GT L1 Basics

Summary of Key Differences

Game Type	Timing of Moves	Information Structure	Solution Concept	Example
Static with Complete Information	Simultaneous	All players know payoffs/types	Nash Equilibrium	Prisoner's Dilemma
Static with Incomplete Information	Simultaneous	Some players have private info	Bayesian Nash Equilibrium	Sealed-bid auction
Dynamic with Complete Information	Sequential	All players know payoffs/types	Subgame Perfect Equilibrium	Ultimatum Game
Dynamic with Incomplete Information	Sequential	Some players have private info	Perfect Bayesian Equilibrium	Signaling game

Strategic Interdependence #flashcard
The payoff of an individual depends not only on her own decisions, but also on the decisions made by the others (or expected decisions in the future)/
Non-Cooperative Game Theory studies situations with strategic interdependence. The uni of analysis is an individual.

Well-Defined Extensive Form Representation of a Game #flashcard
4 elements:

- Players
  - Randomness if played by an extra player called Nature
- Rules
- Outcomes
- Payoffs
  Tree Representations:
- Initial nodes
- Branches
- Decision nodes
- Information sets
- Terminal nodes
  For the game to be well-defined:
- There has to be a clear sequence of the game
  - A single starting point (initial node)
  - A single node after each action
  - Each terminal node has a unique outcome
  - No cycles
- The set of possible actions at every node in an information set should be the same. (If not, then the player can tell where she is)
- Each information set and each action should be allocated to a single player

Common Knowledge #flashcard
The structure of the game (including payoffs) is known by all players.
All players know that the others know the structure of the game.

Perfect Information #flashcard

At each decision node, the players know perfectly what has happened till then.

If there exists information set which contains more than 1 node, then the game has imperfect information.

Note that perfect information allows the movement of nature/uncertainty as long as it's observed by players.

Complete/Symmetric Information #flashcard

The structure of the game and the relationship between outcomes and payoffs are known for all. (but players may not observe previous moves of the others)

Perfect Recall #flashcard

Players will not forget their previous decisions.

Strategy #flashcard

A strategy for player i, denoted as s_i is a function mapping all her information sets I_i to a completely contingent action plan A.

Actions at information sets that will not be played have to be included as well.

Simultaneous Move Game #flashcard

All players move only once and at the same time.

In this case, a strategy coincides with an action.

Any game can be considered as a simultaneous-move game where players choose full contingent action plans at the beginning.

Normal Form Representation of a Game #flashcard

The normal formal representation of a game contains:

$$\Gamma_N = [N(\text{players}), \{S_i\}(\text{set of strategies}), \{u_i(.)\}(\text{set of utilities})]$$

Correspondence between Normal and Extensive Form #flashcard

- $\forall$ extensive form game, there exists a unique normal form representation
- A normal form game may represent different extensive games

Pure/Mixed/Behavioural Strategies #flashcard

- **Pure Strategy**: a deterministic action after each information set
- **Mixed Strategy**: in plain words, assigning probabilities on each pure strategy (typically used with normal form representations)
    - implicit assumption: players mix independently of each other
- **Behavioural Strategy**: in plain words, assigning probabilities on each action at each information set (typically used with extensive form representations)
- Khun 1953: in games of perfect recall, we can always replicate mixed/behavioural strategies with each other

Player's Rationality #flashcard

We assume players are von Neumann and Morgenstern expected utility maximisers.

Common Knowledge Rationality (CKR) #flashcard

- Players are rational
- Players know others are rational, and they know the others know...

