0 M48 Book Notes

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Chapter 1: Logic, Definitions, and Proofs

1.1 Quantifiers

- 1.1-Quantifiers_Notes - A

Statements

- Definition of Statements: A statement is any declarative sentence which is *either true or false*.
- Examples:
    - $3 + 6 = 9$
    - Pigs can fly
- Non-Examples:
    - Do you like bananas?
    - $2x + 6 = 9$
        - Whether this is true depends on what x is. To turn it into a statement, we either specify a specific substitution or quantify over the variable

Quantifiers

- "Forall/Every": $\forall$ (an *universal* quantifier)
    - example: $\forall x, 2x + 6 = 9$ (false)
- "There exists/is" or "for some": $\exists$ (an *existential* quantifier)
    - example: $\exists x \in \mathbb{R}\ s.t.\ 2x + 6 = 9$ i.e. "there exists a real number such that $2x + 6 = 9$" (true); $\exists x \in \mathbb{Z}\ s.t.\ 2x + 6 = 9$ (false)
- Caveat: when working with quantifiers, always be aware of the domain

Negating Quantifiers

- Definition of Negation: The negation of a statement is something that will be *true exactly when the original statement is false*.
- Negation of forall/every ($\forall$): there exists a counterexample ($\exists$)
- Negation of there exists ($\exists$): for all ... ($\forall$)

Simple Proof with Quantifiers

- $\forall$: forall/every
    - Prove to be true
    - Prove to be false: show the negation is true (we can just use a *counterexample*)
- $\exists$: there exists/is / for some
    - Prove to be true
    - Prove to be false: show the negation is true (cannot just use counterexample)

Double Quantifiers

- When there are double quantifiers, their *order* matters a lot!
- Example:
    - $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\ s.t.\ x > y$ means "every real number is larger than some other real number"
    - $\exists y \in \mathbb{R}\ s.t.\ \forall x \in \mathbb{R}, x > y$ means "There is a real number smaller than all real numbers"
- Proving Statements with Double Quantifiers
    - Example

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } x > y.$$

Proof.

Let $x \in \mathbb{R}$. ← ① fix an arbitrary $x \in \mathbb{R}$

Take $y = x - 1$. ← ② choose y

Notice that since $x \in \mathbb{R}$,

- $y = x - 1 \in \mathbb{R}$ and
- $y = x - 1 < x$.

← ③ Check that your y works.

1.2 Conditional Statements

- 1.2-Conditionals_Notes - A

Intro

- Definition of Conditional Statements: Given statements P and Q, a conditional statement (also known as an implication) is a statement is the form:

$$\text{if } \underbrace{P}_{\text{hypothesis}}, \text{then } \underbrace{Q}_{\text{conclusion}}$$

- *Notation*: $P \implies Q$
  - *Reads*: P implies Q / if P, then Q
- Meaning: whenever P is true, Q must also be true.

When is A Conditional Statement True?

- The statement "if P, then Q" is *true* when:
  - P is true and Q is true
  - P is false and Q is true
  - P is false and Q is false
- The statement "if P, then Q" is *false* only when:
  - P is true and Q is false
- Takeaway: when the hypothesis (P) is false, the statement is always true! (we must take an "innocent until proven guilty" approach)

Hidden Quantifiers in Conditional Statements

- Conditional statements often contain a "hidden" universal quantifier
- When we write "$P(x) \implies Q(x)$", we actually mean "$\forall x, P(x) \implies Q(x)$"

Proving Conditional Statements: Direct Proof

- Workflow: *assumptions -> definitions -> proof*
- Example: Theorem: odd + odd = even

Proof.

Assume that x and y are odd integer numbers. Start by assuming our hypothesis

Then, by the definition of odd number, there exist $m, n \in \mathbb{Z}$ such that

$$x = 2m + 1 \quad \text{and} \quad y = 2n + 1.$$

Therefore,

$$\begin{aligned} x + y &= (2m+1) + (2n+1) \\ &= 2m + 2n + 2 \\ &= 2(m + n + 1) \end{aligned}$$

Let $q = m + n + 1$. Then $q \in \mathbb{Z}$ because the sum of two integer number is also an integer (integers are closed under addition) and $x + y = 2q$.

- Hence, $x + y$ is even.

Negating Conditional Statements

- The negation of a conditional statement (if..., then...) is *NEVER* another conditional statement (if..., then...)!

Contrapositive, Converse, and Biconditioinal

- Definition of Contrapositive: the contrapositive of a conditional statement "*if P, then Q*" is another conditional statement "*if not Q, then not P*"
  - Contrapositive is *equivalent* to the original conditional statement
- Definition of Converse: the converse of a conditional statement "*if P, then Q*" is another conditional statement "*if Q, then P*"
  - Converse is *not logically equivalent* to the original conditional statement
- Definition of Biconditional: a biconditional statement is the conjunction of a conditional statement with its converse and it is written in the form "*P if and only if Q*"
  - Notation: $P \iff Q$ or "P iff Q"
  - Meaning: P and Q are both true or they are both false (*P is equivalent to Q*)

1.3 Working with Definitions

- 1.3-Working-with-Definitions_Notes - A

Steps for Working with New Definitions

1. Understand the rigorous definition (not just the intuition)
2. Write some examples

3. Write some non-examples
4. Write the negation of the definition

An Example

- Definition of an Increasing Function:

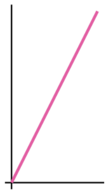
$$\forall x_1, x_2 \in \mathbb{D}, \text{ if } x_1 < x_2, \text{ then } f(x_1) < f(x_2)$$

  or equivalently:

$$\text{if } x_1, x_2 \in \mathbb{D} \text{ and } x_1 < x_2, \text{ then } f(x_1) < f(x_2)$$

- Example and proof:

  $f(x) = 2x$ is increasing on $\mathbb{R}$.



We need to check that this function satisfies the definition!

We have to show that

$$\forall x_1, x_2 \in \mathbb{R},\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Proof.

Let $x_1, x_2 \in \mathbb{R}$.

Assume that $x_1 < x_2$.

Then, $\underbrace{2x_1}_{= f(x_1)} < \underbrace{2x_2}_{= f(x_2)}$.

Therefore, $f(x_1) < f(x_2)$.

😃

- Negation:

$$\exists x_1, x_2 \in \mathbb{R}\ s.t.\ x_1 < x_2 \text{ and } f(x_1) \geq f(x_2)$$

1.4 Proofs

- 1.4 Proofs_NOTES - A

Indirect Proof

Proof by Contrapositive

- This only applies to conditional statements!
- *Method*: Instead of proving "*If P, then Q*" directly, we prove the equivalent contrapositive "*If not Q, then not P*"

- *Proof Structure/Steps*:
  1. Assume that the negation of Q is true (assume that Q is false)
  2. (continue as in a direct proof)
  3. Conclude that the negation of P is true (assume that P is false)

Proof by Contradiction

- This applies to all statements but is especially *handy to prove statements written in a negative way* (e.g. ... does not exist)
- *Method*: instead of showing that the statement is true, we show that the *statement is not false*
- *Proof Structure/Steps*:
  1. Assume that your statement is false (assume the *negation is true*)
     - "assume for the sake of contradiction that ..."
  2. Proceed as your would with a direct proof
  3. Arrive at a contradiction
     "this contradicts with the assumption that..."
  4. State that because of the contradiction, it can't be the case that the statement is false, so it must be true

Proof by Contrapositive vs Contradiction

- Proof by contrapositive only works for conditional statements; proof by contradiction works for any kind of statements
- When proving conditional statements ($P \implies Q$):

Contrapositive	Contradiction
1 assumption to work with (not Q)	2 assumptions to work with P and (not Q)
Clear goal (not P)	no clear goal

- We need to be careful when using proof by contradiction: wrong step could lead to wrong contradictions

Caveats for Using Indirect Proofs

- A direct proof is usually preferred over an indirect proof
- Always let the reader know what proof technique you are using
- Be careful when negating statements (critical stepstone!)
- When proving $P \implies Q$, if you assume (not Q) and this leads to (not P). This is a proof by contrapositive, NOT a proof by contradiction

Direct Proof

Proof by Induction

- A proof by induction involves 2 *Steps*:
    1. *Base case*: prove that $P(1)$ is true
    2. *Induction step*: prove that $\forall n \geq 1, P(n) \Rightarrow P(n+1)$
- *When using proof by induction, always check whether we can induct to the domain we want*
- Example

  Example. Prove that for every $n \in \mathbb{N}$, $1+3+\cdots+(2n-1) = n^2$.

  Proof. We will prove this statement by induction.
  Let $P(n)$ be the statement "$1+3+\cdots+(2n-1) = n^2$".

  Base Case: $(n=1)$ $\quad 1 = 1^2$ ✓

  Induction Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true, that is

  $$1+3+\cdots+(2n-1) = n^2. \qquad \text{(This is our induction hypothesis.)}$$

  We want to show that $P(n+1)$ is true, that is $1+3+\cdots+(2(n+1)-1) = (n+1)^2$.
  Notice that

  $$\begin{aligned} 1+3+\cdots+(2(n+1)-1) &= 1+3+\cdots+(2n-1)+(2(n+1)-1) \\ &= n^2+(2(n+1)-1)) \qquad \text{(by induction hypothesis)} \\ &= n^2+2n+1 \\ &= (n+1)^2. \end{aligned}$$

  state explicitly

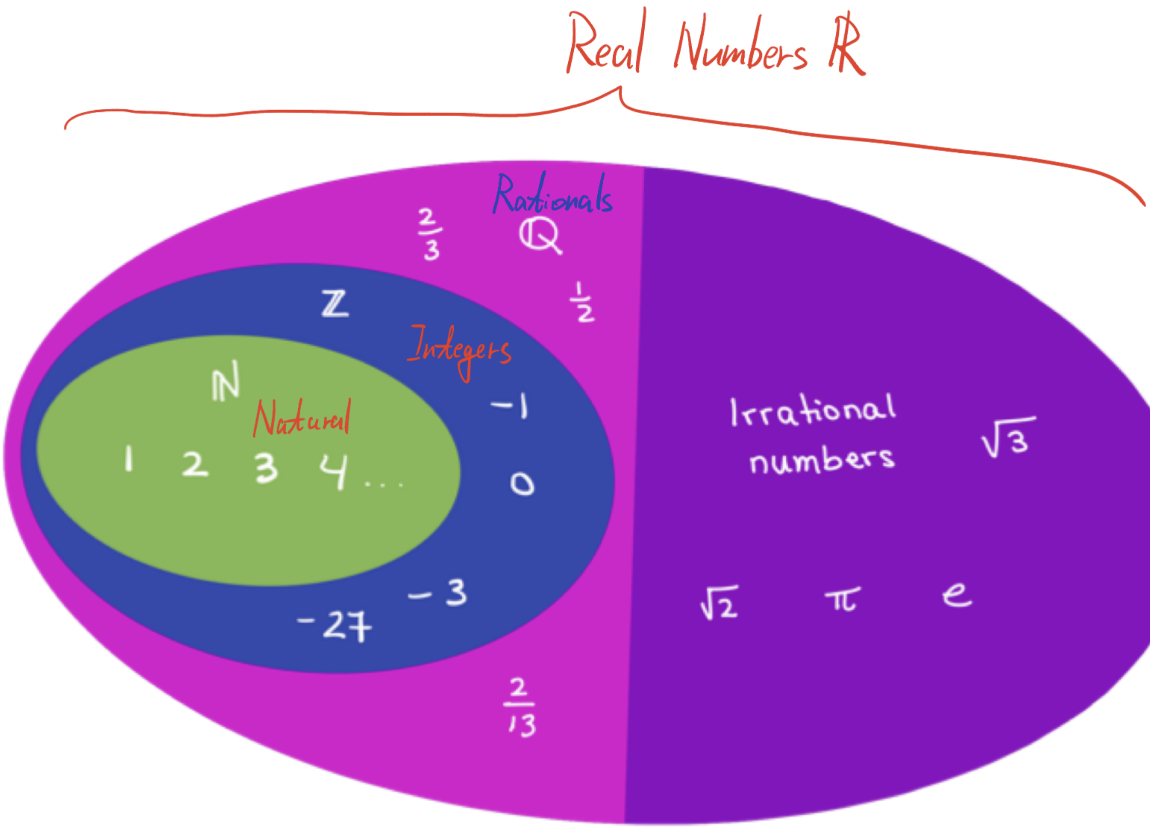
  Hence, $P(n) \Rightarrow P(n+1)$. ✓
  Thus, by the principle of induction, we conclude that $P(n)$ holds for every positive integer n.
  Therefore, $1+3+\cdots+(2n-1) = n^2$ for all $n \in \mathbb{N}$. 😃

Chapter 2: The Real Numbers

2.1 The Set of Real Numbers

- 2.1 Real numbers_NOTES - A

Rational and Irrational Numbers



-
- Rational Numbers $\mathbb{Q}$
  - $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$
  - Integers $\mathbb{Z}$
    - $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
    - Natural Numbers $\mathbb{N}$
      - $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
      - 不包括0
      - Natural Number Operations
        - Addition: $m, n \in \mathbb{N} \rightsquigarrow m + n \in \mathbb{N}$
        - Multiplication: $m, n \in \mathbb{N} \rightsquigarrow m \cdot n \in \mathbb{N}$
- With them, we can solve all equations of the form: $a \cdot x = b$
- Irrational Numbers $\mathbb{P}$

Axioms for the Real Numbers

- axioms: we take them as true without proof

Field/Algebraic Axioms

Field Axioms		
Axioms of addition	**Axioms of multiplication**	
$\forall a,b \in \mathbb{R}, a+b \in \mathbb{R}$	$\forall a,b \in \mathbb{R}, ab \in \mathbb{R}$	(closure)
$\forall a,b \in \mathbb{R}, a+b = b+a$	$\forall a,b \in \mathbb{R}, ab = ba$	(commutativity)
$\forall a,b,c \in \mathbb{R}, (a+b)+c = a+(b+c)$	$\forall a,b,c \in \mathbb{R}, (ab)c = a(bc)$	(associativity)
$\exists 0 \in \mathbb{R}$ s.t. $\forall a \in \mathbb{R}, a+0 = a$	$\exists 1 \in \mathbb{R}$ s.t. $\forall a \in \mathbb{R}, a \cdot 1 = a$	(identity)
$\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R}$ s.t. $a+(-a) = 0$	$\forall a \in \mathbb{R}$ $\{0\}, \exists a^{-1} \in \mathbb{R}$ s.t. $aa^{-1} = 1$	(inverse)
$\forall a,b,c \in \mathbb{R}, a(b+c) = ab+ac$ (distributivity of multiplication over addition)		

-tx-

	Axioms of *Addition*	Axioms of *Multiplication*
Closure	$\forall a, b \in \mathbb{R}, a+b \in \mathbb{R}$	$\forall a, b \in \mathbb{R}, ab \in \mathbb{R}$
Commutativity	$\forall a, b \in \mathbb{R}, a+b = b+a$	$\forall a, b \in \mathbb{R}, ab = ba$
Associativity	$\forall a, b, c \in \mathbb{R}, (a+b)+c = a+(b+c)$	$\forall a, b, c \in \mathbb{R}, (ab)c = a(bc)$
Identity	$\exists 0 \in \mathbb{R}\ s.t.\ \forall a \in \mathbb{R}, a+0 = a$	$\exists 1 \in \mathbb{R}\ s.t.\ \forall a \in \mathbb{R}, a \cdot 1 = a$
Inverse	$\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R}\ s.t.\ a+(-a) = 0$	$\forall a \in \mathbb{R}\{0\}, \exists a^{-1} \in \mathbb{R}\ s.t.\ aa^{-1} = 1$
Distributivity of multiplication over addition	$\forall a, b, c \in \mathbb{R}, a(b+c) = ab+ac$	

- *Propositions* (Elementary consequences of the field axioms)
  - we cannot use them without proof

    1. 0 and 1 are unique
    2. The additive inverse and the multiplicative inverse (of a nonzero x) are unique
    3. $\forall x \in \mathbb{R},\ x \cdot 0 = 0$
    4. $\forall x, y, z \in \mathbb{R},\ x+z = y+z \Rightarrow x = y$
    5. $\forall x \in \mathbb{R},\ -x = (-1)x$
    6. $\forall x, y \in \mathbb{R},\ (-x)y = -(xy)$
    7. $\forall x, y \in \mathbb{R},\ (-x)(-y) = xy$
    8. $\forall x, y, z \in \mathbb{R},\ xz = yz$ and $z \neq 0 \Rightarrow x = y$
    9. $\forall x, y \in \mathbb{R},\ xy = 0 \Rightarrow x = 0$ or $y = 0$

We will prove (3): $\forall x \in \mathbb{R},\ x \cdot 0 = 0$

Proof.
Let $x \in \mathbb{R}$. Then

$$\begin{aligned}
x \cdot 0 &= x \cdot 0 + 0 && \text{(additive identity)} \\
&= x \cdot 0 + (x \cdot 0 + (-x \cdot 0)) && \text{(additive inverse)} \\
&= (x \cdot 0 + x \cdot 0) + (-x \cdot 0) && \text{(associativity)} \\
&= x \cdot (0 + 0) + (-x \cdot 0) && \text{(distributivity)} \\
&= x \cdot 0 + (-x \cdot 0) && \text{(additive identity)} \\
&= 0 && \text{(additive inverse)}
\end{aligned}$$

Exercise: prove the other properties

Ordering Axioms

Ordering Axioms	
$\forall a, b \in \mathbb{R}$, exactly one of $a < b$, $a = b$ and $a > b$ is true	(trichotomy)
$\forall a, b, c \in \mathbb{R}$ if $a < b$ and $b < c$, then $a < c$	(transitivity)
$\forall a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$	
$\forall a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ca < cb$	

- Trichotomy: $\forall a, b, \in \mathbb{R}$, exactly one of $a < b$, $a = b$, and $a > b$ is true
- Transitivity: $\forall a, b, c \in \mathbb{R}$ if $a < b$ and $b < c$, then $a < c$
- $\forall a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$
- $\forall a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ca < cb$

- Notation: $x \leq y$ means that either $x < y$ or $x = y$
- *Propositions*

Proposition. For all $x, y, z, w \in \mathbb{R}$,

1. $x < y \iff y - x > 0$,
2. $x > 0 \iff -x < 0$,
3. $x < y \iff -y < -x$,
4. $xy > 0 \iff x > 0, y > 0$ or $x < 0, y < 0$,
5. $x \neq 0 \iff x^2 > 0$ (in particular $1 > 0$),
6. $x < y$ and $z < 0 \Rightarrow xz > yz$.
7. $x > 0 \iff \frac{1}{x} > 0$,
8. $x^2 < y^2,\ x \geq 0, y > 0 \Rightarrow x < y$
9. $x < y$ and $z < w \Rightarrow x + z < y + w$

2.2 Supremum and Completeness Axiom

- 2.2 Supremum and Completenss NOTES - A

Maximum, Upper/Lower Bound, Supremum/Infimum, Bounded Above/Below

- Maximum: definition: let $S \subset \mathbb{R}$. We say M is the maximum of S if $M \in S$ and $\forall x \in S, x \leq M$
  - intuition: the *largest element of the set*
- Not evert set has a maximum: e.g. the open interval $(0, 1)$ does not have a maximum
- Upper Bound: definition: Let $S \in \mathbb{R}$. Let $b \in \mathbb{R}$. We say b is an upper bound of S if and only if $\forall x \in S, x \leq b$
- Supremum / Least Upper Bound: definition: Let $S \in \mathbb{R}$. Let $b \in \mathbb{R}$. We say b is the supremum / least upper bound of S if and only if
  1. b is an upper bound of S
  2. if c is an upper bound of S, then $b \leq c$
  - notation: $\sup S$
    - prove the supremum of a set S is x:
      1. prove x is an upper bound for S （一般直接根据set的定义）
      2. prove $M < x \implies M$ is not an upper bound of S ($\iff$ x is the least upper bound / supremum)
  - Equivalent definition (useful later): S is an upper bound of A and:

9 $\forall \varepsilon > 0, \quad \exists x \in A \quad \text{such that} \quad S - \varepsilon < x.$

10 $\forall \varepsilon > 0, \quad \exists x \in A \quad \text{such that} \quad S - \varepsilon < x \leq S.$

- Bounded Above: definition: A set is bounded above if it has at least one upper bound
- Example

set	bnd above?	upper bounds	sup	max
$\{2,3,5,7\}$	Yes	$9, 11, 27 \cdots$	7	7
$[0,1]$	Yes	$1, 2, \pi, \cdots$	1	1
$(0,1)$	Yes	$1, 2, \pi, \cdots$	1	DNE
$(0,\infty)$	No	DNE	DNE	DNE

- Note that *if the maximum of a set S exists, then* $\max S = \sup S$
  - My trial proof for this: let b be the maximum of S. By definition of maximum, we know that $b \in S$ and $\forall x \in S, x \leq b \implies b$ satisfies the definition of an upper bound.

Let $M < b$, $b < M$ and $b \in S$, so M is not an upper bound. Hence, b is the least upper bound $\implies$ supremum

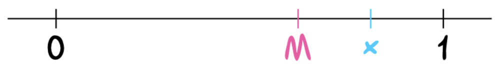
- Lower Bound: definition: Let $S \subset \mathbb{R}$. Let $b \in \mathbb{R}$. We say b is a lower bound of S if and only if $\forall x \in S, x \geq b$
- Infimum / Greatest Lower Bound: definition: Let $S \in \mathbb{R}$. Let $b \in \mathbb{R}$. We say b is the infimum / greatest lower bound of S if and only if
  1. b is a lower bound of S and
  2. if c is a lower bound of S, then $b \geq c$
- Bounded Set: definition: a set is bounded if it is both bounded above and below
- What is $\inf \phi$ (infimum of an empty set)?
  - let A be a set
  - L is a lower bond of A if $\forall x \in A, L \leq x$
  - Then any real number satisfies this
  - $\implies$ *any real number is a lower bound of the empty set*
  - $\inf \phi$ is the greatest lower bound
  - $\implies$ $\inf \phi = +\infty$ (*infinity is not a number, but an abstract concept to show it's larger than any real number*)
- Proving $\sup(0,1) = 1$:

  Proof.

  ① Since $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$, by definition $x \leq 1$ for all $x \in (0,1)$.

  Hence, 1 is an upper bound of $(0,1)$. ✓

  ② Let $M < 1$. We need to show that M is not an upper bound.
  - If $M \leq 0$, then $M < 0.5$ and $0.5 \in (0,1)$. Therefore, M is not an upper bound.
  - If $0 < M < 1$, let $x = \frac{M+1}{2}$.



