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2024 W1 Preference and Utility

Lecture 1: Preference, Utility, and Choice

Preference Symbols #flashcard

- Strict preference relation \succ is defined by:

$$x \succ y \iff x \succeq y \text{ but not } y \succeq x$$

- Indifference relation \sim is defined by:

$$x \sim y \iff x \succeq y \text{ and } y \succeq x$$

Rational Preference Relation #flashcard

Definition: Rational Preference Relation

The preference relation \succsim is said to be *rational* if it is complete and transitive:

- *Completeness*: for all $x, y \in X$, we either have $x \succsim y$ or $y \succsim x$ (or both).
- *Transitivity*: for all $x, y, z \in X$, whenever we have $x \succsim y$ and $y \succsim z$ then we have $x \succsim z$.

If \succsim is rational, we have the following restrictions on \succ and \sim (see Hw1):

1. \succ is both *irreflexive* (i.e., $x \succ x$ cannot hold) and transitive.
2. \sim is both *reflexive* (i.e., $x \sim x$ always holds) and transitive.
3. If $x \succ y$ and $y \succsim z$ then $x \succ z$.

Utility Function and Its Ordinal Property #flashcard

Definition: Utility Function

A function $u : X \rightarrow \mathbb{R}$ is a utility function representing the preference relation \succsim if, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow u(x) \geq u(y)$$

Definition: Ordinal property

A property of a function is ordinal if it is preserved under any strictly increasing transformation of this function.

- **Claim:** If u represents \succeq and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function, then $v(x) = f(u(x))$ represents \succeq as well.

Proof: $x \succsim y \Leftrightarrow u(x) \geq u(y)$ (since u represents \succeq)
 $\Leftrightarrow f(u(x)) \geq f(u(y))$ (f strictly increasing)
 $\Leftrightarrow v(x) \geq v(y)$

- Thus, "preference representation function" would be a much more accurate and useful (albeit clunky) term than "utility function".

- Existence of a utility function \implies Rationality
- Rationality does not necessarily implies the existence of a utility function e.g. lexicographic preference
- From lect 2 slides:

Theorem: Debreu's Theorem

If a rational preference relation \succeq is continuous on \mathbb{R}_+^L , then it admits a utility representation $u : \mathbb{R}_+^L \rightarrow \mathbb{R}$. Even more, it admits a *continuous* utility representation.

Weak Axiom of Revealed Preference (WARP) #flashcard

Definition: WARP

A choice structure $(\mathcal{B}, C(\cdot))$ satisfies WARP if whenever $x \in C(A)$ for some $A \in \mathcal{B}$ and $y \in C(B)$ for some $B \in \mathcal{B}$ such that $x, y \in A \cap B$, we also have $x \in C(B)$.

- In words, if x is ever chosen while y was available, there cannot be any choice set where both are available and y is chosen but x is not.

Another way of stating WARP is by defining a binary relation \succeq^* from observed choice behavior $C(\cdot)$:

Definition: Revealed Preference Relation

$$x \succeq^* y \iff \text{there is some } B \in \mathcal{B} \text{ such that } x, y \in B \text{ and } x \in C(B).$$

- $x \succeq^* y$ can be read as "x revealed at least as good as y"
- Note that \succeq^* does not have to complete or transitive.
- We could also read $x \in C(B)$ and $y \notin C(B)$ for $x, y \in B$ as "x revealed preferred to y"
- Restatement of WARP: "If x is revealed at least as good as y, then y cannot be revealed preferred to x".

WARP and Rationality #flashcard

- Rationality \implies WARP
- WARP + the set of choice experiments B contains all subset of X up to 3 elements

Lecture 2: Consumer Choice

NOT YET ARCHIVED

Relation between Preference/Utility Function on Convexity

If \succeq is representable by some $u(\cdot)$, then convexity of preference $\succeq \iff$ quasi-concavity of $u(\cdot)$

2024 W2 Classical Demand Theory

Lecture 1 (Monday)

Goods Classification

Goods Classification #flashcard

- By income effect on Marshalian demand / income elasticity:

$$\begin{cases} \frac{\partial x_l}{\partial w} > 0 & \iff \epsilon_{lw} > 0 \iff \text{Normal Good} \\ \frac{\partial x_l}{\partial w} < 0 & \iff \epsilon_{lw} < 0 \iff \text{Inferior Good} \end{cases}$$

- By income effect on budget share / income elasticity:

$$\begin{cases} \frac{\partial b_l}{\partial w} > 0 & \iff \epsilon_{lw} > 1 \iff \text{Luxury Good} \in \text{Normal Good} \\ \frac{\partial b_l}{\partial w} < 0 & \iff \epsilon_{lw} < 1 \iff \text{Necessity} \supset \text{Inferior Goods} \end{cases}$$

- and the good's budget share increases with its price if its own price elasticity $> -1: \frac{\partial b_l}{\partial p_l} > 0 \iff \epsilon_{lw} > -1$

- By own price effect:

$$\frac{\partial x_l}{\partial p_l} > 0 \iff \text{Giffen Good} \in \text{Inferior Good}$$

- Giffen goods have to be strongly inferior good as implied by [[0x Y1 Core Micro MT W1-3 Demand#^c42e8f|Slutsky Equation]]

Goods Classification COPY #flashcard

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- and the good's budget share increases with its price if its own price elasticity $> -1: \frac{\partial b_l}{\partial p_l} > 0 \iff \epsilon_{lp_l} > -1$

- By own price effect:

$$\frac{\partial x_l}{\partial p_l} > 0 \iff \text{Giffen Good} \in \text{Inferior Good}$$

- Giffen goods have to be strongly inferior good as implied by [[0x Y1 Core Micro MT W1-Demand#^c42e8f|Slutsky Equation]]

Slutsky Substitution Matrix and Symmetry #flashcard

- For L goods, we can define an $L \times L$ Slutsky substitution matrix:

$$S(p, w) = \begin{bmatrix} s_{11}(p, w) & s_{12}(p, w) & \dots & s_{1L}(p, w) \\ s_{21}(p, w) & s_{22}(p, w) & \dots & s_{2L}(p, w) \\ \vdots & \vdots & \ddots & \vdots \\ s_{L1}(p, w) & s_{L2}(p, w) & \dots & s_{LL}(p, w) \end{bmatrix}$$

where $s_{lk}(p, w) = \frac{\partial h_l(p, \bar{u})}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w)$ can be observed from market demands

- This matrix is symmetric due to symmetry:

$$\underbrace{\frac{\partial h_l(p, w)}{\partial p_k}}_{s_{lk}(p, w)} = \underbrace{\frac{\partial h_k(p, w)}{\partial p_l}}_{s_{kl}(p, w)}$$

- Violation of this property can be used to reveal failures of rationality in the traditional sense

Lecture 2 (Wednesday)

Money-metric Measures of Welfare (CV/EV)