Cournot (Quantity) Competition (GT PS1 Q5)

Finding the Cournot quantity and the number of firms in NE when there's a fixed cost `#flashcard`

- Cournot quantity:
    - Fix the quantity of the other firms Q_{-i} and find the optimal response $_i$.
    - Use symmetry: $_i = {}^* \forall i$ to solve for *
- Effects of merger $\to$ perfectly competitive:

$$\lim_{n\to\infty} p^* = c,\ \lim_{n\to\infty} \pi^* = 0$$

- Number of firms in NE
  - Check the incentive to remain:

$$\pi > F$$

which typically gives the upper limit of the number of firms.
  - Key: firms staying will produce the Cournot quantity.
  - Check no incentive to enter:

$$\pi | \text{if enter} > F$$

which typically gives the lower bound of the number of firms.
  - Key: the entrant's optimal quantity is not the Cournot one, we need to reoptimise it given then existing firms producing the Cournot quantity.

Number of NEs in Finite, Generic, Normal Form Games (GT PS1 Q5) `#flashcard`

- *Finite, generic, normal form games tend to have an odd number of Nash equilibria*, due to deep results from topology and fixed-point theory.

Games of Complete Information

GT L2 Games of Complete Information - Static Games (Dominance and Rationalisability)

Dominance

Strictly/Weakly Dominant Strategy `#flashcard`

- A pure strategy s_i is a strictly/weakly dominant strategy for player $i \iff$ for that player, playing any other strategy leads to a strictly/weakly lower utility regardless of other players' strategies
- A mixed strategy can never be a strictly dominant strategy, since payoff of the mixed strategy is a convex combination of that of pure strategies.
- If each player has a strictly dominant strategy, assuming CKR $\implies$ predicting outcome.

Strictly/Weakly Dominated Strategy `#flashcard`

A pure strategy s_i is strictly/weakly dominated for player $i \iff$ there exists another strategy (can be a mixed strategy) that generates strictly/weakly higher utility no matter other players' strategies.

- It is enough to consider only the pure strategies of other players, but mixed strategies of the player herself have to be incorporated.
- This concept can be extended to mixed strategies.

Strictly Dominated Strategies will Never be Played `#flashcard`

If s_i is strictly dominated, then any mixed strategy which places a positive probability on s_i is also dominated.
(Strictly dominated strategies will never be played by a rational player, even in mixed strategies.)

Procedures to Check whether a Pure Strategy s_i is Strictly Dominated `#flashcard`

1. Does there exist a combination of *other players'* pure strategies (s_{-i}) such that our strategy s_i will generate weakly more utility?
   - If yes $\implies$ s_i is NOT strictly dominated
2. If not, then: if there exists another pure/mixed strategy that can generate higher utility regardless of other players' strategy?
   - If yes $\implies$ s_i is strictly dominated

Iterated Deletion of Strictly Dominated Strategies (IDSDS) `#flashcard`

CKR $\implies$ We can iteratively eliminate strictly dominated strategies.
The set of strategies surviving IDSDS is independent of the order if deletion.

- However, this is not true if we delete weakly dominated strategies.

Dominance-Solvable Games `#flashcard`

If IDSDS $\rightsquigarrow$ a unique strategy for each player, then the game is dominance-solvable.
CKR can uniquely predict the outcome.

Best Responses

Best Response / Never a Best Response `#flashcard`

A strategy σ_i is a **best response** for player i to other players' particular strategy σ_{-i} $\iff$ it can generate weakly higher utilities than any other pure strategies in response to σ_i:

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}) \,\forall\, s_i \in S_i$$

- Only compare with other pure strategies of play i is sufficient
  A strategy σ_i is **never a best response** $\iff$ we cannot find a particular strategy (can be mixed) for other players in response to which σ_i is a best response.
  CKR $\implies$ players will never play a never-a-best-response strategy.
  A strategy is strictly dominated $\implies$ it is never a best response.

Rationalisable Strategy

3 Definitions of Rationalisable Strategy `#flashcard`

- Survivors of iterated deletion of never-a-best-response strategies.
- Strategies that are k-rationalisable $\forall\, k$
  - a 1-rationalizable strategy as one for which there is a profile of others' strategies that makes it a best response.
  - a 2-rationalizable strategy as one for which there is a profile of others' 1-rationalizable strategies that makes it a best response.
  - and so on...

