0 ECON113 Adv. Econ of Finance Index CONCISE

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Week 0: Notation and Choice Under Uncertainty

Lecture Notes Choice Under Uncertainty

- Notation convention
- Choice Under Uncertainty: Expected Utility
    - expected utility
    - Jensen's inequality: If the function f is concave ($f'' \leq 0$), then for any random variable x:

$$f(E[x]) \geq E[f(x)]$$

with strict inequality if the function is strictly concave ($f'' < 0$)

- The role of curvature
  - preference risk premium
  - certainty equivalent of a random payoff x is the fixed amount an individual (with utility function $u(.)$ and wealth w is willing to exchange for the random payoff x:

$$u(w + CE(u, w, x)) = E[u(w + x)]$$

- The role of uncertainty
  - coefficient of absolute risk aversion:

$$A(u, w) = -\frac{u''(w)}{u'(w)}$$

  - coefficient of relative risk aversion:

$$R(u, w) = -w\frac{u''(w)}{u'(w)} = wA(u, w)$$

  - probability premium
  - risk-adjusted probability
- Summarizing risk aversion
- Time-separable utilities

- Link to Finance
  - Demand for risky asset
  - Demand for Insurance

Week 1: FAPE and SDF

Lecture Slides

- 01 LectureSlides1 - A
  - First derivation of the Fundamental Asset Pricing Equation (Utility Maximization)
    - FAPE
    - Stochastic Discount Factor (SDF)
  - FAPE in return space
  - Crucial Properties of Prices and Returns
    - Prices are additive and linear in scale of payoffs
    - Returns are not additive and scale-invariant
  - We Have Not Assumed

    - Basic Intuition: marginal benefit vs marginal cost
    - Implications of the FAPE
        - Risk-less Bond
        - Excess Return / Risk Premia
        - Risky Bond
    - The SDF captures the time and risk dimension
    - Key Implication: risk-adjustment
- 01 LectureSlides1 Live - A
    - Week 0 Rev: Role of Preferences
    - Core Asset Pricing Intuition
    - Sample Questions

Lecture Notes FAPE from Simple Consumtpion-Saving Problem

Derive FAPE in A Simple 2-Period Setup

- Simple 2 Period Setup
- Maximisation:

$$\max_{q} = U(c_t, c_{t+1}) = u(c_t) + E_t[\delta u(c_{t+1})] \; s.t. \; \begin{cases} c_t = e_t - qp_t \\ c_{t+1} = e_{t+1} + qx_{t+1} \end{cases}$$

- Assume binding/equality budget constraints
- FOC/2-Period FAPE:

$$p_t = E_t\left[\underbrace{\frac{\delta u'(c_{t+1})}{u'(c_t)}}_{\text{IMRS or SDF}} x_{t+1}\right] = E_t[m_{t,t+1}x_{t+1}]$$

    - SDF: $m_{t,t+1} = \frac{\delta u'(c_{t+1})}{u'(c_t)}$
        - Stochastic: it is uncertain (we don't know what future consumption will be because either/both endowment or asset's payoff are uncertain)
        - Discount Factor: t translates the payoff in future dates $(t+1)$ to a present value (t)

Prices vs Returns

- Prices vs Returns
- Definitions
    - Gross return: $R_{x,t+1} \equiv \frac{x_{t+1}}{p_{x,t}}$; Net return: $R = 1 + r$
- FAPE (another representation):

$$p_t = E_t[m_{t,t+1}]E_t[x_{t+1}] + \underbrace{Cov_t[m_{t,t+1}, x_{t+1}]}_{-RP}$$

- Divide both sides by p_t: $1 = E_t[m_{t,t+1}R_{t+1}]$
- Expand and rearrange, we get the FAPE (in Return Space):

$$E_t[R_{t+1}] = \underbrace{\frac{1}{E_t[m_{t,t+1}]}}_{R_t^f}(1 - Cov_t[m_{t,t+1}, R_{t+1}])$$

- Prices vs Returns: FAPE indicates that
  - *Asset prices are additive functions of payoffs and are linear on the scale of payoffs*
  - *Returns are not additive and are scale invariant*

Implications

- Implications

Risk-Free Rates

- Risk-free rates
- FAPE applies to risk-free bond (know payoff $x_{t+1} = 1$):

$$p_t^f = E_t[m_{t,t+1}]$$

- Thus, the Risk-free Rate is:

$$R_t^f = \frac{1}{E_t[m_{t,t+1}]}$$

- This is the pure *time dimension* captured by the SDF -- just time discounting between periods
- Cases
  - *If there's no consumption growth*, then $m_{t,t+1} = \delta = \frac{1}{R_t^f} \Rightarrow R_t^f = \frac{1}{\delta}$
  - *If the consumption growth is g (certain) and $u(x) = \ln x$*, then $m_{t,t+1} = \frac{\delta}{1+g} = \frac{1}{R_t^f} \Rightarrow R_t^f = \frac{1+g}{\delta}$
- Determinants of risk-free rate:
  - Consumption growth: $g \uparrow R_t^f \uparrow$
  - RRA: $R(u, c) = -c\frac{u''(c)}{u'(c)} \uparrow R_t^f \uparrow$
  - Impatience: $\delta \uparrow R_t^f \uparrow$
  - Uncertainty: $E_t[(c_{t+1} - c_t)^2] \uparrow R_t^f(\text{typically}) \uparrow$

Excess Returns and Risk Premia

- Excess Returns and Risk Premia
- Risk Premium:

$$RP = \frac{E_t[x_{t,t+1}]}{R_t^f} - p_{x,t} = -Cov_t(m_{t,t+1}, x_{t+1})$$

- We define it as the negative of the covariance term, so positive and larger RPs always depress prices of risky assets and increase their returns
- Risk premium (the covariance between the SDF and the payoff) completely determines the risk adjustment. This is the *risk dimension* captured by the SDF
- FAPE (with Excess Returns):

$$E_t[R^e_{t+1}] = -R^f_t \frac{Cov_t[m_{t,t+1}, x_{t+1}]}{p_t} = -R^f_t Cov_t[m_{t,t+1}, R_{t+1}]$$

- fundamental intuition for risk compensation: what matters is whether the asset adds to the aggregate risk -- *to buy an asset that performs poorly when you really need it (in periods of low consumption/wealth), you require extra return relative to a risk-less bond*.
  - Excess returns will be higher for pro-cyclical assets (positive covariance of payoff with consumption) as they add to uncertainty:

$$E_t[R^e_{t+1}] \propto Cov_t[c_{t+1}, R_{t+1}]$$

Default Risk

- Default Risk
- Consider a risky bond:

$$x_{t+1} = \begin{cases} 1 & \text{, if no default, with probability} = 1 - \pi_\delta \\ \delta_{t+1}(<1) & \text{, otherwise(default), with probability} = \pi_\delta \end{cases}$$

- The excess return of risky bond is *ambiguous*:

$$E_t[R^e_{t+1}] = -R^f_t \frac{Cov_t[m_{t,t+1}, x_{t+1}]}{p_t} \underbrace{\Longrightarrow}_{\text{Law of Total Variance}} -R^f_t \frac{E\Big[Cov_t[m_{t,t+1}, \delta_{t+1}|default]\Big] + Cov\Big[I}{p_t}$$

  - If default is positively correlated with SDF (default in bad times), then investors will demand a positive excess return in compensation
- The price of risky bond is *always lower* than risk-free bond:

$$p_t = (1-\pi_\delta)E_t[m_{t,t+1}|\text{no default}] + \underbrace{\pi_\delta E_t[m_{t,t+1}\delta_{t+1}|\text{default}]}_{<\pi_\delta E_t[m_{t,t+1}|\text{default}]} < p^f_t$$

- If default is uncorrelated with the SDF, the risky bond will have the risk-free return, but a lower price (only adjust for expected default loss)

Summarising FAPE

Summarising FAPE

- Summarising FAPE
- SDF captures both time and risk dimensions:

- *Time dimension: expected value of the SDF / risk-free rate* (see Risk-free Rates):

$$R_t^f = \frac{1}{E_t[m_{t+1}]}$$

- *Risk dimension: covariance with the SDF* (see Excess Returns and Risk Premia):

$$p_t - \frac{E_t[x_{t+1}]}{R_t^f} = Cov_t[m_{t,t+1}, x_{t+1}]$$

Assumptions

- What are the assumptions?
- We have *Not assumed*:
  - Markets are complete
  - Any specific distributions
  - Any specific utility functions
  - Anything about the economy structure

Week 2: Arrow-Debreu Securities: FAPE in General Equilibrium

Lecture Slides

- 02 LectureSlides2 - A
  - Arrow-Debreu Economies
    - Assumptions
    - Arrow-Debreu securities
    - Equilibrium
    - Perfect Rank Correlation
  - Risk-neutral Pricing
    - 'risk-neutral' (risk-adjusted expectations) version of FAPE
    - Intuition
    - Complete Market, Efficiency, and Risk Sharing
  - Risk sharing and aggregate risk
  - Equity Premium Puzzle & Risk-free Rate Puzzle
- 02 LectureSlides2_Live - A
  - AD securities
  - Insights from GE
    - Risk Sharing
    - Complete Markets?
    - FAPE

- Risk-adjusted probabilities
- Equivalent FAPE Representations
- Macro-finance Puzzles

