Francesco Zanetti - Growth

Prelim

Kaldor's Long-Term Facts on Macroeconomy #flashcard

- Roughly constant:capital output ratio, return to capital, capital/labour share of income, consumption/investment to GDP
- Grows: output per worker, capital per worker, real wages

Overlapping Generations (OLG) Model

Baseline OLG: 2-Generation OLG with Cobb-Douglas PF and CRRA Utility

- **Workflow**:
  - Household Optimisation → Euler Equation
  - Firm Optimisation → FOCs
  - Combine them → Capital LOM
  - Combine with Market Clearing → Full Equilibrium Results
- **Setup**:
  - Household:

$$\max_{C_t^0, K_{t+1}^1} U(C_t^0) + \beta U(C_{t+1}^1)$$

  - s.t.

$$\begin{cases} BC_0: & K_{t+1}^1 + C_t^0 = w_t \\ BC_1: & C_{t+1}^1 = r_{t+1} K_{t+1}^1 \end{cases}$$

  - and:$U(c) = \ln c$
  - Firm:

$$\max_{t, K_t} (K_t, {}_t) - w_{t t} - r_t K_t$$

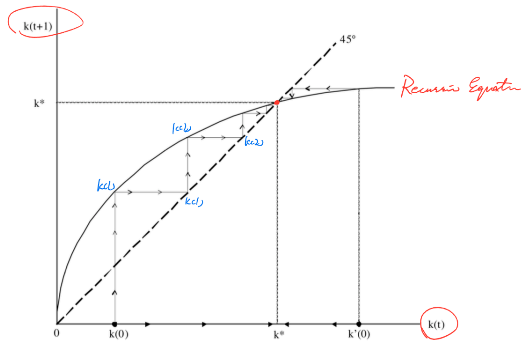
  - and $(K,) = K^{\alpha 1-\alpha}$ and normalise $= 1$ #flashcard
- **Solving the model**:
  - Household optimisation:
    - In this simple setup, we do not need a Lagrangian. Simply combine 2 BCs to eliminate K_{t+1}^1 and take the FOC for C_t
    - $\implies$ Consumption Euler Equation:

$$U\left(C_t^0\right) = \beta r_{t+1} U\left(C_{t+1}^1\right)$$

      which can be further calculated using the log-utility assumption
  - Firm's optimisation:
    - Take the FOCs
  - Combining the Euler Equation + Firm's FOC (together with our functional form assumptions), we can already get the Capital LOM:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}(1-\alpha)\left(K_t^1\right)^\alpha$$

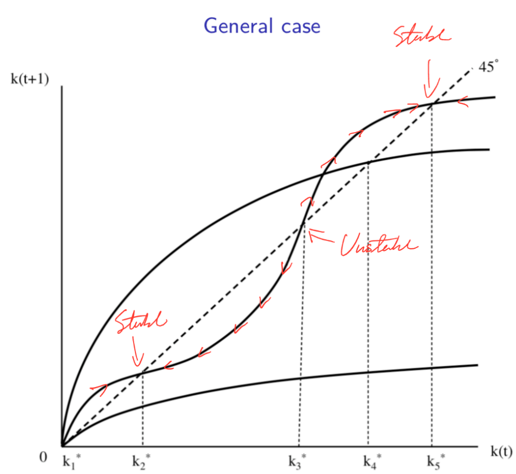
    - 2 Generations + CDPF $\implies$ Convergence to a SS capital level k^*:



- To solve the full model, we also need the market clearing condition:

$$C_t^0 + C_t^1 + K_{t+1}^1 = (K_t^1, {}_t)$$

- **Comments**:
  - Physical capital forms the basis of intertemporal link in neoclassical models.
  - CDPF ensures convergence towards a steady state.
  - CDPF + 2 Generations with CRRA Utilities $\Longrightarrow$ Convergence towards the unique globally stable steady state.
  - There's possibility of dynamic inefficiency where capital stock exceeds the Pareto efficient level (equilibrium does not ensures efficiency).
  - In the general case with more generations, there could be multiple equilibria.
  -



OLG with Taxes / Government Fiscal Policies

- **Workflow**:
  - Household Optimisation $\rightarrow$ Euler Equation
  - Firm Optimisation $\rightarrow$ FOCs
  - Combine them $\rightarrow$ Capital LOM
  - *Combine with specific Government Budget Constraint $\rightarrow$ Different Results*
  - Combine with Market Clearing $\rightarrow$ Full Equilibrium
- **Setup**:
  - Household:

$$\max_{C_t^0, K_{t+1}^1} U(C_t^0 + G_t) + \beta U(C_{t+1}^1)$$

  - s.t.

