1 General Method

Continuous-time Optimal Control with Discounting

- **Objective**:

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u\left(c(t), s(t)\right) dt \; s.t. \; \dot{s} = \phi\left(c(t), s(t)\right)$$

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- **Current-value Approach**:
  - **CV Hamiltonian**:

$$H_{cv} = u(c, s) + \lambda(t)\phi(c, s)$$

  - CV Maximum Principle:
    - Hamiltonian Maximisation:

$$\forall\, t, c_t \text{ maximises } H_{cv} \; s.t. \; \underbrace{G(c, s) \leq 0}_{\text{Per-period Constraint}}$$

    - Co-state Equation:

$$-\frac{\partial H_{cv}}{\partial s} = \dot{\lambda}_t - \rho\lambda_t$$

    - State Equation / Law of Motion:

$$\dot{s}(t) = \phi(x, s)$$

    - Transversality Condition:

$$\lim_{t \to \infty} e^{-\rho t}\lambda(t)s(t) = 0$$

  - "Current-value" indicates that we evaluate the problem with the value at each time of decision. Here the CV co-state variable λ measures the current-period shadow price (not discounted):

$$\lambda(t) = \frac{\partial u}{\partial c}$$

- **Present-value Approach**:
  - **PV Hamiltonian**:

$$H_{pv} = e^{-\rho t}u(c, s) + e^{-\rho t}v(t)\phi(c, s)$$

  - PV Maximum Principle:
  - Hamiltonian Maximisation:

$$\forall\, t, c_t \text{ maximises } H_{pv} \; s.t. \; \underbrace{G(c, s) \leq 0}_{\text{Per-period Constraint}}$$

  - Co-state Equation:

$$\dot{v}(t) = -\frac{\partial H_{pv}}{\partial s}$$

  - State Equation / Law of Motion:

$$\dot{s}(t) = \phi(x, s)$$

- Transversality Condition:

$$\lim_{t \to \infty} v(t)s(t) = 0$$

- "Present-value" indicates that we evaluate the problem with the value at $t = 0$. Here the PV co-state variable v measures the present shadow price (already discounted):

$$v(t) = e^{-\rho t} \frac{\partial u}{\partial c_t}$$

Log Linearisation

- Start from an equation:

$$f(x, y, z) = g(x, y, z)$$

with steady state values:

$$x^*, y^*, z^* \text{ s.t. } f(x^*, y^*, z^*) = g(x^*, y^*, z^*)$$

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- Method 1: Taylor Expansion

1. Take 1st Order Taylor Expansion around (x^*, y^*, z^*):

$$f(x^*, y^*, z^*) \ + f_x(x^*)(x - x^*) + f_y(y^*)(y - y^*) + f_z(z^*)(z - z^*) = g(x^*, y^*, z^*) \ + g_x(x^*)(x - x^*) + g_y(y^*)(y - y^*) + g_z(z^*)(z - z^*)$$

2. Apply the formula:

$$= \ln\left(\frac{y}{y^*}\right) = \frac{\ -\ ^*}{^*} \qquad -\ ^* = \ ^*$$

to get:

$$f_x(x^*)x^*x + f_y(y^*)y^*y + f_z(z^*)z^*z = g_x(x^*)x^*x + g_y(y^*)y^*y + g_z(z^*)z^*z$$

3. Collect terms (typically by divide through) if it's a good market equilibrium

- Method 2: Shortcut Formula:
  - Apply the shortcut formula directly:

$$x \qquad (+x)x^*xy \qquad (+x+y)x^*y^*xy \qquad (+x+y)x^*y^*$$

- Example:
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Example: Log-Linear Approximation of Goods Market Equilibrium

- Non-linear equilibrium condition

$$\underbrace{A_t K_{t-1}^{\alpha} N_t^{1-\alpha}}_{f(A_t, K_{t-1}, N_t)} = \underbrace{C_t + K_t - (1-\delta)K_{t-1} + G_t}_{g(C_t, K_t, K_{t-1}, G_t)}$$

- First-order Taylor expansion of both sides

$$\cancel{AK^{\alpha}N^{1-\alpha}} + K^{\alpha}N^{1-\alpha}(A_t - A) + \alpha AK^{\alpha-1}N^{1-\alpha}(K_{t-1} - K) + (1-\alpha)AK^{\alpha}N^{-\alpha}(N_t - N)$$
$$= \cancel{(C + \delta K + G)} + (C_t - C) + (K_t - K) - (1-\delta)(K_{t-1} - K) + (G_t - G)$$

Same in Equilibrium

- Simplify steady state and use $\widehat{X}_t \approx (X_t - X)/X$ $X_t - X = \hat{X}_t X$

$$AK^{\alpha}N^{1-\alpha}\widehat{A}_t + \alpha AK^{\alpha}N^{1-\alpha}\widehat{K}_{t-1} + (1-\alpha)AK^{\alpha}N^{1-\alpha}\widehat{N}_t = C\widehat{C}_t + K\widehat{K}_t - (1-\delta)K\widehat{K}_{t-1} + G\widehat{G}_t$$

- Divide through by steady state output $Y = AK^{\alpha}N^{1-\alpha}$

$$\widehat{A}_t + \alpha\widehat{K}_{t-1} + (1-\alpha)\widehat{N}_t = \frac{C}{Y}\widehat{C}_t + \frac{K}{Y}[\widehat{K}_t - (1-\delta)\widehat{K}_{t-1}] + \frac{G}{Y}\widehat{G}_t$$