Lecture Notes Arrow-Debreu Securities: FAPE in General Equilibrium

Arrow-Debreu Economy

- Arrow-Debreu Economy

Setup

- Arrow-Debreu securities / state-contingent securities / state claims: assets pay 1 unit of consumption good in only one of the states of nature
- Assume only AD securities are traded and all states are tradable, each agent k has the following optimisation problem:

$$\max_{\{c_{k,t},\{c_{k,t+1}(s)\}_{s=1:S}\}} u_k(c_{k,t}) + \delta_k \sum_{s=1:S} \pi_s u_k(c_{k,t+1}(s)) \; s.t. \; \begin{cases} c_{k,t} + \sum_{s=1:S} pc_t(s) c_{k,t+1}(s) \leq e_{k,t} + \sum_{s=} \\ c_{k,t}, \{c_{k,t+1}(s)\}_{s=1:S} \geq 0 \end{cases}$$

- Further assume equality budget constraint, and denote $q_k(s) = c_{k,t+1}(s) - e_{k,t+1}(s)$, each agent k solves:

$$\max_{\{q_k(s)\}_{s=1:S}} u_k\left(e_{k,t} - \sum_{s=1:S} pc_t(s) q_k(s)\right) + \delta_k \sum_{s=1:S} \pi_s u_k\Big(q_k(s) + e_{k,t+1}(s)\Big)$$

Equilibrium

- Equilibrium: the above optimisation is satisfied with aggregate feasibility:

$$\begin{cases} \sum_{k=1:K} c_{k,t} \leq \sum_{k=1:K} e_{k,t} \\ \sum_{k=1:K} c_{k,t+1}(s) \leq \sum_{k=1:K} e_{k,t+1}(s) \; \forall \; s \end{cases}$$

- Results for GE FOC:
  - Arrow-Debreu Prices (FAPE on AD with IMRS as a valid SDF):

$$pc_t(s) = \delta_k \frac{\pi_s u_k'(c_{k,t+1}(s))}{u_k'(c_{k,t})} = \pi_s m_{t,t+1}(s) \; \forall \; k, s$$

  - Common IMRS/SDF across agents:

$$m_{t,t+1}(s) = \underbrace{\delta_{k_1} \frac{u_{k_1}'(c_{k_1,t+1}(s))}{u_{k_1}'(c_{k_1,t})}}_{m_{t,t+1,k_1}} = \underbrace{\delta_{k_2} \frac{u_{k_2}'(c_{k_2,t+1}(s))}{u_{k_2}'(c_{k_2,t})}}_{m_{t,t+1,k_2}}$$

- Common MRS between states across agents:

$$\frac{u'_{k_1}(c_{k_1,t+1}(s_1))}{u'_{k_1}(c_{k_1,t+1}(s_2))} = \frac{u'_{k_2}(c_{k_2,t+1}(s_1))}{u'_{k_2}(c_{k_2,t+1}(s_2))}$$

- FAPE must hold in GE
  - FAPE $\implies$ AD FOC: FAPE implies the FOC of the A-D economy for each A-D security
  - AD FOC $\implies$ FAPE: FOC of the A-D economy for each A-D security implies the FAPE

Risk Sharing and Aggregate Risk

- Risk sharing and aggregate risk
- Aggregate risk: variation in the sum of available resources per state $\sum_{k=1:K} e_{k,t+1}(s)$, which is inevitable irrespective of how markets share risk
- Idiosyncratic risk: variation at individual level, which can be diversified by trading if that state is traded

Perfect Rank Correlation

- AD FOC/FAPE -> Common MRS -> perfect rank correlation:

$$Corr(u'_{k_1}(c_{k_1,t+1}), u'_{k_2}(c_{k_2,t+1})) = 1$$

  - *If you rank consumption across all states of the world for each agent, their consumption ranks are perfectly correlated*
  - Combined with aggregate feasibility, this implies that, for all traded states, *only aggregate risks matter for IMRS/SDF hence risk premia*:
    - Only aggregate risk determines consumption patterns, or conversely, agents fully insure idiosyncratic risk
    - The less risk-averse provide insurance to the more risk averse, or equivalently, bear a disproportionate amount of risk
    - Risk is shared "proportionately": the low/high consumption states are the same for all agents when there is aggregate risk, and fully insured (fixed consumption across states) when there is no aggregate risk
- Example

Asset Market Participation

- Asset Market Participation
- *AD FOC / FAPE only holds for agents with FPF*

Complete Markets

Complete Markets and FAPE

- Complete markets: an economy has complete markets if every state of nature can be traded
- With the full set of AD securities, we can construct any payoffs by AD portfolios, and price them using FAPE/LOOP
- FAPE (AD/Contingent Pricing):

$$p_{x,t} = \sum_s pc_t(s)x_{t+1}(s)$$

- SDF (defined by Contingent/AD Prices):

$$m_{t,t+1}(s) \equiv \frac{pc_t(s)}{\pi_s} \iff pc_t(s)$$

- AD Prices:

$$pc_t(s) \equiv \pi_s m_{t,t+1}(s)$$

- Market completeness and FAPE: *the existence of an SDF and the fact it applies to every traded asset does not depend on market completeness*
    - If we want to price a new asset (i.e. not yet traded): then incompleteness means we cannot use the SDF obtained from existing assets. In a complete market we can.

Pareto Optimality

- Pareto Optimality
- *Market Completeness matters for Pareto Optimality*: In complete markets, the equilibrium is Pareto optimal
    - The FOC for a social planner will be identical for individuals in a GE in complete markets, so the market allocation is efficient
- Example continued

Risk-Neutral/Risk-Adjusted Pricing

Equations

- Risk-neutral pricing
- Risk-neutral/adjusted Probability:

$$\pi_s^Q \equiv \frac{\pi_s m_{t,t+1}(s)}{\sum_s \pi_s m_{t,t+1}(s)} = \frac{\pi_s m_{t,t+1}(s)}{E_t[m_{t,t+1}]} = \frac{pc_t(s)}{\sum_s pc_t(s)}$$

- Risk-adjusted/neutral FAPE:

$$p_{x,t} = \frac{1}{R_t^f} E_t^Q[x_{t+1}]$$

- Risk-adjusted/neutral Returns:

$$\begin{cases} E_t^Q[R_{x,t+1}] = R_t^f \\ E_t^Q[R_{x,t+1}^e] = 0 \end{cases}$$

Intuition

- Intuition:

$$\frac{\pi_s^Q}{\pi_s} \propto m_{t,t+1}(s) \propto u_k'(c_{k,t+1}(s))$$

- We wrap risks into probability by *overweighting the probability of the bad states*, so asset prices can be thought as the expected payoff of payoffs for a *risk-neutral but pessimistic* investor
    - Cochrane: "risk aversion is equivalent to paying more attention to unpleasant states relative to their actual probability of occurrence"
- Alternative derivation of risk-neutral probabilities by AD prices

Equity Premium Puzzles & Disaster Risk

- The macro - asset pricing link: the risk-free and equity premium 'puzzles'
    - Consumption is too smooth (low growth volatility and low comovement with stock markets) to justify the observed excess returns for equities, unless the RRA is implausibly large (>20)
- Possible explanation: Disaster Risk
    - If a disaster is big enough, even it has very small actual probability, it can have significant influence on risk-neutral expectations because its probability is exaggerated
    - The distribution may have an extreme negative skewness (and higher moments) relative to a lognormal consumption growth

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Week 3: No-Arbitrage and the Fundamental Theorem of Asset Pricing (FTAP)

Lecture Slides

- 03 LectureSlides3 - A
    - Law of One Price (LOOP)
    - No-Arbitrage
    - Assuming free portfolio formation, no-arbitrage implies the LOOP
    - LOOP does not imply no-arbitrage
    - Fundamental Theorem of Asset Pricing

- Applications of No-Arbitrage Condition
    - Complete Markets and Uniqueness of SDF
    - Limits to Arbitrage: liquidity and impediments to trading
- 03 LectureSlide3_Live
    - Problem Set 1
    - FTAP
        - Role of No-Arbitrage (NA)
        - FTAP with AD securities
        - Generality of FAPE
        - Complete vs. incomplete markets
    - Limits to Arbitrage
    - Problem Set 1 2019
    - Problems LN 3

Lecture Notes No-Arbitrage and the Fundamental Theorem of Asset Pricing

Free Portfolio Formation (FPF), Law of One Price (LOOP), and No-Arbitrage

- Law of one price and no-arbitrage condition
- Free portfolio formation: agents can freely trade every asset in M, which means they can form any portfolio they want by freely combining existing assets.
    - This means that for any $x, y \in M$, if $z = ax + by$ then $z \in M$ for any (constant) scalars $a, b \in R$
    - Equivalent to *no trade restrictions of any kind*
- Law of One Price (LOOP): for any $x, y \in M$ if $z = ax + by$ then $(z \in M)$ $p_z = ap_x + bp_y$
    - Purely about *relative pricing*
    - FPF + NA implies LOOP; LOOP does NOT imply NA (e.g. all prices are 0)
- No-arbitrage: for any $x \in M$, if $x \geq 0$ and $x(s) > 0$ with $\pi_s > 0$ for at least some state s, then $p_x > 0$

Fundamental Theorem of Asset Pricing (FTAP)

- Fundamental Theorem of Asset Pricing: Assuming Free Portfolio Formation, the No-Arbitrage condition holds if and only if there $\exists m > 0$ such that for $\forall x \in M : p_x = E[mx]$
    - i.e. *No Arbitrage + FPF -(FTAP)-> FAPE holds with a positive SDF*
    - FAPE = NA are necessary conditions for any equilibrium (otherwise, with $u' > 0$, someone will exploit the arbitrage opportunity)
    - Intuitive proof of the theorem

