high-cost seller. To get the high-cost seller to trade with the high-value buyer we have to promise them a price of at least 60 half the time. But this means that the low-cost seller, by pretending to be the high-cost seller, can get at least $25 = 60 - 35$ half the time. So the very least the low-cost seller can get from cheating on truthtelling is $0.5(60 - 35) = 12.5$. But we want the low-cost seller to tell the truth. So we must promise the low-cost seller at least 12.5 if they are to tell the truth. Half of the time the seller is high cost (and we must give them some value) half of the time the seller is low cost and we must give them at least 12.5. So we must expect to give at least

$$\frac{1}{2}0 + \frac{1}{2}12.5 = 6.25$$ for the seller to tell the truth

of the surplus from trade to the seller.

Now suppose we begin by thinking about the low-value buyer. To get the low-value buyer to trade with the low-cost seller we have to promise them a price of at most 40 half the time. But this means that the high-value buyer, by pretending to be the low-value buyer, can get at least $25 = 65 - 40$ half the time. So the least the high-value buyer can get from cheating on truthtelling is $0.5(65 - 40) = 12.5$. But we want the high-value buyer to tell the truth. So we must promise the high-value buyer at least 12.5 if they are to tell the truth. Half of the time the buyer is low value (and we must give them some surplus) half of the time the buyer is high value and we must give them at least 12.5. So we must expect to give at least (Same process)

$$\frac{1}{2}0 + \frac{1}{2}12.5 = 6.25$$ for the buyer to tell the truth

of the surplus from trade to the buyer.

Combining these two calculations we have to give at least 6.25 to the seller to get truthtelling and at least 6.25 to the buyer to get truthtelling and there is only 10 in surplus to be shared. It is clearly impossible to get truthtelling because it requires resources of 12.5 and there is only 10 available. This result has profound implications for many economic arguments. The idea that individual bargaining or negotiation leads to efficiency is *not* true when the bargainers have private (payoff relevant) information. Arguments such as the Coasian solution to the externality problem, or many Chicago-style analyses are vulnerable to this issue.

What we can do under this circumstance:

| | $L(v=40)$ | $H(v=65)$ |
|---|---|---|
| $E(c=35)$ | $(P-\alpha c, \alpha v-P)$ | $(Q-\beta c, \beta v-Q)$ |
| $H(c=60)$ | $\underbrace{(R-\gamma c, \gamma v-R)}_{\substack{\text{We want this to be 0} \\ (0,0)}}$ | $(S-\delta c, \delta v-S)$ |

Tell the prices they have to pay and probabilities of trade.

If $\alpha=\beta=\delta=1 \rightarrow$ Impossible (M-S Principle)

We want they to participate: IRCs:

$$\begin{cases} P+Q \geq (\alpha+\beta)35 & \text{(Low Cost Seller)} \\ R+S \geq (\gamma+\delta)60 & \text{(High Cost Seller)} \\ 40(\alpha+\gamma) \geq P+R & \text{(Low Value Buyer)} \\ 65(\beta+\delta) \geq Q+S & \text{(High Value Buyer)} \end{cases}$$

We want they to tell the truth: ICCs:

(Tell the truth to be a dominant strategy)

$$\begin{cases} 60(\alpha-\gamma) \geq P-R \geq 35(\alpha-\gamma) \\ 60(\beta-\delta) \geq Q-S \geq 35(\beta-\delta) \\ 65(\beta-\alpha) \geq Q-P \geq 40(\beta-\alpha) \\ 65(\delta-\gamma) \geq S-R \geq 40(\delta-\gamma) \end{cases}$$

Conclusion: $\begin{cases} \alpha \geq \gamma,\ \beta \geq \delta,\ \beta \geq \alpha,\ \delta \geq \gamma \\ P \geq R,\ Q \geq S,\ Q \geq P,\ S \geq R \end{cases}$

Q is the largest

## 2. Mechanisms and the Revelation Principle

A *mechanism* is a device for describing a possible economic processes where information about individuals has to be collected and aggregated and then an economic decision must then be taken.

