Topic 1 : Budget constraints and consumer demand

Summary: *Consumer choice is restricted by affordability as captured by the budget constraint and this on its own restricts the nature of demand functions in important ways. Dependence on the parameters of the budget constraint are the basis for several ways to classify demands.*

$\vec{q}$ (vector) $= \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \cdots \\ q_M \end{pmatrix}$

1.1 Budget constraint

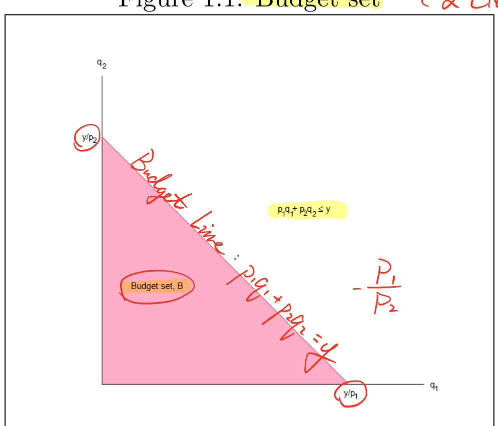
(physical constraints) $q_i > 0$

Consumers purchase M goods q, lying in some set of possible values called the commodity space, typically $q \geq 0$. Affordability constrains them to choose from from within a budget set B of affordable bundles. In the standard model, prices p are constant and total spending has to remain within budget so that $p'q \leq y$ where y is total budget. (economic constraint) Maximum affordable quantity of any commodity is y/p_i and slope $\partial q_i/\partial q_j|_B = -p_j/p_i$ is constant and independent of total budget as in Figure 1.1. Budget Constraint

Linear Budget Constraint : Fixed Budget (y) + Fixed Prices $\vec{P} = \begin{pmatrix} P_1 \\ P_2 \\ \cdots \\ P_M \end{pmatrix}$

Physical Constraint -> Commodity Space
Economic Constraint -> Budget Set

Figure 1.1: Budget set (2 Goods Case)



$\vec{p}'\vec{q} \leq y$

$\sum_i p_i q_i \leq y$

Slope $= \frac{\partial q_i}{\partial q_j}|_B = -\frac{p_j}{p_i}$

Change in total budget y -> Parallel shift of the budget line
Change in prices -> Changes in one price : Rotation

In practical applications budget constraints are frequently kinked or discontinuous as a consequence, for example, of taxation or non-linear pricing. If the price of a good rises with the quantity purchased (say because of taxation

$q\uparrow$ $p\uparrow$ (Smoothly) → Convex Budget Set / Concave Budget Line

$q\uparrow$ $p\downarrow$ (Smoothly) OR A Jump in Prices → Non-Convex Budget Set

above a threshold) but without any discontinuity then the budget set is convex whereas if it falls (say because of a bulk buying discount) or there is a jump (say because a price change applies to units below the threshold) then the budget set is not convex. Examples are shown in Figures 1.2 and 1.3.

Figure 1.2: Nonlinear budget sets: rising price (e.g. tax above certain threshold)

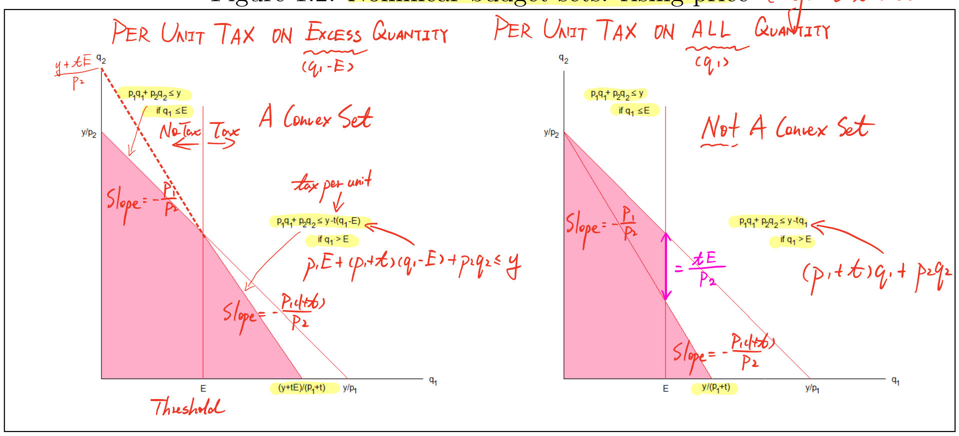
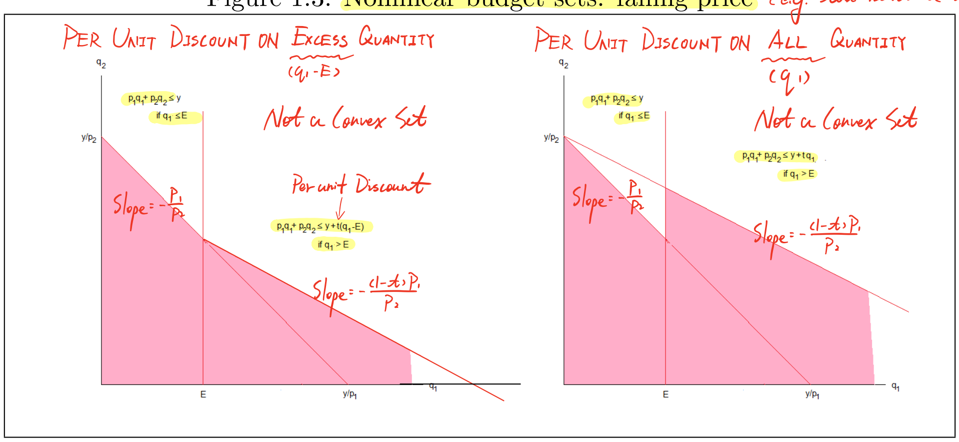


Figure 1.3: Nonlinear budget sets: falling price (e.g. tax below a threshold / bulk buying discount)



1.2 Marshallian demands

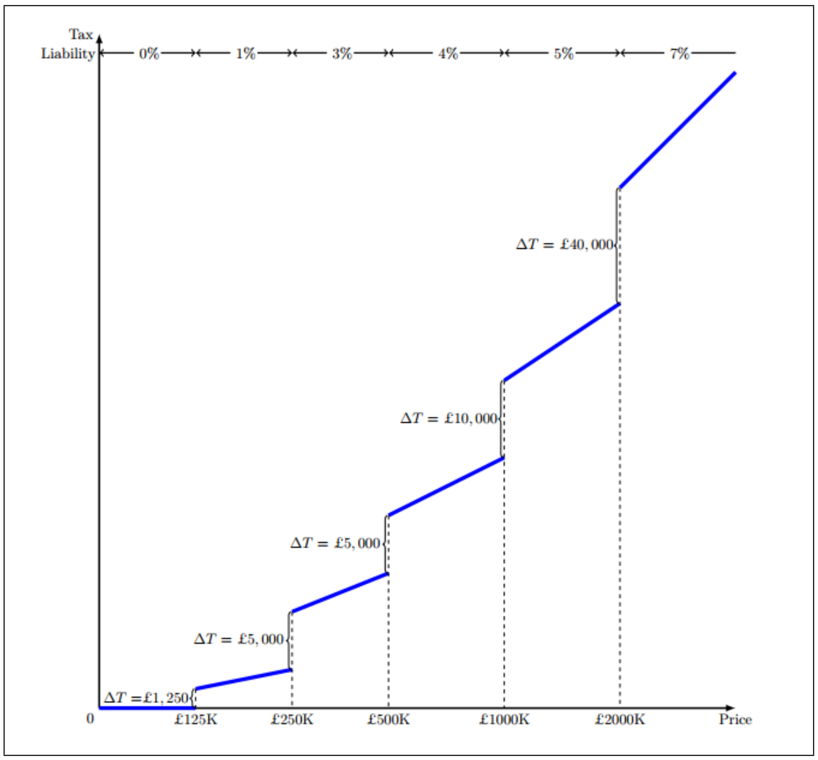
The consumer's chosen quantities written as a function of y and p are the Marshallian demands or uncompensated demands $q = f(y, p)$. $(\vec{p})$

Consider the effects of changes in y and p on demand for, say, the ith good:

choosen quantity = f(income, prices)

Case Study 1|1 : Budget constraints: Stamp Duty Land Tax

Residential property transactions in England are subject to a tax known as stamp duty land tax. Prior to late 2014, if the value of the transaction was below \$125K the transaction was exempt but, once it exceeded that value, tax was due at 1% on the whole of the value of the transaction. This meant that as the value passed \$125K not only did the after-tax price of owner-occupied housing increase but also the tax that was liable jumped by \$1.25K. At \$250K another threshold was passed at which the rate of tax increased to 3%, again on the whole of the value so that there was a jump of \$5K in the liability. There were further discrete jumps for similar reasons at higher values. When translated into a budget constraint between housing and other wealth this created jumps (or "notches") at these points and there is evidence that house sales showed bunching at values just below these notches. Objections to the "badly designed" form of the tax led to the announcement of reforms smoothing out the schedule in the Autumn Statement of 2014.



[Source: M. Best and H. Kleven, 2013, Housing Market Responses to Transaction Taxes: Evidence From Notches and Stimulus in the UK, LSE Working Paper.]

Total Budget Effects

- total budget y
  - the path traced out by demands as y increases is called the income expansion path (seen in Figure 1.4) whereas the graph of $f_i(y, p)$ as a function of y is called the Engel curve (seen in Figure 1.5) (下一页)
  - we can summarise dependence in the total budget elasticity

ϵ_i Total Budget Elasticity
$$\boxed{\epsilon_i = \frac{y}{q_i}\frac{\partial q_i}{\partial y} = \frac{\partial \ln q_i}{\partial \ln y}}$$
Proportional Δ in quantity consumed in response to proportional Δ in y.

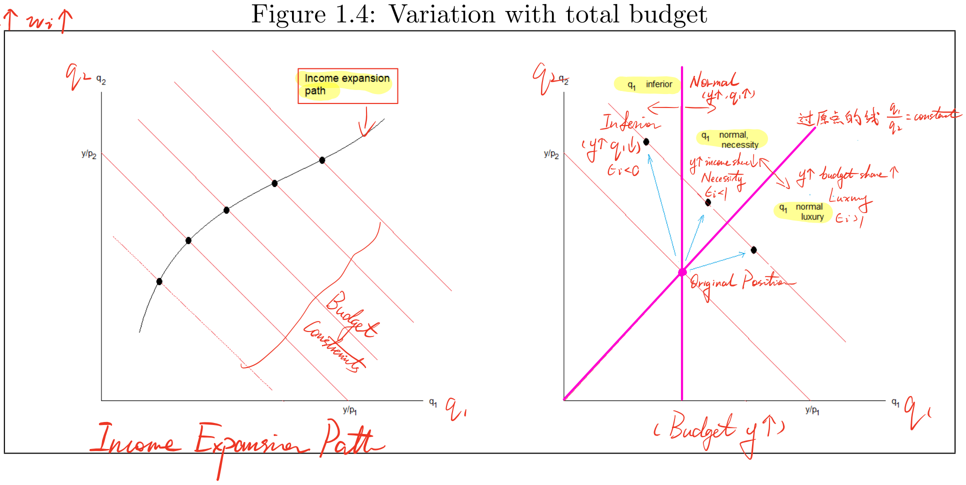
$\epsilon_i < 0$: Inferior
$\epsilon_i > 0$: Normal

$\epsilon_i < 1$: Necessity (Income-Inelastic) $y\uparrow\ w_i\downarrow$
$\epsilon_i > 1$: Luxury (Income Elastic) $y\uparrow\ w_i\uparrow$

∴ All inferior are necessity
All luxury are normal

  - if demand for a good rises with total budget, $\epsilon_i > 0$, then we say it is a normal good and if it falls, $\epsilon_i < 0$, we say it is an inferior good $y\uparrow q\downarrow$ ($\epsilon_i<0$) $y\uparrow q\uparrow$ ($\epsilon_i>0$)
  - if budget share of a good, $w_i = p_i q_i / y$, rises with total budget, $\epsilon_i > 1$, then we say it is a luxury or income elastic good and if it falls, $\epsilon_i < 1$, we say it is a necessity or income inelastic good. (These different cases are shown in Figure 1.4.)

Figure 1.4: Variation with total budget



Price Effects (Own)

- own price p_i (Holding total budget y constant)
  - the path traced out by demands as p_i increases is called the offer curve (seen in Figure 1.6) whereas the graph of $f_i(y, p)$ as a function of p_i is called the demand curve (seen in 1.7) 下一页 下两页
  - we can summarise dependence in the (uncompensated) own price elasticity

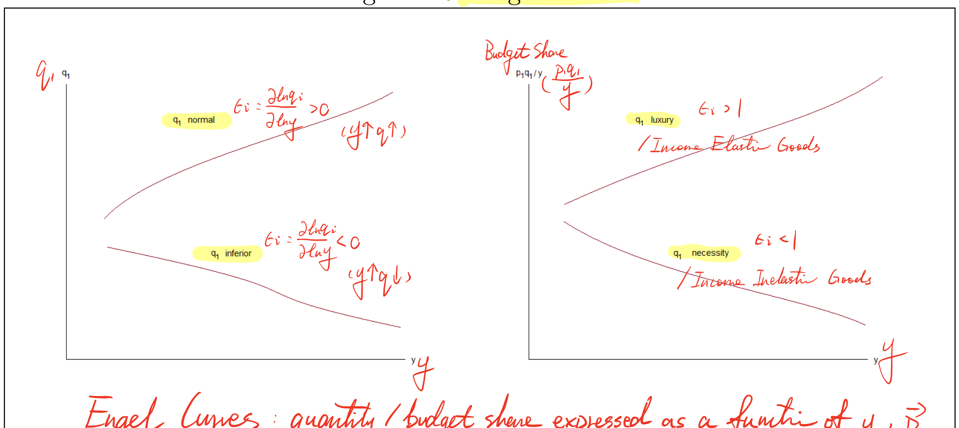
η_{ii} Uncompensated Own Price Elasticity
$$\boxed{\eta_{ii} = \frac{p_i}{q_i}\frac{\partial q_i}{\partial p_i} = \frac{\partial \ln q_i}{\partial \ln p_i}}$$
η_{ii} ← Price of which good are we changing; ↑ Quantity of which good are we looking at

  - if uncompensated demand for a good rises with own price, $\eta_{ii} > 0$, then we say it is a Giffen good
  - if budget share of a good rises with price, $\eta_{ii} > -1$, then we say it is a price inelastic good and if it falls, $\eta_{ii} < -1$, we say it is a price elastic good. (These different cases are shown in Figure 1.6.)

$\eta_{ii} > 0$ Giffen Goods $p\uparrow q\uparrow$
$\eta_{ii} < 0$ Non-Giffen Goods $p\uparrow q\downarrow$

$\eta_{ii} > -1$ Price Inelastic Goods $p\uparrow$ income share $\uparrow$
$\eta_{ii} < -1$ Price Elastic Goods $p\uparrow$ income share $\downarrow$

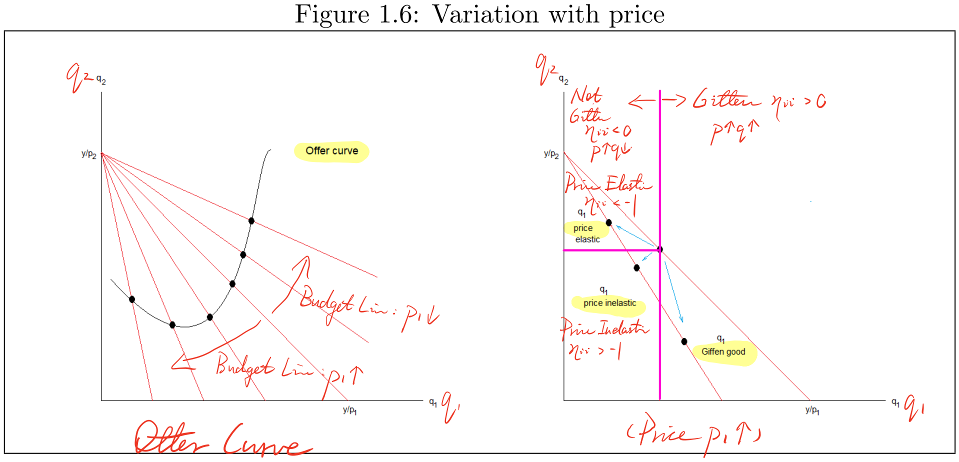
Figure 1.5: Engel curves



Engel Curves

Engel Curves: quantity / budget share expressed as a function of $y, \vec{p}$

Figure 1.6: Variation with price



Price Effects (Others)

- other price p_j, $j \neq i$
  - we can summarise dependence in the (uncompensated) cross price elasticity

η_{ij} Uncompensated Cross Price Elasticity

$$\boxed{\eta_{ij} = \frac{p_j}{q_i}\frac{\partial q_i}{\partial p_j} = \frac{\partial \ln q_i}{\partial \ln p_j}}$$

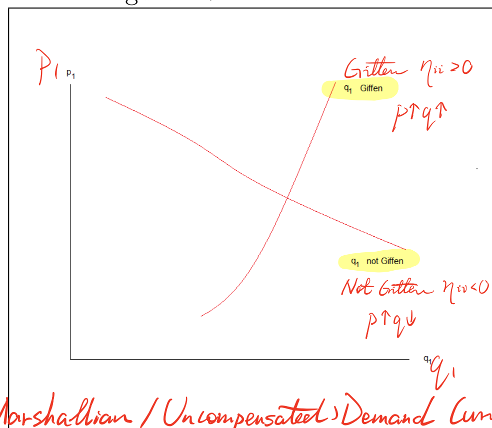
η_{ij} ← Price of price of good j
↑ Quantity of Good i

  - if uncompensated demand for a good rises with the price of another, $\eta_{ij} > 0$, then we can say it is an (uncompensated) substitute whereas if it falls with the price of another, $\eta_{ij} < 0$, then we can say it is an (uncompensated) complement. These are not the best definitions of complementarity and substitutability however since they may not be symmetric, in other words q_i could be a substitute for q_j while q_j is a complement for q_j. A better definition, guaranteed to be symmetric, is one based on the concept of compensated demands to be introduced below.

$\eta_{ij} > 0$ i, j are (uncompensated) substitutes $p_j\uparrow q_i\uparrow$

$\eta_{ij} < 0$ i, j are (uncompensated) complements $p_j\uparrow q_i\downarrow$

Figure 1.7: Demand curve



(Marshallian / Uncompensated) Demand Curve

1.3 Adding up

We know that demands must lie within the budget set:

$$p'f(y,p) \le y. \qquad \sum_i p_i f_i(y,p) \le y$$

If consumer spending exhausts the total budget then this holds as an equality,

① Adding Up :

$$p'f(y,p) = y, \qquad \sum_i p_i f_i(y,p) = y \qquad \frac{p'f(y,p)}{y} = 1$$

which is known as adding up.

$y\uparrow$ $f(y,p)$ 总有上升的吧

→ at least 1 normal goods cannot be all inferior

If we differentiate with respect to y then we get a property known as Engel aggregation

② Engel Aggregation :

$$\sum_i p_i \frac{\partial f_i}{\partial y} = \sum_i w_i \epsilon_i = 1.$$

$$\sum_i \underbrace{\left(\frac{p_i q_i}{y}\right)}_{\text{Budget Share}} \underbrace{\left(\frac{\partial f_i}{\partial y} \frac{y}{q_i}\right)}_{\text{Own Price Elasticity}} = 1$$

Budget-share-weighted (w_i) average of the total budget elasticity ϵ_i $= 1$

$\therefore$ ϵ_i cannot be all <0 / >1 / <1

By adding up

- not all goods can be inferior $(\epsilon_i < 0)$ (at least 1 normal)
- not all goods can be luxuries $(\epsilon_i > 1)$
- not all goods can be necessities $(\epsilon_i < 1)$

Restriction on ϵ_i

Also certain specifications are ruled out for *demand systems*. It is not possible, for example, for all goods to have constant income elasticities unless these elasticities are all 1. Otherwise $p_i q_i = A_i y^{\alpha_i}$, say, and $1 = \sum_i A_i y^{\alpha_i - 1}$ for all budgets y which is impossible unless all $\alpha_i = 1$. This does not rule out constant elasticities for individual goods.

Restriction on p_i

There are also restrictions on price effects. The property derived by differentiating with respect to an arbitrary price p_j is known as Cournot aggregation

$\sum_i p_i f_i(y,p) = y$ total differentiated with respect to p_j

③ Cournot Aggregation :

$$\boxed{f_j + \sum_i p_i \frac{\partial f_i}{\partial p_j} = 0 \quad \Rightarrow \quad w_j + \sum_i w_i \eta_{ij} = 0}$$

Budget Share $_j$ + $\sum_i$ Budget Share $_i \times$ PED $_{ij}$ $= 0$

As a consequence, for example, if the price of some good goes up then purchases of at least some good must be reduced.

$$\frac{p_j q_j}{y} + \sum_i \left(\frac{p_i q_i}{y}\right)\left(\frac{\partial f_i}{\partial p_j} \times \frac{p_j}{q_i}\right) = 0$$

i.e. The budget-share-weighted (w_i) sum of elasticities with respect to p_j (η_{ij}) = − budget share of j

If all demands are known except one, the unknown one can be derived :

2 goods example : $p_1 f_1(y,p) + p_2 f_2(y,p) = y$

$$f_2(y,p) = \frac{y - p_1 f_1(y,p)}{p_2}$$

Otherwise, if all ϵ_i are the same :

$\ln q_i = c_i + \alpha_i \ln y$

$q_i = B_i y^{\alpha_i}$

$p_i q_i = A_i y^{\alpha_i}$

$\frac{p_i q_i}{y} = A_i y^{\alpha_i - 1}$

1.4 Homogeneity

Multiplying y and p by the same factor does not affect the budget constraint and, if it does not affect motivations for choice within budget sets, choices should not be affected either. In other words, if all prices are scaled by the same factor so that all relative prices are unchanged and if total budget is similarly scaled so that all previously affordable bundles remain affordable then there is no reason for the consumer to change choices. This implies a property known as homogeneity.

Marshallian demands should therefore be *homogeneous* of degree zero:

4 Homogenity

$$f(\lambda y, \lambda p) = f(y, p)$$

True for every individual demand.

for any $\lambda > 0$.

Differentiating with respect to λ and choosing to consider the case $\lambda = 1$ gives

$f_i(y,p) = f_i(\lambda y, \lambda p)$

~~Differentiated~~ with λ

5

$$\boxed{y\frac{\partial f_i}{\partial y} + \sum_j p_j \frac{\partial f_i}{\partial p_j} = 0 \quad \Rightarrow \quad \epsilon_i + \sum_j \eta_{ij} = 0}$$

$0 = y \cdot \frac{\partial f_i}{\partial \lambda}(\lambda y, \lambda p) + \sum_j p_j \frac{\partial f_i}{\partial \lambda}(\lambda y, \lambda p)$

(True for all λ)

for every good i.

Total Budget Elasticity (ϵ_i) + $\sum_j$ Cross PED$_{ij}$ (η_{ij}) = 0

If $y_A = \lambda y_B$, $p_A = \lambda p_B$ $\rightarrow$ Budget Set: $p^{A'} q \le y_A$ (Budget Set will be the same)

$\cancel{\lambda} p^{B'} q \le \cancel{\lambda} y_B$ Marshallin Demand will also be the same.

$f(y,p) = f(\lambda y, \lambda p)$

Worked Example 1|A : Adding up and homogeneity

Consider the demand system specified by

$$p_i f_i(y,p) = a_i y + \sum_j b_{ij} p_j + c_i \qquad i = 1, 2, \dots, M$$

for appropriate parameters a_i, b_{ij} and c_i, $i, j = 1, 2, \dots, M$. For obvious reasons this is called the *linear expenditure system*.

Adding up requires $\sum_i p_i f_i(y,p) = y$ for all possible values of y and p. Hence we need

$$\left(\sum_i a_i - 1\right) y + \sum_i \sum_j b_{ij} p_j + \sum_i c_i = 0$$

for all possible values of y and p. This can only be true if

$$\sum_i a_i = 1 \qquad \sum_i b_{ij} = 0 \qquad \sum_i c_i = 0 \qquad j = 1, 2, \dots, M.$$

These are the requirements of adding up.

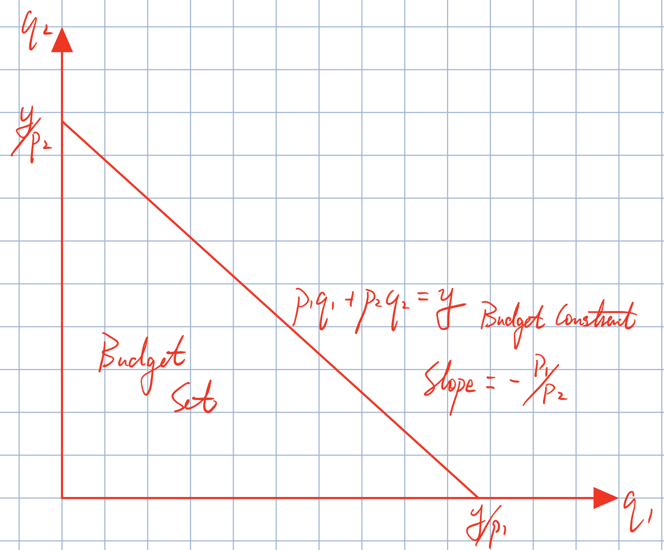
To satisfy homogeneity requires $f_i(\lambda y, \lambda p) = f_i(y,p)$ for $i = 1, 2, \dots, M$ and any $\lambda > 0$. So

$$a_i \frac{\lambda y}{\lambda p} + \sum_j b_{ij} \frac{\lambda p_j}{\lambda p_i} + c_i \frac{1}{\lambda p_i} = a_i \frac{y}{p} + \sum_j b_{ij} \frac{p_j}{p_i} + c_i \frac{1}{p_i}$$

which is true if and only if $c_i = 0$ for every good.

Consumer Choice

Goods $q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$



Affordability captured by budget constraints

Fixed prices $p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_3 \end{pmatrix}$

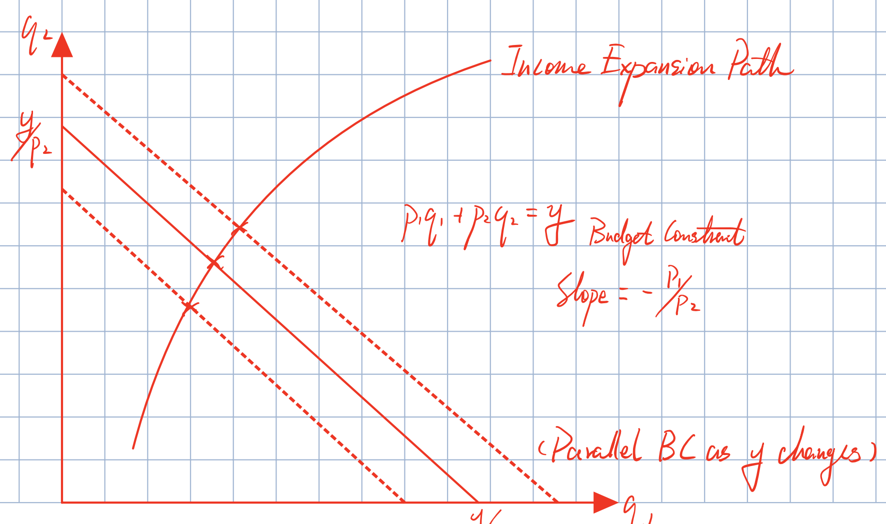
Fixed total budget y

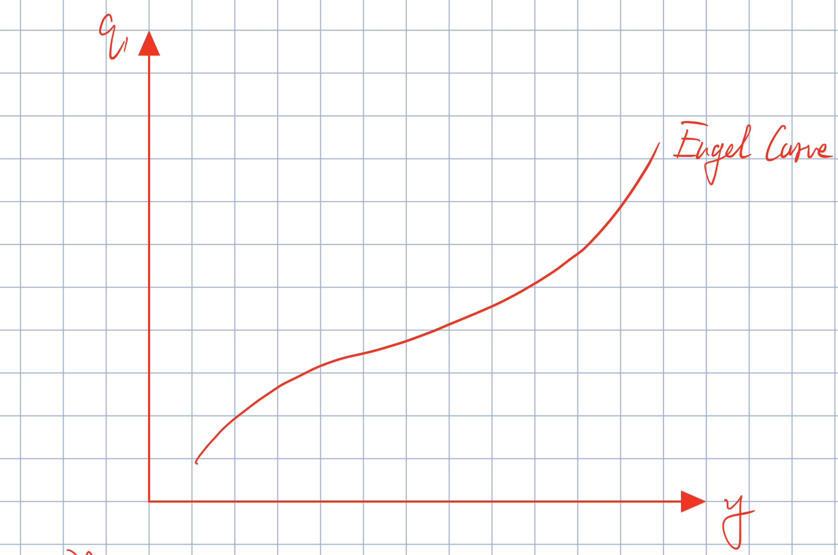
$$\sum_i p_i q_i = p'q \le y$$

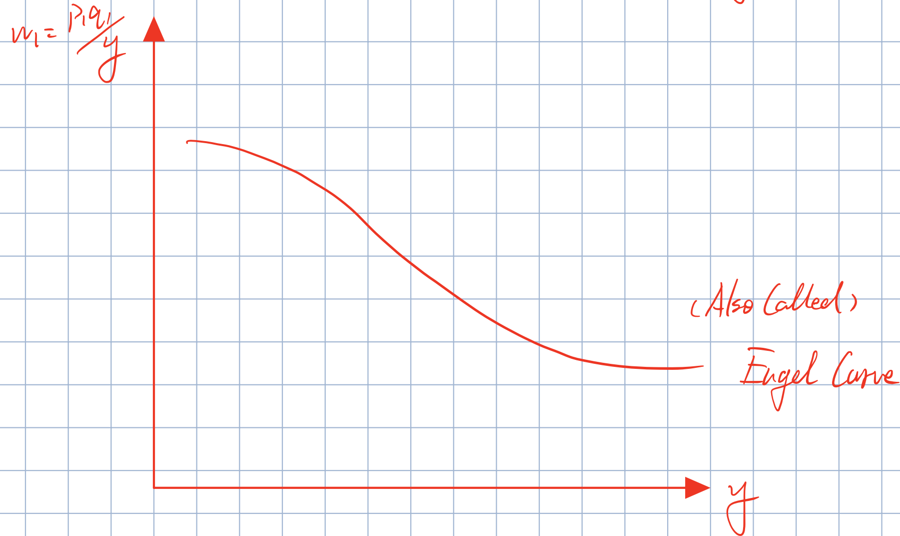
Non-Linear Budget Constraint (arises when price depends on quantity)

We can write consumer's choice as a function of y, p

$q = f(y, p)$ Marshallian (Uncompensated) Demand Function





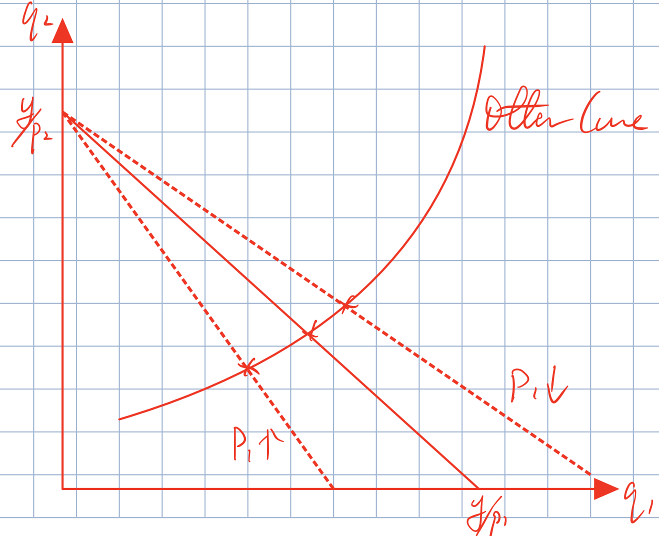


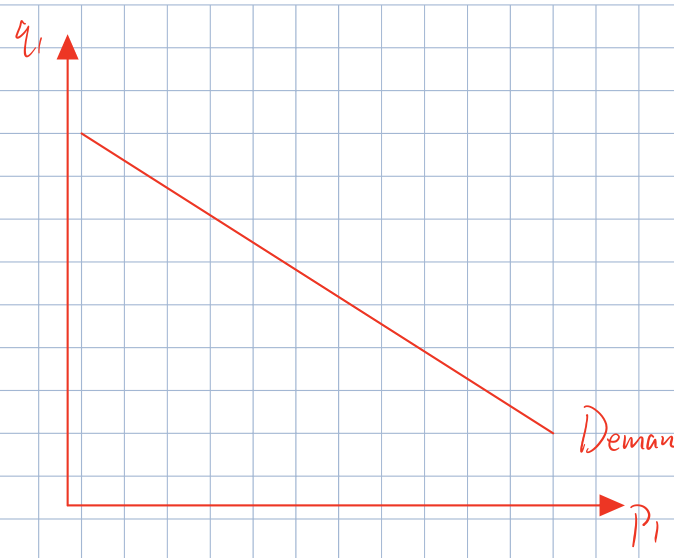
If q is increasing in y → Normal good

- If not → Inferior good.