- NA + FPF -> FAPE (with positive SDF): NA -> LOOP + strictly positive AD prices for traded states (isolated by FPF) -> we can construct a strictly positive SDF by AD prices
- FAPE (with positive SDF) -> NA: FAPE -> LOOP and positive AD prices -> NA

- Application and testing of no-arbitrage condition
- Checking No-Arbitrage: Risk-neutral probabilities, contingent claim prices and SDF
  - Check whether there are *valid risk-neutral probabilities / valid AD prices* (requires no knowledge about the true probabilities)
  - or Check whether there is a *valid SDF* (requires the true probabilities)

Complete Markets, FTAP, and Uniqueness of SDF

- Complete markets: full-rank payoff matrix (invertible, hence we can crate AD securities for all states) -> unique AD pricing by LOOP; unique SDF / AD prices / risk-neutral probabilities
- Incompleteness: for at least 1 state, we can find a range of SDF / AD prices / risk neutral probabilities consistent with no arbitrage
  - A non-redundant asset will not be uniquely priced, and the GE could be completely changed
  - Example
- Uniqueness of SDF and complete markets
  - FPF + NA + Completeness $\rightsquigarrow$ Unique SDF
  - FPF + NA + Incompleteness $\rightarrow$ A range of SDF, but with *a unique projection* on prices of traded assets/states
- (Week 10) Effective Uniqueness: we can redefine the possible states as those that are traded, so that on that subset of all possible states we have the complete market paradigm; in other words, if for trading/pricing purposes we cannot distinguish between two states, it is only semantics to call them different states

Limits to Arbitrage

- Limits to arbitrage: arbitrageur wealth, transaction costs and liquidity
- *Trading frictions (no FPF) do not mean FAPE breaks down, just that SDF will include different or additional factors*

Week 4: Introducing Dynamics: The Multiperiod FAPE

Lecture Slides

- 04 LectureSlides4 - A

Lecture Notes 04 LN [Multiperiod FAPE] - A

Conditional Expectation and LIE

- Conditional Expectations and the Law of Iterated Expectations
- LIE
    - $E_t[E_{t+k}[X]] = E_t[X]$
- Randomness
    - $Var_t[E_{t+k}[X]] > 0$ unless $E_{t+k}[X] = X$

Dynamic/Multiperiod FAPE

- Simple Discrete Setup

Multiperiod FAPE

- Introducing Dynamics in FAPE
- Multiperiod FAPE:

$$p_{x,t} = E_t\left[\sum_{j=1}^{J} m_{t,t+j} x_{t+j}\right] = \sum_{j=1}^{J} \frac{E_t[x_{t+j}]}{R^f_{j,t}} + \underbrace{\sum_{j=1}^{J} Cov_t\left[m_{t,t+j}, x_{t+j}\right]}_{-RP}$$

where $m_{t,t+j}$ is the Multiperiod SDF:

$$m_{t,t+j} = \prod_{s=1}^{j} m_{t+s-1,t+s}$$

- One-period HPR:

$$E_t[R_{x,t\to t+1}] = E_t\left[\frac{x_{t+1} + p_{x,t+1}}{p_{x,t}}\right] = R^f_{1,t} - R^f_{1,t} Cov_t[m_{t,t+1}, R_{x,t\to t+1}]$$

- Next period price:

$$p_{x,t+1} = E_{t+1}\left[\sum_{j=1}^{J-1} m_{t+1,t+1+j} x_{t+1+j}\right]$$

- One period HPR:

$$R_{x,t\to t+1} = \frac{x_{t+1} + p_{x,t+1}}{p_{x,t}}$$

Derivation

- Iterating Two-Period Case
- Exercise 9 LN1 (another method with additional assumption)
- Simple Example: 3 periods

Empirical Implications

- Empirical Implications

Random Walk Hypothesis

- Random Walk?

$$RWH : E_t[p_{x,t+k}] = p_{x,t} \ \forall\ k \geq 1$$

  - Generally not true because of: RP, interim cashflow, and Time Discounting:

$$E_t[p_{x,t+1}] = \frac{p_{x,t}}{E_t[m_{t,t+1}]} - E_t[x_{t+1}] + RP$$

- *RWH only holds when there's no interim cashflow, no RP, and risk-free rate = 0*

Testing FAPE

- Testing FAPE
- Ex post Realised Returns can be different from ex ante Expected Returns (i.e for one-period HPR: $R_{x,t\rightarrow t+1} = \frac{x_{t+1}+p_{x,t+1}}{p_{x,t}} \neq E_t[R_{x,t\rightarrow t+1}] = \frac{E_t[x_{t+1}+p_{x,t+1}]}{p_{x,t}}$) because of:
    1. Unexpected Payoff (Cash Flow News.1): $x_{t+1} \neq E_t[x_{t+1}]$
    2. Price Changes $\underbrace{E_{t+1}\left[\sum_{j=1}^{J-1} m_{t+1,t+1+j}x_{t+1+j}\right]}_{p_{x,t+1}} \neq \underbrace{E_t\left[E_{t+1}\left[\sum_{j=1}^{J-1} m_{t+1,t+1+j}x_{t+1+j}\right]\right]}_{E_t[p_{x,t+1}]}$

       due to:
        1. Cash Flow News.2: $E_{t+1}[x_{t+1+j}] \neq E_t[E_{t+1}[x_{t+1+j}]]$
        2. Discount Factor News (typically affects the whole economy):
            - SDF changes: $E_{t+1}[m_{t+1,t+1+j}] \neq E_t\left[E_{t+1}[m_{t+1,t+1+j}]\right]$
            - Covariance with SDF changes: $Cov_t[m_{t+1,t+1+j}, x_{t+1+j}] \neq Cov_{t+1}[m_{t+1,t+1+j}, x_{t+1+j}]$ for 1 or more periods j
    3. Failure of FAPE
- Therefore, we always suffer from the joint hypothesis problem when testing FAPE (when data reject our model, we don't know whether it's a failure of FAPE or misspecification of SDF, since we cannot observe RP or expectations)
- Risk Premia Changes (additional complication)
    - A risk premia shock in any period will have the opposite effect on that period than what is expected for future periods
    - RP in $t+k$ increases (RP shock, such as change in risk aversion or aggregate risk) $\implies$ Returns has be higher while Expected cash flow paths haven't changed (cannot have a higher overall price path) $\implies$ Current price has to decrease

Dynamics with Parametric Distributions

- Introducing dynamics with parametric distributions
    - Rf with Lognormal Consumption Growth
        - Implication of the Lognormal Assumption

$$z \sim N(a, b^2) \Rightarrow E[\exp(z)] = \exp\left(a + \frac{b^2}{2}\right)$$

or

$$\ln z \sim N(a, b^2) \Rightarrow E[z] = E[\exp(\ln z)] = \exp\left(a + \frac{b^2}{2}\right)$$

    - Rf with General Consumption Growth Uncertainty (derivations not required)

- R^f with General Consumption Growth Uncertainty

$$r_t^f = \delta + \gamma\mu_{c,t} - \frac{\gamma^2}{2}\sigma_{c,t}^2 + \frac{\gamma^3}{3!}skew_{c,t}\sigma_{c,t}^3 - \frac{\gamma^4}{4!}xkurt_{c,t}\sigma_{c,t}^4 + h.o.t.$$

consumption's even moments $\uparrow$ / odd moments $\downarrow$ -> risk $\uparrow$ -> r_t^f $\downarrow$

Week 5: The Term Structure of Interest Rates

Lecture Slides

- 05 LS [Term Structure of Interest Rate] - A
  - COMPOUNDING
    - continuously compounded annual interest rate
  - Term Structure of Interest Rates
    - Term premia
    - Jensen term
    - Risk-neutral version
    - Expectation Hypothesis
  - Forward Rates
  - GADTSM
- 05 LS L [Term Structure of Interest Rate] - A
  - Standard FAPE Application
  - (Real) Term Premium
  - Intuition TP Sign
  - Excess Returns
  - Forwards

Lecture Notes The Term Structure of Interest Rates

Measuring IR

- Measuring Interest Rates
- compounding
- continuously compounded annual interest rate

$$R_{n,t}^f = e^{nr_{n,t}^f}$$

where $r_{n,t}^f$ is the per year CCIR of a n-year risk-free bond

Pricing Zero-Coupon Bonds

- Price and CCIR for k-period zero-coupon (non-defaultable) bond (using multiperiod FAPE)

$$p^f_{k,t} = \frac{1}{R^f_{k,t}} = e^{-kr^f_{k,t}} = E_t[m_{t,t+k}]$$

$$r^f_{k,t} = -\frac{1}{k}\ln E_t[m_{t,t+k}]$$

Term Premium

- Term premium

2 or 1 Period Bonds

- 2-period bond relative to the two 1-period bond: $TP_{2,1,t}$
  - $TP_{2,1,t}$ in price space

$$p^f_{2,t} = p^f_{1,t}E_t\left[p^f_{1,t+1}\right] + \underbrace{Cov_t(m_{t,t+1}, p^f_{1,t+1})}_{-TP_{2,1,t}}$$

  - $TP_{2,1,t}$ in return space

$$\frac{1}{R^f_{2,t}} = \frac{1}{R^f_{1,t}}E_t\left[\frac{1}{R^f_{1,t+1}}\right] + \underbrace{Cov_t\left(m_{t,t+1}, \frac{1}{R^f_{1,t+1}}\right)}_{-TP_{2,1,t}}$$

