L1: Social Choice

Basic Concepts on Social Welfare Functions #flashcard

- X: the set of all possible states/alternatives
- Agents: $h = 1, \ldots, h$
- Individual preference relation: R_h, assumed to be complete and transitive on X (same as $\succeq_h$)
- Individual strict preference relation: P_h (same as $\succ_h$)
- Social ordering / preference relation: R, assumed to complete and transitive on X
- Social welfare functional: $f : \{\text{individual preference}\} \to \text{Social Ordering}$

Arrow's 4 Requirements on Social Welfare Functionals and Arrow's Impossibility Theorem #flashcard

- **Unrestricted Domain (U)**: the domain of f must include all possible profiles of individual preference ordering on X
- **Weak Pareto Principle (WP)**: $\forall\, x, y \in X, xP_h y \,\forall\, h \implies xPy$
  - All individual prefer x to $y \implies$ the society prefer x to y
- **Independence of Irrelevant Alternative (IIA)**: the social ranking between two alternatives x and y should depend only on how individuals rank x and y, not on their rankings of other alternatives.
  - Adding or removing a third option z should not change the ranking between x and y.
- **Non-Dictatorship (ND)**: there is no agent h such that $\forall\, x, y \in X, xP_h y \implies xPy$ regardless of others' preferences.
  - No single individual should completely determine the social ranking, regardless of others' preferences. There must be at least some influence from multiple individuals.
- **Arrow's Impossibility Theorem**: if there are at least 3 states in X, then there is NO social welfare functional satisfying all 4 requirements simultaneously.
- Additional assumption: **Transitivity**: there is no cycle in the social ordering.
- Similar assumptions can be extend to social welfare functions.

Borda Count

- A method to aggregating voters' rankings #flashcard
- Steps:
  1. For each voter, assign value 1 to her first choice, 2 to her second choice and so on.
  2. Rank each state/candidate by the ascending order of the sum of voter's rankings.
- Properties:
  - It violates IIA

Strategy-Proof and the Gibbard-Satterthwaite Theorem On Social Choice Function #flashcard

- **Strategy-Proof**: the social welfare functional yields an equilibrium where each agent reporting their true preference as a dominant strategy.
  - You can submit your true voting without having to worry about other voters' votes.
- **Gibbard-Satterthwaite Theorem**: if there are at least 3 states in X, then if the social choice functional is strategy-proof, then it is dictatorial.

Utility Possibility Sets and Social Welfare Function #flashcard

- The **Utility Possibility Set** is the set of all achievable combinations of agents' utilities:

$$U = \{(u_1, \ldots, u_h) : u_1 \leq \bar{u}_1(x)\}, \ldots, u_H \leq \bar{u}_H(x), \; \forall x \in X$$

- A **(Bergson-Samuelson) Social Welfare Function** is a function mapping individual utilities to a social utility level:

$$W : U \to \mathbb{R}, (u_1, \ldots, u_H) \;\; u_{social} \in \mathbb{R}$$

Invariance Requirements on Social Welfare Functions and Impossibility Results #flashcard
The social welfare functional is invariant subject to transformations on individual utilities G

- **Ordinal Non-Comparability (ONC)**: $G = \{g : g_h \text{ strictly increasing, potentially different}\}$ (essentially goes back to pure ordinality)
  - Stronger: **Ordinal Level-Comparability (OLC)**: $G = \{g : g_h \text{ strictly increasing, same for all } h\}$
    - Rawlsian SWF ($U_{social} = \min\{u_i\}$) satisfies this.
- **Cardinal Non-Comparability (CNC)**: $G = \{g : g_h(u_h) = \alpha_h u_h + \beta_h, \alpha_h > 0, \text{potentially different}\}$ (cannot compare utilities across agents)
  - Stronger: **Cardinal Unit-Comparability (CUC)**: $G = \{g : g_h(u_h) = \alpha u_h + \beta_h, \alpha_h > 0, \}$
  - Utilitarian SWF ($U_{social} = \sum_i u_i$) satisfies this.
  Possibility Results:
- If X has at least 3 elements:
  - No SWF can satisfy 4 Arrow requirements and ONC.
  - No SWF can satisfy 4 Arrow requirements and CNC.
- In general: ONC/CNC do not help; OLC/CUC help.