If w is increasing in y → Luxury

If w is decreasing in y → Necessity





If q_1 increases as p_1 increases → Giffen goods

If $p_1 q_1$ increases as p_1 increases → Price inelastic

If $p_1 q_1$ decreases as p_1 increases → Price elastic

Adding Up

$$\sum_i p_i f_i(y, p) \leq y$$

- Many implications, restrictions

Homogeneity

change the p and y by a common factor λ → No change in demand

$$f(y, p) = f(\lambda y, \lambda p)$$

Topic 2 : Revealed preference and the law of demand

Summary: *Revealed preference restrictions can be seen as requirements of consistency for choice across different budget sets. They impose further important restrictions on the nature of consumer demands including justifying the Law of Demand.*

2.1 Revealed preference

Suppose the consumer would choose q^A at prices p^A when q^B is no more expensive than q^A:

R_D

$$p^{A\prime} q^A \geq p^{A\prime} q^B.$$

and still choose q^A → q^A R_D q^B Revealed Preferred Directly

We say that this is a case of direct revealed preference: q^A is directly revealed preferred to q^B (written $q^A \ R_D \ q^B$. The choice made by the consumer shows that they are willing to buy q^A when they could have bought q^B.

WARP

The Weak Axiom of Revealed Preference (WARP) says that if $q^A \ R_D \ q^B$ then there are no prices p^B such that q^A would be cheaper than q^B

$$p^{B\prime} q^A < p^{B\prime} q^B$$

Personal Understanding: q^A R_D q^B → no p^B that $q^B R_D q^A$

and at which the consumer would choose q^B. This is a sort of consistency requirement on behaviour. Cases in which choices do or do not comply with WARP are shown in Figure 2.1.

R_I

It is possible that no choices directly reveal one bundle q^A as preferred to another q^C but that there is indirect revealed preference in the sense that q^A is directly revealed preferred to q^B and q^B is directly revealed preferred to q^C (or some longer chain of direct revealed preference holds). If we say simply that one bundle q^A is revealed preferred to another q^B, written $q^A \ R \ q^B$ then we mean it is either directly or indirectly revealed preferred (without specifying which). Figure 2.2 shows an example. The Strong Axiom of Revealed Preference (SARP) applies the same sort of consistency requirement to revealed preference as WARP applies to direct revealed preference. To be precise, it says that there should be no *cycles* in revealed preference so, for example, we should never find q^A revealed preferred to q^B, q^B revealed preferred to q^C and yet q^C chosen when q^A is cheaper (or any longer cycle). SARP: No Cycle

q^A R q^B
$\begin{cases} q^A R_D q^B \\ (\text{or}) \ q^A R_I q^B \end{cases}$

SARP

Below we develop a full theory of consumer optimisation based on preference satisfaction but essentially this is doing no more than teasing out the implications of SARP.

SARP is equivalent to the assumption of consumer optimization.

Figure 2.1: WARP

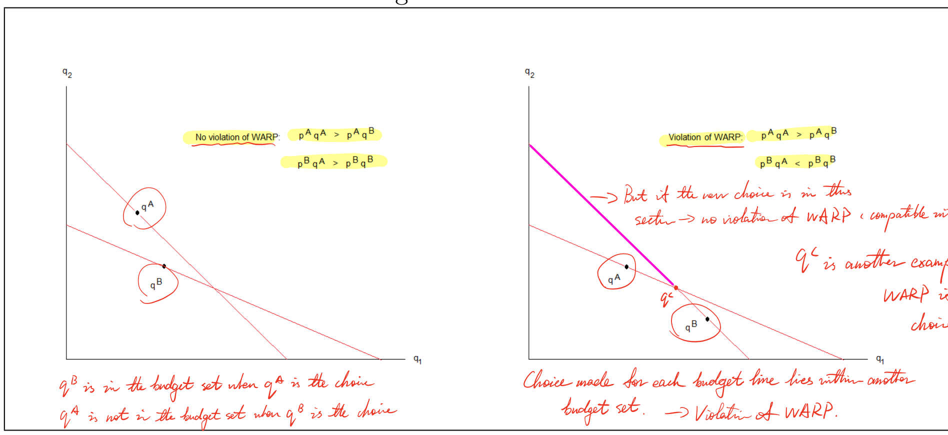
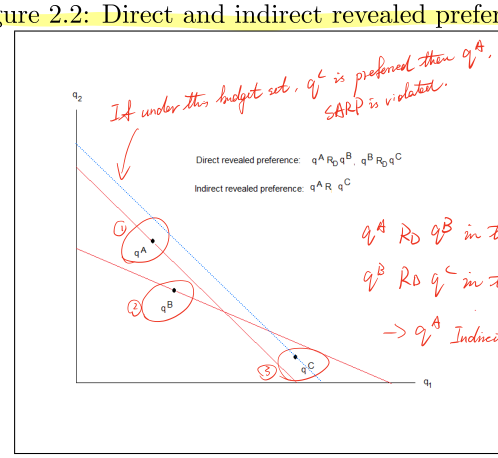


Figure 2.2: Direct and indirect revealed preference

* If only 2 goods, any violation of SARP implies a violation of WARP.
(More than 2 goods, SARP is stronger than WARP)



q^A R_D q^B in the first budget set
q^B R_D q^c in the second budget set
→ q^A Indirectly Revealed Preferred to q^c
(the chain can even be longer)

2.2 Negativity

$\vec{p}_A \to \vec{p}_B = \vec{p}_A + \Delta\vec{p}$
$\vec{q}_A \to \vec{q}_B = \vec{q}_A + \Delta\vec{q}$

e.g. an increase in P_1 → Slutsky Compensation $= \Delta p_1 \times q_1^0$

Negativity

Suppose that as prices change from $\vec{p}^A$ to $\vec{p}^B$ the consumer is compensated in the sense that their total budget is adjusted to maintain affordability of the original bundle. This is known as Slutsky compensation. Slutsky Comp. → Then choices before and after satisfy $p^{B\prime}q^A = p^{B\prime}q^B$. But the later choice cannot then have been cheaper at the initial prices or the change would violate WARP. Hence $p^{A\prime}q^A \leq p^{A\prime}q^B$. By subtraction we get negativity:

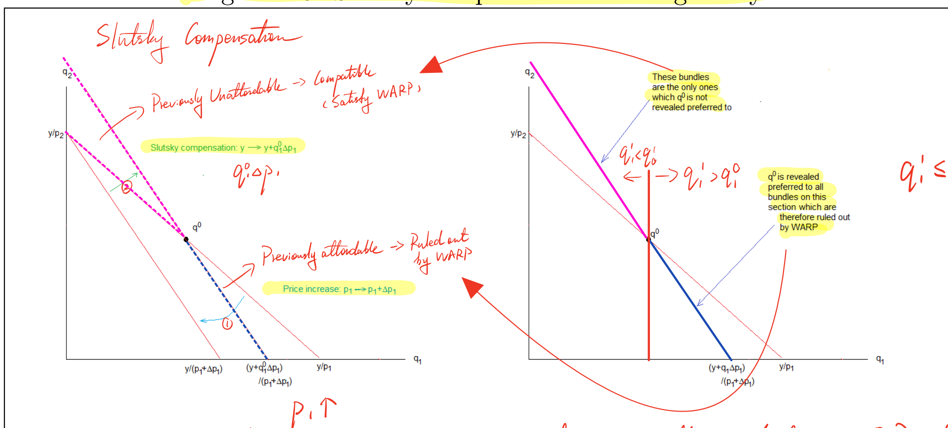
Slutsky Comp. → $\vec{p}_B{}'\vec{q}_A = \vec{p}_B{}'\vec{q}_B$
WARP → $\vec{p}_A{}'\vec{q}_A \le \vec{p}_A{}'\vec{q}_B$
Subtract
$\Delta\vec{p}'q^A \ge \Delta\vec{p}'q^B$

①
$$(p^B - p^A)'(q^B - q^A) \leq 0.$$

$\Delta\vec{p}' \times \Delta\vec{q} \le 0$ i.e. $\sum_i \Delta p_i \Delta q_i \le 0$

This shows a sense in which price changes and quantity changes must move, on average, in opposite directions if the consumer is compensated. Figure 2.3 illustrates the argument graphically for the case of two goods.

Figure 2.3: Slutsky compensation and negativity



$q_1' \leq q_1^0$ to satisfy WARP

Slutsky-Compensated Demand

$p_1 \uparrow$

* If the consumer is Slutsky compensated for an increase in price for q_1, they satisfy WARP if and only if they consume no more of q_1.

If we consider the case where the price of only one good changes then we see that this implies that Slutsky-compensated effects of own price rises must be negative. In other words, Slutsky-compensated demand curves necessarily slope down. $P\uparrow Q\downarrow$

Note the weakness of the assumptions needed for this conclusion. (Only WARP)

2.3 Slutsky equation

The Slutsky-compensated demand function given initial bundle q^A is defined by

$$\underset{\text{(Slutsky)}}{g(q^A, p)} = \underset{\text{(Marshallian)}}{f(p'q^A, p)} \qquad y = \vec{p}'q^A$$

- that is to say it is the demand if budget is constantly adjusted to keep initial choice q^A affordable. Differentiating establishes a relationship between Slutsky-compensated and Marshallian price effects known as the Slutsky equation

② Slutsky Equation:

$$\underbrace{\frac{\partial g_i}{\partial p_j}}_{\text{Compensated/Sub Effect}} = \underbrace{q_j^A \frac{\partial f_i}{\partial y}}_{-\text{ Income Effect}} + \underbrace{\frac{\partial f_i}{\partial p_j}}_{\text{Uncompensated / Total Effect}}.$$

(demand of i, price of j)

The difference between the uncompensated effect $\partial f_i/\partial p_j$ and compensated effect (or substitution effect) $\partial g_i/\partial p_j$ is known as the income effect $-q_j^A \partial f_i/\partial y$.

We can write the equation also in terms of elasticities as

$$\frac{p_j}{q_i}\cdot\frac{\partial g_i}{\partial p_j} = \frac{p_j}{q_i}\frac{\partial f_i}{\partial q_j} + \left(\frac{p_j q_j}{y}\right)\left(\frac{\partial f_i}{\partial y}\cdot\frac{y}{q_i}\right)$$

③
$$\eta_{ij}^* = \eta_{ij} + w_j^A \epsilon_i$$

when $i = j$: $\eta_{ii}^* = \eta_{ii} + w_i \epsilon_i$

Compensated Price Elasticity $_{ij}$ ($\eta_{ij}^* = \frac{\partial \ln g_i}{\partial \ln p_j}$) = Uncompensated Price Elasticity $_{ij}$ ($\eta_{ij} = \frac{\partial \ln f}{\partial \ln p_j}$) + Total Budget Share $_j$ × Total Budget Elasticity $_i$

where $\eta_{ij}^* = \partial \ln g_i / \partial \ln p_j$ is the compensated price elasticity.

Since the term $q_j^A \partial f_i/\partial y$ is positive if the good is normal and the Slutsky-compensated effect has been shown to be necessarily negative by WARP, so must be the Marshallian effect for normal goods. This is the Law of Demand. $\eta_{ii}^* < 0$ (WARP), if $\eta_{ii} < 0$ (Normal Good) $\rightarrow \epsilon_i > 0$

Giffen goods are therefore possible for a consumer satisfying WARP only if

$$0 > \eta_{ij}^* > w_j^A \epsilon_i.$$

$\eta_{ii}^* < 0$ (WARP), if $\eta_{ii} > 0 \rightarrow \epsilon_i < 0$ and $w_i \epsilon_i < 0$

(Budget share w_i needs to be large enough)

$\eta_{ii} + w_i^A \epsilon_i < 0$

Case Study 2|1 : Revealed Preference Tests

Revealed preference ideas can be applied to data on consumer outcomes to test compatibility of behaviour with economic models in ways that are not restrictively tied to any particular specification of preferences. Famulari, for example, took data on over 4000 households in 25 metropolitan districts of the US between 1982 and 1985. She grouped them into 43 demographic types, to control for taste heterogeneity, and looked for revealed preference violations in pairwise comparisons involving consumption of nine categories of items. For the median demographic type her results showed violations in only 0.70% of comparisons. However, violations were much more likely in comparisons involving similar total expenditure. When total expenditures were less than 20% apart the rate of violation was 2.67% and when less than 5% this rose to 5.13%. Violations of this sort could reflect inconsistent behaviour but could also arise from other explanations such as taste change, issues of within-household decision-making, problems of data aggregation or other inappropriateness of the model applied.

[Source: M. Famulari, 1995, A household-based, non-parametric test of demand theory, *Review of Economics and Statistics* 77, 372-382.]

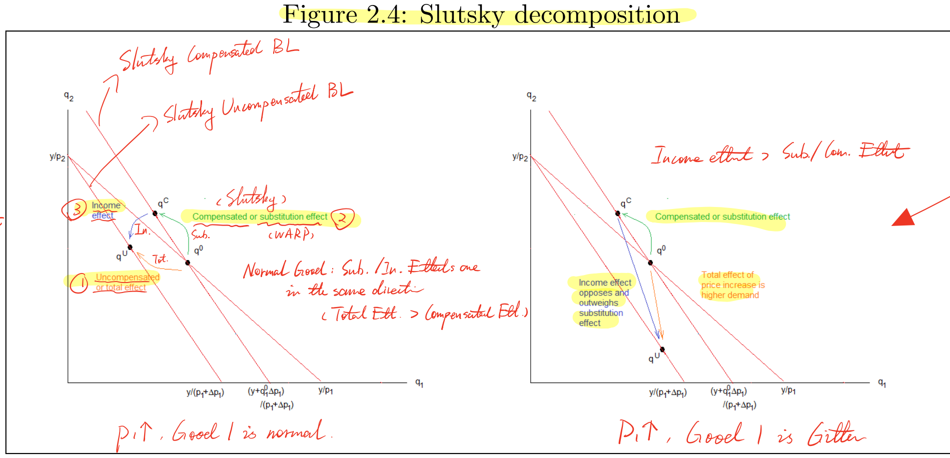
Since a negative Marshallian effect requires that the income effect not only have the opposite sign to the substitution effect but also be large enough to offset it, the Giffen good would need not only to be inferior but also to be consumed in large enough quantity for the price change to have a material impact on the budget set. Figure 2.4 breaks down the uncompensated impact of a price change graphically into substitution and income effects for both an ordinary and a Giffen good.

The Slutsky equation is highly useful. Its importance is that it allows testing of restrictions regarding compensated demands since it shows how to calculate compensated effects from the sort of uncompensated effects estimated in applied demand analysis.

Figure 2.4: Slutsky decomposition

Uncompensated / Total Effect

q^0 Compensated / Substitution Effect q^c Income Effect q^u



∴ For normal goods, the Marshallian demand curve is also downward sloping.
(Law of Demand)

Normal Good $(\varepsilon_i > 0)$: In. Effect and Sub. Effect are in the same direction. → Marshallian D must slope down

Inferior Good $(\varepsilon_i < 0)$: In. Effect and Sub. Effect are in the opposite directions. → Marshallian D can slope up or down

- Special Inferior Good: Giffen good $(\varepsilon_i < 1)$: { In. Eff and Sub. Eff are in opposite directions; In. Eff > Sub. Eff } → Marshallian D must slope up

*All under assumption of WARP!

Worked Example 2|A : Negativity and the Slutsky equation

Consider the linear expenditure system again and assume that homogeneity has been imposed so $c_i = 0$ for all goods. Demand for the ith good is

$$f_i(y, p) = \frac{a_i y}{p_i} + \sum_j b_{ij} \frac{p_j}{p_i}.$$

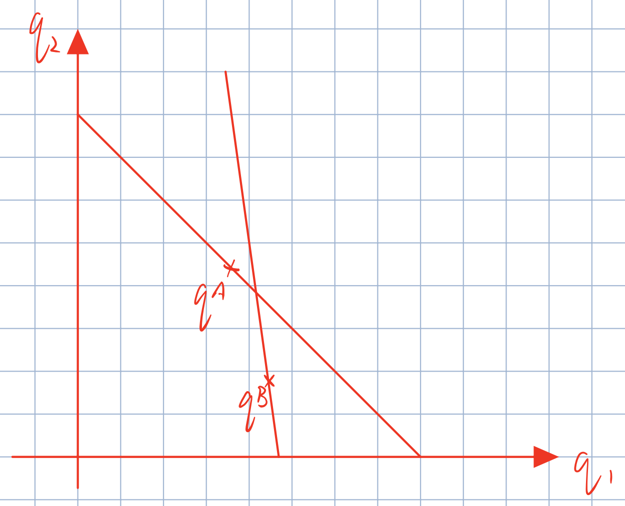
Using the Slutsky equation the compensated own price effect is

$$\frac{\partial g_i}{\partial p_i} = f_i \frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial p_i} = \frac{b_{ii} - (1 - a_i) q_i}{p_i}$$

so compensated demand curves slope down only if

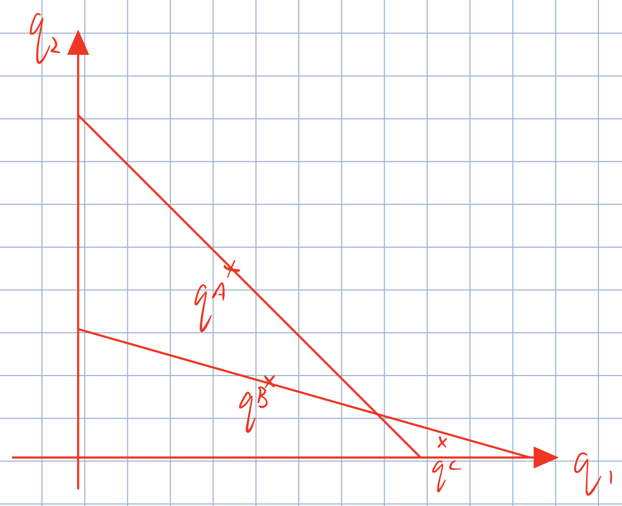
$$f_i(y, p) > \frac{b_{ii}}{1 - a_i}.$$

Direct Revealed Preference



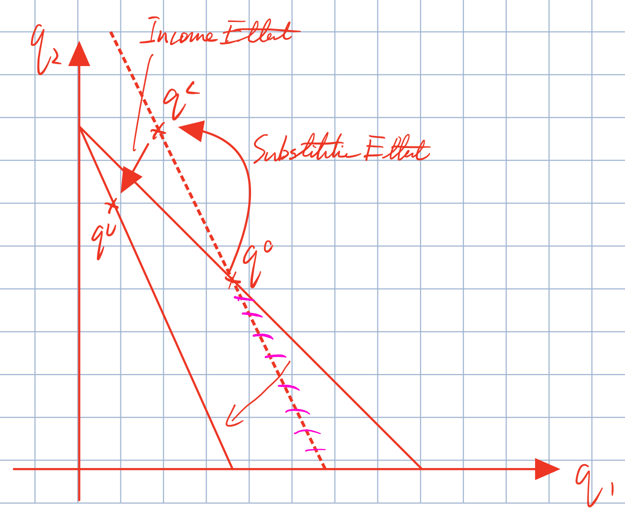
WARP

Indirect Revealed Preference



SARP: the same idea as the consumer is optimizing

Using WARP to draw conclusions about price responses



- Consider $p_1 \uparrow$

- Combine this with a total budget y so that the original bundle is affordable again.
  (Slutsky Compensation)

- WARP: if we increase the price of one good with compensate, consumers will purchase less of that good.

Law of Demand: if the good is normal, $p \uparrow$ uncompensated demand $\downarrow$

If the good is not normal: $P \uparrow$ uncompensated demand may $\uparrow$ (Giffen Good)

Slutsky Equation

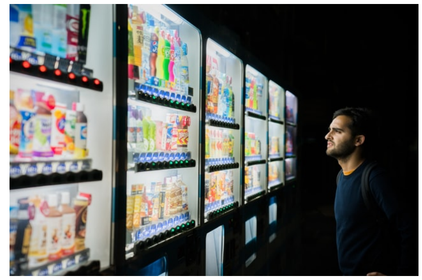
- Define Slutsky Compensated Demand

$$g(p, q^0) = f(p'q^0, p)$$

- $$\frac{\partial g_i}{\partial p_j} = \frac{\partial f_i}{\partial p_j} + q_j^0 \frac{\partial f_i}{\partial y}$$

$$\frac{\partial f_i}{\partial p_j} = \underbrace{\frac{\partial g_i}{\partial p_j}}_{\text{Sub. Effect}} - q_j^0 \underbrace{\frac{\partial f_i}{\partial y}}_{\text{Inc. Effect}}$$

Topic 3 : Consumer preferences



Summary: *Specifying which bundles the consumer prefers to which others provides the foundation for a fuller modelling of choice. The nature of these preferences can be restricted by assumptions which vary from what is minimally necessary for a viable theory to assumptions which seriously constrain the nature of choice.*

3.1 Preferences

Bundle q^A is weakly preferred to bundle q^B

/no worse than

Suppose the consumer has a preference relation $\succsim$ where $q^A \succsim q^B$ means q^A *is at least as good as* q^B. For the purpose of modelling demand this can be construed as an inclination to choose the bundle q^A over the bundle q^B. For modelling welfare effects, the interpretation needs to be strengthened to include a link to consumer wellbeing.

A weak preference relation suffices to define strict preference $\succ$ and indifference $\sim$ if we let

- $\succsim$ and $\precsim$ be equivalent to $\sim$, Indifference: $q^A \succsim q^B$ and $q^B \succsim q^A \to q^A \sim q^B$
- $\succsim$ and $\not\precsim$ be equivalent to $\succ$. Strictly Preference: $q^A \succsim q^B$ and $q^B \not\succsim q^A \to q^A \succ q^B$

For any bundle q^A we can define

- $R(q^A)$ the weakly preferred set (orupper contour set) $R(q^A)$ as all bundles q^B such that $q^B \succsim q^A$ $\{q^B \mid q^B \succsim q^A\}$
- $L(q^A)$ the weakly dispreferred set (or lower contour set) $L(q^A)$ as all bundles q^B such that $q^A \succsim q^B$ $\{q^B \mid q^A \succsim q^A\}$
- $I(q^A)$ the indifference set $I(q^A)$ as all bundles q^B such that $q^B \sim q^A$ (or, in other words, as the intersection of $R(q^A)$ and $L(q^A)$) $\{q^B \mid q^A \sim q^B\}$

$I(q^A) = R(q^A) \cap L(q^B)$

3.2 Consumer Rationality

We want the preference relation to provide a basis to consistently identify a set of most preferred elements in any possible budget set and for this we need assumptions.

- **Completeness** Either $q^A \succsim q^B$ or $q^B \succsim q^A$. This ensures that choice is possible in any budget set.

The consumer can always make a decision.

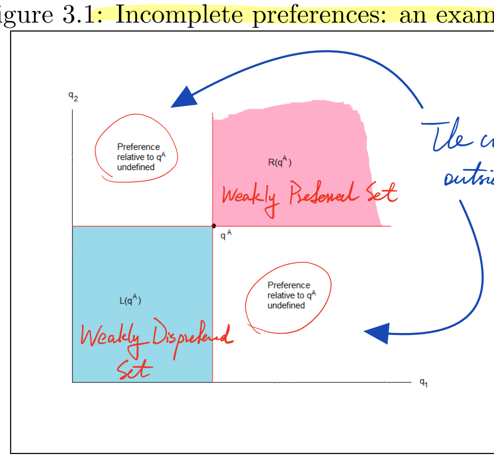
This requires that $R(q)$ and $L(q)$ fill up the consumer space.

Counterexample to Completeness :

Consumer preferences are incomplete if, say, there are two goods q_1 and q_2 and $q^A \succsim q^B$ if and only if both $q_1^A \geq q_1^B$ and $q_2^A \geq q_2^B$. With these preferences, the consumer simply cannot decide between two bundles if each bundle has more of one of the two goods. This counterexample is illustrated in Figure 3.1.

Consumer preferences would also be incomplete if specified using a criterion which is undefined for part of the consumption set. For example, if there are two goods q_1 and q_2 and $q^A \succsim q^B$ if and only if $\ln\left(q_1^A - 1\right) + \ln\left(q_2^A - 1\right) \geq \ln\left(q_1^B - 1\right) + \ln\left(q_2^B - 1\right)$ then the criterion for comparison would be undefined if the quantity of either good in either bundle were less than or equal to 1. Of course, in a case like this, if the budget set allows the consumer to consume quantities of all goods above 1 then it would probably be sensible to assume the consumer would never want to consume such quantities and simply redefine the consumption set to exclude the problematic bundles.

Figure 3.1: Incomplete preferences: an example



The consumer cannot make a decision outside $L(q^A)$ and $R(q^A)$.

q^A $\precsim$ $\succsim$ $q^B \leq q^C$

- **Transitivity** $q^A \succsim q^B$ and $q^B \succsim q^C$ implies $q^A \succsim q^C$. This ensures that there are no *cycles* in preferences within any budget set.

  Graphically this requires nesting of weakly preferred sets. If one bundle q^B is in the weakly preferred set of q^A then its own weakly preferred $R(q^B)$ set must lie entirely within $R(q^A)$ as illustrated in Figure 3.2.

Consumer Rationality { Completeness, Transitivity → Preference ordering in any case

Together these two assumptions are often referred to as *consumer rationality*. They ensure that the preference relation is a preference ordering.

To make preferences mathematically well behaved we make the technical assumption:

- **Continuity** If $q^A \succsim q^B$ and $q^B \succsim q^C$ then there is a bundle indifferent to q^B on any path joining q^A to q^C. This rules out discontinuous jumps in preferences. (见下页)

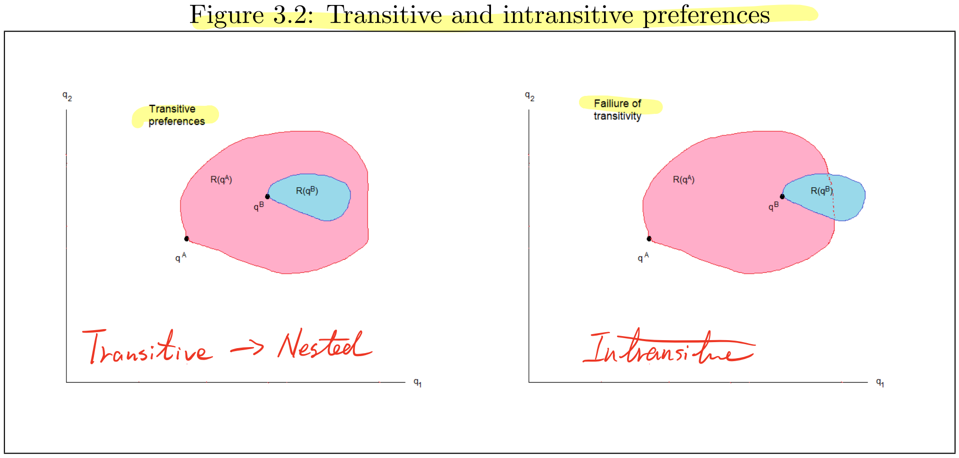
Continuity requires that weakly preferred sets contains its boundaries. ($R(q)$) (Closed Sets)

Counterexample to Transitivity :

Consumer preferences are intransitive if, say, there are two goods q_1 and q_2 and $q^A \succsim q^B$ if and only if either $q_1^A \geq q_1^B$ or $q_2^A \geq q_2^B$. Take bundles q^A and q^C such that both $q_1^A > q_1^C$ and $q_2^A > q_2^C$. Clearly $q^A \succ q^C$. But if we take another bundle q^B such that $q_1^B < q_1^C$ but $q_2^B > q_2^A$ then $q^B \succsim q^A$ and $q^C \succsim q^B$. Hence a preference cycle exists and transitivity is violated. Figure 3.3 illustrates. (T-Ws)

To take another example, suppose there are three goods and preferences are such that $q^A \succsim q^B$ if and only if the number of goods i for which $q_i^A > q_i^B$ is at least as many as the number of goods for which $q_i^A < q_i^B$. If we take the bundles $q^A = (0,1,2)$, $q^B = (2,0,1)$ and $q^C = (1,2,0)$ then we see that $q^A \succ q^B \succ q^C \succ q^A$ which clearly violates transitivity. (These are the sort of preferences that might arise for a collectivity through members voting entirely rationally but differing in their opinions of which goods are most important. A majority voting cycle like this is called a Condorcet cycle.) (no less of 2 of the 3 goods)

Figure 3.2: Transitive and intransitive preferences



If preferences are continuous, utility functions exist to describe preferences. (Guaranteed)

3.3 Utility functions

Order of utility functions = Order of preferences

A utility function $u(q)$ is a representation of preferences such that $q_A \succsim q_B$ if and only if $u(q_A) \geq u(q_B)$. A utility function exists if preferences give a continuous ordering.

Utility functions are not however unique since if $u(q)$ represents certain preferences then any strictly increasing transformation $\phi(u(q))$ also represents the same preferences. We say that utility functions are ordinal.

Utility function is a function that $u(q^A) \geq u(q^B)$ if and only if $q^A \succsim q^B$.

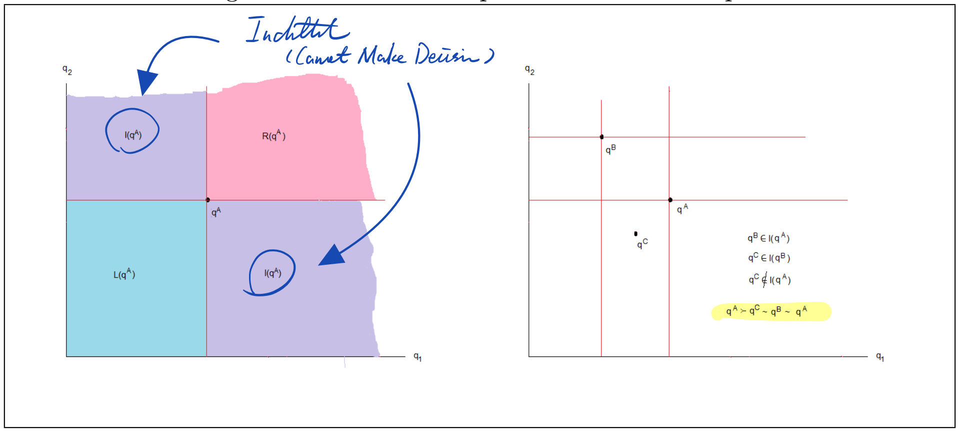
3.4 Nonsatiation, monotonicity and convexity

A further assumption rules out consumers ever being fully satisfied:

- **Nonsatiation** Given any bundle there is always some direction in which changing the bundle will make the consumer better off.

i.e. for any bundle q^A, there is always another bundle q^B arbitrarily close to q^A such that $q^B \succsim q^A$.