CV and EV #flashcard

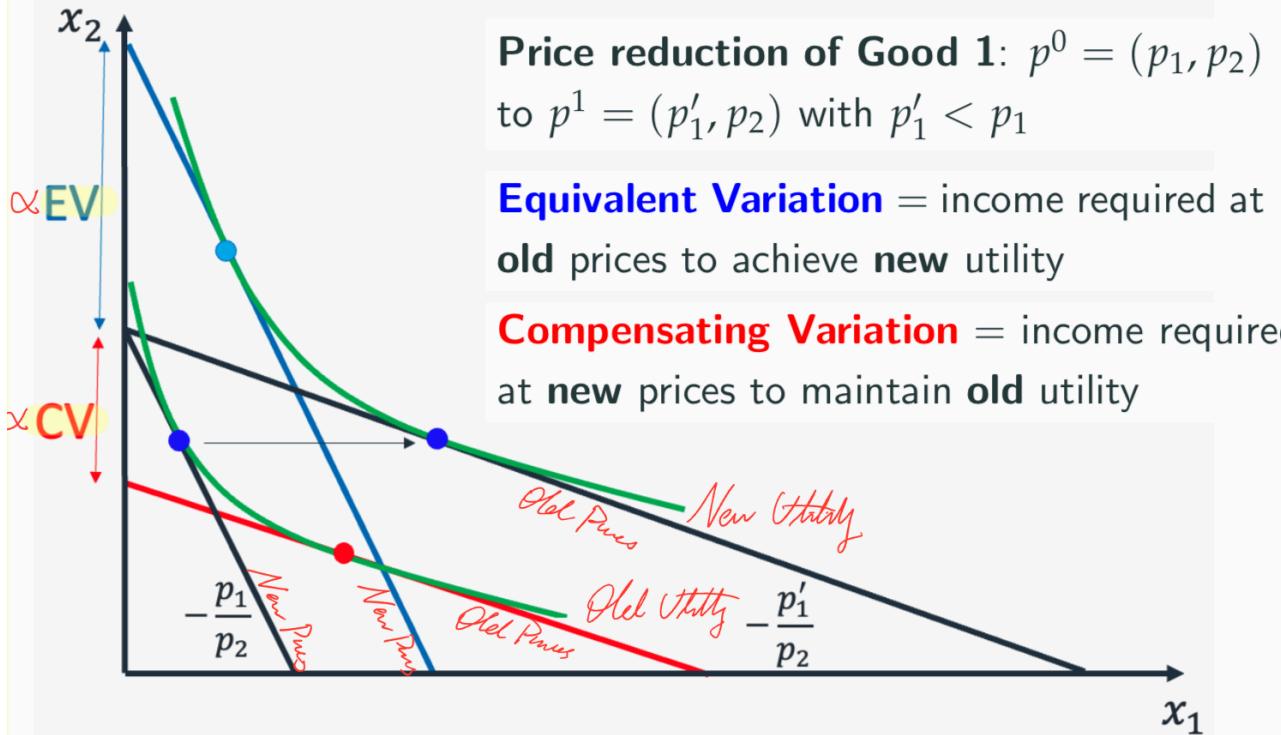
- Key: w is unchanged, but u can change $u^0 \rightarrow u^1$

Compensating Variation (CV): income required to *Maintain old utility at new prices*:

$$CV(p^0, p^1, w) = \underbrace{e(p^1, u^1) \text{ or } e(p^0, u^0)}_w - e(p^1, u^0) = w - e(p^1, u^0)$$

Equivalent Variation (EV): income required to *Achieve new utility at old prices*:

$$EV(p^0, p^1, w) = e(p^0, u^1) - \underbrace{e(p^1, u^1) \text{ or } e(p^0, u^0)}_w = e(p^0, u^1) - w$$



EV, CV aren't directly observable, but recoverable.

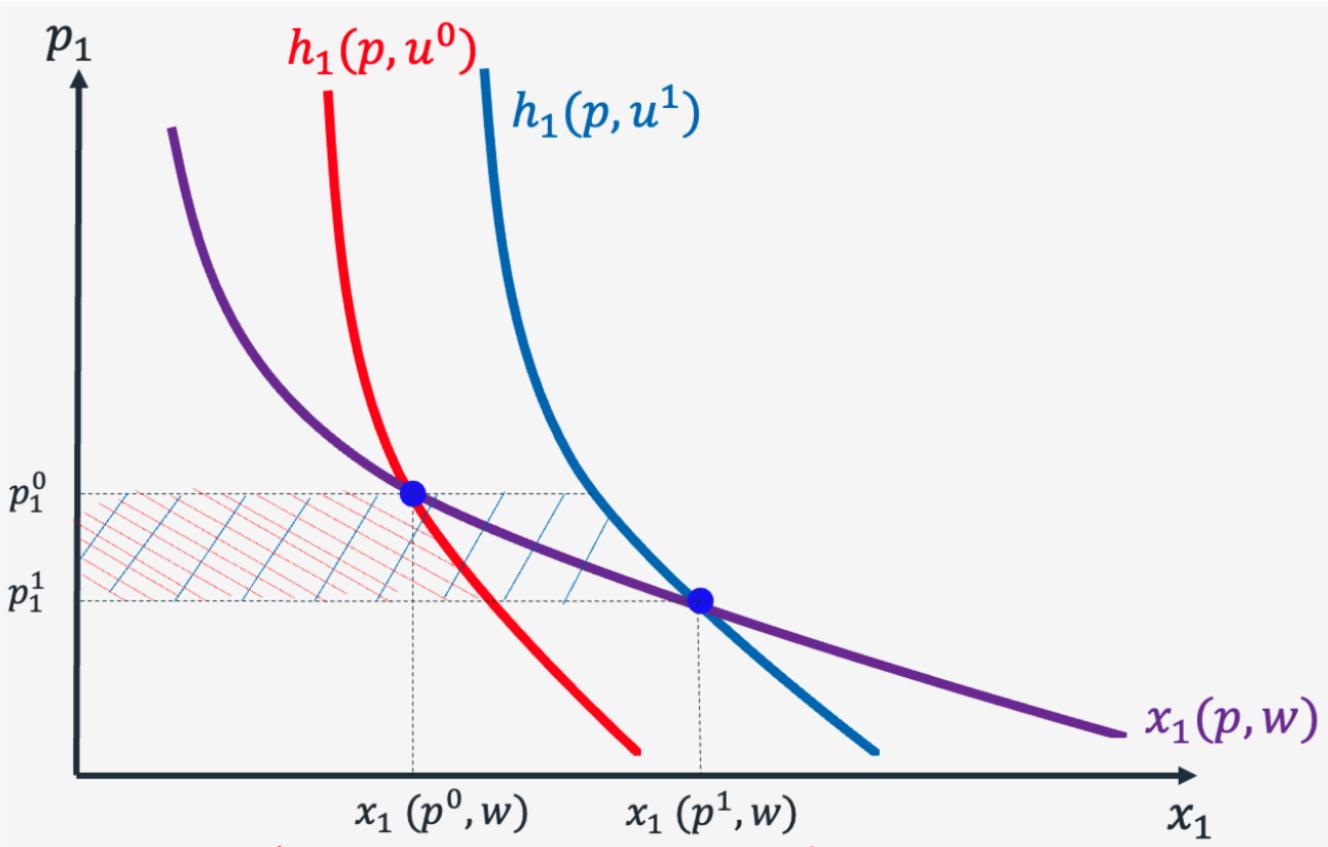
Interpretation of CV/EV as WTP/WTA #flashcard

THIS SEEMS TO BE WRONG (GPT is giving the opposite answer which I agreed)

depends on whether the policy change is positive or negative ($u^1 > u^0$ or $u^1 < u^0$):

	Compensating Variation	Equivalent Variation
+ Decrease in price of normal good ($u^1 > u^0$)	WTP (CV > 0) <i>Willingness to Pay</i>	WTA (EV > 0) <i>Willingness to Accept</i>
- Increase in price of normal good ($u^1 < u^0$)	WTA (CV < 0)	WTP (EV < 0)
	$v(p^1, w - CV) = v(p^0, w) = u^0$	$v(p^0, w + EV) = v(p^1, w) = u^1$
	Money transfer after the policy change that would leave the agent <i>just as well off as before</i> .	Money transfer that, without the policy change occurring , would leave the agent <i>just as well off as if it occurred</i> .

