Ox Y1 Core Micro - Demand

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2024 W1 Preference and Utility

Lecture 1: Preference, Utility, and Choice

Preference Symbols #flashcard

- Strict preference relation $\succ$ is defined by:

$$x \succ y \iff x \succeq y \text{ but not } y \succeq x$$

- Indifference relation $\sim$ is defined by:

$$x \sim y \iff x \succeq y \text{ and } y \succeq x$$

Rational Preference Relation #flashcard

Definition: Rational Preference Relation

The preference relation $\succsim$ is said to be *rational* if it is complete and transitive:

- *Completeness*: for all $x, y \in X$, we either have $x \succsim y$ or $y \succsim x$ (or both).
- *Transitivity*: for all $x, y, z \in X$, whenever we have $x \succsim y$ and $y \succsim z$ then we have $x \succsim z$.

If $\succsim$ is rational, we have the following restrictions on $\succ$ and $\sim$ (see Hw1):

1. $\succ$ is both *irreflexive* (i.e., $x \succ x$ cannot hold) and transitive.
2. $\sim$ is both *reflexive* (i.e., $x \sim x$ always holds) and transitive.
3. If $x \succ y$ and $y \succsim z$ then $x \succ z$.

Utility Function and Its Ordinal Property #flashcard

Definition: Utility Function

A function $u : X \to \mathbb{R}$ is a utility function representing the preference relation $\succsim$ if, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow u(x) \geq u(y)$$

Definition: Ordinal property

A property of a function is ordinal if it is preserved under any strictly increasing transformation of this function.

- **Claim**: If u represents $\succeq$ and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $v(x) = f(u(x))$ represents $\succeq$ as well.

  Proof: $x \succsim y \Leftrightarrow u(x) \geq u(y)$ (since u represents $\succeq$)
  $\Leftrightarrow f(u(x)) \geq f(u(y))$ (f strictly increasing)
  $\Leftrightarrow v(x) \geq v(y)$
- Thus, "preference representation function" would be a much more accurate and useful (albeit clunky) term than "utility function".

- Existence of a utility function $\implies$ Rationality
- Rationality does not necessarily implies the existence of a utility function e.g. lexicographic preference
- From lect 2 slides:
  -

Theorem: Debreu's Theorem

If a rational preference relation $\succeq$ is continuous on $\mathbb{R}^L_+$, then it admits a utility representation $u : \mathbb{R}^L_+ \to \mathbb{R}$. Even more, it admits a *continuous* utility representation.

Weak Axiom of Revealed Preference (WARP) #flashcard

Definition: WARP

A choice structure $(\mathcal{B}, C(\cdot))$ satisfies WARP if whenever $x \in C(A)$ for some $A \in \mathcal{B}$ and $y \in C(B)$ for some $B \in \mathcal{B}$ such that $x, y \in A \cap B$, we also have $x \in C(B)$.

- In words, if x is ever chosen while y was available, there cannot be any choice set where both are available and y is chosen but x is not.

Another way of stating WARP is by defining a binary relation $\succeq^*$ from observed choice behavior $C(\cdot)$:

Definition: Revealed Preference Relation

$x \succeq^* y \iff$ there is some $B \in \mathcal{B}$ such that $x, y \in B$ and $x \in C(B)$.

- $x \succeq^* y$ can be read as "x revealed at least as good as y"
- Note that $\succeq^*$ does not have to complete or transitive.
- We could also read $x \in C(B)$ and $y \notin C(B)$ for $x, y \in B$ as "x revealed preferred to y"
- Restatement of WARP: "If x is revealed at least as good as y, then y cannot be revealed preferred to x".

WARP and Rationality #flashcard

- Rationality $\implies$ WARP
- WARP + the set of choice experiments $\mathbb{B}$ contains all subset of $\mathbb{X}$ up to 3 elements

Lecture 2: Consumer Choice

NOT YET ARCHIVED

Relation between Preference/Utility Function on Convexity

If $\succeq$ is representable by some $u(.)$, then convexity of preference $\succeq \iff$ quasi-concavity of $u(.)$

2024 W2 Classical Demand Theory

Lecture 1 (Monday)

Goods Classification

Goods Classification #flashcard

- By income effect on Marshalian demand / income elasticity:

$$\begin{cases} \frac{\partial x_l}{\partial w} > 0 & \iff \epsilon_{lw} > 0 & \iff \text{Normal Good} \\ \frac{\partial x_l}{\partial w} < 0 & \iff \epsilon_{lw} < 0 & \iff \text{Inferior Good} \end{cases}$$

- By income effect on budget share / income elasticity:

$$\begin{cases} \frac{\partial b_l}{\partial w} > 0 & \iff \epsilon_{lw} > 1 & \iff \text{Luxury Good} \in \text{Normal Good} \\ \frac{\partial b_l}{\partial w} < 0 & \iff \epsilon_{lw} < 1 & \iff \text{Necessity} \supset \text{Inferior Goods} \end{cases}$$

- and the good's budget share increases with its price if its own price elasticity > -1:

$$\frac{\partial b_l}{\partial p_l} > 0 \iff \epsilon_{ll} > -1$$

- By own price effect:

$$\frac{\partial x_l}{\partial p_l} > 0 \iff \text{Giffen Good} \in \text{Inferior Good}$$

- Giffen goods have to be strongly inferior good as implied by Slutsky Equation

Goods Classification COPY #flashcard

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Slutsky Substitution Matrix and Symmetry #flashcard

- For L goods, we can define an $L \times L$ Slutsky substitution matrix:

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & s_{12}(p,w) & \dots & s_{1L}(p,w) \\ s_{21}(p,w) & s_{22}(p,w) & \dots & s_{2L}(p,w) \\ \dots & \dots & \dots & \dots \\ s_{L1}(p,w) & s_{L2}(p,w) & \dots & s_{\mathcal{L}}(p,w) \end{bmatrix}$$

  where $s_{lk}(p,w) = \frac{\partial h_l(p,\bar{u})}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$ can be observed from market demands
- This matrix is symmetric due to symmetry:

$$\underbrace{\frac{\partial h_l(p,w)}{\partial p_k}}_{s_{lk}(p,w)} = \underbrace{\frac{\partial h_k(p,w)}{\partial p_l}}_{s_{kl}(p,w)}$$

- Violation of this property can be used to reveal failures of rationality in the traditional sense

Lecture 2 (Wednesday)

Money-metric Measures of Welfare (CV/EV)