  Then $x \in (0,1)$ and $x > M$, which shows that M is not an upper bound. ✓

  - Hence, 1 is the supremum of $(0,1)$.
    - 1. show 1 is an upper bound; 2. show any $M < 1$ is not an upper bound

Completeness Axiom and Consequences

- Completeness Axiom: let $S \subseteq \mathbb{R}$. If: 1. S is not empty; 2. S is bounded above, then S has a least upper bound (supremum) $b \in \mathbb{R}$
  - Unbounded above $\iff$ supreme does not exist (ignore empty sets)

- Unbounded below $\iff$ infimum does not exist (ignore empty sets)

Consequences of the Completeness Axiom

- Archimedean Property: $\mathbb{N}$ is not bounded above
  - *Equivalent to Archimedean Property*:
    - For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$
    - For every $x > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$
- Floor Function: For every $x \in \mathbb{R}$, there exists an $n \in \mathbb{Z}$ such that $n \leq x < n+1$
  - Intuition: every real number is either an integer or between two integers
- Existence of Square Roots: Let D be a natural number. Then, there exists a positive real number x such that $x^2 = D$
  - $D \in \mathbb{N} \implies \exists x \in \mathbb{R}^+ \ s.t. \ x^2 = D$
- Proof for the Archimedean Property:

  Proof.

  Assume, for the sake of contradiction, that $\mathbb{N}$ is bounded above.

  Since $\mathbb{N}$ is not empty and $\mathbb{N} \subset \mathbb{R}$, $\mathbb{N}$ has a least upper bound (sup) by the Completeness Axiom.

  Let $\alpha = \sup \mathbb{N}$.

  By the definition of supremum, this means that $\alpha - 1$ is not an upper bound for $\mathbb{N}$.

  Therefore, there is some integer k with $\alpha - 1 < k$.

  But then $\alpha < k+1$ and $k+1 \in \mathbb{N}$.

  This contradicts the fact that α is an upper bound for $\mathbb{N}$!

  Hence, $\mathbb{N}$ is not bounded above.

Summary of All 16 Axioms for Real Numbers

Algebraic Axioms	
Axioms of addition	**Axioms of multiplication**
(A1) $\forall a,b \in \mathbb{R}, a+b \in \mathbb{R}$ (closure)	(A6) $\forall a,b \in \mathbb{R}, ab \in \mathbb{R}$ (closure)
(A2) $\forall a,b \in \mathbb{R}, a+b = b+a$ (commutativity)	(A7) $\forall a,b \in \mathbb{R}, ab = aba$ (commutativity)
(A3) $\forall a,b,c \in \mathbb{R}, (a+b)+c = a+(b+c)$ (associativity)	(A8) $\forall a,b,c \in \mathbb{R}, (ab)c = a(bc)$ (associativity)
(A4) $\exists 0 \in \mathbb{R}$ s.t. $\forall a \in \mathbb{R}, a+0 = a$ (existence of add. identity)	(A9) $\exists 1 \in \mathbb{R}$ s.t. $\forall a \in \mathbb{R}, a \cdot 1 = a$ (existence of mult. identity)
(A5) $\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R}$ s.t. $a+(-a) = 0$ (existence of add. inverses)	(A10) $\forall a \in \mathbb{R}\ \{0\}, \exists a^{-1} \in \mathbb{R}$ s.t. $aa^{-1} = 1$ (existence of mult. inverses)
(A11) $\forall a,b,c \in \mathbb{R}, a(b+c) = ab+ac$ (distributivity of multiplication over addition)	

field

Ordering Axioms
(A12) $\forall a,b \in \mathbb{R}$, exactly one of $a<b$, $a=b$ and $a>b$ is true (trichotomy)
(A13) $\forall a,b,c \in \mathbb{R}$ if $a<b$ and $b<c$, then $a<c$ (transitivity)
(A14) $\forall a,b,c \in \mathbb{R}$, if $a<b$, then $a+c<b+c$
(A15) $\forall a,b,c \in \mathbb{R}$, if $a<b$ and $c>0$, then $aca < cb$

ordered field

Completeness Axiom
(A16) Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$

complete ordered field

Rational Number Is Dense in Real Numbers

- $\mathbb{Q}, \mathbb{P}$ are Dense in $\mathbb{R}$: For every $x, y \in \mathbb{R}$ such that $x < y$, there exists $t \in \mathbb{Q}, s \in \mathbb{P}$ such that $x < t < y, x < s < y$
  - Intuition: the real axis is a continuous line without "holes"
- Example use:

Example

Let $S = \{x \in \mathbb{Q} : x^2 < 2\}$. Prove that $\sup S = \sqrt{2}$

Proof.

1. Notice that $x^2 < 2 \Rightarrow x < \sqrt{2}$.
   Therefore, $\sqrt{2}$ is an upper bound for S.

2. Now, let $H < \sqrt{2}$.
   ① If $H \leq 0$, then H is not an upper bound for S because $1 \in S$.
   ② If $0 < H < \sqrt{2}$, then there exists $r \in \mathbb{Q}$ such that

$$0 < H < r < \sqrt{2}$$

because the rational numbers are dense in $\mathbb{R}$.

Therefore, $r^2 < 2$ and $r \in \mathbb{Q}$.

Hence, $r \in S$ and since $H < r$, we have that H is not an upper bound for S.

Thus, $\sup S = \sqrt{2}$.

Chapter 3: Sequences

3.1 Limits of Sequences

- 3.1 Sequences NOTES - A

Sequences

Standard Definitions (from 1 / any integer)

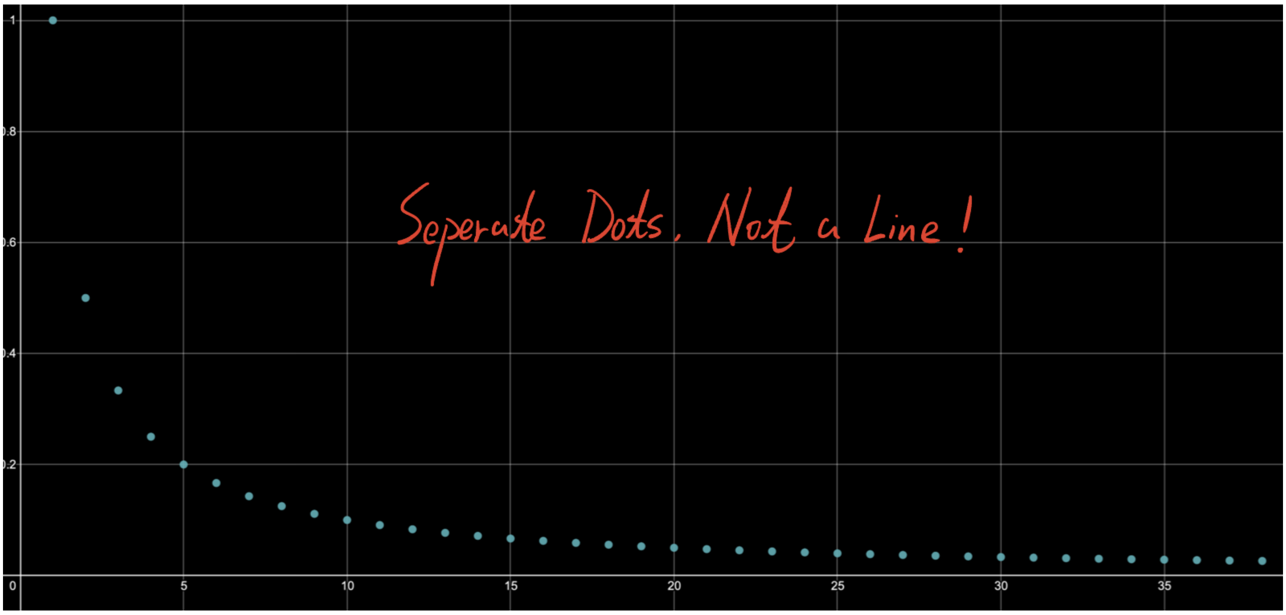
- Definition of Sequences: A sequence is a function from $\mathbb{N}$ to $\mathbb{R}$:

$$a : \mathbb{N} \to \mathbb{R}, a(n) = a_n$$

  - *Intuition*: a sequence is an infinite list of real numbers written in a specific order
  - *Notation*: $(a_n), (a_n)_{n=1}^{\infty}, \{a_n\}_{n=1}^{\infty}, \{a_n\}_{n\geq 1}, \{a_n\}$

$(a_n)_{n=1}^{\infty}$ NOTATION	GENERAL TERM	TERMS OF THE SEQUENCE
$\left(\frac{1}{n^2}\right)_{n=1}^{\infty}$	$a_n = \frac{1}{n^2}$	$1, \frac{1}{4}, \frac{1}{9}, \ldots$
$\left(\frac{(-1)^n}{2n}\right)_{n=1}^{\infty}$	$a_n = \frac{(-1)^n}{2n}$	$-\frac{1}{2}, \frac{1}{4}, -\frac{1}{6}, \ldots$

- Sequences are plotted as separate points, not a line



- It does not matter where we start
  - we can start a sequence from any integer number $N_0 \in \mathbb{Z}$

- Example. The sequence 3, 4, 5, 6, 7, ... can be expressed in many different ways depending on where we choose to start the index:

  $$(n+2) = (n+2)_{n=1}^{\infty}, \quad \text{or} \quad (n)_{n=3}^{\infty}, \quad \text{or} \quad (n-5)_{n=8}^{\infty} \ldots$$

- Definition of Sequence (General Version): a sequence is a function from a subset of integers starting with some integer N_0 to the set of real numbers:

  $$a : D \to \mathbb{R}, a(n) = a_n$$

  where $D = \{n \in \mathbb{Z} : n \geq N_0\} = \{N_0, N_0 + 1, N_0 + 2, \ldots\}$ for some fixed $N_0 \in \mathbb{Z}$
  - *Notation* for starting at some integer N_0 other than $n = 1$, we can use:$(a_n)_{n=N_0}^{\infty}$
  - but (n) *needs to be consecutive*!

Recursive Definition

- Recursive Definition of Sequence: a sequence is defined recursively by giving:
  1. the value (or) values of the initial term (or terms)
  2. a rule (called the *recursion formula*) for calculating any later term from terms that precede it
- Example:

  Example

  The famous Fibonacci sequence (f_n) is defined recursively by the following initial conditions

  $$f_1 = 0 \quad \text{and} \quad f_2 = 1,$$

  and the recursion formula

  $$f_n = f_{n-1} + f_{n-2} \qquad (n \geq 3)$$

  In other words, each term is the sum of the previous two terms.

  So, the first few terms are

  $$0, 1, 1, 2, 3, 5, 8, 13, \cdots$$

Limit of A Sequence

- Definition of Limit of A Sequence: the sequence (a_n) converges to L if and only if:

  $$\forall \epsilon > 0, \exists N \in \mathbb{R} \; s.t. \; n > N \implies |a_n - L| < \epsilon$$

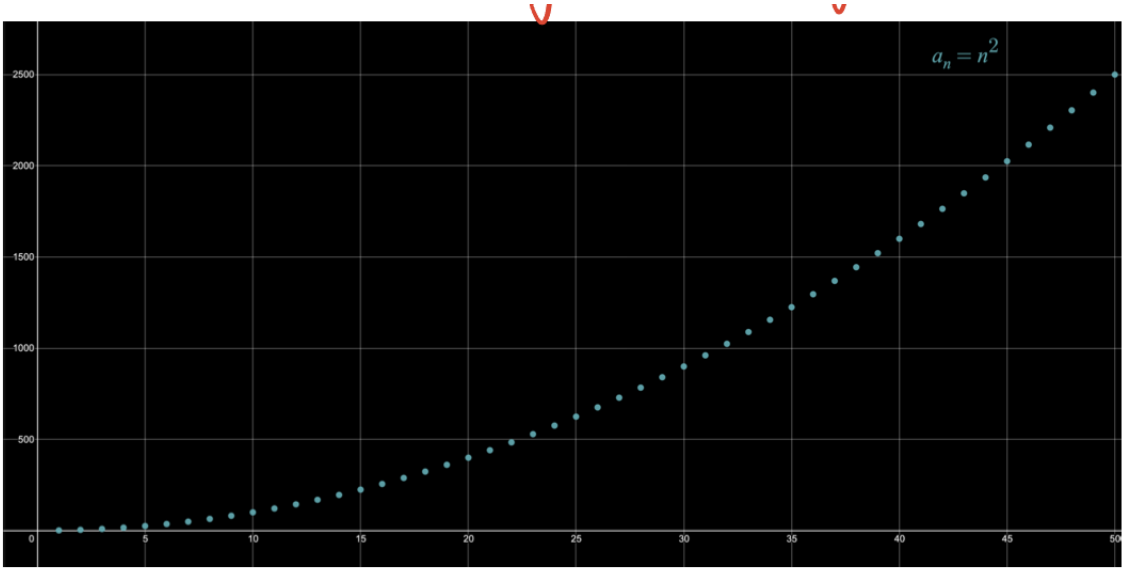
$\forall \varepsilon > 0,$	$\exists N \in \mathbb{R}$	s.t.	$n > N$	$\Rightarrow$	$\lvert a_n - L \rvert < \varepsilon.$
no matter how small ε is	there is a point	s.t	after that point		all terms a_n are within ε of L

  - *Intuition*: $\lim_{n\to\infty} a_n = L$ means if n is large enough, we can make a_n as close as we like to L
  - *Notations*: $\lim_{n\to\infty} a_n = L, a_n \to L, a_n \to L \; as \; n \to \infty$

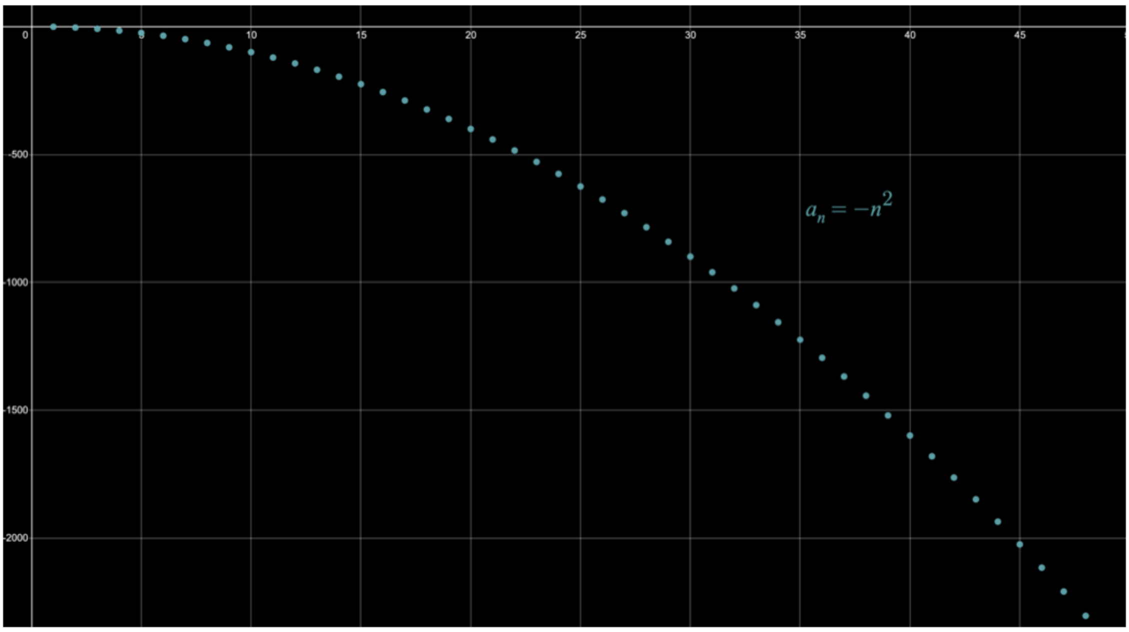
- Definition of A Convergent Sequence: a sequence is convergent if it has a limit. That is a sequence (a_n) is convergent if:

$$\exists L \in \mathbb{R}\ s.t.\ \forall \epsilon > 0, \exists N \in \mathbb{R}\ s.t.\ n > N \implies |a_n - L| < \epsilon$$

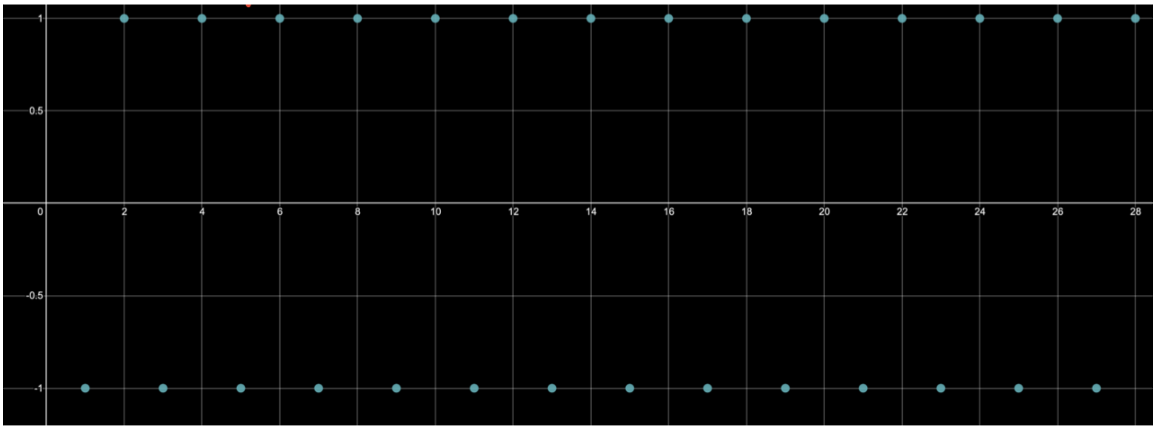
  - Note that convergence implies $\exists M \in \mathbb{R}\ s.t.\ |a_n| < M\ \forall\ n$
- Definition of A Divergent Sequence: a sequence is divergent if it does not have a limit.
  - $\forall L \in \mathbb{R}, \exists \epsilon > 0\ s.t.\ \forall n_0 \in \mathbb{N}, \exists n > n_0\ s.t.\ |a_n - L| \geq \epsilon$
    - I *cannot trap my sequence no matter how far I go*
  - 3 main types of divergence:
    1. Diverge to infinity $a_n \to \infty$



      - because ∞ is not a real number hence not a limit, so we say "diverge to infinity" instead on "converge to infinity"
      - $\forall \epsilon > 0, \exists n_0 \in \mathbb{R}\ s.t.\ n > n_0 \implies a_n > \epsilon$
    2. Diverge to negative infinity $a_n \to \infty$



3. (a_n) is oscillating



- my trial definition: (don't know..)

Proving the Limit of a Sequence

Structure

- Proof for $\lim_{n\to\infty} a_n = L$:
  - We want to show: $\forall \epsilon > 0, \exists N \in \mathbb{R}\ s.t.\ n > N \implies |a_n - L| < \epsilon$
  1. let $\epsilon > 0$ be arbitrary
  2. Take $N =????$ (usually find N by manipulating $|a_n - L| < \epsilon$)
  3. Show that your N works
     - Assume $n > N$
     - Conclude that $|a_n - L| < \epsilon$

Examples

Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Want to show

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ \text{s.t.}\quad n > N \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon.$$

-

- How to find N:

Rough Work (Finding N) 

Let $\varepsilon > 0$ be arbitrary. We want to show that there exists N s.t.

$$n > N \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon.$$

Notice that

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n}$$

so we just need to find $N \in \mathbb{R}$ s.t. for all $n > N$

$$\frac{1}{n} < \varepsilon \quad \Leftrightarrow \quad n > \frac{1}{\varepsilon}.$$

So we should take $N = \frac{1}{\varepsilon}$.

- Formal proof:

Proof.

Let $\varepsilon > 0$.

Take $N = \frac{1}{\varepsilon}$.

Assume that $n > N$. Then

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have proved that

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \quad n > N \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon.$$

Example

Prove that $\lim\limits_{n \to \infty} \frac{2n^2 - 1}{n^2 + 1} = 2.$

Want to show

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \quad n > N \Rightarrow \left|\frac{2n^2 - 1}{n^2 + 1} - 2\right| < \varepsilon.$$

-

- How to find N:

Let $\varepsilon > 0$ be arbitrary. We want to show that there exists N s.t.

$$n > N \Rightarrow \left| \frac{2n^2-1}{n^2+1} - 2 \right| < \varepsilon.$$

Notice that

$$\left| \frac{2n^2-1}{n^2+1} - 2 \right| = \left| \frac{2n^2-1-2n^2-2}{n^2+1} \right| = \left| \frac{-3}{1+n^2} \right| = \frac{3}{1+n^2}$$

so we just need to find $N \in \mathbb{R}$ s.t. for all $n > N$

$$\frac{3}{1+n^2} < \varepsilon \iff n^2 > \frac{3}{\varepsilon} - 1 \overset{?}{\iff} n > \sqrt{\frac{3}{\varepsilon} - 1} \quad \left(\text{only if } \frac{3}{\varepsilon} - 1 \geq 0 \right)$$

Take $N = \begin{cases} \sqrt{\frac{3}{\varepsilon} - 1} & \text{if } \frac{3}{\varepsilon} - 1 \geq 0 \\ 1 & \text{if } \frac{3}{\varepsilon} - 1 < 0 \end{cases}$

- My trial formal proof:
  - Let $\epsilon > 0$
  - Take $N = \begin{cases} \sqrt{\frac{3}{\epsilon} - 1} & if\ \frac{3}{\epsilon} - 1 \geq 0 \\ 1 & if\ \frac{3}{\epsilon} - 1 < 0 \end{cases}$
  - Assume $n > N$
  - If $\frac{3}{\epsilon} - 1 \geq 0$:
    - $\left| \frac{2n^2-1}{n^2+1} - 2 \right| = \left| \frac{3}{1+n^2} \right| < \left| \frac{3}{1+\frac{3}{\epsilon}-1} \right| = \epsilon$
  - If $\frac{3}{\epsilon} - 1 < 0$:
    - $\left| \frac{2n^2-1}{n^2+1} - 2 \right| = -1.5 < 0 < \epsilon$
  - Since $\epsilon > 0$ was arbitrary, we have prove that $\exists N \in \mathbb{R}\ s.t.\ \forall \epsilon > 0, n > N \implies \left| \frac{2n^2-1}{n^2+1} - 2 \right| < \epsilon$

Combining Convergent Sequences

Limit Laws for Sequences

- If (a_n) and (b_n) are convergent sequences such that

$$\lim_{n\to\infty} a_n = L, \lim_{n\to\infty} b_n = M$$

- then:
  - Limit of the sum is the sum of limit

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$$

  - $$\forall c \in \mathbb{R}, \lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cL$$

- Limit of the product is the product of limit

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$$

- Limit of the quotient is the quotient of the limit (but the denominator cannot be 0):

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M} \text{ if } \forall n, b_n \neq 0 \text{ and } M \neq 0$$

Proving the Law for Sums

If $a_n \to L$ and $b_n \to M$, then $(a_n + b_n) \to L + M$.