$$BC_0: \quad K_{t+1}^1 + B_{t+1}^1 + C_t^0 = w_t - {}_t^0$$

$$BC_1: \quad C_{t+1}^1 = r_{t+1}(K_{t+1}^1 + B_{t+1}^1) - \cancel{{}_{t+1}^1} \;\; 0 \text{ (assmption)}$$

- and:$U(c) = \ln c$
- Firm:

$$\max_{t,K_t} (K_t, t) - w_t t - r_t K_t$$

- and $(K,) = K^{\alpha 1-\alpha}$ and normalise $= 1$ #flashcard

- **Solving the model**:
  - Household optimisation:
  - In this simple setup, we do not need a Lagrangian. Simply combine 2 BCs to eliminate K_{t+1}^1 and take the FOC for C_t
  - $\implies$ Consumption Euler Equation:

$$U(C_t^0 + G_t) = \beta r_{t+1} U(C_{t+1}^1)$$

  which can be further calculated using the log-utility assumption
  - Firm's optimisation
  - Take the FOCs
  - Combining the Euler Equation + Firm's FOC (together with our functional form assumptions), we can already get the Capital LOM:

$$K_{t+1} = \frac{\beta}{1+\beta}\left(\frac{\partial(1, K_t^1)}{\partial} - {}_t^0 + G_t\right) - B_{t+1}^1$$

  - Combine with specific Government Budget Constraint to get the actual Capital Dynamics:
  - Consider 1-period government spending at t:

$$\begin{cases} G_t & = G \quad \text{or } t \\ G_{t+} & = 0 \quad \text{or } \ 0 \end{cases}$$

  - **Balanced Budget $\implies$ Ricardian Equivalence**
  - The young generation is taxed at t to finance t $\iff$ Government keeps a balanced budget per period; no gov debt:

$$\begin{cases} {}_t^0 & = G_t \\ B_{t+1}^1 & = 0 \end{cases}$$

  - This implies the Capital LoM:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}\left(\frac{\partial(1, K_t^1)}{\partial}\right)$$

  which is exactly the same as the baseline OLG model $\implies$ no fiscal distortion (neutral fiscal policy).
  - **Deficit Financed Government Spending $\implies$ Distortion**
  - The government issues bond B_{t+1}^1 to finance its spending at t and tax the young at $t+1$ to repay debt:

$$\begin{cases} {}_t^0 & = 0 \\ B_{t+1}^1 & = G_t \\ {}_{t+1}^0 & = (1 + r_{t+1}) G_t \end{cases}$$

  - Substituting into the Capital LoM, we will see a distortion in $t, t+1, t+$ but no distortion from $t+$
  - On capital:
  - In period $t+1$, government debt decreases capital through the Euler Equation $t \to t+1$
  - In period $t+$, interest on debt decreases capital through the Euler Equation $t+1 \to t+$
  - From $t+$ onward, no effect.
- **Insights**:
  - Balanced government budget per period $\implies$ Ricardian Equivalence / No Distortion
  - Deficit (Bond) Financed $\implies$ Distortion

OLG with Pension System

- **Workflow**:

- (same, just different setup)
- Household Optimisation $\rightarrow$ Euler Equation
- Firm Optimisation $\rightarrow$ FOCs
- Combine them $\rightarrow$ Capital LOM
- Combine with Market Clearing $\rightarrow$ Full Equilibrium Results

- **Setup**:
  - Household:

$$\max_{C_t^0, K_{t+1}^1} U(C_t^0) + \beta U(C_{t+1}^1)$$

  - s.t.

$$\begin{cases} BC_0: & C_t^0 + {}_t + {}_t = w_t \\ BC_1: & C_{t+1}^1 = (1 + r_{t+1}){}_t + B_{t+1} \end{cases}$$

  where ${}_t$ is saving, ${}_t$ is the contribution to the pension system, and B_{t+1} is the benefit got from the pension system at $t+1$
  .
  - and:$U(c) = \ln c$
  - Firm:

$$\max_{{}_t, K_t} (K_t, {}_t) - w_t{}_t - r_t K_t$$

  - and $(K,) = K^{\alpha} {}^{1-\alpha}$ and normalise $= 1$
  - Capital:

$$K_{t+1} = {}_t + {}_t$$

- **Solving the model**: #flashcard
  - **Fully-Funded (Self-Founded) Pension System $\implies$ No Distortion**
  - Benefits when old is financed by the contribution when young:

$$B_{t+1} = (1 + r_t){}_t$$

  - Essentially, ${}_t$ play the same role as ${}_t$, and we will have exactly the same Capital LoM:

$$K_{t+1}^1 = \frac{\beta}{1+\beta} \left(\frac{\partial(1, K_t^1)}{\partial} \right)$$