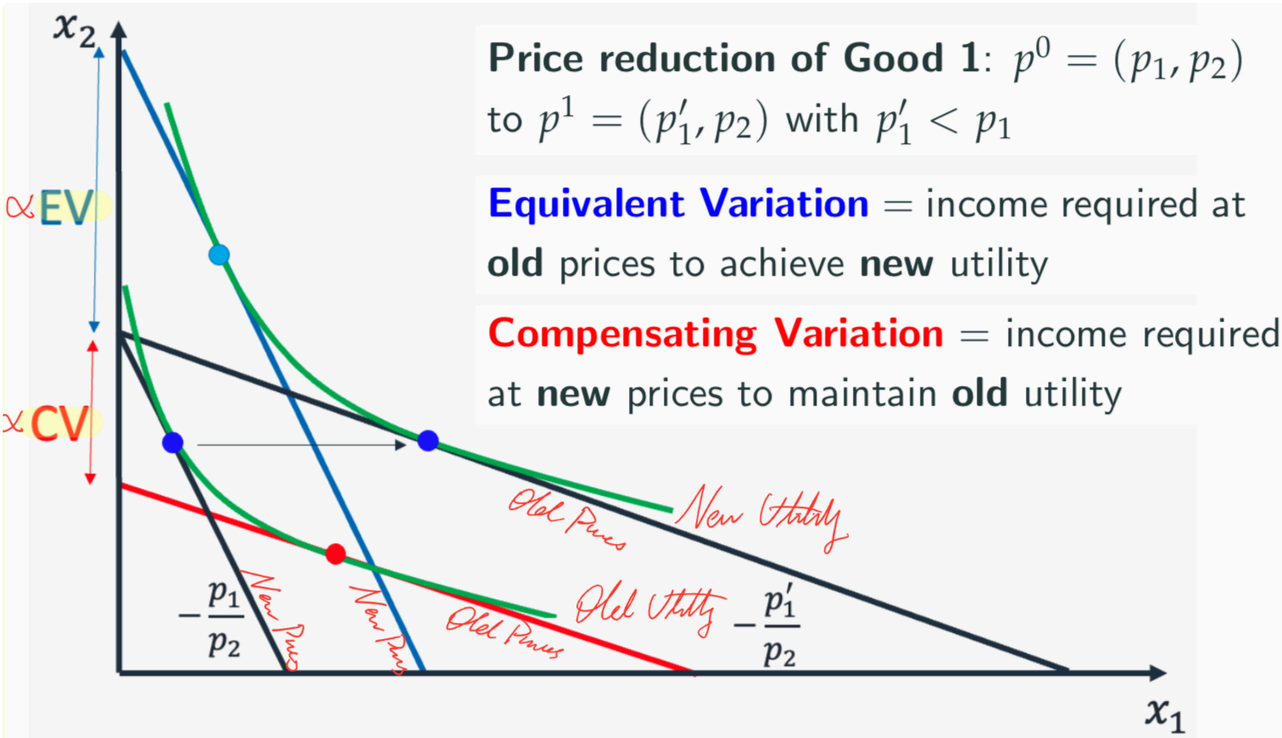
CV and EV #flashcard

- Key: w is unchanged, but u can change $u^0 \to u^1$
  Compensating Variation (CV): income required to *maintain old utility at new prices*:

$$CV(p^0,p^1,w) = \underbrace{e(p^1,u^1) \text{ or } e(p^0,u^0)}_{w} - e(p^1,u^0) = w - e(p^1,u^0)$$

  Equivalent Variation (EV): income required to *achieve new utility at old prices*:

$$EV(p^0,p^1,w) = e(p^0,u^1) - \underbrace{e(p^1,u^1) \text{ or } e(p^0,u^0)}_{w} = e(p^0,u^1) - w$$



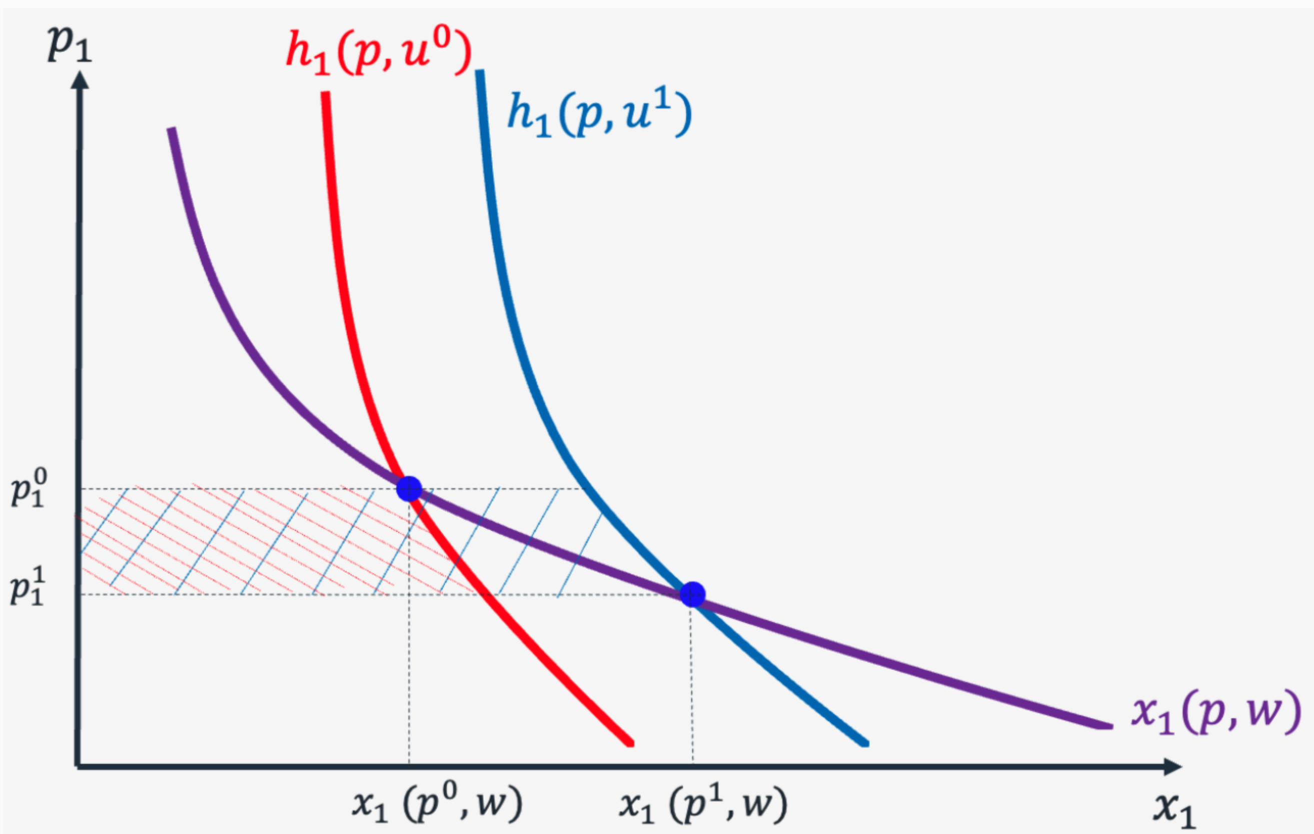
EV, CV aren't ***directly*** *observable, but recoverable.*

Interpretation of CV/EV as WTP/WTA #flashcard

depends on whether the policy change is positive or negative ($u^1 > u^0$ or $u^1 < u^0$):

	Compensating Variation	Equivalent Variation
+ Decrease in price of normal good $(u^1 > u^0)$	WTP (CV > 0) Willingness to Pay	WTA (EV > 0) Willingness to Accept
− Increase in price of normal good $(u^1 < u^0)$	WTA (CV < 0)	WTP (EV < 0)
	$v(p^1, w - CV) = v(p^0, w) = u^0$	$v(p^0, w + EV) = v(p^1, w) = u^1$
	Money transfer **after the policy change** that would leave the agent *just as well off as before.*	Money transfer that, **without the policy change occurring**, would leave the agent *just as well off as if it occurred.*

If there's only price change in one good ($p_1^0 \to p_1^1$), then CV/EV corresponds to area under corresponding Hicksian demand curves:



△ Area under old h △ Area under new h

$$\text{CV} = \int_{p_1^1}^{p_1^0} h_1(p, u^0)dp_1 < \int_{p_1^1}^{p_1^0} h_1(p, u^1)dp_1 = \text{EV}$$

Due to the existence of income effects, for normal goods:

$$CV \le \Delta CS \le EV$$

Unit-free Measure of Welfare (Consumer Indices)

Konus/Laspeyres/Paasche Cost-of-living Indices #flashcard

Consider a price change: $p^0 \to p^1$

- **Konus CoL Index** compare costs to achieve a fixed level of utility:

$$P_K(p^0, p^1, u) = \frac{e(p^1, u)}{e(p^0, u)}$$

- **Laspeyres CoL Index** compares costs to maintain the old utility:

$$P_L(p^0, p^1, u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)} = \frac{e(p^1, u^0)}{w^0}$$