If there's only price change in one good ($p_1^0 \rightarrow p_1^1$), then CV/EV corresponds to area under corresponding Hicksian demand curves:



Due to the existence of income effects, for normal goods:

$$CV \leq \Delta CS \leq EV$$

Unit-free Measure of Welfare (Consumer Indices)

Konus/Laspeyres/Paasche Cost-of-living Indices #flashcard

Consider a price change: $p^0 \rightarrow p^1$

- **Konus CoL Index** compare costs to achieve a fixed level of utility:

$$P_K(p^0, p^1, u) = \frac{e(p^1, u)}{e(p^0, u)}$$

- **Laspeyres CoL Index** compares costs to maintain the old utility:

$$P_L(p^0, p^1, u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)} = \frac{e(p^1, u^0)}{w^0}$$

- **Paasche CoL Index** compares costs to maintain new utility:

$$P_P(p^0, p^1, u^1) = \frac{e(p^1, u^1)}{e(p^0, u^1)} = \frac{w^1}{e(p^0, u^1)}$$

- If preferences are homothetic, then Paasche and Laspeyres indices provide 2-side bounds on the unique Konus index:

$$\frac{p^1 x^1}{p^0 x^1} \leq \frac{e(p^1, u)}{e(p^2, u)} \leq \frac{p^1 x^0}{p^0 x^0}$$

Quantity Indices #flashcard

No detailed discussion this year, so adapted from the last year:

- ① Allen Quantity indices/real income measures are also index numbers but these hold the prices fixed and compare the indifference curves (different utility)

$$Q(p, u^0, u^1) = \frac{e(p, u^1)}{e(p, u^0)}$$

- If we choose either p^0 or p^1 as the reference price the idea is identical to an EV/CV but expressed in ratios instead of a cash amount

- For example the Laspyeres and Paasche quantity indices provide approximations/bounds:

True QIs *Approximate QIs*

$$Q_L(u^0, u^1, p^0) \equiv \frac{e(p^0, u^1)}{e(p^0, u^0)} = \frac{e(p^0, u^1)}{w^0} \leq \frac{p^0 \cdot x^1}{p^0 \cdot x^0} \equiv Q_L(x^0, x^1, p^0)$$

$$Q_P(u^0, u^1, p^1) \equiv \frac{e(p^1, u^1)}{e(p^1, u^0)} = \frac{w^1}{e(p^1, u^0)} \geq \frac{p^1 \cdot x^1}{p^1 \cdot x^0} \equiv Q_P(x^0, x^1, p^1)$$

- The quality of these approximations depends on the same substitutability ideas etc.

Aggregation

Aggregation of Rational Individuals #flashcard

Assume all individuals are rational in an economy.

Generally, the AD will depend on prices and the entire distribution of wealth.

Exceptions: when utility functions take form of:

$$v_i(p, w_i) = a_i(p) + b(p)w_i$$

i.e. utility functions take the Gorman-Polar form (including homothetic and quasi-linear ones)

2024 W3 Risks

Lecture 1: Decisions under Risk

Definition: Independence

A preference relation \succsim on the space of simple lotteries \mathcal{L} satisfies independence if for all lotteries $L, L', L'' \in \mathcal{L}$ and $0 < \alpha < 1$,

$$L \succsim L' \Leftrightarrow \alpha L + (1 - \alpha)L'' \succsim \alpha L' + (1 - \alpha)L''$$

- In words: the agent's preferences over two lotteries (L and L') completely determine which they'd prefer to have as part of a compound lottery, regardless of the other possible outcome of this compound lottery L''

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Theorem: Expected Utility Theorem

Suppose preferences \succsim over lotteries are rational, continuous and satisfy the Independence Axiom. Then there exists a utility representation $U : \mathcal{L} \rightarrow \mathbb{R}$ for \succsim which takes the expected utility form:

$$L \succsim L' \Leftrightarrow \sum_{n=1}^N p_n u_n \geq \sum_{n=1}^N p'_n u_n$$

in which $U(L^n) = u_n$ and L^n yields z_n with probability 1.

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in which $U(L^n) = u_n$ and L^n yields z_n with probability 1.

Creating Utility Functions Using EUT #flashcard

- Find the best and worst outcome
- For each choice, find the probability p such that the individual is indifferent between the option and the mixed lottery of $p \times \text{best} + (1 - p) \times \text{worst}$
- Define that probability as its utility
- Note: since utility has cardinal meanings here, the utility function is unique up to positive affine transformations.

Lecture 2: Risk Aversion

Certainty Equivalent #flashcard

CE is defined as:

$$u(CE) = \mathbb{E}[u(x)] = \int u(x)f(x)dx = \int u(x)dF(x)$$

i.e. the sure amount the consumer is willing to get in exchange of the lottery

Risk Premium #flashcard

PR is defined as:

$$u(\mathbb{E}[x] - RP) = \mathbb{E}[u(x)]$$

and

$$\mathbb{E}[x] = CE + RP$$

i.e. the amount which the agent is willing to give up in order to eliminate the risk

Absolute Risk Aversion and Relative Risk Aversion #flashcard

ARA:

$$ARA = -\frac{u''(x)}{u'(x)} > 0 \text{ if risk averse}$$

- Constant ARA \iff Exponential utility ($u(x) = \frac{1-e^{-rx}}{r}$ where r is ARA) \iff Constant RP \forall initial wealth

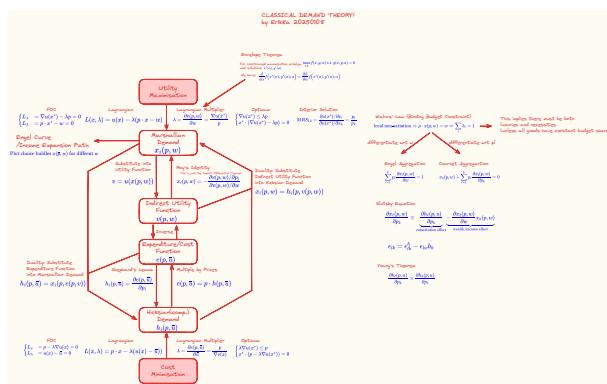
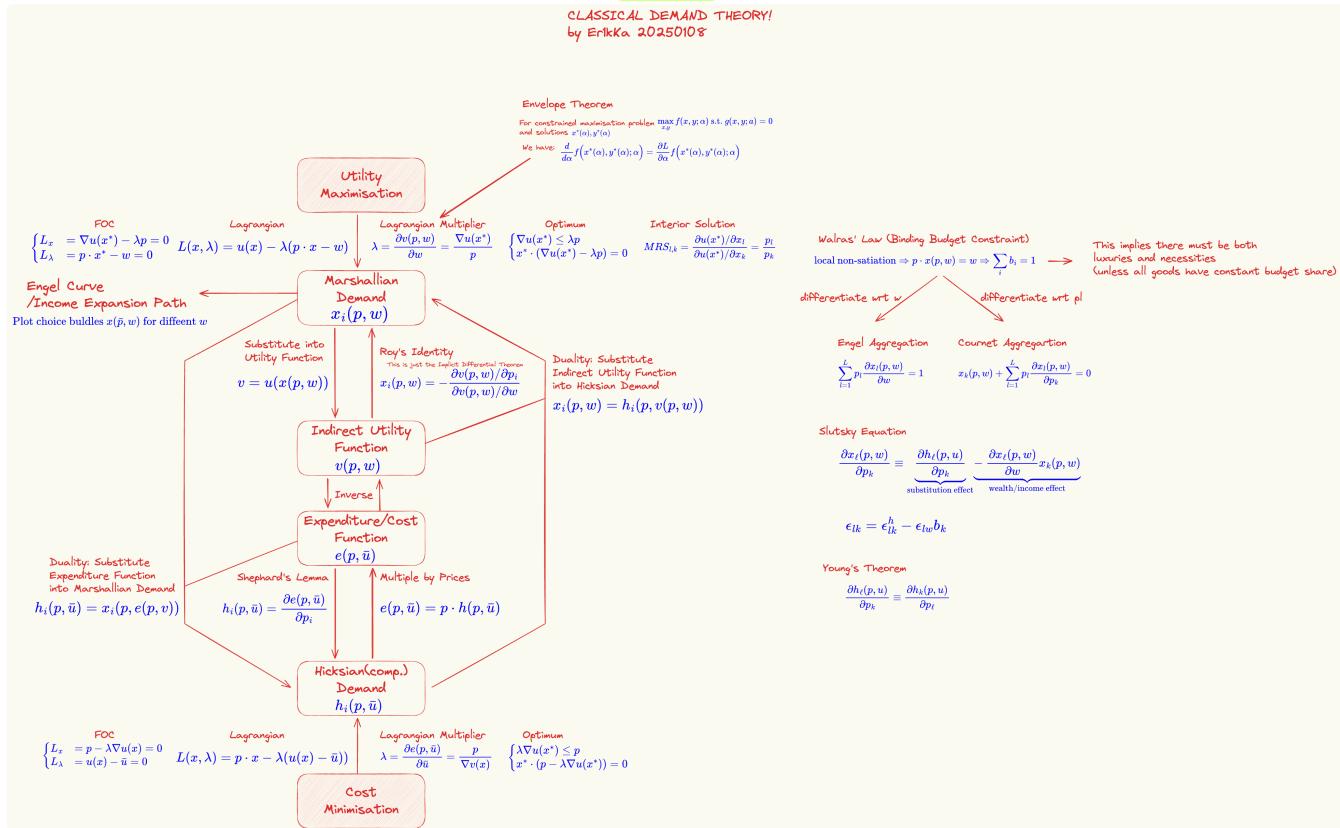
RRA:

$$-x \frac{u(x)}{u'(x)} > 0 \text{ if risk averse}$$

- Constant RRA $\iff u(x) = \frac{x^{1-r}-1}{1-r}$ and log utility $\ln(x)$ is a special case of $r \rightarrow 1$

2023 MT W5-6 Demand

Demand System: All Equations (Update 2025-04-05) #flashcard



Common Utility Functions and Their Properties #flashcard

FILL THIS LATER!!!!!! #notes/lbd

W5L1 Preference and Utility

Preference Relation

Rationality #flashcard

The preference relation \lesssim is said to be *rational* if it is complete and transitive:

- Rational*
- **Completeness**: for all $x, y \in X$ we either have $x \lesssim y$ or $y \lesssim x$ (or both).
 - **Transitivity**: for all $x, y, z \in X$, whenever we have $x \lesssim y$ and $y \lesssim z$ then we have $x \lesssim z$.

Utility

Utility Function #flashcard

- choice set*
- A function $u : X \rightarrow \mathbb{R}$ is a utility function representing the preference relation \lesssim if, for all $x, y \in X$,
$$x \lesssim y \Leftrightarrow u(x) \geq u(y)$$
- \Updownarrow
- $u(x)$ assigns scalar numerical value to each element $x \in X$ such that $x \lesssim y \Leftrightarrow u(x) \geq u(y)$
 - u is not unique, eg, $\hat{u}(x) \equiv u(x) + 1$ works just as well, i.e., the utility function is “ordinal” not “cardinal”

Existence of Utility Function \implies Rationality (Completeness + Transitivity)

but the Reverse is NOT True: Lexographic preference is rational but there's no corresponding $U(\cdot)$

Consumer Choice

Walrasian Budget Set and Budget Hyperplane

Budget Set:

$$B_{p,w} = \{s \in \mathbb{R}_+^n \text{ s.t. } p \cdot s \leq w\}$$

this is a convex set

Budget Hyperplane:

$$\{x \in \mathbb{R}_+^n \text{ s.t. } p \cdot x = w\}$$

Demand Functions

Basic Properties of Walrasian/Marshallian Demand Function #flashcard

- Homogeneous of degree 0
- Satisfies Walras's Law (non-satiation):

$$p \cdot x(p, w) = w$$

Engel Curve / Income Expansion Path

- For fixed prices p , the bundle demanded as a function of wealth w , $x(p, w)$, is the consumer's *Engel curve* (or *wealth/income expansion path*)

Price/Income Elasticities #flashcard

$$\varepsilon_{\ell k} = \frac{\partial x_\ell(p, w)}{\partial p_k} \frac{p_k}{x_\ell(p, w)} ; \quad \varepsilon_{\ell w} = \frac{\partial x_\ell(p, w)}{\partial w} \frac{w}{x_\ell(p, w)}$$

↓ ↓
price elasticity of good ℓ wrt. p_k *income/wealth elasticity of good ℓ*

Cournot Aggregation and Engel Aggregation

- Walras Law, differentiated wrt. $p_k \implies$ Cournot Aggregation:

$$x_k(p, w) + \sum_{l=1}^L p_l \frac{\partial x_l(p, w)}{\partial p_k} = 0$$

- Walras Law, differentiated wrt. $w \implies$ Engel Aggregation:

$$\sum_{l=1}^L p_l \frac{\partial x_l(p, w)}{\partial w} = 1$$

W5L2 Utility Maximisation

Preference Relations

Indifference Curve, Upper Contour Set, Lower Contour Set

- the *indifference curve* for x is $\{y \in X \text{ such that } y \sim x\}$
- the *upper contour set* for x is $\{y \in X \text{ such that } y \succsim x\}$
- the *lower contour set* for x is $\{y \in X \text{ such that } x \succsim y\}$

(Strict) Convexity of Preference

- Preferences \succsim are *convex* if the upper contour sets are convex,
i.e., if $y, z \succsim x$ then $\alpha y + (1 - \alpha)z \succsim x$ for any $0 \leq \alpha \leq 1$

Any combinations of 2 bundles in the upper contour set will be in the upper contour set.

- Preferences are *strictly convex* if $y, z \succsim x$ and $y \neq z$ implies
that $\alpha y + (1 - \alpha)z \succ x$ for $0 < \alpha < 1$

(Strict) Monotonicity of Preferences

- \succsim is *strongly monotonic* if y and x are 2 bundles such that
 $y \neq x$ and $y_\ell \geq x_\ell$ for each ℓ then $y \succ x$. Means all goods are
“good” (not bad) and indifference curves cannot be “thick”.
- \succsim is *weakly monotonic* if y and x are two bundles such that if
 $y_\ell \geq x_\ell$ for each ℓ then $y \succsim x$. This allows indifference curves
to be “thick”. *there could be a mettinent set*

Monotonicity corresponding to increasingness of the utility function, downward sloping indifference curves, and negative MRS.

Local Non-Satiation

- \succsim is *strongly monotonic* if y and x are 2 bundles such that
 $y \neq x$ and $y_\ell \geq x_\ell$ for each ℓ then $y \succ x$. Means all goods are
“good” (not bad) and indifference curves cannot be “thick”.
- \succsim is *weakly monotonic* if y and x are two bundles such that if
 $y_\ell \geq x_\ell$ for each ℓ then $y \succsim x$. This allows indifference curves
to be “thick”. *there could be a mettinent set*

Continuity of Preference

- A final—somewhat technical—assumption usually made is that preferences are *continuous*, by which we mean that the upper and lower contour sets are *closed sets*.
- This means that if y_n is a sequence of bundles which converges to bundle y , and each $y_n \succsim x$, then the limit point y also satisfies $y \succsim x$. *we can not escape from the upper / lower contour in the limit*
- Another way to say this is that if $y \succ x$, then all bundles sufficiently close to y are also strictly preferred to x .
- The usual example of preferences which are not continuous (but strongly monotonic) is the *lexicographic ordering*: there are two goods, and $x \succsim y$ if either $x_1 > y_1$ or $\{x_1 = y_1 \text{ and } x_2 \geq y_2\}$.
- In practice it's hard to work with " \succsim "; it's much easier to use a utility function which represents these preferences.