- Fixed point definition
-

> ## Definition
>
> An strategy $s_i \in S_i$ is rationalizable in $\Gamma = [N, (S_i), (u_i)]$ if for each $j \in N$, there exists a $Z_j \subset S_j$ such that:
>
> - $s_i \in Z_i$
> - every action $s_j \in Z_j$ is a best response to a σ_{-j} whose support is in Z_{-j}.

GT L3: Nash Equilibrium

Nash Equilibrium

Nash Equilibrium in Pure Strategies #flashcard
Assumptions:

- CKR
- Common/correct beliefs
  Definition 1: A strategy profile of all players $s = (s_1, \ldots, s_n)$ is a NE in pure strategies $\iff$ for every player $i = 1, \ldots, n : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \, \forall \, s'_i \in S_i$
  Definition 2: Everyone is playing their best responses to others' strategies
- Formally: define a player i's **best response correspondence** $B_i : S_{-i} \to S_i$ as a mapping from each $s_{-i} \in S_{-i}$ to the set $B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})\}$
- A strategy profile of all players $s = (s_1, \ldots, s_n)$ is a NE in pure strategies $\iff s_i \in B_i(s_{-i}) \, \forall \, i = 1, \ldots, n$
  Theorems:
- Strategies in an NE are all rationalisable (but not all the rationalisable strategies are part of a NE)
- If a game is dominance-solvable, then the solution is a unique NE
- If there exists a predictable outcome for the game, it must be a NE. Otherwise, the prediction will not be consistent.

Nash Equilibrium in Mixed Strategies #flashcard
Assumptions:

- CKR
- Common/correct belief
  Definition 1: A strategy profile of all players $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a NE $\iff$ for every player $i = 1, \ldots, N : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \, \forall \, s'_i \in S_i$
  Definition 2: Everyone is playing their best responses to others' strategies
- Formally: define a player i's **best response correspondence** $B_i : \Delta(S_{-i}) \to \Delta(S_i)$ as a mapping from each $\sigma_{-i} \in \Delta(S_{-i})$ to the set $B_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})\}$
- A strategy profile of all players $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a NE in pure strategies $\iff \sigma_i \in B_i(\sigma_{-i}) \, \forall \, i = 1, \ldots, n$

Indifference for NE in Mixed Strategies #flashcard

- Intuition: if a player randomise between 2 or more actions, those actions must give her the same payoff
- Formally: a mixed strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a NE $\iff$:
  - For all s_i, s'_i played with positive probability, they must generate the same utility given others' strategies in the NE: $u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i})$
  - For all s_i played and s'_i not played, the played strategy must generate a weakly higher utility than the not played given others' strategies in the NE: $u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})$

Existence of Pure/Mixed Strategy Nash Equilibrium `#flashcard`

- **Existence of Pure Strategy NE**
  - The game Γ_N has a NE in pure strategies if:
    - $\forall\, i \in \{1, \ldots, N\}$:
      - S_i is non-empty, convex, and compact
      - $u_i(s)$ is continuous in s and quasi-concave in s_i
- **Existence of Mixed Strategy NE**
  - Every finite game has a mixed strategy NE.

Trembling-Hand Perfect Equilibrium

Trembling-Hand Perfect Equilibrium (THPE) `#flashcard`
Definition: A mixed-strategy profile σ is trembling-hand perfect equilibrium if there is a sequence of complete mixed strategies $\{\sigma^k\}$ that:

- converges to σ
- for each player i, σ_i is a best response to $\{\sigma_i^k\}$ for every k
  Notes:
- THPE is refined NE, robust to the possibility of small mistakes ($THPE \in NE$)
  - It rules our equilibria that rely on assigning 0 probabilities the strategies that could occur
- We only need to find 1 sequence of completely mixed strategies
  Theorems:
- In a THPE, no weakly dominated strategy can be played with positive probability.

