

Ox Core Macro - General Method

1 General Method

Continuous-time Optimal Control with Discounting

- **Objective:**

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(c(t), s(t)) dt \text{ s.t. } \dot{s} = \phi(c(t), s(t))$$

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- **Current-value Approach:**

- **CV Hamiltonian:**

$$H_{cv} = u(c, s) + \lambda(t)\phi(c, s)$$

- CV Maximum Principle:

- Hamiltonian Maximisation:

$$\forall t, c_t \text{ maximises } H_{cv} \text{ s.t. } \underbrace{G(c, s)}_{\text{Per-period Constraint}} \leq 0$$

- Co-state Equation:

$$-\frac{\partial H_{cv}}{\partial s} = \dot{\lambda}_t - \rho\lambda_t$$

- State Equation / Law of Motion:

$$\dot{s}(t) = \phi(x, s)$$

- Transversality Condition:

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) s(t) = 0$$

- "Current-value" indicates that we evaluate the problem with the value at each time of decision. Here the CV co-state variable λ measures the current-period shadow price (not discounted):

$$\lambda(t) = \frac{\partial u}{\partial c}$$

- **Present-value Approach:**

- **PV Hamiltonian:**

$$H_{pv} = e^{-\rho t} u(c, s) + e^{-\rho t} v(t) \phi(c, s)$$

- PV Maximum Principle:

- Hamiltonian Maximisation:

$$\forall t, c_t \text{ maximises } H_{pv} \text{ s.t. } \underbrace{G(c, s)}_{\text{Per-period Constraint}} \leq 0$$

- Co-state Equation:

$$\dot{v}(t) = -\frac{\partial H_{pv}}{\partial s}$$

- State Equation / Law of Motion:

$$\dot{s}(t) = \phi(x, s)$$

- Transversality Condition:

$$\lim_{t \rightarrow \infty} v(t) s(t) = 0$$

- "Present-value" indicates that we evaluate the problem with the value at $t = 0$. Here the PV co-state variable v measures the present shadow price (already discounted):

$$v(t) = e^{-\rho t} \frac{\partial u}{\partial c_t}$$

Log Linearisation

- Start from an equation:

$$f(x, y, z) = g(x, y, z)$$

with steady state values:

$$x^*, y^*, z^* \text{ s.t. } f(x^*, y^*, z^*) = g(x^*, y^*, z^*)$$

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- Method 1: Taylor Expansion

1. Take 1st Order Taylor Expansion around (x^*, y^*, z^*) :

$$\begin{aligned} & \underbrace{f(x^*, y^*, z^*)}_{f(A_t, K_t, N_t)} + f_x(x^*)(x - x^*) + f_y(y^*)(y - y^*) + f_z(z^*)(z - z^*) \\ &= \underbrace{g(x^*, y^*, z^*)}_{g(C_t, K_t, N_t)} + g_x(x^*)(x - x^*) + g_y(y^*)(y - y^*) + g_z(z^*)(z - z^*) \end{aligned}$$

2. Apply the formula:

$$\hat{k} = \ln \left(\frac{y}{y^*} \right) = \frac{k - k^*}{k^*} \iff k - k^* = k^* \times \hat{k}$$

to get:

$$f_x(x^*)x^*\hat{x} + f_y(y^*)y^*\hat{y} + f_z(z^*)z^*\hat{z} = g_x(x^*)x^*\hat{x} + g_y(y^*)y^*\hat{y} + g_z(z^*)z^*\hat{z}$$

3. Collect terms (typically by divide through Y) if it's a good market equilibrium

- Method 2: Shortcut Formula:

- Apply the shortcut formula directly:

$$\begin{cases} x & \approx (1 + \hat{x})x^* \\ xy & \approx (1 + \hat{x} + \hat{y})x^*y^* \\ x^\alpha y^\beta & \approx (1 + \alpha\hat{x} + \beta\hat{y})x^{*\alpha}y^{*\beta} \end{cases}$$

- Example:

Example: Log-Linear Approximation of Goods Market Equilibrium

- Non-linear equilibrium condition

$$\underbrace{A_t K_{t-1}^\alpha N_t^{1-\alpha}}_{f(A_t, K_t, N_t)} = \underbrace{C_t + K_t - (1-\delta)K_{t-1} + G_t}_{g(C_t, K_t, K_{t-1}, G_t)}$$

- First-order Taylor expansion of both sides

$$\begin{aligned} & AK^\alpha N^{1-\alpha} + K^\alpha N^{1-\alpha}(A_t - A) + \alpha AK^{\alpha-1} N^{1-\alpha}(K_{t-1} - K) + (1-\alpha)AK^\alpha N^{-\alpha}(N_t - N) \\ &= (C + \delta K + G) + (C_t - C) + (K_t - K) - (1-\delta)(K_{t-1} - K) + (G_t - G) \end{aligned}$$

Some in Equilibrium

- Simplify steady state and use $\hat{X}_t \approx (X_t - X)/X$ $X_t - X = \hat{X}_t X$

$$AK^\alpha N^{1-\alpha} \hat{A}_t + \alpha AK^\alpha N^{1-\alpha} \hat{K}_{t-1} + (1-\alpha)AK^\alpha N^{1-\alpha} \hat{N}_t = C \hat{C}_t + K \hat{K}_t - (1-\delta)K \hat{K}_{t-1} + G \hat{G}_t$$

- Divide through by steady state output $Y = AK^\alpha N^{1-\alpha}$

$$\hat{A}_t + \alpha \hat{K}_{t-1} + (1-\alpha) \hat{N}_t = \frac{C}{Y} \hat{C}_t + \frac{K}{Y} [\hat{K}_t - (1-\delta) \hat{K}_{t-1}] + \frac{G}{Y} \hat{G}_t$$