    - Reasons for the failure of EH $R^f_{2,t} \neq R^f_{1,t}E_t\left[R^f_{1,t+1}\right]$:
      - Jensen Term: $E_t\left[\frac{1}{R^f_{1,t+1}}\right] \neq \frac{1}{E_t\left[R^f_{1,t+1}\right]}$ (difference increases with uncertainty/volatility of IR)
      - Covariance Term / RP: $Cov_t\left(m_{t,t+1}, \frac{1}{R^f_{1,t+1}}\right)$

- Intuition in Price space
  - TP and Autocorrelation of consumption growth

$$\begin{aligned} TP_{2,1,t} = R^f_{2,t} - R^f_{1,t}E_t\left[R^f_{1,t+1}\right] &= -Cov_t(m_{t,t+1}, p^f_{1,t+1}) \\ &= -Cov_t(m_{t,t+1}, E_{t+1}[m_{t+1,t+2}]) \\ &\propto -Cov_t(m_{t,t+1}, m_{t+1,t+2}) \\ &\propto -Cov_t\left(\frac{c_{t+1}}{c_t}, \frac{c_{t+2}}{c_{t+1}}\right) \end{aligned}$$

  - The 2-period bond is still risky because we can resell it in the intermediate period. *Positive autocorrelations in consumption growth* means 2-period bonds will have a higher resell price in the intermediate period if consumption is low in that intermediate period, offering a good hedge and having a *negative TP*. (*Long bonds have lower returns* than expected return of buying short bonds successively)

- Intuition in Return space:

$$Cov_t\left(\frac{c_{t+1}}{c_t}, \frac{c_{t+2}}{c_{t+1}}\right) > 0 \Leftrightarrow Cov_t\left(\frac{c_{t+1}}{c_t}, R^f_{1,t+1}\right) > 0$$
$$\Leftrightarrow \frac{c_{t+1}}{c_t} \downarrow, R^f_{1,t+1} \downarrow$$

- The short rate in the bad state will be lower, so 2-period bond provides a good hedge and has negative TP

N Period Bonds

- Generalizing to N-period Bond
  - $RP_{n,n-1,t}$ (n period & $n-1$ period bonds)
    - $RP_{n,n-1,t}$ in price space

$$p^f_{n,t} = \underbrace{E_t[m_{t,t+1}]}_{p^f_{1,t}} E_t\left[p^f_{n-1,t+1}\right] + \underbrace{Cov_t(m_{t,t+1}, p^f_{n-1,t+1})}_{-RP_{n,n-1,t}}$$

    - $RP_{n,n-1,t}$ in return space

$$\frac{1}{R^f_{n,t}} = \frac{1}{R^f_{1,t}} E_t\left[\frac{1}{R^f_{n-1,t+1}}\right] + \underbrace{Cov_t\left(m_{t,t+1}, \frac{1}{R^f_{n-1,t+1}}\right)}_{-RP_{n,n-1,t}}$$

    - $RP_{n,n-1,t}$ in CCIR

$$exp(-nr_{n,t}) = exp(-r_{1,t})E_t\Big[exp\big(-(n-1)r_{n-1,t+1}\big)\Big] - \underbrace{Cov_t\Big(m_{t,t+1}, exp\big(-(n-1)r}_{-RP_{n,n-1,t}}$$

    - *Persistent SDF / persistent consumption growth (in our consumption view of the SDF)* $\rightsquigarrow$ *Negative $RP_{n,n-1,t}$ (long bonds have lower returns)*
  - $RP_{n,1,t}$ (n period & 1 period bonds)
    - $RP_{n,1,t}$ in price space

$$p^f_{n,t} = \prod_{s=0}^{n-1} E_t\left[p^f_{1,t+s}\right] + \underbrace{\sum_{i=0}^{n-2} Cov_t(m_{t+i,t+i+1}, m_{t+i+1,t+n}) \prod_{j=i,i\geq 1}^{n} E_t[m_{t+j-1,t+j}]}_{-RP_{n,1,t}}$$

    - $RP_{n,1,t}$ in return space

$$\frac{1}{R^f_{n,t}} = \prod_{s=0}^{n-1} E_t\left[\frac{1}{R^f_{1,t+s}}\right] - RP_{n,1,t}$$

- $RP_{n,1,t}$ in CCIR

$$exp(-nr_{n,t}) = \prod_{s=0}^{n-1} E_t\left[exp(-r_{1,t+s})\right] - RP_{n,1,t}$$

- The expression for $RP_{n,1,t}$ is more complicated and related to the autocorrelation function: if the autocorrelation changes signs with maturity, $RP_{n,1,t}$ will not be monotonic growing/shrinking with maturity

Expectation Hypothesis

- Expectation Hypothesis: long-term CCIRs are average of expected short-term CCIRs

$$\begin{aligned} r^f_{n,t} &= -\frac{1}{n}\ln\left(\prod_{s=1}^{n} E_t\left[exp(-r_{t+s-1})\right]\right) \\ &= E_t\left[\frac{1}{n}\sum_{s=1}^{n} r_{t+s-1}\right] + \text{Jensen Term} \end{aligned}$$

- EH is *generally invalid* when there is a risk premia (due to interest rate uncertainties): long-term bonds are not entirely risk-free due to possibility of reselling (at an uncertain price because changing IR)
- EH implies that Actual Expectations = Risk-Adjusted Expectation (see below Risk-Adjusted Expectations)

Holding Period Excess Return

- Holding-period Excess Returns
- Expected k-period Expected Excess HPR (EExHPR) of an n-period risk-free bond

$$\underbrace{E_t\left[R^{(f)}_{n,t,t\to t+k}\right] - R^f_{k,t}}_{\text{k-period EExHPR}} = -R^f_{k,t} Cov_t\left[m_{t,t+k}, R^{(f)}_{n,t,t\to t+k}\right]$$

where $R^{(f)}_{n,t,t\to t+k} = \frac{p^f_{n-k,t+k}}{p^f_{n,t}}$ is the return of selling the n-period bond at $t+k$

- Can be shown that *TP on an n-period bond is the EExHPR over the life of the bond*

Forward Rates & Spot Rates

- Forward Interest Rates $f_{t,\text{(maturity)},\text{(starting period)}}$
- Forward Rates: implied future short rates known now

$$f_{t,1,1} = 2r_{2,t} - r_{1,t}$$

$$f_{t,1,k} = (k+1)r_{k+1,t} - kr_{k,t}$$

- Spot Rates: current long-term rates are the average of short-term forward rates (all known today)

$$r_{n,t} = \frac{1}{n} \sum_{s=0}^{n-1} f_{t,1,s}$$

Risk-Adjusted Expectations

- risk-adjusted expectations
- Risk-adjusted prices

$$p_{n,t}^f = E_t^Q \left[\prod_{k=0}^{n-1} \frac{1}{R_{1,t+k}^f} \right] = E_t^Q \left[\exp \left(-\sum_{s=0}^{n-1} r_{1,t+s} \right) \right]$$

- The current long-term return is the average of *risk-adjusted* future short rates

$$r_{n,t}^f = -\frac{1}{n} \ln E_t^Q \left[\exp \left(-\sum_{s=0}^{n-1} r_{1,t+s} \right) \right] = \frac{1}{n} \sum_{s=0}^{n-1} E_t^Q [r_{1,t+s}] + \text{Jensen Term}$$

Others

- Decomposing the Term Structure of either spot/forward rates:

  $$\text{Nominal IR} = \text{Expected Real Rate} + \text{Expected Inflation} + \text{Real Term Premium} + \text{Inflation Te}$$

- Introducing dynamics with parametric distributions

Week 6: Equity Pricing: Beta Representation and Factor Models

Lecture Slides

- 06 LS [Equity Pricing_ Beta Representation and Factor Models] - A
  - Challenge with FAPE
  - DDM
  - Gordon Growth Model
  - Beta representation for returns
    - Conditional
    - Unconditional
  - Factor Pricing Models
    - Equivalence with beta representations
    - Beta representation with Returns
    - In excess returns space
    - Number of factors
    - Conditional vs. unconditional factor models
  - Empirical application of factor models

    - time series regression
    - cross section regression
    - Model evaluation
    - Active vs Passive
  - Consumption Capital Asset Pricing Model (CCAPM) and Capital Asset Pricing Model (CAPM)
  - Arbitrage Pricing Theory
- 06 LS L [Equity Pricing_ Beta Representation and Factor Models] - A
  - Standard FAPE Intractable for Equities
  - Beta Representation
  - Unconditional Beta Representation
  - Linear Factor Models
  - Beta with Factor Models
  - Empirical Applications
  - Problem Set 3 2019
  - Exercises Lecture Notes 6

Lecture Note Equity Pricing: Beta Representation and Factor Models

Problems with FAPE and DDM/GGM

FAPE

- Equities and FAPE
- Intractable: need to specify payoffs and comovements with SDF to infinity and hard to solve analytically

Dividend Discount Model & Gordon Growth Model

- DDM and the Gordon Growth Model
- Dividend Discount Model (DDM)

$$p_{i,t} = \underbrace{\sum_{k=1}^{K} \frac{D_{i,t+k}}{R_{i,t,t+k}}}_{\text{Discounted Future Dividend}} + \underbrace{\frac{p_{i,t+K}}{R_{i,t,t+K}}}_{\text{Residual Value}} = \sum_{k=1}^{\infty} \frac{D_{i,t+k}}{R_{i,t,t+k}}$$

- Gordon Growth Model (a particular case of DDM): Assume constant growth of dividends: $g = \frac{d_{t+1}}{d_t} \; \forall \, t$, we can derive the Gordon Formula:

$$p_{i,t} = d_{i,t} \times \frac{1+g}{r-g}$$

- Problem: with a reasonable IR (5%), the residual value will be still high even after 20 years.