  - **Pay-As-You-Go Pension System $\implies$ Distortion**
  - Benefits of the old is paid by the current-period young:

$$B_t = {}_t$$

  - Budget Constraint:

$$\begin{cases} C_t^0 + {}_t + {}_t & = w_t \\ C_{t+1}^1 & = (1 + r_{t+1}){}_t + {}_{t+1} \end{cases}$$

  - Combine BC:

$$C_t^1 = (1 + r_{t+1})(w_t - {}_t - C_t^0) + {}_{t+1}$$

  - Same Euler Equation, but different expressions for $C \implies$ Distorted Capital LoM:

$$K_{t+1} = \frac{\beta}{1+\beta} \left(\frac{\partial(1, K_t^1)}{\partial} - {}_t + \frac{1}{\beta(1 + r_{t+1})} {}_{t+1} \right)$$

  - The initial generation enjoys, but the latter generations suffer from lower capital accumulation.

OLG with Technological Progress #flashcard

- **Hicks Neutral TP**:

$$Y_t = {}_t(K_t, {}_t)$$

- **Capital-Augmenting TP**:

$$Y_t = ({}_t K_t, {}_t)$$

- **Labour-Augmenting TP**:

$$Y_t = (K_t, {}_t {}_t)$$

  - This leads to a BGP that is consistent with the Kaldor Facts.
  - Key: write the capital LoM in terns of $\frac{K_{t+1}}{{}_{t+1}}$. Everything else remains the same.

Ramsey Model

Stock Variables and Flow Variables #flashcard

- **Stock Variables** are not affected by the length of time
  - e.g. Capital
- **Flow Variables** have to be adjusted for the length of time
  - e.g. Investment, Depreciation

Baseline Ramsey Model

- **Setup**
  - Household:

$$\max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)\, dt \ \text{ s.t. } \ \dot{k}_t = w_t + r_t k_t + {}_t - c_t - k_t$$

  - Firms:

$$\max_{k_t, {}_t} y_t - w_t {}_t - r_t k_t \ \text{ s.t. } \ y_t = f(k_t, {}_t)$$

- **Solving the Ramsey Model** #flashcard
  - Household Optimisation
  - CV Hamiltonian:

$$H_{cv} = u(c_t) + \lambda(t) w_t + r_t k_t + {}_t - c_t - k_t$$

  - CV Maximum Principle
  - Hamiltonian Maximisation:

$$\frac{\partial H_{cv}}{\partial c_t} = 0$$

  - Co-state Equation:

$$-\frac{\partial H}{\partial k_t} = \dot{\lambda}_t - \rho \lambda_t$$

  - State Equation:

$$\dot{k}_t = w_t + r_t k_t + {}_t - c_t - k_t$$

  - Transversality Condition:

$$\lim_{t \to \infty} e^{-\rho t} \lambda_t k_t = 0$$

- Differentiate the Hamiltonian Maximisation equation wrt t and combine with the Co-state Equation. Then, substitute the definition of RRA (don't forget the negative sign!!!) to get Consumption Euler Equation:

$$\frac{\dot{c}_t}{c_t} = \frac{{}_t - - \rho}{(c_t)}$$

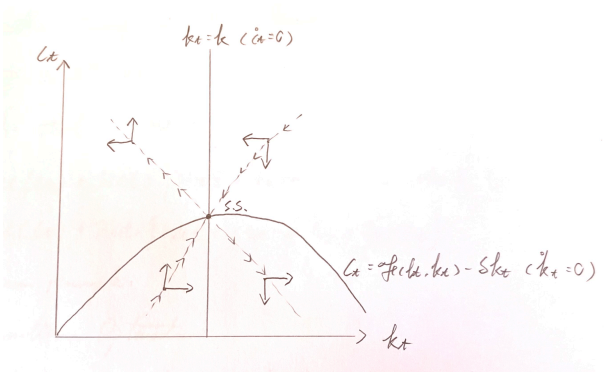
- Firm Optimisation
- FOCs:

$$\begin{aligned} w_t &= \frac{\partial f}{\partial_t} \\ r_t &= \frac{\partial f}{\partial k_t} \end{aligned}$$