Figure 3.3: Intransitive preferences: an example



> *Counterexample to Continuity* :
>
> Continuity is violated by the example of lexicographic preferences. Say that there are two goods q_1 and q_2 and that the consumer prefers one bundle to another if and only if it either has more of q_1 or the same amount of q_1 and more of q_2. The consumer is indifferent between no bundles that differ in any way and indifferent sets are single points. Given any three different bundles it is easy to draw a path from the least to the most preferred that does not pass through any bundle indifferent to the third, since it is only necessary to avoid the path passing directly through the third bundle itself. Figure 3.4 illustrates the notion of continuity and shows the anomalous nature of weakly preferred and dispreferred sets under lexicographic preferences.

先比较第一个，相同则比较第二个

下一页

If this is true then indifferent sets have no thick regions to them and we can visualise them as indifference curves. There are also no local maxima to utility or bliss points. Figure 3.5 illustrates the sorts of things ruled out. (下一页)

The next assumption takes this further by specifying the direction of increasing preference:

- **Monotonicity** Larger bundles are preferred to smaller bundles.

i.e. $q^A > q^B$ whenever q^A contains more of one good and no less of any other

Given monotonicity, indifference curves must slope down (as seen in Figure 3.6). This slope is known as the marginal rate of substitution (MRS). (下两页)

Monotonicity → $u(q)$ must be increasing in every good's quantity. $MU = \frac{\partial u}{\partial q_i} > 0 \ \forall i$

Monotonicity corresponds to increasingness of the utility function $u(q)$. An indifference curve is defined by $u(q)$ being constant and therefore the MRS is given by

Marginal Rate of Substitution :

$$\text{MRS} = \left.\frac{dq_2}{dq_1}\right|_{u(q)=\bar{u}} = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2} < 0$$

the rate at which the consumer is willing to give up one good for another while remaining on the same level of satisfaction.

which is obviously negative if $\underbrace{\partial u/\partial q_1, \ \partial u/\partial q_2 > 0}_{\text{Ensured by Monotonicity}}$.

Although $u(q)$ is not unique, MRS will always have the same expression :

$\phi(u(q))$ where $\phi' > 0$

$MRS = -\frac{\partial u/\partial q_1 \cdot \phi'(u)}{\partial u/\partial q_2 \cdot \phi'(u)}$

$\partial u/\partial q_1$

Derivation :

An indifference curve is defined by $u(q_1, q_2) = \bar{u}$

~~Differentiate~~ with respect to q_1 but allow q_2 to vary so as to keep the consumer on the IC

$= -\frac{\partial u/\partial q_1}{\partial u/\partial q_2}$

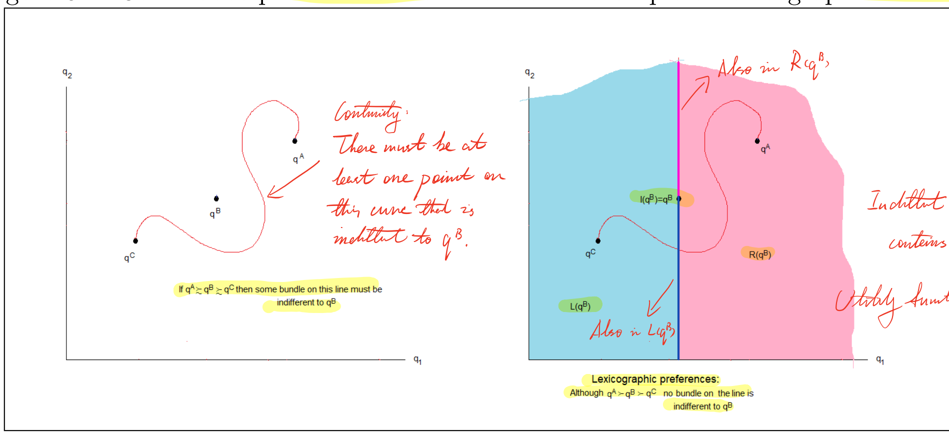
$$\frac{\partial u}{\partial q_1} + \frac{\partial u}{\partial q_2} \times \frac{\partial q_2}{\partial q_1}\bigg|_{u(q)=\bar{u}} = 0$$

$$\downarrow$$

$$MU_1 + MU_2 \cdot MRS = 0$$

Must be negative $\rightarrow MRS = -\frac{MU_1}{MU_2} \leftarrow$ Both positive by Monotonicity

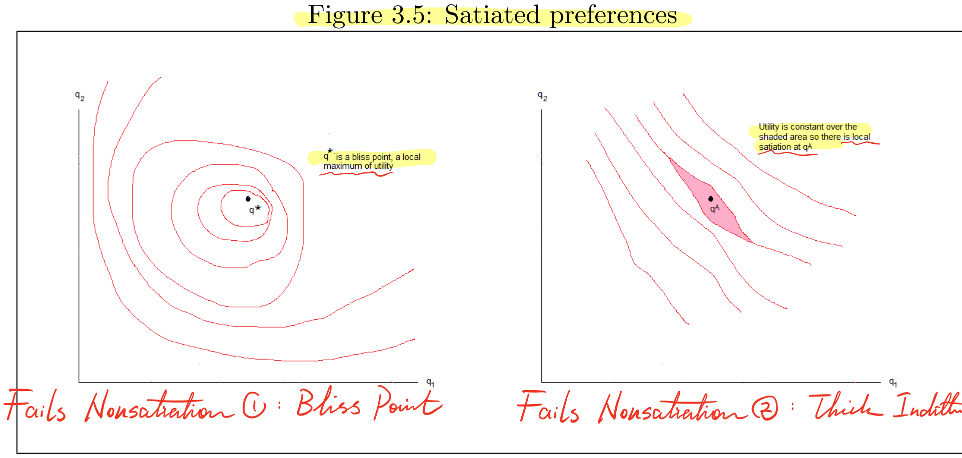
Figure 3.4: Continuous preferences with the counterexample of lexicographic ordering



Indifferent Set $I(q^B)$ only contains itself and no other bundle.

Utility function does not exist for this.

Figure 3.5: Satiated preferences



Fails Nonsatiation ①: Bliss Point

Fails Nonsatiation ②: Thick Indifferent Set + Local Maximum

> *Counterexample to Nonsatiation and monotonicity :*
>
> Suppose preferences are represented by utility function $u(q) = -(q-\gamma)'(q-\gamma) = -\sum_i (q_i - \gamma_i)^2$ for some bundle γ. Indifference curves for such preferences are circles around the "bliss point" γ. Nonsatiation is violated since there is a bundle, γ, from which there is no direction in which it is possible to increase consumer satisfaction. Utility is increasing in all quantities provided that $q_i < \gamma_i$ for all i but not otherwise.

- **Convexity** $\lambda q^A + (1-\lambda)q^B \succsim q^B$ if $q^A \succsim q^B$ and $1 \geq \lambda \geq 0$.

This says that weakly preferred sets are convex or, equivalently, MRS is diminishing. Convex preferences look like those illustrated in Figure 3.7 (T-Q2)

i.e. for any two bundles $q^A \sim q^B$, then for any bundle $q^c = \lambda q^A + (1-\lambda)q^B$

$\forall\ 0 < \lambda < 1,\ q^c \succsim q^A, q^B$

To test Convexity: Check whether MRS is diminishing e.g. $u(q) = q_1 + \ln q_2$

$MRS = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2} = -\frac{1}{1/q_2} = -q_2$ seems to be not related to q_1, but NOT the case! because this does not capture the fact that the consumer has to be on the indifference curve.

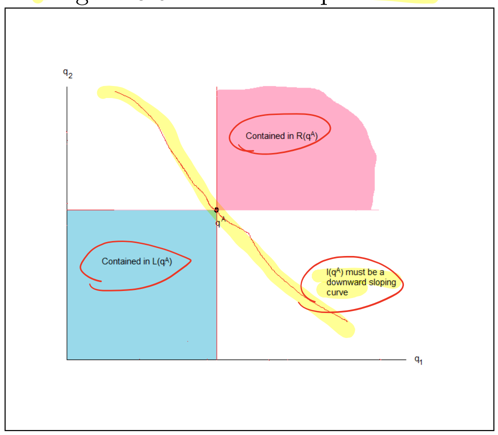
Holding $u(q) = \bar{u}$: $q_1 + \ln q_2 = \bar{u}$

$q_2 = e^{\bar{u}} - q_1$

What we should do instead

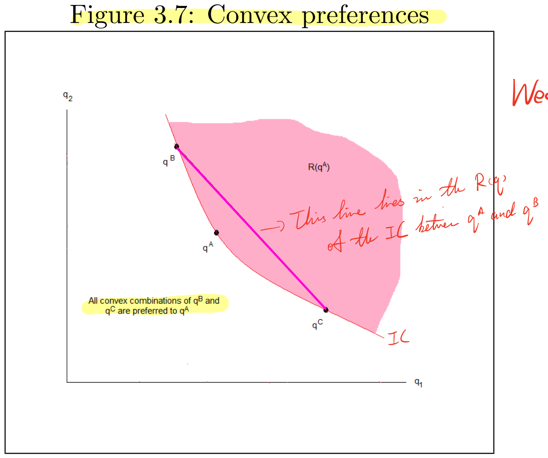
$\frac{\partial q_2}{\partial q_1} = -e^{\bar{u}-q_1}$

Figure 3.6: Monotonic preferences



Under monotonicity, there is an indifference map consisting which of downward sloping indifference curves.

Figure 3.7: Convex preferences



Weakly preferred sets have to be Convex Sets

Counterexample to Convexity :

Suppose preferences are represented by utility function $u(q) = q'q = \sum_i q_i^2$. Preferences are monotonic but indifference curves are quarter-circles centred on the origin and MRS is increasing rather than diminishing.

Convexity can be interpreted as capturing taste for variety. It says that a consumer will always prefer to mix any two bundles between which they are indifferent. The corresponding property of the utility function is known as *quasiconcavity*.

3.5 Homotheticity and quasilinearity

Homotheticity and quasilinearity are both strong restrictions expressing different ways in which indifference curves within an indifference map can share a common shape.

does not impose constrains on individual IC

If preferences are homothetic, MRS will be constant on any ray through the origin.
i.e. MRS is homogenous of degree 0 on quantities
MRS only depends on the ratio of the 2 goods
e.g. in 2 goods case, only depends on $\frac{q_1}{q_2}$

- **Homotheticity** If $q^A \sim q^B$ then $\lambda q^A \sim \lambda q^B$ for any $\lambda > 0$.

Budget Share will not depend on y or v

Graphically this means that higher indifference curves can be constructed from lower ones by magnifying from the origin. This is a strong restriction that would rarely be made in practice but it is useful to consider as a reference case. It is not a restriction on the shape of any one indifference curve but on the relationship between indifference curves within an indifference map.

If preferences are homothetic then marginal rates of substitution are constant along rays through the origin. This is only true for homothetic preferences and this is usually an easy way to check whether given preferences are homothetic.

Homogenous $u(q)$ → Homothetic preferences

If there exists a homogeneous utility representation $u(q)$ where $u(\lambda q) = \lambda u(q)$ then preferences can be seen to be homothetic. Since increasing transformations preserve the properties of preferences, then any utility function which is an increasing function of a homogeneous utility function also represents homothetic preferences. In fact all utility functions representing homothetic preferences are of this form.

Specifically, preferences are homothetic iff utility is an increasing transformation of a linearly homogenous function of q.

- **Quasilinearity (with respect to the first good)** If $\begin{pmatrix} q_1^A \\ q_2^A \\ q_3^A \\ \vdots \end{pmatrix} \sim \begin{pmatrix} q_1^B \\ q_2^B \\ q_3^B \\ \vdots \end{pmatrix}$ then $\begin{pmatrix} \boxed{q_1^A + \lambda} \\ q_2^A \\ q_3^A \\ \vdots \end{pmatrix} \sim \begin{pmatrix} \boxed{q_1^B + \lambda} \\ q_2^B \\ q_3^B \\ \vdots \end{pmatrix}$ for any λ.

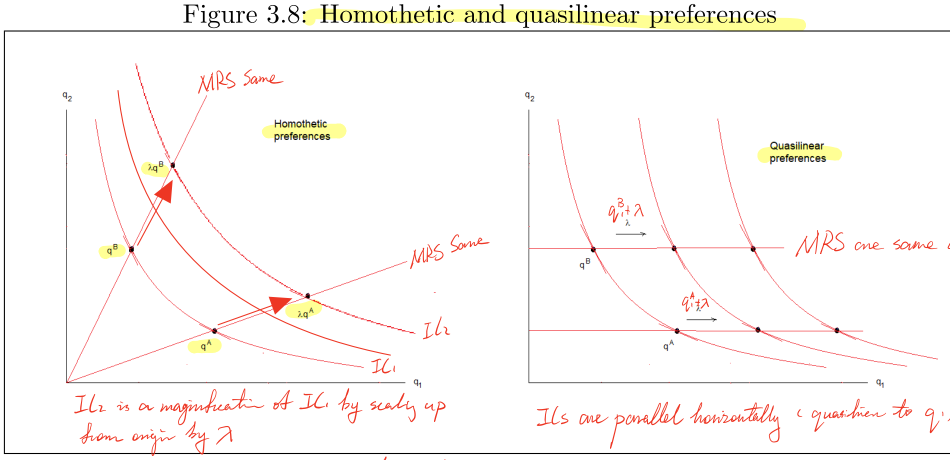
i.e. MRS only depends on other goods (used to check)

Quasilinearity is another strong restriction based on a similar idea. It says that adding the same amount to one particular good preserves indifference. This means that higher indifference curves are parallel translations of lower ones. In this case, marginal rates of substitution are constant along lines parallel to axes.

If there exists a utility representation $u(q)$ such that $u(q_1, q_2, \ldots) = q_1 + F(q_2, q_3, \ldots)$, say, then preferences are quasilinear. This is also true of any utility functions which are increasing transformations of functions with this property.

Both homothetic and quasilinear preferences are illustrated for the case of two goods in Figure 3.8.

Figure 3.8: Homothetic and quasilinear preferences



MRS are same along horizontal lines

All indifference curves have the same shape and any IC can be derived from one another by magnification and shrink.

Worked Example 3|A : Preferences

Some examples of preference structures illustrate some of these properties:

- **Perfect substitutes** $u(q) = \sum_i \alpha_i q_1$: The MRS between the ith and jth good is $-\alpha_i/\alpha_j$ and is constant. In two dimensions, indifference curves are parallel straight lines as illustrated in Figure 3.9. Weakly preferred sets are therefore (weakly but not strictly) convex. These are the only continuous nonsatiated preferences which are homothetic *and* quasilinear.

- **Perfect complements** $u(q) = \min[\alpha_1 q_1, \alpha_2 q_2, \ldots, \alpha_M q_M]$: In two dimensions, indifference curves are L-shaped with the kinks lying on a ray through the origin of slope α_1/α_2 as in Figure 3.9. Weakly preferred sets are convex. These preferences are homothetic but not quasilinear.

- 也可以写为 $q_1^\alpha q_2^\beta \ldots$，这里只是取了log transformatin

  Cobb-Douglas preferences: $u(q) = \sum_i \alpha_i \ln q_i$: Preferences are homothetic, indifference curves are smooth and weakly preferred sets are convex as in Figure 3.10. MRS is $-\alpha_i q_j/\alpha_j q_i$. Typically, the expression for utility would be scaled so that $\sum_i \alpha_i = 1$.

  If there are two goods and we look along the indifference curve corresponding to utility υ then $q_2 = e^{\upsilon/(1-\alpha)} q_1^{-\alpha/(1-\alpha)}$. So the MRS is

  $$MRS = -\frac{\alpha}{1-\alpha} e^{\upsilon/(1-\alpha)} q_1^{-1/\alpha}$$

  which is diminishing in q_1.

- **Stone-Geary preferences**: $u(q) = \sum_i \alpha_i \ln(q_i - \gamma_i)$: This is a simple modification to Cobb-Douglas under which preferences are no longer homothetic but indifference curves remain smooth and MRS $-\alpha_i (q_j - \gamma_j) / \alpha_j (q_i - \gamma_i)$ is still diminishing.

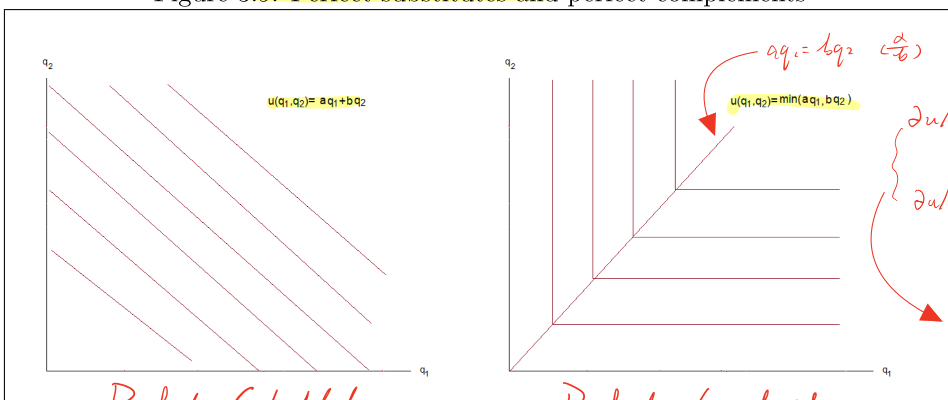
- **A quasilinear example**: $u(q) = \alpha_1 q_1 + \sum_i \alpha_i \ln q_i$: Preferences are quasilinear, indifference curves are smooth and weakly preferred sets are convex as in Figure 3.10. MRS between the first good and the jth is $-\alpha_1 q_j/\alpha_j$.

  If there are two goods and we look along the indifference curve corresponding to utility υ then $q_2 = e^{(\upsilon - \alpha_1 q_1)/\alpha_2}$. So the MRS is

  $$MRS = -\frac{\alpha_1}{\alpha_2} e^{(\upsilon - \alpha_1 q_1)/\alpha_2}$$

  which is diminishing in q_1.

Figure 3.9: Perfect substitutes and perfect complements

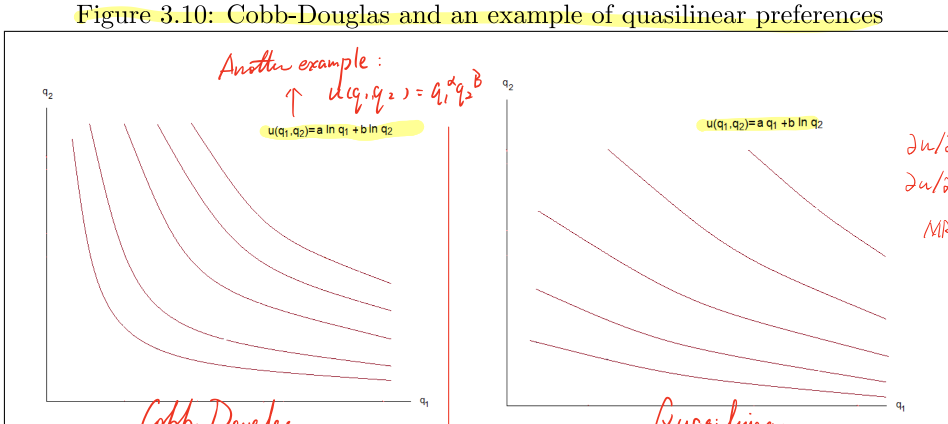


Perfect Substitutes   Perfect Complements

$\partial u/\partial q_1 = 0$ if $aq_1 > bq_2$
$= a$ if $aq_1 < bq_2$
$\partial u/\partial q_2 = 0$ if $bq_2 > aq_1$
$= b$ if $bq_2 < aq_1$

$MRS = 0$ if $aq_1 > bq_2$
$= -\infty$ if $aq_1 < bq_2$

Figure 3.10: Cobb-Douglas and an example of quasilinear preferences



Cobb-Douglas   Quasilinear

$\partial u/\partial q_1 = a$
$\partial u/\partial q_2 = b/q_2$
$MRS = -aq_2/b$

$\partial u/\partial q_1 = a/q_1$
$\partial u/\partial q_2 = b/q_2$
$MRS = -\frac{aq_2}{bq_1} = -\frac{a}{b}\cdot\frac{q_2}{q_1}$

Preference Relation $\succsim$

$q^A \succsim q^B$ Weak Preference

$q^A \succ q^B$ Strict Preference

$\sim$ Indifference

Weakly Preferred Set: all bundles weakly preferred to q

Indifferent Set: all bundles that are indifferent to q

Properties: Consumer Rationality:

Completeness: no gaps in preferences
Transitivity: no loops in preferences

} Preference relation is a preference ordering

Continuity: no jumps in preferences → Exists a utility function to represent preferences

$u(q^A) > u(q^B) \Longleftrightarrow q^A \succsim q^B$

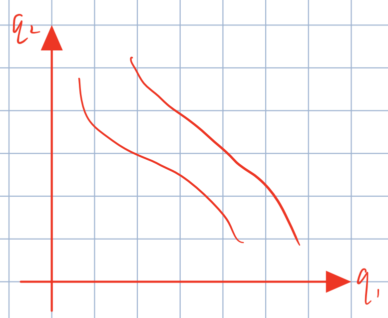
- Not unique

Further Assumptions:

Non-Satiation: there's always a direction in which consumers move towards a higher preference

→ No flat sections of utility / no local maxima

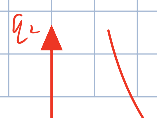
→ Indifference curves

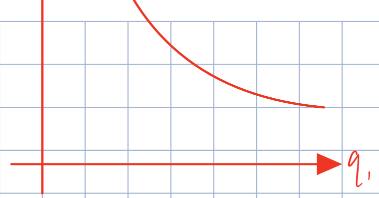


Monotonicity: consumers prefer more

→ Indifference curves slope down

Convexity: weakly preferred sets are convex (consumers prefer variety)



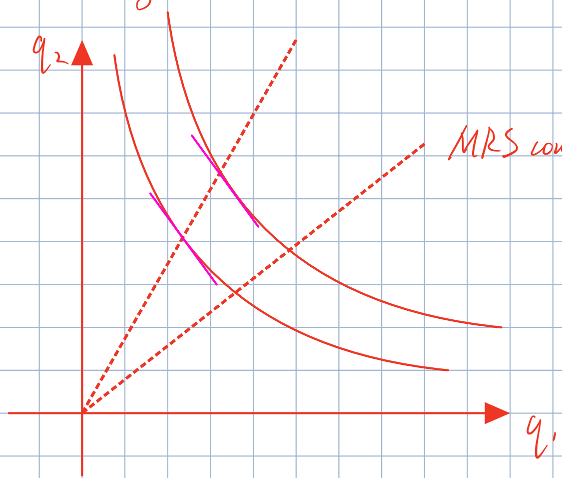


Slope of indifference curves : $MRS = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2}$

MRS are unique

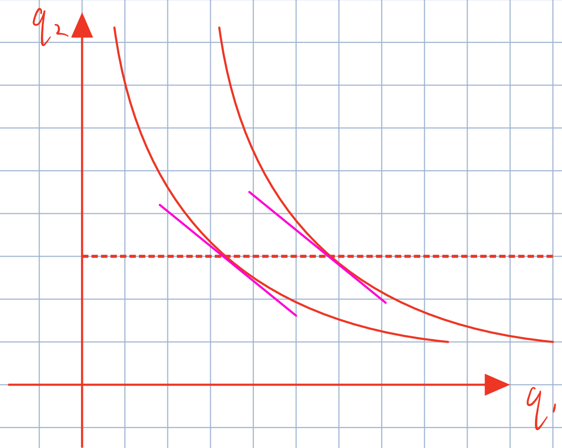
Classes of Preferences

Homotheticity

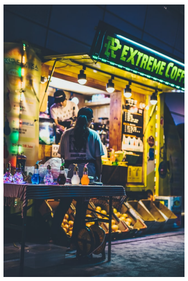


MRS constant along rays from the origin

Quasilinearity



MRS constant along horizontal lines



Topic 4 : Consumer choice

Summary: *If the consumer consistently chooses the most preferred among affordable bundles then it is possible to give a complete description of the properties that demands must satisfy. To see this, it is illuminating to think of consumers also as expenditure minimisers subject to utility constraints.*

Consumer Optimization

4.1 Consumer optimisation

Suppose that the consumer chooses q so as to best satisfy their preferences within the budget set, which is to say to solve

Max utility function s.t. budget set

$$\max_q u(q) \quad \text{s.t.} \quad p'q \leq y.$$

Solution will define Marshallian Demand $q = f(y, p)$

Then choices satisfy *WARP*. In fact, the assumption of consumer optimisation is equivalent to *SARP*. Utility maximising choices therefore satisfy negativity, homogeneity and, assuming non-satiation, adding up.

Tangency

4.2 Tangency condition

If preferences are monotonic and convex then the solution to the consumer's optimisation problem is at a tangency between an indifference curve and the boundary of the budget set.

If the budget set is linear then this can occur either in the interior of the budget set or at a corner where one or more of the goods is not consumed. At interior solutions with smooth convex indifference curves the MRS is equal to the price ratio. Corner solutions occur when no such point exists. Figure 4.1 shows both cases.

Interior : Both are consumed $MRS = -\frac{p_1}{p_2}$
Corner : Only 1 good is consumed

That optima occur at tangencies is true even for non-linear budget sets. However if the budget set is linear (or indeed simply convex) then we know that such a tangency is unique so finding one guarantees that we have found the best choice for the consumer. If the budget set is not convex, on the other hand, then there can be multiple tangencies and the optimum can typically be found only by comparing the level of utility at each of them. Figure 4.2 shows the problem.

The nature of Marshallian demands can then be inferred by moving the budget constraint to capture changes in y and p and tracing out movement of the tangency. In this way both income expansion paths and offer curves can be constructed as in Figure 4.3.

How to find tangencies for a kinked budget set

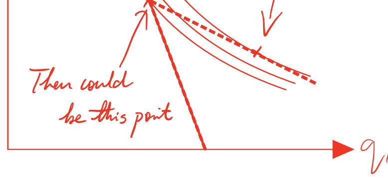


Figure 4.1: Consumer optimisation with a linear budget set

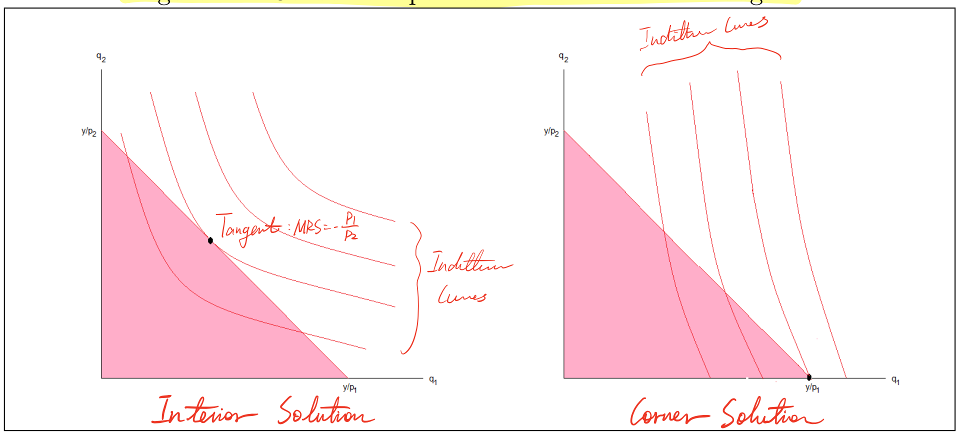
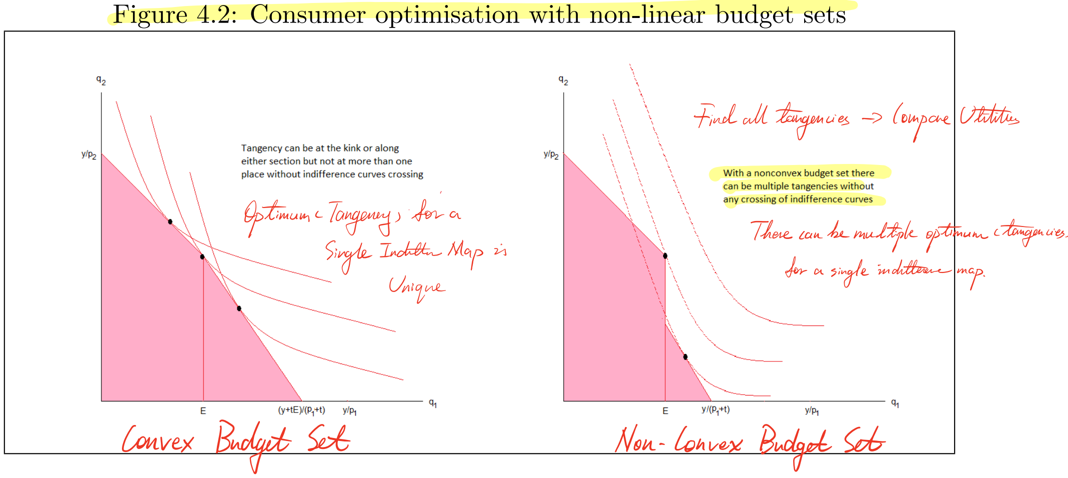


Figure 4.2: Consumer optimisation with non-linear budget sets



4.3 Demand under homotheticity and quasilinearity

As income increases. slopes of budget constraints do not change. Income expenditure paths traced out by the tangencies as incomes are increased therefore all occur at points with the same MRS.

Homotheticity and quasilinearity are each characterised by the nature of paths along which MRS is constant and therefore each give rise to income expansion paths of particular shapes.

Every Good's Total budget elasticity = 1 (Only in this case)

- **Homotheticity** MRS is constant along rays through the origin so income expansion paths are rays through the origin. Quantities consumed are proportional to total budget y given prices and budget shares are independent of y. Marshallian D: $q = f(p, y) = a(p)y$ Budget share will be constant.

- **Quasilinearity** MRS is constant along lines parallel to one of the axes so income expansion paths are parallel to one of the axes. Quantities demanded of all but one of the goods are independent of y (so long as all goods are chosen in positive quantities). If total budget falls low enough then a corner solution is reached with quantity of that good equal to zero.

$$\begin{cases} q_2 = f_2(y, p) = b(p) \text{ only depend on prices} \\ q_1 = f_1(y,p) = (y - p_2 b(p))/p_1 \end{cases}$$

This is the only kind of preferences under which demand for a good is independent of income.

Income Expansion Paths are paths along which MRS is constant $= -\frac{p_1}{p_2}$

Figure 4.3: Income expansion paths and offer curves

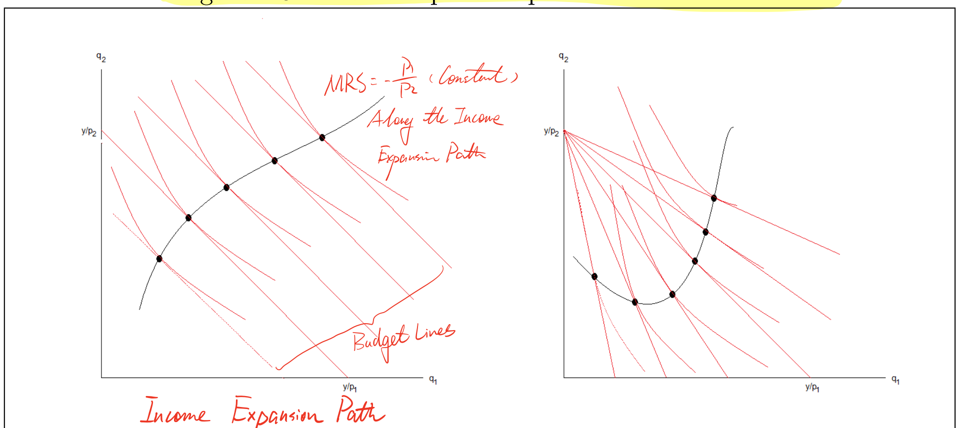


Figure 4.4 illustrates income expansion paths for both classes of preferences.