L2: Coalitional Bargaining

The Shapley Value and Axioms in a Coalitional Game with Transferrable Utility #flashcard
4 Axioms:

- **Efficiency**: a solution concept ϕ is efficient if for every coalition game $(N; v)$:

$$\sum_{i \in N} \phi_i(N; v) = v(N)$$

- **Symmetry**: a solution concept ϕ is symmetric if for every coalition game $(N; v)$ and for each pair of symmetric player i, j:

$$\phi_i(N; v) = \phi_j(N; v)$$

  - i.e. "equal treatment of equals"
- **Dummy**: a solution concept ϕ satisfies the dummy property if for every coalition game $(N; v)$ and every dummy player i:

$$\phi_i(N; v) = 0$$

- **Additivity**: a solution concept ϕ satisfies the additivity property if for every pair of coalition games $(N; v)$ and $(N; w)$:

$$\phi(N; v + w) = \phi(N; v) + \phi(N; w)$$

  The only point solution that satisfied efficiency, symmetry, dummy, and additivity is the **Shapley Value**, which is defined as the *average marginal contribution to the coalition across all possible permutations*.
- i.e. Randomly order players in the grand coalition, the expected marginal value of the player.

Imputations and Core of a Coalition Game #flashcard
An **imputation** for a coalition structure B is a vector $x \in \mathbb{R}$ that is:

- **individually rational**: for every player $i \in N, x_i \geq v(\{i\})$

- **efficient**: for every coalition $S \in B, \sum_{i \in S} x_i = v(S)$
  The **core** of a coalition game $(N; v)$ is defined as:

$$C(N;v) := \left\{ x \in X(N;v) : \sum_{i \in S} x_i \geq v(S),\ \forall\ S \subseteq N \right\}$$

- The core also has to be *individually rational, efficient, and coalitional stable*.
- The core is a convex set (proved in PS1).
- i.e. no coalition $S \subseteq N$ has an incentive to deviate from the grand coalition by having a higher aggregate payoff for its members. $\implies$ stability of the core
- Note that the Shapley value is not always contained by the core, since the core does not always exists, but the Shapley value can always be calculated.

Existence of a Non-empty Core and Containment of Shapley Value in a Coalition Game #flashcard

- A coalitional game $(N; v)$ is **convex** if for every pair of coalition S and T, we have that:

$$v(S) + v(T) \leq v(S\ \ T) + v(S\ \ T)$$

  which is stronger than monotonicity.
  - $\implies$ the contribution of a player (or coalition) to any coalition increases as the coalition grows.
- *If a coalition game is convex, then its core is non-empty and always contains the Shapley Value.*

L3: Matching Markets

Matching Properties (Pareto Efficient / Strategyproof / Individual Rational / Non-wasteful / No Justified Envy / Stable / Student-Optimal) #flashcard

- A matching is **Pareto efficient** if there is no other matching that can make all students weakly better off and at least one student strictly better off.
- A mechanism is **strategyproof** if reporting the true preference is the dominant strategy for everyone.
- A matching is **individual rational** if participating in the matching is weakly preferred to being unmatched.
- A matching is **non-wasteful** $\iff$ if a student prefers another school to her matched school, then that school must have filled its capacity.
- A matching has **no justified envy** $\iff$ if a student prefers a school to her matched school, then all students matched to the preferred school must have a higher priority than that student.
- A matching is **stable** if it satisfies individual rationality, non-wastefulness, and no justified envy.
- A *stable* matching is **student-optimal** if it is weakly preferred by all students to any other stable matching.

Immediate Acceptance (IA) / Boston Algorithm #flashcard

- Procedure:
  - In each round:
    - Each student proposes to her first-choice school.
    - Each school immediately accepts the highest-priority proposing student up to its quota and rejects the left
    - Rejected students move to the next round
- IA outcome is:
  - Pareto efficient
  - Not strategyproof (easily manipulatable)
  - Not stable

Top Trading Cycles (TTC) Algorithm #flashcard

- Procedure:
  - Draw an arrow from each student to her most preferred school.
  - Draw an arrow from each school to its highest priority student.
  - There must be at least one cycle.
  - Each student in the cycle is assigned to her preferred school.
  - Move to the next round...
- TTC outcome is:
  - Pareto efficient
  - Strategyproof
  - May NOT be Stable: may have justified envy

Deferred Acceptance (DA) Algorithm #flashcard

- Procedure:
  - Each student proposes to her first-choice school
  - Each school holds temporarily its highest-priority student up to its quota and permanently rejects the left.
  - Rejected students propose to their second-choice school.
  - ...
- DA outcome is:
  - Stable (individual rationality, non-wastefulness, and no justified envy)
  - Student optimal
  - Strategyproof
  - May NOT be Pareto Efficient