Want to show

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \quad n > N \Rightarrow |(a_n + b_n) - (L+M)| < \varepsilon.$$

Proof Strucutre

1. Let $\varepsilon > 0$ be arbitrary
2. Take N = ???
3. Show that your N works
   - Assume that $n > N$
   - Conclude that $|(a_n + b_n) - (L+M)| < \varepsilon$

- How to think about this proof (find N)?

Rough Work (Finding N)

Let $\varepsilon > 0$ be arbitrary.
We need to find $N \in \mathbb{R}$ s.t. $n > N \Rightarrow |(a_n + b_n) - (L+M)| < \varepsilon$.

$$|(a_n+b_n)-(L+M)| = |(a_n - L) + (b_n - M)| \overset{?}{\le} \varepsilon$$
$$\le \underbrace{|a_n - L|}_{<\frac{\varepsilon}{2}} + \underbrace{|b_n - M|}_{<\frac{\varepsilon}{2}}$$

We know:

- $a_n \to L$:
  for the value $\varepsilon' = \frac{\varepsilon}{2}$, $\exists N_a \in \mathbb{R}$ s.t. $n > N_a \Rightarrow |a_n - L| < \frac{\varepsilon}{2}$
- $b_n \to M$:
  for the value $\varepsilon' = \frac{\varepsilon}{2}$, $\exists N_b \in \mathbb{R}$ s.t. $n > N_b \Rightarrow |b_n - M| < \frac{\varepsilon}{2}$

The N's could be different!

- Take $N = \max\{N_a, N_b\}$.

- Formal proof:

Let $\varepsilon > 0$ be arbitrary.

Since $a_n \to L$ and $b_n \to M$, applying the definition of limit with $\varepsilon' = \frac{\varepsilon}{2}$, we have that

- $\exists N_a \in \mathbb{R}$ s.t. $n > N_a \Rightarrow |a_n - L| < \frac{\varepsilon}{2}$. (1)
- $\exists N_b \in \mathbb{R}$ s.t. $n > N_b \Rightarrow |b_n - M| < \frac{\varepsilon}{2}$. (2)

Take $N = \max\{N_a, N_b\}$.

Assume that $n > N$. Then $n > N_a$ and $n > N_b$. Therefore,

- $|a_n - L| < \frac{\varepsilon}{2}$ by (1)
- $|b_n - M| < \frac{\varepsilon}{2}$ by (2)

Then
$$\begin{aligned}|(a_n+b_n)-(L+M)| &= |(a_n - L) + (b_n - M)| \\ &\le |a_n - L| + |b_n - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |(a_n + b_n) - (L+M)| < \varepsilon$.

Useful Tricks for Proofs

- Triangle Inequality:

$$|x+y| \le |x| + |y|$$

- Using it, we can bound the quantity that we want to show is "small" by the sum of two quantities each of which is know to be small
- Reverse Triangle Inequality:

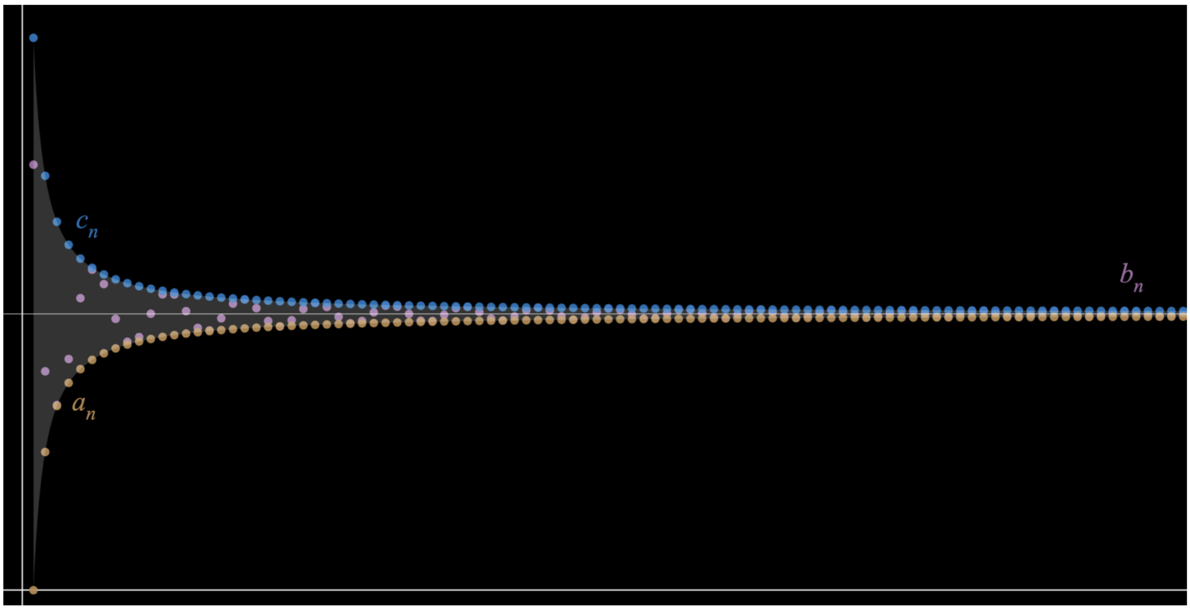
$$\big||x| - |y|\big| \leq |x - y|$$

- *The $\frac{\epsilon}{2}$ Trick*:
  - Applying the definition of limit with $\epsilon' = \frac{\epsilon}{2}$ in place of ϵ so as to end up with an overall bound for ϵ
- *The $N = \max\{N_a, N_b\}$ Trick*
  - Choose N to be the larger of all the N-values arising in the proof to ensure that any $n > N$ satisfies both $n > N_a$ and $n > N_b$

Proving Convergence vs Using Convergence

Proving $x_n \to L$	Using $x_n \to L$
We are given an arbitrary $\epsilon > 0$ (we cannot choose it)	We can choose any value of ϵ we like
We must find a value N that works for this ϵ	It is guaranteed that there is a N that works for my choice of ϵ
I CANNOT CHOOSE ϵ and have to say HOW TO PRODUCE N	*I CAN CHOOSE any value of ϵ I want and the N will be GIVEN to me*

Sandwich Theorem / Squeeze Theorem

- Sandwich/Squeeze Theorem: if $a_n \leq b_n \leq c_n$ for $n \geq N_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$
- 

- Proving the sandwich theorem:
  - Idea

- Want to show

  $$\forall \varepsilon > 0,\ \exists N \in \mathbb{R} \text{ s.t. } \quad n > N \implies |b_n - L| < \varepsilon.$$

  Proof Strucutre

  1. Let $\varepsilon > 0$ be arbitrary
  2. Take $N = ???$
  3. Show that your N works
     - Assume that $n > N$
     - Conclude that $|b_n - L| < \varepsilon$

- How to find N?

  Rough Work (Finding N)

  Let $\varepsilon > 0$. We need to find $N \in \mathbb{R}$ s.t. $\quad n > N \implies |b_n - L| < \varepsilon$.

  $$|b_n - L| < \varepsilon \iff L - \varepsilon < b_n < L + \varepsilon$$

  We know

  - $\forall n \geq N_0, \quad a_n \leq b_n \leq c_n$
  - $a_n \to L$: $\quad \exists N_1 \in \mathbb{R}$ s.t. $\quad n > N_1 \implies L - \varepsilon < a_n < L + \varepsilon$
  - $c_n \to L$: $\quad \exists N_2 \in \mathbb{R}$ s.t. $\quad n > N_2 \implies L - \varepsilon < c_n < L + \varepsilon$

  Take $N = \max\{N_0, N_1, N_2\}$.

- Formal proof

Proof.

Let $\varepsilon > 0$.

Since $a_n \to L$: $\exists N_1 \in \mathbb{R}$ s.t. $n > N_1 \Rightarrow |a_n - L| < \varepsilon \Rightarrow L - \varepsilon < a_n$ (1)

Since $c_n \to L$: $\exists N_2 \in \mathbb{R}$ s.t. $n > N_2 \Rightarrow |c_n - L| < \varepsilon \Rightarrow c_n < L + \varepsilon$ (2)

We also know that $a_n \leq b_n \leq c_n$ for $n \geq N_0$. (3)

Take $N = \max\{N_0, N_1, N_2\}$.

Assume that $n > N$. Therefore,

- $n > N_1$ so by (1), $L - \varepsilon < a_n$
- $n > N_0$ so by (3), $a_n \leq b_n \leq c_n$
- $n > N_2$ so by (2), $c_n < L + \varepsilon$

Hence, $L - \varepsilon < b_n < L + \varepsilon \iff |b_n - L| < \varepsilon$.

- Example use:

Example

Determine whether $\left(\frac{\cos n}{n^2}\right)_{n=1}^{\infty}$ is convergent or divergent.

Solution

We know that $-1 \leq \cos n \leq 1$ for all $n \in \mathbb{N}$.

Since $n^2 > 0$, we have that

$$\frac{-1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{-1}{n^2} = 0$, by the Sandwich Theorem we have that

$$\lim_{n\to\infty} \frac{\cos n}{n^2} = 0.$$

3.2 Bounded and Monotone Sequences

- 3.2 Bounded and Monotone Sequences_NOTES - A

Bounded Sequences

- *Definition* of a Bounded Above Sequence: the sequence (a_n) is bounded above if and only if $\exists B \in \mathbb{R}\ s.t.\ \forall n \in \mathbb{N}, a_n \leq B$

- *Definition* of a Bounded Below Sequence: the sequence (a_n) is bounded below if and only if $\exists B \in \mathbb{R}\ s.t.\ \forall n \in \mathbb{N}, a_n \geq B$
- *Definition* of a Bounded Sequence: the sequence (a_n) is bounded if and only if it is both bounded above and bounded below
    - i.e. $\exists A, B \in \mathbb{R}\ s.t.\ \ \forall n \in \mathbb{N}, A \leq a_n \leq B$
    - Equivalent: $\exists M > 0\ s.t.\ \forall n \in \mathbb{N}, |a_n| \leq M$

Monotone Sequences

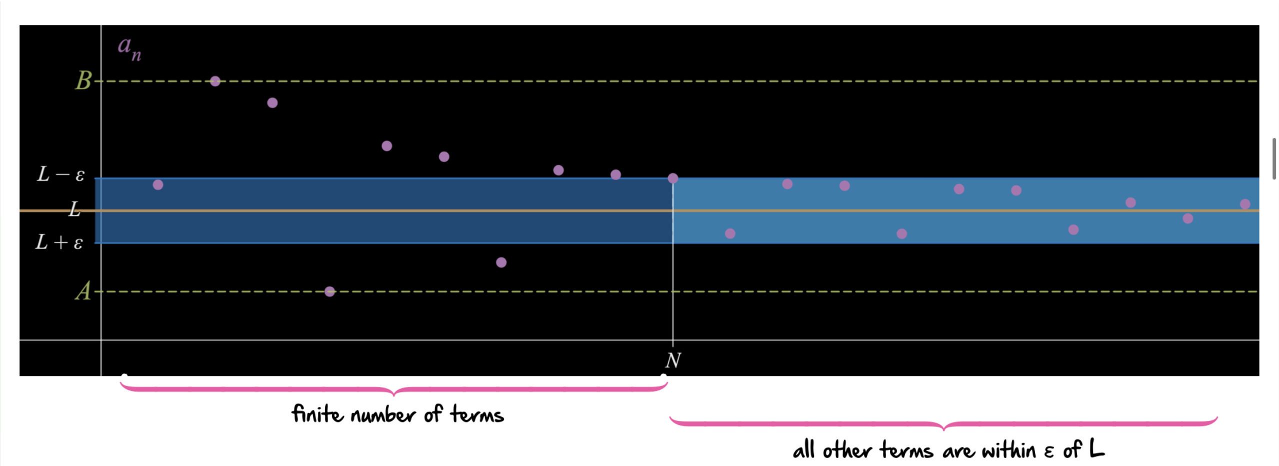
- *Definition* of an Increasing Sequence: a sequence (a_n) is increasing if $\forall n \in \mathbb{N}, a_{n+1} \geq a_n$
    - equivalently: if $\forall m, n \in \mathbb{N}, m < n \implies a_m \leq a_n$
- *Definition* of a Decreasing Sequence: a sequence (a_n) is decreasing if $\forall n \in \mathbb{N}, a_{n+1} \leq a_n$
    - equivalently: if $\forall m, n \in \mathbb{N}, m < n \implies a_m \geq a_n$
- *Definition* of a Monotone Sequence: a sequence (a_n) is monotone if it is increasing or decreasing
    - (in math, "or" includes "both")
- *Definition* of Strictly Increasing: a sequence (a_n) is strictly increasing if $\forall n \in \mathbb{N}, a_{n+1} > a_n$
- *Definition* of Strictly Decreasing: a sequence (a_n) is strictly decreasing if $\forall n \in \mathbb{N}, a_{n+1} < a_n$

Convergence $\implies$ Bounded

- Theorem: Convergent $\implies$ Bounded: if the sequence (a_n) is convergent, then (a_n) is bounded

Proof

- Idea: use the definition of convergent to split the sequence into a finite segment and an infinite but bounded segment



- Formal proof:

  Assume that the sequence (a_n) converge to L.

  Let $\varepsilon = 1$ in the definition of $a_n \to L$. Then

  $$\exists N_0 \in \mathbb{N} \quad \text{s.t.} \quad \forall n \in \mathbb{N} \quad n > N_0 \Rightarrow L - 1 < a_n < L + 1 \qquad (1)$$

  Take

  $A = \min\{L - 1, a_1, a_2, \ldots, a_{N_0}\}$

  $B = \max\{L + 1, a_1, a_2, \ldots, a_{N_0}\}$

  Let $n \in \mathbb{N}$,

  - if $n > N_0$, then $A \le L - 1 < a_n < L + 1 \le B$ by (1)
  - if $n \le N_0$, then $A \le a_n \le B$ by the definition of A and B.

Some Examples

> Exercise: For each combination find an example or convince yourself that it is impossible.

Monotone + Bounded => ~~Convergent~~

		convergent	divergent
monotone	bounded	$a_n = 1 \ \forall n \in \mathbb{N}$	DNE
	not bounded	DNE	$a_n = n^2 \ \forall n \in \mathbb{N}$
not monotone	bounded	$a_n = (-1)^n \frac{1}{n} \ \forall n \in \mathbb{N}$	$a_n = (-1)^n \forall n \in \mathbb{N}$
	not bounded	DNE	$a_n = (-n)^n \ \forall n \in \mathbb{N}$

Convergent => Bounded

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Monotone Convergence Theorem (MCT)

- Theorem: Increasing + Bounded Above $\implies$ Convergent: if the sequence (a_n) is increasing and bounded above, then (a_n) is convergent
- Theorem: Decreasing + Bounded Below $\implies$ Convergent: if the sequence (a_n) is decreasing and bounded below, then (a_n) is convergent

Proof

- Ideas

WANT TO SHOW:

(a_n) is convergent, i.e. we want to show that $a_n \to L$ for some $L \in \mathbb{R}$.

$$\exists L \in \mathbb{R} \quad \text{s.t.} \quad \forall \varepsilon > 0, \quad \exists N_0 \in \mathbb{N} \quad \text{s.t.} \quad n > N_0 \Rightarrow L - \varepsilon < a_n < L + \varepsilon$$

Proof Structure

1. Take $L = ???$ (we have to guess this one)
2. Fix an arbitrary $\varepsilon > 0$
3. Take $N_0 = ???$
4. Assume that $n > N_0$
5. Conclude that $L - \varepsilon < a_n < L + \varepsilon$

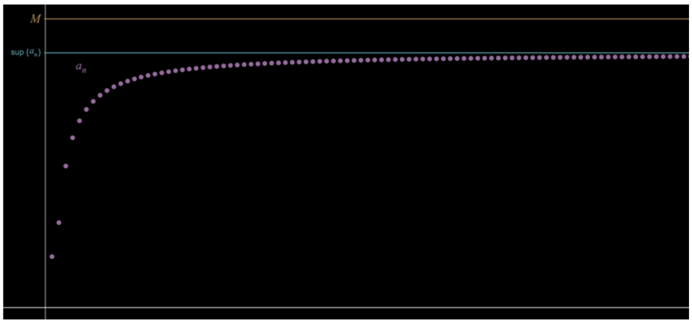
- Pack the sequence into a set -> use the completeness axiom (supreme exists) -> use the definition of supremum (below the supremum cannot be an upper bound)

Main Ideas:

Let $S = \{a_n : n \in \mathbb{N}\}$. Then
- S is not empty
- S is bounded above

Therefore, the supremum exists!

Completeness Axiom



TAKE: $L = \sup S$

WILL SHOW: $a_n \to L$.

def of supremum

- For $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound of S:

$$\exists N_0 \in \mathbb{N} \quad \text{s.t.} \quad L - \varepsilon < a_{N_0}$$

- (a_n) is increasing $\Rightarrow a_{N_0} \leq a_n \ \forall n > N_0$
- L is an upper bound of $S \Rightarrow a_n \leq L \ \forall n$

- Formal proof

Proof.

Let (a_n) be an increasing and bounded above sequence.

Let $S = \{a_n : n \in \mathbb{N}\}$. This set is not empty and bounded above because the sequence is bounded above.

Therefore, by the completeness axiom, the supremum exists.

Let $L = \sup S$. We will show that $a_n \to L$.

Let $\varepsilon > 0$. By the definition of supremum, there exists $N_0 \in \mathbb{N}$ such that

$$L - \varepsilon < a_{N_0}. \qquad (1)$$

Assume that $n > N_0$. Then

- Since the sequence is increasing, we have that $a_n \geq a_{N_0}$ (2)
- Since L is an upper bound of S, we also know that $a_n \leq L$ (3)

Combining (1), (2), and (3) we have

$$L - \varepsilon < a_{N_0} \leq a_n \leq L < L + \varepsilon \quad \Rightarrow \quad |a_n - L| < \varepsilon.$$

Use the Monotone Convergence Theorem to Find Limit

- MCT is especially handy to look for the limit of a recursively defined sequence
- Example:
  - Find the limit of the sequence given by

    $$a_1 = 1 \quad \text{and} \quad a_{n+1} = 3 - \frac{1}{a_n}$$

  - Steps:
    1. Show the sequence is increasing (by induction)

       (1) a_n is increasing. We need to show that $a_n \leq a_{n+1} \quad \forall n \in \mathbb{N}$.

       We will show this by induction.

       Base case: $a_1 = 1$ and $a_2 = 3 - \frac{1}{1} = 2$. Therefore, $a_1 \leq a_2$.

       Inductive Step: Assume that $a_n \leq a_{n+1}$. Then, $\frac{1}{a_{n+1}} \leq \frac{1}{a_n}$. Therefore,

       $$a_{n+2} = 3 - \frac{1}{a_{n+1}} \leq 3 - \frac{1}{a_n} = a_{n+1}.$$

       Hence, mathematical induction $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

2. Show the sequence is bounded above (by induction)

a_n is bounded above. We will show that $a_n < 3 \quad \forall n \in \mathbb{N}$. (Still induction)

Base case: $a_1 = 1 < 3$.

Inductive Step: Assume that $a_n < 3$. Then

$$\frac{1}{a_n} > \frac{1}{3} \Rightarrow -\frac{1}{a_n} < -\frac{1}{3} \Rightarrow 3 - \frac{1}{a_n} < 3 - \frac{1}{3}.$$

Therefore, $a_{n+1} < 3$.

This shows, by mathematical induction, that $a_n < 3$ for all $n \in \mathbb{N}$.

3. Find the limit by exploiting the recurrence relation

Convince yourself that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$. Then

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(3 - \frac{1}{a_n}\right) = 3 - \frac{1}{\lim_{n\to\infty} a_n}$$

This give us the following equation

$$L = 3 - \frac{1}{L} \Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{5}}{2}$$

4. Pick the correct limit (using the first term + increasing property)

Which one should we pick?

$$L_1 = \frac{3+\sqrt{5}}{2} > 1 \quad \text{and} \quad L_2 = \frac{3-\sqrt{5}}{2} < 1$$

💡 The sequence is increasing and $a_1 = 1 \Rightarrow a_n \geq 1 \quad \forall n$

So the terms of the sequence cannot get arbitrarily close to a number strictly less than 1.

Therefore, $L = \dfrac{3+\sqrt{5}}{2}$ 😃

3.3 Subsequences

Subsequences

- *Definition* of A Subsequence: a subsequence of (a_n) is a sequence (b_n) that defined by $b_n = a_{j_n}$, where $j_1 < j_2 < \ldots$ is a strictly increasing sequence of natural numbers
  - we want to maintain the original order
- Theorem: Subsequences of convergent sequences converge to the same limit

- Proof: (use the "n" of the original sequence)

  Proof.
  Let $\varepsilon > 0$. Since $a_n \to L$,

  $$\exists N_0 \in \mathbb{R} \quad \text{s.t.} \quad n > N_0 \Rightarrow |a_n - L| < \varepsilon.$$

  Claim: If (j_n) is a strictly increasing sequence of natural numbers, then $j_n \geq n$ for all $n \in \mathbb{N}$.
  Let $n > N_0$. Then, by the previous claim, we have that $j_n \geq n > N_0$. Therefore,

  $$|a_{j_n} - L| < \varepsilon.$$

  - Remark: term number of subsequence: let (b_n) be a subsequence of (a_n) and $b_n = a_{j_n}$, then $j_n \geq n \,\forall\, n \in \mathbb{N}$
  - Proof of the claim used:

    Proof of Claim. We will prove this statement by induction.

    Base Case: $j_1 \in \mathbb{N} \Rightarrow j_1 \geq 1$ ✓

    Inductive Step: Suppose that $j_n \geq n$. Then since (j_n) is strictly increasing

    $$j_{n+1} > j_n \geq n \quad \Rightarrow \quad j_{n+1} \geq n+1.$$

- *Consequences*: shifting a convergent sequence by a finite natural number does not alter its limit

  $$a_n \to L \implies a_{n+1} \to L$$

  and more generally:

  $$\forall k \in \mathbb{N}, a_n \to L \implies a_{n+k} \to L$$

- *Contrapositive (equivalent)* of the Theorem above: *IF (a_n) is a sequence that EITHER has a subsequence that diverges OR two convergent subsequences with different limits, THEN (a_n) is divergent*

The Bolzano-Weierstrass Theorem

- Bolzano-Weierstrass Theorem: every bounded sequence has a convergent subsequence
- Proof
  - Step 1: prove *every sequence has a monotone subsequence*
    - Introduce the concept of *summit*

      We say that a_n is a summit if it's at least as high as every later point. That is, $a_n \geq a_m$ for all $m \geq n$.