Beta Representation

- Beta-representation
  - the nature of the (con/uncon.) beta representation is that the (con/uncon.) return of any asset is a linear function of it's covariance with the SDF
- FAPE <-> Conditional Beta Representation (Full equivalence, no information loss)

$$E_t[R_{i,t+1}] = \underbrace{\frac{1}{E_t[m_{t,t+1}]}}_{\gamma_t} + \underbrace{\left(-\frac{Var_t[m_{t,t+1}]}{E_t[m_{t,t+1}]}\right)}_{\lambda_t} \times \underbrace{\frac{Cov_t[m_{t,t+1}, R_{i,t+1}]}{Var_t[m_{t,t+1}]}}_{\beta_{[R_i,m],t}}$$

  - only $\beta_{[R_i,m],t}$ is asset-specific, other parameters can change over time but are the same across assets
- Unconditional Beta Representation (Gain Tractability Step 1)
  - derived from FAPE: $1 = E_t[m_{t,t+1}R_{i,t+1}] \Rightarrow^{(LIE)} 1 = E[m_{t,t+1}R_{i,t+1}]$
  - $$E[R_{i,t+1}] = \underbrace{\frac{1}{E[m_{t,t+1}]}}_{\gamma} + \underbrace{\left(-\frac{Var[m_{t,t+1}]}{E[m_{t,t+1}]}\right)}_{\lambda} \times \underbrace{\frac{Cov[m_{t,t+1}, R_{i,t+1}]}{Var[m_{t,t+1}]}}_{\beta_{[R_i,m]}}$$
  - Only $\beta_{[R_i,m]}$ is asset-specific, other parameters are all constant across assets/time
- Conditional vs. unconditional beta representation
  - *Conditional beta-rep does not imply Unconditional beta-rep* (because variance/covariance do not condition down)
  - If we really want to consider the unconditional expectation of the condition B-rep, it becomes a 3-factor semi-beta model:

$$\begin{aligned} E\Big[E_t[R_{i,t+1}]\Big] &= E[\gamma_t] + E[\lambda_t]E[\beta_{[R_i,m],t}] + Cov\left[\lambda_t, \beta_{[R_i,m],t}\right] \\ &= E[\lambda_t] + Cov\left[\lambda_t, \beta_{[R_i,m],t}\right] + E[\lambda_t]\left(E[\beta_{[R_i,m,t]}] - \beta_{[R_i,m]}\right) + E[\lambda_t]\beta_{[R_i,m]} \end{aligned}$$

  - *Restrictive assumptions* have to be made, such as SDF and returns have valid unconditional distributions, and conditional moments are constant over time
  - *Information loss*

Factor Pricing Models

- Factor pricing models
- Linear Factor Models (Gain Tractability Step 2: solve the problem that SDF is unobserved)
  - Assume SDF is linear in some pricing factors:

$$m_{t,t+1} = a + b' f_{t+1}$$

    this is equivalent to (both directions, proofs are given in the LN):

$$E[R_{i,t+1}] = \gamma + \lambda' \beta_{[R_i,f]}$$

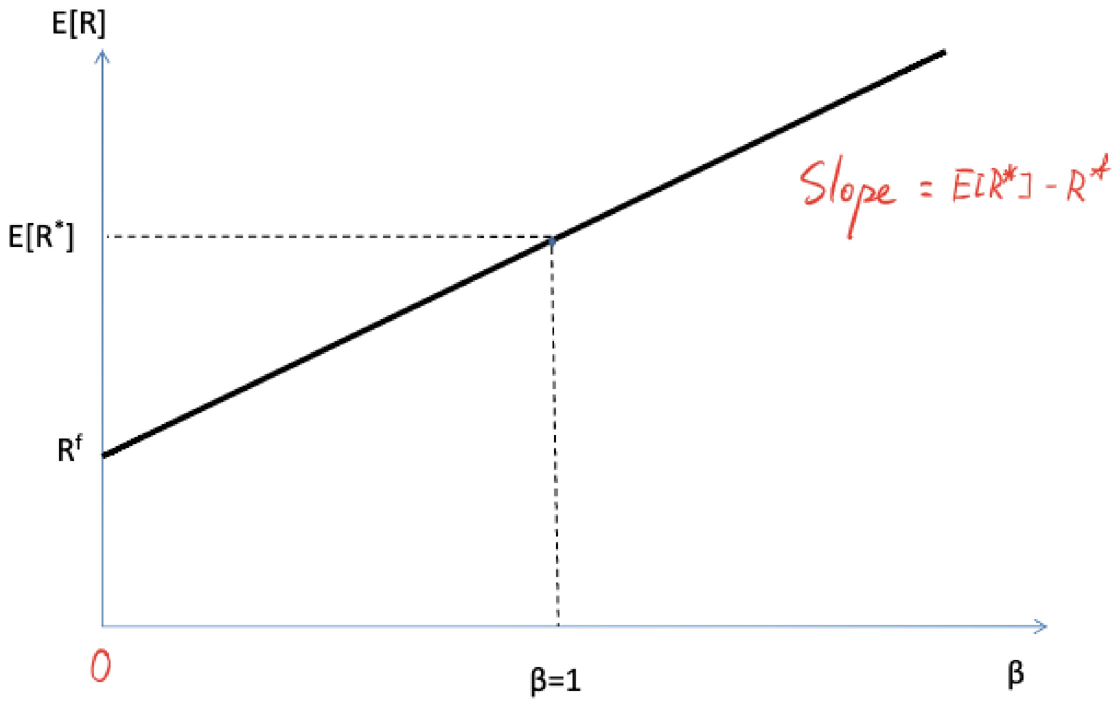
- a, b, γ, λ are *constant over time*
- only constraint: positivity: $m_{t,t+1} = a + b' f_{t+1} > 0$

- Price and quantity of risk
  - $\beta_{[R_i,f]}$ measures an asset's *exposure (quantity) to risk* f (asset-specific)
  - λ_{f_i} measures the *price (additional return needed) for risk* f (constant across asset/time)
- beta of a portfolio
  - simply the weighted average of components' betas:

$$\beta_{[z,f]} = \sum_i w_i \beta_{[x_i,f]} \text{ where } w_i \text{ are portfolio weights: } z = \sum_i w_i x_i$$

- Returns as factors
  - Define the unconditional risk-free rate R^f as the *zero-beta return*, then $\gamma = R^f$
  - If a return R^* is chosen as a risk factor:

$$E[R_i] = R^f + \beta_{[R_i,R^*]} \left[E[R^*] - R^f \right]$$



- Scale and mean of pricing factors are not important
- Multivariate case: only constraints:
  - positivity: $m_{t,t+1} = a + b' f_{t+1} > 0$
  - no perfect multicollinearity
- Number of factors does not matter because we can always combine them into one factor (SDF), but separating them allows us to study what drives SDF
- Conditional vs. unconditional factor models

Empirical Application of Factor Models

- Empirical application of factor models

Two-Step Regression

- Two-Step regression
    - Time Series Regression for Each Asset i (to determine exposures (β) to risk factors)

$$R_{i,t} = \underbrace{a_i}_{Not\ \alpha_i} + \beta'_{[R_i,f]} f_t + \epsilon_{i,t} = \underbrace{a_i}_{Not\ \alpha_i} + \sum_j \beta_{[R_i,f_j]} f_{j,t} + \epsilon_{i,t},\ t = 1, 2, \ldots, T$$

    - Cross Section Regression across assets (to estimate γ and λ)

$$E[R_i] = \gamma + \lambda' \beta_{[R_i,f]} + \alpha_i = \gamma + \sum_j \lambda_{f_j} \beta_{[R_i,f_j]} + \alpha_i,\ i = 1, 2, \ldots, N$$

Evaluations

- Model evaluation
    - small α and large R^2 are desirable
    - large α is a rejection of model
    - but models with small R^2 but significant relationships are still useful (to identify sources of risks)
- Performance measurement
    - Performance cannot be assessed solely based on returns, but also their riskiness
    - From the model's perspective, *$\alpha > 0$ means underpricing; $\alpha \neq 0$ is an arbitrage opportunity ($E[R_i] \neq \gamma + \sum_j \lambda_j \beta_{[R_i,f_i]}$)*
    - But this could also be caused by *unexpected return* ($R_{i,t} \neq E_{t-1}[R_{i,t}]$) with reasons discussed in Testing FAPE or *model misspecification*
- Active vs. passive investment
    - Active investment: we can find $\alpha > 0$; Passive: after risk adjustment, all $\alpha = 0$
    - Literature: subtracting fees, active investment does not outperfrom

Famous Single Factor Models

- The Consumption Capital Asset Pricing Model (CCAPM): consumption growth $g_{c,t+1} = \frac{c_{t+1}}{c_t}$ (or its transformation) is the risk factor
- The Capital Asset Pricing Model (CAPM): return of the total wealth portfolio R^m is the risk factor (next week)

Arbitrage Pricing Theory (APT)

- Arbitrage Pricing Theory (APT): more agnostic about underlying structure, just posits that there must be a factor structure to returns