A mechanism takes as its input a basic model of an economy:

(1) There are individuals in the economy: $1, 2, \ldots$

(2) A set of possible allocations, or social states, $x \in X$. (Allocations)

(3) Each individual has payoff/utility function $u_i(x, \theta_i)$, that depends on the social state $x$ chosen and on their private "type" which could be knowledge ~~or information~~ $\cancel{\theta_i}$. We don't know $\theta$

(4) Individual ~~$i$'s type, $\theta_i$, is selected from a set $\Theta_i$ by a known probability distribution.~~

In the example of the previous section there was: (1) Players: buyer and seller, (2) States: trade or no trade and an exchange of money, (3) Types: high or low cost and high or low value, (4) 50:50 probabilities. There are many other economic questions that fit this basic structure.

Examples:

- Should a bridge be built or not?
- Who should be sold a good?

- How much tax each person should pay?
- What firm should produce a good?
- How much pollution individuals should be allowed to produce and how much each of them should be made to pay for the right to pollute?

Mechanism

Signal $S_i$ ↓ Rule $\gamma$ Allocation $x \in X$

A mechanism describes a process for deciding what allocation $x$ to choose in this model. It has two elements. The first is a set of signals $S_i$ for each person to send. (In the examples above this could be: a vote, a bid, a tax return, a quote, a technology.) The second is a policy rule $\gamma(s_1, s_2, \dots, s_N)$ that takes the signals sent (one from each player) and maps them to an allocation $x \in X$. (The the policy rule might take individuals tax returns and map them into a tax bill and a decision on who to audit.) Other examples of mechanisms would be: (1) A message the buyer and seller could send and a decision on whether trade occurs and who pays what. (2) In the case of pollution the message could be how much you are willing to reduce your pollution emissions and the rule determines what you have to pay or be paid. (3) The message could also be how much you value the bridge and the rule could be whether to build the bridge or not and who pays for it.

Once a mechanism $(S_1, \dots, S_N; \gamma)$ is determined, the individuals in the world will find themselves playing a game. First, they will observe their utility and their type. Then they must decide what message $s_i \in S_i$ to send. And once all the messages are sent the allocation $\gamma(s_1, \dots, s_N)$ gets decided. Typically the messages that are sent will depend on an individual's type. The individuals are playing a game, because what message an individual wishes to send will depend on the messages others send.

Players do not know others' types, so they declare messages based on their own type and the mechanism

Summary: Choosing a mechanism is like choosing a game for people to play, the designer want to choose the mechanism to produce an allocation that satisfies their objectives.

In this game, messages are actions.

Equilibrium in Mechanism Games

2.1. **Equilibrium of the Mechanism Game and Implementation.** The strategies in the game determined by a given mechanism are functions: $s_i^*(\theta_i)$, $i = 1, 2, \dots, N$. These functions tell individuals what message to send when they are type $\theta_i$. This is just like the bidding function in an auction, (in an auction a player must decide what bid to submit for each value they might have).

$\theta_i \xrightarrow{S_i^*} S_i$

First, we might think about different kinds of equilibrium of the mechanism. The profile of equilibrium strategies, $(s_1^*(\theta_1), \dots, s_N^*(\theta_N))$, might be an equilibrium in dominant strategies. This would clearly be very robust and give people strong incentives to send the messages $s_i^*$. It might just be an ordinary (Bayes) Nash equilibrium, where incentives could be less robust. When we design a mechanism we ought to think about whether the equilibrium is nice and robust (in dominant strategies) or perhaps less robust.

When the equilibrium of the game is played out the players send the messages $(s_1^*(\theta_1), \dots, s_N^*(\theta_N))$ and the mechanism designer takes the decisions $\gamma(s_1^*(\theta_1), \dots, s_N^*(\theta_N))$. We can think of this as a three step process. First the types of players are determined, next the signals are sent, and finally a decision is taken

$$(\theta_1, \dots, \theta_N) \mapsto (s_1^*(\theta_1), \dots, s_N^*(\theta_N)) \mapsto \gamma(s_1^*(\theta_1), \dots, s_N^*(\theta_N))$$

Types -> Messages -> Implementations

If we look at the ends of this process, ignoring the intermediate step, we get a relationship between the players' types and the decision that gets taken

$$(\theta_1, \dots, \theta_N) \mapsto \gamma(s_1^*(\theta_1), \dots, s_N^*(\theta_N)) \to X$$

Implementation

$$(\theta_1, \dots, \theta_N) \mapsto \Gamma(\theta_1, \dots, \theta_N) \to X$$

Types → Implementation (Eventual decision)

This tells us that if the players are the type $(\theta_1, \dots, \theta_N)$ it is possible to take the decision $\Gamma(\theta_1, \dots, \theta_N) \equiv \gamma(s_1^*(\theta_1), \dots, s_N^*(\theta_N))$. This is called *Implementation.* That is, there exists a mechanism that allows you to take the decision $\Gamma(\theta_1, \dots, \theta_N)$ when the types are $\theta_1, \dots, \theta_N$.