Utility Functions

Existence of Utility Function #flashcard

If a preference is:

- Rational (complete + transitive)
- Strongly monotonic
- Continuous

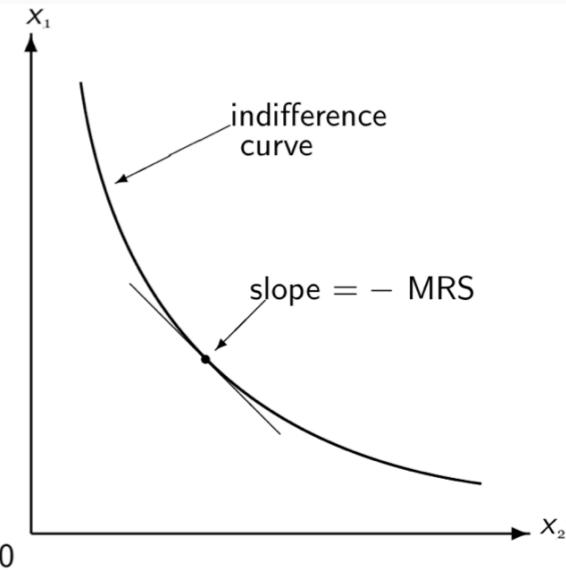
Then there exists a continuous function u s.t. $u(y) \geq u(x) \iff y \succsim x$

- A utility function is a representation of preferences which assigns real numbers to bundles: if $y \succsim x$ then y is assigned a weakly higher number than x .
- Utility functions are not unique since if u represents certain preferences then any positive monotonic transformation $\varphi(u)$ also represents the same preferences. We say that utility functions are “ordinal”.

Marginal Rate of Substitution (MRS) #flashcard

MRS between good 1 and good 2 is defined as the quantity of good 2 willing to give up in exchange of one unit of good 1,

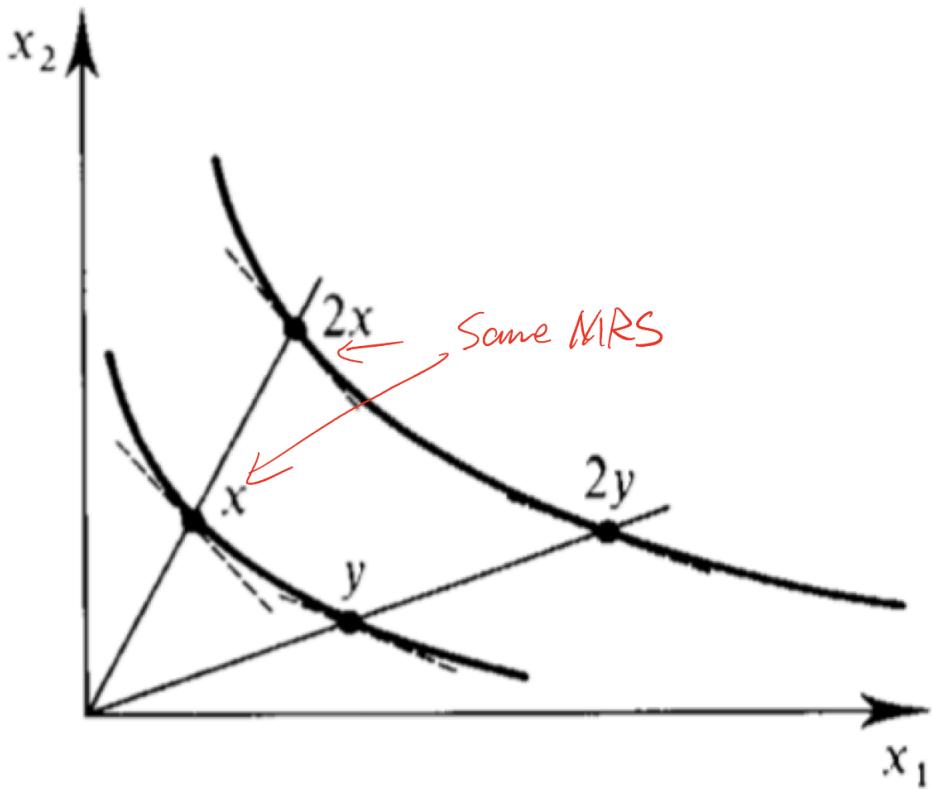
keeping the utility level unchanged.



$$MRS_{1,2} = \frac{dx_2}{dx_1}_u = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$

Homothetic Preferences #flashcard

- **Homothetic preferences:** if $x \sim y$ then $kx \sim ky$ for any positive scaling factor k
- Graphically: higher indifference curves are magnified versions of lower ones from the origin.
MRS will be the same on a straight line from the origin.
- Not a restriction on shape of any one indifference curve but on relationship between indifference curves within an indifference map
- In words: if you've seen one indifference curve, you've seen them all.



MWG Figure 3.B.5: Example of Homothetic Preferences

- Utility has “constant returns to scale”
- MRS is unaffected by scaling factor k . In other words, if preferences are homothetic then MRS's are constant along rays through the origin.
- This is only true for homothetic preferences and is usually an easy way to check whether given preferences are homothetic
- Implies constant budget shares and income elasticities equal to one

- If the preferences are homothetic then the expenditure function and the indirect utility functions take the linear forms

Homothetic Preference $\Rightarrow \begin{cases} e(p, u) = b(p)u \\ v(p, w) = \frac{1}{b(p)}w \end{cases}$

where $b(p)$ is a h.o.d.1 function of prices

- By Roy's Identity (discussed below) this then implies that the Marshallian demands take the separable form

$$x_\ell(p, w) = \beta_\ell(p)w$$

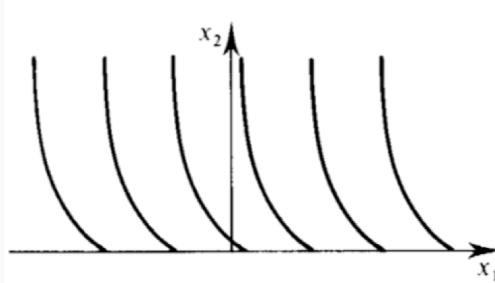
where $\beta_\ell(p) = \frac{1}{b(p)} \partial b(p) / \partial p_\ell$

- Engel curves/income expansion paths are thus linear rays from the origin

Quasi-Linear Preference

- Quasi-linear preferences:* one product (e.g., good 1) can be consumed in positive or negative quantities, so $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$, and if $x \sim y$ then for any k we have $(x_1 + k, x_2, \dots, x_L) \sim (y_1 + k, y_2, \dots, y_L)$.
- In words: adding the same amount to one particular good preserves indifference. (*between other goods*)
- Utility functions which represent these preferences take the form $x_1 + u(x_2, \dots, x_L)$.
- The “linear” good is often interpreted as “money”. It is sometimes called the “outside good”.
- It is often convenient to choose it as the numéraire.

- In two-goods case, each indifference curve is just a “sideways” translation of any other, and MRS is only a function of x_2
- All income effects are concentrated on x_1 (the other goods have no income effects)
- Provides a foundation for partial equilibrium analysis



MWG Figure 3.B.6: Quasilinear Preferences

W6L1:

Special case of Gorman Polar form

With quasi-linear preferences, consider for simplicity the case of two goods, and utility is $y + u(x)$ where $u(\cdot)$ is a strictly concave function and y is the numéraire (“money”). The consumer maximizes $y + u(x)$ subject to $y + px \leq w$, so x chosen to maximize $u(x) - px$.