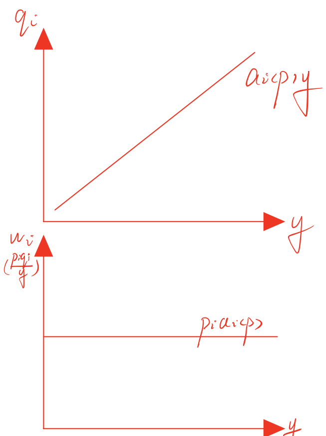
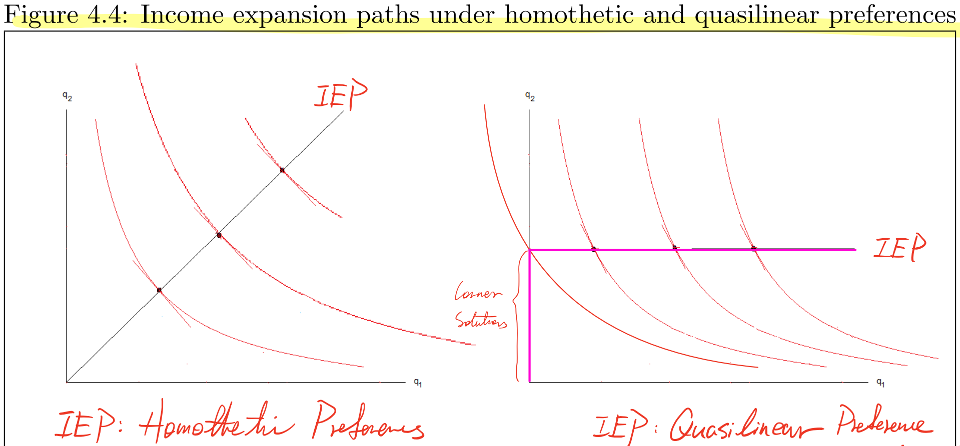
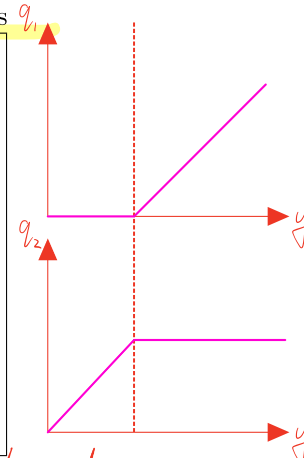


Figure 4.4: Income expansion paths under homothetic and quasilinear preferences





Ray through the origin: $y \uparrow$ q_1, q_2 expand with a fixed proportion, budget share will be constant. Total budget elasticity of every good = 1.

Demand for good 2 is constant when y changes
$q_2 = f_2(y,p) = b(p)$ only depend on prices
$q_1 = f_1(y,p) = (y - p_2 b(p))/p_1$

4.4 Utility maximisation

Mathematically, Marshallian demands solve the utility maximisation problem

$$\max_q u(q) \qquad \text{s.t.} \qquad p'q \leq y$$

$p'q - y \leq 0$

This can be solved by finding stationary points of the Lagrangean

$$u(q) - \lambda(p'q - y).$$

For interior solutions, first order conditions require

FOC: $\frac{\partial u}{\partial q_i} - \lambda p_i = 0$ for all i →

$$\boxed{\frac{\partial u}{\partial q_i} = \lambda p_i}$$

which imply

$$\boxed{\text{MRS} = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{p_i}{p_j}.}$$

This is a confirmation of the tangency condition - the slope of indifference curve and budget constraint are equal at interior solutions.

Expenditure Minimization

4.5 Expenditure minimisation

In comparison with the utility maximisation problem

$$\max_q u(q) \qquad \text{s.t.} \qquad p'q \leq y$$

consider now the alternative problem of expenditure minimisation, minimising the amount it is necessary to spend to reach a given utility

$$\min_q p'q \qquad \text{s.t.} \qquad u(q) \geq v$$

Hicksian Demand

(Not Slutsky)

Solutions to this problem are Hicksian demands or compensated demands $q = g(v, p)$. Homogenous of Degree 0 $g(v, \lambda p) = g(v, p)$

The problem can be solved by finding stationary points of the Lagrangean

$$p'q - \mu(u(q) - v).$$

which, for interior solutions, gives first order conditions requiring FOC: $p_i - \mu\frac{\partial u}{\partial q_i} = 0$

$$\boxed{p_i = \mu\frac{\partial u}{\partial q_i}}$$

and therefore

MRS ← Slope

$$\boxed{\text{MRS} = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{p_i}{p_j}.}$$

Note that this is exactly the same tangency condition encountered in solving the utility maximisation problem.

We can define important functions giving the values of the two problems. The value of the maximised utility function as a function of y and p can be found by substituting the Marshallian demands back into the direct utility function $u(q)$. We call this the *indirect utility function*

Indirect Utility Function

$$v(y, p) = u(\underbrace{f(y, p)}_{\text{Marshallian Demand}}) = \max_q u(q) \text{ s.t. } p'q \leq y$$

Direct Utility Function

Highest attainable utility with budget y and prices p

The value of the minimised cost as a function of v and p can be found by costing the Hicksian demands. We call this the expenditure function or cost function

Expenditure/Cost Function

$$c(v, p) = p'g(v, p) = \min_q p'q \text{ s.t. } u(q) \geq v.$$

Prices Hicksian Demand

Lowest expenditure necessary to reach utility u with prices p

Hicksian D = Marshallian D (Cost Function , Prices)

The link between the two problems can be expressed by noting the equality of the quantities solving the two problems

$$f(c(v,p),p) = g(v,p) \qquad f(y,p) = g(v(y,p),p)$$

Marshallian D = Hicksian D (Indirect U Function , Prices)

or noting that $v(y,p)$ and $c(v,p)$ are inverses of each other in their first arguments

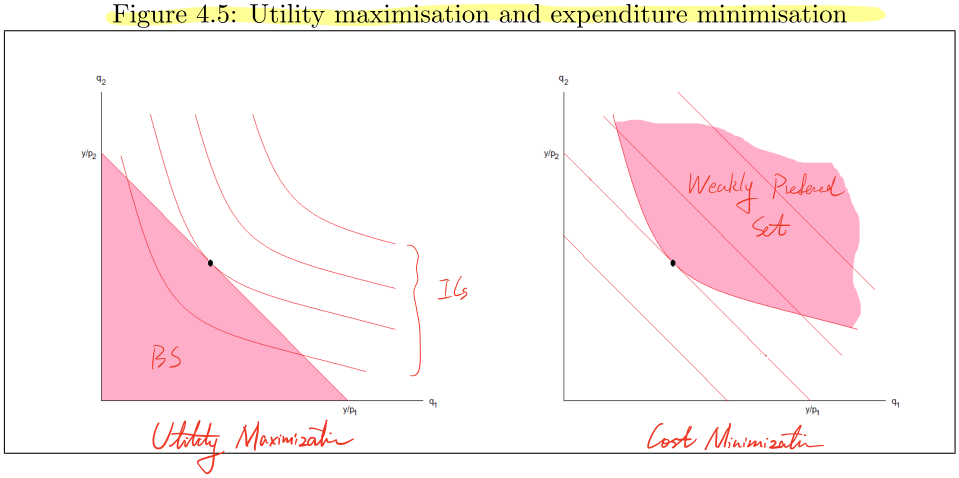
Indirect Utility Function Cost Function

$$v(c(v,p),p) = v \qquad c(v(y,p),p) = y.$$

Cost Function Indirect Utility Function

Figure 4.5 shows how the two problems are related.

Figure 4.5: Utility maximisation and expenditure minimisation



Expenditure / Cost Function $c(v,p)$

$c(v,p) = \min p'q$ s.t. $u(q) \geq v$

4.6 Expenditure function

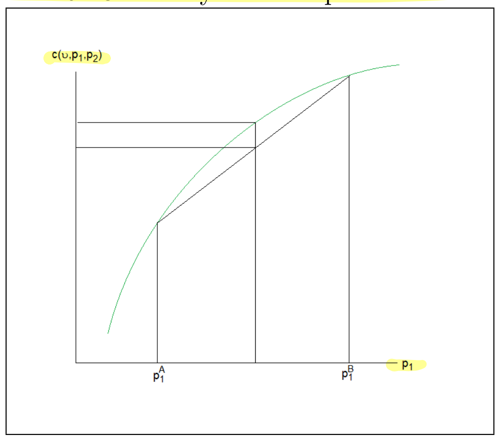
The expenditure function $c(v,p)$ has the properties that

- it is increasing in every price in p (assuming the good is consumed) and in v
- it is homogeneous of degree one in prices p, $c(v,\lambda p) = \lambda c(v,p)$. The Hicksian demands are homogeneous of degree zero in prices so the total cost of purchasing them must be homogeneous of degree one

$$c(v,\lambda p) = \lambda p' \underbrace{g(v,\lambda p)}_{\text{Hicksian Demand}} = \lambda p' g(v,p) = \lambda c(v,p)$$

- it is concave in prices p, $c(v,\lambda p^A + (1-\lambda)p^B) \geq \lambda c(v,p^A) + (1-\lambda)c(v,p^B)$. This follows simply from the fact that the cost-minimising choices at prices p^A and p^B no longer minimise costs at $\lambda p^A + (1-\lambda)p^B$. Figure 4.6 shows the implied shape with respect to variation in any one of the prices.

Figure 4.6: Concavity of the expenditure function



Indirect Utility Function

$v(y,p)$

4.7 Indirect utility function

The properties of the indirect utility function $v(y,p)$ correspond exactly to those of the expenditure function given that the two are inverses of each other. In particular it is

- increasing in y and decreasing in each element of p
- homogeneous of degree zero in y and p:

$$v(\lambda y, \lambda p) = v(y,p)$$

$v(\lambda y, \lambda p) = u(f(\lambda y, \lambda p)) = u(f(y,p)) = v(y,p)$

This should be apparent also from the homogeneity properties of Marshallian demands.

(The property corresponding to concavity of the expenditure function is known as *quasiconvexity* of the indirect utility function.)

4.8 Shephard's Lemma

Among the most useful features of these functions are their simple links to the associated demands. For example, it is possible to get from the expenditure function to the Hicksian demands simply by differentiating.
Since $c(\upsilon,p) = p'g(\upsilon,p)$

$\sum_i p_i g_i(v,p)$ Expenditure Function $/ \partial p_i$

$$\boxed{\frac{\partial c(\upsilon,p)}{\partial p_i}} = g_i(\upsilon,p) + sum_i p_i \frac{\partial g(\upsilon,p)}{\partial p_i}$$

$\frac{\partial \text{ Cost Function}}{\partial p_i}$ = Hicksian Demand (i) (compensated)

$$= g_i(\upsilon,p) + \mu \boxed{\sum_i \frac{\partial u}{\partial q_i}\frac{\partial g(\upsilon,p)}{\partial p_i}}$$

$=0$ because it measures the effect of changes in p_i on total utility, which is constant in cost minimization. (From FOC)

Hicksian Demand

$$= g_i(\upsilon,p)$$

using the first order condition for solving the cost minimisation problem and the fact that utility is held constant in that problem.

This is known as Shephard's lemma. Its importance is that it allows compensated demands to be deduced simply from the expenditure function.

It's an application of the envelope theorem.

4.9 Roy's Identity

Since $v(c(v,p),p) = v$

$c(v,p) = y$

$g_i(v,p) = f_i(y,p)$ (Shephard Lemma)

$$\frac{\partial v(y,p)}{\partial p_i} + \frac{\partial v(y,p)}{\partial y}\boxed{\frac{\partial c(u,p)}{\partial p_i}} = 0$$

$$\Rightarrow -\frac{\partial v(y,p)/\partial p_i}{\partial v(y,p)/\partial y} = g_i(v(y,p),p)$$

$$= f_i(y,p)$$

$-\dfrac{\text{Indirect Utility Function} / \partial p_i}{\text{Indirect Utility Function} / \partial y}$

Marshallian Demand [i]

using Shephard's Lemma.

This is Roy's identity and shows that uncompensated demands can be deduced simply from the indirect utility function by differentiation.

In many ways it is easier to derive a system of demands by beginning with well specified indirect utility function $v(y,p)$ or expenditure function $c(v,p)$ and differentiating than by solving a consumer problem directly given a direct utility function $u(q)$.

4.10 Slutsky equation, again

Since $g(v,p) = f(c(v,p),p)$ Differentiate Hicksian Demand of good i with Respect to Price of Good j

$$\frac{\partial g_i(v,p)}{\partial p_j} = \frac{\partial f_i(y,p)}{\partial p_j} + \frac{\partial f_i(y,p)}{\partial y}\frac{\partial c(v,p)}{\partial p_j}$$

$$= \frac{\partial f_i(y,p)}{\partial p_j} + \frac{\partial f_i(y,p)}{\partial y} f_j(y,p)$$

Notice that this is the same as the Slutsky equation derived earlier for Slutsky-compensated demands. Hicks-compensated price derivatives are the same as Slutsky-compensated price derivatives since the two notions of compensation coincide at the margin.

This means the results derived earlier can simply be carried over. Hicksian demands therefore also satisfy *negativity* at the margin. In particular

$$\frac{\partial g_i(v,p)}{\partial p_i} \leq 0.$$

$\dfrac{\partial g_i(v,p)}{\partial p_i} = \dfrac{\partial c/\partial p_i}{\partial p_i} = \dfrac{\partial^2 c}{\partial p_i^2} \leq 0$

Another way to look at this is to see that it follows directly from concavity of the expenditure function and the fact, from Shephard's Lemma, that compensated demands are derivatives of the expenditure function..

(Negativity can actually be stated slightly more strongly than this, involving also restrictions on cross-price effects, but this is the most important implication).

4.11 Slutsky symmetry

There is one more property of Hicksian demands that can now be deduced.

From Shephard's Lemma

$$\frac{\partial g_i(v,p)}{\partial p_j} = \frac{\partial^2 c(v,p)}{\partial p_i p_j} = \frac{\partial g_j(v,p)}{\partial p_i}$$

Compensated Cross Price Effects are symmetric

$\dfrac{\partial c(v,p)}{\partial p_i} / \partial p_j$ $\quad \dfrac{\partial c(v,p)}{\partial p_j} / \partial p_i$

Compensated cross-price derivatives are therefore also *symmetric*. Holding utility constant, the effect of increasing the price of one good on the quantity chosen of another is numerically identical to the effect of increasing the price of the other good on the quantity chosen of the first good.

This shows that notions of complementarity and substitutability are consistent between demand equations if using compensated demands and provides a strong argument for defining complementarity and substitutability in such terms. This would *not* be true if using uncompensated demands because income effects are not symmetric.

4.12 Integrability

To summarise, if demands are consistent with utility maximising behaviour then they have the following properties Demands are compatible with optimizing behaviour if and only if they satisfy:

- Adding up: Demands must lie on the budget constraint and therefore

$$
\begin{aligned}
p'f(y,p) &= y \\
p'g(\upsilon,p) &= c(\upsilon,p)
\end{aligned}
$$

- Homogeneity: Increasing all incomes and prices in proportion leaves the budget constraint and therefore demands unaffected

$$
\begin{aligned}
f_i(y,p) &= f_i(\lambda y,\lambda p) \\
g_i(\upsilon,p) &= g_i(\upsilon,\lambda p)
\end{aligned}
$$

- Negativity of compensated own price effects: In particular, a compensated increase in any good's price can only reduce demand for that good

$$\partial g_i/\partial p_i = \partial f_i/\partial p_i + f_i \partial f_i/\partial y \leq 0$$

- (Slutsky) Symmetry of compensated cross price effects:

$$\partial g_i/\partial p_j = \partial g_j/\partial p_i$$

If demands satisfy these restrictions then there is a utility function $u(q)$ which they maximise subject to the budget constraint. We say demands are *integrable*. These are *all* the restrictions required by consumer optimisation.

We know a system of demands is integrable if any of the following hold

- They were derived as solutions to the utility maximisation or expenditure minimisation problem given a well specified direct utility function
- They were derived by Shephard's Lemma from a well specified cost function or they were derived by Roy's Identity from a well specified indirect utility function
- They satisfy adding up, homogeneity, symmetry and negativity.

The connections between the concepts discussed can be summarised in the diagram below:

$$\begin{array}{ccc}
\text{Utility maximisation problem} & & \text{Expenditure minimisation problem} \\
\max_q u(q) \quad s.t. \quad p'q \le y & \longleftrightarrow & \min_q \sum p'q \quad s.t. \quad u(q) \ge \upsilon \\
\downarrow & & \downarrow \\
\text{Uncompensated demands} & & \text{Compensated demands} \\
q_i = f_i(y,p) \qquad i = 1, 2, \ldots & \longleftrightarrow & q_i = g_i(\upsilon, p) \qquad i = 1, 2, \ldots \\
\updownarrow & & \updownarrow \\
\text{Indirect utility function} & & \text{Expenditure function} \\
v(y,p) & \longleftrightarrow & c(\upsilon, p)
\end{array}$$

3 Routes to Specifying An Integrable Demand System :

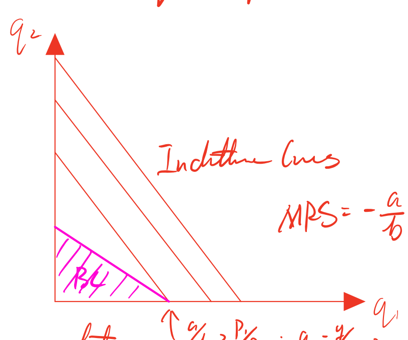
① Write down a utility function and solve the utility maximization / cost minimization problem

② Write down an indirect utility function or expenditure function and then use the Shephard's Lemma or Roy's Identity.

③ Write the demand function directly and impose restrictions to satisfy the 4 properties.

Additional Example: Perfect Substitutes

$u = aq_1 + bq_2$



Usually get corner solutions $\uparrow \; a/b > P_1/P_2 : \left.\begin{array}{l} q_1 = y/P_1 \\ q_2 = 0 \end{array}\right\}$ Marshallian Demand

Indirect Utility Function: $v(y,p) = \max\left[\frac{ay}{P_1}, \frac{by}{P_2}\right] = y \cdot \max\left[\frac{a}{P_1}, \frac{b}{P_2}\right]$
$= y/\min\left[\frac{P_1}{a}, \frac{P_2}{b}\right]$

Check Roy's Identity: $\partial v/\partial y = 1/\min\left[\frac{P_1}{a}, \frac{P_2}{b}\right]$

$\partial v/\partial p_1 = \begin{cases} -ay/P_1^2 & \text{if } a/b > P_1/P_2 \\ 0 & \text{if } a/b < P_1/P_2 \end{cases}$

$-\frac{\partial v/\partial p_1}{\partial v/\partial y} = y/P_1 \quad \text{if } a/b > P_1/P_2$

Expenditure Function: $c(v,p) = v \min[P_1/a, P_2/b]$

Hicksian Demand: $g_1(v,p) = \begin{cases} \frac{\partial c}{\partial p_1} = \frac{u}{a} & \text{if } a/b > P_1/P_2 \\ 0 & \text{if } a/b < P_1/P_2 \end{cases}$

Worked Example 4|A : Demands under Stone-Geary preferences

Suppose the direct utility function is $u(q) = \sum_i \alpha_i \ln(q_i - \gamma_i)$ with $\sum_i \alpha_i = 1$. This is Stone-Geary preferences as introduced earlier.
The tangency condition defining optimum consumer choice is

$$MRS = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{\alpha_i}{\alpha_j}\left(\frac{q_j - \gamma_j}{q_i - \gamma_i}\right) = -\frac{p_i}{p_j} \qquad i, j = 1, 2, \ldots, M$$

Hence for each good $p_i q_i = p_i \gamma_i + \frac{\alpha_i}{\alpha_1}(p_1 q_1 - p_1 \gamma_1)$ and substituting into the budget constraint

$$y = \sum_i p_i q_i = \sum_i p_i \gamma_i + \frac{1}{\alpha_1}(p_1 q_1 - p_1 \gamma_1).$$

So $p_1 q_1 = p_1 \gamma_1 + \alpha_1 (y - \sum_i p_i \gamma_i)$ and by a similar argument we establish all Marshallian demands

$$f_i(y, p) = \gamma_i + \frac{\alpha_i}{p_i}\left(y - \sum_j p_j \gamma_j\right) \qquad i = 1, 2, \ldots, M.$$

Budget shares are $w_i = \alpha_i + \left(p_i \gamma_i - \alpha_i \sum_j p_j \gamma_j\right)/y$ so those goods are necessities for which $p_i \gamma_i$ is greater than $\alpha_i \sum_j p_j \gamma_j$.
Substituting into the direct utility function gives the indirect utility function

$$\begin{aligned} v(y, p) &= \sum_i \alpha_i \ln(f_i(y, p) - \gamma_i) \\ &= \ln\left(y - \sum_i p_i \gamma_i\right) - \sum_i \alpha_i \ln p_i + \sum_i \alpha_i \ln \alpha_i \end{aligned}$$

Inverting $v(y, p)$ in y then gives the expenditure function

$$c(v, p) = \sum_i p_i \gamma_i + e^v \prod_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}$$

The compensated demands are then most easily found by differentiating $c(v, p)$ (using Shephard's Lemma) or by substituting $c(v, p)$ into the uncompensated demands

$$g_i(v, p) = \gamma_i + e^v \prod_j \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j} \qquad i = 1, 2, \ldots, M.$$

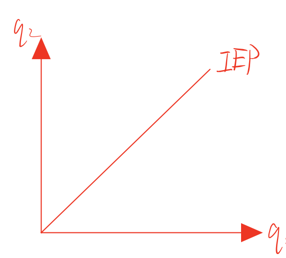
+ Addition Example: Cobb-Douglas: $u = a \ln q_1 + b \ln q_2$ (Homothetic Preferences)

Define $\alpha = \frac{a}{a+b}$

Replace with: $u = \alpha \ln q_1 + (1-\alpha) \ln q_2$

$MRS = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2} = -\frac{\alpha/q_1}{(1-\alpha)/q_2} = -\frac{\alpha}{1-\alpha}\frac{q_2}{q_1}$ (Homothetic)

Equate to the slope of b.c.



$\frac{\alpha}{1-\alpha}\frac{q_2}{q_1} = \frac{p_1}{p_2} \rightarrow \frac{p_2 q_2}{1-\alpha} = \frac{p_1 q_1}{\alpha}$

Marshallian Demand: $y = p_1 q_1 + p_2 q_2 = p_1 q_1 \times (1 + \frac{1-\alpha}{\alpha})$

$q_1 = f_1(y,p) = \alpha y / p_1$, $w_1 = p_1 q_1 / y = \alpha$

$q_2 = f_2(y,p) = (1-\alpha) y / p_2$, $w_2 = 1-\alpha$

Constant Budget Shares

Indirect Utility Function: $v(y,p) = \alpha \ln(\alpha y/p_1) + (1-\alpha)\ln((1-\alpha)y/p_2)$

$= \ln y - \alpha \ln p_1 - (1-\alpha)\ln p_2 + \underbrace{\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)}_{\text{Constant } (A)}$

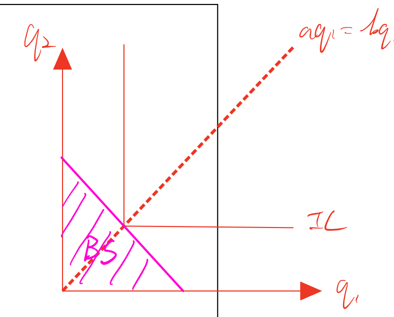
Expenditure Function: $\ln c(v,p) = v + \alpha \ln p_1 + (1-\alpha)\ln[(1-\alpha)y/p_2] - A$

$c(v,p) = e^{v-A} p_1^{\alpha} p_2^{1-\alpha}$

Hicksian Demand: $g_1(v,p) = \alpha e^{v-A}(p_2/p_1)^{1-\alpha}$

Homogenous of degree 0 of prices

Worked Example 4|B : Demands under perfect complements



Suppose direct utility is

$$u(q_1, q_2) = \min[a_1 q_1, a_2 q_2]$$

Goods are perfect complements and at the optimum

$$\begin{aligned} a_1 q_1 &= a_2 q_2 \\ \Rightarrow a_1 f_1(y,p) &= a_2 f_2(y,p) = \frac{a_1 a_2 y}{a_2 p_1 + a_1 p_2} \end{aligned}$$

Optimal

Marshallian Demand

$q_1 = f(y,p) = a_2 y/(a_2 p_1 + a_1 p_2)$

$q_2 = f(y,p) = a_1 y/(a_2 p_1 + a_1 p_2)$

Substituting into the direct utility function gives the indirect utility function

$$\begin{aligned} v(y,p) &= \min[f_1(y,p), f_2(y,p)] \\ &= \boxed{\frac{a_1 a_2 y}{a_2 p_1 + a_1 p_2}} \end{aligned}$$

Inverting in y gives the expenditure function

$$c(v,p) = \frac{1}{a_1 a_2} v(a_2 p_1 + a_1 p_2)$$

Differentiating or substituting gives the compensated demands

$$g_i(v,p) = \frac{\partial c(v,p)}{\partial p_i} = v/a_i \qquad i = 1, 2$$

Worked Example 4|C : Demands under quasilinear preferences

Suppose direct utility takes the quasilinear form

$$u(q_1, q_2) = \alpha \ln q_1 + q_2$$

The tangency condition is

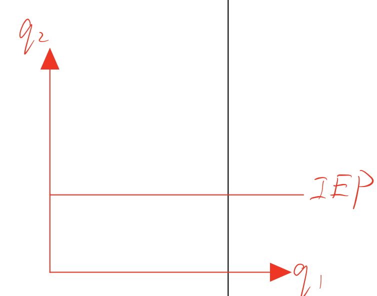
$$MRS = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2} = -\frac{\alpha}{q_1} = -\frac{p_1}{p_2} \quad \Rightarrow p_1 q_1 = \alpha p_2$$

This defines an interior optimum assuming $y > \alpha p_2$.
Hence, directly and by substituting into the budget constraint, uncompensated demands are

Marshallian Demand:

$$\begin{aligned} f_1(y,p) &= \alpha p_2/p_1 \\ f_2(y,p) &= (y/p_2) - \alpha \end{aligned}$$

→ Does not depend on y



Uncompensated demand for the first good is independent of total budget y.
Substituting into the direct utility function gives the indirect utility function

$$\begin{aligned} v(y,p) &= \alpha \ln f_1(y,p) + f_2(y,p) \\ &= \alpha \ln(\alpha p_2/p_1) + (y/p_2) - \alpha \end{aligned}$$

Inverting in y gives the expenditure function

$$c(v,p) = p_2(v - \alpha \ln(\alpha p_2/p_1) + \alpha)$$

Differentiating or substituting then gives the compensated demands

Hicksian Demand:

$$\begin{aligned} g_1(v,p) &= \frac{\partial c(v,p)}{\partial p_1} = f_1(c(v,p),p) = \alpha p_2/p_1 \\ g_2(v,p) &= \frac{\partial c(v,p)}{\partial p_2} = f_2(c(v,p),p) = v - \alpha \ln(\alpha p_2/p_1) \end{aligned}$$

The compensated demand for the first good is independent of v.

If $y \leq \alpha p_2$ then nothing is spent on the second good, $f_1(y,p) = y/p_1$ and $f_2(y,p) = 0$. Hence $v(y,p) = \alpha \ln(y/p_1)$ and $c(v,p) = p_1 \mathrm{e}^{v/\alpha}$.

Worked Example 4|D : Demands with a nonconvex budget set

Continue considering the case of quasilinear direct utility

$$u(q_1, q_2) = \alpha \ln q_1 + q_2$$

but now suppose that the price of the first good falls from p_1 to γp_1 after q_1 reaches a threshold E (where $\gamma < 1$). Suppose moreover that the lower price is paid not just on the excess $q_1 - E$ but on the whole of spending so there is a discontinuity in the budget constraint. The budget constraint has two sections

$$\begin{aligned} q_2 &= y/p_2 - p_1 q_1/p_2 && \text{if } q_1 < E \\ q_2 &= y/p_2 - \gamma p_1 q_1/p_2 && \text{if } q_1 \geq E \end{aligned}$$

A tangency on the first section, if it existed, would be at

$$q_1 = \alpha p_2/p_1 \qquad q_2 = (y/p_2) - \alpha$$

whereas a tangency on the second section would be at

$$q_1 = \alpha p_2/\gamma p_1 \qquad q_2 = (y/p_2) - \alpha.$$

There are three possible cases to consider.

- If $E < \alpha p_2/p_1$ then the only tangency is on the second section and demand is $f_1(y, p_1, p_2, E) = \alpha p_2/\gamma p_1$
- If $\alpha p_2/p_1 < E < \alpha p_2/\gamma p_1$ then there are tangencies on both sections. Since both of these involve the same quantity of the second good $(y/p_2) - \alpha$, the bundle on the second section is preferred so, again, $f_1(y, p_1, p_2, E) = \alpha p_2/\gamma p_1$
- If $\alpha p_2/\gamma p_1 < E$ then the only tangency is on the first section. The most preferred bundle on the second section is at the threshold where $q_1 = E$, $q_2 = (y - \gamma p_1 E)/p_2$. We need to compare utilities and utility is higher at the tangency on the first section than at the discontinuity if and only if

  $$\alpha \ln(\alpha p_2/p_1) + (y/p_2) - \alpha > \alpha \ln E + (y - \gamma p_1 E)/p_2$$

  which is true if and only if

  $$\ln(\alpha p_2/p_1 E) < 1 - \gamma p_1 E/\alpha p_2.$$

  Hence for lower values of $p_2/p_1 E$ at which this does not hold $f_1(y, p_1, p_2, E) = E$ whereas for higher values at which it does hold $f_1(y, p_1, p_2, E) = \alpha p_2/p_1$. The demand function jumps discontinuously as it passes between the two.