- Equilibrium Dynamics
- Combine with the Capital Accounting Equation, Firm FOCs, and the TVC, we have the equilibrium dynamics:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{\frac{\partial f}{\partial k_t} - - \rho}{(c_t)} \\ \dot{k}_t & = f(k_t,) - c_t - k_t \\ \lim_{t \to \infty} & e^{-\rho t} \lambda_t k_t = 0 \end{cases}$$

- (c_t) is often assumed to be constant $\iff$ CRRA utility
- Use the concavity of production function to get conditions for $\dot{c}_t \;\; 0, \dot{k}_t \;\; 0$and draw the Phase Diagram on c_t to k_t plane



- Key points:
  - Don't forget the NEGATIVE SIGN and c_t in the definition of RRA
  - Use the basic accounting equation for the capital dynamics (no need for the equilibrium equation)
- Comments:
  - This is a purely deterministic model with no stochasticity $\Longrightarrow$ everything is pinned down at time 0 and no need to model expectations

Ramsey with Capital Taxation and Rebate

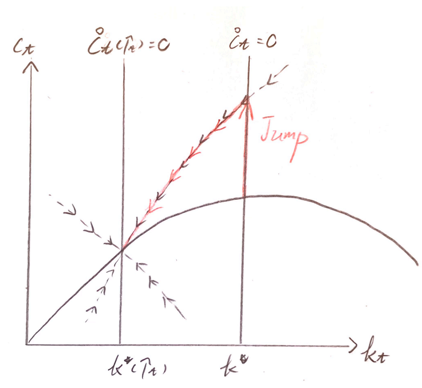
- **Modification**:
  - The government imposes a tax ${}_t$ on capital returns and rebate the tax revenue ${}_t r_t k_t$ to households
  - Household Optimisation becomes

$$\max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)\, dt \;\text{ s.t. }\; \dot{k}_t = w_t + (1 - {}_t) r_t k_t + {}_t - c_t - k_t + {}_t$$

  where ${}_t = {}_t r_t k_t$ #flashcard
- **Results**
  - Solve the model exactly the same as the Baseline Ramsey
  - Resulting Dynamics:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{(1 - {}_t) \frac{\partial f}{\partial k_t} - - \rho}{(c_t)} \\ \dot{k}_t & = f(k_t,) - c_t - k_t \\ \lim_{t \to \infty} & e^{-\rho t} \lambda_t k_t = 0 \end{cases}$$

- (c_t) is often assumed to be constant $\Longleftrightarrow$ CRRA utility
- Phase Diagram and Transition Dynamics:
-



- Transition Dynamics:
- *TVC implies that: in the period where the tax is imposed, the economy immediately jumps upwards onto the saddle path of the new SS and converges to the new SS continuously*.
- In the new SS, *both capital and consumption are lower than before*.

Ramsey with Technological Progress: Exogenous Growth

- **Motivation**: to match Kaldor facts
  - Standard Ramsey model implies convergence towards SS.
    - Over time:
      - Capital converges to a fixed amount k^*
      - Interest rate falls $\to$ 0
      - Growth rate falls $\to$ 0
      - Capital-labour ration grows and gradually approaches a fixed quantity
  - However, Kaldor found no change over time in interest rates, output growth, capital share, capital-labour ration, capital-output ratio, etc.
- **Modification**
  - Labour-Augmenting Technological Progress:

$$y_t = f({}_t, k_t)$$

  where ${}_t$ grows exogenously at a constant rate:

$$\frac{\dot{}_t}{{}_t} =$$

  #flashcard
- **Results**:
- Solve the model exactly the same as the Baseline Ramsey
- Equilibrium Dynamics
- Combine with the Capital Accounting Equation, Firm FOCs, and the TVC, we have the equilibrium dynamics:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{\frac{\partial f({}_t, k_t)}{\partial k_t} - - \rho}{(c_t)} \\ \dot{k}_t & = f(k_t, {}_t) - c_t - k_t \\ \lim_{t\to\infty} & e^{-\rho t}\lambda_t k_t = 0 \end{cases}$$

- (c_t) is often assumed to be constant $\Longleftrightarrow$ CRRA utiltiy
- Stationarise the variables:

$$\begin{aligned} k_t &= \frac{k_t}{{}_t} \\ c_t &= \frac{c_t}{{}_t} \end{aligned}$$

- Differentiate the new variables wrt t to get:

$$\begin{cases} \frac{\dot{k}_t}{t} & = \left(\dot{k_t}\right) + k_t \\ \frac{\dot{c}_t}{t} & = (\dot{c_t}) + c_t \end{cases}$$