Kesten's Theorem in Pareto Efficiency, Strategyproof, and Stability #flashcard

- There is no Pareto-efficient and strategyproof mechanism that selects Pareto-efficient and stable matching whenever it exists:

$$\text{Pareto-efficient } + \text{Strategyproof} \implies \text{Not Stable}$$

Property Summary of IA, DA, TTC #flashcard

Mechanism	**Stability**	**Pareto Efficiency**	**Strategyproofness**
Immediate Acceptance (IA) (Boston Mechanism)	❌ Not stable (students may prefer another available school)	✅ Pareto efficient (no student can improve without harming another)	❌ Not strategyproof (strategic ranking is often required)
Top Trading Cycles (TTC)	❌ Not stable (blocking pairs can exist)	✅ Pareto efficient (no student can improve without harming another)	✅ Strategyproof (truthful reporting is optimal)
Deferred Acceptance (DA) (Gale-Shapley)	✅ Stable (no blocking pairs)	❌ Not always Pareto efficient (stability may not maximize welfare)	✅ Strategyproof for students (truthful reporting is optimal)

L4: Externalities

Solutions to the Externality Problem and Coase Theorem #flashcard

- Aim/Optimal level produced:

$$\text{Marginal Cost} + \text{Marginal (Negative) Externality} = \text{Marginal Benefit}$$

- Solutions:
  - **Firms Merge**: merge the producer and the firm suffering externality
  - **Pigovian Taxes**: impose tax equal to marginal externality
  - **Quota**: imposing a maximum level of output e equals to the socially optimal quantity
  - **Create a Market for Externality**: for each unit produced, the firm producing must buy a permit from the firm suffering from externality
  - **Assign Property Rights**: assign property right to either the firm producing or the firm suffering the externality:
  - Rights given to the firm producing $\implies$ Firm suffering has no surplus and giving all surplus above $-\bar{d}$ to the firm producing $\rightsquigarrow$ Firm producing internalises profits and externality
  - Rights given to the firm suffering $\implies$ Firm producing has no surplus and giving all profits to the firm suffering $\rightsquigarrow$ Firm suffering internalises profits and externality
  - **Coase Theorem**: if property rights are assigned so that trade in externality can occur, efficiency will be ensured through bargaining regardless of to whom the property rights are assigned.
  - This only holds when there is:
  - no cost of bargaining
  - we have quasi-linear utility
  - the planner has complete information (for Pigovian taxes and quota)
  - if we do not have complete information $\implies$ impossibility of efficient bilateral bargaining

Impossibility of Efficient Bilateral Bargaining (Myerson and Satterthwaite 1983) #flashcard

There exists no mechanism for bilateral trading that satisfies Interim Individual Rationality, Balance, Efficiency, and Bayesian Incentive Compatibility.

- **Interim IR**: willing to participate in the mechanism, having learned your value
- **Balance**: net transfers to the agents add up to zero
- **Efficiency**: ex-post allocative efficiency
- **Bayesian IC**: there is a Bayesian Nash Equilibrium where each agent reports their value truthfully.
  In short, incomplete information bargaining might not be efficient.

L5: Public Goods

Public Goods #flashcard

Public goods are a type of externality with 2 characteristics:

- **Non-rival**: the amount consumed by one agent does not affect the amount available to others
- **Non-excludable**: agents cannot be prevented from consuming

Efficient Public Goods Provision Condition #flashcard

- Efficient Public Goods Provision Condition:

$$\sum_i MRS_i = MRT$$

- Duality between private and public goods efficiency conditions:
  -

	Private Good	Public Good
Efficiency	$MRS_i = MRT \quad \forall i$	$\sum_i MRS_i = MRT$
Market Clearing	$\sum_i q_i = Q$	$q_i = Q \quad \forall i$

Lindahl Equilibrium #flashcard

- Idea: set a personal market with personal price (p_i) for each consumer.
- In a MRS MRT setup, this involves every consumer individually choosing the quantity such that

$$MRS_i = MRT \; \forall \; i$$

- In a market setup, the consumer i solves:

$$\max_{x_i} b_i(x) - p_i x_i \implies b_i'(x_i) = p_i$$

- The firm solves:

$$\max_X \left(\sum_i p_i\right) X - c(X) \implies \sum_i p_i = c'(X)$$

- In equilibrium, since the good is a public good:

$$x_i = X \; \forall \; i$$

- Efficiency:

$$\sum_i b_i'(X) = c'(X)$$

- Discussion:
  - elegant but ignores free-rider problem: each consumer has an incentive to report 0 marginal benefit to pay 0 price and still enjoy the public good