- **Paasche CoL Index** compares consts to maintain new utility:

$$P_P(p^0, p^1, u^1) = \frac{e(p^1, u^1)}{e(p^0, u^1)} = \frac{w^1}{e(p^0, u^1)}$$

- If preferences are homothetic, then Paasche and Laspeyres indices provide 2-side bounds on the unique Konus index:

$$\frac{p^1 x^1}{p^0 x^1} \leq \frac{e(p^1, u)}{e(p^2, u)} \leq \frac{p^1 x^0}{p^0 x^0}$$

Quantity Indices #flashcard

No detailed discussion this year, so adapted from the last year:

- ① Allen Quantity indices/real income measures are also index numbers but these hold the prices fixed and compare the indifference curves (different utility)

$$\boxed{Q(p, u^0, u^1) = \frac{e(p, u^1)}{e(p, u^0)}}$$

- If we choose either p^0 or p^1 as the reference price the idea is identical to an EV/CV but expressed in ratios instead of a cash amount

- For example the ② Laspyeres and ③ Paasche quantity indices provide approximations/bounds:

True QIs   Approximate QIs

$$Q_L(u^0, u^1, p^0) \equiv \frac{e(p^0, u^1)}{e(p^0, u^0)} = \frac{e(p^0, u^1)}{w^0} \leq \frac{p^0 \cdot x^1}{p^0 \cdot x^0} \equiv Q_L(x^0, x^1, p^0)$$

$$Q_P(u^0, u^1, p^1) \equiv \frac{e(p^1, u^1)}{e(p^1, u^0)} = \frac{w^1}{e(p^1, u^0)} \geq \frac{p^1 \cdot x^1}{p^1 \cdot x^0} \equiv Q_P(x^0, x^1, p^1)$$

- The quality of these approximations depends on the same substitutability ideas etc.

Aggregation

Aggregation of Rational Individuals #flashcard

Assume all individuals are rational in an economy.
Generally, the AD will depend on prices and the entire distribution of wealth.
Exceptions: when utility functions take form of:

$$v_i(p, w_i) = a_i(p) + b(p)w_i$$

i.e. utility functions take the Gorman-Polar form (including homothetic and quasi-linear ones)

2024 W3 Risks

Lecture 1: Decisions under Risk

The Independence Axiom #flashcard

Definition: Independence

A preference relation $\succsim$ on the space of simple lotteries $\mathcal{L}$ satisfies independence if for all lotteries L, L', $L'' \in \mathcal{L}$ and $0 < \alpha < 1$,

$$L \succsim L' \Leftrightarrow \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

- In words: the agent's preferences over two lotteries (L and L') completely determine which they'd prefer to have as part of a compound lottery, regardless of the other possible outcome of this compound lottery L''

The Independence COPY #flashcard

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Expected Utility Theorem (EUT) #flashcard

Theorem: Expected Utility Theorem

Suppose preferences $\succsim$ over lotteries are rational, continuous and satisfy the Independence Axiom. Then there exists a utility representation $U : \mathcal{L} \to \mathbb{R}$ for $\succsim$ which takes the expected utility form:

$$L \succsim L' \Leftrightarrow \sum_{n=1}^{N} p_n u_n \geq \sum_{n=1}^{N} p'_n u_n$$

in which $U(L^n) = u_n$ and L^n yields z_n with probability 1.

Expected Utility Theorem (EUT) COPY #flashcard

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Creating Utility Functions Using EUT #flashcard

- Find the best and worst outcome
- For each choice, find the probability p such that the individual is indifferent between the option and the mixed lottery of $p \times \text{best} + (1-p) \times \text{worst}$
- Define that probability as its utility
- Note: since utility has cardinal meanings here, the utility function is unique up to positive affine transformations.

Lecture 2: Risk Aversion

Certainty Equivalent #flashcard

CE is defined as:

$$u(CE) = \mathbb{E}\left[u(x)\right] = \int u(x) f(x)\, dx = \int u(x)\, dF(x)$$

i.e. the sure amount the consumer is willing to get in exchange of the lottery

Risk Premium #flashcard

PR is defined as:

$$u\left(\mathbb{E}\left[x\right] - RP\right) = \mathbb{E}\left[u(x)\right]$$

and

$$\mathbb{E}[x] = CE + RP$$

i.e. the amount which the agent is willing to give up in order to eliminate the risk

Absolute Risk Aversion and Relative Risk Aversion #flashcard

ARA:

$$ARA = -\frac{u''(x)}{u'(x)} > 0 \text{ if risk averse}$$

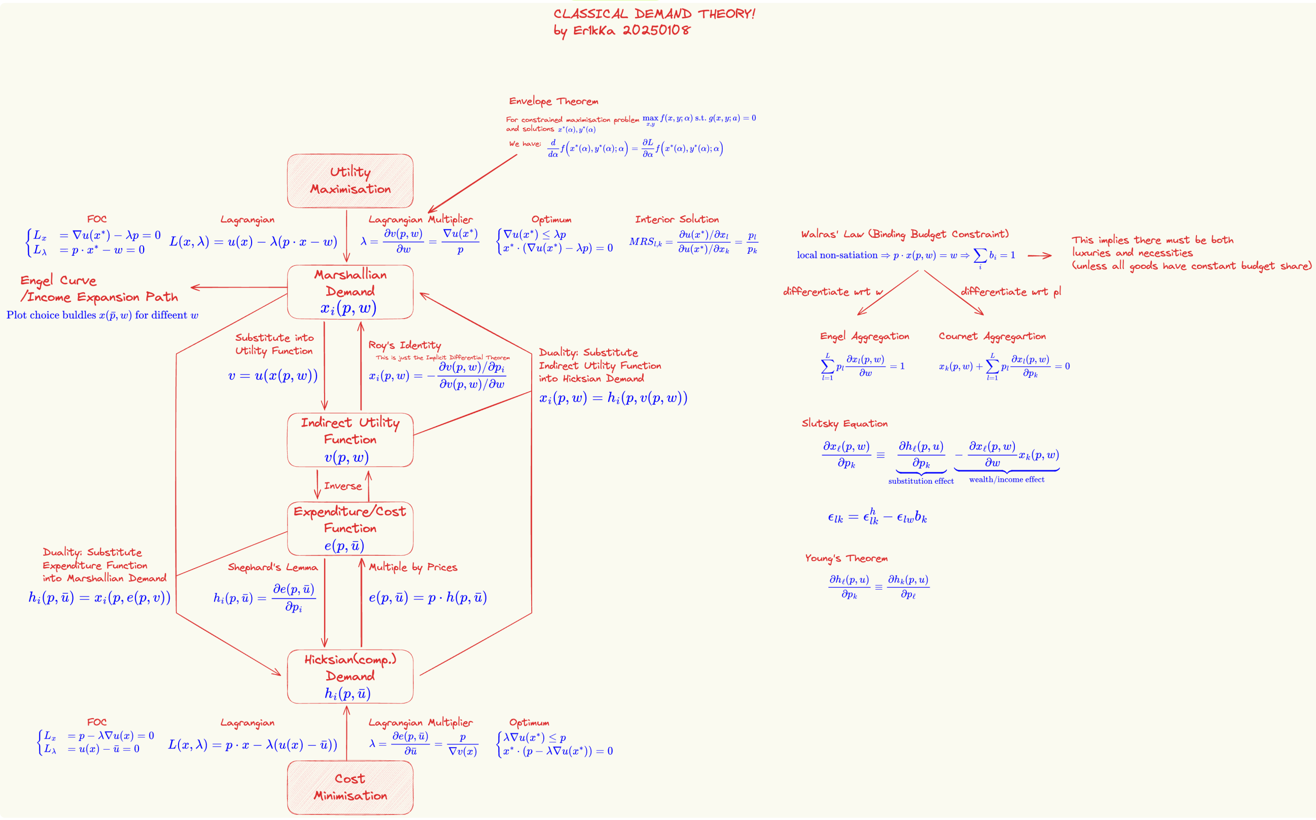
- Constant ARA $\iff$ Exponential utility ($u(x) = \frac{1-e^{-rx}}{r}$ where r is ARA) $\iff$ Constant RP $\forall$ initial wealth
  RRA:

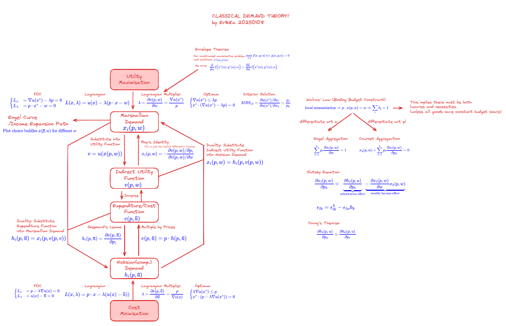
$$-x\frac{u''(x)}{u'(x)} > 0 \text{ if risk averse}$$

- Constant RRA $\iff$ $u(x) = \frac{x^{1-r}-1}{1-r}$ and log utility $\ln(x)$ is a special case of $r \to 1$

2023 MT W5-6 Demand

Demand System: All Equations (Update 2025-04-05) #flashcard





Common Utility Functions and Their Properties #flashcard

FILL THIS LATER!!!!!! #notes/tbd

W5L1 Preference and Utility

Preference Relation

Rationality #flashcard

The preference relation $\succsim$ is said to be *rational* if it is complete and transitive:

Rational
①
- *Completeness*: for all $x, y \in X$ we either have $x \succsim y$ or $y \succsim x$ (or both).

②
- *Transitivity*: for all $x, y, z \in X$, whenever we have $x \succsim y$ and $y \succsim z$ then we have $x \succsim z$.

Utility

Utility Function #flashcard

choice set

- A function $u : X \to \mathbb{R}$ is a utility function representing the preference relation $\succsim$ if, for all $x, y \in X$, $x \succsim y \Leftrightarrow u(x) \geq u(y)$

$\Updownarrow$

- $u(x)$ assigns scalar numerical value to each element $x \in X$ such that $x \succsim y \Leftrightarrow u(x) \geq u(y)$
- u is not unique, eg, $\hat{u}(x) \equiv u(x) + 1$ works just as well, i.e., the utility function is "ordinal" not "cardinal"

Existence of Utility Function $\implies$ Rationality (Completeness + Transitivity)
but the Reverse is NOT True: Lexographic preference is rational but there's no corresponding U(.)

Consumer Choice

Walrasian Budget Set and Budget Hyperplane
Budget Set:

$$B_{p,w} = \{s \in \mathbb{R}^{+} \; s.t. \; p \cdot x \leq w\}$$

this is a convex set
Budget Hyperplane:

$$\{x \in \mathbb{R}_{+}^{L} \; s.t. \; p \cdot = w\}$$

Demand Functions

Basic Properties of Walrasian/Marshallian Demand Function #flashcard

- Homogeneous of degree 0
- Satisfies Walras's Law (non-satiation):

$$p \cdot x(p, w) = w$$

Engel Curve / Income Expansion Path

- For fixed prices p, the bundle demanded as a function of wealth w, $x(p,w)$, is the consumer's *Engel curve* (or *wealth/income expansion path*)

Price/Income Elasticities #flashcard

$$\boxed{\varepsilon_{\ell k} = \frac{\partial x_\ell(p,w)}{\partial p_k}\frac{p_k}{x_\ell(p,w)} \ ; \ \varepsilon_{\ell w} = \frac{\partial x_\ell(p,w)}{\partial w}\frac{w}{x_\ell(p,w)}}$$

price elasticity of good ℓ wrt. p income/wealth elasticity of good ℓ

Cournet Aggregation and Engel Aggregation

- Warlas Law, differentiated wrt. $p_k \implies$ Cournet Aggregation:

$$x_k(p,w) + \sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} = 0$$

- Warlas Law, differentiated wrt. $w \implies$ Engel Aggregation:

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1$$

W5L2 Utility Maximisation

Preference Relations

Indifference Curve, Upper Contour Set, Lower Contour Set

- the *indifference curve* for x is $\{y \in X \text{ such that } y \sim x\}$
- the *upper contour set* for x is $\{y \in X \text{ such that } y \succsim x\}$
- the *lower contour set* for x is $\{y \in X \text{ such that } x \succsim y\}$

(Strict) Convexity of Preference

- Preferences $\succsim$ are *convex* if the upper contour sets are convex, i.e., if $y, z \succsim x$ then $\alpha y + (1-\alpha)z \succsim x$ for any $0 \le \alpha \le 1$

Any combinations of 2 bundles in the upper contour set will be in the upper contour set.

- Preferences are *strictly convex* if $y, z \succsim x$ and $y \neq z$ implies that $\alpha y + (1-\alpha)z \succ x$ for $0 < \alpha < 1$

(Strict) Monotonicity of Preferences

- $\succsim$ is *strongly monotonic* if y and x are 2 bundles such that $y \neq x$ and $y_\ell \geq x_\ell$ for each ℓ then $y \succ x$. Means all goods are "good" (not bad) and indifference curves cannot be "thick".

- $\succsim$ is *weakly monotonic* if y and x are two bundles such that if $y_\ell \geq x_\ell$ for each ℓ then $y \succsim x$. This allows indifference curves to be "thick". there could be a indifferent set

Monotonicity corresponding to increasingness of the utility function, downward sloping indifference curves, and negative MRS.

Local Non-Satiation

- $\succsim$ is *strongly monotonic* if y and x are 2 bundles such that $y \neq x$ and $y_\ell \geq x_\ell$ for each ℓ then $y \succ x$. Means all goods are "good" (not bad) and indifference curves cannot be "thick".

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Continuity of Preference

- A final–somewhat technical–assumption usually made is that preferences are *continuous*, by which we mean that the upper and lower contour sets are *closed* sets.
- This means that if y_n is a sequence of bundles which converges to bundle y, and each $y_n \succsim x$, then the limit point y also satisfies $y \succsim x$. we can not escape from the upper / lower contour in the limit
- Another way to say this is that if $y \succ x$, then all bundles sufficiently close to y are also strictly preferred to x.
- The usual example of preferences which are not continuous (but strongly monotonic) is the *lexicographic* ordering: there are two goods, and $x \succsim y$ if either $x_1 > y_1$ or $\{x_1 = y_1 \text{ and } x_2 \geq y_2\}$.
- In practice it's hard to work with "$\succsim$" preferences; it's much easier to use a utility function which represents these preferences.

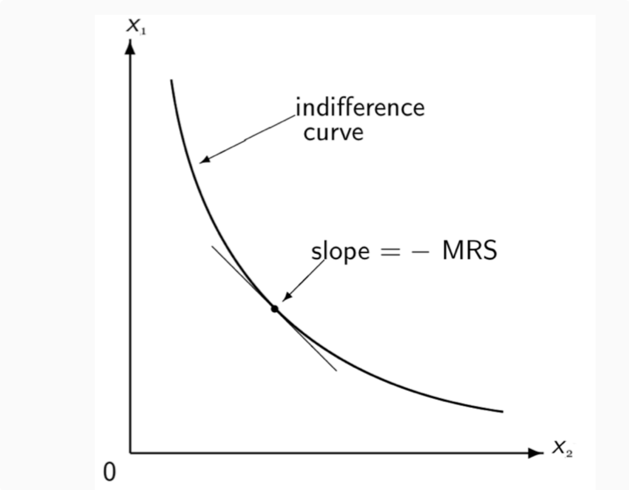
Utility Functions

Existence of Utility Function #flashcard
If a preference is:

- Rational (complete + transitive)
- Strongly monotonic
- Continuous
  Then there exists a continuous function u s.t. $u(y) \geq u(x) \iff y \succsim x$

- A utility function is a representation of preferences which assigns real numbers to bundles: if $y \succsim x$ then y is assigned a weakly higher number than x.
- Utility functions are not unique since if u represents certain preferences then any positive montonic transformation $\varphi(u)$ also represents the same preferences. We say that utility functions are "ordinal".