For example, one decision rule we would like to implement is the rule that obliges the buyer and seller in to trade whenever it is efficient. We know this is impossible, by the Myerson & Satterthwaite result. Another decision rule is to oblige the buyer and seller to trade only if the buyer is high value and the seller is low cost. We know this is possible—just choose an appropriate fixed price.

Revelation Principle

The main result we have on implementation is that in many situations we do not need to give the agents complicated message spaces to get optimal behaviour. *The Revelation Principle* states that if the strategies $s_i^*(\theta)$ are an equilibrium in dominant strategies. It is enough to allow the agents to just tell you their type $S_i = \Theta_i$. No more complicated messages are necessary. The idea behind this is to take any arbitrary mechanism with an equilibrium $(s_1^*(\theta_1), \dots, s_N^*(\theta_N))$ and add on to it an initial stage where players are asked to report their type and if they report $\theta_i$ the initial stage then sends the message $s_i^*(\theta_i)$ to the mechanism. This will not change the players' incentives if $(s_1^*(\theta_1), \dots, s_N^*(\theta_N))$ was an equilibrium of the original mechanism.

Groves - Clark Mechanism

## 3. The Groves-Clark Mechanism

Suppose the set of social states is as follows. A project gets undertaken and the individuals in the society get values $(v_1, v_2, \dots, v_N)$. Or a project does not get undertaken and the individuals get values $(0, 0, \dots, 0)$. The values $v_i$ are private information. The Groves-Clark Mechanism describes a process to get individuals to truthfully reveal their values and to get the project built if and only if it is socially efficient to do so.

The Groves-Clark Mechanism works as follows

(1) Individuals report their value for the project, say $\tilde{v}_i$. This report does not necessarily have to be truthful. $\tilde{v}_i$ may not $= v_i$
(2) The reported values are summed $\sum_i \tilde{v}_i$.
(3) Let $C > 0$ be the cost of the project. If the sum of individuals' reports is greater than the cost, then

$$\sum_i \tilde{v}_i \geq C.$$

and the project is done. If the sum of individuals' reports is less than the cost then

$$\sum_i \tilde{v}_i < C.$$

and the project is not done.

(4) In the final stage payments are determined. Suppose we sum the reported values of all individuals apart from individual $j$ and found that these reported values are less than the cost: $\sum_{i\neq j}\tilde{v}_i < C$. But when we also add in individual $j$'s report the values are more than the cost $\sum_i \tilde{v}_i > C$. Such individual is called a *decisive* or *pivotal* individual. Decisive or pivotal individuals must pay the sum

His decision changed the decision

$$C - \sum_{i\neq j}\tilde{v}_i$$

cost - All others' value

to the decision taker. All other individuals make no payment.

For this mechanism, the Nash equilibrium is for each individual to truthfully report their value. In fact it is weakly dominant to truthfully report your value. Thus, the decision taker will have the right information on whether to do the project and will take a socially optimal decision. (However, the money raised by the mechanism will be different from the cost of building the project.)

To see that it is optimal to for individual $j$ tell the truth we will consider several cases. A key variable $U = \sum_{i\neq j}\tilde{v}_i$ the sum of everyone else's reports. (You will notice that this is very much like the proof that it is optimal to bid truthfully in a second-price auction.)

**If $U > C$.** Then the project will be built whatever individual $j$ reports, they don't care what they report, so it is optimal for individual $j$ to report truthfully. (Your $\tilde{v}_i$ doesn't matter)

**Now suppose that $U + v_j > C > U$.** In this case reporting truthfully $\tilde{v}_j = v_j$, will result in the project being built and individual $j$ is pivotal. They will then have to pay $C - U$, thus reporting truthfully gets them the utility $v_j - (C - U) > 0$ (as $v_j + U > C$). But if they understate their value the project will not be built and they get value zero. Truthful reporting is better than under reporting.

**Finally $C > U + v_j$.** In this case reporting truthfully results in the project not being built and the individual getting zero utility. Exaggerating the value and reporting $\tilde{v}_j > C - U$ will get the project built, but because the individual is pivotal they must pay $C - U$ which is greater than their true value.