GT L4 Subgame Perfect Nash Equilibrium

Sequential Rationality and Backward Induction `#flashcard`
NEs may not be sequentially rational. We need to perform backward induction to ensure sequential rationality.
Zermelo's Theorem:

- Every finite game of perfect information has a pure strategy NE that can be derived through backward induction.
- If there are no ties in the payoffs of the terminal nodes, there is a unique equilibrium that can be derived by backward induction.
  Backward Induction: (works well in finite games of perfect information)
- Start by the final decision node and choose the optimal action at those nodes
- Reduce the game by replacing the final decision nodes by terminal nodes with the payoff induced by the optimal action
- Iterate upward
- (Note that backward induction is in fact a process of iterated elimination of weakly dominated strategies, but the order of elimination is determined by the sequentiality of the game.)
- (This can be extended with SPNE later)

Subgame and Subgame Perfect Nash Equilibrium (SPNE) `#flashcard`
Subgame: a subgame of an extensive game is the subset of the game such that:

- begins at an information set that contains a single decision node, and contains all the successors of this node, and only those
- if the decision node x is in the subgame, every node in the information set $I(x)$ is also in this subgame
  Subgame Perfect Nash Equilibrium: a strategy profile σ in an extensive form game is a SPNE if it induces a NE in

every subgame.
Theorems:
- Any SPNE is a NE
- Any SPNE induces a SPNE in all subgames
- Every finite extensive form game has a SPNE

Backward Induction with SPNE #flashcard

- Start in the final subgames and identify the NE of those games
- Replace those subgames by the outcome that the NE prescribe
- Re-apply the above step

Subgame #flashcard
A **subgame** of an extensive game is the subset of the game such that:

- *begins at* an information set that contains *a single decision node*,
- *contains all the successors of this node*, *and only those*
- if the decision node x is in the subgame, every node in the information set $I(x)$ is also in this subgame

EXAMPLES ARE SKIPPED FOR NOW

GT L5 Repeated Complete Information Games (Multiple-stage Games with Observed Actions)

Terminologies in Multi-stage Games with Observed Actions (MSGOA) #flashcard
History: $h^t = (a^1, \ldots, a^{t-1})$ all actions taken before stage t
Pure Strategy: a pure strategy for player i is a contingent plan of how to play in each stage after every possible history:

$$s_i = \left\{ s_i^t \right\}_t^T \ s.t. \ s_i^t(h^t) \in A_i(h^t) \ \forall \ h^t \in H^t$$

The **payoff** of player i is then a function $u_i : H^{T+1} \to \mathbb{R}$.
A strategy profile s will determine a **path** of actions and generate a particular payoff.

Subgame Perfection in MSGOA #flashcard
A strategy profile of s of a MSGOA is a SPNE $\iff$ for every h^t, the strategy profile after h^t implied by the strategy profile ($s|h^t$) is a NE of is a NE of the subgame following h^t ($G(h^t)$).

- In simple words, at every history, we have a NE next.

Repeated Games #flashcard
(a particular example of concatenated games) We repeat a simultaneous-move game at each stage, and the payoff for player i of the whole game is:

$$u_i(s) = \sum_{t=1}^T u_i^t(a_s^t)$$

Finite Concatenation Cannot Change **Unique** NEs #flashcard

- Consider a concatenated game Γ_E where n players play T (*finite*) concatenated simultaneous-move games Γ_t and there's a *unique* NE for each Γ_t
  $\implies$ The overall game Γ_E will have a unique SPNE where players play NE in each individual game.

- i.e. If each individual game has a *unique* NE, *finitely* concatenating them together will not alter players' behaviour.
- Moreover, the strategies in this unique SPNE cannot be history-dependent.