Worked Example 4|E : Integrability

Conditions for adding up, homogeneity and negativity for the linear expenditure system have been discussed above. Now consider symmetry. Using the Slutsky equation again

$$\frac{\partial g_i}{\partial p_j} = f_j \frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial p_j} = \frac{b_{ij} + a_i q_j}{p_i} \qquad i \neq j.$$

Hence $\partial g_i/\partial p_j = \partial g_i/\partial p_j$ requires

$$p_j b_{ij} + a_i a_j y + a_i \sum_k b_{jk} p_k = p_i b_{ji} + a_j a_i y + a_j \sum_k b_{ik} p_k.$$

By equating terms on each side for each price, this can be seen to be true if

$$\begin{aligned} (a_i - 1) b_{ji} &= a_j b_{ii} \qquad i \neq j \\ a_i b_{jk} &= a_j b_{ik} \qquad i \neq j \neq k \end{aligned}$$

Introduce a new notation γ_i for $b_{ii}/(1 - a_i)$. Then the first condition implies $b_{ij} = -a_i \gamma_j$ for all $i \neq j$ in which case the second set of conditions are all automatically satisfied. So the system reduces to

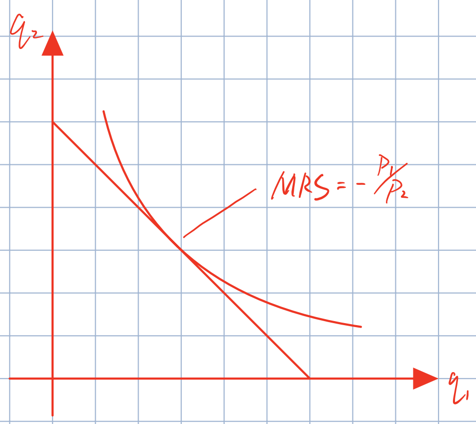
$$p_i f_i(y, p) = p_i \gamma_i + a_i \left(y - \sum_j p_j \gamma_j \right) \qquad i = 1, 2, \ldots, M$$

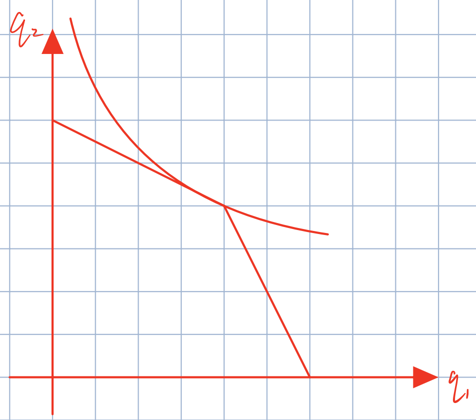
It is clear that the linear expenditure system with integrability restrictions imposed is exactly the same as the Stone-Geary demand system with $a_i = \alpha_i$.

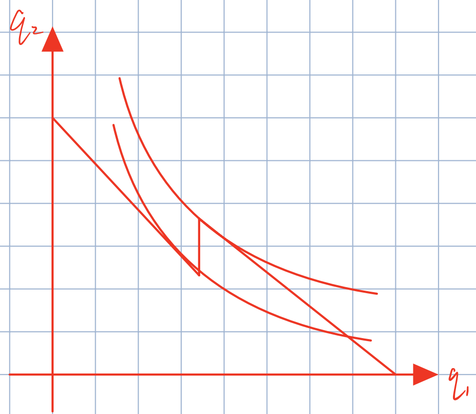
Lecture 4 Part II

Consumer Choice: consumers choose the most preferred bundle in the budget set.

$$\max_{q} \; u(q) \quad s.t. \quad p'q \le y$$



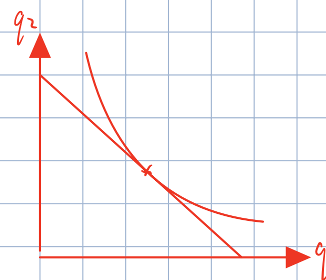




Lecture 5 Part II

Consider the problem:

$\max u(q)$ s.t. $p'q \le y$



Solution is the Marshallian D $f(y,p)$

Substitute back to utility function

$u(f(y,p)) \rightarrow$ Indirect Utility Function

Property: Increasing in y $\longrightarrow$ Then, we can invert it: $v(y,p) = v$

Decreasing in p

Homogenous of degree 0 of y,p together

$\rightarrow y = c(v,p)$ Expenditure Function

Minimum budget needed to reach certain utility given p.

i.e. $\min_q p'q$ s.t. $u(q) \ge v$

Demands solve this are Hicksian Demands $q = g(v,p)$

Therefore: if $y = c(v,p)$ or $v = v(y,p)$

then $f(y,p) = g(v,p)$

Properties of expenditure function: - Increasing in v - Increasing in p - Homogenous of degree 1 in p - Concave in prices

Shephard's Lemma

$$g_i(v,p) = \frac{\partial c}{\partial p_i}$$

Roy's Identity

$$f_i(y,p) = -\frac{\partial v / \partial p_i}{\partial v / \partial y}$$

$\leftarrow$ Common Scaling Factor

Slutsky's Equation

$$\frac{\partial g_i}{\partial p_j} = \frac{\partial f_i}{\partial p_j} + q_j \frac{\partial f_i}{\partial y}$$

Some Implications: From Shephard's Lemma

$$\partial g_i / \partial p_i = \partial^2 c / \partial p_i^2 < 0$$

$\downarrow$

Hicksian D is decreasing in price (Negativity)

$$\partial g_i / \partial p_j = \partial^2 c / \partial p_i \partial p_j = \partial g_j / \partial p_i \quad \text{(Symmetry)}$$

+ Homogeneity

+ Adding Up

= Integrability (required for consumer optimization)

Recall that: $\begin{cases} \text{Indirect Utility Function } v(y,p) = \max_q u(q) \text{ s.t. } p'q \leq y \\ \text{Expenditure Function } c(v,p) = \min_q p'q \text{ s.t. } u(q) \geq v \end{cases}$

Topic 5 : Consumer welfare

Summary: *The view of consumers as choosing in their own best interests suggests natural ways to measure the effects of prices on consumer welfare.*

5.1 Consumer surplus

By Roy's identity the effect of a small change in price p_i on utility is proportional to the quantity consumed

Roy's Identity: $f_i(y,p) = -\dfrac{\partial v/\partial p_i}{\partial v/\partial y}$ $\xrightarrow{\text{Rearrange}}$

$$\boxed{\frac{\partial v}{\partial p_i} = -\frac{\partial v}{\partial y} f_i(y,p).}$$

If marginal utility of income $\partial v/\partial y$ is (roughly) constant as price changes then the quantity demanded can therefore be (roughly) interpreted as an indicator of the marginal welfare cost as price changes. Integrating as price increases, say from p_i^A to p_i^B, we can therefore interpret the area traced out to the left of the demand curve as a measure of the resulting welfare loss, $\mathcal{A} + \mathcal{B}$ in Figure 5.1. Here

- $\mathcal{A}$ is the increase in the price of those units still consumed after the change q_i^B
- $\mathcal{B}$ is the excess of the consumer's willingness to pay $\mathcal{B} + \mathcal{C}$ over what was previously paid $\mathcal{C}$ on those items no longer consumed $q_i^A - q_i^B$

If we imagine price increased to the point $\bar{p}_i$ where none of the good is demanded, only the second area is left and the welfare loss is the triangular area labelled as consumer surplus in the second panel.

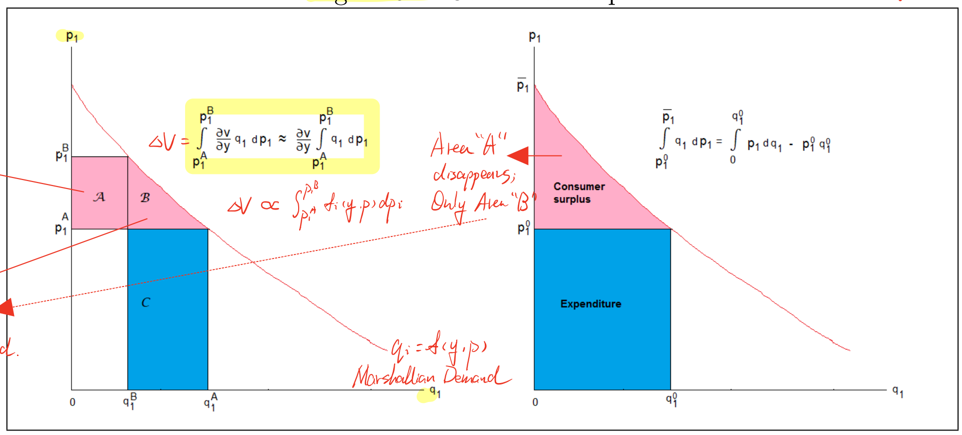
Alternatively we can note that consumer optimisation sets marginal utilities proportional to prices. $\partial u/\partial q_i = \lambda p_i$. The price which the consumer is willing to pay is therefore an indicator of the marginal welfare cost of reducing quantity. If the factor of proportionality (in other words, the Lagrange multiplier on the budget constraint in the utility maximisation) is constant then the effect of decreasing the quantity to the point where none of the good is demanded is therefore a trapezoidal area underneath a demand curve. If we subtract from this the cost of buying the good $p_i q_i$ then we reach consumer surplus again as a measure of consumer welfare.

How reasonable then is the assumption that $\partial v/\partial y$ (or the Lagrange multiplier on the budget constraint) remains constant? It can be shown that this will be true if preferences are quasilinear (with respect to the other good). Consumer surplus arguments using conventional demand curves are therefore rigorously justified if preferences are quasilinear. Even if preferences are not quasilinear then the general idea behind calculating

Proportionality valids under assumption that $\frac{\partial v}{\partial y}$ stays the same across prices: Quasilinear Preference with respect to the other good. Reason: all changes in y at the margin are spent on the other good, therefore don't depend on the price of this one. For other types of preferences, this "roughly" true.

Figure 5.1: Consumer surplus

$\Delta V \propto$ Increase in expenditure on the quantity consumers continue to consume. + Total excess of willingness to pay over what they actually paid for units no longer consumed.



consumer surplus as the area under a demand curve usually still gives a reasonable approximation to a good measure of welfare.

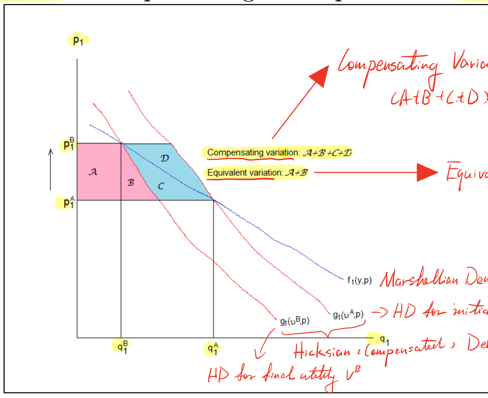
For a different perspective we can consider constructing a similar measure of welfare loss using a compensated rather than uncompensated demand curve. Along a compensated demand curve, utility is held constant by construction and $g_i(v,p) = \partial c(v,p)/\partial p_i$. The quantity on the compensated demand curve is therefore interpretable as the increase in the cost of reaching that given utility as a price changes. The similar area under the curve can therefore be regarded as a measure of the cost to the consumer in that sense

Shephard's Lemma: $\frac{\partial c(v,p)}{\partial p_i} = g_i(v,p)$

$$\boxed{c(v,p^B) - c(v,p^A) = \int_{p_i^A}^{p_i^B} \frac{\partial c(v,p)}{\partial p_i} \mathrm{d}p_i = \int_{p_i^A}^{p_i^B} g_i(v,p) \mathrm{d}p_i}$$

If the fixed level of utility is initial utility then this is called the compensating variation. If the fixed level of utility is final utility then this is called the equivalent variation. Both of these are recognised measures of welfare, not requiring any assumption about constancy of marginal utility to sustain the interpretation rigorously. The difference between them is illustrated in Figure 5.2. Note that if preferences are quasilinear with respect to q_2 then the compensated demands do not depend on v and the uncompensated demand does not depend on y so the three demand curves in the diagram coincide. Compensating and equivalent variation are therefore identical to each other and to the approximate measure of welfare loss based on the uncompensated demand curve.

Figure 5.2: Compensating and equivalent variation



Compensating Variation: Increase in expenditure needed to reach initial utility (V^A)
(A+B+C+D)
$= \text{Total Area} = \int_{p_1^A}^{p_1^B} g_1(V^A,p)\,dp_1 = c(V^A,p_1^B) - c(V^A,p_1^A)$

Equivalent variation: Increase in expenditure needed to reach final utility (V^B)
(A+B)

A+B+C: Welfare Loss

Marshallian Demand Curve

→ HD for initial utility V^A

Hicksian (Compensated) Demand (Steeper than Marshallian Demand Curve because there's no income effect)

HD for final utility V^B

If there's no income effect: $f_1(y,p) = g_1(V^A,p) = g_2(V^A,p)$
(Quasilinear preference with respect to the other good)

Worked Example 5|A : Compensating and equivalent variation

Take the example of quasilinear preferences $u(q) = \alpha \ln q_1 + q_2$ as analysed earlier with expenditure function

$$c(\upsilon, p) = p_2(\upsilon - \alpha \ln(\alpha p_2/p_1) + \alpha)$$

Consider an increase in the price of good 1 from p_1^A to p_1^B. Compensating and equivalent variation are equal

$$\begin{aligned} c(\upsilon^A, p_1^B, p_2) - c(\upsilon^A, p_1^A, p_2) &= c(\upsilon^B, p_1^B, p_2) - c(\upsilon^B, p_1^A, p_2) \\ &= p_2\alpha\left[\ln p_1^A - \ln p_1^B\right] \end{aligned}$$

5.2 Cost of living indices

The expenditure function is the ideal concept for comparing cost of living. We can define a true cost of living index (or Konüs cost of living index) as the ratio of the minimum cost of reaching a given utility in two periods. Say that we are comparing current prices p^A with prices in a base period p^B. Then the cost of living index is the ratio of expenditure functions

True Cost of Living Index:

$$\boxed{T(\upsilon, p^A, p^B) = \frac{c(\upsilon, p^A)}{c(\upsilon, p^B)}}$$

Notice that such a cost of living index depends on the utility level υ at which we make the comparison. Must this be so? There are only two case in which not: (Not depend on v)

$(c(v,p^A) = c(v,\lambda p^B) = \lambda c(v,p^B))$

- If prices are proportional $p^A = \lambda p^B$ then the cost of living index is equal to λ at all υ, whatever preferences, $T = \lambda$
- If preferences are homothetic then the cost of living index is equal at all υ, whatever prices.

Homothetic Preferences: → $c(v,p) = v A(p)$
(Budget shares are independent of v)

$T(v,p^A,p^B) = \frac{v A(p^A)}{v A(p^B)} = \frac{A(p^A)}{A(p^B)}$ → Independent of v

Two common approximations to the true index are used. Both compare the cost of purchasing a fixed bundle of goods.

The Laspeyres index is the ratio of the costs of purchasing the base period bundle q^B (Base Bundle)

Laspeyres Index

$$\boxed{L(p^A, p^B) = \frac{p^{A\prime} q^B}{p^{B\prime} q^B} = \sum_i w_i^B \left(\frac{p_i^A}{p_i^B} \right)}$$

$q^B \to q^A$
Base $\to$ Final

where the second expression shows that the Laspeyres index can be conveniently written as a budget-share-weighted average of price ratios.

If $L(p^A, p^B) < 1$ and consumer's total expenditure is unchanged then the consumer can afford the base bundle and cannot be worse off.

We also know that $p^{B\prime} q^B = c(\upsilon^B, p^B)$ where υ^B is the base period utility and also that $p^{A\prime} q^B$ cannot be less than the minimum cost of attaining u^B in the current period (since q^B gives utility υ^B but not necessary most cheaply at current prices). Hence the Laspeyres index is greater than the true cost of living index at base period utility,

$p^{A\prime} q^B > c(v^B, p^A)$
$p^{B\prime} q^B = c(v^B, p^A)$

$$L(p^A, p^B) \geq T(\upsilon^B, p^A, p^B).$$

(Overestimates the true index)

This is so because consumers are free to substitute away from goods which become more expensive and therefore evaluating the cost at a fixed bundle exaggerates the impact on cost of living.

The Paasche index is the ratio of the costs of purchasing the current period bundle q^A (Final Bundle)

Paasche Index

$$\boxed{P(p^A, p^B) = \frac{p^{A\prime} q^A}{p^{B\prime} q^A} = 1 / \left[\sum_i w_i^A \left(\frac{p_i^B}{p_i^A} \right) \right]}.$$

By similar arguments the consumer must be worse off if $P(p^A, p^B) > 1$ and total expenditure is unchanged. Likewise the Paasche index can be shown to be less than the true cost of living index at current utility υ^A,

$$P(p^A, p^B) \leq T(\upsilon^A, p^A, p^B).$$

If preferences are homothetic then the true index T is the same at all utilities so we can write simply

Homothetic : $P \leq T \leq L.$ In other preferences, we can't make such direct comparison.

$p^A = \lambda p^B$: $P = T = L$

Worked Example 5|B : Price indices

Take the example of Stone-Geary preferences $u(q) = \sum_i \alpha_i \ln(q_i - \gamma_i)$ as analysed earlier.
Base period budget shares are

$$w_i^B = \alpha_i + \left(s_i^B - \alpha_i\right) \sum_j p_j^B \gamma_j / y^B \qquad i = 1, 2, \ldots, M$$

as shown earlier, where we define $s_i^B = p_i^B \gamma_i / \sum_j p_j^B \gamma_j$.
Suppose that the first good is a necessity at base prices, so $s_1^B > \alpha_1$, and say that between the base and final period the price of good 1 doubles and other prices stay the same, so we can express the Laspeyres index as

$$\begin{aligned} L(p^A, p^B) &= \sum_i w_i^B \left(\frac{p_i^A}{p_i^B}\right) \\ &= \sum_i w_i^B + w_1^B \\ &= 1 + \alpha_1 + \left(s_1^B - \alpha_1\right) \sum_j p_j^B \gamma_j / y^B. \end{aligned}$$

The Laspeyres index is therefore higher for poorer households, those with lower y^B.
The true index evaluated at base utility is

$$\begin{aligned} T(\upsilon^B, p^A, p^B) &= \frac{c(\upsilon^B, p^A)}{c(\upsilon^B, p^B)} \\ &= \frac{\sum_i p_i^A \gamma_i + \mathrm{e}^{\upsilon^B} \prod_i \left(\frac{p_i^A}{\alpha_i}\right)^{\alpha_i}}{\sum_i p_i^B \gamma_i + \mathrm{e}^{\upsilon^B} \prod_i \left(\frac{p_i^B}{\alpha_i}\right)^{\alpha_i}} \\ &= \left(\sum_i p_i^B \gamma_i + p_1^B \gamma_1 + \left(y^B - \sum_i p_i^B \gamma_i\right) 2^{\alpha_1}\right) / y^B \\ &= 2^{\alpha_1} + \left(1 + s_1^B - 2^{\alpha_1}\right) \sum_i p_i^B \gamma_i / y^B \end{aligned}$$

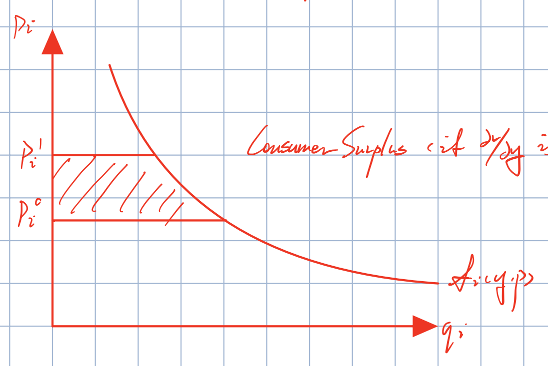
Since $s_1^B > \alpha_1$ and $1 + \alpha_1 - 2^{\alpha_1} > 0$ for all $0 < \alpha_1 < 1$ the true index is also higher for poorer households, those with lower y^B.
Note also that the Laspeyres index does indeed always exceed this true index

$$L(p^A, p^B) - T(\upsilon^B, p^A, p^B) = (1 + \alpha_1 - 2^{\alpha_1}) \left(1 - \sum_i p_i^B \gamma_i / y^B\right) > 0$$

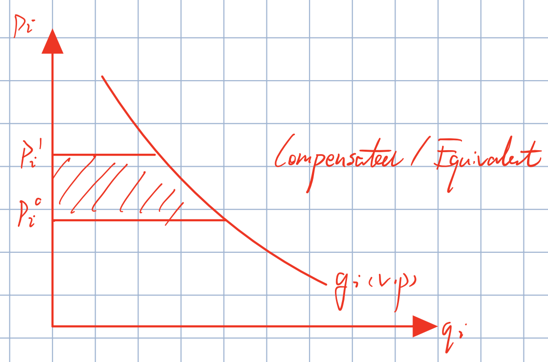
Lecture 5 Part III

Roy's Identity: $x_i(y,p) = -\dfrac{\partial v/\partial p_i}{\partial v/\partial y}$

$\partial v/\partial p_i = -x_i(y,p) \times \partial v/\partial y$



Shephard Lemma: $\dfrac{\partial c}{\partial p_i} = q_i(v,p)$



Cost of Living Indices

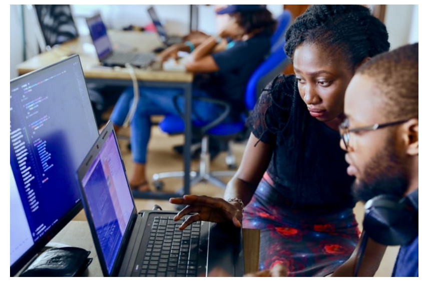
$T(v,p^1,p^0) = \dfrac{C(v,p^1)}{C(v,p^0)}$

$L(p^1,p^0) = \dfrac{p^{1\prime} q^0}{p^{0\prime} q^0}$ $\quad L \geq T(v^0)$

When Δ cost of living will be identical for different v?

either $p^1 = \lambda p^0$ or preferences are homothetic

Topic 6 : Labour supply and demand with endowments



Summary: *Extending the analysis to recognise the fact that consumers may begin with endowments and buy as well as sell goods leads to a richer model. In particular, it suggests a way to model supply of labour and to understand why responses to wage changes may not be as simple as responses to other prices.*

6.1 Buying and selling

Suppose an individual has endowments $\omega = (\omega_1, \omega_2, ...)$ of goods. The consumer problem becomes

$$\max_q u(q) \qquad \text{s.t.} \qquad p'q \leq y + p'\omega$$

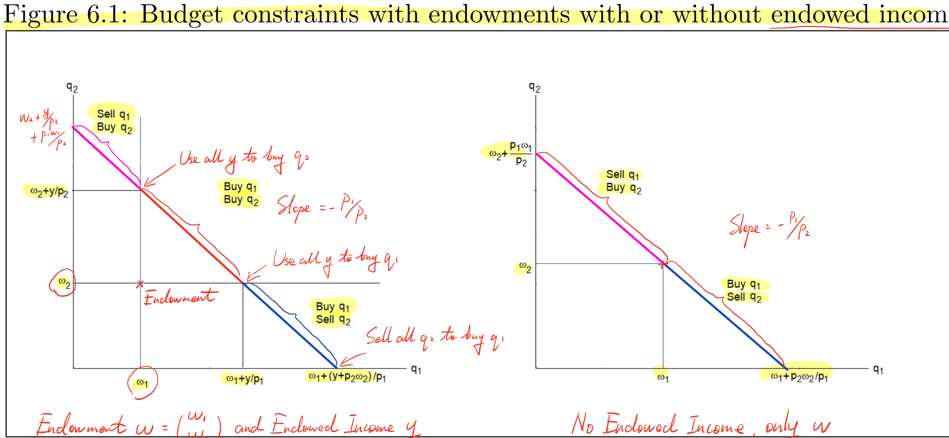
$\vec{\omega}$ (omiga) = $\begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \end{pmatrix}$

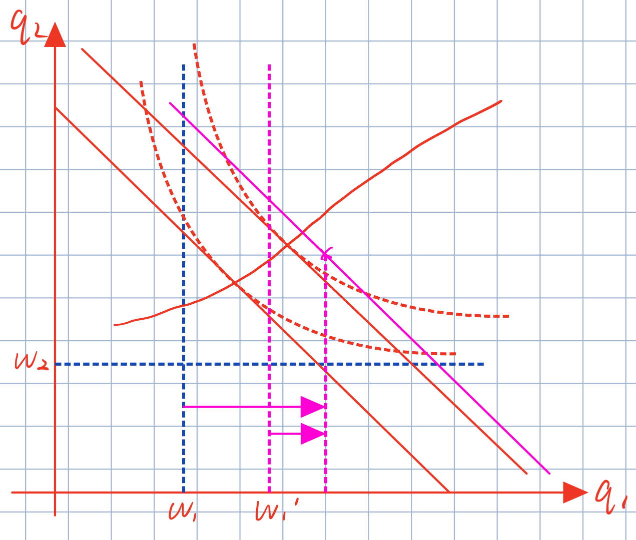
Endowed Income

What is different now is the budget constraint. Figure 6.1 illustrates cases with and without endowed income y. If there is no endowed income then individuals can either sell one good and buy the other or sell the second and buy the first. As prices change the slope of the budget constraint changes but the endowment point where the consumer simply consumes their endowment remains fixed. Adding endowed income adds the possibility of being a net buyer of both goods.

no change in preferences, but changes in the budget constraint

Figure 6.1: Budget constraints with endowments with or without endowed income





Change in Y
$(Y = y + w'p)$

Changes in Endowed Income:

$y\uparrow$ Income Exp. Path

(Parallel Shifts of Budget Constraint)

$$\frac{\partial q_i}{\partial y} = \frac{\partial \phi_i}{\partial y} = \frac{\partial \phi_i}{\partial y}$$

Change in Endowsement:

Still parallel changes in budget constraints

A change in w affects Y the same way as a change in $y = \Delta w'p$

$$\frac{\partial \phi_i}{\partial w_j} = \frac{\partial q_i}{\partial w_j} = \frac{\partial \phi_i}{\partial Y} \times p_j \quad [i \neq j]$$

$$\frac{\partial \phi_i}{\partial w_i} = \frac{\partial q_i}{\partial w_i} - 1 = \frac{\partial \phi_i}{\partial Y} p_i - 1$$

Net / Excess Demand $z = q - w = f(Y, p) - w = \phi(y, w, p)$ $\begin{cases} >0 & \text{Buying} \\ <0 & \text{Selling} \end{cases}$

Gross Demand q_i: $q_i = g_i(v, p)$ $i = 1, 2, \dots$ (Hicksian Compensated Demand) Exactly the same

Demands are now

$$q_i = f_i(y + p'\omega, p) \qquad i = 1, 2, \dots$$

(Marshallian Demand)

or $f_i(Y, p)$, where $Y = y + p'w$

where $f_i(.)$ is the standard uncompensated demand function. It is sometimes convenient to draw a distinction between *gross demands* q and *net demands* or *excess demands* $z = q - \omega$. Let $\phi(y, p, \omega) = f(y + p'\omega, p) - \omega$ define a new function giving net demands as a function of y, p and ω.

Net Demand Function

Changes in endowments have effects like income effects. Changes in prices have the usual effects plus an effect due to the change in the value of the individual's endowment - the endowment income effect. Specifically

$$\frac{\partial q_i}{\partial p_j} = \frac{\partial \phi_i}{\partial p_j} = \frac{\partial f_i}{\partial p_j} + \boxed{\frac{\partial f_i}{\partial y}\omega_j}$$
$$= \frac{\partial g_i}{\partial p_j} - (q_j - \omega_j)\frac{\partial f_i}{\partial y}$$

Endowment Income Effect

$$\underbrace{\left[\frac{\partial g_i}{\partial p_j} - q_j \frac{\partial f_i}{\partial Y}\right]}_{\frac{\partial f_i}{\partial p_j} \text{ Slutsky's Equation}} + \frac{\partial f_i}{\partial Y}\omega_j$$

where $g_i(v, p)$ is the usual compensated demand function. This extends the Slutsky equation to the case of demand with endowments. Note that, written in terms of net demands, the equation is unchanged

Slutsky's Equation with Endowments:

$$\frac{\partial q_i}{\partial p_j} = \frac{\partial \phi_i}{\partial p_j} = \frac{\partial g_i}{\partial p_j} - z_j \frac{\partial \phi_i}{\partial y}.$$

Notice that the sign of the income effect depends upon whether the individual is a net buyer ($q_i > \omega_i$) as in the usual case or a net seller ($q_i < \omega_i$). An increase in the price of a normal good can now increase demand if the individual is a net seller and the endowment income effect is strong enough. The Slutsky decomposition with endowment income effect is illustrated in Figure 6.2.

Figure 6.2: Endowment income effect

Price ↑ for good i:

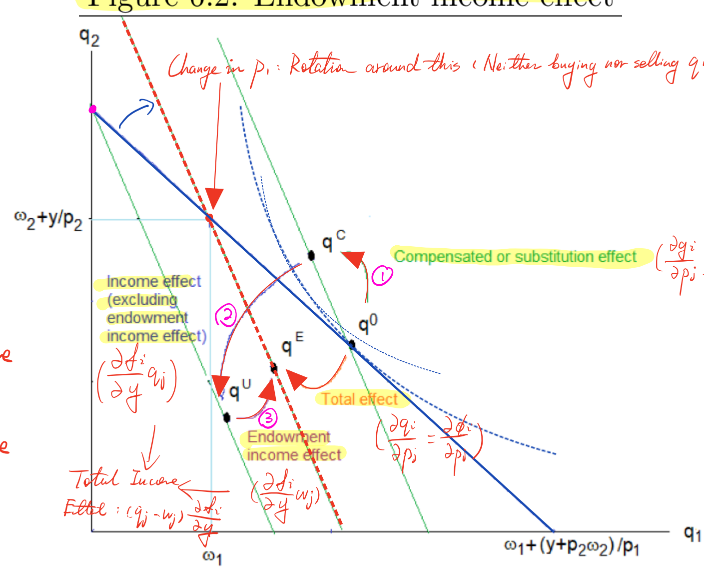
$$\frac{\partial q_i}{\partial p_i} = \frac{\partial g_i}{\partial p_i} - \underbrace{(q_i - w_i)}_{(z_i)} \frac{\partial f_i}{\partial Y}$$

Negative by WARP

Positive if buyer
Negative if seller

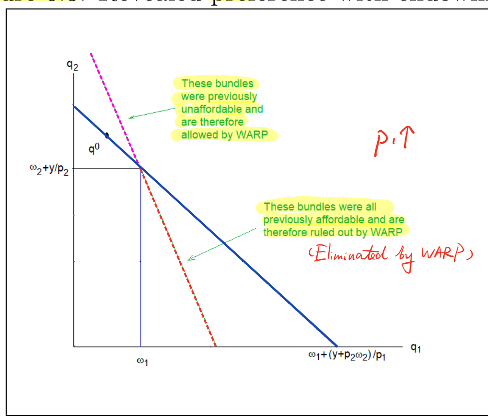
Positive if normal

For buyers of a normal good, demand curve slopes down.
For sellers of a normal good, demand curve slopes up or down.



Note that there is an important revealed preference argument relating to uncompensated demands in this situation. A seller will never become a buyer if the price rises and a buyer will never become a seller if the price falls. In each case, such a change is not possible since it would involve consuming a bundle available before the change when the bundle then chosen remains affordable, as shown in Figure 6.3.

Figure 6.3: Revealed preference with endowments



$p \uparrow$ Seller will never switch to be buyer
$p \downarrow$ Buyer will never switch to be seller
(by WARP)

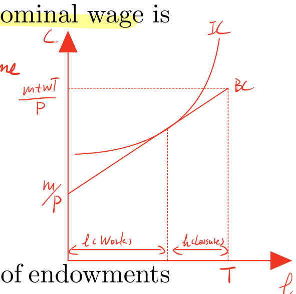
consumption good c
time : endowment T — sell hours of work l; hours not working $h = T - l$

6.2 Labour supply

The prime example of the importance of considering demand with endowments is the analysis of labour supply. Suppose an individual has preferences over hours not working ("leisure") h and consumption c. They have unearned income of m and endowment of time T. The price of consumption is p and the nominal wage is w. The individual's budget constraint is

Consumption Leisure | Unearned Income (Endowed Budget) — Endowment Time } Full Income

$$pc + wh \leq m + wT \equiv M$$



which may appear more familiar if written in terms of hours worked $l = T - h$:

$$pc \leq m + wl$$

where the value of consumption is bounded by the sum of unearned and earned income. The value of endowments in this context $M = m + wT$ is referred to as full income.