      Let's make a list of all of the summits. Let's say they're $a_{n_1}, a_{n_2}, \dots$.

- 2 possible scenarios
  - Infinitely many summits

    Case 1: there are infinitely many summits.

    In this case we have an infinite sequence $a_{n_1}, a_{n_2}, a_{n_3}, \dots$.

    - a_{n_1} is a summit and $n_2 > n_1$, so $a_{n_1} \geq a_{n_2}$
    - a_{n_2} is a summit and $n_3 > n_2$, so $a_{n_2} \geq a_{n_3}$
    - and so on

    We end up with $a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \dots$

    This is a decreasing subsequence! So we're done 😃
  - Finitely many (or 0) summits

    Case 2: there are finitely many summits (including the possibility of no summit at all).

    Let a_M be the last summit.

    Put $a_{m_1} = a_{M+1}$. (If there's no summit at all, put $a_{m_1} = a_1$.)

    - a_{m_1} isn't a summit, so there must be some later higher point, say a_{m_2} with $a_{m_2} \geq a_{m_1}$
    - a_{m_2} isn't a summit, so there must be some later higher point, say a_{m_3} with $a_{m_3} \geq a_{m_2}$
    - and so on

    We get an infinite increasing subsequence $a_{m_1}, a_{m_2}, a_{m_3}, \dots$

    So we're done in this case too! 😃
- Step 2: since the sequence is bounded, any subsequence is bounded
- Step 3: MCT: bounded + monotone $\implies$ convergent
- 2&3:

  Let (a_n) be a bounded sequence. Then there exists a monotone subsequence (a_{m_n})

  Since (a_n) is bounded, there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

  In particular, $|a_{m_n}| \leq M$ for all $n \in \mathbb{N}$.

  Therefore, (a_{m_n}) is bounded and monotone.

  Hence, by the monotone convergence theorem, (a_{m_n}) converge.

Chapter 4: Series

4.1 Series (Introduction)

- 4.1_Series_NOTES - A

Series

- *Definition* of Series: a series is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \ldots$$

  where (a_n) is the *sequence of terms of the series*
- *Definition* of Partial Sum of A Series: given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$, its k^{th} *partial sum* is the sum:

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k$$

- The sequence (S_k) is referred to as the Sequence of Partial Sums:
- If the sequence (S_k) is convergent with limit S, then the series $\sum a_n$ is convergent, and we write:

$$\sum_{n=1}^{\infty} a_n = S = \lim_{k \to \infty} S_k = \lim_{k \to \infty} \sum_{n=1}^{k} a_n$$

  - i.e. *a convergent series is just the limit of its sequence of partial sums*

Telescoping Series

- Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

  What's the k^{th} partial sum?

  Note: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

  $$\begin{aligned} S_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{k}} - \frac{1}{k+1}\right) \\ &= 1 - \frac{1}{k+1} \end{aligned}$$

  $$\lim_{k \to \infty} S_k = \lim_{k \to \infty} \left(1 - \frac{1}{k+1}\right) = 1$$

  Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

- Another example 0 M48 Live Notes > $ sum_{ n=1 } { infty } ln frac{n}{n+1}$: $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

2 Examples of Divergent Series

- Diverge to infinity

  Consider the series

  $$\sum_{n=1}^{\infty} n = 1 + 2 + \cdots + n + \cdots$$

  Let's take a look at the k^{th} partial sum:

  $$S_k = \sum_{n=1}^{k} n = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

  ①

  Notice that $\frac{k(k+1)}{2} \to \infty$ in the limit as $k \to \infty$.

  Therefore the series $\sum_{n=1}^{\infty} n$ diverges.

- Oscillating

Consider the series

$$\sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+1-1+1-1+\cdots$$

Note that

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1-1=0 \\ S_3 &= 1-1+1=1 \\ S_4 &= 1-1+1-1=0 \\ &\vdots \\ S_k &= \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

Thus, the sequence of partial sums (S_k) is divergent, and so the given series is divergent.

Geometric Series

- A Geometric Series is a series of the form:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \ldots$$

where x is a real number (often called the *ratio of the geometric series*)

  - Note that the summation starts at $n = 0$ NOT 1!

- This is equal to the limit of the partial sum if convergent:

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} 1 + x + x^2 + \cdots + x^{k-1}$$

  - If $x = 1$, then $S_k = 1 + \cdots + 1 = k \to \infty$ (the geometric series diverges to infinity)
  - If $x \neq 1$, then: $S_k = \frac{1-x^k}{1-x}$

Taking the k-th partial sum (the sum of the first k terms, starting at n=0)

$$S_k = 1 + x + x^2 + \cdots + x^{k-1},$$

and multiplying both sides by k, we get

$$xS_k = x + x^2 + x^3 + \cdots + x^k$$

Subtracting these two equations from each other, we get

$$(1-x)S_k = S_k - xS_k = 1 - x^k$$

Dividing both sides by $\underbrace{(1-x)}_{\neq 0}$, we obtain:

$$S_k = \frac{1-x^k}{1-x}.$$

- $\sum_{n=0}^{\infty} x^n$ converges $\iff$ $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-x^k}{1-x}$ exists:

$$\sum_{n=0}^{\infty} x^n \text{ converges} \iff \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-x^k}{1-x} \text{ exists.}$$

- If $-1 < x < 1$, then $x^k \to 0$ as $k \to \infty$, so

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-x^k}{1-x} = \frac{1}{1-x}.$$

Thus, in this case, the geometric series is convergent.

- If $x > 1$, (x^k) diverges to ∞. Therefore, (S_k) diverges.

- If $x < -1$, (x^k) contains the subsequence (x^{2k}) which diverges to ∞. Therefore, (S_k) diverges.

- If $x = -1$, the subsequences (S_{2k}) and (S_{2k-1}) converge to different limits. Therefore, (S_k) diverges.

- *Summary*:

$$\sum_{n=0}^{\infty} x^n \begin{cases} = \dfrac{1}{1-x} & \text{if } |x| < 1 \\ \text{diverges} & \text{if } |x| \geq 1 \end{cases}$$

4.2 Series of Non-negative Numbers

- 4.2_non-negative_Series_NOTES - A

Series of Non-Negative Numbers

- In general

In general,

$$\text{A series may be} \begin{cases} \text{convergent} \\ \text{divergent} \begin{cases} \text{diverges to } \infty \\ \text{diverges to } -\infty \\ \text{oscilates} \end{cases} \end{cases}$$

- For non-negative series ($\forall n \in \mathbb{N}, a_n \geq 0$) or eventually non-negative series ($a_n \geq 0$ after some $N_0 \in \mathbb{N}$), there are only 2 possibilities:

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{converges} \\ \text{diverges to } \infty \end{cases}$$

  - Reason: the sequence of partial sums will be increasing, if it's also bounded above. Then, by MCT, it's convergent

Assume that $a_n \geq 0$ for all $n \in \mathbb{N}$.

From the definition, we know that

$$\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k \quad \text{where} \quad S_k = a_1 + \cdots + a_k.$$

Notice that

$$S_{k+1} = S_k + a_{k+1} \geq S_k.$$

Therefore, the sequence (S_k) is increasing.

There are two options:

- (S_k) is bounded above

  By Monotone Convergence Theorem, (increasing + bounded above) $\Rightarrow$ convergent

- (S_k) is unbounded above

  We also know that (increasing + unbounded above) $\Rightarrow$ divergent to ∞

Harmonic Series

- Harmonic Series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots$$

- It *diverges to infinity*

- Idea: divide the series into groups whose sums $\geq \frac{1}{2}$

Consider the harmonic series (Divide them into groups whose sums $\geq \frac{1}{2}$)

$$1+\underbrace{\frac{1}{2}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{3}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{4}+\frac{1}{5}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{\geq\frac{1}{2}}+\cdots$$

$$S_2 = 1+\frac{1}{2} \qquad \geq \frac{1}{2}+\frac{1}{2} \qquad \geq 2\cdot\frac{1}{2}$$

$$S_4 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \qquad \geq \frac{1}{2}+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{=\frac{1}{2}} \qquad \geq 3\cdot\frac{1}{2}$$

$$S_8 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \geq \frac{1}{2}+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{=\frac{1}{2}}+\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{=\frac{1}{2}} \geq 4\cdot\frac{1}{2}$$

$$S_{2^k} = 1+\underbrace{\frac{1}{2}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{3}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{4}+\frac{1}{5}}_{\geq\frac{1}{2}}+\underbrace{\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geq\frac{1}{2}}+\cdots+\underbrace{\left(\frac{1}{2^{k-1}+1}+\cdots+\frac{1}{2^k}\right)}_{\geq 2^{k-1}\cdot\frac{1}{2^k}=\frac{1}{2}} \qquad \geq \frac{n+1}{2}$$

> Therefore,
>
> $$S_{2^n} \geq \frac{n+1}{2} \quad \forall n \in \mathbb{N}.$$
>
> By the Archimedean property, we have that (S_{2^k}) is not bounded above.
>
> Therefore, (S_k) is not bounded above. and it's increasing
>
> Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

- Consider this series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

- This one *converges!*
  - Idea: It's obviously increasing, then we can show it's bounded above by dividing them into groups whose sum $< \frac{1}{2^k}$:

We will show that the sequence of partial sums (S_k) is bounded above.

$$S_2 = 1 + \frac{1}{2^2} \qquad\qquad < 1+1$$

$$S_4 = 1 + \frac{1}{2^2} + \left(\frac{1}{3^2} + \frac{1}{4^2}\right) < 1 + 1 + \underbrace{\frac{1}{2^2} + \frac{1}{2^2}}_{=\frac{1}{2}} < 1 + 1 + \frac{1}{2}$$

$$S_8 = 1 + \frac{1}{2^2} + \left(\frac{1}{3^2} + \frac{1}{4^2}\right) + \left(\frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2}\right) < 1 + 1 + \underbrace{\frac{1}{2^2} + \frac{1}{2^2}}_{=\frac{1}{2}} + \underbrace{\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}}_{=4\cdot\frac{1}{4^2}=\frac{1}{4}} < 1 + 1 + \frac{1}{2} + \frac{1}{4}$$

$$S_{2^k} = \underbrace{1}_{<1} + \underbrace{\frac{1}{2^2}}_{<1} + \underbrace{\left(\frac{1}{3^2} + \frac{1}{4^2}\right)}_{<\frac{1}{2}} + \underbrace{\left(\frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2}\right)}_{<\frac{1}{2^2}} + \cdots + \underbrace{\left(\frac{1}{(2^{k-1}+1)^2} + \cdots + \frac{1}{(2^k)^2}\right)}_{<2^{k-1}\cdot\frac{1}{(2^{k-1})^2} = \frac{1}{2^{k-1}}}$$

Therefore,

$$\begin{aligned} S_{2^k} &< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2^{k-1}} \\ &< 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= 1 + \frac{1}{1-\frac{1}{2}} \\ &= 3 \end{aligned}$$

By Bernoulli's inequality, we know that $k < 2^k$ for all $k \in \mathbb{N}$.
Since (S_k) is a strictly increasing sequence, we have that

$$k < 2^k \Rightarrow S_k < S_{2^k} < 3 \text{ for all } k \in \mathbb{N} \Rightarrow S_k \text{ is bounded above}$$

Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (MCT)

- *Theorem* - Bernoulli's Inequality: $\forall\, x \geq -1,\ \forall\, n \in \mathbb{N}, (1+x)^n \geq 1 + nx$
  - This implies $k < 2^k\ \forall\ k \in \mathbb{N}$

p-Series

- $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges if and only if $a > 1$
- $\sum_{n=1}^{\infty} \frac{1}{(n+k)^a}$ converges if and only if $a > 1$

4.3 Properties of Convergent Series

- 4.3_Convergent_series_NOTES - A

The Vanishing Condition

- Necessary Condition for Convergence of Series: if the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$
  - Proof:

    Proof.

    Assume that the series $\sum_{n=1}^{\infty} a_n$ is convergent.

    Then, the sequence of partial sums (S_n) converges to a real number S.

    Notice that for every $n \in \mathbb{N}$

    $$S_n - S_{n-1} = (a_1 + \cdots + a_{n-1} + a_n) - (a_1 + \cdots + a_{n-1}) = a_n$$

    Therefore,

    $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

- *Warning! The Converse is NOT true!* If $\lim_{n\to\infty} a_n = 0$, we CANNOT conclude anything about the series.
  - Example: the harmonic series diverges while the $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

    Note that

    $$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n^2} = 0$$

    and

    - The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
    - The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

- The Necessary Condition Test is usually used to determine divergent series: (the contrapositive of the original statement): *If $\lim_{n\to\infty} a_n \neq 0$ (either converges to a non-0 number, or does not exist), then the series $\sum_{n=1}^{\infty} a_n$ is divergent.*
  - 2 examples

- Consider the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$.

$$a_n = \frac{n}{n+1} \to 1 \neq 0.$$

Therefore, by the Necessary Condition Test the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

- Consider the series $\sum_{n=1}^{\infty} (-1)^n$.
Then, the limit of the sequence $a_n = (-1)^n$ does not exist. In particular, it is not 0.

Therefore, by the Necessary Condition Test the series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

4.4 Comparison Tests (Only Apply to Non-Negative Series)

- 4.4_Comparison_Tests_NOTES - A

Basic Comparison Test

- Basic Comparison Test: Suppose that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then:
  - If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent
  - If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent
- Proof:

(1) Assume that

(a) $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$
(b) $\sum_{n=1}^{\infty} b_n$ is convergent

Let $S_k = \sum_{n=1}^{k} a_n$ and $T_k = \sum_{n=1}^{k} b_n$. Then by (a), $0 \leq S_k \leq T_k$ for all $k \in \mathbb{N}$.

Since $a_n \geq 0$, the sequence (S_k) is increasing.

We will now show that (S_k) is bounded.

Since $\sum b_n$ is convergent, we have that the sequence of its partial sums (T_k) is convergent.
Therefore, (T_k) is bounded.
This means that there exists $T > 0$ such that $T_k \leq T$ for all $k \in \mathbb{N}$.

Hence, $S_k \leq T_k \leq T$ for all $k \in \mathbb{N}$.

Thus, (S_k) is a bounded monotone sequence.

So, by the Monotone Convergence Theorem, (S_k), and hence $\sum_{n=1}^{\infty} a_n$ converges.

  - (2) This is the contrapositive of (1).

- Basic Comparison Test - Improved Version: Suppose that $0 \leq a_n \leq b_n$ for all $n \geq N$ for some positive number N. Then:
  - If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent
  - If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent
- Examples:
  - Example 1

Example 1. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2n^2+n}$ converges or diverges.

Observe that

$$\frac{1}{2n^2+n} < \frac{1}{2n^2} \text{ for all } n \geq 1$$

We know that the series $\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{2n^2+n}$ is convergent by the Comparison Test.

  - Example 2

Example 2.

If $p < 1$, then the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

We will only consider the case $p \in \mathbb{Q}$.

Claim. If $p < 1$ ($p \in \mathbb{Q}$), then $n^p \leq n$ for all $n \in \mathbb{N}$.

Since $n^p \leq n$, we have that $0 \leq \frac{1}{n} \leq \frac{1}{n^p}$.

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Therefore, by the Basic Comparison Test the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

- Claim. If $p < 1$ $(p \in \mathbb{Q})$, then $n^p \leq n$ for all $n \in \mathbb{N}$.

  Proof of Claim.

  Let $n \in \mathbb{N}$.

  Let $p \in \mathbb{Q}$ such that $p < 1$. Then, there exists $m \in \mathbb{Z}$, $k \in \mathbb{N}$ s.t $p = \frac{m}{k}$.

  Suppose for the sake of a contradiction that $n^p > n$. Then

  $$\begin{aligned} n^{m/k} > n &\Rightarrow n^m > n^k \\ &\Rightarrow 1 > n^{k-m}. \end{aligned}$$

  Since $p = \frac{m}{k} < 1$, we have that $k - m > 0$.

  This implies that $n < 1$, which is a contradiction.

  Therefore, $n^p \leq n$ for all $n \in \mathbb{N}$.

Limit Comparison Test

- Limit Comparison Test: Suppose that $a_n, b_n > 0$ for all $n \in \mathbb{N}$ and

  $$\lim_{n\to\infty} \frac{a_n}{b_n} = L \text{ for some } 0 < L < \infty$$

  Then, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges
  - *Extension (PP5)*:
    - If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
    - If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges
- Proof:

Suppose that $\sum_{n=1}^{\infty} b_n$ converges.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, using $\varepsilon = 1$ in the definition of limit, we have that $\exists N_0 \in \mathbb{N}$ s.t.

$$\left|\frac{a_n}{b_n} - L\right| < 1 \quad \text{for all } n > N_n.$$

Therefore, for $n > N_0$

$$\frac{a_n}{b_n} = \left|\frac{a_n}{b_n}\right| = \left|\frac{a_n}{b_n} - L + L\right| \le \left|\frac{a_n}{b_n} - L\right| + |L| < 1 + |L|$$

Hence, $a_n < (1+|L|)\, b_n$ for all $n > N_0$.

Therefore, by the Basic Comparison Test, the series $\sum_{n=1}^{\infty} a_n$ converges.

- The other direction uses $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{L}$

- *Building Blocks to Comparison Test (can be used directly)*
  - $\sum_{n=1}^{\infty} \frac{1}{a^n}$ converges if and only if $|a| > 1$ (geometric series)
  - $\sum_{n=1}^{\infty} a^n$ converges if and only if $|a| < 1$ (geometric series)
  - $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges if and only if $a > 1$
  - $\sum_{n=1}^{\infty} n^a$ converges if and only if $a < -1$

- *Enough Information?*
  - We know $\forall n \in \mathbb{N}, a_n > 0$ and the series $\sum_{n=1}^{\infty} a_n$ is convergent
  - Determine:
    - $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ is divergent because $a_n \to 0 \implies \frac{1}{1+a_n} \to 1$, so the necessary condition test fails
    - $\sum_{n=1}^{\infty} \sqrt{a_n}$ not enough information
      - Divergent example: $a_n = \frac{1}{n^2}, \sqrt{a_n} = \frac{1}{n}$
      - Convergent example: $a_n = \frac{1}{n^4}, \sqrt{a_n} = \frac{1}{n^2}$
    - $\sum_{n=1}^{\infty} (a_n)^2$ is convergent (comparison test: $\exists N > 0 \; s.t. \; n > N \implies a_n < 1 \implies a_n > (a_n)^2 > 0$)
  - Caveat: always think about $\sum_{n=1}^{\infty} \frac{1}{n^a}$

- *Quick Trick for Determining Convergence*
  - Determining whether

$$\sum_{n=1}^{\infty} \frac{a_0 n^{P_0} + a_1 n^{P_1} + \cdots + a_k n^{P_k}}{b_0 n^{Q_0} + b_1 n^{Q_1} + \cdots + b_m n^{Q_m}}$$

where $0 \le P_0 < P_1 < \cdots < P_k$ and $0 \le Q_0 < Q_1 < \cdots < Q_m$

- Compare it with $b_n = \frac{a_k n^{P_k}}{b_m n^{Q_m}}$ (highest power terms) in the limit comparison test
- Converges if $P_k - Q_m > 1$ (cannot use directly)

4.5 Alternating Series Test

- 4.5_Alternating_Series-NOTES - A
- *Definition* of Alternating Series: An alternating series is a series whose terms are alternatively positive and negative. In general, the n^{th} term of an alternating series is of the form:

$$a_n = (-1)^{n-1} b_n \text{ or } a_n = (-1)^n b_n$$

where $b_n > 0 \quad (b_n = |a_n|)$

- Alternating Series Test: If the alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ with $b_n > 0$ for all n, satisfies:
  - $\lim_{n \to \infty} b_n = 0$
  - $b_{n+1} \leq b_n$ for all $n \geq N$ for some positive number N
  - Then, the alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges
- Example:
  - The *Alternating Harmonic Series* converges!

The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is an alternating series with $b_n = \frac{1}{n}$ and satisfies

- $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$
- $b_{n+1} \leq b_n$ for all $n \geq 1$ because $\frac{1}{n+1} < \frac{1}{n}$

Therefore, by the Alternating Series Test, the alternating harmonic series is convergent. But the normal harmonic series is divergent

4.6 Absolute and Conditional Convergence

- 4.6 Absolute and Conditional Convergence NOTES - A

Absolute Convergence and Conditional Convergence

- *Definition* of Absolute Convergence: A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series of the absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent (this is stronger than convergence)
- We can then apply the basic comparison test / limit comparison test
- *Definition* of Conditional Convergence: A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it is convergent but not absolutely convergent
    - e.g. the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$
- Absolute Convergent Test: If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent
    - First check this test -- if it passes, we immediately know that the series is convergent
    - The converse is not true! Absolute convergent is stronger than convergent!
    - Proof:

Assume that $\sum |a_n|$ converges.