- However, *if there are multiple NEs, we can construct SPNE where conditional equilibrium selection is involved and achieve outcomes other than NEs in single-stage games!*

Concatenation and Multiple NEs #flashcard

- If individual games Γ_i in a concatenated game Γ_E have *several equilibria*, then the *overall SPNE doe NOT necessarily involves playing NE in each individual game*.
- In the last round, players must play NEs, but they can choose which to play conditional on the history. (see PS2 Q2)

One-Deviation Principle in finite MSGOA #flashcard
In a finite MSGOA, a strategy profile $s = (s_i, s_{-i})$ is a SPNE $\iff$ no player i can gain higher utility with one-shot deviation (deviation only at one stage) from s_i.

- In other words, for any one-shot deviation $\hat{s}_i$, if we consider the subgame starting at the deviation point $G(h^t)$, $\hat{s}_i$ cannot be a better response than s_i.

## One Deviation Principle in Infinite MSGOA with Discounting #flashcard
In an infine multi-stage game with observed actions and discounting, strategy profile σ is a SPNE $\iff$ there is no profitable one-shot deviation from σ.

Feasible and Supportable Payoffs (Nash Reversion Folk Theorem) in Infinite MSGOA with Discounting #flashcard
Feasible Payoffs: the convex hull of the payoffs of the pure strategies. Those payoffs can be achieved by mixing pure strategies.
(A subset of) Supportable Payoffs - Nash Reversion Folk Theorem: Let s be a NE of the stage game Γ with payoffs u. Then, $\forall$ feasible payoff profile u s.t. $u_i > \underline{u}_i$ for all players $i = 1, \ldots, N$, there exists a discount rate $\underline{\delta} \in (0,1)$ $s.t.$ any $\delta \in (\underline{\delta}, 1)$ supports a SPNE with payoff u in an *infinitely repeated* game. [See Folk Theorem for the full set of supportable payoffs]

- i.e. Any payoff strictly benefiting anyone than any NE can be supported by a δ
- **How to?**
    - Play the strategies that generate our desired u
    - If someone deviates, then play the stage Nash Equilibrium forever
- Example:
    -
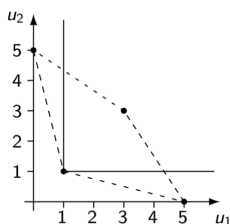
Minmax Payoffs #flashcard
The minmax payoff for player i is the lowest payoff that the other players can impose on player i (the lowest punishment for player i):

$$\underline{u}_i = \min_{\sigma_{-i}} \left\{ \max_{s_i} u_i(s) \right\}$$

- Some this minmax payoff is achieved by mixed strategies of other players σ_{-i}

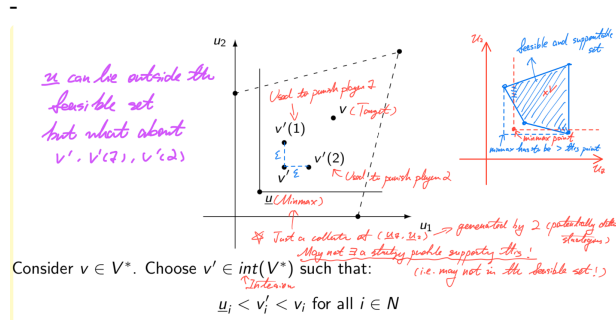**Individual Rationality, Folk Theorem, and 3-Phase Strategy** `#flashcard`

Individual Rational: a payoff u_i is individually rational for player i if it's higher than minmax payoff: $u_i > \underline{u}_i$

Folk Theorem: suppose that the set of feasible and individually rational payoffs V has full-dimensionality. Then, for any $v \in V$, there exists a $\underline{\delta} \in (0,1)$ such that for every $\underline{\delta} < \delta < 1$, there exists a SPNE of *infinitely repeated* game with payoff v.