Marginal Rate of Substitution (MRS) #flashcard
MRS between good 1 and good 2 is defined as the quantity of good 2 willing to give up in exchange of one unit of good 1, keeping the utility level unchanged.

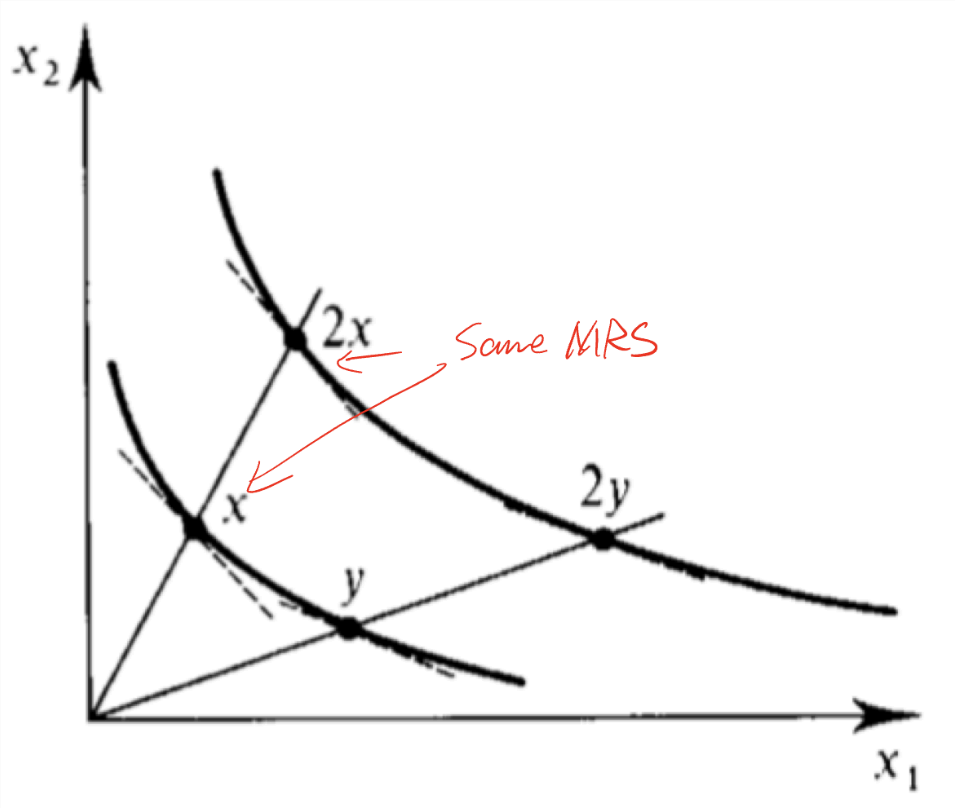


$$MRS_{1,2} = \left.\frac{dx_2}{dx_1}\right|_u = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$

Homothetic Preferences #flashcard

- *Homothetic preferences*: if $x \sim y$ then $kx \sim ky$ for any positive scaling factor k
- Graphically: higher indifference curves are magnified versions of lower ones from the origin.

  MRS will be the same on a straight line from the origin.
- Not a restriction on shape of any one indifference curve but on relationship between indifference curves within an indifference map
- In words: if you've seen one indifference curve, you've seen them all.



MWG Figure 3.B.5: Example of Homothetic Preferences

- Utility has "constant returns to scale"
- MRS is unaffected by scaling factor k. In other words, if preferences are homothetic then MRS's are constant along rays through the origin.
- This is only true for homothetic preferences and is usually an easy way to check whether given preferences are homothetic
- Implies constant budget shares and income elasticities equal to one

W6L1:

- If the preferences are homothetic then the expenditure function and the indirect utility functions take the linear forms

Homothetic Preference $\Rightarrow$

$$\begin{cases} e(p,u) = b(p)u \\ v(p,w) = \dfrac{1}{b(p)}w \end{cases}$$

Inversion

where $b(p)$ is a h.o.d.1 function of prices

- By Roy's Identity (discussed below) this then implies that the Marshallian demands take the separable form

$$x_\ell(p,w) = \beta_\ell(p)w$$

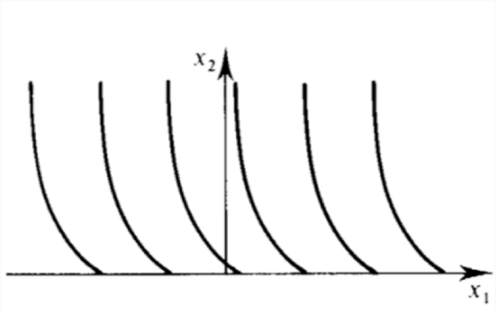
where $\beta_\ell(p) = \frac{1}{b(p)}\partial b(p)/\partial p_\ell$

- Engel curves/income expansion paths are thus linear rays from the origin

Quasi-Linear Preference

- *Quasi-linear preferences*: ① one product (e.g., good 1) can be consumed in positive or negative quantities, so $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$, and ② if $x \sim y$ then for any k we have $(x_1 + k, x_2, ..., x_L) \sim (y_1 + k, y_2, ..., y_L)$.
- In words: adding the same amount to one particular good preserves indifference. (between other goods)
- Utility functions which represent these preferences take the form $x_1 + u(x_2, ..., x_L)$.
- The "linear" good is often interpreted as "money". It is sometimes called the "outside good".
- It is often convenient to choose it as the numéraire.

- In two-goods case, each indifference curve is just a "sideways" translation of any other, and MRS is only a function of x_2
- All income effects are concentrated on x_1 (the other goods have no income effects)
- Provides a foundation for partial equilibrium analysis



MWG Figure 3.B.6: Quasilinear Preferences

W6L1:

Special case of Gorman Polar form

With quasi-linear preferences, consider for simplicity the case of two goods, and utility is $y + u(x)$ where $u(\cdot)$ is a strictly concave function and y is the numéraire ("money"). The consumer maximizes $y + u(x)$ subject to $y + px \leq w$, so x chosen to maximize $u(x) - px$.

- The Marshallian demands are

Quasi-linear Preferences =>

$$\left\{ \begin{aligned} x(p,w) &= x(p) \\ y(p,w) &= w - px(p) \end{aligned} \right.$$

no wealth effect on non-numeraire goods

- Indirect utility and expenditure functions are special cases of the Gorman Polar form:

$$v(p,w) = w + v(p)$$
$$e(p,u) = u - v(p)$$

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Perfect Substitute, Perfect Complement, Cobb-Douglas Preferences

1. Perfect substitutes $u(x_1, x_2) = ax_1 + bx_2$: The MRS is $-a/b$ and is constant. Indifference curves are parallel straight lines. These are the only preferences which are homothetic and quasilinear.
2. Perfect complements $u(x_1, x_2) = min[ax_1, bx_2]$: Indifference curves are L-shaped with the kinks lying on a ray through the origin of slope a/b. These preferences are homothetic but not quasilinear.
3. Cobb-Douglas: $u(x_1, x_2) = alnx_1 + blnx_2$: Preferences are homothetic, indifference curves are smooth and MRS ax_2/bx_1 is diminishing.