- The Marshallian demands are

Quasi-linear Preferences \Rightarrow

$$\begin{cases} x(p, w) = x(p) & \text{no restriction on non-numeraire goods} \\ y(p, w) = w - px(p) \end{cases}$$

- Indirect utility and expenditure functions are special cases of the Gorman Polar form:

$$\begin{aligned} v(p, w) &= w + v(p) \\ e(p, u) &= u - v(p) \end{aligned}$$

1. Perfect substitutes $u(x_1, x_2) = ax_1 + bx_2$: The MRS is $-a/b$ and is constant. Indifference curves are parallel straight lines. These are the only preferences which are homothetic and quasilinear.
2. Perfect complements $u(x_1, x_2) = \min[ax_1, bx_2]$: Indifference curves are L-shaped with the kinks lying on a ray through the origin of slope a/b . These preferences are homothetic but not quasilinear.
3. Cobb-Douglas: $u(x_1, x_2) = alnx_1 + blnx_2$: Preferences are homothetic, indifference curves are smooth and MRS ax_2/bx_1 is diminishing.

Separability of Preferences

- Partition the bundle of goods into two sub-bundles so that the consumption bundle is (x, z)
- Partition the price vector analogously into (p_x, p_z)
- Preferences are said to be separable in the x -goods if preferences have the property that

$$(x, z) \succeq (x', z) \text{ if and only if } (x, z'') \succeq (x', z'')$$

for all x, x', z and z'' .

- In words: if x is preferred to x' given some quantities of the other goods, then x is preferred to x' for any quantities of the other goods; preferences for the x -goods are independent of the z -goods.

- If this holds then the utility function can be written as

$$u(v(x), z)$$

where $v(x)$ is a “sub-utility function” and $u(v, z)$ is an increasing function of v

- Overall utility can be written as a function of the sub-utility of x and the level of consumption of the z -goods
- What this means is that MRS's within the separable x -goods are independent of the z -goods
- The structure $u(v(x), z)$ is said to be “weakly separable” in the x -goods
- The general structure $u(v(x), w(z))$ is said to be “weakly separable” *separable in all types of goods*
- The structure $u(x, z) = v(x) + w(z)$ is said to be “additively separable”

Utility Maximisation

(see [Ox Micro Demand System Illustration.excalidraw](#))

CES Preferences

- CES Preferences:

$$U(x) = \left[\sum_{\ell}^L \beta_{\ell} x_{\ell}^{\theta} \right]^{1/\theta}$$

- The only preferences that are both additively separable and homothetic
- First-order condition \Rightarrow Frisch demands: $x_{\ell} = \left(\frac{\lambda p_{\ell}}{\theta \beta_{\ell}} \right)^{\frac{1}{\theta-1}}$
(no need to derive)

W6L1 Classical Demand Theory Part 1

Expenditure Minimisation

(see [Ox Micro Demand System Illustration.excalidraw](#))

Compensated Law of Demand

$$(p' - p) \cdot (h(p', u) - h(p, u)) \leq 0$$

Price and Hicksian demand must move in the opposite directions.

Gorman Polar Form

- The expenditure and indirect utility functions are affine (have an intercept)

Gorman-Polen

form

\Rightarrow

$$\left\{ \begin{array}{l} e(p, u) = a(p) + b(p)u \\ v(p, w) = \frac{w - a(p)}{b(p)} \end{array} \right.$$

Inversion

where both $a(p)$ and $b(p)$ are h.o.d.1

- By Roy's Identity this then implies that the Marshallian demands take the separable form

$$x(p, w) = \alpha_l(p) + \beta_l(p)w$$

where $\alpha_\ell(p) = \partial a(p)/\partial p_\ell - \beta_\ell(p)a(p)$ and
 $\beta_\ell(p) = \frac{1}{b(p)}\partial b(p)/\partial p_\ell$

- Engel curves/income expansion paths are linear but don't necessarily pass through the origin

Slutsky Equation #flashcard

$$\frac{\partial x(p, w)}{\partial p_k} \equiv \underbrace{\frac{\partial h(p, u)}{\partial p_k}}_{\text{substitution effect } < 0} - \underbrace{\frac{\partial x(p, w)}{\partial w} x_k(p, w)}_{\text{wealth/income effect}}$$

- The *substitution effect* captures how demand for good ℓ changes as the price for good k rises, staying on the same indifference curve u (must be negative)
- The *wealth/income effect* takes account of how the consumer's wealth changes when p_k rises; when p_k rises by dp_k , she spends an extra amount $x_k(p, w) \times dp_k$ on that good and her wealth falls correspondingly and this affects x_ℓ via $\partial x_\ell / \partial w$

$$\frac{\partial x_\ell(p, w)}{\partial p_k} \equiv \underbrace{\frac{\partial h_\ell(p, u)}{\partial p_k}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)}_{\text{wealth/income effect}} \quad (< 0 \text{ if normal})$$

- The Slutsky equation implies that $\partial x_\ell / \partial p_k$ is below $\partial h_\ell / \partial p_k$ if good ℓ is a *normal* good and above $\partial h_\ell / \partial p_k$ if good ℓ is an *inferior* good

$$\ell \text{ is } \begin{cases} \text{Normal: } \frac{\partial x_\ell}{\partial p_k} < \frac{\partial h_\ell}{\partial p_k} \\ \text{Inferior: } \frac{\partial x_\ell}{\partial p_k} > \frac{\partial h_\ell}{\partial p_k} \end{cases}$$
- Since the own-price effect is always negative for Hicksian demands, we deduce that a Giffen good must be a (strongly) inferior good *LARGE wealth effect*

Hicksian Substitutes and Complements

- Since Hicksian demand $h(p, u)$ is the derivative of the concave function $e(p, u)$ we deduce two things:
 - Since the matrix of second derivatives of $e(p, u)$ is symmetric, we have:

$$\frac{\partial h_\ell(p, u)}{\partial p_k} \equiv \frac{\partial h_k(p, u)}{\partial p_\ell}$$

This is essentially **Young's Theorem** (the order of partial differentiation doesn't matter). In words: effect of price change for product k on compensated demand for product ℓ is the same as price change for ℓ on product k .
 - The matrix of (price) derivatives of $h(p, u)$ is negative semi-definite; in particular, the own-price effect $\partial h_\ell / \partial p_\ell$ is negative.

- Two distinct products ℓ and k are:
 - *Hicksian Substitutes* if $\partial h_\ell(p, u) / \partial p_k \geq 0$
 - *Hicksian Complements* if $\partial h_\ell(p, u) / \partial p_k \leq 0$
- Compensated cross-price effects are symmetric:

$$\partial h_\ell(p, u) / \partial p_k = \partial h_k(p, u) / \partial p_\ell$$

Welfare

Basic Ideas of Comparing Welfare

- {
1. the distance between tangent lines (based on the expenditure function and expressed in cash amounts)
 2. the radial distance (based on the distance function and unit-free)

- Money difference: EV/CV
- Unit-free ratios: Cost-of-living indices, Quantity Indices

Equivalent Variation and Compensating Variation

1. Equivalent variation:

Δ minimal spending needed
to reach the new and old
utility levels given old prices

$$EV = e(p^0, v(p^1, w^1)) - e(p^0, v(p^0, w^0))$$

$= u_0$

new max utility old max utility
↑ ↑
original prices

2. Compensating variation:

Δ minimal spending needed
to reach the new and old
utility levels given new prices

$$CV = e(p^1, v(p^1, w^1)) - e(p^1, v(p^0, w^0))$$

now max utility old max utility
↑ ↑
= w, now prices

Given that the consumer is optimising, these are equivalent to:

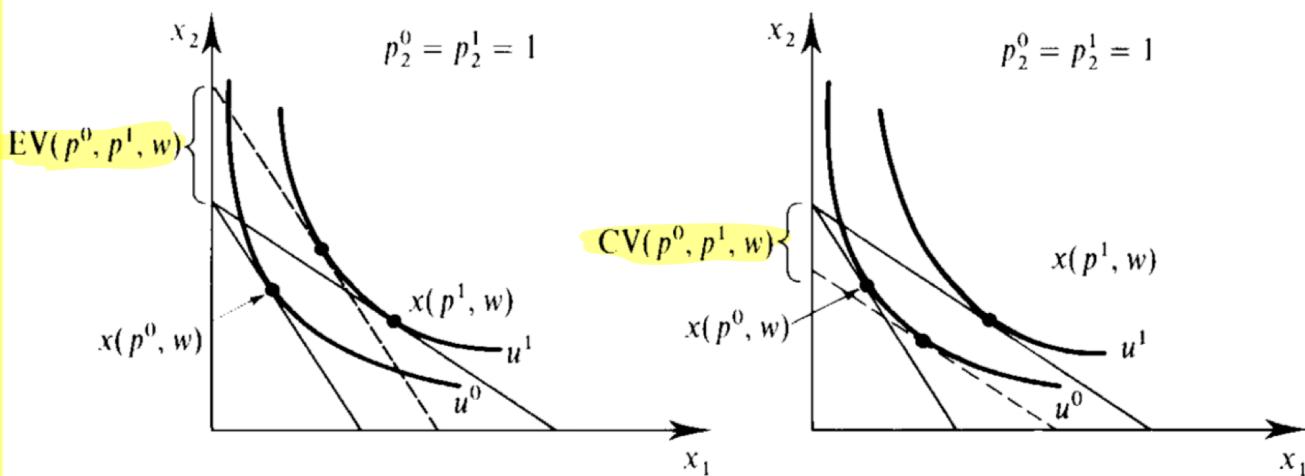
1. Equivalent variation:

$$EV = e(p^0, v(p^1, w^1)) - w^0$$

2. Compensating variation:

$$CV = w^1 - e(p^1, v(p^0, w^0))$$

If there's only price change but no wealth change, here's a graphical illustration:



Special Case when Only Change Taking Place is to a Single Good

NOT YET FIGURED OUT

P41-43 in SLIDES

W6L2 Classical Demand Theory Part 2

Welfare Economics (cont.)

Konus, Laspeyres, and Paasche Cost-of-living Indices

- Kionus (True) Cost-of-living Index

- Konüs (Cost-of-living indices) are ratios of expenditure functions:

$$P(p, p', u) = \frac{e(p', u)}{e(p, u)}$$

(*u to be defined*)

- Compare prices on a fixed indifference curve
(index cost)

- Laspeyres and Paasche Cost-of-living Indices

- These give us the Laspeyres and the Paasche cost-of-living indices:

Laspeyres : original utility

$$P_L(p^0, p^1, u^0) = \frac{e(p^1, u^0)}{e(p^0, u^0)} = \frac{e(p^1, u^0)}{w^0}$$

Paasche : new utility

$$P_P(p^0, p^1, u^1) = \frac{e(p^1, u^1)}{e(p^0, u^1)} = \frac{w^1}{e(p^0, u^1)}$$

Approximated Laspeyres and Paasche Cost-of-living Indices

- Approximated Laspeyres CoL Index overestimates the true Laspeyres CoL Index

- Often we may only have immediate access to prices and demands before and after some change:

$$\{p^0, x^0\} \rightarrow \{p^1, x^1\}$$

- These can be used to form approximate welfare measures
- For example the Laspeyres price index approximates the corresponding cost-of-living index from above: *(overstate)*

$$P_L(p^0, p^1, x^0) = \frac{p^1 \cdot x^0}{p^0 \cdot x^0} \geq \frac{e(p^1, u^0)}{e(p^0, u^0)} = \underbrace{P_L(p^0, p^1, u^0)}_{\text{True Laspeyres PI}}$$

(ignores neoptimisation)

Approximated Laspeyres PI

- Approximated Paasche CoL Index underestimates the true Paasche CoL Index

- There is a similar result for the Paasche price index

$$\frac{p^1 \cdot x^1}{p^0 \cdot x^1} \stackrel{\text{same}}{\leq} \frac{e(p^1, u^1)}{e(p^0, u^1)}$$

- The Paasche index understates the corresponding true change in the cost-of-living

CoL Indices when Preference is Homothetic

- In general a cost-of-living index depends on the reference indifference curve (*u*)
- An important exception occurs when preferences are homothetic, in which case the expenditure function is linear

$$e(p, u) = b(p)u$$

- Hence the cost-of-living index is independent of the reference *u*:

$$\frac{e(p^1, u)}{e(p^0, u)} = \frac{b(p^1)}{b(p^0)}$$

- Reason: under homotheticity indifference curves are radial expansions of each other ("if you've seen one ...")

- ⇒ The Paasche/Laspeyres are the same & u*
- If preferences are homothetic, the Paasche and Laspeyres indices give two-sided bounds on the unique index

$$\underbrace{\frac{p^1 \cdot x^1}{p^0 \cdot x^1}}_{\text{Approx. Paasche Index}} \leq \underbrace{\frac{e(p^1, u)}{e(p^0, u)}}_{\text{True K/P/L CoL Index}} \leq \underbrace{\frac{p^1 \cdot x^0}{p^0 \cdot x^0}}_{\text{Approx. Laspeyres Index}}$$

Quality of Approximation of QoL Indices

- If prices don't vary much between periods 0 and 1, or if prices move proportionally, or if there is little substitution, then the substitution effects will tend toward zero and the approximation will be better
- If there are no substitution effects then the Paasche/Laspeyres index numbers will be "exact"

Exact and Superlative Indices

NOT YET FIGURED OUT

SEE SLIDES P11-15

Allen, Laspeyres, and Paasche Quantity Indices (with Approximation)

- ① Allen Quantity indices/real income measures are also index numbers but these hold the prices fixed and compare the indifference curves *(different utility)*

$$Q(p, u^0, u^1) = \frac{e(p, u^1)}{e(p, u^0)}$$

- If we choose either p^0 or p^1 as the reference price the idea is identical to an EV/CV but expressed in ratios instead of a cash amount

- ② For example the Laspyeres and Paasche quantity indices provide approximations/bounds:

$$\begin{aligned} Q_L(u^0, u^1, p^0) &\equiv \frac{e(p^0, u^1)}{e(p^0, u^0)} = \frac{e(p^0, u^1)}{w^0} \leq \frac{p^0 \cdot x^1}{p^0 \cdot x^0} \equiv Q_L(x^0, x^1, p^0) \\ Q_P(u^0, u^1, p^1) &\equiv \frac{e(p^1, u^1)}{e(p^1, u^0)} = \frac{w^1}{e(p^1, u^0)} \geq \frac{p^1 \cdot x^1}{p^1 \cdot x^0} \equiv Q_P(x^0, x^1, p^1) \end{aligned}$$

- The quality of these approximations depends on the same substitutability ideas etc.

Laspyeres Index uses original standards, Paasche Index uses new standards.

Price/Quantity Index Pairs and Implicit Quantity Index

- A general definition of a price and quantity index is that they are a pair of functions which satisfy the functional equation

$$P(p^0, p^1, x^0, x^1) Q(p^0, p^1, x^0, x^1) = \frac{w^1}{w^0}$$

- Therefore if you know, for example, the price index you can work out the implicit quantity index:

$$Q(p^0, p^1, x^0, x^1) = \frac{w^1}{w^0} \frac{1}{P(p^0, p^1, x^0, x^1)}$$

P, Q indices appear in pairs

- If we apply this to the Laspeyres cost-of-living and Paasche Allen indices then we have:

$$\frac{e(p^1, u^0)}{e(p^0, u^0)} \frac{e(p^1, u^1)}{e(p^1, u^0)} = \frac{e(p^1, u^1)}{e(p^0, u^0)} = \frac{w^1}{w^0}$$

- It follows that the implicit quantity index defined using a Laspeyres index is the Paasche index:

Laspeyres PI $\left\{ \frac{e(p^1, u^1)}{e(p^1, u^0)} = \frac{w^1}{w^0} \frac{1}{\frac{e(p^1, u^0)}{e(p^0, u^0)}} \right\}$ *Paasche QI*

- And vice versa.