Let

$$a_n^+ = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases} \quad \text{and} \quad a_n^- = \begin{cases} -a_n & \text{if } a_n < 0 \\ 0 & \text{if } a_n \geq 0 \end{cases}$$

- $a_n = a_n^+ - a_n^-$
- $0 \leq a_n^+ \leq |a_n|$ and $0 \leq a_n^- \leq |a_n|$
- By the Basic Comparison Test $\sum a_n^+$ and $\sum a_n^-$ are both convergent

Therefore, by the linearity of convergent series $\sum a_n$ converges and

$$\sum a_n = \sum (a_n^+ - a_n^-) = \sum a_n^+ - \sum a_n^-$$

    - where we used the Linearity of Convergent Series: we can split and combine convergent series
    - *Theorem*: $\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\iff$ $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are both convergent
- *Three types of series*:

	$\sum \|a_n\|$ convergent	$\sum \|a_n\|$ divergent
$\sum a_n$ convergent	1 Absolutely convergent	2 Conditionally convergent
$\sum a_n$ divergent	IMPOSSIBLE (by the contrapositive of the ACT)	3 Divergent

- In other words:

$$\text{A series}: \begin{cases} \text{Divergent}: & \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} |a_n| \text{ both divergent} \\ \text{Convergent} & \begin{cases} \text{Absolutely Convergent}: & \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} |a_n| \text{ both convergent} \\ \text{Conditionally Convergent}: & \sum_{n=1}^{\infty} a_n \text{ convergent}, \sum_{n=1}^{\infty} |a_n| \text{ diverg} \end{cases} \end{cases}$$

Conditionally Convergent Series are Naughty

- *Definition* of Reordering: A reordering of a given series is another series containing precisely the same terms as the original series, but in a different order
- *Theorem* - We Lost Commutativity of A Conditionally Convergent Series: If a series is conditionally convergent, then it can be reordered to obtain any result we want. It can be:
  - convergent to any real number we want
  - divergent to ∞
  - divergent to $-\infty$
- How to reorder to get any real number S?
  1. Add positive terms until a partial sum goes above S
  2. Add negative terms until a partial sum goes below S
  3. Go back to step 1
  - Since $\lim_{n \to \infty} a_n = 0$, each time that the partial sum goes above/below S, they get closer to S. Therefore, the limit of the sequence of partial sums is S
- We can do this because *Theorem* - Some Further Properties of Conditionally Convergent Series: If the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then:
  - The series of positive terms diverges to ∞: $\sum_{n=1}^{\infty} a_n^+ = \infty$
  - The series of negative terms diverges to $-\infty$: $\sum_{n=1}^{\infty} (-a_n^-) = -\sum_{n=1}^{\infty} a_n^- = -\infty$
  - $\lim_{n \to \infty} |a_n| = 0$

Absolutely Convergent Series are Nice

- *Theorem* - Reordering Absolutely Convergent Series: If a series is absolutely convergent, then reordering its terms doesn't change the value of the sum
  - In other words: if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\sum_{n=1}^{\infty} b_n$ is any reordering of $\sum_{n=1}^{\infty} a_n$, then $\sum_{n=1}^{\infty} b_n$ is absolutely convergent and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$
  - Proof:

Proof.

(Case $a_n \geq 0$ for all $n \in \mathbb{N}$)

Assume that $a_n \geq 0$ for all $n \in \mathbb{N}$.

Let $\sum a_n$ be an absolutely convergent series and let $\sum b_n$ be a reordering of $\sum a_n$.

Let $k \in \mathbb{N}$.

Choose M such that all the terms $b_1, b_2, \cdots, b_k$ are included among the $a_1, a_2, \cdots, a_M$. Then

$$\sum_{n=1}^{k} b_n \leq \sum_{n=1}^{M} a_n \leq \sum_{n=1}^{\infty} a_n.$$

Therefore the partial sums of $\sum_{n=1}^{\infty} b_n$ are bounded above. So $\sum_{n=1}^{\infty} b_n$ converges. In addition,

$$\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n.$$

Reversing the role of a_n and b_n in the previous argument, we have

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

Hence,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n.$$

(General case)

Recall: $\sum a_n$ is absolutely convergent $\Longleftrightarrow$ $\sum a_n^+$ and $\sum a_n^-$ are both convergent.

Let $\sum a_n$ be an absolutely convergent series. Then

$$\sum a_n^+ \quad \text{and} \quad \sum a_n^- \text{ both converge.}$$

If $\sum b_n$ is a reordering of $\sum a_n$, then

$$\sum b_n^+ \quad \text{and} \quad \sum b_n^- \quad \text{are reorderings of} \quad \sum a_n^+ \quad \text{and} \quad \sum a_n^- \text{ respectively.}$$

It follows from what we've just proved that they converge and

$$\sum_{n=1}^{\infty} a_n^+ = \sum_{n=1}^{\infty} b_n^+ \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^- = \sum_{n=1}^{\infty} b_n^-$$

Therefore, $\sum b_n$ is absolutely convergent and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} b_n^+ - \sum_{n=1}^{\infty} b_n^- = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- = \sum_{n=1}^{\infty} a_n.$$

4.7 Ratio Test

- 4.7 Ratio Test NOTES - A

Ratio Test

- *Theorem* - Ratio Test: Suppose that $a_n \neq 0 \; \forall \, n \in \mathbb{N}$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ (including the possibility of $L = \infty$)

    - If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent
    - If $L > 1$ (including the possibility of $L = \infty$), then the series $\sum_{n=1}^{\infty} a_n$ is divergent
    - If $L = 1$, then the Ratio Test is inconclusive
- *Factorials in both numerator and denominator $\rightsquigarrow$ ratios test is likely to be helpful!*
- Example:

Example Determine whether the following series is conditionally convergent, absolutely convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}.$$

Solution: We will use the Ratio Test with $a_n = \dfrac{(-1)^n n}{3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{(-1)^{n+1}(n+1)}{3^{n+1}}\right|}{\left|\frac{(-1)^n n}{3^n}\right|} = \frac{\frac{(n+1)}{3^{n+1}}}{\frac{n}{3^n}} = \frac{3^n(n+1)}{3^{n+1} n} = \frac{1}{3}\frac{n+1}{n} \to \frac{1}{3} \quad \text{as } n \to \infty$$

Since, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1$, by the Ratio Test, the given series converges absolutely

Ratio Test - Proof

- We prove the ratio test case by case (suppose $a_n \neq 0 \; \forall \; n \in \mathbb{N}$)
    - Prove $\left|\frac{a_{n+1}}{a_n}\right| \to L < 1 \implies \sum_{n=1}^{\infty} a_n$ converges by comparing with a self-constructed geometric series

1. Assume that $\left|\frac{a_{n+1}}{a_n}\right| \to L < 1$

💡 Compare the given series with a convergent geometric series!

Since $L < 1$, we can choose a number r such that $L < r < 1$.

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, taking $\varepsilon = r - L$, we have that there exists $N_0 \in \mathbb{N}$ s.t.

$$\forall n \geq N_0 \qquad \left|\frac{a_{n+1}}{a_n}\right| < L+\varepsilon = L+r-L = r$$

Therefore, $|a_{n+1}| < |a_n| r \quad \forall n \geq N_0$.

Using this we have

$$\begin{aligned} |a_{N_0+1}| &< |a_{N_0}| r \\ |a_{N_0+2}| &< |a_{N_0+1}| r < |a_{N_0}| r^2 \\ |a_{N_0+3}| &< |a_{N_0+2}| r < |a_{N_0}| r^3 \end{aligned}$$

and in general

- $$|a_{N_0+k}| < |a_{N_0}| r^k \quad \text{for all } k \geq 1$$

Now the series $\sum_{k=1}^{\infty} |a_{N_0}| r^k$ is convergent because it is a geometric series with $0 < r < 1$.

Since $|a_{N_0+k}| < |a_{N_0}| r^k$ for all $k \geq 1$, by the Comparison Test,

$$\sum_{n=N_0+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N_0+k}|$$

is also convergent.

It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent (a finite number of terms doesn't affect convergence.)

- Therefore $\sum a_n$ is absolutely convergent.

- Prove $\left|\frac{a_{n+1}}{a_n}\right| \to L > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges by necessary condition test

2. Assume that $\left|\frac{a_{n+1}}{a_n}\right| \to L > 1$ or $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$

In both cases, there exists $N_0 \in \mathbb{N}$ s.t.

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \qquad \forall n \geq N_0.$$

Therefore, $|a_{n+1}| > |a_n|$ for all $n \geq N_0$.

Since $|a_{N_0}| > 0$, we have that

$$\lim_{n\to\infty} a_n \neq 0.$$

Therefore, $\sum a_n$ diverges. (necessary condition test fails)

- Prove $\left|\frac{a_{n+1}}{a_n}\right| \to L = 1 \implies$ inconclusive by taking examples

3.

- Can you think of a convergent series such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$?

The series $\sum \frac{1}{n^2}$ converges and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} \to 1 \quad \text{as } n \to \infty$$

- Can you think of a divergent series such that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$?

The series $\sum \frac{1}{n}$ diverges and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \to 1 \quad \text{as } n \to \infty$$

Series - Summary

Toolbox for Series

- Definition of convergent series (sequence of partial sums converges)
- Convergence conditions for Geometric Series
- Whether Series of Non-Negative Numbers is bounded above
- Harmonic series is divergent
- Necessary Condition Test
- Basic Comparison Test
- Limit Comparison Test
- Alternating Series Test

- Absolute Convergent Test
- Ratio Test

A Collection of Series

- $\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$ (Harmonic Series diverges to infinity)
- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (Alternating Harmonic Series converges)
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges
- $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges if and only if $a > 1$ (p-Series)
- $\sum_{n=1}^{\infty} \frac{1}{(n+k)^a}$ converges if and only if $a > 1$ (p-Series)
- $\sum_{n=1}^{\infty} n^a$ converges if and only if $a < -1$
- $\sum_{n=1}^{\infty} a^n$ converges to $\frac{a}{1-a}$ if and only if $|a| < 1$ (Geometric Series)
- $\sum_{n=1}^{\infty} \frac{1}{a^n}$ converges if and only if $|a| > 1$ (Geometric Series)
- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1$ (Telescoping Series)

Chapter 5: Power Series

- 5. Power Series - A

5.1 Power Series

Power Series

- *Definition* of the Power Series: A power series is a series of the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

  where x is a variable and the a_n's are constants called the coefficients of the series
- Note that
  - Here we adopt the convention that $x^0 = 1$ even when $x = 0$
  - *The power series is guaranteed to be convergent (to a_0) when $x = 0$*
- Example:

Example

For what values of x is the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ convergent?

Solution: We use the Ratio Test with $a_n = \frac{x^n}{n!}$.

↑ Ratio test is very helpful when studying power series

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \frac{n!}{(n+1)!}\frac{|x|^{n+1}}{|x|^n} = \frac{n!}{(n+1)\,n!}|x| = \frac{1}{n+1}|x|$$

Therefore

x is just a constant here

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1}|x| = 0 < 1 \qquad \text{for all } x \in \mathbb{R}.$$

Hence, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$.

Remark: We define the *exponential function* as

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Power Series: Example

Example

Find all the values of $x \in \mathbb{R}$ so that the series $\sum_{n=0}^{\infty} \frac{x^n}{n2^n}$ converges.

Solution: We use the Ratio Test with $a_n = \frac{x^n}{n2^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{x^{n+1}}{(n+1)2^{n+1}}\right|}{\left|\frac{x^n}{n2^n}\right|} = \frac{n\,2^n}{(n+1)\,2^{n+1}}\frac{|x|^{n+1}}{|x|^n} = \frac{1}{2}\frac{n}{n+1}|x|$$

Therefore,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\frac{n}{n+1}|x| = \frac{|x|}{2}$$

- If $|x| < 2$, then $L = \dfrac{|x|}{2} < 1$.

  So, by the Ratio Test, the series $\sum_{n=0}^{\infty} \dfrac{x^n}{n2^n}$ converges absolutely.

- If $|x| > 2$, then $L = \dfrac{|x|}{2} > 1$.

  So, by the Ratio Test, the series $\sum_{n=0}^{\infty} \dfrac{x^n}{n2^n}$ diverges.

- If $|x| = 2$, then $L = \dfrac{|x|}{2} = 1$.

  The Ratio Test is inconclusive! 😣 $\Rightarrow$ we need to do something else.

- If $x = 2$, we have

$$\sum_{n=0}^{\infty} \frac{x^n}{n2^n} = \sum_{n=0}^{\infty} \frac{2^n}{n2^n} = \sum_{n=0}^{\infty} \frac{1}{n}$$

  so the series diverges (this is the harmonic series).

- If $x = -2$, we have

$$\sum_{n=0}^{\infty} \frac{x^n}{n2^n} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

  so the series converges (this is the alternating harmonic series).

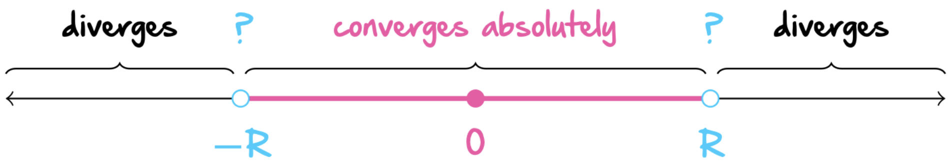
Therefore, the series $\sum_{n=0}^{\infty} \dfrac{x^n}{n2^n}$ converges for $-2 \leq x < 2$.

Interval of Convergence: $[-2, 2)$.

Radius of Convergence = 2.

Convergence of Power Series (Theorem)

- *Theorem*: Convergence of Power Series: The set of values x for which the power series $\sum_{n=0}^{\infty} a_n x^n$ converges is an interval centred at 0

- Possibilities: $(-R, R), (-R, R], [-R, R), [-R, R], \mathbb{R}, \{0\}$
- Even more, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $x \in (-R, R)$
  - but the theorem does not tell us what happens when $x = R$ or $x = -R$
- The interval on which the series converges is called the *interval of convergence*
- R is called the *radius of convergence* $(0 \leq R \leq \infty)$
- 

- $\implies$ if the power series converges for some $x_0 \in \mathbb{R}$, then it converges absolutely for every $x \in \mathbb{R}$ with $|x| < |x_0|$

Workflow: Finding the Interval of Convergence of a Power Series

- 1. Apply the Ratio Test to find the interval of x for which $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
- 2. Check the values of x for which the Ratio Test is inconclusive $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$
  - (Using other tests, usually one end uses Alternating Series Test)

Chapter 6: Limits and Continuity

6.1 Limits

- 6.1 Limits_NOTES - A

Intuitive Idea of Limit

- $\lim_{x\to c} f(x)$ is not necessarily $f(c)$
  - and *A function doesn't have to be defined at a point for the limit to exist*
  - e.g. $\lim_{x\to 1} \frac{x^2-3x+2}{x-1} = \lim_{x\to 1}(x-2) = -1$

One-Sided Limits

- Consider the function:

$$f(x) = \begin{cases} x^2 & if\ x \leq 0 \\ x-1 & if\ x > 0 \end{cases}$$

- $\lim_{x\to 0} f(x)$ does not exist, because we approach different points from the left and from the right
- But we can say that:
  - If x is very close to 0 with $x > 0$, then $f(x)$ is very close to 1:

$$\lim_{x\to 0^+} f(x) = 1$$

- If x is very close to 0 with $x < 0$, then $f(x)$ is very close to 0:

$$\lim_{x \to 0^-} f(x) = 0$$

The Limit Does Not Exist Examples

- Intuition: $\lim_{x\to c} f(x)$ exists if $\exists L \in \mathbb{R}\ s.t.\ \lim_{x\to c} f(x) = L$
- Examples:
  - $f(x) = \sin\left(\frac{\pi}{x}\right)$
    - $\lim_{x\to 0} f(x)$ DNE: For x close to 0, $f(x)$ does not stay close to a single value
  - $f(x) = \frac{1}{|x-2|}$
    - $\lim_{x\to 2} f(x)$ DNE: For x close to 2, $f(x)$ becomes arbitrarily large
      - We say that $\lim_{x\to 2} f(x) = \infty$

Definition of Limits of Functions

- *Definition* of Limit of a Function: Let f be a function defined at least on an open interval $(c-p, c+p)$ except possibly at c itself. We say that

$$\lim_{x\to c} f(x) = L$$

if

$$\forall \epsilon > 0, \exists \delta > 0\ s.t.\ 0 < |x-c| < \delta \implies |f(x) - L| < \epsilon$$

- Intuition: *no matter how close to L I want $f(x)$ to be, I can find x sufficiently close to c such that $f(x)$ is as close to L as I wanted*

> ### *Trick for $\epsilon - \delta$ Proof*
> - If you have something of the form $|x-a| \cdot f(x)$, then assume $|x-a| < 1$ and find an upper bound for $|f(x)|$.
> - Sometimes, you need to have $|x| < B$ for $B \leq 1$ to avoid problems
> - Then, let $\delta = \min\{B, \text{something depends on } \epsilon\}$

Limit of Functions and Sequences

- *Theorem* - Characterisation of Limits Via Sequences: Suppose $a < c < b$, and f is a function defined for all numbers in the interval (a, b), with the possible exception at c. Then:

$$\lim_{x\to c} f(x) = L$$

if and only if ($\iff$) for all sequences (x_n) with $x_n \in (a,b), x_n \neq c\ \forall n \in \mathbb{N}$,

$$x_n \to c \implies f(x_n) \to L$$

- Proof:

Assume f is defined on the interval (a, b), except possibly at $c \in (a, b)$, and let $L \in \mathbb{R}$.

The two statements that we want to prove equivalent are:

- Statement A: $\lim_{x \to c} f(x) = L$
- Statement B: "For all sequences (x_n) with $x_n \in (a, b)$ and $x_n \neq c$ for every $n \in \mathbb{N}$ and $x_n \to c$, we have $f(x_n) \to L$."

We first prove $A \Rightarrow B$.

Assume $\lim_{x \to c} f(x) = L$, and that (x_n) is a sequence of numbers in (a, b) satisfying $x_n \neq c$ for all n and $x_n \to c$.

We want to show $f(x_n) \to L$, which would mean that for any given $\varepsilon > 0$, there exists a number $N \in \mathbb{R}$ such that

$$n > N \Rightarrow |f(x_n) - L| < \varepsilon.$$

Since $\lim_{x \to c} f(x) = L$, we know that for any such $\varepsilon > 0$, there exists a number $\delta > 0$ with

$$\underbrace{|x - c| < \delta \text{ and } x \neq c}_{0 < |x-c| < \delta} \Rightarrow |f(x) - L| < \varepsilon.$$

Further, since $\delta > 0$, the convergence $x_n \to c$ guarantees the existence of a number $N \in \mathbb{R}$ such that

$$n > N \Rightarrow |x_n - c| < \delta.$$

Since $x_n \in (a, b)$ and $x_n \neq c$ for all n, combining we have

$$n > N \Rightarrow |x_n - c| < \delta \Rightarrow |f(x_n) - L| < \varepsilon,$$

which proves $f(x_n) \to L$.

- We now prove $B \Rightarrow A$.

  In order to prove $B \Rightarrow A$, we will prove the contrapositive (not $A \Rightarrow$ not B):

  Statement A means:

  For every $\varepsilon > 0$, there exists $\delta > 0$ such that
  for all $x \neq c$, $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

  Negation of A:

  There exists a number $\varepsilon > 0$ such that for every $\delta > 0$, there exists $x \neq c$ with $|x - c| < \delta$ but $|f(x) - L| \geq \varepsilon$.

  Statement B means:

  For all sequences (x_n) with $x_n \in (a, b)$ and $x_n \neq c$ for every $n \in \mathbb{N}$ and $x_n \to c$, we have $f(x_n) \to L$.

  Negation of B:

  There exists a sequence (x_n) with $x_n \in (a, b)$ and $x_n \neq c$ for every $n \in \mathbb{N}$ and $x_n \to c$, but $f(x_n) \not\to L$.

- Now assume the negation of A is true.

  Choose $n \in \mathbb{N}$ large enough so that $(c - \frac{1}{N}, c + \frac{1}{N}) \subset (a, b)$.
  Then there exists $\varepsilon > 0$ such that for all integers $n \geq N$, one can find a number $x_n \in (a, b)$ with $x_n \neq c$ and $|x_n - c| < 1/n$ but $|f(x_n) - L| \geq \varepsilon$.

  For $n = 1, \cdots, N - 1$, define x_n to be arbitrary numbers in (a, b) not equal to c.

  This gives a sequence (x_n) with $x_n \in (a, b)$ for all n and, for all $n \geq N$,

  $$c - \frac{1}{n} < x_n < c + \frac{1}{n},$$

  so the sandwich theorem implies $x_n \to c$.

  But we also have $|f(x_n) - L| \geq \varepsilon$ for all $n \geq N$,
  which implies that $(f(x_n))$ cannot converge to L.

  This sequence thus establishes the negation of statement B.

Properties of Limits of Function

- *Theorem* - Limit Laws for Functions: Suppose $\lim_{x \to c} f(x) = A$ and $\lim_{x \to c} g(x) = B$. Then:

- Limit of the sum is the sum of the limit

$$\lim_{x \to c}[f(x) + g(x)] = A + B$$

- Limit of the product is product of the limit:

$$\lim_{x \to c}[f(x)g(x)] = AB$$

- Limit of the quotient is quotient of the limit: If $B \neq 0$ and $g(x) \neq 0$ for all x sufficiently close to c, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}$$

- Proof of the first one (*using the characterisation of limits via sequences*)
  - Use characterisation of limits via sequence for arbitrary sequence $(x_n) \to c$; Apply limit laws for sequences; Use characterisation of limits via sequence again to go back to functions

(1) Let (x_n) be an arbitrary sequence with $x_n \neq c$ for all n and $x_n \to c$.

Using the characterisation of limits via sequences we have that

$$f(x_n) \to A \quad \text{and} \quad g(x_n) \to B.$$

Applying the sum rule for limits of sequences, this implies

$$f(x_n) + g(x_n) \to A + B.$$

Since this is true for every sequence $x_n \to c$ with $x_n \neq c$ for all n, by the characterisation of limits via sequences this implies

$$\lim_{x \to c}[f(x) + g(x)] = A + B.$$

- *Theorem* - Sandwich Theorem for Functions: Suppose f, g, h are 3 functions defined on an interval (a, b), except possibly at the point $c \in (a, b)$, and satisfying:

$$f(x) \leq g(x) \leq h(x) \; \forall \; x \in (a, c) \cup (c, b)$$

Suppose also that:

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

Then:

$$\lim_{x \to c} g(x) = L$$

- Proof (again, characterisation by sequences):

Proof.

Let (x_n) be an arbitrary sequence with $x_n \in (a,b)$ and $x_n \neq c$ for all n, and $x_n \to c$.
Using the characterisation of limit of function via sequences we have that

$$f(x_n) \to L \quad \text{and} \quad h(x_n) \to L,$$

Moreover, since $x_n \in (a,c) \cup (c,b)$, we have that $f(x_n) \leq g(xn) \leq h(x_n)$ for all n.

Therefore, by the sandwich theorem for limits of sequences we have that $g(x_n) \to L$.

Since (x_n) was an arbitrary sequence with $x_n \in (a,c) \cup (c,b)$ for all n, and $x_n \to c$, the characterisation of limits of functions via sequences implies that $\lim_{x \to c} g(x) = L$.