- Equilibrium vs APT

Week 7: Mean-Variance Efficient Frontier and CAPM

Lecture Slides

- 07 LS [Mean Variance Efficient Frontier and CAPM] - A
    - Role of Diversification
        - simple two asset case
        - General case
    - Derivation of risky asset minimum variance frontier (RA-MVF)
    - Mean-Variance Bounds from FAPE
    - Mean-variance efficient frontier (MVEF)
        - Why the tangent portfolio?
    - Mean-Variance Approach
    - Two-fund Portfolio Separation Theorem
    - CAPM
        - Capital Market Line
        - Security Market Line
    - Derivation of CAPM from Consumption-based model
    - ICAPM
    - Mean-variance vs SDF

Lecture Notes 07 LN [Mean Variance Efficient Frontier and CAPM] - A

Portfolio Diversification

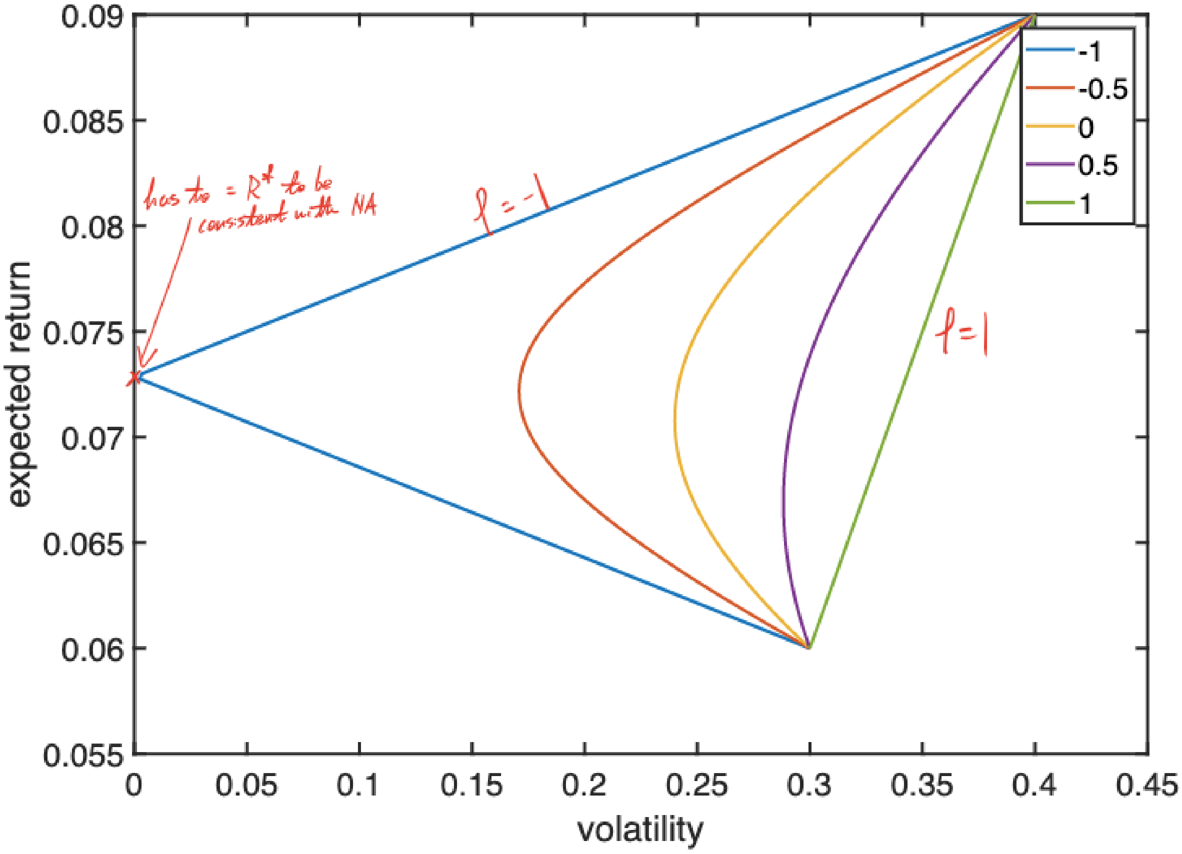
- Portfolio diversification: including many assets that are imperfectly correlated in a portfolio will reduce the portfolio's volatility without sacrificing return proportionally
- Expected return is the weighted average

$$\mu_z = w\mu_x + (1-w)\mu_y$$

- Variance is smaller than the weighted average

$$\begin{aligned}\sigma_z^2 &= w^2\sigma_x^2 + (1-w)^2\sigma_y^2 + 2w(1-w)\rho_{xy}\sigma_x\sigma_y \\ &= [w\sigma_x + (1-w)\sigma_y]^2 + 2w(1-w)\underbrace{(\rho_{xy}-1)}_{\leq 0}\sigma_x\sigma_y \\ &\leq [w\sigma_x + (1-w)\sigma_y]^2\end{aligned}$$

- minimum variance frontier (MVF)



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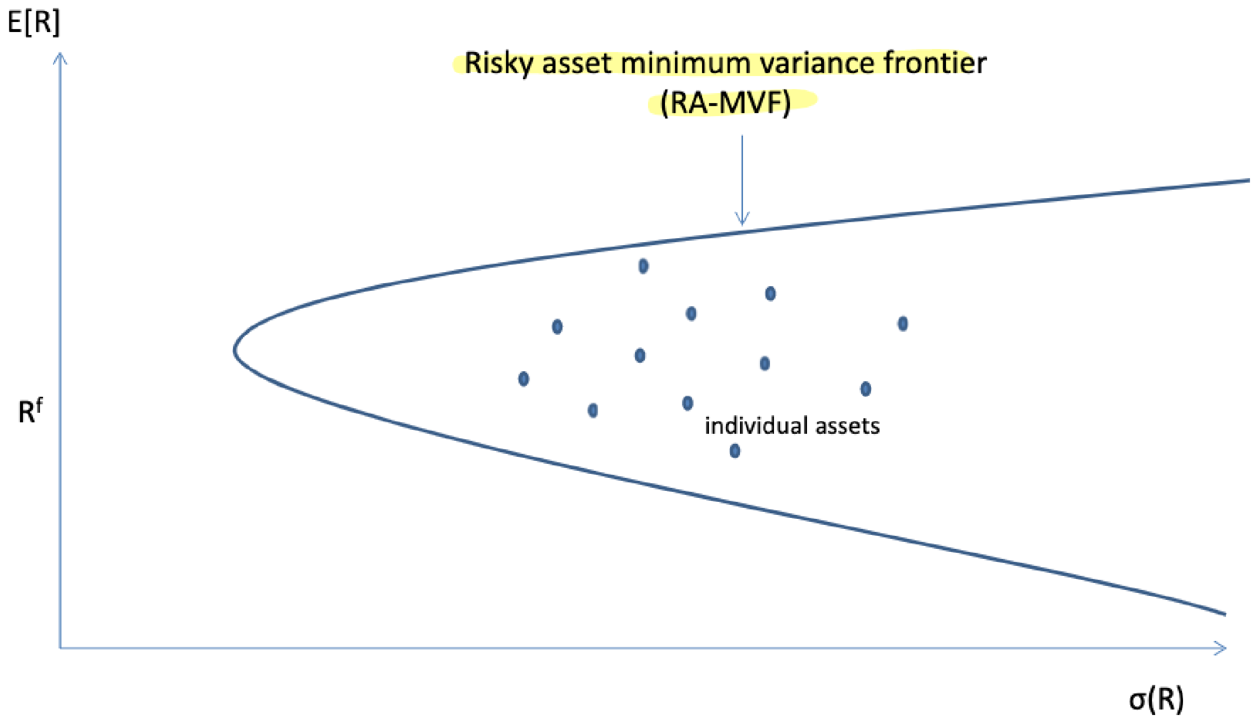
$$
\sigma_z \begin{cases} < \sum_i w_i \sigma_i & , \text{if } \rho \in (-1, 1) \\ = \frac{1}{N} \sqrt{\sum_i \sigma_i^2} & , \text{if } \rho = 0 \\ = \frac{1}{\sqrt{N}} \sigma & , \text{if } \rho = 0, \sigma_i = \sigma \\ = \sum_i w_i \sigma_i & , \text{if } \rho = 1 \\ = 0 & , \text{if } \rho = -1 \end{cases}
$$

RA-MVF (Risky Asset - Minimum Variance Frontier)

- Risky asset minimum variance frontier (RA-MVF)