- *Equivalence scale*: a measure of the cost of living of a household of a given size and demographic composition, relative to the cost of living of a reference household (usually a single adult), when both households attain the same level of utility or standard of living

- Consider a consumer (an individual or a household) with a vector of demographic characteristics z
- We can define the conditional expenditure cost function $e(p, u, z)$ which equals the minimum expenditure necessary to attain a utility level u , conditional on the consumer having characteristics z
- An equivalence scale compares costs conditional on different demographic characteristics

$$\frac{e(p, u, z^1)}{e(p, u, z^0)}$$

- By revealed preference theory, demand data identifies the shapes and rankings of a consumer's indifference curves over bundles of goods, but not the actual utility level associated with each indifference curve

- Thus equivalence scales, important as they are, cannot be identified from behaviour
- Another way to put it is to say that if a researcher estimates a demand system $x(p, w, z)$ and recovers the conditional expenditure function associated with it $e(p, w, z)$ then the implied equivalence scale

$$\frac{e(p, u, z^1)}{e(p, u, z^0)}$$

is essentially arbitrary

Two Stage Budgeting

Two Stage Budgeting

- Two-stage budgeting is a form of rational “mental accounting” which simplifies decision-making. It is also known as “decentralisation”.

Allocate wealth to each class → allocate wealth to each specific good within the class

- To simplify the bottom stage only requires weak separability of preferences across the broad groups
- This means that there are no cross-price effects with goods outside of this group
- As a result the demand for the x goods only depends on the prices of the x goods and the x budget

- Let $x(p_x, p_z, w)$ and $z(p_x, p_z, w)$ denote the demands for the x and the z goods respectively
- Then if the overall utility function is separable in the x goods we can find the optimal choice of the x goods by solving the sub-problem:

Lower Stage : $\max_x v(x) \text{ subject to } p_x \cdot x = p_x \cdot x(p_x, p_z, w)$

where $p_x \cdot x(p_x, p_z, w) \equiv w_x$ is the part of the budget allocated to the x goods

- In other words, we can write the demands for the x goods as a function of the prices of the x goods and the x budget

$$x(p_x, p_z, w) = x(p_x, w_x)$$

- The demand function $x(p_x, w_x)$ is sometimes called a conditional demand function — it tells us optimal demand given the prices of the goods in this group conditional on the level of the budget allocated to the group

- The between group allocation problem is more complex. In general the allocation of the x budget will depend on all of the prices of all of the goods in all of the groups.
- But this allocation problem too can be simplified such that the allocation across groups depends on a price index for each group which only depends on the prices of the goods in the groups.
- For example, the allocation of budget to food versus services versus non-durables becomes a three-good problem which depends on a price aggregate for the food group, the services group and the non-durables group.

Aggregation

General Properties of Aggregate Demand

- aggregate demand is h.o.d. 0
- aggregate demand satisfies Walras' Law
- if the individual demands are continuous, aggregate demand will be continuous; even if individual demands are discrete then with heterogeneous individuals aggregate demand will become continuous as $I \rightarrow \infty$

When is AD a Function of Prices and Aggregate Wealth?

- This must require that aggregate demand is invariant to mean-preserving/aggregate wealth-preserving changes in the wealth distribution.

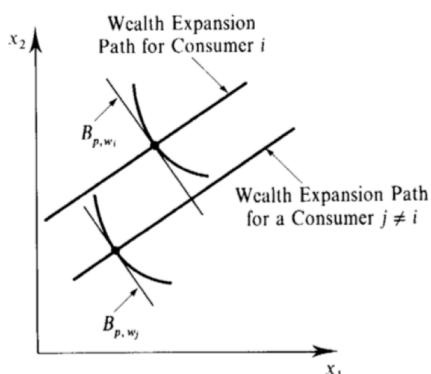
- That is

$$\sum_{i=1}^I \frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} dw_i = 0 \quad \forall \ell$$

- This can be true for all redistributions (dw_1, \dots, dw_L) such that $\sum_{i=1}^I dw_i = 0$ and all initial wealth distributions (w_1, \dots, w_L) iff

$$\frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} = \frac{\partial x_{\ell j}(p, w_j)}{\partial w_j} \quad \forall \ell$$

i.e. the slopes of the Engel curves are the same for everyone.



- This requires that the slopes of the wealth expansion paths/Engel curves are the same at all levels of wealth
- In other words that everyone's preferences are quasi-homothetic/ Gorman Polar form

[MWG 4.B.1]

- The necessary and sufficient condition for parallel straight Engel curves is that their indirect utility functions are of the form

$$v_i(p, w_i) = a_i(p) + b(p)w_i$$

- Preference heterogeneity is confined to the intercepts of these affine functions. The slopes are the same.
- Note that homothetic and quasi-linear preferences are special cases.

Individual Rationality and Aggregate Rationality

Individual rationality does not imply aggregate rationality.
Aggregate rationality does not imply individual rationality.

Problem Sets

Quasi-concavity of Utility Function and Convexity of Indifference Curves #flashcard

- (Strict) quasi-concavity of utility function \iff (Strict) convexity of indifference curves
- Mathematical definitions
- Upper-Level Sets Are Convex:** For every $\alpha \in \mathbb{R}$, the set $\{x : f(x) \geq \alpha\}$ is convex.
 - min-Preserving Inequality:** For every $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$,
$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}.$$
 - No “Dip Below” for Convex Combinations:** If $f(x) \geq \alpha$ and $f(y) \geq \alpha$, then $f(\lambda x + (1 - \lambda)y) \geq \alpha$ for all $\lambda \in [0, 1]$.

All of these say, in different words, that f **never goes below** its own “low points” on straight lines between any two points in its domain. That property ensures its upper contour sets are convex, which is the essence of **quasi-concavity**.

Misc

Lagrangian Method: Constrained Maximisation #flashcard

Our Problem:

$$\max_x u(x) \text{ s.t. } G(x) \leq 0, x \geq 0$$

where x is a vector

#flashcard

- This applies for constrained minimisation as well, just convert add a negative sign to the objective function.
- **Lagrangian:**

$$L(x, \lambda) = u(x) - \lambda G(x)$$

- Minus sign to punish disobeying of \leq constraints.

KKT: If:

- $u(\cdot)$ is quasi-concave ($\iff \{x : u(x) \geq \bar{u}\}$ is convex)
- the constraint set $\{x : G(x) \leq 0\}$ is convex with a non-empty interior

Then, the global optimum satisfies:

- $\frac{\partial L}{\partial x} \leq 0$ and $x \geq 0$ with CS: $\frac{\partial L}{\partial x} x = 0$ (beware of the signs: at optimal, x can only have negative marginal effects $\frac{\partial L}{\partial x} < 0$ if we hit the non-negativity constraint, which means we have a corner solution $x = 0$)
- $\frac{\partial L}{\partial \lambda} \geq 0 \iff G(x) \leq 0$ and $\lambda \geq 0$ with CS: $\frac{\partial L}{\partial \lambda} \lambda = 0 \iff G(x)\lambda = 0$ (beware of the signs: at optimal, as long as the constraint value has positive marginal effect $\lambda > 0$, we should have it binding $\frac{\partial L}{\partial \lambda} = 0$)