Payment ≤ Benefit received

Auctions

## 4. Auctions

Auctions are a very popular way of selling an object when the seller does not know the demand function. In the classical model of monopoly a monopolist knows its demand function and should just set an optimal price (this is called the Posted-Price Model). When a monopolist doesn?t know demand it may make a big mistake by behaving in this way (and not sell or sell too cheaply). Having an auction allows you to learn demand (by observing the bids) and set a good price at the same time. Thus the monopolist is finding a mechanism to reveal its demand and to determine its price. The idea here is that competition amongst buyers gets them to tell you the true demand.

**4.1. There are 5 different types of auction.**

5 Different Types of Auction

(1) First-Price Sealed Bid
- The buyers submit bids in secret.
- The seller awards the object to the highest bidder.
- The highest bidder pays their bid to the seller.

(2) Second-Price Sealed Bid
- The buyers submit bids in secret.
- The seller awards the object to the highest bidder.
- The highest bidder pays the second highest bid to the seller. [Option to bid truthfully]

(3) Dutch Auction
- The auctioneer announces a very high price and slowly reduces it (using a clock).
- The first buyer to press a button stops this clock. Similar to 1st price auctions.
- They get the object at the price on the clock.
-

(4) English Auction
- The auctioneer announces a very low price and slowly increases it in response to bids.
- The price stops increasing when the last but one buyer drops out. Similar to 2nd price auctions.
- The last buyer in gets the object at the price where the last but one buyer was eliminated.

(5) Double Auction
- There are $K$ sellers each with one unit to sell and many buyers each with one unit to buy.
- Sellers and buyers simultaneously submit bids.
- The $K$ highest bids get an object whether they come from a seller or a buyer. No1. ~ K. BIDs get
- The price is determined by an average of the $K$ and $K + 1$ highest bids.
- Sellers with bids above the price get to keep their objects, sellers with bids below the price sell their objects at the price. $P = \frac{P_K + P_{K+1}}{2}$
- Low buyers don't get an object high buyers do.

**4.2. A Model of a Single Unit Auction.** We will study auctions where there is one indivisible unit of a good for sale, like a car, and $N$ buyers with unknown values: $(v_1, \ldots, v_N)$. It is these values, $(v_1, \ldots, v_N)$, that determine the demand curve. We will assume that these values are drawn independently from a distribution on $[0, 1]$.

$U = v_i - P$ (if win)
$U = 0$ if lose

The order of play in this game will be first individuals observe their own value $v_i$ and no-one else's. Then they submit a bid $b_i(v_i)$. The rules of the auction then determines who gets the good and who pays what. Notice that a player's strategy is a function that maps their value to a bid.

There are many assumptions about the probability that generates the vector $(v_1, \ldots, v_N)$ which have technical names:

4 Different Ways of

① **Independent Symmetric Private Values:** $v_i$ is drawn independently from the density $f(v)$ on $[0, 1]$. (Same distribution)

☆ Here, we only study Independent Symmetric Private Values.

* Symmetric: Everyone's distribution are the same.

Generating Values



(2) **Independent Private Values:** $v_i$ is drawn independently from the density $f_i(v)$ on $[0, 1]$. Different Distribution
(3) **Correlated Private Values:** The vector $(v_1, \ldots, v_N)$ is drawn from a full support density on $[0, 1] \times [0, 1] \times \cdots \times [0, 1]$. Joint Distribution, My value includes info about others' values.
(4) **Common Values:** $v_1 = v_2 = \cdots = v_N$ but players do not know what $v_i$ is. They do have private signals on it. Same value, but unknown. (Financial asset)

(Bayes-Nash Equilibrium)

Bayes-Nash Equilibria in Auctions

4.3. **Equilibrium in Second Price Auctions.** In an auction where the price paid is the second highest bid, the strategy $b_i(v_i) = v_i$ (that is submit a bid equal to your value), weakly dominates all other strategies.

To see this suppose you have a value $v_i$ and consider your payoffs for a bid $b' > v_i$. Your payoff only depends on the highest bid from the other players call this $B$. WE have already shown that whatever the value $B$ it is weakly dominant to submit a bid equal to your value.

Bidders' strategies: $b_i : [0,1] \to [0, \infty]$, $v_i \to b_i(v_i)$

Assumption 1: * Symmetric: everyone's strategies are the same.