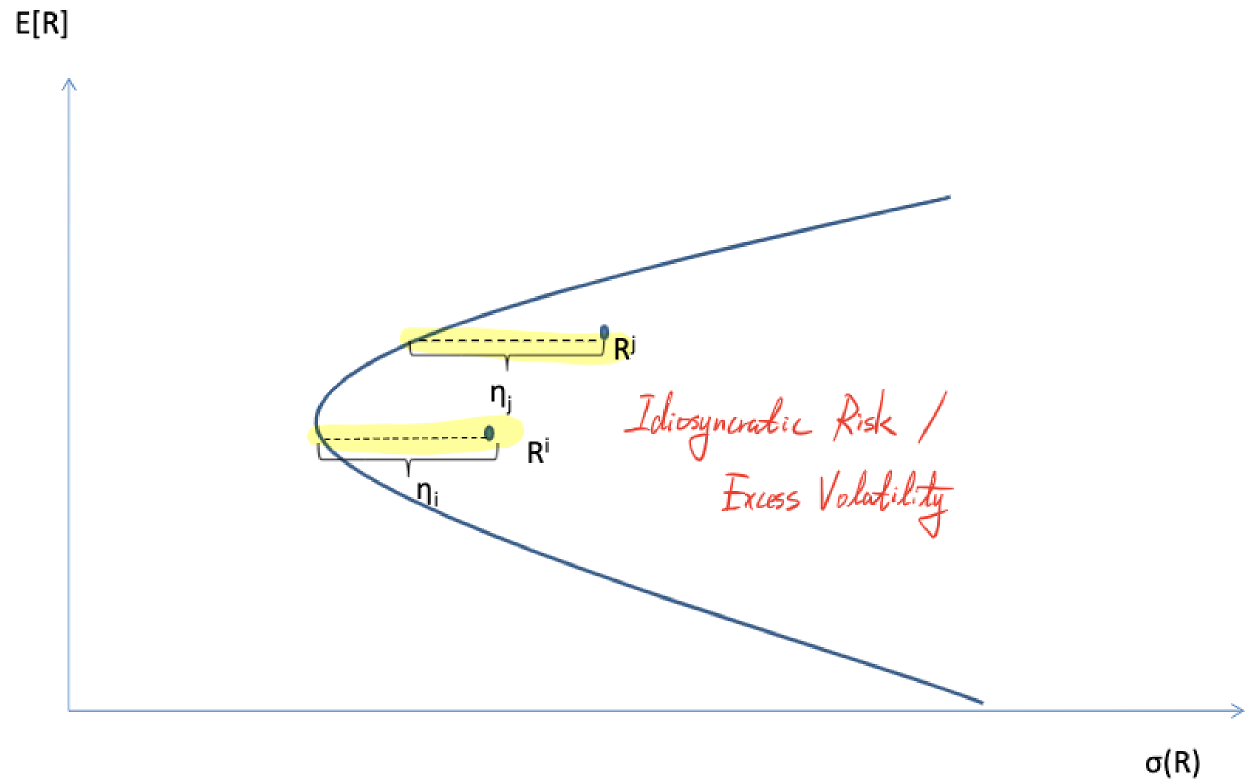
- Weights solving the RA-MVF is linear on the target expected return, so we can retrieve the whole RA-MVF given any two points on it
  - *given 2 portfolios on RA-MVF, we can retrieve the whole curve by just changing their weights because they have no diversifiable risk*
- RA-MVF (2 Assets)

$$w_1 = a + b\bar{\mu}$$

where $a = \frac{\mu_2}{\mu_1 - \mu_2}, a = \frac{1}{\mu_1 - \mu_2}$ and $\bar{\mu}$ is the given return level

Diversifiable Risk

- Diversifiable/Idiosyncratic Risk: can be diversified (excess volatility of any return relative to the point on the RA-MVF with same expected return)

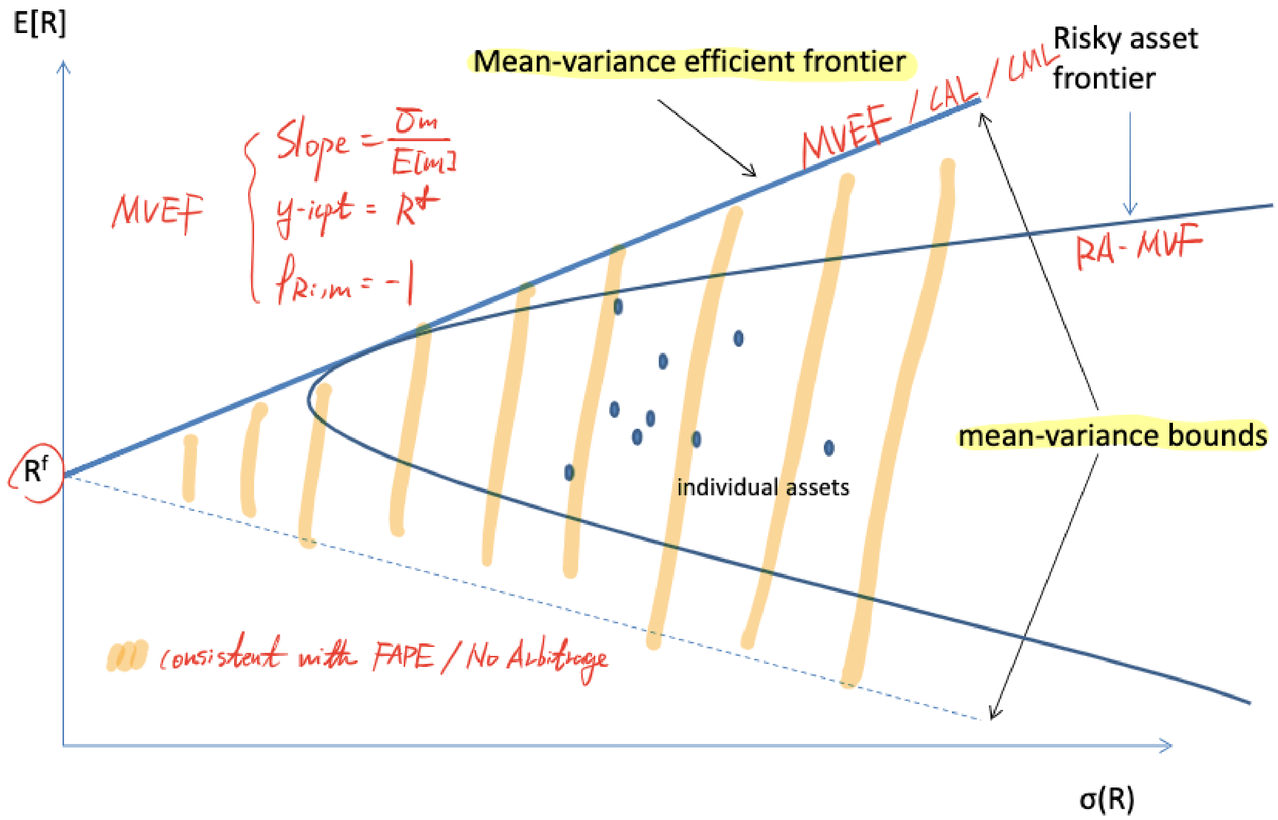


- The term "Diversification" only applies to idiosyncratic risks (not correlated with the SDF and hence not priced), systematic risks cannot be simply diversified out.

MVEF (Mean Variance Efficient Frontier)

FAPE -> MVEF

- Mean-variance efficient frontier: FAPE meets RA-MVF: the maximum return we can achieve (consistent with FAPE/NA) for a given level of volatility



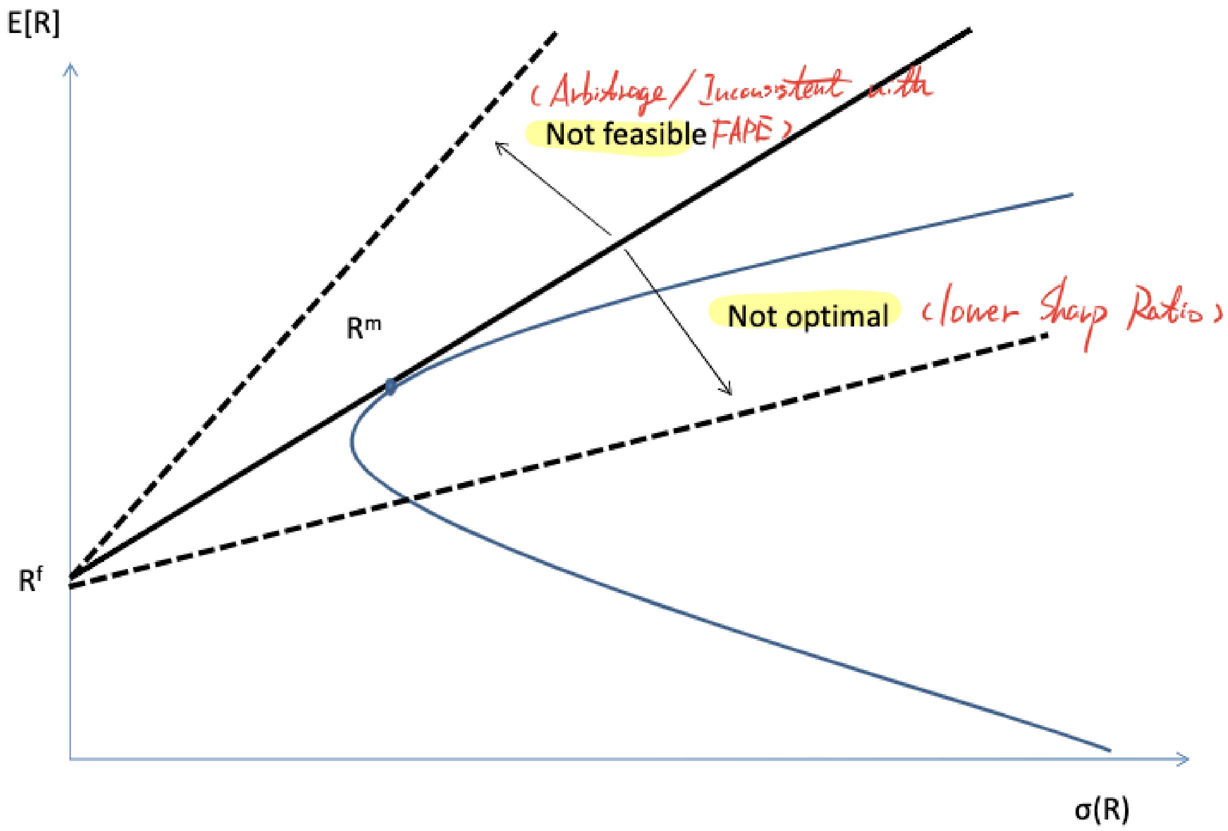
- Unconditional FAPE -> Bounds for Returns (MVEF)

$$R^f - \frac{\sigma_m}{E[m]}\sigma_{R_i} \leq \underbrace{E[R_i] \leq R^f + \frac{\sigma_m}{E[m]}\sigma_{R_i}}_{\text{MVEF/CAL/CML}}$$