Suppose the individual is free to choose any hours of work, subject only to their budget constraint. Demand for leisure can be written as an uncompensated demand function, dependent on full income, wage and output price

Uncompensated Demand for Leisure
$$h = f(M, w, p)$$

or as a compensated demand function

Compensated Demand for Leisure
$$h = g(v, w, p).$$

Gross Demand for Leisure

Leisure

The Slutsky equation for leisure is

Slutsky Equation for Leisure
$$\boxed{\frac{\partial f}{\partial w} = \frac{\partial g}{\partial w} - (h - T)\frac{\partial f}{\partial M}.}$$

Since the individual *sells* time ($h \leq T$) the income effect of a wage change is opposed to the compensated effect if leisure is normal.

$$\frac{\partial f}{\partial w} = \boxed{\frac{\partial g}{\partial w}} + \boxed{\underset{\substack{+\\ \text{Seller}}}{(T-h)} \underset{+}{\frac{\partial f}{\partial M}}}$$

Substitution Effect $-$ Income Effect $+$

Labour

Rephrasing in the more familiar terms of labour supply l we can define an uncompensated labour supply function

$\ell = T - h = -\phi(m, T, w, p)$

Uncompensated Labour Supply
$$l = L(m, w, p) \equiv T - f(m + wT, w, p)$$

$$\frac{\partial L}{\partial w} = -\left.\frac{\partial h}{\partial w}\right|_m$$
$$\frac{\partial \Lambda}{\partial w} = -\frac{\partial g}{\partial w}$$
$$\frac{\partial L}{\partial m} = -\frac{\partial f}{\partial M}$$

and a compensated labour supply function

Compensated Labour Supply
$$l = \Lambda(v, w, p) \equiv T - g(v, w, p)$$

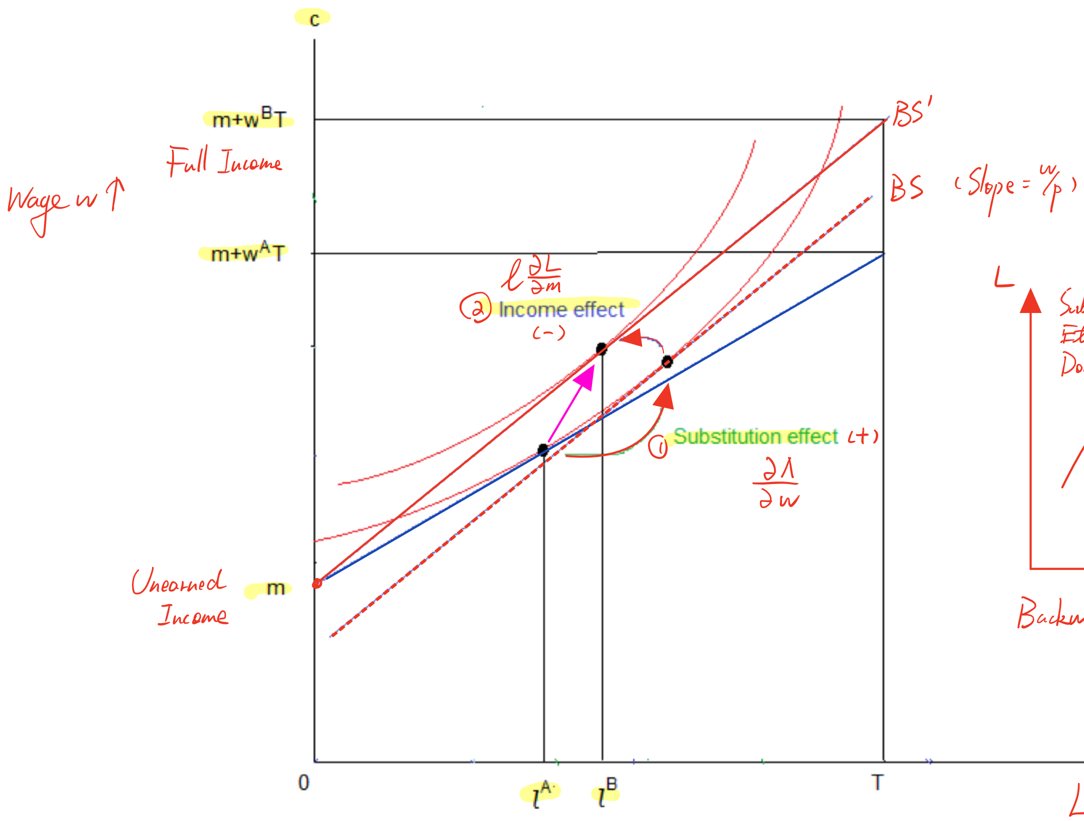
and it follows that

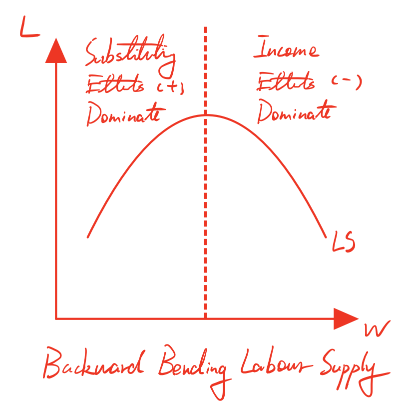
Slutsky Equation for Labour Supply
$$\boxed{\frac{\partial L}{\partial w} = \underset{+}{\frac{\partial \Lambda}{\partial w}} + \underset{+}{l}\underset{-}{\frac{\partial L}{\partial m}}.}$$

Opposite Sub/Inc. Effect because the individual is a seller of labour

The opposition of income and substitution effects is still there and the direction of the uncompensated wage effect on chosen hours of work depends upon the balance between the two. An example in which the substitution effect dominates so that a wage increase leads to an increase in labour supply is shown in Figure 6.4. Since that balance can differ at different wage rates it is quite possible for a labour supply curve to slope upwards at some wages and downwards at others as in a so-called backward-bending labour supply curve.

Figure 6.4: Labour supply





Worked Example 6|A : Labour supply

Suppose the utility function defined over leisure h and consumption c has a Cobb-Douglas form

$$u(c, h) = \alpha \ln h + (1 - \alpha) \ln c.$$

The budget constraint is $wh + pc = wT + m$ which we can rewrite in terms of labour supplied $l = T - h$ to express the consumer's problem as

$$\max_l \alpha \ln(T - l) + (1 - \alpha) \ln \frac{wl + m}{p}$$

.
The first order condition for solution $\alpha/(T - l) = (1 - \alpha)w/(wl + m)$ is solved by

$$l = (1 - \alpha)T - \alpha m/w.$$

Labour supply is increasing in w for all values of $m > 0$ so the substitution effect always dominates the income effect for such preferences.

Case Study 6|1 : Labour Supply

Practical study of labour supply is complicated by a number of factors. Taxation and benefit systems make actual budget constraints linking hours of work to labour income much more complicated than the simple linear budget constraint considered here. Corner solutions at zero hours may be optimal for certain types of households with low wages and participation responses to changes in budget constraints can be as important as adjustments in hours worked. Also individual labour supply decisions of individuals within households need to be modelled jointly to capture interdependence.
Lone mothers are a particularly common focus of policy interest. Over the 1980s and 1990s several changes in US tax policy were aimed at encouraging them to work and these were accompanied by unprecedented increases in employment and hours. Meyer and Rosenbaum compare the behaviour of single women with and without children to assess the contribution of different policy changes using a large population survey. They find that over 60% of the change was accounted for by responses to changes in tax credits, a smaller but still significant proportion to changes in welfare programs and little to changes in Medicaid, training and child care programs. They conclude that policies aimed at "making work pay" rather than penalising not working are most effective in achieving the desired effects.

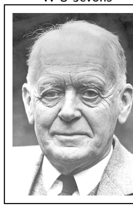
[Source: B. Meyer and D. Rosenbaum, 2001, Welfare, the Earned Income Tax Credit, and the labor supply of single mothers, *Quarterly Journal of Economics* 116, 1063-1114.]

History Note 6|I : Labour supply

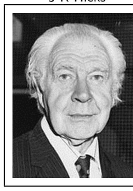
Recognition that labour supply can be analysed as a consumer choice problem involving the weighing of disutility of labour against the utility of consumption can be found in marginalist works, for example, of Stanley Jevons (1835-1882), Alfred Marshall (1842-1924) and even Hermann Gossen (1810-1858).

The possibility that the opposition of income and substitution effects given the endowment of time might create ambiguity in the slope of labour supply curves and even backward-bending behaviour was clarified by work of British economists Lionel Robbins (1898-1984) and John Hicks (1904-89, Nobel 1972) in the 1930s.

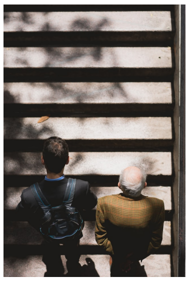
W S Jevons



J R Hicks



L Robbins



Topic 7 : Saving and borrowing

Summary: *Allocation of spending over time can be seen as a further example of the theory of demand with endowments. Particular features of the intertemporal setting suggest restrictions on preferences which put natural structure on intertemporal demand behaviour. If we allow a variety of assets then in the simplest models their investment decision is separable from the decision about when to consume the returns.*

7.1 Intertemporal choice

Another example of demand with endowments is analysis of intertemporal choice. Suppose an individual has preferences over consumption when young c_0 and consumption when old c_1. They have endowed income of y_0 and y_1 in the two periods. (If necessary, bequests received can be treated as part of y_0 and bequests given as part of c_1). Assume no uncertainty about the future. If the real interest rate on bonds linking the two periods is equal to r for both lending and borrowing then the budget constraint is

Budget Constraint :

$$\underbrace{c_0 + \frac{c_1}{1+r}}_{\text{PV Consumption}} = \underbrace{y_0 + \frac{y_1}{1+r}}_{\text{PV Income}}.$$

which implies that the present value of consumption must equal the present value of income.

Demand for current consumption is

Present Discounted Value of Lifetime Income

Demand for c_0 :

$$c_0 = f_0\left(y_0 + \frac{y_1}{1+r}, r\right)$$

Interest Rate

An example is illustrated in Figure 7.1.

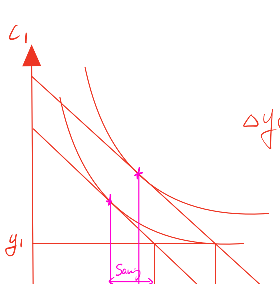
The effect of interest rate changes clearly depend upon whether the individual is a saver or a borrower since this determines the sign of the income effect. Note that an interest rate rise will never induce a saver to become a borrower and an interest rate fall will never induce a borrower to become a saver. (WARP)

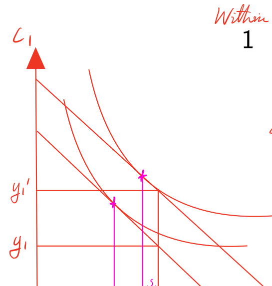
Often it is assumed that the utility function can be written as the sum of utility contributions from the different periods with similar within-period utility functions but with future utility discounted. Thus

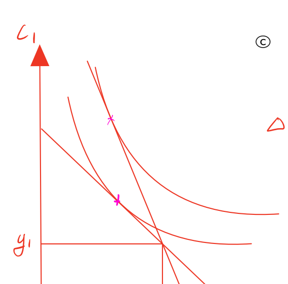
$$u(c_0, c_1) = \nu(c_0) + \frac{1}{1+\delta}\nu(c_1)$$

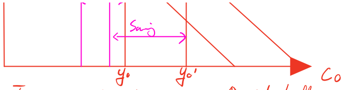
Subjective Discount Rate

Within Period Utility Function

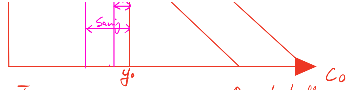




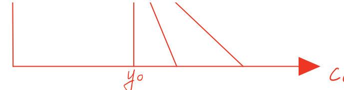




Increase in y_0: $c_0, c_1 \uparrow$ if both are normal goods.
(Current Income↑) Saving↑ or Borrowing↓



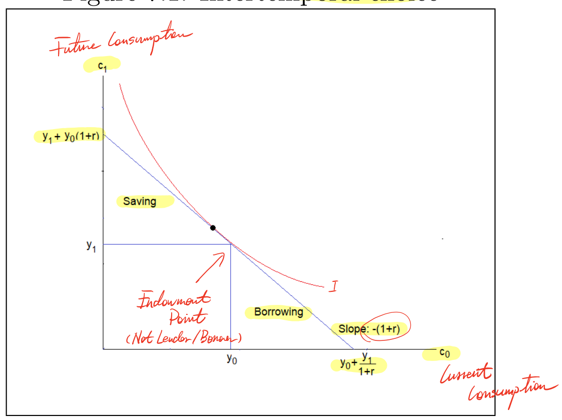
Increase in y_1: $c_0, c_1 \uparrow$ if both are normal goods.
(Future Income↑) Saving↓ or Borrowing↑

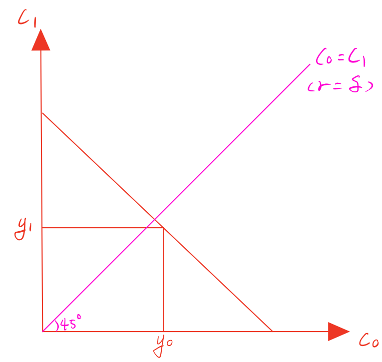


Increase in r: Income & Substitution Effect
- Substitution E: $c_0\downarrow$ $c_1\uparrow$ as current consumption is more "expensive"
- Income E: If the individual is a net saver ($c_0 < y_0$): $c_0, c_1 \uparrow$
  If the individual is a net borrower (c_0 y_0): $c_0, c_1 \downarrow$

†WARP still applies: $r\uparrow$ Savers will not switch to borrowers. v.v.

Figure 7.1: Intertemporal choice





$r = \delta \rightarrow c_0 = c_1$

where $\nu(.)$ is the within-period utility function and δ is a subjective discount rate. Convexity of preferences, which amounts here to a desire to smooth consumption over the life-cycle, requires $\nu(.)$ to be concave, $\nu''(.) < 0$.

Maximising such a utility function subject to the lifetime budget constraint

$$\max_{c_0} \quad \nu(c_0) + \frac{1}{1+\delta}\, \nu(y_1 + (y_0 - c_0)(1+r))$$

$MRS = -\frac{\nu'(c_0)}{\nu'(c_1)}(1+\delta)$

Slope of BC $= -(1+r)$

gives first order condition

FOC / Consumption Euler Equation:

$$\frac{\nu'(c_0)}{\nu'(c_1)} = \frac{1+r}{1+\delta}.$$

This is known as the consumption Euler equation. Given concavity of $\nu(.)$, if $r = \delta$, so that subjective discounting matches the market interest rate and impatience cancels out the market incentive to save, then $c_0 = c_1$ and the consumption stream is flat. If $r > \delta$ then $c_0 < c_1$ and if $r < \delta$ then $c_0 > c_1$. In all of these cases c_0 and c_1 will both be increasing functions of lifetime resources $y_0 + y_1/(1+r)$.

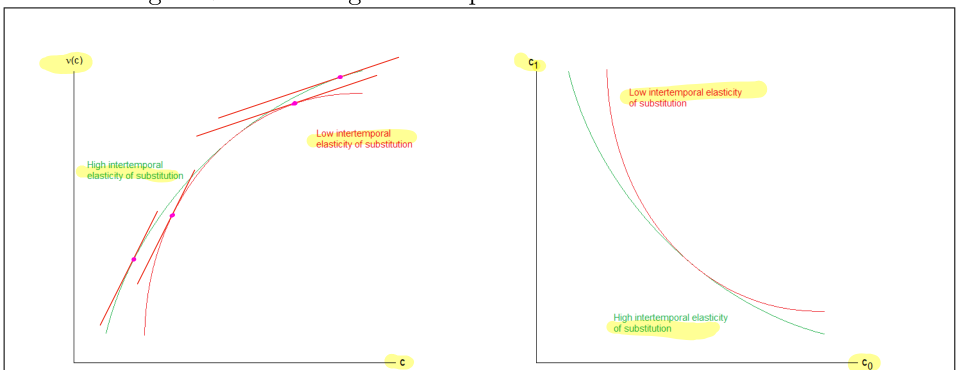
The responsiveness of consumption decisions to changes in the interest rate r depends critically in such a context on the concavity in the within-period utility function $\nu(.)$. The more concave is $\nu(.)$ the more sensitive is the intertemporal marginal rate of substitution to differences in consumption between periods (as seen in Figure 7.2) and therefore the less dramatically do consumption decisions need to respond to bring the MRS into harmony with the interest rate (as seen in Figure 7.3).

The intertemporal elasticity of substitution is a measure of how responsive the steepness of consumption paths is to the interest rate, defined by

$$\sigma = \frac{\partial \ln(c_1/c_0)}{\partial \ln(1+r)} = -\frac{\partial \ln(c_1/c_0)}{\partial \ln(\nu'(c_1)/\nu'(c_0))}.$$

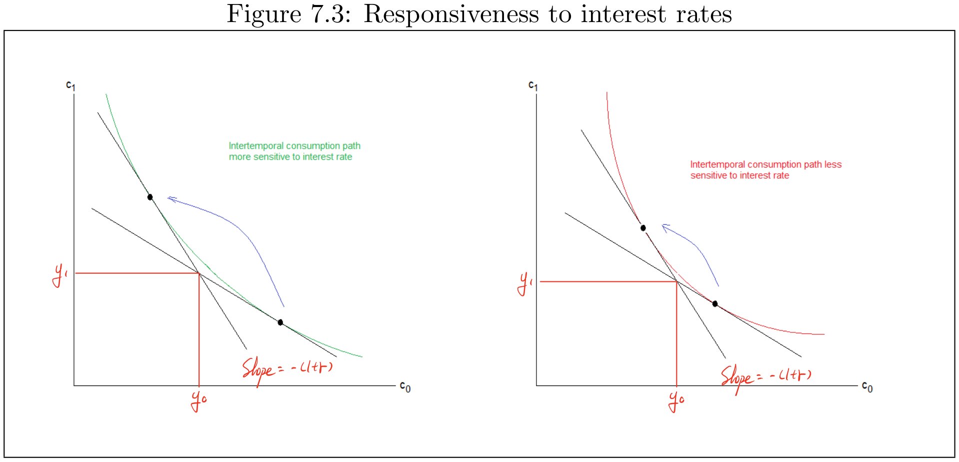
A high intertemporal elasticity of substitution is therefore associated with a low concavity of $\nu(.)$.

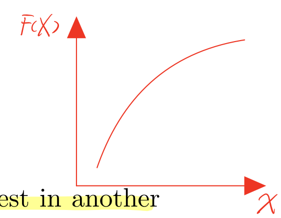
Figure 7.2: Differing intertemporal elasticities of substitution



Marginal Utility is highly sensitive to Consumption <-> Less Δc needed to achieve $\Delta V'$ <-> Rapidly changing slope IC
<-> Response of c to r is less <-> Low Intertemporal Elasticity of Substitution

Figure 7.3: Responsiveness to interest rates





7.2 Asset choice

Suppose that as well as investing in bonds with fixed return of r the individual can also invest in another asset - say a family enterprise. If X is placed in the family enterprise in the first period then suppose $F(X)$ is returned in the second period.

The optimisation problem now has two dimensions B.C. $c_1 = y_1 + F(X) + (y_0 - c_0 - X)(1+r)$

$$\boxed{\max_{c_0, X} \quad \nu(c_0) + \frac{1}{1+\delta}\,\nu(y_1 + F(X) + (y_0 - c_0 - X)(1+r))}$$

and first order conditions (assuming an interior solution) are

$$\text{FOC:}\quad \left\{ \begin{array}{rcl} \dfrac{\nu'(c_0)}{\nu'(c_1)} & = & \dfrac{1+r}{1+\delta} \\ F'(X) & = & (1+r). \end{array} \right.$$

Consumption Still Determined by Consumption Euler Equation

Investment Decision is Independent of Preferences of Consumption

If $F'(0) < 1+r \rightarrow X = 0$ (Corner Solution)

Case Study 7|1 : Intertemporal Choice

The top two panels of the figure below show how mean consumption and income vary over the life cycles of American households. The figure is made by taking data from the US Consumer Expenditure Survey over the 1980s and early 1990s, grouping observed households into generational cohorts according to their date of birth and plotting the means of these variables against age. The humped shape to both profiles reflects to some extent the pattern of household size as captured in the profiles for numbers of adults and children below.
Regressing mean changes in log consumption on interest rates (with controls for seasonality and demographic change) gives an estimate of the Euler equation suggesting an intertemporal elasticity of substitution of 0.637 (with a standard error of 0.333).

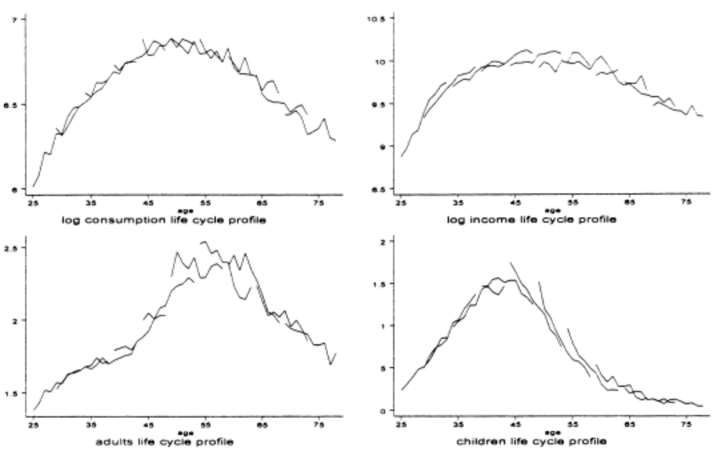
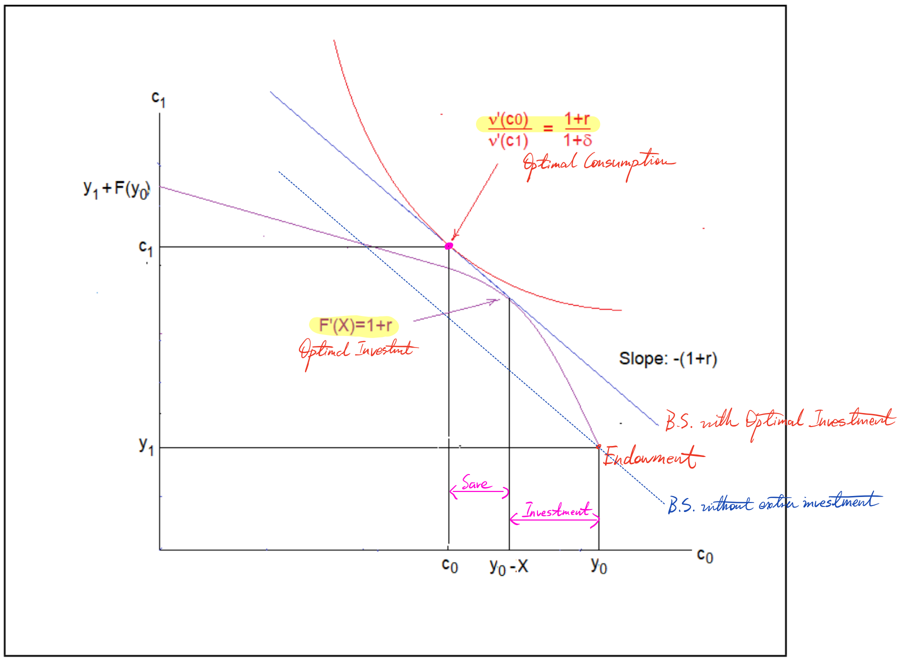


Figure 1. Life-Cycle Profiles.

[Source: O. Attanasio, J. Banks, C. Meghir and G. Weber, 1999, Humps and bumps in lifetime consumption, Journal of Business and Economic Statistics 17, 22-35.]

Note that the solution to the financial decision is independent of intertemporal preferences. The individual invests in the family enterprise until the marginal rate of return falls to the market interest rate. This maximises the present value of the individual's asset portfolio and the first order condition for optimum consumption choice, given that present value, is as in the simpler problem above. The separation between the two decisions is shown in Figure 7.4.

The simplicity of the investment decision is a consequence of assuming away issues concerning risk, liquidity and so on.



Worked Example 7|A : Constant intertemporal elasticity of substitution

Suppose an individual lives for two periods with life-time preferences

$$u(c_0, c_1) = \nu(c_0) + \frac{1}{1+\delta}\nu(c_1)$$

where $\nu_t(c_t) = \frac{1}{1+\gamma}c_t^{1+\gamma}$ and $\gamma < 0$ (and $\nu(c_t) = \ln c_t$ in the special case that $\gamma = 1$).
The consumption Euler equation takes the form

$$\left(\frac{c_1}{c_0}\right)^{-\gamma} = \frac{1+r}{1+\delta}$$

so the intertemporal elasticity of substitution is

$$\sigma = \frac{\partial \ln (c_1/c_0)}{\partial \ln (1+r)} = -\frac{1}{\gamma}$$

which is a constant depending only on γ. The higher is γ the less responsive are consumption streams to market incentives.

Worked Example 7|B : Life-cycle spending with age-dependent needs

Suppose an individual lives for two periods with life-time preferences

$$u(c_0, c_1) = \ln (c_0 - \gamma_0) + \frac{1}{1+\delta}\ln (c_1 - \gamma_1)$$

where γ_0 and γ_1 are period-specific consumption needs. This is similar to the additive model considered above but with within-period utility functions allowed to vary with needs of the period, $\nu_t(c_t) = \ln(c_t - \gamma_t)$. Note that this is an intertemporal version of Stone-Geary preferences. Utility-maximising demands take the form

$$\begin{aligned} c_0 &= \gamma_0 + \frac{1+\delta}{2+\delta}\left[y_0 - \gamma_0 + \frac{y_1 - \gamma_1}{1+r}\right] \\ c_1 &= \gamma_1 + \frac{1+r}{2+\delta}\left[y_0 - \gamma_0 + \frac{y_1 - \gamma_1}{1+r}\right] \end{aligned}$$

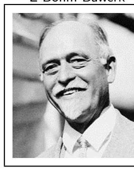
The life-cycle path for consumption is independent of the path of incomes given the discounted present value of lifetime income. A higher value for the interest rate r leads to a more steeply rising path and a higher value of the impatience parameter δ to a less steeply rising one. Even if $r = \delta$ the consumption path will still not be flat if $\gamma_0 \neq \gamma_1$.

History Note 7|I : Intertemporal Choice

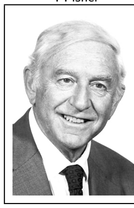
The application of marginalist ideas to questions of intertemporal allocation was taken up in depth in the 1870s by the Austrian civil servant and economist Eugen von Böhm-Bawerk (1851-1914), a follower of Carl Menger (1840-1921). His ideas led to development of the two period model of income and consumption outlined here, laid out as a model of intertemporal choice in the work of the American economist Irving Fisher (1867-1947) in the early twentieth century and making clear the links between interest rates, intertemporal rates of substitution, intertemporal rates of return and consumer time preference.

Relationships between paths of income and consumption were a prominent aspect of the life-cycle model of Franco Modigliani (1918-2003, Nobel 1985) and permanent income model of Milton Friedman (1912-2006, Nobel 1975) developed in the US in the 1950s and 1960s.

E Böhm-Bawerk



I Fisher



F Modigliani

Topic 8 : Choice under uncertainty

Summary: *Choice under uncertainty can be modelled similarly in many ways to intertemporal choice, with the distinction between states of the world mirroring that between periods of time and aversion to risk mirroring the desire to smooth consumption over time. The theory is able to describe important aspects of behaviour but also fails to capture certain anomalies.*

8.1 Uncertainty

Extending the standard analysis to the case of uncertainty involves regarding quantities consumed in different uncertain states of the world as different goods. Preferences will depend on perceived probabilities of states of the world occurring. Budget constraints depend on the mechanisms available for managing risk.

Some examples of budget constraints in circumstances involving risk are:

- An individual with an asset worth A faces a probability π of losing it. He can purchase insurance of K at a cost of γK. Consumption in case of loss is $c_1 = (1-\gamma)K$ and in the case of no loss is $c_0 = A - \gamma K$. The budget constraint for the individual is $c_0 = A - \frac{\gamma}{1-\gamma}c_1$. Assuming overinsurance $K > A$ is ruled out (for legal reasons, for instance) the budget constraint looks like Figure 8.1.

  without insurance $\begin{cases} c_0 = A & \text{No loss} \\ c_1 = 0 & \text{Loss occurs} \end{cases}$ π does not affect the budget constraint

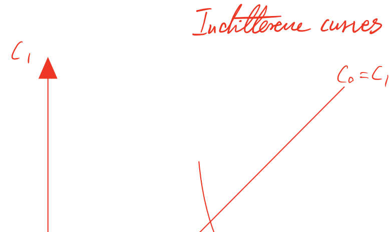
- An individual with income of y has a true tax liability of T but tries to evade an amount D by underdeclaration. There is probability π of being audited in which case he pays the full liability T plus a fine fD. Consumption in case of audit is $c_1 = y - T - fD$ and in the case of no audit is $c_0 = y - T + D$. The budget constraint for the individual is $c_1 = (1+f)(y-T) - fc_0$.

In both of these cases the budget constraint is linear, downward sloping and independent of the probability π.

Preferences are defined over quantities consumed in the different states $(c_0, c_1 ...)$ and depend on perceived probabilities of the states occurring $(\pi_0, \pi_1, ...)$. Under certain assumptions it may be reasonable to regard the consumer as maximising expected utility

$$\max_{c_0 \cdots} E[U] = u(c_0, c_1 ..., \pi_0, \pi_1, ...) = \sum \pi_i \nu(c_i)$$

Indifference curves depend on $\begin{cases} \text{Attitude to risk} \\ \text{Perceived } \pi \text{ (Subjective)} \end{cases}$



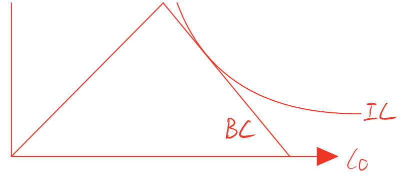
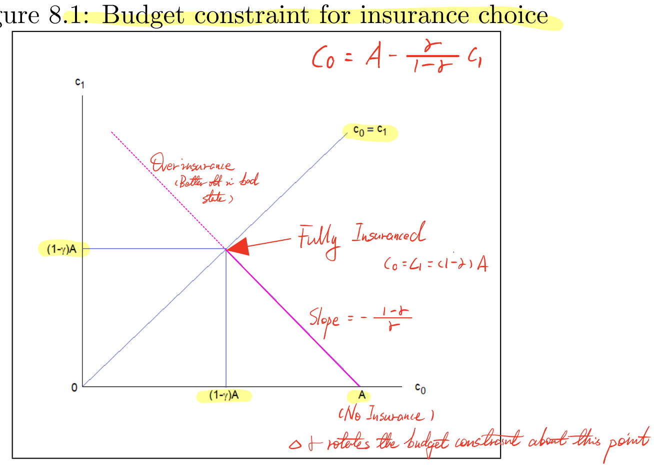


Figure 8.1: Budget constraint for insurance choice



Lists of consumption in different states and associated probabilities are called Lotteries.

for some state-specific utility function (or Bernoulli utility function) $\nu(.)$. We refer to $u(\cdot)$ as a *vNMu*. Von Neumann – Morgenstern Utility Function

The most controversial assumption required to justify an expected utility formulation is the sure thing principle (or the closely related *strong independence axiom*). Consider the following two choices:

Choose the same option in 2 cases

$$\text{Choice 1}: \quad \left[\begin{array}{lccc} & & \text{Probability} & \\ & \pi_1 & \pi_2 & 1-\pi_1-\pi_2 \\ \boxed{\text{Option } A_1} & \alpha_1 & \alpha_2 & \gamma \\ \text{Option } B_1 & \beta_1 & \beta_2 & \gamma \end{array}\right]$$

Sure Thing

$$\text{Choice 2}: \quad \left[\begin{array}{lccc} & & \text{Probability} & \\ & \pi_1 & \pi_2 & 1-\pi_1-\pi_2 \\ \boxed{\text{Option } A_2} & \alpha_1 & \alpha_2 & \delta \\ \text{Option } B_2 & \beta_1 & \beta_2 & \delta \end{array}\right]$$

In each case the two options deliver the same outcome as each other with probability $1 - \pi_1 - \pi_2$ (though these outcomes differ between the two choices). It might therefore be argued that the choice should be driven only by the different outcomes occurring in the other columns. However these outcomes are the same in the two choices. Therefore if A_1 is preferred to B_1 it is argued that A_2 should be preferred to B_2. This is the sure thing principle. Combined with other less controversial axioms extending choice to uncertain situations with multiple outcomes it implies that the MRS between consumption in any two states is independent of outcomes in any other state.