- i.e. every feasible and individually rational payoff is achievable in infinitely repeated games
- **How to? 3-Phase Strategy**
  - Find the following points:
  -

Consider $v \in V^*$. Choose $v' \in int(V^*)$ such that:

$$\underline{u}_i < v'_i < v_i \text{ for all } i \in N$$

And choose $\epsilon > 0$ such that

$$v'(i) = (v'_i, v'_{-i} + \epsilon) \in V^* \quad and \quad v'(i) < v \quad \text{for all } i \in N$$

- Phase 1: Collaboration
- Play profile $s : u(s) = v$
- If a single player j deviates from s, then go to phase II_j
- Phase II_j: Punishing the deviating player
- Play the minmax strategy $m^j : u_j(m^j) = \underline{u}_j$
- This last for n periods: the period has to be long enough to make deviations not profitable:

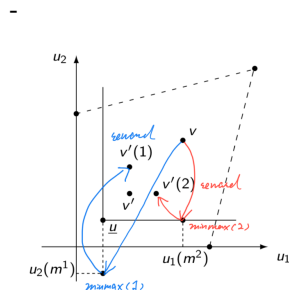$$n > \frac{\max\{u_i\} - v'_i}{v'_i - \underline{u}_i}$$

- After n periods, go to phase III_j
- During this phase, if another player k deviates, go to phase II_k
- Phase III_j: Rewarding punishers
- Play $s(j) : u(s(j)) = v'(j)$
- Graphical illustration
-

GT L6: Dynamic Complete Information Games - Sequential Equilibrium

Weak Perfect Bayesian Equilibrium (WPBE) `#flashcard`

An assessment (σ, μ) is a **Weak Perfect Bayesian Equilibrium (WPBE)** if:

- The strategy profile σ is **sequentially rational** given the system of beliefs μ

- **Sequential rationality**: a behavioural strategy σ is sequentially rational given a system of belief μ if: for each player i and any information set I:

$$\mathbb{E}\left[u_i | I, \mu, \sigma_i, \sigma_{-i}\right] \geq \mathbb{E}\left[u_i | I, \mu, \sigma_i', \sigma_{-i}\right] \forall \sigma_i' \in \Delta(S_i)$$

- The system of beliefs μ is **consistent** with the strategy profile σ
  - **Consistency with strategies**: the system of beliefs μ is consistent with the strategy profile σ if: for any information set I reached with positive probability under σ, we have:

$$\mu(x) = \frac{Pr(x|\sigma)}{Pr(I|\sigma)} \forall x \in I$$

Comments:
- In WPBE, a player's *belief system must be consistent with the strategies actually played by others — but only at information sets that are reached with positive probability under those strategies!!!*
- WPBE is *weak* because it allows *arbitrary beliefs off-path*: *at information sets that are not reached with positive probability under the equilibrium strategies (i.e. off the equilibrium path), players are allowed to hold beliefs that are inconsistent with the actual strategies.*
- *The system of beliefs needs to cover all information sets -- even those that won't be reached (but there will be no restriction on off-path beliefs)*
- WPBE $\in$ NE, but:
  - a WPBE may not be a SPNE
  - a NE may not be a WPBE
  Practical Steps to Find WPBE: (?)
- Start from the last information set
- Calculate the belief μ consistent with each possible action in the final node
- Update belief
- Check whether the updated beliefs are consistent with strategy
- ...

Sequential Equilibrium #flashcard
An assessment (σ, μ) is a sequential equilibrium if:

- The strategy profile σ is **sequentially rational** given the system of beliefs μ
- a sequence of *completely mixed* strategies $\{\sigma^k\}_{k=1}^{\infty}$ with $\lim_{k \to \infty} \sigma^k = \sigma$ such that $\lim_{k \to \infty} \mu^k = \mu$ where μ^k are the beliefs implied by strategy σ^k using Bayes' rule
  Comments:
- This is a refinement of WPBE: SE $\in$ WPBE
- **Practical Steps**: lectured by Yangyang
  1. Find WPBEs
  2. Consider the strategy in the WBPE assessment: add some perturbs ϵ to the strategy so it becomes completely mixed: $\{\sigma^k\}_{k=1}^{\infty}$
  3. Find the belief μ^k consistent with $\{\sigma^k\}_{k=1}^{\infty}$
  4. Find the belief system in the limit $\lim_{\epsilon \to 0} \mu^k = \mu$
  5. Check whether the strategy specified in WPBE assessment is optimal for the new belief system μ