6.2 Continuity

- 6.2 Continuity_NOTES - A

Definition of Continuity

- *Definition* of Continuity at a Point: Let $c \in \mathbb{R}$. Let f be a function defined at least on an open interval centred at c. We say f is continuous at c if:

$$\lim_{x \to c} f(x) = f(c)$$

  In other words (ϵ, δ Definition):

$$\forall \epsilon > 0, \exists \delta > 0 \; s.t. \; |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$$

- This definition demands 3 things:
  - $\lim_{x \to c} f(x)$ exists
  - $f(c)$ is defined
  - $\lim_{x \to c} f(x) = f(c)$
- ϵ, δ *Definition* of Continuity at a Point: Let $c \in \mathbb{R}$. Let f be a function defined at least on an open interval centred at c. We say f is continuous at c if:

$$\forall \epsilon > 0, \exists \delta > 0 \; s.t. \; |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$$

- Intuitively, f is continuous at c means that "if x is close to c, then $f(x)$ is close to $f(c)$"
- *Definition* of Left and Right Continuity:
  - A function is called left-continuous at c if

$$\lim_{x \to c^-} f(x) = f(c)$$

  - A function is called right-continuous at c if

$$\lim_{x \to c^+} f(x) = f(c)$$

- *Definition* of Continuity on an Open Interval: f is continuous on the open interval (a, b) if f is continuous at $c \; \forall c \in (a, b)$
- *Definition* of Continuity on a Closed Interval: f is continuous on the closed interval $[a, b]$ if:
  - f is continuous on (a, b)
  - f is right-continuous at a
  - f is left-continuous at b

Combining Continuous Functions

- *Theorem* - Combining Continuous Functions: If f, g are continuous at c, then:
  - $f \pm g$ is continuous at c
  - αf is continuous at $c \; \forall \alpha \in \mathbb{R}$
  - fg is continuous at c
  - $\frac{f}{g}$ is continuous at c provided $g(c) \neq 0$
- Proof (just limit laws):
  - Proof.

    Each follows immediately from the Limit Laws. Let us prove (c) as an example.

    Since f, g are both continuous at c, we have

    $$\lim_{x \to c} f(x) = f(c) \text{ and } \lim_{x \to c} g(x) = g(c).$$

    Since both limits exist, we can apply the Limit Law "limit of product is product of limits".

    $$\lim_{x \to c} [f.g](x) = \lim_{x \to c} [f(x)g(x)] = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)] = f(c)g(c) = [f.g](c).$$
- Use this theorem, we can derive:
- *Theorem* - Polynomials are continuous on $\mathbb{R}$: Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and let $c \in \mathbb{R}$:

$$\lim_{x \to c} P(x) = P(c)$$

- Proof:($\lim_{x\to c} x = c, \lim_{x\to c} a_0 = a_0$ then induction to show that)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and let $c \in \mathbb{R}$.

- It's easy to prove that $\lim\limits_{x\to c} x = c$ and $\lim\limits_{x\to c} a_0 = a_0$
- Use part (c) and induction to show that for every $k \in \mathbb{N}$,

$$\lim_{x\to c} x^k = \left(\lim_{x\to c} x\right)^k = c^k$$

Therefore, using the previous theorem, we have

$$\lim_{x\to c} P(x) = \lim_{x\to c}(a_n x^n + \cdots + a_1 x + a_0) = a_n c^n + \cdots + a_1 c + a_0 = P(c).$$

- *Theorem* - Rational Functions are Continuous: Let $f(x) = \frac{P(x)}{Q(x)}$ where $P(x), Q(x)$ are polynomials and $Q \not\equiv 0$. Let $c \in \mathbb{R}$:

$$\lim_{x\to c} f(x) = f(c)$$

- Proof:

If f is defined at c (i.e. if $Q(c) \neq 0$) then

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} = f(c).$$

6.3 Limits and Composition

- 6.3 Limits and Composition_NOTES - A

Continuity and Composition

- *Theorem* - Composition:
  - Let f, g be functions such that:
  - $\lim_{x\to c} g(x) = L$
  - *f is continuous at L*
  - Then:

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$$

- Proof:

Proof.

Since $\lim_{x\to c} g(x) = L$, it is sufficient to prove that $\lim_{x\to c} f(g(x)) = f(L)$.

Let $\varepsilon > 0$, we need to find $\delta > 0$ such that

$$\text{if } 0 < |x - c| < \delta \text{ then } |f(g(x)) - f(L)| < \varepsilon.$$

- Since f is continuous at L, there exists $\delta_1 > 0$ such that for every $y \in \mathbb{R}$,

$$\text{if } |y - L| < \delta_1 \text{ then } |f(y) - f(L)| < \varepsilon. \qquad (1)$$

- We also know that $\lim_{x\to c} g(x) = L$.

So for the $\delta_1 > 0$ obtained above, there exists $\delta > 0$ such that for every $x \in \mathbb{R}$,

$$\text{if } 0 < |x - c| < \delta \text{ then } |g(x) - L| < \delta_1. \qquad (2)$$

Combining (1) and (2), and by setting $y = g(x)$, we have for every $x \in \mathbb{R}$,

$$0 < |x - c| < \delta \Rightarrow |g(x) - L| < \delta_1 \Rightarrow |f(g(x)) - f(L)| < \varepsilon.$$

Therefore, if $0 < |x - c| < \delta$ then $|f(g(x)) - f(L)| < \varepsilon$ as required.

- *Theorem* - Composition of Continuous is Continuous:
  - If f, g are functions such that
  - g is continuous at c
  - f is continuous at $g(c)$
  - Then: $f \circ g$ is continuous at c ($f \circ g$ means the composition of f, g: $f \circ g = f(g(x))$)

$$\lim_{x\to c} f(g(x)) = f(g(c))$$

- Proof:

Proof.

Using the previous theorem, we have

$$\begin{aligned}\lim_{x\to c} f(g(x)) &= f(\lim_{x\to c} g(x)) && (f \text{ is continuous at } g(c) = \lim_{x\to c} g(x)) \\ &= f(g(c)) && \underbrace{(\lim_{x\to c} g(x) = g(c))}_{\text{since } g(x) \text{ is continuous at } c}\end{aligned}$$

- Example use:

  Example The function $F(x) = |x| + \frac{1}{5-\sqrt{x^2+16}}$ is continuous everywhere except at $x = \pm 3$ where it is not defined.

  To see this, note that $F = f_1 + f_2 \circ f_3 \circ f_4 \circ f_5$ where

  $$f_1(x) = |x|, \quad f_2(x) = \frac{1}{x}, \quad f_3(x) = 5 - x, \quad f_4(x) = \sqrt{x}, \quad f_5(x) = x^2 + 16.$$

  Note that

  - f_1, f_3 and f_5 are continuous on $\mathbb{R}$;
  - f_2 is evaluated only at nonzero numbers and
  - f_4 is evaluated only at positive numbers.

  Hence applying the previous theorem we see that F is continuous. *

Limits and Composition

- $\lim_{x\to c} g(x) = L, \lim_{y\to L} f(y) = M$ Does NOT imply $\lim_{x\to c} f(g(x)) = M$
  - Counterexample:

    Let

    $$f(y) = \begin{cases} 2 & \text{if } y = 1 \\ 3 & \text{if } y \neq 1 \end{cases} \qquad \text{and} \qquad g(x) = 1 \text{ for all } x \in \mathbb{R}$$

    - $\lim_{x\to 0} g(x) = 1$
    - $\lim_{y\to 1} f(y) = 3$
    - $\lim_{x\to 0} f(g(x)) = ???$

    For $x \in \mathbb{R}$, $f(g(x)) = f(1) = 2 \quad \Rightarrow \quad \lim_{x\to 0} f(g(x)) = 2!$

- Problem here:

(1) $\lim_{x \to c} g(x) = L$ means $\underbrace{\left\{ \begin{array}{c} x \text{ close to } c \\ x \neq c \end{array} \right\} \Rightarrow g(x) \text{ is close to } L}_{\text{allows for } g(x) = L \text{ around } c}$

(2) $\lim_{y \to L} f(y) = M$ means $\left\{ \begin{array}{c} y \text{ close to } L \\ y \neq L \end{array} \right\} \Rightarrow f(y) \text{ is close to } M$

We can't concatenate the two implications 😣

- Circumvent this issue:
- *Theorem* - Limits and Composition:
  - Variant 1 (remove the condition $y \neq L$)
    - If:
      - $\lim_{x \to c} g(x) = L$
      - f is continuous at L
    - Then: $\lim_{x \to c} f(g(x)) = f(L)$
  - Variant 2 (remove the possibility that $g(x) = L$ around c)
    - If:
      - $\lim_{x \to c} g(x) = L$
      - $g(x) \neq L$ for $x \in (c - \delta, c + \delta)$ for some $\delta > 0$ except possibly at c
      - $\lim_{y \to L} f(y) = M$
    - Then: $\lim_{x \to c} f(g(x)) = M$

6.4 Intermediate Value Theorem and Extreme Value Theorem

- 6.4 IVT and EVT_NOTES - A

Intermediate Value Theorem

- *Theorem* - Intermediate Value Theorem (IVT):
  - Let f be *continuous on a close interval* $[a, b]$ and K be any real number between $f(a)$ and $f(b)$
  - Then there is at least one number c between a and b for which $f(c) = K$
  - In other words:

$$\forall K \in (f(a), f(b)), \exists c \in (a, b) \; s.t. \; f(c) = K$$

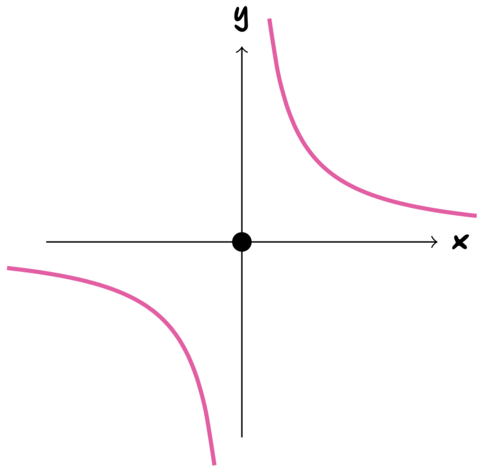
- Caveat: always check the hypothesis/condition that $f(x)$ is continuous on a closed interval $[a, b]$ first

- Intuition: a function that is continuous on an interval has no gaps and hence cannot skip over values
    - Non-example:

Consider the function $f(x) = \frac{4}{x}$.

We have $f(-1) = -4 < 0$ and $f(1) = 4 > 0$.

Is there a number $c \in (-1, 1)$ such that $f(c) = 0$?



No!

This does not contradict the IVT because this function is not continuous on $[-1, 1]$.

- Example use:
    - Using IVT to locate zeros of a function

Show that there is a solution to the equation $x^2 = \sin x + 2\cos x$ on the interval $(0, \frac{\pi}{2})$.

Let $f(x) = \sin x + 2\cos x - x^2$.

$$f(c) = 0 \iff \text{the given equation has } x = c \text{ as a solution.}$$

Note f is a continuous function and

$$f(0) = 2 > 0 \quad \text{and} \quad f(\frac{\pi}{2}) = 1 - \frac{\pi^2}{4} < 0.$$

Therefore by IVT, there exists $c \in (0, \frac{\pi}{2})$ such that $x^2 = \sin x + 2\cos x$.

Extreme Value Theorem

- *Theorem* - Extreme Value Theorem (EVT):
    - If f is continuous on a *bounded closed interval* $[a, b]$
    - Then, on that interval f attains both a maximum and a minimum value
- Caveat: in the Extreme Value Theorem, *all the hypotheses are needed*:

- If the interval is not bounded: $f(x) = x^3$ is continuous but has no maximum on $[0, \infty)$
- If the interval is not closed: $f(x) = x^2$ is continuous but has no maximum on $(-1, 1)$
  - There is a supremum, but not a maximum
- If the function is not continuous: $f(x) = \frac{1}{x}$ does not attain a maximum on the closed and bounded interval $[0, 1]$

Mapping

- *Theorem* - Mapping:
  - IVT $\rightsquigarrow$ continuous functions map intervals to intervals
  - IVT + EVT $\rightsquigarrow$ continuous functions map closed and bounded intervals to closed and bounded intervals

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Chapter 7: Derivatives

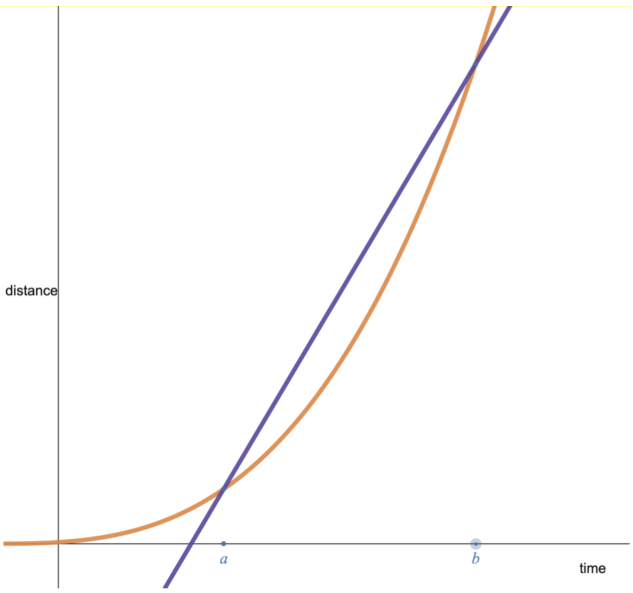
7.1 Derivatives (Definition)

- 7.1 Derivative (definition) NOTES - A

Derivative (Defintion)

- *Definition* - Secant Line: Let f be a function and $a < b$ be two real numbers. The unique straight line passing through the points $(a, f(a)), (b, f(b))$ is called the scant line from a to b

  -



- The slope of this scant line is given by:

$$m_{ab} = \frac{f(b) - f(a)}{b - a}$$

- *Definition* - Derivative at a Point: Let f be a function and a be a number in the domain of f.
  - We say that f is differentiable at a if:

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \text{ exists}$$

  - If this limit exists, it is called the derivative of f at a and denoted by $f'(a)$
- Example:

Example Find the derivative of the function $f(x) = x^2$ at $x = a$.

From the definition of derivative, we have

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \\ &= \lim_{x \to a} \frac{x^2-a^2}{x-a} \\ &= \lim_{x \to a} \frac{(x+a)(x-a)}{x-a} \\ &= \lim_{x \to a}(x+a) \\ &= 2a. \end{aligned}$$

Derivative of a Function

- *Definition* - Derivative of a Function: Let f be a function. For each x where f is differentiable at x, there is a value $f'(x)$ associated with x. We can view the derivative as a function with

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$$

  - we say that f' is the derivative of f
  - The domain of f' is the set $\{x : f'(x) \text{ exists}\} = \{x : f \text{ is differentiable at } x\}$
  - The domain of f' may be smaller than the domain of f
- Example

Example
For $f(x) = \sqrt{x}$, find f'.

Since $f'(x)$ is a two-sided limit, we don't expect the derivative to exist for $x \le 0$.

For $x > 0$, we have

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}.\end{aligned}$$

Therefore $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$.

Thus we see that $f'(x)$ for all $x > 0$, hence the domain of f' is $(0,\infty)$.
This is smaller than the domain of f, which is $[0,\infty)$.

Derivative of Piecewise Defined Functions

Example Find $f'(1)$ where

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x-1 & \text{if } x > 1. \end{cases}$$

By definition

$$f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1}.$$

Since f is defined by different formulas on each side of 1, we evaluate this by calculating the one-sided limits.

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{x^2-1}{x-1} = \lim_{x\to 1^-}\frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1^-}(x+1) = 2$$

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{2x-1-1}{x-1} = \lim_{x\to 1^+}\frac{2(x-1)}{x-1} = 2.$$

Hence $f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = 2$.

7.2 Continuity and Differentiability

- 7.2 Differentiablity and Continuity_NOTES - A

Differentiable Functions are Continuous

- *Theorem* - Differentiable $\implies$ Continuous: If f is differentiable at a, then f is continuous at a
  - *Contrapositive*: If f is not continuous at a, then f is not differentiable at a
  - The *converse is not true*! Continuous does not imply differentiable.
- Proof:
  - Let f be differentiable at the point a, then

    $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L \text{ exists and is finite.}$$

    To show that f is continuous at a, we must show that

    $$\lim_{x \to a} f(x) = f(a) \quad \text{or equivalently} \quad \lim_{x \to a}[f(x) - f(a)] = 0.$$

    If we multiply and divide $(f(x) - f(a))$ by $(x - a)$ we get

    $$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a) \text{ for } x \neq a.$$
  - Therefore

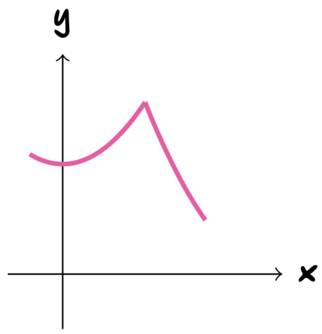
    $$\begin{aligned}\lim_{x \to a}[f(x) - f(a)] &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) \quad \text{(by limit laws)} \\ &= L \cdot 0 \\ &= 0.\end{aligned}$$

    Hence $\lim_{x \to a} f(x) = f(a)$.

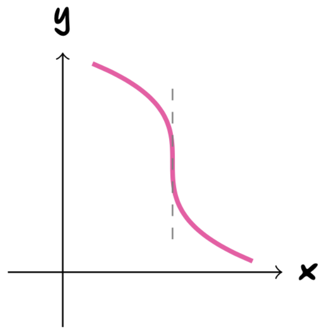
Continuous but NOT Differentiable

- *Theorem* - 3 Types of Non-Differentiable: There are three ways in which a function fail to be differentiable:
  - a corner or cusp (non-differentiable but continuous)
  - a vertical tangent line (non-differentiable but continuous)

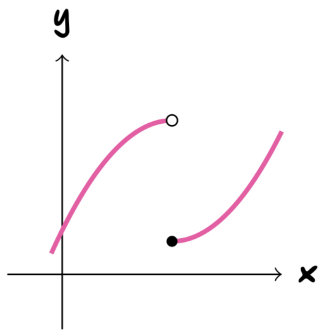
- a discontinuity (non-differentiable, non-continuous)



(a) A corner or cusp



(b) A vertical tangent line



(c) A discontinuity

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- Corner example

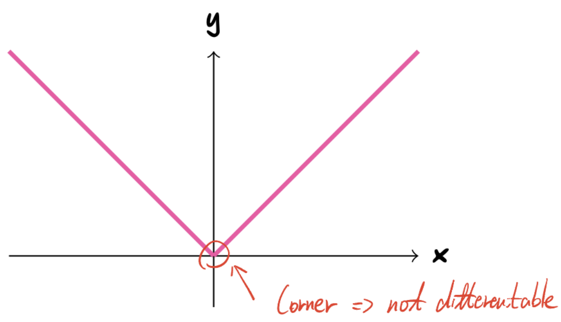
Example The function $f(x) = |x|$ is not differentiable at 0.

Note that

$$\frac{f(0+h) - f(0)}{h} = \frac{|0+h| - |0|}{h} = \frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0. \end{cases}$$

Therefore $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = -1$ and $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = 1.$

Hence $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$ does not exist.



- Vertical tangent line example

Example The function $f(x) = x^{\frac{1}{3}}$ is not differentiable at $x = 0$.

Note that

$$\frac{f(0+h)-f(0)}{h} = \frac{h^{\frac{1}{3}}-0}{h} = \frac{1}{h^{\frac{2}{3}}},$$

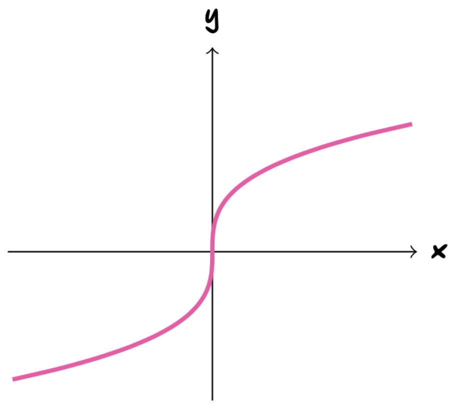
so we have that

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \infty.$$

Thus f is not differentiable at 0

At $x = 0$, the slope of the tangent line is infinite $\rightsquigarrow$ the graph of has a vertical tangent line at the origin.

$\Downarrow$ not differentiable



- ## *Differentiable Functions (Consequences)*
  - If a function f is differentiable everywhere on $\mathbb{R}$, and let $a \in \mathbb{R}$. Then:
    - $f(a)$ is defined
    - $\lim_{x\to a} f(x)$ exists
    - f is continuous at a
    - $f'(x)$ exists
  - However:
    - f' may not be continuous at a $\iff$ $\lim_{x\to a} f'(x)$ DNE
      - The only possibility that $f(x)$ is differentiable at a point, but its derivative is not continuous at that point is: $\lim_{x\to a} f(x)$ D NE
  - Example:

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & ,\text{if } x \neq 0 \\ 0 & ,\text{if } x = 0 \end{cases} \implies f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & ,\text{if } x \neq 0 \\ 0 & ,\text{if } x = 0 \end{cases}$$

$$\implies f'(0) = 0 \text{ but } \lim_{x\to 0} f'(x) \text{ DNE}$$

$$\implies f'(x) \text{ is not contiuous at } 0$$

7.3 Some Differentiation Rules

- 7.3-Differentiation Rules_NOTES - A

- *Theorem* - Derivative of Sums and Scalar Multiples: Let $\alpha \in \mathbb{R}$. If f and g are differentiable at x, then $f+g$ and αf are differentiable at x, and
    - $(f+g)'(x) = f'(x) + g'(x)$
    - $(\alpha f)'(x) = \alpha f'(x)$
- Proof:

For (1), using the definition of derivative, we have

$$\begin{aligned}
(f+g)'(x) &= \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} && ((f+g)(x) = f(x) + g(x)) \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} && \text{(by the limit laws)} \\
&= f'(x) + g'(x).
\end{aligned}$$

- *Theorem* - The Product Rule: If f and g are differentiable at x, then so is their product and

$$(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x)$$

- Proof: key: add and subtract $f(x+h)g(x)$

We form the difference quotient.

$$\frac{(f\cdot g)(x+h)-(f\cdot g)(x)}{h}=\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$=f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}.$$

Since f is differentiable, we know that f is continuous. Therefore,

$$\lim_{h\to0} f(x+h)=f(x).$$

Since

$$\lim_{h\to0}\frac{g(x+h)-g(x)}{h}=g'(x) \quad \text{and} \quad \lim_{h\to0}\frac{f(x+h)-f(x)}{h}=f'(x),$$

using the limit laws we obtain

$$(f\cdot g)'(x)=\lim_{h\to0}\frac{(f\cdot g)(x+h)-(f\cdot g)(x)}{h}$$

(↓ limit law)

$$=\lim_{h\to0} f(x+h)\cdot\lim_{h\to0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to0} g(x)\cdot\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=f(x)g'(x)+g(x)f'(x).$$

- *Theorem* - The Reciprocal Rule: If g is differentiable at x and $g(x)\neq0$, then $\frac{1}{g}$ is differentiable at x and

$$\left(\frac{1}{g}\right)'(x)=-\frac{g'(x)}{[g(x)]^2}$$

- *Theorem* - The Quotient Rule: If f and g are differentiable at x and $g(x)\neq0$, then the quotient $\frac{f}{g}$ is differentiable at x and

$$\left(\frac{f}{g}\right)'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$

- Trick for direct proof: add and subtract $f(a)g(a)$ in the denominator

-

Proof Remarks (see PPT)

- Checklist for proving the quotient rule:
  - Did you use the definition of derivative?
  - Are there words or only equations?
  - Does every step follow logically?
  - Did you only assume things you can assume?
  - Did you assume at some point that a function was differentiable?
  - Did you assume at some point that a function is continuous? If so, did you justify it?