Note that the function $\nu(.)$ is *not* ordinal. Preferences are changed by arbitrary increasing transformations of $\nu(.)$. However $u(.)$ is still ordinal.

Violation in reality: Allais Paradox

		1%	10%	89%		
Choice 1	A_1	£1m	£1m	£1m	(£1m with certainty)	Most people choose A_1
	B_1	0	£5m	£1m		
Choice 2	A_2	£1m	£1m	0	(£1m with 11%)	Most people choose B_2
	B_2	0	£5m	0	(£5m with 10%)	

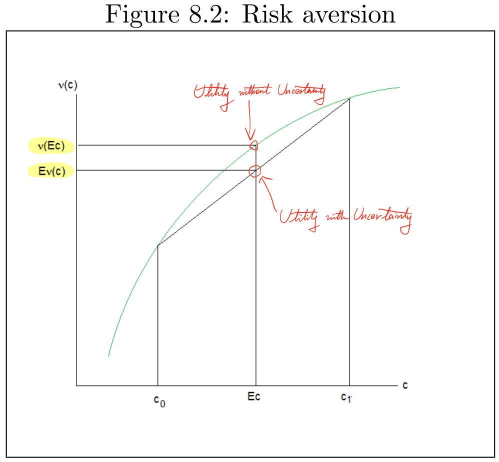
8.2 Risk aversion

To capture risk aversion we need to capture the fact that risk averse individuals prefer to receive the expected value of any gamble with certainty to undertaking the gamble. Thus

$$\nu((1-\pi)c_0 + \pi c_1) > (1-\pi)\nu(c_0) + \pi\nu(c_1).$$

For this always to be true requires that $\nu(.)$ be a concave function as in Figure 8.2. The degree of concavity is an indicator of the strength of aversion to risk (in a similar way to that in which concavity of within-period utility is linked to a desire for consumption smoothing in intertemporal preferences) as in Figure 8.3. The more concave is the within-state utility function the more averse the individual is to risk. IF $\nu(.)$ is linear then the individual is indifferent between participating in the gamble and taking the expected value and is said to show risk neutrality.

Figure 8.2: Risk aversion



Consider the insurance case again. An expected utility maximising consumer chooses K to maximise

$$\max_K E[U] = (1-\pi)\nu(A-\gamma K) + \pi\nu((1-\gamma)K).$$

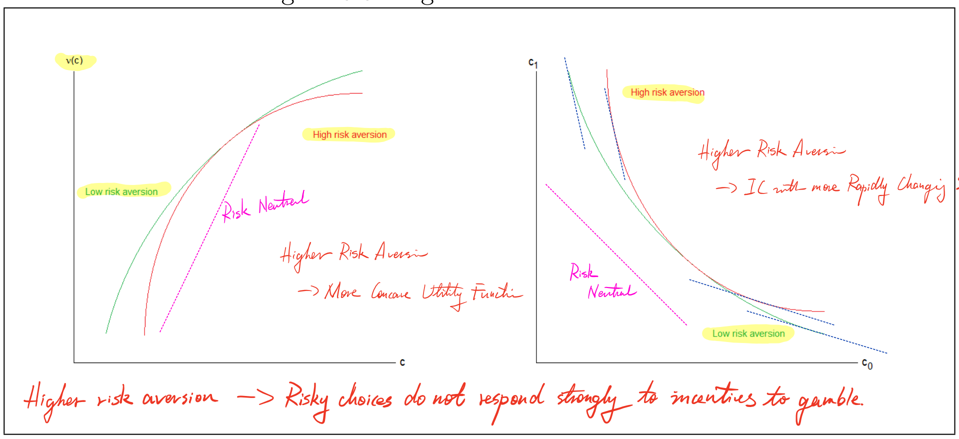
The first order condition requires

$$(1-\pi)\gamma\nu'(A-\gamma K) = (1-\gamma)\pi\nu'((1-\gamma)K).$$

If insurance is actuarially fair then $\pi = \gamma$ and therefore $\nu'(A-\gamma K) = \nu'((1-\gamma)K)$. $MU_0 = MU_1$ If the individual is risk averse then $\nu'(.)$ is a decreasing function and therefore $A = K$ so there is full insurance. This is a typical illustration of behaviour under risk. The fairness of insurance means that risk can be eliminated without compromising expected consumption and a risk averse individual chooses therefore to eliminate risk. The situation is as in Figure 8.4.

† Actuarially Fair: Neither the insurer nor the insuree is making expected profit.

Figure 8.3: High and low risk aversion



If insurance is better than fair $\pi > \gamma$ then the individual would like to overinsure but if not able to (because it might be legally prohibited) will remain at full insurance. If insurance is less than fair $\pi < \gamma$ then there is underinsurance. The extent of underinsurance depends on the concavity of $\nu(.)$. That is to say, the responsiveness of chosen exposure to risk to the departure from actuarial fairness will depend on the individual's aversion to risk (in just the same way that responsiveness of chosen steepness of consumption streams to interest rate depends on concavity of within-period utility function). The two cases are shown in Figure 8.5.

$\gamma > \pi$ FOC implies: $v'(R-\gamma k) > v'(A-\gamma k)$ $A > K$ Underinsurance.

$\gamma < \pi$ FOC implies: $v'(R-\gamma k) < v'(A-\gamma k)$ $A < K$ Over-insurance.

The extent of under/over-insurance depends on: The excessive of the premium $\gamma - \pi$; Risk aversion

Gambling: π: probability of success

C_0: $A - B$ (Unsuccessful)

C_1: $A + Bz$ (Successful)

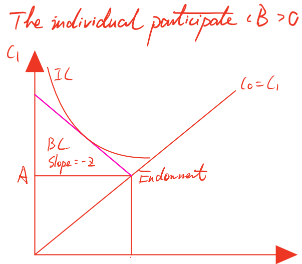
$E[U] = (1-\pi)v(A-B) + \pi v(A+Bz)$

Maximize this with respect to B:

$-(1-\pi)v'(A-B) + \pi z v'(A+Bz) = 0$

The individual participate $(B > 0)$ only if $\pi z - (1-\pi) > 0$

$z > (1-\pi)/\pi$



Note that, for a lottery, $z = \frac{\text{reward} - \text{price}}{\text{price}}$

Figure 8.4: Full insurance under actuarial fairness

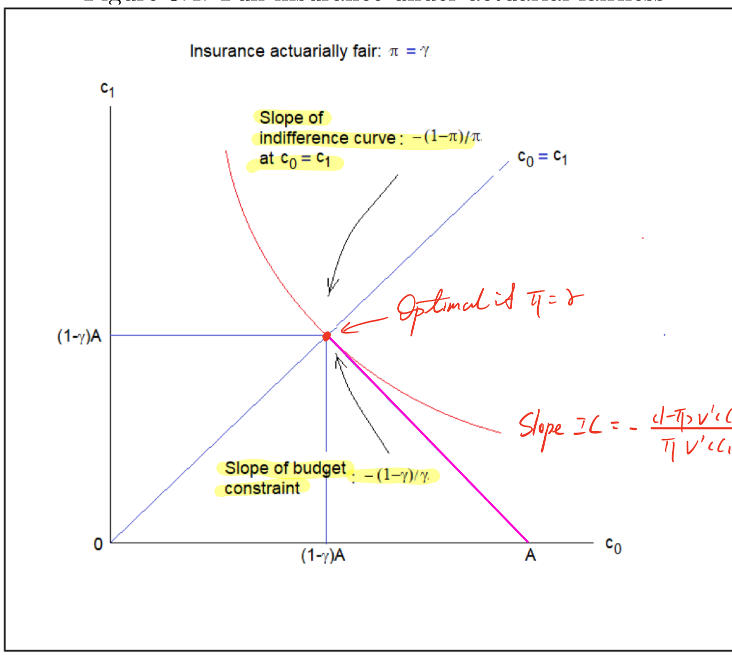
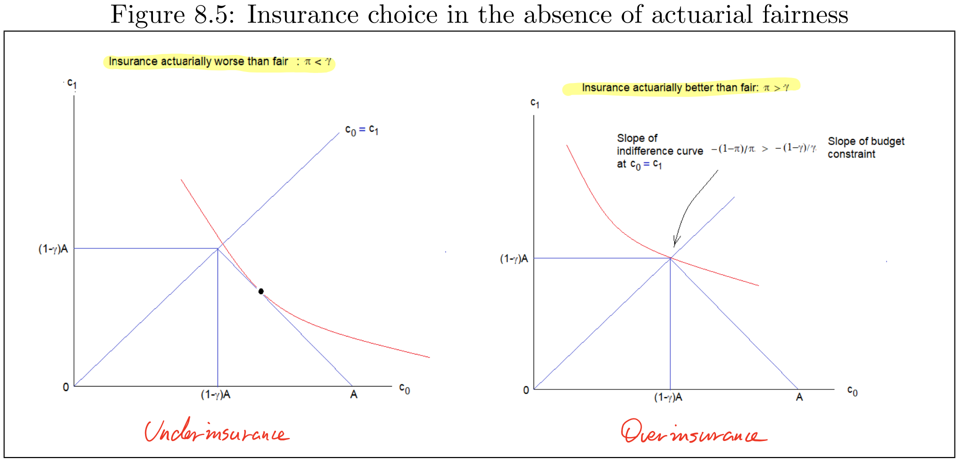


Figure 8.5: Insurance choice in the absence of actuarial fairness



Case Study 8|1 : Choice under Uncertainty

Theoretical consumers behaving as expected utility maximisers seem to show a perhaps unnaturally high proficiency in their understanding and ability to work with probabilities. In practice there is abundant evidence that individuals understand probabilities poorly. Even where probabilities are described clearly their behaviour often fails to satisfy axioms upon which the theory is based. They attach greater importance to the status quo than is explicable and their decisions depend upon how the situation is framed when described to them.
A particularly well known example in which behaviour fails to fit is the so-called Allais paradox. Consider the following two choices:

$$\text{Choice 1 :} \quad \begin{bmatrix} & & \text{Probability} & \\ & 0.01 & 0.33 & 0.66 \\ \text{Option } A_1 & 2400 & 2400 & 2400 \\ \text{Option } B_1 & 0 & 2500 & 2400 \end{bmatrix}$$

$$\text{Choice 2 :} \quad \begin{bmatrix} & & \text{Probability} & \\ & 0.01 & 0.33 & 0.66 \\ \text{Option } A_2 & 2400 & 2400 & 0 \\ \text{Option } B_2 & 0 & 2500 & 0 \end{bmatrix}$$

Kahneman and Tversky report the outcome of presenting these to 72 students. In the first choice 82% chose A_1 whereas in the second choice 83% chose B_2. 61% of students made the modal choice in both cases but this combination of choices violates the sure thing principle. Presumably the attraction of certainty in the first choice has something to do with this.
Expected utility theory captures some features of behaviour that we would want to include, such as aversion to risk, and continues to be widely assumed in much work but it has clear inadequacies in many contexts and exploration of such deviations is an important part of ongoing research.

[Source: D. Kahneman and A. Tversky, 1979, Prospect theory: an analysis of decision under risk, *Econometrica* 47, 263-292.]

Worked Example 8|A : Insurance

Suppose that an individual is an expected utility maximiser with within-state utility function given by $\nu(c) = \ln c$. Since this is concave the individual is risk-averse. Their objective actually has the form of Cobb-Douglas utility with the probabilities taking the place of preference coefficients

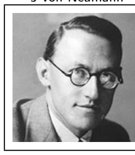
$$u(c_0, c_1) = (1-\pi)\ln c_0 + \pi \ln c_1.$$

The first order condition for optimal insurance choice

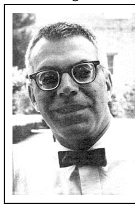
$$\frac{(1-\pi)\gamma}{A-\gamma K} = \frac{(1-\gamma)\pi}{(1-\gamma)K}$$

implies $K = \pi A/\gamma$. As expected this is less than A unless $\gamma \leq \pi$. Wealth levels in the two states are $c_0 = (1-\pi)A$ and $c_1 = (1-\gamma)\pi A/\gamma$.

J von Neumann



O Morgenstern

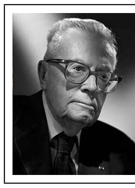


L J Savage

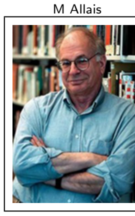
History Note 8|I : Choice under Uncertainty

Development of ideas of probability in the eighteenth century was tied closely to development of ideas about choice under uncertainty. The St Petersburg paradox which seemed to suggest an implausible case in which it would be rational to gamble unlimited sums of money was resolved by the postulation of logarithmic utility by Daniel Bernoulli (1700-82), alluded to in earlier sections.

The expected utility theorem was proved by the Hungarian-American polymath John von Neumann (1903-57) and German-American collaborator Oskar Morgenstern (1902-77) in connection with their foundational work on game theory. The sure thing principle was put forward by the American statistician Leonard Savage (1917-71) as part of a proposed axiomatisation of subjective decision theory. Measurement and comparison of risk aversion in an expected utility framework was explored in the 1960s by American economists Kenneth Arrow (b.1921, Nobel 1972) and John Pratt (b.1931).



M Allais

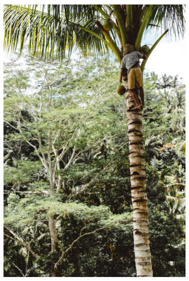


D Kahnemann

A Tversky

History Note 8|II : Behavioural Economics

The Allais paradox, attributed to French economist Maurice Allais (1911-2010, Nobel 1988), was one of several departures of behaviour from the postulates of expected utility theory that have been apparent since the 1950s. Paradoxes such as these have done much to encourage the growth of the field of behavioural economics associated, for example, with the Israeli-American and Israeli psychologists, Daniel Kahneman (b. 1934, Nobel 2002) and Amos Tversky (1937-96).



Topic 9 : General equilibrium

Summary: *If we put consumers together and allow them to trade then we can ask whether prices will exist that lead all markets to clear and, if so, whether this provides a sensible or illuminating theory of price determination. Adding profit-maximising producers to the picture turns out not to complicate matters unmanageably.*

9.1 Exchange equilibrium

(Pure change economy)

Suppose the economy consists of H households. The hth household has endowment ω^h and consumes a bundle q^h. An allocation of goods is said to be *feasible* if the aggregate amount consumed of each good equals the aggregate endowment

$$\sum_h q_i^h = \sum_h \omega_i^h \quad i = 1, 2, ..., H$$

The initial endowments obviously constitute one feasible allocation.

A common diagrammatic representation of exchange equilibrium for a two-person two-good exchange economy is the Edgeworth-Bowley box. This is a rectangular box whose horizontal and vertical dimensions are set to the economy-wide endowments of the two goods so that feasible allocations correspond to points within the box. Consumptions of the two individuals are read from opposite corners of the box and any point in the box, including the initial endowment point, represents a feasible allocation in the economy as illustrated in Figure 9.1.

Preferences of the two individuals can be represented by drawing indifference curves in the Edgeworth-Bowley box. In general, there will be allocations that leave both individuals better off than with their initial endowments, which is to say allocations Pareto superior to the endowment point, as can be seen in the left panel of Figure 9.2. If so then individuals can gain from trade if it allows them to move into this area.

Setting a price vector for the economy defines a common budget constraint passing through the endowment point representing trading possibilities for the two individuals. Individuals' desired trades are defined by tangencies between this budget constraint and their indifference curves as in the right hand panel. Typically these desired trades will not match so there will be excess supply of some goods and excess demand for others at arbitrarily chosen prices.

General Case:

Demands if prices are p are $q_i^h = f_i^h(p'\omega^h, p)$, $i = 1, 2, ..., H$ where $p'\omega^h$ is the value of the individual's endowment. (Note that the no assumption is being made that different households have the same preferences.) Market demand

Figure 9.1: Feasible allocations in an Edgeworth-Bowley box

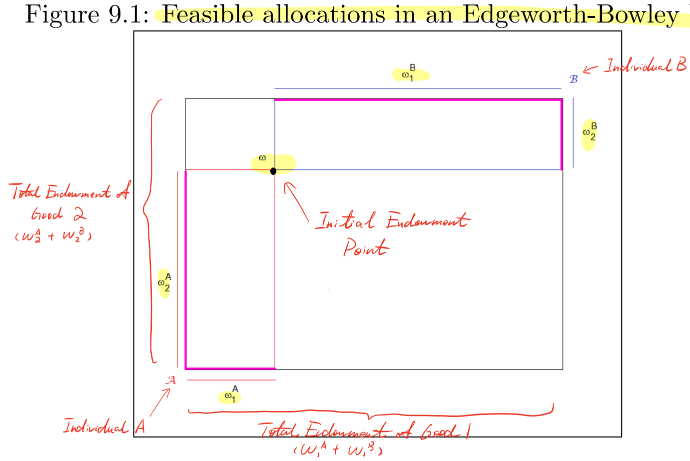
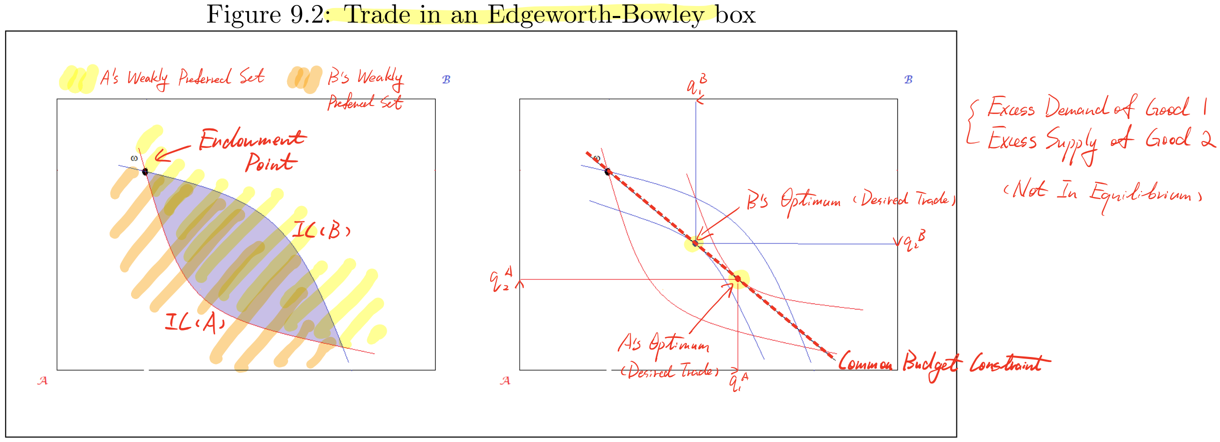


Figure 9.2: Trade in an Edgeworth-Bowley box



is found by adding the demands across individuals

Aggregate / Market Gross Demand

$$\boxed{Q_i(p'\omega^1, p'\omega^2, ..., p) = \sum_h f_i^h(p'\omega^h, p) \quad i = 1, 2, ...}$$

Note the dependence on the complete distribution of endowments.

Let z_i^h denote the excess demand from the ith household. The aggregate excess demand z_i is given by the excess of market demand over the sum of endowments

Aggregate / Market Excess Demand

$$\boxed{Z_i(p) = \sum_h z_i^h = \sum_h [f_i^h(p'\omega^h, p) - \omega_i^h] \quad i = 1, 2, ...}$$

General equilibrium - referred to also as market equilibrium, competitive equilibrium or Walrasian equilibrium - is a set of prices such that aggregate excess demand is zero on all markets.

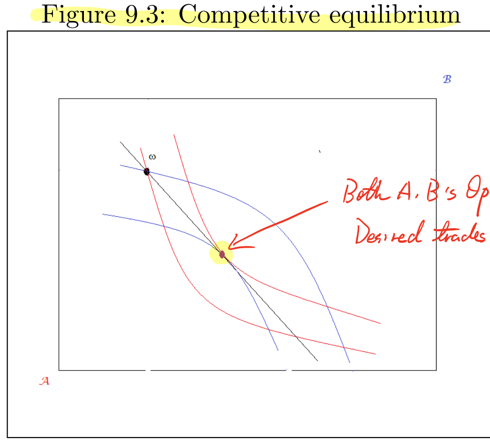
General Equilibrium

$$\boxed{Z_i = \sum_h z_i^h = 0 \quad i = 1, 2, ...}$$

$Z_i(p^*)$

A competitive allocation, if it exists, is another example of a feasible allocation. Figure 9.3 shows competitive equilibrium in a two good economy.

Figure 9.3: Competitive equilibrium



Both A, B's Optimum
Desired trades are matched

9.2 Walras' Law

(Adding Up)

If there are M goods then this seems to define M equations in M unknown prices. However this is misleading. The fact that each household must be on its budget constraint implies that the value of that household's excess demand is zero

$$\boxed{\sum_i p_i q_i^h = \sum_i p_i \omega_i^h \quad \Rightarrow \quad \sum_i p_i z_i^h = 0.}$$

for each household

Adding this equation over households establishes that the value of aggregate excess demand is also zero

$$\boxed{\sum_i p_i Z_i(p) = 0.}$$

→ Clearing all but 1 markets also clears the remaining market.

This is Walras' law and is true for any prices (not only the equilibrium prices). It implies that the M excess demands are not independent - in fact there are only $M - 1$ independent excess demands to set to zero. → M-1 Equations

However, since demands are homogeneous, multiplying all prices by any positive number will give the same excess demands. If any prices constitute a Walrasian equilibrium, then so therefore do any positive multiple of those prices. It is therefore only relative prices which are determined by the equilibrium conditions. → M-1 R. Prices

There are therefore actually $M - 1$ independent equations determining $M - 1$ relative prices.

In 2 consumers, 2 goods situations, only need to solve 1 market clearing with 1 relative price.

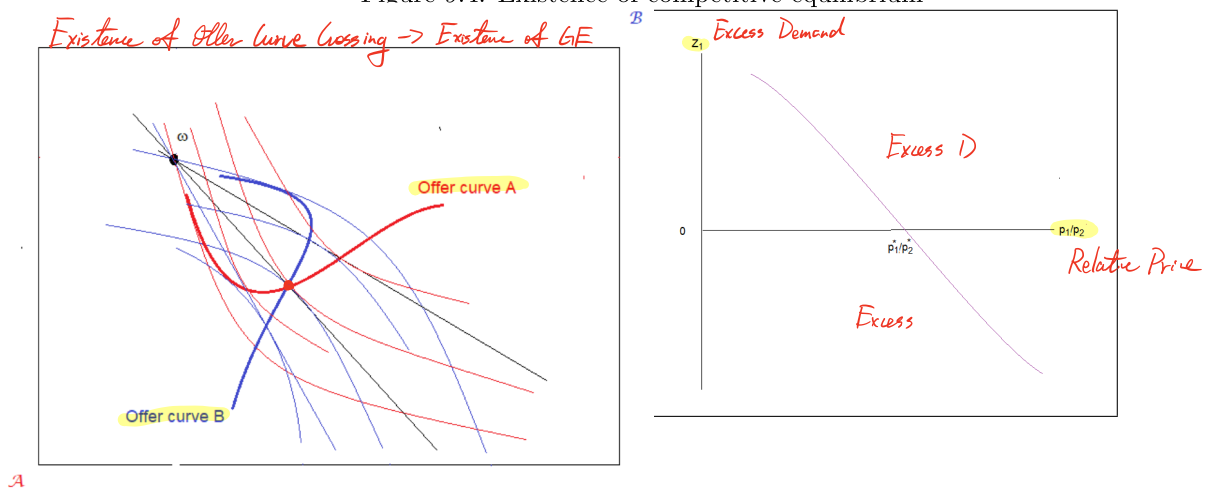
9.3 Existence, uniqueness and stability

Nothing said so far ensures existence of a Walrasian equilibrium but if aggregate demands vary continuously as a function of prices then it can be proved that at least one equilibrium must exist. (Individual preferences are strictly convex)

Hold for all number of goods, people

Figure 9.4 shows, for instance, why equilibrium needs to exist in the two good case. At a low enough relative price p_1/p_2 there will be excess demand for good 1 whereas at a high enough p_1/p_2 there would be excess supply. If demands vary continuously with prices then there is necessarily some intermediate price ratio at which excess demand is zero. As the price ratio varies over this range the individuals move along their offer curves and equilibrium occurs at a crossing. This will be so if, for instance, individuals all have convex downward-sloping indifference curves (as they will if preferences satisfy continuity, monotonicity and convexity) since then individual excess demands will be continuous and so therefore will aggregate excess demands.

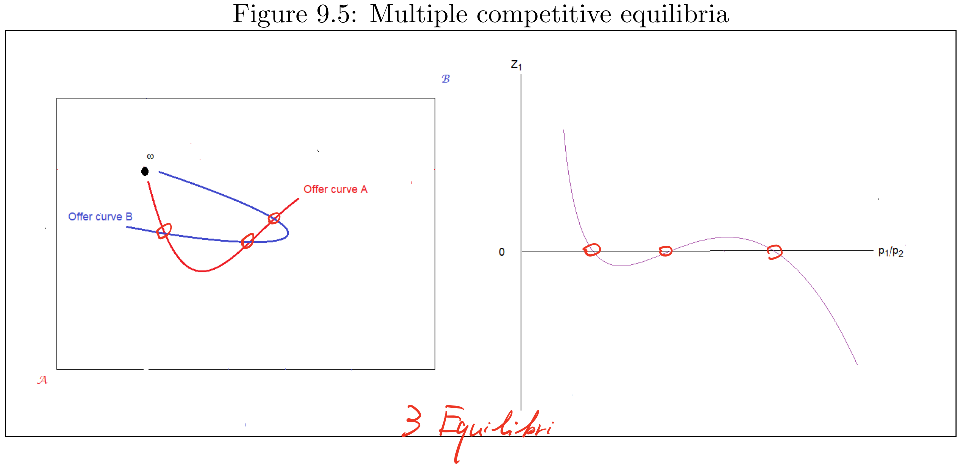
Figure 9.4: Existence of competitive equilibrium



There is no guarantee however that the equilibrium will be unique without further assumptions on preferences and indeed economies with multiple equilibria are easily illustrated, as in Figure 9.5 where offer curves cross three times.

In general, GE needs not to be uni:

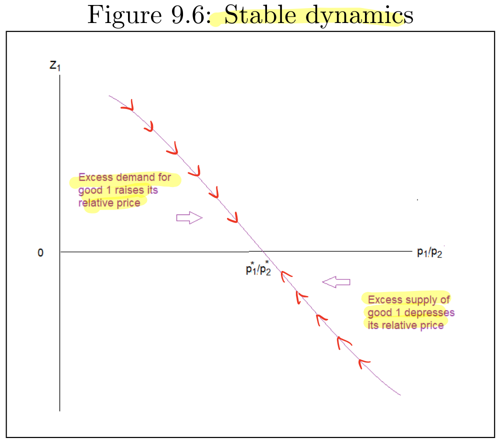
Figure 9.5: Multiple competitive equilibria



We do not discuss stability in this cour

Whether or not prices in an economy out of equilibrium will tend to move so as to take it towards equilibrium is a question that cannot be answered without a theory of what happens out of equilibrium. If not all demands can be met from the economy's endowments then what happens and how do prices adjust? It is possible to tell artificial and simple stories demonstrating the ability of a hypothetical auctioneer to find equilibrium prices under appropriate assumptions about preferences but these arguably tell us little about the stability of equilibrium in actual economies with trading taking place out of equilibrium. Figure 9.6 shows how simple dynamics in a two good economy could justify an assumption of stability if equilibrium is unique.

Figure 9.6: Stable dynamics



9.4 Equilibrium in economies with production

We can introduce production into the economy by assuming the existence of K firms, each taking prices as given and choosing production plans so as to maximise profits given their own technology

$$\boxed{\pi^k = \max_{y^k} p'y^k \qquad \text{s.t.} \qquad y^k \text{ technologicallyfeasible}}$$

y is the net output (negative if used as input)

for $k = 1, 2, ..., K$. Note that firms are not being assumed to share a common technology.

Inputs to production come from consumers' net supply of endowments (and particularly labour supply). Net outputs of firms supplement consumers' endowments as sources of resources for consumption.

Firms are assumed to be owned by consumers so that profits are returned to them. If the hth household owns a share θ_{hk} of the kth firm then its budget constraint is therefore

$$\boxed{p'q^h \leq p'\omega^h + \sum_k \theta_{hk}\pi^k.}$$

All profits are assumed to be distributed in this way so that $\sum_h \theta_{hk} = 1$ for all $k = 1, 2, \ldots$.

Worked Example 9|A : Exchange equilibrium

Demands if prices are p are $q_i^h = f_i^h(Y^h, p)$, $i = 1, 2, ..., M$ where $Y^h = p'\omega^h$ is the value of the individual's endowment. We need to find a price vector p solving the market clearing equations

$$\sum_h f_i^h(p'\omega^h, p) = \sum_h \omega_i^h \quad i = 1, 2, ..., M$$

From Walras' law we need only solve for $M-1$ relative prices achieving market clearing on $M-1$ of the M markets.
As an example, suppose there are only two goods so we need to find only one relative price to clear one market. Since we can solve only for the relative price we normalise the price of good 2 to be 1 and let P be the price of the first. Let there be two consumers A and B who have Cobb-Douglas demands over the two goods. Individual h therefore has demands

$$q_1^h = \alpha^h Y^h/P \qquad q_2^h = (1-\alpha^h)Y^h \qquad h = A, B$$

where α^h is an individual-specific taste parameter.
Individual endowments are $\omega^h = (\omega_1^h, \omega_2^h)$. Therefore, by substituting the value of the endowments, demands are

$$q_1^h = \alpha^h(P\omega_1^h + \omega_2^h)/P \qquad q_2^h = (1-\alpha^h)(P\omega_1^h + \omega_2^h)$$

To find equilibrium, we know from Walras' law that we need only find the price to clear one market. Take the first. Market clearing requires

$$\omega_1^A + \omega_1^B = q_1^B + q_1^B = \alpha^A(P\omega_1^A + \omega_2^A)/P + \alpha^B(P\omega_1^B + \omega_2^B)/P$$

Solving for P gives

$$P = \frac{\alpha^A\omega_2^A + \alpha^B\omega_2^B}{(1-\alpha^A)\omega_1^A + (1-\alpha^B)\omega_1^B}.$$

Note that this is increasing in endowments of the second good and decreasing in endowments of the first. Note also that it is increasing in the demand parameters α^A and α^B. These are readily intelligible demand and supply effects for this example.