Existence of SE/WPBE #flashcard
Kreps and Wilson 1982: every finite game has a sequential equilibrium (hence WPBE). The strategy profile in a SE constutes a SPNE.
Comments:

- Any completely mixed NE is a SE
- Any THPE is a SE, and the converse is almost true

Games of Incomplete Information

GT L7: Static Incomplete Information Games - Bayesian Nash Equilibrium

Bayesian Game #flashcard

A Bayesian Game $\Gamma = [N, \{S_i\}, \{_i\}, p, \{u_i\}]$ consists of:

- A finite set of players $N = \{1, \ldots, n\}$
- Each player i has a set of action S_i
- Each player i has a set of types $_i$
- There is a joint distribution of all players' types $p(\theta_1, \theta_2, \ldots, \theta_n)$
- Each player i has a vNM utility function $u_i : S \times \ \to \mathbb{R}$
  Some particular cases:
- **Independent Types**: one player's type does not reveal information about other players' types
- **Private Types**: preference of a player only depends on her own type and actions, but not on others' types
  Pure Strategy in Bayesian Games: for player i, this is a function $s_i : \ _i \to S_i$ mapping her type realisation θ_i to a strategy $s_i(\theta_i) \in S_i$

Bayesian Nash Equilibrium and Harsanyi Equivalence #flashcard

- A **Bayesian Nash Equilibrium (BNE)** is a profile of strategies $(\sigma_1(.), \ldots, \sigma_N(.))$ such that no player can have a profitable deviation contingent on her type.
    - Formally: $\forall i \in \{\text{Players}\}, \theta_i \in \ _i, s_i \in S_i$:

    $$\mathbb{E}_{\theta_{-i}}\left[u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})); \theta | \theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i(s_i, \sigma_{-i}(\theta_{-i})); \theta | \theta_i\right]$$

    - Intuition:
        - In a BNE, *each player's strategy is a best response to the strategy functions of the other players*.
        - In a BNE, *players are best responding to others' actual strategy functions (not guessed or perceived ones)*, but they *compute expected payoffs based on their beliefs about others' types*, since types are private.
- **Harsanyi Equivalence**: a profile of Bayesian strategies $(\sigma_1(.), \ldots, \sigma_n(.))$ is a Bayesian Nash Equilibrium of a Bayesian Game $\Gamma = [N, \{S_i\}, \{_i\}, p, \{u_i\}] \iff$ it's a Nash Equilibrium of the Normal Game $[N, \{S_i\}, \{\tilde{u}_i\}]$ where $\tilde{u}_i = \mathbb{E}_\theta[u_i(.)]$
  - In other words, for all i and for all $s_i \in S_i$:

  $$\mathbb{E}_\theta\left[u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}); \theta)\right] \geq \mathbb{E}_\theta\left[u_i(s_i, \sigma_{-i}(\theta_{-i}); \theta)\right]$$

  - Practical implication: with common priors, we can transform Incomplete Bayesian Game $\to$ Complete Information Game by introducing a random move by "Nature" at the beginning of the game. Nature assigns types to players according to a known probability distribution.

WPBE and BNE #flashcard

Feature	Bayesian Nash Equilibrium (BNE)	Weak Perfect Bayesian Equilibrium (WPBE)
Game form	Bayesian (normal-form)	Extensive-form with imperfect information
Beliefs over	Types (private info)	Nodes within information sets
When beliefs matter	Only at the start (before actions)	Throughout the game (at each info set)
Off-path beliefs	Not an issue	Allowed to be arbitrary (weaker requirement)
Focus	Optimal strategies given types	Optimal strategies at every decision point

- BNE is for static games with private info.

- WPBE is for dynamic (extensive-form) games with beliefs at every stage.
- WPBE generalises BNE to games with richer structure and unfolding information.