Separability of Preferences

- Partition the bundle of goods into two sub-bundles so that the consumption bundle is (x, z)
- Partition the price vector analogously into (p_x, p_z)
- Preferences are said to be separable in the $x-$goods if preferences have the property that

$$\boxed{(x, z) \succeq (x', z) \text{ if and only if } (x, z'') \succeq (x', z'')}$$

for all x, x', z and z''.

- In words: if x is preferred to x' given some quantities of the other goods, then x is preferred to x' for any quantities of the other goods; preferences for the $x-$goods are independent of the $z-$goods.

- If this holds then the utility function can be written as

$$\boxed{u(v(x), z)}$$

where $v(x)$ is a "sub-utility function" and $u(v, z)$ is an increasing function of v

- Overall utility can be written as a function of the sub-utility of x and the level of consumption of the $z-$goods
- What this means is that MRS's within the separable $x-$goods are independent of the $z-$goods

- The structure $u(v(x), z)$ is said to be "weakly separable" in the $x-$goods
- The general structure $u(v(x), w(z))$ is said to be "weakly separable" separable in all types of goods
- The structure $u(x, z) = v(x) + w(z)$ is said to be "additively separable"

Utility Maximisation

(see Ox Micro Demand System Illustration.excalidraw)

CES Preferences

- *CES* Preferences:

$$\boxed{U(x) = \left[\sum_{\ell}^{L} \beta_\ell x_\ell^\theta\right]^{1/\theta}}$$

- The only preferences that are *both* additively separable and homothetic

- First-order condition $\Rightarrow$ Frisch demands: $x_\ell = \left(\frac{\lambda p_\ell}{\theta \beta_\ell}\right)^{\frac{1}{\theta - 1}}$

(no need to derive)

W6L1 Classical Demand Theory Part 1

Expenditure Minimisation

(see Ox Micro Demand System Illustration.excalidraw)

Compensated Law of Demand

$$(p' - p) \cdot (h(p', u) - h(p, u)) \leq 0$$

Price and Hicksian demand must move in the opposite directions.

Gorman Polar Form

- The expenditure and indirect utility functions are affine (have an intercept)

Gorman-Polar Form =>

$$\begin{cases} e(p,u) = a(p) + b(p)u \\ v(p,w) = \dfrac{w - a(p)}{b(p)} \end{cases}$$

Inversion

where both $a(p)$ and $b(p)$ are h.o.d.1

- By Roy's Identity this then implies that the Marshallian demands take the separable form

$$x(p,w) = \alpha_l(p) + \beta_l(p)w$$

where $\alpha_\ell(p) = \partial a(p)/\partial p_\ell - \beta_\ell(p)a(p)$ and $\beta_\ell(p) = \frac{1}{b(p)}\partial b(p)/\partial p_\ell$

- Engel curves/income expansion paths are linear but don't necessarily pass through the origin

Slutsky Equation #flashcard

$$\frac{\partial x_\ell(p,w)}{\partial p_k} \equiv \underbrace{\frac{\partial h_\ell(p,u)}{\partial p_k}}_{\text{substitution effect } <0} \underbrace{- \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)}_{\text{wealth/income effect}}$$

- The *substitution effect* captures how demand for good ℓ changes as the price for good k rises, staying on the same indifference curve u (must be negative)

- The *wealth/income effect* takes account of how the consumer's wealth changes when p_k rises; when p_k rises by dp_k, she spends an extra amount $x_k(p,w) \times dp_k$ on that good and her wealth falls correspondingly and this affects x_ℓ via $\partial x_\ell/\partial w$

$$\frac{\partial x_\ell(p,w)}{\partial p_k} \equiv \underbrace{\frac{\partial h_\ell(p,u)}{\partial p_k}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)}_{\text{wealth/income effect } (<0 \text{ if normal})}$$

- The Slutsky equation implies that $\partial x_\ell / \partial p_k$ is *below* $\partial h_\ell / \partial p_k$ if good ℓ is a *normal* good and *above* $\partial h_\ell / \partial p_k$ if good ℓ is an *inferior* good

  ℓ is $\begin{cases} \text{Normal: } \frac{\partial x_\ell}{\partial p_k} < \frac{\partial h_\ell}{\partial p_k} \\ \text{Inferior: } \frac{\partial x_\ell}{\partial p_k} > \frac{\partial h_\ell}{\partial p_k} \end{cases}$

- Since the own-price effect is always negative for Hicksian demands, we deduce that a Giffen good must be a (strongly) inferior good Large wealth effect

Hicksian Substitutes and Complements

- Since Hicksian demand $h(p,u)$ is the derivative of the concave function $e(p,u)$ we deduce two things:

1. Since the matrix of second derivatives of $e(p,u)$ is symmetric, we have:

$$\boxed{\frac{\partial h_\ell(p,u)}{\partial p_k} \equiv \frac{\partial h_k(p,u)}{\partial p_\ell}}$$

   This is essentially Young's Theorem (the order of partial differentiation doesn't matter). In words: effect of price change for product k on compensated demand for product ℓ is the same as price change for ℓ on product k.

   Concavity of $e(p,u) \Rightarrow \nabla^2 e(p,u)$ is NSD

2. The matrix of (price) derivatives of $h(p,u)$ is negative semi-definite; in particular, the own-price effect $\partial h_\ell / \partial p_\ell$ is negative.

- Two distinct products ℓ and k are:
    - *Hicksian Substitutes* if $\partial h_\ell(p,u)/\partial p_k \geq 0$
    - *Hicksian Complements* if $\partial h_\ell(p,u)/\partial p_k \leq 0$

- Compensated cross-price effects are symmetric:

$$\boxed{\partial h_\ell(p,u)/\partial p_k = \partial h_k(p,u)/\partial p_\ell}$$

Welfare

Basic Ideas of Comparing Welfare

1. the distance between tangent lines (based on the expenditure function and expressed in cash amounts)

2. the radial distance (based on the distance function and unit-free)

- Money difference: EV/CV
- Unit-free ratios: Cost-of-living indices, Quantity Indices

Equivalent Variation and Compensating Variation

1. Equivalent variation:

Δ minimal spending needed to reach the new and old utility levels given old prices

$$\boxed{EV = e(p^0, \overbrace{v(p^1,w^1)}^{\text{new max utility}}) - \overbrace{e(p^0, \overbrace{v(p^0,w^0)}^{\text{old max utility}})}^{= w_0}}$$

original prices

2. Compensating variation:

Δ minimal spending needed to reach the new and old utility levels given new prices

$$\boxed{CV = \underbrace{e(p^1, \overbrace{v(p^1,w^1)}^{\text{new max utility}})}_{= w_1} - e(p^1, \overbrace{v(p^0,w^0)}^{\text{old max utility}})}$$

new prices

Given that the consumer is optimising, these are equivalent to:

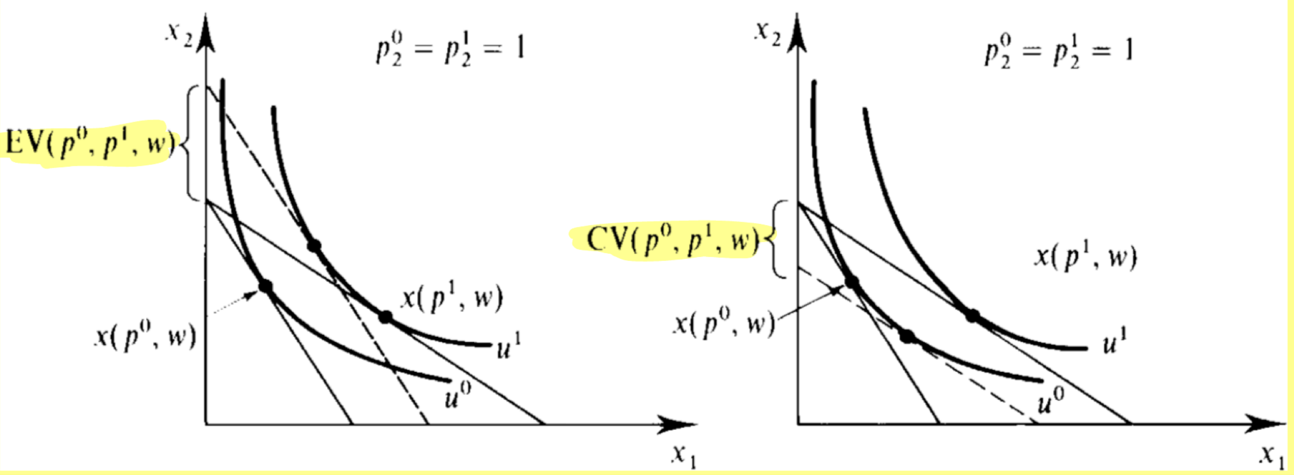
1. Equivalent variation:

$$EV = e(p^0, v(p^1, w^1)) - w^0$$

2. Compensating variation:

$$CV = w^1 - e(p^1, v(p^0, w^0))$$

If there's only price change but no wealth change, here's a graphical illustration:



Special Case when Only Change Taking Place is to a Single Good

NOT YET FIGURED OUT

P41-43 in SLIDES

W6L2 Classical Demand Theory Part 2

Welfare Economics (cont.)

Konus, Laspeyres, and Paasche Cost-of-living Indices

- **Kionus (True) Cost-of-living Index**
  - (Konüs) Cost-of-living indices are ratios of expenditure functions:

$$\boxed{P(p, p', u) = \frac{e(p', u)}{e(p, u)}}$$

(u to be defined)

  - Compare prices on a fixed indifference curve

(indeed cost)

- **Laspeyres and Paasche Cost-of-living Indices**

- These give us the Laspeyres and the Paasche cost-of-living indices:

Laspeyres: (original utility)

$$P_L(p^0, p^1, u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)} = \frac{e(p^1, u^0)}{w^0}$$

Paasche: (new utility)

$$P_P(p^0, p^1, u^1) = \frac{e(p^1, u^1)}{e(p^0, u^1)} = \frac{w^1}{e(p^0, u^1)}$$

Approximated Laspeyres and Paasche Cost-of-living Indices

- Approximated Laspeyres CoL Index overestimates the true Laspeyres CoL Index
  - Often we may only have immediate access to prices and demands before and after some change:

$$\{p^0, x^0\} \rightarrow \{p^1, x^1\}$$

  - These can be used to form approximate welfare measures
  - For example the Laspeyres price index approximates the corresponding cost-of-living index from above: (overstate)

$$\underbrace{P_L\left(p^0, p^1, x^0\right) = \frac{p^1 \cdot x^0}{p^0 \cdot x^0}}_{\text{Approximated Laspeyres PI}} \geq \frac{e(p^1, u^0)}{e(p^0, u^0)} = \underbrace{P_L\left(p^0, p^1, u^0\right)}_{\text{True Laspeyres PI}}$$

(ignores reoptimisation)

- Approximated Paasche CoL Index underestimates the true Paasche CoL Index
  - There is a similar result for the Paasche price index

same

$$\frac{p^1 \cdot x^1}{p^0 \cdot x^1} \leq \frac{e(p^1, u^1)}{e(p^0, u^1)}$$

$\geq$

  - The Paasche index understates the corresponding true change in the cost-of-living

CoL Indices when Preference is Homothetic

- In general a cost-of-living index depends on the reference indifference curve (u)

- An important exception occurs when preferences are homothetic, in which case the expenditure function is linear

$$e(p,u) = b(p)u$$

- Hence the cost-of-living index is independent of the reference u:

$$\frac{e(p^1,u)}{e(p^0,u)} = \frac{b(p^1)}{b(p^0)}$$

- Reason: under homotheticity indifference curves are radial expansions of each other ("if you've seen one ...")

$\Rightarrow$ True Paasche/Laspeyres are the same $\forall u$

  - If preferences are homothetic, the Paasche and Laspeyres indices give two-sided bounds on the unique index

$$\underbrace{\frac{p^1 \cdot x^1}{p^0 \cdot x^1}}_{\text{Approx. Paasche Index}} \leq \underbrace{\frac{e(p^1,u)}{e(p^0,u)}}_{\text{True K/P/L CoL Index}} \leq \underbrace{\frac{p^1 \cdot x^0}{p^0 \cdot x^0}}_{\text{Approx. Laspeyres Index}}$$

Quality of Approximation of QoL Indices

- If prices don't vary much between periods 0 and 1, or if prices move proportionally, or if there is little substitution, then the substitution effects will tend toward zero and the approximation will be better

- If there are no substitution effects then the Paasche/Laspeyres index numbers will be "exact"

Exact and Superlative Indices

NOT YET FIGURED OUT

SEE SLIDES P11-15

Allen, Laspeyres, and Paasche Quantity Indices (with Approximation)

- (1) Allen Quantity indices/real income measures are also index numbers but these hold the prices fixed and compare the indifference curves (different utility)

$$\boxed{Q(p,u^0,u^1) = \frac{e(p,u^1)}{e(p,u^0)}}$$

- If we choose either p^0 or p^1 as the reference price the idea is identical to an EV/CV but expressed in ratios instead of a cash amount

- For example the (2) Laspyeres and (3) Paasche quantity indices provide approximations/bounds:

True QIs   Approximate QIs

$$\boxed{\begin{aligned} Q_L(u^0,u^1,p^0) &\equiv \frac{e(p^0,u^1)}{e(p^0,u^0)} = \frac{e(p^0,u^1)}{w^0} \leq \frac{p^0\cdot x^1}{p^0\cdot x^0} \equiv Q_L(x^0,x^1,p^0) \\ Q_P(u^0,u^1,p^1) &\equiv \frac{e(p^1,u^1)}{e(p^1,u^0)} = \frac{w^1}{e(p^1,u^0)} \geq \frac{p^1\cdot x^1}{p^1\cdot x^0} \equiv Q_P(x^0,x^1,p^1) \end{aligned}}$$

- The quality of these approximations depends on the same substitutability ideas etc.

Laspyeres Index uses original standards, Paasche Index uses new standards.

Price/Quantity Index Pairs and Implicit Quantity Index

- A general definition of a price and quantity index is that they are a pair of functions which satisfy the functional equation

$$\boxed{P\left(p^0, p^1, x^0, x^1\right) Q\left(p^0, p^1, x^0, x^1\right) = \frac{w^1}{w^0}}$$

- Therefore if you know, for example, the price index you can work out the implicit quantity index:

$$\boxed{Q\left(p^0, p^1, x^0, x^1\right) = \frac{w^1}{w^0} \frac{1}{P\left(p^0, p^1, x^0, x^1\right)}}$$

P, Q indices appear in pairs

- If we apply this to the Laspeyres cost-of-living and Paasche Allen indices then we have:

$$\frac{e\left(p^1, u^0\right)}{e\left(p^0, u^0\right)} \frac{e\left(p^1, u^1\right)}{e\left(p^1, u^0\right)} = \frac{e\left(p^1, u^1\right)}{e\left(p^0, u^0\right)} = \frac{w^1}{w^0}$$

- It follows that the implicit quantity index defined using a Laspeyres index is the Paasche index:

$$\underbrace{\frac{e\left(p^1, u^1\right)}{e\left(p^1, u^0\right)}}_{\text{Laspeyres PI}} = \frac{w^1}{w^0} \underbrace{\frac{1}{\frac{e(p^1,u^0)}{e(p^0,u^0)}}}_{\text{Paasche QI}}$$

- And vice versa.