We can now redefine aggregate excess demand for any good as the excess of aggregate consumption over endowments *and* production:

$$Z_i(p) = \sum_h (q_i^h - \omega_i^h) - \sum_k y_i^k.$$

→ firm's net production

Given that $\sum_h \theta_{hk} = 1$, note that Walras' law still holds since

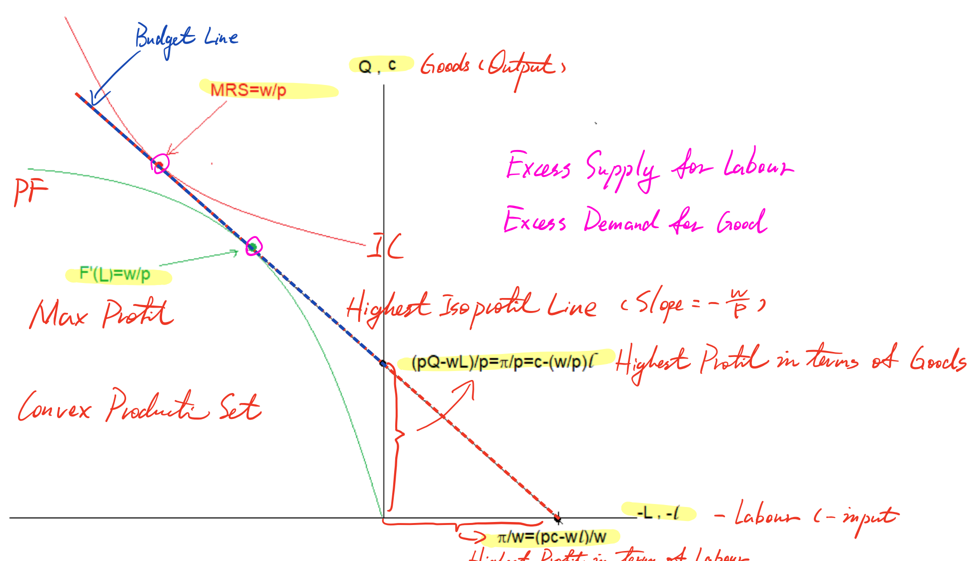
$$\sum_i p_i Z_i(p) = 0.$$

Just as we have the Edgeworth-Bowley box as a useful visual guide to intuition for an exchange economy, there is a helpful diagram also for the case of a competitive economy with production. For diagrammatic reasons we keep to two goods, but one is now an input (labour, L when demanded and l when supplied) (price = w) and the other an output good (Q when produced and c when consumed) (price = 1). There are also two agents, one a firm and one a consumer, but since the firm's profits are all passed to the consumer as income we can regard these as two aspects of the behaviour of a single person and the economy is therefore referred to as a Robinson Crusoe economy. (The fact that Robinson Crusoe acts a price taker in exchanges with himself under these two roles is slightly odd but it is best to think of this as a heuristic abstraction to capture what might be a more reasonable description of interactions in a multi-person economy.)

We can assume that technology takes the form of a production function, $Q = F(L)$. Given a wage w and output price p, profits are therefore $pF(L) - wL$ and a profit maximising firm sets the marginal product $F'(L)$ equal to the real wage w/p. Consumer optimisation equates the marginal rate of usbtitution between consumption and labour to the same real wage w/p Such an economy is illustrated in Figure 9.7. For the given wage w and output price p illustrated there is excess supply of labour and excess demand for output.

Firm: $\max_L \; pQ - wL$ s.t. $Q = F(L)$

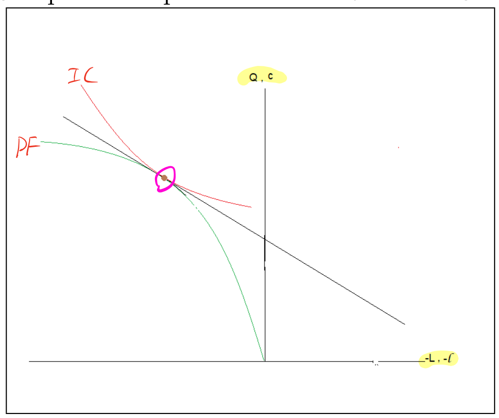
Then profit is returned to the consumer



Walrasian equilibrium is defined in the same way as in an exchange economy as a price vector which ensures excess demands are zero on all markets. Existence of at least one equilibrium is guaranteed, as in an exchange economy, if aggregate net demands are continuous functions of prices. This will be so, for instance, if, besides the earlier assumption of convex, downward-sloping indifference curves for consumers, we also have strict convexity of firms' production possibilities. Equilibrium in the Robinson Crusoe economy is shown in Figure 9.8.

Equilibrium is defined for p^* such that $Z_i(p^*) = 0$ for all good

Figure 9.8: Competitive equilibrium in a Robinson Crusoe economy



9.5 Limitations of general equilibrium analysis

Models of general equilibrium describe idealised economies without market power for individual agents, without frictions which might prevent clearing of markets and in which all interdependence between agents can be accommodated through the price mechanism. They are interesting and important benchmarks because of the relatively well understood nature of equilibria and their interesting welfare properties, discussed below, but many of the most interesting questions in economics are about the properties of economies which do not fit such a simplified description.

Worked Example 9|B : General equilibrium with production

We now need to find a price vector p solving the more complicated market clearing equations

$$\sum_h f_i^h(p'\omega^h + \sum_k \theta_{hk}\pi^k(p), p) = \sum_h \omega_i^h \ + \ \sum_k y_i^k(p) \quad i = 1, 2, ..., M$$

where $\pi^k(p)$ and $y_i^k(p)$ are functions giving profits and net supplies given p. From Walras' law we still only need only solve for $M-1$ relative prices achieving market clearing on $M-1$ of the M markets.

To take a typically simple example, suppose there is one firm, one consumer, one output and one input which is labour time. Suppose the only endowment is one unit of time held by the consumer. The consumer receives all profits, $\theta_{11} = 1$.

We can solve only for the relative price so we normalise the price of output to be 1 and let W be the price of labour (which is to say, the real wage). Full income for the consumer is $W + \pi(W)$ where $\pi(W)$ is the firm's profits returned to the consumer as unearned income.

Let the consumer have Cobb-Douglas preferences so that their demand for time is $h = \alpha(W + \pi(W))/W$ where α is a preference parameter. Supply of labour is therefore $l = 1 - h = (1-\alpha) - \alpha\pi(w)/W$.

Let the firm have a technology which allows it to produce output $\sqrt{L}$ with labour input L. Profit is therefore $\sqrt{L} - WL$. Maximising this gives labour demand $L = 1/4W^2$ and maximised profit $\pi(W) = 1/4W$. Equating supply of labour to demand for labour requires

$$1/4W^2 \ = \ (1-\alpha) - \alpha/4W^2$$

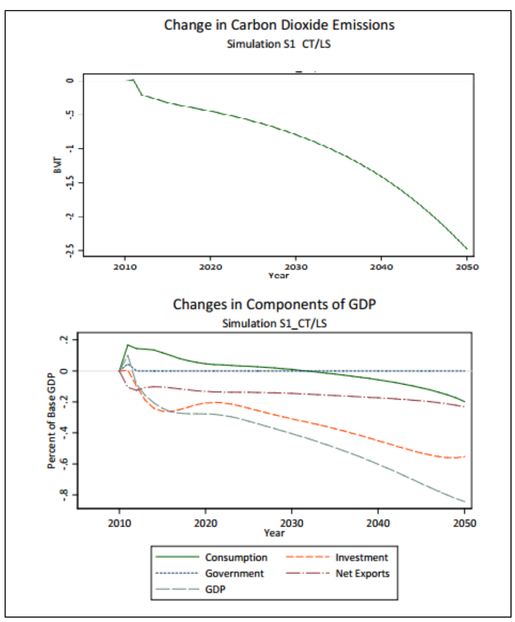
which implies equilibrium real wage

$$W \ = \ \frac{1}{2}\sqrt{\frac{1+\alpha}{1-\alpha}}.$$

A stronger preference for leisure tightens supply of labour and increases the equilibrium real wage in this simple economy.

Case Study 9|1 : Computable General Equilibrium

One of the things to be learned from general equilibrium models is the way in which changes in conditions in one market can ramify into effects on others. This can be particularly important, for example, in analysing the impact of tax changes which affect goods which are inputs into production of many others. General equilibrium effects therefore seem particularly important in considering the effect of carbon taxes proposed as policy responses to global warming. Several institutions have developed computable multiple sector models to assist in analysis. An example is G-Cubed, a nine-geographical-region twelve-industrial-sector model of the world economy developed at the Brookings Institution. The figure below shows projections for the impact of a gradually introduced fossil fuel tax on the US economy (with revenue returned to households in the form of lump sum rebates). by raising the price of coal, natural gas and oil, the policy reform achieves significant abatement of carbon dioxide emissions. Underlying this is a complex pattern of sectoral adjustments which feed through into changing composition of GDP as illustrated in the lower panel. Scenarios in which the tax revenue is used differently can look quite different. The virtue of this sort of modelling is that it can deliver a sophisticated picture of the effects of nuanced reforms. The weakness is the sometimes opaque complex dependence on details of the model's formulation, estimation of its parameters and assumptions about smoothness of equilibriation.



[Source: W. McKibbin, A. Morris, P. Wilcoxen and Y. Cai, 2012, The Potential Role of a Carbon Tax in US Fiscal Reform, Brookings Institution, Climate and Energy Economics Discussion Paper.
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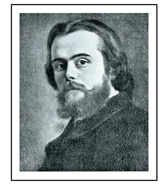
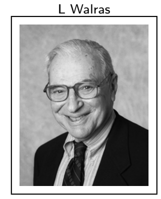
History Note 9|I : General equilibrium

Determination of prices in exchange is a topic that has been of longstanding interest to economics. Léon Walras (1834-1910) was particularly successful in formulating a mathematical description of a competitive equilibrium with multiple markets and with consumption and production of goods, inspired in part by notions of equilibrium in physics. The Edgeworth-Bowley box used to illustrate equilibrium geometrically was a construction introduced by the Irish economist Francis Ysidro Edgeworth (1845-1926) and popularised by the English statistician Arthur Bowley (1869-18) (both sometime lecturers at UCL) after refinement by the Italian social thinker Vilfredo Pareto (1848-1923).

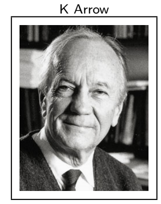
Rigorous proofs of theorems on existence and uniqueness of equilibrium for an economy in which complete markets are hypothesised to cover all uncertain contingencies were made in the work of the 1950s by the American economists Lionel McKenzie (1919-2010) and Kenneth Arrow (b.1921, Nobel 1972) and his French coauthor Gérard Debreu (1921-2004, Nobel 1983), work widely regarded as a highpoint of its brand of mathematical economics.



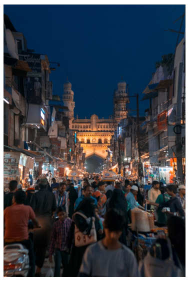
L Walras



K Arrow



L McKenzie



Topic 10 : Welfare theorems and public goods

Summary: *Competitive equilibrium exhausts the scope for mutual gains from trade. Outcomes are efficient in the sense that it is impossible to make everyone better off. Moreover any efficient allocation is supportable as a competitive equilibrium if government can redistribute endowments appropriately. If individual demands are interdependent, as there are when there are public goods or externalities, then efficiency can be promoted by public intervention.*

10.1 Fundamental welfare theorems for exchange equilibria

1st Welfare Theorem

Walrasian equilibrium in exchange economies have the general property of being Pareto efficient (or Pareto optimal). This means that there is no feasible allocation such that all consumers are better off (or some are better off without any being any worse off).

P

To prove this suppose it were not the case. Then there would exist a feasible allocation r^1, r^2,... such that r^1 was preferred to q^1, r^2 was preferred to q^2 and so on. But then these bundles could not be affordable at the equilibrium prices p or the consumers would have purchased them. Thus

$$\sum_i p_i r_i^h \geq \sum_i p_i q_i^h = \sum_i p_i \omega_i^h \quad h = 1, 2, ..., Ha$$

with at least one of these inequalities being strict. (Someone is better off without any other being worse off)

Adding across consumers gives

$$\sum_i p_i \sum_h r_i^h > \sum_i p_i \sum_h \omega_i^h.$$

(Aggregate value of alternative allocation must be higher, so it is infeasible)

But this conflicts with feasibility which requires $\sum_h r_i^h = \sum_h \omega_i^h$, $i = 1, 2, ..., M$ Hence there can be no such alternative allocation.

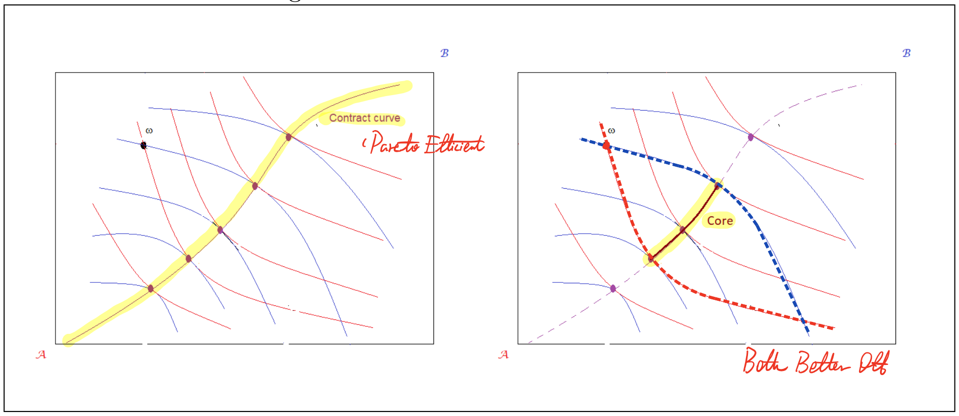
This is the First Fundamental Theorem of Welfare Economics. Walrasian equilibrium is always Pareto efficient.

Note that this says nothing about desirability in other respects.

There are many Pareto efficient allocations and any Walrasian equilibrium that results from particular initial endowments will be just one. The locus of Pareto efficient allocations is known as the contract curve and is illustrated for a two good economy using the Edgeworth-Bowley box in Figure 10.1. Those allocations on the contract curve that are Pareto superior to the initial endowment allocation are known as the core.

Considerations such as distributional equity may give good grounds for regarding certain allocations on the contract curve to be socially preferred to others and whether or not these are attained will inevitably depend, for example, upon the equity in the allocation of initial endowments.

Figure 10.1: Contract curve and core



2nd Welfare Theorem

The Second Fundamental Theorem of Welfare Economics tells us conversely that, under certain further assumptions, any Pareto efficient allocation can be sustained as a Walrasian equilibrium given the right allocation of initial endowments. In particular, this is true if we assume all agents have convex preferences. Pareto Efficient <-> Competitive Equilibrium

10.2 Efficiency of exchange equilibria

Take an economy with total endowments $\Omega = \sum_h \omega^h$ and consider choosing the allocation across consumers to maximise the utility of one consumer, say $h = 1$, subject to specified utilities, $\bar{u}^h$, $h = 2, 3, \ldots, H$, for all of the others. Solutions to this problem define the set of Pareto efficient allocations for the economy. The quantities consumed by the first household are what is left over after subtracting the consumption of all the others from the aggregate endowment

$$\boxed{q^1 = \Omega - \sum_{h=2,3,\ldots,H} q^h}$$

so the problem is:

$$\boxed{\begin{aligned} \max_{q^2,\, q^3,\, \ldots} \; & u^1\left(\Omega - \sum_{h=2,3,\ldots,H} q^h\right) \\ \text{s.t.} \quad & u^h(q^h) = \bar{u}^h, \; h = 2, 3, \ldots, H \end{aligned}}$$

First order conditions for internal optima require

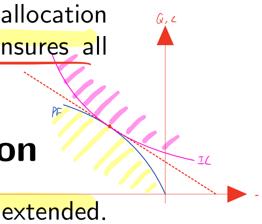
$$\boxed{\frac{\partial u^1}{\partial q_i^1} = \lambda^h \frac{\partial u^h}{\partial q_i^h}, \; h = 2, 3, \ldots, H}$$

where λ^h is the Lagrange multiplier on the utility constraint for the hth household. These imply

$$\boxed{\frac{\partial u^h / \partial q_i^h}{\partial u^h / \partial q_j^h} = \frac{\partial u^g / \partial q_i^g}{\partial u^g / \partial q_j^g}}$$

Common MRS

for every pair of households $\{g, h\}$ and every pair of goods $\{i, j\}$. Unless marginal rates of substitution are equated across consumers then a reallocation can improve some individuals utilities without harming others and the allocation is not Pareto optimal. Competitive equilibrium achieves Pareto efficiency because consumer choice ensures all individuals equate their MRS to a common price ratio.



1st & 2nd Welfare Theorem can be extended to an economy with production

10.3 Fundamental welfare theorems for equilibria with production

Both fundamental theorems still hold for economies with production if assumptions are appropriately extended. In particular Walrasian equilibria are still Pareto efficient.

Proof of 1st WT still holds

To prove this, assume, as when proving efficiency in an exchange economy, that it were not so. Then there would exist productions plans x^1, x^2,... and a feasible allocation r^1, r^2,... such that r^1 was preferred to q^1, r^2 was preferred to q^2 and so on. But, as in the earlier proof, these bundles could not be affordable at the equilibrium prices p, given equilibrium profits or the consumers would have purchased them. Thus

$$\sum_i p_i r_i^h \geq \sum_i p_i \omega_i^h + \sum_k \theta_{hk} \sum_i p_i y_i^k \quad h = 1, 2, ..., H$$

with at least one of these inequalities being strict.

Adding across consumers gives

$$\sum_i p_i \sum_h r_i^h > \sum_i p_i \left[\sum_h \omega_i^h + \sum_k y_i^k \right]$$

since $\sum_h \theta_{hk} = 1$.

But if this alternative allocation is feasible under the alternative production plans then

$$\sum_h r_i^h = \left[\sum_h \omega_i^h + \sum_k x_i^k \right].$$

and therefore

$$\sum_i p_i \sum_k x_i^k > \sum_i p_i \sum_k y_i^k$$

which means that, contrary to assumption, the equilibrium production plans cannot be maximising profits.

10.4 Efficiency of equilibria with production

Extending the earlier reasoning, consider choosing both consumption allocation and production plans to maximise the utility of one consumer, again say $h = 1$, subject to specified utilities, $\bar{u}^h$, $h = 2, 3, \ldots$, for all of the others but also subject to technical feasibility.

We need to represent technical feasibility so as to write it as a constraint to this problem so introduce functions $\Gamma^k(y^k)$ which take the value zero along each firm's production possibility frontier. In other word's $\Gamma^k(y^k) = 0$ for all and only those production plans which are feasible for firm k. (For a firm producing one output, say Q^k, with inputs, say L^k, according to a production function $F(L^k)$ then we could write $y^k = \left(Q^k, -L^k\right)$ and $\Gamma^k\left(Q^k, -l^k\right) =$

$Q^k - F\left(l^k\right)$ but the more general formulation is useful in allowing for multiple outputs, for instance.) Then the Pareto efficiency problem is:

$$\boxed{\begin{aligned} \max_{q^2,\, q^3,\, \ldots} \; u^1 & \left(\sum_k y^k - \sum_{h=2,3,\ldots,H} q^h \right) \\ \text{s.t.} \qquad u^h(q^h) &= \bar{u}^h, \; h = 2, 3, \ldots, H \\ \Gamma^k(y^k) &= 0, \; k = 1, 2, \ldots, K \end{aligned}}$$

First order conditions for internal optima now require

$$\boxed{\frac{\partial u^1}{\partial q_i^1} = \lambda^h \frac{\partial u^h}{\partial q_i^h} = \mu^k \frac{\partial \Gamma^k}{\partial y_i^k}}$$

where μ^k is the Lagrange multiplier on the technology constraint for the kth firm and these conditions hold for all households h, $h = 2, 3, \ldots, H$, and for all firms k, $k = 1, 2, \ldots, K$, which use or produce both goods in question.

These imply

$$\boxed{\frac{\partial u^h / \partial q_i^h}{\partial u^h / \partial q_j^h} = \frac{\partial \Gamma^k / \partial y_i^k}{\partial \Gamma^k / \partial y_j^k}}$$

Equal MRS = MRT for all individuals

for every household h, every firm k using or producing both goods and every pair of goods $\{i, j\}$.

The ratios $\left(\partial \Gamma^k / \partial y_i^k\right) / \left(\partial \Gamma^k / \partial y_j^k\right)$ are marginal rates of transformation. They describe the rates at which firms can substitute between inputs in production, substitute between alternative outputs or convert inputs into outputs. For

Pareto efficiency to hold, marginal rates of substitution need not only to be equated across consumers but also to be equated to common marginal rates of transformation among firms. Competitive equilibrium still achieves this because firms face the same price ratios as consumers. MRS = MRT = Price Ratio

10.5 Public goods

Public goods are goods that are usually characterised as *nonrival* and *nonexcludable*. Nonrivalry means that the benefits of consumption enjoyed by one person do not compromise those enjoyed by another. Nonexcludability means that noone can be prevented from enjoying the benefits once the good is provided. These characteristics are conceptually distinct and there are nonrivalrous goods that are excludable (broadcasting) and rivalrous goods from the benefits of which it is difficult to exclude people (common-pool resources like fisheries). Provision of public goods raises problems both at the national level but also at much smaller levels such as within individual households.

For simplicity, let us assume there is only a single private good, q^h, and extend the specification of utilities for individuals within the population $h = 1, 2, \ldots, H$ so as to include dependence not only on this good but also a collectively consumed public good Q, $u^h = u^h(q^h, Q)$. Let us also simplify production technology so that there is a constant marginal rate of transformation between the two goods at which P units of the private good can be converted into one unit of the public good. In other words $\Omega = \sum_h q^h + PQ$ for some fixed Ω.

Worked Example 10|A : Contract Curve

Consider the two-person two-good Cobb-Douglas exchange economy described earlier. Pareto efficiency requires that the two individuals have the same MRS

$$\begin{aligned}\left(\frac{\alpha^A}{1-\alpha^A}\right)\frac{q_2^A}{q_1^A} &= \left(\frac{\alpha^B}{1-\alpha^B}\right)\frac{q_2^B}{q_1^B}\\ &= \left(\frac{\alpha^B}{1-\alpha^B}\right)\frac{\Omega_2-q_2^A}{\Omega_1-q_1^A}\end{aligned}$$

Rearranging, we get an equation for the contract curve

$$\left(\frac{\alpha^B}{1-\alpha^B}\right)\frac{q_1^A}{\Omega_1-q_1^A} = \left(\frac{\alpha^A}{1-\alpha^A}\right)\frac{q_2^A}{\Omega_2-q_2^A}$$

or, equivalently,

$$\frac{q_2^A}{\Omega_2} = \frac{\theta q_1^A/\Omega_1}{1-(1-\theta)q_1^A/\Omega_1}$$

where $\theta = \alpha^B(1-\alpha^A)/\alpha^A(1-\alpha^B)$.

Note that if preferences are identical, $\alpha^A = \alpha^B$, then $q_1^A/\Omega_1 = q_2^A/\Omega_2$ so the contract curve is simply the diagonal of the Edgeworth-Bowley box.

Every one of the allocations on the contract curve is sustainable as a competitive equilibrium.

10.6 Efficient supply of public goods

Finding the condition for efficient supply of the public good and allocation of private goods requires solving

$$\boxed{\begin{array}{l} \max\limits_{q^2,\, q^3,\, \dots} u^1\left(\Omega - \sum\limits_{h=2,3,\dots,H} q^h - PQ, Q\right) \\ \text{s.t.} \quad u^h(q^h, Q) \;=\; \bar{u}^h,\; h = 2, 3, \dots, H \end{array}}$$

First order conditions for internal optima now require

$$\begin{aligned} \frac{\partial u^1}{\partial q^1} &= \lambda^h \frac{\partial u^h}{\partial q^h},\; h = 2, 3, \dots, H \\ P\frac{\partial u^1}{\partial q^1} &= \frac{\partial u^1}{\partial Q} + \sum_{h=2,3,\dots,H} \lambda^h \frac{\partial u^h}{\partial Q} \end{aligned}$$

which together imply

Samuelson condition
$$\boxed{P \;=\; \sum_h \frac{\partial u^h/\partial Q}{\partial u^h/\partial q^h}}$$

Optimal provision requires not that the marginal rate of substitution between the public and private good be equated for each consumer to P but rather that the sum be equated to P. This is sometimes known as the Samuelson condition. Provision of the public good benefits all consumers and these benefits need to be totalled up when considering efficiency of provision, not considered separately.

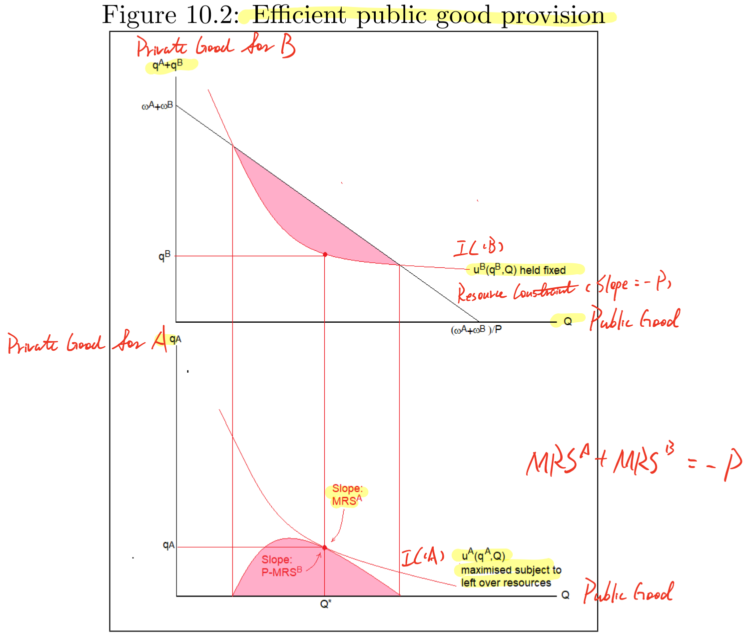
The condition is illustrated for a two person economy in Figure 10.2. In the top panel the utility of one of the individuals has been fixed. Imposing the implied indifference curve onto the economy's resource constraint we note that the resources left over for the other individual are given, for each possible value of the public good Q, by the vertical distance between the two and plot this in the lower panel where the highest possible utility for the other individual is found at a tangency between this curve and an indifference curve. At the tangency $P - MRS^B = MRS^A$ in accordance with the Samuelson condition. What is clear also from the diagrammatic exposition is that, since a different choice of fixed utility for the first individual could lead to a different optimal level of Q, the condition describes a set of efficient allocations not a single specific optimal provision of the public good.

10.7 Private and public provision of public goods

How then should public goods be provided? Suppose that individuals start with endowments of the private good from which they can contribute individually towards a common pool to be used for provision of the public good. Demands are now interdependent and we need to take that into account in deciding how to model decision making. Suppose that individuals take the contributions of others as given and consider the Nash equilibrium in private contributions G^h. Each individual solves

$$\boxed{max_{G^h} u^h\left(\omega^h - PG^h, G^h + \sum_{g \neq h} G^g\right)}$$

Figure 10.2: Efficient public good provision



Some will not contribute at all but those who do will choose such that

$$\boxed{P = \frac{\partial u^h/\partial Q}{\partial u^h/\partial q^h},}$$

Efficiency requires $\sum_i \frac{\partial u^i/\partial Q}{\partial u^i/\partial q^i} = P$

equating individual MRS to P. It is clear that the condition for efficiency will not be satisfied. The sum of MRS will typically be much greater than P. The public good will therefore be underprovided as individuals fail to take account of the benefits to others from private provision.

This is usually taken as a case for collective public provision, financed by taxes and determined by some mechanism of collective choice. The appropriate nature of taxes and the elicitation of individual valuations of public provision is a topic pursued in courses on public economics.

It is the dependence of individual wellbeing on levels provided by others that generates the inefficiency in private provision. Similar problems will arise from other forms of interdependence. One person's consumption of a good can affect other's enjoyment negatively (for example, noise pollution) and there can be positive or negative interdependence between activities of different producers or between consumers and producers. These are all examples of the more general phenomenon of externalities.

Worked Example 10|B : Public Goods

Consider an economy of N individuals consuming a public good Q and a private good q^h, $h = 1, 2, \ldots, N$. All have preferences described by Cobb-Douglas utility $u(q^h, Q) = \alpha \ln q^h + (1-\alpha) \ln Q$. Individuals begin with endowments of the private good ω^h but there is technology through which endowments can be transformed into the public good. The economy has a constant marginal rate of transformation P between the two goods such that its production possibilities are $\sum_h q^h + PQ = \Omega$ where $\Omega = \sum_h \omega^h$.

Pareto optimal supply requires the sum of individual marginal rates of substitution equal the marginal rate of transformation

$$\left(\frac{1-\alpha}{\alpha}\right) \frac{\sum_h q^h}{Q} = P$$

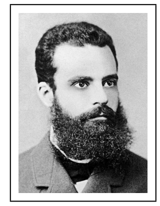
Substituting into the production possibility frontier gives $PQ = (1-\alpha)\Omega$.

If the public good is supplied privately then, assuming each individual contributes positively, each individual supplies only up to the point where

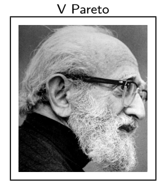
$$\left(\frac{1-\alpha}{\alpha}\right) \frac{q^h}{Q} = P.$$

Each individual consumes the same quantity of the private good, $q^h = \alpha PQ/(1-\alpha)$, and substitution of this into the production possibility frontier gives $PQ = (1-\alpha)\Omega/(1+(N-1)\alpha)$. The public good is inefficiently undersupplied.

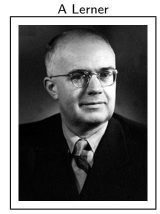
If any individual has endowment below the common private consumption so that $\omega^h < \alpha\Omega/(1+(N-1)\alpha)$ then they will not contribute. Equilibrium provision will be determined by decisions among those who are contributors.



V Pareto



A Lerner

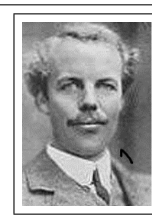


O Lange

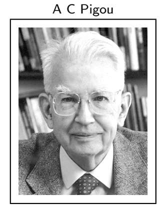
History Note 10|I : Welfare in General Equilibrium

The ethical defensibility of market outcomes has been a persistent matter of political contention that has attracted the interest of economists for centuries. The interpretation of the welfare theorems as vindicating the observations of Adam Smith (1723-90) on the workings of an "invisible hand" in market economies is common, if arguably strained.

The Pareto criterion for welfare comparison is owed to the Italian social thinker Vilfredo Pareto (1848-1923) and fits naturally with the withdrawal from willingness to make interpersonal comparisons of utility associated with his recognition that consumer theory relies only on ordinal description of preferences. Pareto himself understood that exchange equilibrium exhausted the scope for Pareto improvements. Proof of the first and second fundamental welfare theorems are associated principally with work in the 1930s and 1940s by Abba Lerner (1903-82) and Oskar Lange (1904-65), extended in the 1950s by Kenneth Arrow (b.1921, Nobel 1972). Lange and Lerner were both socialists by inclination concerned with the possibility for decentralising efficient decision making under public ownership, Lerner a Russian-born economist who worked in the UK then the US and Lange a Polish economist who worked also in the UK and US before returning to communist Poland after the Second World War.



A C Pigou



R Coase

E Ostrom

History Note 10|II : Public Goods, Externalities and Collective Action

Recognition that collective consumption creates a case for state intervention can be found in Adam Smith (1723-90). Conditions for optimum supply of a public good were proved in the 1950s by American economist Paul Samuelson (1915-2009, Nobel 1970). The economic concept of externalities, the difficulties posed by them for market allocation and the possibility of remedies through taxation were discussed by English economist Arthur Cecil Pigou (1877-1959) in the 1920s. The possibility that well-defined property rights over externality-producing activities would nonetheless permit the achievement of efficient solutions through bargaining was discussed by the British economist Ronald Coase (1910-2013, Nobel 1991) in the 1960s. Elinor Ostrom (1933-2012, Nobel 2009) has studied the practice of economic governance of common property resources.