- Divide both sides of the original dynamics equations by $_t$, and use the CRS/HoD1 assumption on PF ($\implies$ $(_t, k_t) = (, k_t)$, $\frac{\partial(_t, k_t)}{\partial k_t} = \frac{\partial(, k_t)}{\partial k_t}$):

$$\begin{cases} \frac{(\dot{c_t})}{c_t} & = \frac{\frac{\partial(, k_t)}{\partial k_t} - - \rho -}{} \\ \left(\dot{k_t}\right) & = (, k_t) - c_t - (+)k_t \end{cases}$$

- *These imply a SS in transformed variables* $\implies$ *BGP where c_t, k_t grow at constant rate .*
- *Transition Dynamics: converging to BGP.*

Endogenous Growth: AK Model

Why AK Model Has Endogenous Growth? $\to$ Inada Conditions #flashcard

- **Inada Conditions**: essentially diminishing marginal returns
  - $$\lim_{K \to 0} {}_K(K,) = \infty, \quad \lim_{K \to \infty} {}_K(K,) = 0$$
  - $$\lim_{\to 0} (K,) = \infty, \quad \lim_{\to \infty} (K,) = 0$$
- Inada conditions prevent *endogenous* growth from capital accumulation along: we will suffer from diminishing marginal returns to capital, and the marginal utility of consuming will eventually outweigh saving/investing.
  - Cobb-Douglas PF satisfies Inada conditions $\implies$ Models with CDPF can only have exogenous growth (e.g. from TFP or from population growth)
- Instead, AK PF demonstrates constant marginal return to capital $\implies$ violates Inada conditions $\implies$ allows for long-term endogenous growth through capital accumulation

Baseline AK Model

- **Setup**
  - Household:

$$\max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)\, dt \ \text{ s.t. } \ \dot{k}_t = r_t k_t + {}_t - c_t - k_t \qquad \text{(no laor)}$$

  - Firms:

$$\max_{k_t} y_t - r_t k_t \ \text{ s.t. } \ y_t = k_t$$

    - Key: Inada condition is violated due to constant marginal return to capital!:

$$\lim_{K \to 0} {}_K(K,) = , \quad \lim_{K \to \infty} {}_K(K,) =$$

- **Solving the AK Model** #flashcard
  - Use the exact same method as Ramsey:
  - Results:

$$\begin{cases} \frac{\dot{c}_t}{c_t} & = \frac{-\rho-}{(c_t)} \\ \frac{\dot{k}_t}{k_t} & = - - \frac{c_t}{k_t} \\ \lim_{t \to \infty} & e^{-\rho t}\lambda(t) k_t = 0 \end{cases}$$

- Insights:
- We *always have positive and exogenously determined rate of consumption growth if* $- \quad \rho$ *(patient households and low depreciation)*
- *Capital growth rate depends not only on , , but also on the consumption-capital ratio, hence on the relative growth of consumption and capital*.
- Consumption grows faster than capital $\implies$ capital depletion (implosion in LR)
- Consumption grows slower than capital $\implies$ inefficiency (violates TVC in LR)
- $\implies$ *only BGP ensures sustainability and efficiency*
- Impose further assumption: CRRA utility $\iff$ (c_t) is constant
- Dynamics:
- Define the Consumption-Capital Ratio:

$${}_t \quad \frac{c_t}{k_t}$$

- Analyse the Consumption-Capital Ratio dynamics:

$$\frac{\dot{}_t}{{}_t} = {}_t - \rho$$

- Balanced Growth Path (BGP)
- BGP is defined as constant consumption-capital ratio $\iff$ constant growth rate of consumption and capital:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{\dot{k}_t}{k_t} = g \\ \dot{}_t &= 0 \end{aligned}$$

- This implies:

$$\frac{\dot{}_t}{{}_t} = {}_t - \rho = 0 \implies {}^*_t = \rho \text{ on te P}$$

- Transitional Dynamics:
- *There is no transitional dynamics in AK model: growth rate of consumption, capital, and output are always constant.*
- We are *always on the BGP* hence:

$$\boxed{\frac{\dot{c}_t}{c_t} = \frac{\dot{k}_t}{k_t} = - \quad - \rho}$$

Microfoundations of AK Models

- The key of AK model is the AK production function:

$$y_t = k_t$$

with exogenous
- There are many microfoundations leading to this production function. #flashcard
- **Leaning by Doing** #notes/tbd
- **1-Sector Human Capital Accumulation**
- **2-Sector Human Capital Accumulation**

Real Business Cycle (RBC) Model