Equivalence Scales and Conditional Expenditure Cost Function

- *Equivalence scale*: a measure of the cost of living of a household of a given size and demographic composition, relative to the cost of living of a reference household (usually a single adult), when both households attain the same level of utility or standard of living

- Consider a consumer (an individual or a household) with a vector of demographic characteristics z
- We can define the conditional expenditure cost function $e(p, u, z)$ which equals the minimum expenditure necessary to attain a utility level u, conditional on the consumer having characteristics z
- An equivalence scale compares costs conditional on different demographic characteristics

$$\boxed{\frac{e(p, u, z^1)}{e(p, u, z^0)}}$$

- By revealed preference theory, demand data identifies the shapes and rankings of a consumer's indifference curves over bundles of goods, but not the actual utility level associated with each indifference curve

- Thus equivalence scales, important as they are, cannot be identified from behaviour

- Another way to put it is to say that if a researcher estimates a demand system $x(p, w, z)$ and recovers the conditional expenditure function associated with it $e(p, w, z)$ then the implied equivalence scale

$$\frac{e(p, u, z^1)}{e(p, u, z^0)}$$

  is essentially arbitrary

Two Stage Budgeting

Two Stage Budgeting

- Two-stage budgeting is a form of rational "mental accounting" which simplifies decision-making. It is also known as "decentralisation".

Allocate wealth to each class $\rightarrow$ allocate wealth to each specific good within the class

- To simplify the bottom stage only requires weak separability of preferences across the broad groups

- This means that there are no cross-price effects with goods outside of this group

- As a result the demand for the x goods only depends on the prices of the x goods and the x budget

- Let $x(p_x, p_z, w)$ and $z(p_x, p_z, w)$ denote the demands for the x and the z goods respectively
- Then if the overall utility function is separable in the x goods we can find the optimal choice of the x goods by solving the sub-problem:

Lower Stage :

$$\boxed{max_x v(x) \text{ subject to } p_x \cdot x = p_x \cdot x(p_x, p_z, w)}$$

where $p_x \cdot x(p_x, p_z, w) \equiv w_x$ is the part of the budget allocated to the x goods

- In other words, we can write the demands for the x goods as a function of the prices of the x goods and the x budget

$$\boxed{x(p_x, p_z, w) = x(p_x, w_x)}$$

- The demand function $x(p_x, w_x)$ is sometimes called a conditional demand function — it tells us optimal demand given the prices of the goods in this group conditional on the level of the budget allocated to the group

- The between group allocation problem is more complex. In general the allocation of the x budget will depend on all of the prices of all of the goods in all of the groups.
- But this allocation problem too can be simplified such that the allocation across groups depends on a price index for each group which only depends on the prices of the goods in the groups.
- For example, the allocation of budget to food versus services versus non-durables becomes a three-good problem which depends on a price aggregate for the the food group, the services group and the non-durables group.

Aggregation

General Properties of Aggregate Demand

- aggregate demand is h.o.d. 0
- aggregate demand satisfies Walras' Law
- if the individual demands are continuous, aggregate demand will be continuous; even if individual demands are discrete then with heterogeneous individuals aggregate demand will become continuous as $I \to \infty$

When is AD a Function of Prices and Aggregate Wealth?

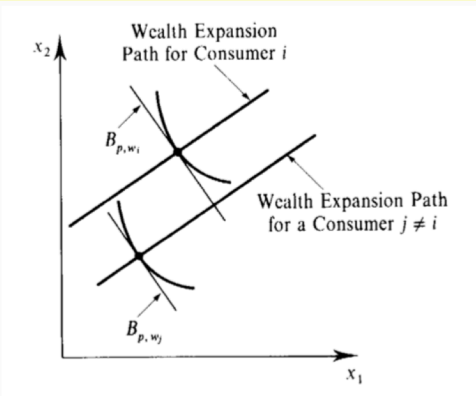
- This must require that aggregate demand is invariant to mean-preserving/aggregate wealth-preserving changes in the wealth distribution.
- That is

$$\boxed{\sum_{i=1}^{I} \frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} dw_i = 0 \quad \forall \ell}$$

- This can be true for all redistributions $(dw_1, ..., dw_L)$ such that $\sum_{i=1}^{I} dw_i = 0$ and all initial wealth distributions $(w_1, ..., w_L)$ iff

$$\boxed{\frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} = \frac{\partial x_{\ell j}(p, w_j)}{\partial w_j} \quad \forall \ell}$$

  i.e. the slopes of the Engel curves are the same for everyone.



[MWG 4.B.1]

- This requires that the slopes of the wealth expansion paths/Engel curves are the same at all levels of wealth
- In other words that everyone's preferences are quasi-homothetic/ Gorman Polar form

- The necessary and sufficient condition for parallel straight Engel curves is that their indirect utility functions are of the form

$$\boxed{v_i(p, w_i) = a_i(p) + b(p)w_i}$$

- Preference heterogeneity is confined to the intercepts of these affine functions. The slopes are the same.

- Note that homothetic and quasi-linear preferences are special cases.

Individual Rationality and Aggregate Rationality

Individual rationality does not imply aggregate rationality.
Aggregate rationality does not imply individual rationality.

Problem Sets

Quasi-concavity of Utility Function and Convexity of Indifference Curves #flashcard

- (Strict) quasi-concavity of utility function $\iff$ (Strict) convexity of indifference curves
- Mathematical definitions

-

1. **Upper-Level Sets Are Convex**: For every $\alpha \in \mathbb{R}$, the set $\{\, x : f(x) \geq \alpha \,\}$ is convex.
2. $\min$-**Preserving Inequality**: For every $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$f\big(\lambda x + (1-\lambda)y\big) \;\geq\; \min\{f(x), f(y)\}.$$

3. **No "Dip Below" for Convex Combinations**: If $f(x) \geq \alpha$ and $f(y) \geq \alpha$, then $f(\lambda x + (1-\lambda)y) \geq \alpha$ for all $\lambda \in [0, 1]$.

All of these say, in different words, that f **never goes below** its own "low points" on straight lines between any two points in its domain. That property ensures its upper contour sets are convex, which is the essence of **quasi-concavity**.

Misc

Lagrangian Method: Constrained Maximisation #flashcard

Our Problem:

$$\max_x u(x) \; s.t. \; G(x) \leq 0, x \geq 0$$

where x is a vector

#flashcard

- *This applies for constrained minimisation as well, just convert add a negative sign to the objective function.*

- **Lagrangian**:

$$L(x, \lambda) = u(x) - \lambda G(x)$$

  - Minus sign to punish disobeying of $\leq$ constraints.

  KKT: If:

- $u(.)$ is quasi-concave ($\iff$ $\{x : u(x) \geq \bar{u}\}$ is convex)
- the constraint set $\{x : G(x) \leq 0\}$ is convex with a non-empty interior
  Then, the global optimum satisfies:
- $\frac{\partial L}{\partial x} \leq 0$ and $x \geq 0$ with CS: $\frac{\partial L}{\partial x} x = 0$ (beware of the signs: at optimal, x can only have negative marginal effects $\frac{\partial L}{\partial x} < 0$ if we hit the non-negativity constraint, which means we have a corner solution $x = 0$)
- $\frac{\partial L}{\partial \lambda} \geq 0 \iff G(x) \leq 0$ and $\lambda \geq 0$ with CS: $\frac{\partial L}{\partial \lambda} \lambda = 0 \iff G(x)\lambda = 0$ (beware of the signs: at optimal, as long as the constraint value has positive marginal effect $\lambda > 0$, we should have it binding $\frac{\partial L}{\partial \lambda} = 0$)