4.4. **Equilibrium in First-Price Auctions.** We will now study an equilibrium of a symmetric first-price auction. In a first-price auction a player wins when their bid is higher than everyone else's. This means the winner of the auction risks over bidding, because there is a chance that others are going to bid a lot less and the price they end up paying will not protect them against this.

It is a lot harder to find the equilibrium of a first-price auction than a second-price auction. We will start by assuming (Assumption 2) that the players' bids are increasing functions of their values. In this case a player will have the highest bid if and only if they they also have the highest value. Thus, they win the auction when they have the highest value. If I have the value $r$ the probability another player $j$ has a value less than $r$ equals $F(r) = \Pr(v_j < r)$ this is the cdf of the random variable $r$. As the players' values are independent, the probability that two players both have a value less than $r$ is $\Pr(v_j < r) \times \Pr(v_j < r) = F(r)^2$. The player that all other players have a value less than $r$ is therefore $F(r)^{N-1}$. Thus the probability that a player with the value $r$ wins a first price auction is

$$F(r)^{N-1}. \quad = [\Pr(v_j < r)]^{N-1}$$

Now suppose that the equilibrium bidding rule of the players is $b^*(v)$. That is, each player bids $b^*(v)$ when they have the value $v$. What is the player's expected utility from playing out the equilibrium? This can be written as

$$E[U] = \text{Probability wins auction} \times \text{Utility from winning} + \text{Probability loses} \times 0$$

The utility from winning is the individual's value less their bid. Putting symbols for these expressions we get

$$E[U] = \text{Expected Utility from Playing the Equilibrium} = F(v)^{N-1}[v - b^*(v)] + 0. \quad = [\Pr(v_j < v_i)]^{N-1}[v - b^*(v)]$$

The individual could also pretend to have another value, say $v'$, and make the bid, $b^*(v')$, that this other type plays. This would change the probability that the individual wins and change the price they would pay

$$E[U] \text{ cheating}: \text{Expected Utility from Playing } v' = F(v')^{N-1}[v - b^*(v')] + 0.$$

$$= [\Pr(v_j < v')]^{N-1}[v - b^*(v')]$$

At an equilibrium the bid $b^*(v)$ is optimal for type $v$ so pretending to be $v'$ and bidding according to this this has to be worse than bidding $b^*(v)$. That is

$$F(v)^{N-1}[v - b^*(v)] \geq F(v')^{N-1}[v - b^*(v')] \qquad \text{for all } v'.$$

E[U] Truth ≥ E[U] Cheating

Another way of saying the same thing is to say the right hand side of the inequality above is maximized at $v' = v$.

$$v' = v \qquad \text{Maximises} \qquad F(v')^{N-1}[v - b^*(v')].$$

If we differentiate this and set it equal to zero we will find the maximum

$$\frac{d}{dv'}\left\{F(v')^{N-1}[v - b^*(v')]\right\} = (N-1)f(v')F(v')^{N-2}[v - b^*(v')] + F(v')^{N-1}[-\frac{db^*}{dv}(v')]$$

Now setting the derivative equal to zero and $v' = v$ we get the differential equation

$$0 = (N-1)f(v)F(v)^{N-2}[v - b^*(v)] - F(v)^{N-1}\frac{db^*}{dv}$$

A little re-arranging gives

$$F(v)^{N-1}\frac{db^*}{dv} + (N-1)f(v)F(v)^{N-2}b^*(v) = (N-1)f(v)F(v)^{N-2}v.$$

The left hand side of this is just the derivative of a product

$$\frac{d}{dv}\left\{F(v)^{N-1}b^*(v)\right\} = (N-1)f(v)F(v)^{N-2}v$$

So integrating both sides allows us to solve for the optimal bidding function

$$F(v)^{N-1}b^*(v) = \int (N-1)f(v)F(v)^{N-2}vdv$$

or

$$\boxed{b^*(v) = \frac{\int_0^v (N-1)f(v)F(v)^{N-2}vdv}{F(v)^{N-1}}}.$$

Pr( The Second Highest Value is the highest )

Pr( Everyone else has a value < v )

This expression tells us that the optimal bid in a first price auction is the bidders best guess about the second highest bid.

Expected Second Highest Value | I have the highest value (Condition)

"If I am really the one with the highest value, what will be my best guess of the next highest value?"

1st Order Auction : Bid* < True Value
2nd Order Auction : Bid* = True Value

In terms of seller's revenue, these designs are the same.

E(Rev) | 1st Order A = E(Rev) | 2nd Order A = 2nd Highest Value