- *Theorem* - Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x, and

$$F'(x) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

  - In using the chain rule we work from the outside to the inside:

$$\frac{d}{dx} f(g(x)) = \underbrace{f'(g(x))}_{\text{derivative of outer evaluated at inner}} \cdot \underbrace{g'(x)}_{\text{derivative of inner}}$$

- Caveat: it is $f'(g(x))$ not $f'(g'(x))$!!!
- Example:

Example Find $F'(x)$ where $F(x) = \sqrt{x^2+1}$.

At the beginning of the section, we expressed F as $F(x) = (f \circ g)(x)$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$.

- Since

$$f'(u) = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 2x,$$

we have, by the chain rule,

$$\begin{aligned} F'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2\sqrt{x^2+1}} \cdot 2x \\ &= \frac{x}{\sqrt{x^2+1}}. \end{aligned}$$

7.4 Local Maximum and Derivatives

- 7.4 Local Maximum_NOTES - A

- *Theorem* - Local Maximum and Local Minimum: Let f be a function defined on an open interval I and let $c \in I$.
  - f has a *maximum* at c if and only if

$$f(c) \geq f(x) \ \forall \ x \in I$$

  - f has a *local maximum* at c if and only if

$$\exists \delta > 0 \ s.t. \ |x-c| < \delta \implies f(c) \geq f(x)$$

- *Theorem* - Local Extreme Value Theorem: Let f be a function defined on an open interval I and let $c \in I$. If f has a local maximum/minimum at c and c is an interior point of I (not an end-point), then $f'(c) = 0$ or $f'(c)$ does not exist.
  - The converse is not true! If $f'(c) = 0$, we cannot conclude that it's a local max/min!
    - Example:

Example

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$

But f has no maximum or minimum at 0!

- Proof:

Assume that f has a local maximum at c.
We want to show that either $f'(c) = 0$ or $f'(c)$ does NOT exist.

We will assume that $f'(c)$ exists and prove that it must be 0.

Since f has a local maximum at c, by the definition,

$$f(c) \geq f(x) \text{ for all } x \text{ in some interval } (c-p, c+p).$$

This implies that

$$\text{if } h > 0 \text{ is so small that } c+h \in (c-p, c+p), \text{ then } f(c) \geq f(c+h)$$

Hence $f(c+h) - f(c) \leq 0$, and since $h > 0$, we have

$$\frac{f(c+h)-f(c)}{h} \leq 0.$$

Thus we have

$$\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0. \tag{1}$$

-

Similarly, if $h < 0$ is small enough, we have $f(c+h) - f(c) \le 0$, but with negative denominator we get

$$\frac{f(c+h) - f(c)}{h} \ge 0.$$

Thus we have

$$\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \ge 0. \tag{2}$$

Combining (1) and (2) and using that f is differentiable at c, we have

$$0 \le \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \le 0.$$

So $f'(c) = 0$.

-

7.5 The Mean Value Theorem

- 7.5 MVT_Notes - A

The Mean Value Theorem

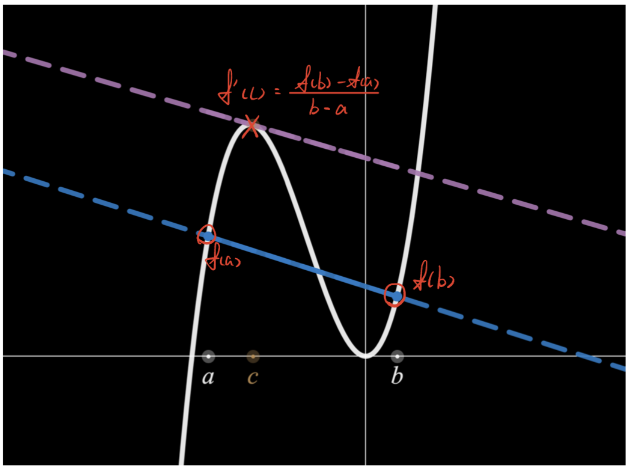
- *Theorem* - Mean Value Theorem (MVT):
  - Let f be a function that satisfies the following hypothesis:
  - f is continuous on the closed interval $[a, b]$
  - f is differentiable on the open interval (a, b)
  - Then, there is at least one number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Geometric interpretation:

The conclusion of the Mean Value Theorem says

$$\exists c \in (a,b) \quad \text{s.t.} \quad f'(c) = \frac{f(b)-f(a)}{b-a}.$$



- $\dfrac{f(b)-f(a)}{b-a}$ is the slope of the secant line between $A(a, f(a))$ and $B(b, f(b))$
- $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$

  - MVT $\implies$ at least one point c where the tangent line to the graph at $x = c$ is parallel to the secant line AB
- Some important applications of the MVT:
  - A function with zero derivative must be constant
  - A function with positive derivative must be increasing
  - The Fundamental Theorem of Calculus
  - L'Hôpital's Rule
  - All integrations methods
  - ...

Rolle's Theorem

- *Theorem* - Rolle's Theorem:
  - Let f be a function that satisfies the following hypothesis:
    - f is continuous on the closed interval $[a, b]$
    - f is differentiable on the open interval (a, b)
    - $f(a) = f(b)$
  - Then, there exists a point $c \in (a, b)$ such that $f'(c) = 0$
- Proof:

- Recall

Definition (Local max and min)

Let f be a function defined on an open interval around c.

- f has a local maximum at c if $f(c) \geq f(x)$ for all x in some open interval containing c.
- f has a local minimum at c if $f(c) \leq f(x)$ for all x in some open interval containing c.

Recall:

Theorem (Extreme Value Theorem)

If f is continuous on a closed and bounded interval $[a, b]$, then f attains both a maximum value and a minimum value on $[a, b]$.

Theorem (Local Extreme Value Theorem)

If f has a local maximum or minimum at c, then $f'(c) = 0$ or $f'(c)$ does NOT exist.

- Proof: EVT + LEVT

Proof.

Since f is continuous, by the Extreme Value Theorem, f attains a maximum and a minimum on $[a, b]$.

Case 1:

strictly in a, b (open intervals)

If f attains a maximum OR a minimum at some point $c \in (a, b)$, then it is a local maximum or a local minimum at c.

By the Local Extreme Value Theorem, $f'(c) = 0$ or $f'(c)$ does NOT exist.

Since f is differentiable at c, $f'(c) = 0$.

Case 2:

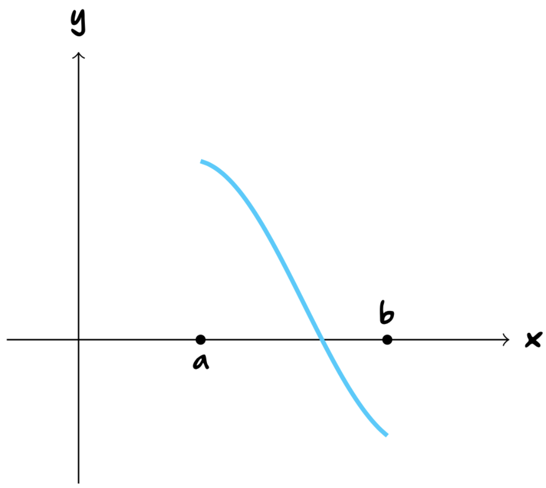
If f attains both its maximum AND minimum at the end-points, since $f(a) = f(b)$, f must be a constant function.

Therefore, for all $x \in (a, b)$, $f'(x) = 0$.

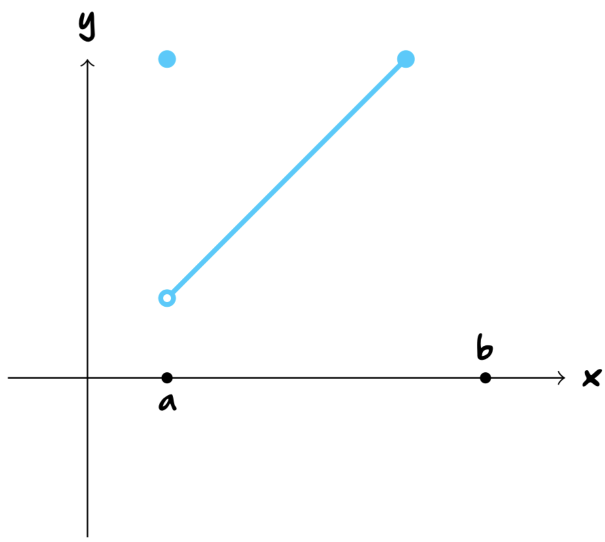
In both cases we have that thre exists $c \in (a, b)$ such that $f'(c) = 0$. 😃

- Note that *all 3 hypotheses are needed in Rolle's Theorem!*

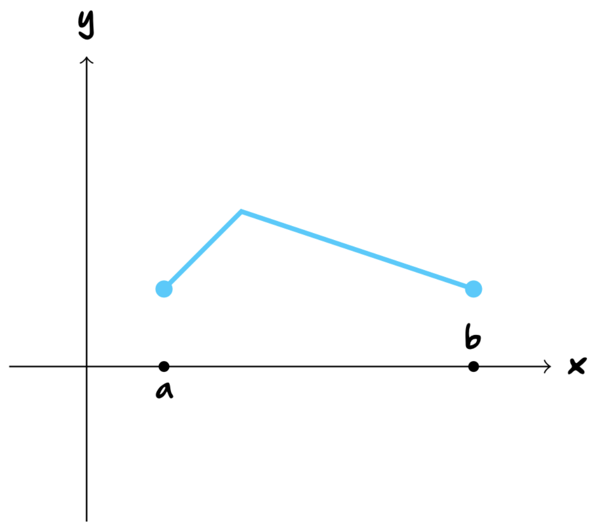
- If $f(a) \neq f(b)$



- If f is not continuous on $[a, b]$



- If f is not differentiable on (a, b)



Implication of Rolle's Theorem: How Many 0's Does a Function Have?

- *Use IVT to find the minimum number of roots, and use MVT/Rolle's theorem to show the maximum number of roots (Rolle's Theorem $\implies$ f has K roots requires f' to have at* *least K-1 zeros and f'' to have at least K-2 zeros)*

Example Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

1. Existence: We first use the Intermediate Value Theorem to show that a root exists.
Let $f(x) = x^3 + x - 1$. Then f is continuous on $\mathbb{R}$ and

$$f(0) = -1 < 0, \quad f(1) = 1 > 0.$$

So by the Intermediate Value Theorem, there exists $c \in (0,1)$ such that $f(c) = 0$.
Thus the given equation has a root.

2. Uniqueness: To show that the equation has no other real root, we use Rolle's Theorem.
Suppose that it had two roots a and b, we aim for a contradiction.

Since f is a polynomial, it is differentiable on (a,b) and continuous on $[a,b]$.

$$f(a) = 0 \quad \text{and} \quad f(b) = 0.$$

Thus by Rolle's Theorem, there is $c \in (a,b)$ such that $f'(c) = 0$. But

$$f'(x) = 3x^2 + 1 \geq 1 \quad \text{for all } x.$$

This gives a contradiction.
Therefore the equation cannot have more than one real root.

What Does f' Say about f?

- *Theorem* - Increasing/Decreasing Test:
  - Suppose that f is differentiable on an open interval (a,b). Then:
  - If $f'(x) > 0 \; \forall \; x \in (a,b)$, then f is increasing on (a,b)
  - If $f'(x) < 0 \; \forall \; x \in (a,b)$, then f is decreasing on (a,b)
- Proof: take any $x_1, x_2 \in (a,b)$, then use MVT to show that $f'(c) > 0 \implies f(x_2) > f(x_1)$

Proof.
We will prove (1).
Let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$.
Since f is differentiable on (a,b), we know that f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By the Mean Value Theorem, there is a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(x) > 0$ for all $x \in (a,b)$, $f'(c) > 0$ and we have

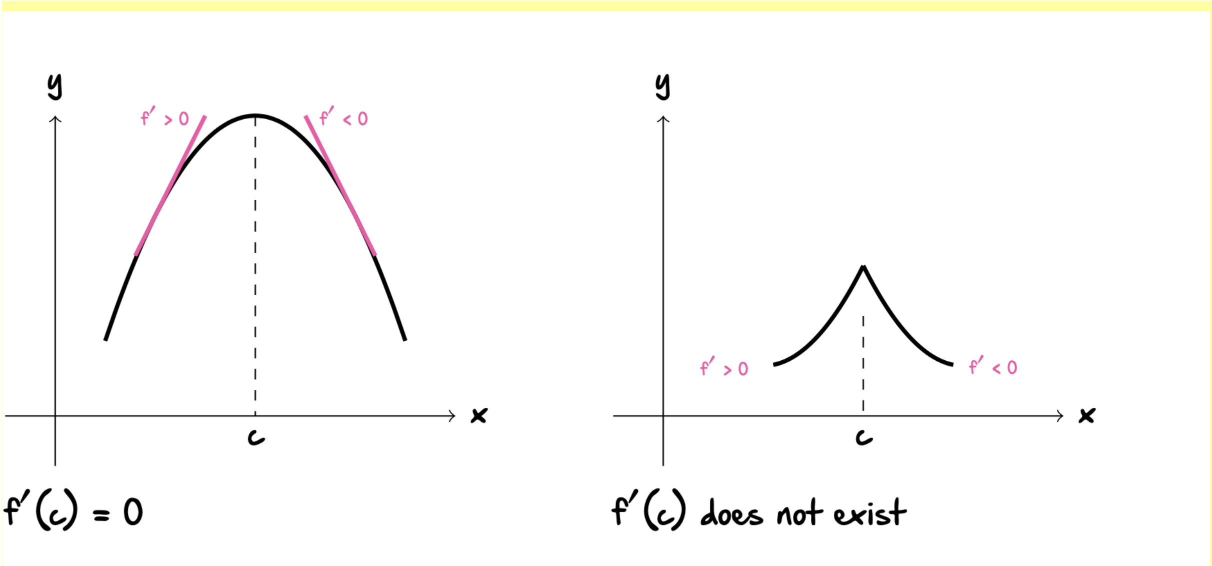
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

Since $x_2 - x_1 > 0$, this implies that $f(x_2) > f(x_1)$.
Therefore, f is increasing on (a,b).

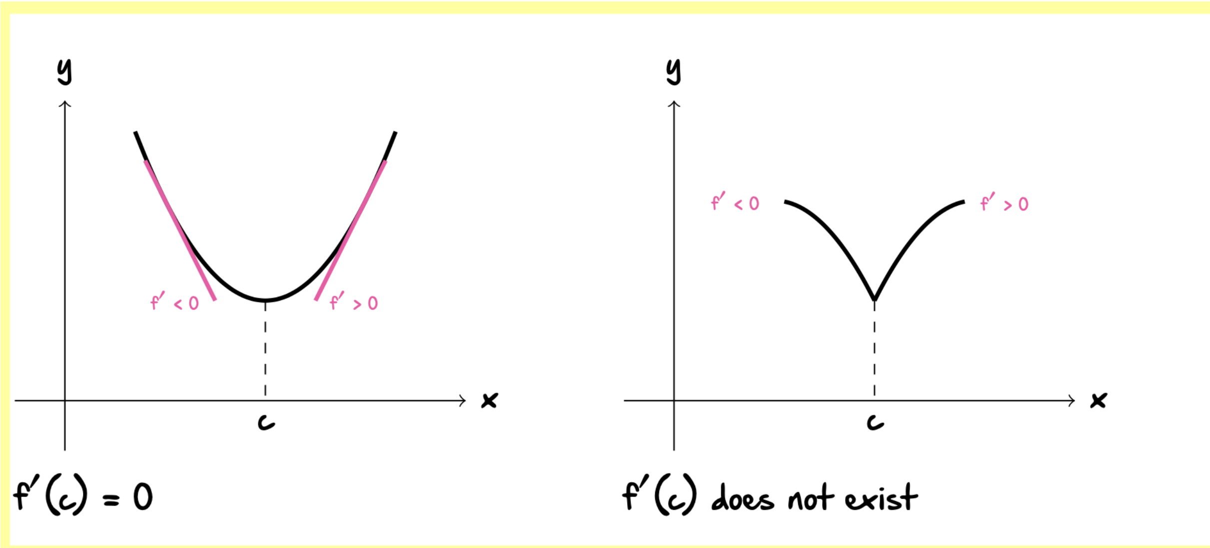
- *Theorem* - Improved Increasing/Decreasing Test:
    - Suppose that f is differentiable on an open interval (a, b) and continuous on the closed interval $[a, b]$. Then:
    - If $f'(x) > 0 \; \forall \; x \in (a, b)$, then f is increasing on $[a, b]$
    - If $f'(x) < 0 \; \forall \; x \in (a, b)$, then f is decreasing on $[a, b]$
- *Theorem* - Zero Derivative Implies Constant:
    - $f'(x) = 0 \; \forall \; x \in (a, b) \iff f$ is constant on (a, b)

First Derivative Test

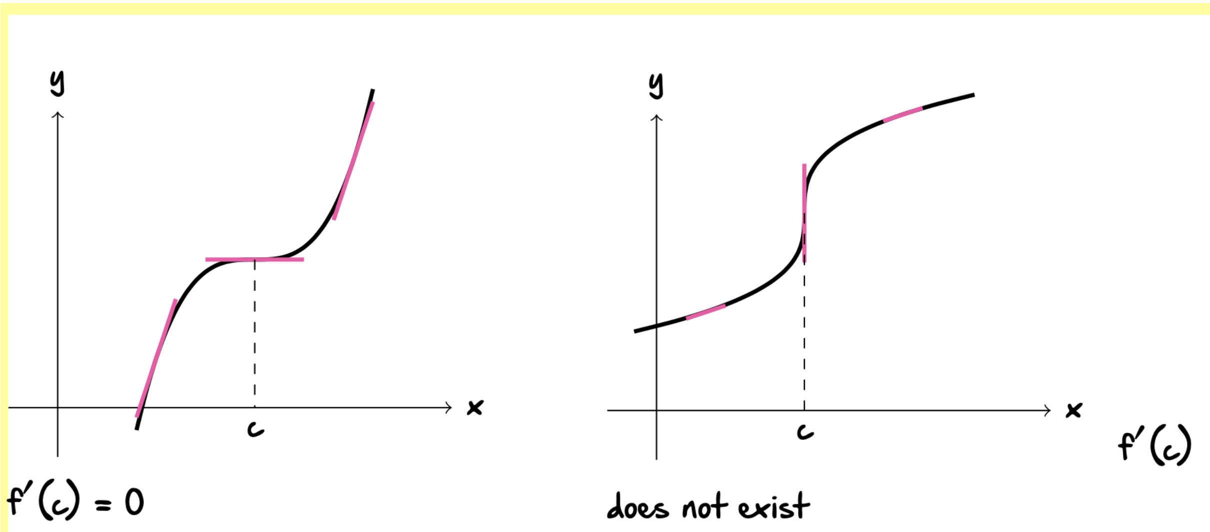
- *Theorem* - First Derivative Test:
    - Suppose that c is a critical point for f (i.e. $f'(c) = 0$ or $f'(c)$ DNE) and f is continuous at c. If there is a positive number δ such that:
    - $f'(x) > 0 \; \forall \; x \in (c - \delta, c)$ and $f'(x) < 0 \; \forall \; x \in (c, c + \delta)$, then f has a local maximum at c
    - $f'(x) < 0 \; \forall \; x \in (c - \delta, c)$ and $f'(x) > 0 \; \forall \; x \in (c, c + \delta)$, then f has a local minimum at c
    - $f'(x)$ does not change sign on $(c - \delta, c) \cup (c, c + \delta)$, then $f(c)$ is not a local extreme value
- Graphs:
    - local maximum



- local minimum



- not a local extreme value



- *Application*: find the local minimum and maximum values for functions
  - Steps:
  - 1. Find all $x\ s.t.\ f'(x) = 0$
  - 2. Check whether $f'(x)$ changes sign
- Example:

Example Find the local minimum and maximum values of the function $f(x) = \frac{x^4}{4} - x^3 + x^2 + 1$.

This function is a polynomial, therefore it is everywhere differentiable. Differentiation gives

$$f'(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2).$$

① find x s.t. $f'(x) = 0$

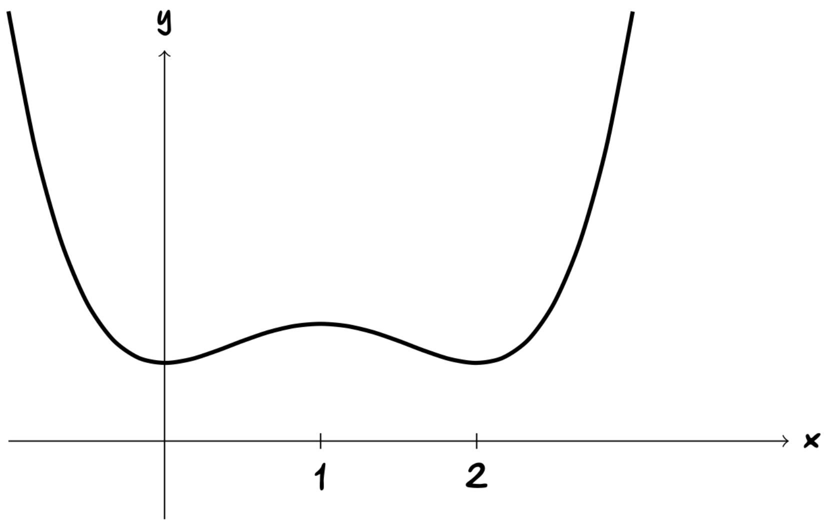
②

Interval	$(-\infty, 0)$	$(0,1)$	$(1,2)$	$(2,\infty)$
x	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Since f is continuous everywhere, f decreases on $(-\infty, 0]$ and $[1,2]$ and increases on $[0,1]$ and $[2,\infty)$.

- $f'(x)$ changes from negative to positive at 0. So $f(0) = 1$ is a local minimum value by the First Derivative Test.
- $f'(x)$ changes from negative to positive at 2, so $f(2) = 1$ is also a local minimum.