The Vickrey-Clarke-Groves (VCG) Mechanism #flashcard

- **Vickrey Mechanism**: produce efficient and truthful outcome
  - Procedures:
  - Each individual reports a valuation $\tilde{b}_i$
  - The government decides:

$$\text{Provide the public good} \iff \sum_i \tilde{b}_i > 0$$

  - If the public good is provides, the government provide a transfer T_i to each individual i the amount of:

$$T_i = \sum_{j \neq i} \tilde{b}_j$$

  - We will see that it is a weakly dominant strategy for each player to report their true valuation, and the decision is efficient. (proof is similar to a second-price auction)
  - problem: may involve large transfers from the government
    - **Groves Mechanism**: reduce government transfer
      - Procedure:
      - based on the Vickrey Mechanism
      - let i pay an additional amount $h_i(\tilde{b}_{-i})$ independent of her own reported value, so the overall payoff for i becomes:

$$\text{Payoff}_i = \begin{cases} b_i + \sum_{j \neq i} \tilde{b}_j + h_i(\tilde{b}_{-i}) & \text{if } \sum_j \tilde{b}_j > 0 \iff \text{Build} \\ h_i(\tilde{b}_{-i}) & \text{if } \sum_j \tilde{b}_j \leq 0 \iff \text{Not Build} \end{cases}$$

- Telling the truth is still the weakly dominant strategy.

- **Clarke Pivotal Mechanism**: lowest possible transfer
  - Procedure:
  - based on Groves Mechanism
  - Choose:

$$h_i(\tilde{b}_{-i}) = \begin{cases} -\sum_{j \neq i} \tilde{b}_j & \text{if } \sum_{j \neq i} \tilde{b}_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

  - Individual i will have the overall payoff:

$$\text{Payoff}_i = \begin{cases} b_i & \text{if } \sum_j \tilde{b} > 0 \quad \sum_{j \neq i} \tilde{b}_j > 0 \quad \text{(Build, Not Pivotal)} \\ b_i + \underbrace{\sum_{j \neq i} \tilde{b}_j}_{\leq 0} & \text{if } \sum_j \tilde{b} > 0 \quad \sum_{j \neq i} \tilde{b}_j \leq 0 \quad \text{(Build, Pivotal)} \\ \underbrace{-\sum_{j \neq i} \tilde{b}_j}_{<0} & \text{if } \sum_j \tilde{b} \leq 0 \quad \sum_{j \neq i} \tilde{b}_j > 0 \quad \text{(Not Build, Pivotal)} \\ 0 & \text{if } \sum_j \tilde{b} \leq 0 \quad \sum_{j \neq i} \tilde{b}_j \leq 0 \quad \text{(Not Build, Not Pivotal)} \end{cases}$$

  - i.e. The **Pivotal Individuals** will have to compensate the aggregate externalities caused to other agents:

$$\text{Extra Payment} = \text{Welfare on others in absence of i} - \text{Welfare on others with i}$$

  - Key assumption: quasi-linear utility
  - Telling the truth is still the weakly dominant strategy.
  - But note that this is not the unique Nash equilibrium. There exist other bad NEs where everyone lies (e.g. everyone reporting a big negative number). This also applies to ascending auctions.
  - Transfer is small, negative, and often not paid.

Public Good Provision Impossibility Theorem #flashcard

- There exists no public good provision mechanism satisfying Interim Individual Rationality, Efficiency, Dominant Strategy Incentive Compatibility, and Budget Balance.
  - **Interim IR**: willing to participate in the mechanism, having learned your value
  - **Dominant Strategy Incentive Compatibility**: everyone telling the truth is a dominant strategy
  - **Efficiency**: ex-post allocative efficiency
  - **Budget Balance**: net transfers add up to zero
- Groves mechanism satisfies: Efficiency, Dominant Strategy Incentive Compatibility, and *either* Budget Balance *or* Interim Individual Rationality.
- Clarke mechanism satisfies: Efficiency, Dominant Strategy Incentive Compatibility, Interim Individual Rationality, but NOT Budget Balance.