Risk Dominant Equilibrium #flashcard

- The **risk-dominant equilibrium** is the NE where players are less vulnerable to losses if the other player deviates.
- A risk-dominant equilibrium may not be Pareto dominant.

Purification #flashcard

- Intuition: A mixed-strategy Nash equilibrium in a game of **complete information** can be interpreted as the limit of pure-strategy Bayesian Nash equilibria in a sequence of games with **slightly perturbed payoffs (incomplete information)**.
    - Purification provides an interpretation of mixed strategies: *it might seem to an outside observer that players are mixing, whereas in fact they are deterministically responding to their private information in a larger game.*
        - **Usual steps to purify a mixed strategy (see PS GT5 Q2)**:
            1. Assume a strategy (typically a cutoff strategy with an undetermined coefficient k) of one player based on her private information
            2. Calculate the BR of the other player and verify the BR is of the same type of the assumed strategy
            3. Use the symmetry to calculate the NE
            4. Verify the limit of their strategies is the mixed-strategy NE we want to purify
- Formally: for almost all complete information game Γ, any NE (including mixed-strategy NE) of Γ can be considered as the limit of $\epsilon \to 0$ of a sequence of pure Bayesian equilibria of the perturbed games Γ_ϵ

GT L8: Dynamic Incomplete Information Games (Signalling Games) - Perfect Bayesian Equilibrium

Perfect Bayesian Equilibrium (PBE) #flashcard

- An extension of BNE to dynamic games.
  A **Perfect Bayesian Equilbrium** of a dynamic game is an assessment (σ, u) such that:
- **Sequential Rationality**:
    - At every information set, players must maximize expected utility given their updated beliefs about other players' types and strategies.
- **Consistent Belief Updating**:
  - Players must update their beliefs using Bayes' rule whenever possible.
  - If an event occurs with zero probability under equilibrium play, beliefs can be specified in an arbitrary but reasonable way (off-path beliefs).

Relations between SE, PBE, SPNE, NE #flashcard
Relations:

$$\{SE\} \ \{PBE\} \ \{SPNE\} \ \{NE\}$$

- SE and PBE:
  - Both require sequential rationality + consistency of beliefs
  - SE introduces a stronger requirement: beliefs must be limits of beliefs from perturbed strategies
  Fudenberg and Tirole 1991:
- If each player has at most 2 types, and there are at most 2 periods, the set of PBE coincides with the set of sequential equilibria.

GT L9: Further Refinements: Forward Induction and Intuitive Criterion

Forward Induction #flashcard

- Forward Induction is a refinement of Sequential Equilibrium. It eliminates non-credible equilibria where players take actions inconsistent with rational belief updates.
- Key idea:
  - Instead of dismissing off-path moves as irrational, forward induction suggests that players should rationalise why such a move occurred.
  - The reasoning is "if a player took an unexpected action, maybe they have private information that makes that action optimal."
  - This is especially useful in signaling games, where a sender's unexpected action can reveal stronger private information than initially assumed.

Intuitive Criterion #flashcard

- Intuitive Criterion is an refinement of Perfect Bayesian Equilibrium used in signalling games. It rules out implausible off-equilibrium beliefs by asking:
  - "Could a type of player possibly benefit from deviating, regardless of how others react?"
- Key Idea: Ruling Out Implausible Signals
  - If some types of a player would never send a certain signal (because it only harms them), then the receiver should believe the sender must be the other type.
  - The receiver eliminates impossible types and updates beliefs accordingly.
- Formal Definition
  -
  Let a sender have types θ_1 and θ_2, and choose a signal s. The receiver assigns beliefs $\mu(\theta_1 \mid s)$ and $\mu(\theta_2 \mid s)$.

  The **Intuitive Criterion** says:

  - If **type θ_1 would never benefit from sending** s for any belief of the receiver, then the receiver **must believe the sender is θ_2**.

  - This eliminates equilibria where a type sends a **self-destructive** signal.