The graph of $f(x)$ is



Chapter 8: Inverse Functions

8 Inverse Functions

- 8. Inverse NOTES - A

Injective, Surjective, Bijective, and Definition of Inverse Function

- *Definition* - Injective:

  - f is said to be injective (or one-to-one) if $\forall\, x_1, x_2 \in X$:

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

  - equivalently:

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

- *Definition* - Surjective:
  - f is said to be surjective (or onto) if $\forall\, y \in Y, \exists x \in X \; s.t. \; y = f(x)$
- *Definition* - Bijective:
  - f is said to be bijective (of f is a bijection / one-to-one correspondence) if f is both injective and surjective
- *Definition* - Inverse Function:

  - If $f : X \to Y$ is a bijection, then the inverse function $f^{-1} : Y \to X$ is the unique function such that:

$$\begin{cases} f^{-1}(f(x)) = x & \forall\, x \in X \\ f^{(}f^{-1}(y)) = y & \forall\, y \in Y \end{cases}$$

  - Intuition: the inverse of f denoted f^{-1} is the function that reverses the effect of f

Inverse of Continuous Functions

- *Theorem* - Continuous + Invertible $\implies$ Strictly Monotonic:
  - If f is continuous on an interval I and has an inverse, then f is strictly monotonic on I
- Proof:

  Proof.
  Suppose for the sake of a contradicition that f is NOT strictly monotonic.
  Then there exists x_1, x_2, x_3 in I such that $x_1 < x_2 < x_3$, but $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$.

  Without loss of generality, assume $f(x_1) < f(x_3) < f(x_2)$.
  By the Intermediate Value Theorem, there exists an x_0 between x_1 and x_2 such that

$$f(x_0) = f(x_3).$$

  So, f is not injective and cannot have an inverse.

  This is a contradiction, so f must be strictly monotonic.

- *Theorem* - Inverse of Continuous Functions is Continuous:
  - Let f be a function whose domain is an interval I. If:
    - f has an inverse and
    - f is continuous on its domain

- Then, f^{-1} is also continuous on its domain

- Proof:
  - **Proof.**

    Because f is continuous, the Intermediate Value Theorem implies that the set $\{f(x) \mid x \in I\}$ forms an interval J. (Continuous functions map intervals to intervals)

    Assume that a is an interior point of J.

    From the previous theorem, we know that f is strictly monotonic. This implies that $f^{-1}(a)$ is an interior point of I.

  - Let $\varepsilon > 0$.

    There exists $0 < \varepsilon_1 < \varepsilon$ such that

    $$I_1 = (f^{-1}(a) - \varepsilon_1, f^{-1}(a) + \varepsilon_1) \subset I.$$

    Because f is strictly monotonic on I_1, the set $\{f(x) \mid x \in I_1\}$ forms an interval $J_1 \subset J$.

    Let $\delta > 0$ such that $(a - \delta, a + \delta) \subset J_1$.

    Finally, if $|y - a| < \delta$, then $|f^{-1}(y) - f^{-1}(a)| < \varepsilon_1 < \varepsilon$. $\lim_{y \to a} f^{-1}(y) = f^{-1}(a)$

    So f^{-1} is continuous at a.

    A similar proof can be given if a is an endpoint.

Inverse Function Theorem

- *Theorem* - Inverse Function Theorem:
  - Let f be a function whose domain is an interval I. If:
    - f has an inverse and
    - f is differentiable on an interval containing a and $f'(a) \neq 0$
  - Then, f^{-1} is differentiable at $f(a)$, and $(f^{-1})'(b) = \frac{1}{f'(a)}$ where $a = f^{-1}(b)$ i.e. $b = f(a)$ (*evaluation point also has to be transformed!*) A better way:

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)} \text{ or } (f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

- Proof:

- Proof.

  By the definition of derivative

  $$(f^{-1})'(b) = \lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b}$$

  where b is in the domain of f^{-1} and $f^{-1}(b) = a$.

- Because f is differentiable on an interval containing a, f is continuous on that interval.

  By the previous theorem, f^{-1} is continuous at b.

  So $y \to b$ implies that $x \to a$, and we have the following:

  $$\begin{aligned}(f^{-1})'(b) &= \lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} \\ &= \lim_{x \to a} \frac{x - a}{f(x) - f(a)} \\ &= \lim_{x \to a} \frac{1}{\left(\frac{f(x)-f(a)}{x-a}\right)} \\ &= \frac{1}{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}} \qquad \text{limit laws} \\ &= \frac{1}{f'(a)}.\end{aligned}$$

  So $(f^{-1})'(a)$ exists and f^{-1} is differentiable at $b = f(a)$.

Chapter 9: Integrations

9.1 The Definition of Integral

- 9.1 Integration_NOTES - A

Definition of the Integral

- *Definition* - Partition:
  - A partition of the interval $[a, b]$ is a set P such that:
    - P is finite
    - $P \subseteq [a, b]$

        - $a \in P, b \in P$
    - Notation: $P = \{x_0, x_1, \ldots, x_N\}$ means $a = x_0 < x_1 < \cdots < x_N = b$
- *Definition* - Lower and Upper Sum:
    - Let f be a bounded function on $[a, b]$
    - let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$
    - For each $i = 1, \ldots, N$, let:
        - m_i be the infimum of f on $[x_{i-1}, x_i]$
        - M_i be the supremum of f on $[x_{i-1}, x_i]$
        - $\Delta x_i = x_i - x_{i-1}$
    - The *P-lower sum of f* is the number: $L(f, P) = \sum_{i=1}^{N} m_i \Delta x_i$
    - The *P-upper sum of f* is the number: $U(f, P) = \sum_{i=1}^{N} M_i \Delta x_i$
- *Definition* - Lower and Upper Integral:
    - Let f be a bounded function on $[a, b]$
    - Let $\mathcal{P}_{[a,b]}$ be the set of all partitions of $[a, b]$.
    - The *lower integral* of f from a to b, denoted by $\underline{\int_a^b} f$, is

$$\underline{\int_a^b} f = \sup \left\{ L(f, P) : P \in \mathcal{P}_{[a,b]} \right\}$$

    - The *upper integral* of f from a to b, denoted by $\overline{\int_a^b} f$, is:

$$\overline{\int_a^b} f = \inf \left\{ U(f, P) : P \in \mathcal{P}_{[a,b]} \right\}$$

- *Definition* - Integrable Function:
    - Let f be a bounded function on $[a, b]$
    - We say that f is integrable on $[a, b]$ if and only if $\underline{\int_a^b} f = \overline{\int_a^b} f$ and that:

$$\int_a^b f(x)\, dx = \underline{\int_a^b} f = \overline{\int_a^b} f$$

- *Definition* - Non-Integrable Function:
    - Let f be a bounded function on $[a, b]$
    - We say that f is non-integrable on $[a, b]$ if and only if $\underline{\int_a^b} f < \overline{\int_a^b} f$ and $\int_a^b f(x)\, dx$ is undefined

Properties of Lower and Upper Sums

- *Definition* - Finer Partition:
    - Let P and Q be partitions of the interval $[a, b]$
    - We say that Q is finer than P if $P \subseteq Q$

- *Theorem* - Properties of Lower and Upper Sums:
  - Property 1:
    - For every partition P of $[a,b]$, $L(f,P) \leq U(f,P)$
  - Property 2: Adding a Point to a Partition
    - Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a,b]$
    - Let $y \in [a,b]$ such that $y \notin P$
    - Let $Q = P \cup \{y\}$
    - Then: $L(f,P) \leq L(f,Q), U(f,P) \geq U(f,Q)$
  - Property 3:
    - Let P and Q be 2 partitions of $[a,b]$
    - If $P \subseteq Q$, then: $L(f,P) \leq L(f,Q), U(f,P) \geq U(f,Q)$
  - Property 4: Lower Sum is Less than Upper Sum
    - Let P, Q be any 2 partitions of $[a,b]$. Then: $L(f,P) \leq U(f,Q)$
- Proof of property 2:

Proof of Property 2

Let P be a partition of $[a,b]$ and let $Q = P \cup \{y\}$ where $x_{r-1} < y < x_r$ for some r, $1 \leq r \leq N$.

For $i \in \{1, \cdots, N\}$, let $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$, and let

$$m_r' = \inf\{f(x) : x_{r-1} \leq x \leq y\} \quad \text{and} \quad m_r'' = \inf\{f(x) : y \leq x \leq x_r\}.$$

Then, $m_r' \geq m_r$ and $m_r'' \geq m_r$.

Hence,

$$\begin{aligned} L(f,Q) &= \sum_{i=1}^{r-1} m_i(x_i - x_{i-1}) + m_r'(y - x_{r-1}) + m_r''(x_r - y) + \sum_{i=r+1}^{N} m_i(x_i - x_{i-1}) \\ &\geq \sum_{i=1}^{r-1} m_i(x_i - x_{i-1}) + m_r(y - x_{r-1}) + m_r(x_r - y) + \sum_{i=r+1}^{N} m_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{r-1} m_i(x_i - x_{i-1}) + m_r(x_r - x_{r-1}) + \sum_{i=r+1}^{N} m_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{N} m_i(x_i - x_{i-1}) \\ &= L(f,P). \end{aligned}$$

Hence, $L(f,P) \leq L(f,Q)$

Exercise: Prove that $U(f,P) \geq U(f,Q)$

- Proof of property 4:

Proof of Property 4

Let P and Q be two partitions of $[a, b]$.
Let $R = P \cup Q$. Then $P \subseteq R$ and $Q \subseteq R$.

$$L(f, P) \underbrace{\leq}_{\text{by Property 3}} L(f, R) \underbrace{\leq}_{\text{by Property 1}} U(f, R) \underbrace{\leq}_{\text{by Property 3}} U(f, Q).$$

- *Theorem* - Lower and Upper Integral:
  - If f is defined and bounded on $[a, b]$, then $\underline{\int_a^b} f \leq \overline{\int_a^b} f$
- Proof:
  - Proof.
    Let $\mathscr{P}_{[a,b]}$ be the set of all partitions of $[a, b]$. Let P and Q be two arbitrary partitions. By Property 4, we have that

    $$L(f, P) \leq U(f, Q).$$

    Since Q is an arbitrary partition, $L(f, P)$ is a lower bound for the set $\{U(f, Q) : Q \in \mathscr{P}_{[a,b]}\}$.
    Therefore, by the definition of infimum,

    $$\overline{\int_a^b} f = \inf\{U(f, Q) : Q \in \mathscr{P}_{[a,b]}\} \geq L(f, P).$$
  - Since $P \in \mathscr{P}_{[a,b]}$ was arbitrary, we have that $\overline{\int_a^b} f \geq L(f, P)$ for all $P \in \mathscr{P}_{[a,b]}$.
    Hence $\overline{\int_a^b} f$ is an upper bound for the set $\{L(f, P) : P \in \mathscr{P}_{[a,b]}\}$.
    By the definition of supremum

    $$\underline{\int_a^b} f = \sup\{L(f, P) : P \in \mathscr{P}_{[a,b]}\} \leq \overline{\int_a^b} f.$$

    Therefore, $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

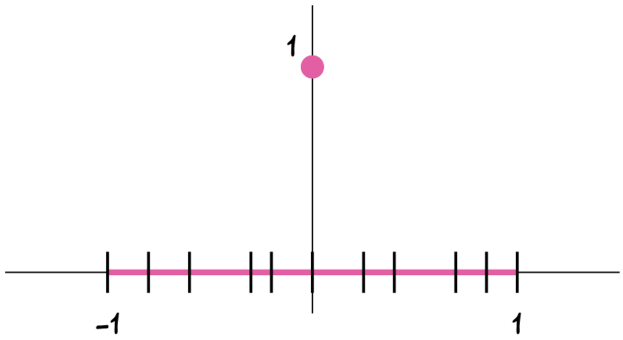
Example: An Integrable Function

- Function:

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

- This function is integrable on $[-1, 1]$
- Lower integral $\underline{\int_{-1}^{1}} f = 0$

Consider any partition of the interval $[-1, 1]$



We first find $\underline{\int_{-1}^{1}} f$

On every subinterval, the infimum of f is $m_i = 0$.

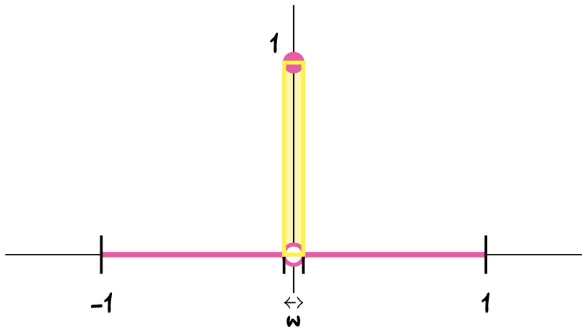
Therefore, for every partition $P = \{x_0, x_1, \cdots, x_N\}$, $L(f, P) = \sum_{i=1}^{N} m_i \Delta x_i = 0$.

Hence,

$$\underline{\int_{-1}^{1}} f = \sup \{ L(f, P) : P \in \mathscr{P}_{[-1,1]} \} = \sup \{ 0 \} = 0$$

- Upper integral $\overline{\int_{-1}^{1}} f = 0$ as well

- Consider any partition of the interval $[-1, 1]$

  We now need to find $\overline{\int_{-1}^{1}} f$

  $U(f, P)$ = area of yellow rectangle(s) $= 1 \cdot w$.

  So $\{U(f,P) : P \in \mathscr{P}_{[-1,1]}\} = (0, 2]$.

  Therefore,

  $$\overline{\int_{-1}^{1}} f = \inf\{U(f,P) : P \in \mathscr{P}_{[-1,1]}\} = \inf(0,2] = 0.$$

- Since

  $$\underline{\int_{-1}^{1}} f = 0 = \overline{\int_{-1}^{1}} f.$$

  We have that f is integrable on $[-1, 1]$ and

  $$\int_{-1}^{1} f(x)\,dx = 0.$$

Example: A Non-Integrable Function

- Function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- This is non-integrable on $[0, 1]$

💡 Key idea The set of rational numbers and the set of irrational numbers are both dense in $\mathbb{R}$:

Given $a, b \in \mathbb{R}$ such that $a < b$,

- $\exists t \in [a,b]$ s.t. $t \in \mathbb{Q}$
- $\exists s \in [a,b]$ s.t. $s \notin \mathbb{Q}$

Let $P = \{x_0, x_1, \cdots, x_N\}$ be any partition of $[0,1]$.

- infimum of g on every subinterval $[x_{i-1}, x_i]$ is 0.
- supremum of g on every subinterval $[x_{i-1}, x_i]$ is 1.

- So $L(f,P) = 0$ and $U(f,P) = 1$.

$$\overline{\int_0^1} f = \inf\{U(f,P) : P \in \mathscr{P}_{[0,1]}\} = \inf\{1\} = 1.$$

$$\underline{\int_0^1} f = \sup\{L(f,P) : P \in \mathscr{P}_{[0,1]}\} = \sup\{0\} = 0.$$

Since $\overline{\int_0^1} f \neq \underline{\int_0^1} f$, g is NOT integrable on $[0,1]$.

Properties of the Integral

- *Theorem* - Linearity of the Integral:
  - Let f, g be bounded functions on $[a,b]$. If f, g are integrable on $[a,b]$ and $c \in \mathbb{R}$, then
  - cf and $f+g$ are integrable on $[a,b]$ and:
  - $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$
  - $$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
- Proof:

Lemma 1. Let f and g be bounded functions on $[a,b]$. Then

$$\sup_{x\in[a,b]} (f+g)(x) \leq \sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x).$$

Proof.
Let $x \in [a,b]$. Then

$$f(x) \leq \sup_{x\in[a,b]} f(x) \quad \text{and} \quad g(x) \leq \sup_{x\in[a,b]} .$$

Therefore

$$f(x) + g(x) \leq \sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x) \quad \text{for all } x \in [a,b].$$

This means that $\sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x)$ is an upper bound for the set

$$\{f(x) + g(x) : x \in [a,b]\}.$$

Hence

$$\sup_{x\in[a,b]} (f+g)(x) \leq \sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x)$$

Lemma 2. If f and g are bounded on $[a,b]$, but not necessarily integrable, then

- $\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g,$
- $\underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g.$

Proof.
Suppose that $P = \{x_0, x_1, \cdots, x_N\}$ is a partition of $[a,b]$. Then

$$\begin{aligned} U(f+g,P) &= \sum_{i=1}^{N} \sup_{x\in[x_{i-1},x_i]} (f+g)(x)\,\Delta x_i \\ &\leq \sum_{i=1}^{N} \sup_{x\in[x_{i-1},x_i]} f(x)\,\Delta x_i + \sum_{i=1}^{N} \sup_{x\in[x_{i-1},x_i]} g(x)\,\Delta x_i \\ &= U(f,P) + U(g,P) \end{aligned}$$

Let $\varepsilon > 0$. Since the upper integral is the infimum of the upper sums, there are partitions Q and R such that

$$U(f, Q) < \overline{\int_a^b} f + \frac{\varepsilon}{2}, \quad \text{and} \quad U(g, R) < \overline{\int_a^b} g + \frac{\varepsilon}{2},$$

Let $P = Q \cup R$, then

$$U(f, P) \leq U(f, Q) < \overline{\int_a^b} f + \frac{\varepsilon}{2}, \quad \text{and} \quad U(g, P) \leq U(g, R) < \overline{\int_a^b} g + \frac{\varepsilon}{2},$$

It follows that

$$\overline{\int_a^b}(f+g) \leq U(f+g, P) \leq U(f, P) + U(g, P) < \overline{\int_a^b} f + \overline{\int_a^b} g + \varepsilon.$$

Since this inequality holds for arbitrary $\varepsilon > 0$, we must have

$$\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g.$$

Similarly, we have $L(f+g, P) \geq L(f, P) + L(g, P)$ for all partitions P, and for every $\varepsilon > 0$, we get

$$\underline{\int_a^b}(f+g) > \underline{\int_a^b} f + \underline{\int_a^b} g - \varepsilon,$$

so

$$\underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g.$$

Proof of Theorem (Linearity of the Integral)

Suppose that f and g are integrable on $[a, b]$, then

$$\overline{\int_a^b} f = \underline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b} g = \underline{\int_a^b} g.$$

Then, using the previous lemma we have

$$\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g = \underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b}(f+g)$$

Since it is always true

$$\underline{\int_a^b}(f+g) \leq \overline{\int_a^b}(f+g)$$

we have

$$\underline{\int_a^b}(f+g) = \overline{\int_a^b}(f+g).$$

Hence $f+g$ is integrable.

- Moreover, there is equality throughout the previous inequality, which proves the result.

- *Theorem* - Monotonicity of the Integral:

  - Let f, g be bounded functions on $[a, b]$. If f, g are integrable on $[a, b]$ and $f(x) \leq g(x)\ \forall\ x \in [a, b]$. Then:

$$\int_a^b f \leq \int_a^b g$$

- *Theorem* - Additivity of the Integral:

  - Let f be bounded functions on $[a, b]$ and $a < c < b$. Then, f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$. In that case:

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

9.2 The Fundamental Theorem of Calculus

- 9.2 FTC_NOTES - A
- *Theorem* - Fundamental Theorem of Calculus, Part 1 (FTC1):
  - If f is continuous on an interval $[a, b]$, then the function g is defined by

$$g(x) = \int_a^b f(t)\, dt \text{ for } a \leq x \leq b$$

  is continuous on $[a, b]$ and differentiable on (a, b)
  - Moreover, $g'(x) = f(x)$

- Intuition: when f is continuous, if we first integrate and then differentiate, we get f back

- Proof:

  - **Proof.**

    Let $x \in [a,b]$. We want to show that $g'(x) = f(x)$.

    $$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \end{aligned}$$

    Hence we need to show that

    $$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt = f(x).$$

  - We may assume that $h > 0$. Let us call

    $$\begin{aligned} M_h &= \text{sup of } f \text{ on } [x, x+h] \\ m_h &= \text{inf of } f \text{ on } [x, x+h] \end{aligned}$$

    Then

    $$\begin{aligned} m_h \cdot h \le &\int_x^{x+h} f(t)dt \le M_h \cdot h \\ m_h \le &\frac{1}{h}\int_x^{x+h} f(t)dt \le M_h \end{aligned}$$

    💡 We want to use the Sandwich Theorem

- Let $h > 0$.
  - Since f is continuous on $[a, b]$ by the Extreme Value Theorem, it has a maximum on $[x, x+h]$.
  - Let M_h = max of f on $[x, x+h]$. Then

$$\forall h > 0,\ \exists c_h \in [x, x+h] \text{ s.t. } M_h = f(c_h)$$

  - $x \le c_h \le x + h$

$$\text{As } h \to 0,\ c_h \to x.$$

  Since f is continuous, we have that

$$M_h = f(c_h) \to f(x)$$

  We can show in a similar way that $\lim_{h\to 0} m_h = 0$.

- We have that

  1. $$m_h \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M_h$$

  2. $\lim_{h\to 0} m_h = \lim_{h\to 0} M_h = f(x)$

  Therefore, by the Sandwich Theorem

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

- *Theorem* - Fundamental Theorem of Calculus, Part 2 (FTC2):
  - If f is continuous on $[a, b]$, then:

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

    where G is an *anti-derivative* of f (i.e. $G' = f$)
  - We often use the following definition:

$$G(x)|_a^b := G(b) - G(a) \quad \text{or} \quad [G(x)]_a^b := G(b) - G(a)$$

- Thus, the equation of FTC2 can be written as:

$$\int_a^b f(x)\,dx = G(x)\big|_a^b = [G(x)]_a^b$$

- Proof:

Proof.

KNOW: $G' = f$.

WANT TO SHOW: $\int_a^b f(x)dx = G(b) - G(a)$

Define $F(x) = \int_a^x f(t)dt$.

Since f is continuous, by FTC 1, $F' = f$.

Therefore, $F' = G'$, so $(F - G)' = 0$. Hence, $F - G$ is constant.

Therefore, there exists $C \in \mathbb{R}$ such that $F(x) = G(x) + C$ for all $x \in [a, b]$.

We now evaluate at $x = a$ to figure out the value of the constant:

$$0 = F(a) = G(a) + C.$$

Hence $C = -G(a)$.

Thus, for every $x \in [a, b]$, $F(x) = G(x) - G(a)$.

Therefore, $\int_a^b f(t)dt = F(b) = G(b) - G(a)$.

Others

Notations

- $\mathbb{Q}$ rational numbers, $\mathbb{R}$ real numbers, $\mathbb{Z}$ integers

Logarithms

- $e^{\ln a^b} = e^{b \ln a}$
- $a^b = e^{\ln a^b} = e^{b \ln a}$

Others

- $\forall \epsilon > 0, L < M + \epsilon \implies L \leq M$