- Points on MVEF has correlation with SDF = -1, and hence their β's are proportional to their σ: $\beta_{[R_i,m]} = \frac{Cov(R_i,m)}{Var(m)} = \underbrace{\rho_{R_i,m}}_{=-1}\sigma_{R_i}\sigma_m^{-1}$

RA-MVF and MVEF

- From RA-MVF to MVEF



- *Maintain FPF, above MVEF - not feasible, below MVEF - not optimal*
- This tangent line to the RA-MVF with intercept equal to the risk-free rate will be the line defining the *maximum achievable expected return for a given level of volatility*. Ignoring technicalities arise with market incompleteness, this will coincide with the MVEF.
- Combining risk-free and risky assets:

$$\begin{cases} \mu_z = R^f + w(\mu_x - R^f) \\ \sigma_z = w\sigma_x \end{cases}$$

where x is portfolio weight of risky asset x (percentage of wealth spent on x)

Tangent Portfolio and Beta Representation

- Tangent portfolio and beta representation

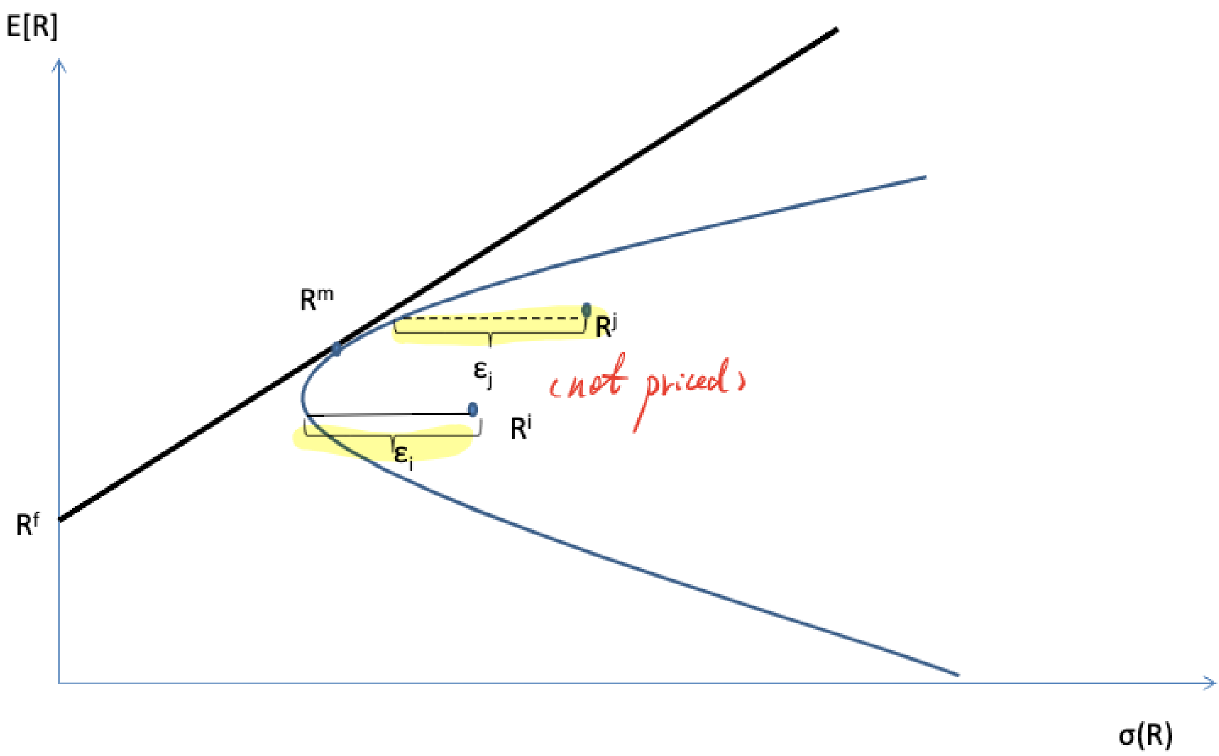
- Because R^m lies on MVEF, it must be perfectly negatively correlated with the SDF, then we can write SDF as $m = a + bR^m, b < 0$
- Using this, we can use its Beta Representation (also know as *Security Market Line (SML)*)

$$\rho_{R^m,m} = -1 \implies m = a + bR^m \implies \underbrace{E[R_i] = R^f + \beta_{[R_i,R^m]}\left\{E[R^m] - R^f\right\}}_{\text{Security Market Line (SML)/CAPM}}$$

and idiosyncratic (excess/diversifiable) volatility uncorrelated with R^m has zero beta (not priced)

- We can think about the idiosyncratic component as ϵ_i such that:

$$R_i^e = \beta_{[R_i,R^m]}\left\{E[R^m] - R^f\right\} + \epsilon_i,\ E[\epsilon_i] = Cov[m, \epsilon_i] = Cov[R^m, \epsilon_i] = 0$$



Mean-Variance Approach

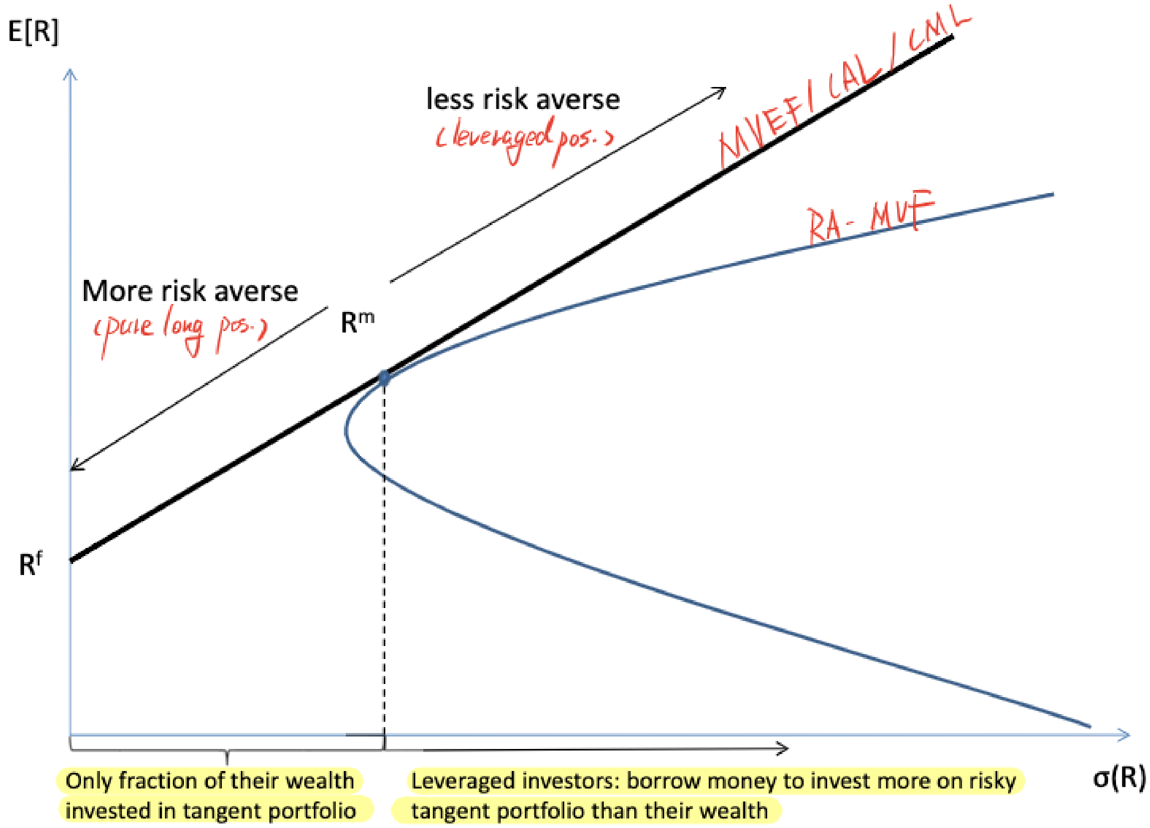
- Mean-variance approach

M-V Approach / CAPM / TFST Assumptions:

1. *Returns are normally distributed or* investors have *quadratic (aka. mean-variance) preferences*
   - $\implies$ Variance/SD gives a full description of the asset's uncertainty
2. *Homogeneous beliefs* (no asymmetric information)
3. All sources of wealth uncertainty are traded assets (*complete markets*)
4. Maintained assumption: *Free portfolio formation*

- Caveats:
  - We can construct MVEF/RAMVF freely without assumption1&2, but to use them as a basis for asset pricing, we need them
  - To use M-V analysis (looking at MVEF/RAMVF) for individual investors, we only need assumption 1

Two-Fund Separation Theorem (TFST)

- Two-fund separation theorem
- (with 4 assumptions above) every investor's portfolio will be a combination of the same two assets, the risk-free asset and the tangent RA-MVF portfolio (tangent portfolio / risky market portfolio)
  - Different risk preferences translate into different points along the MVEF. More risk averse agents will have lower proportions of their wealth in the risky market portfolio, v.v.
  - 

Capital Asset Pricing Model (CAPM)

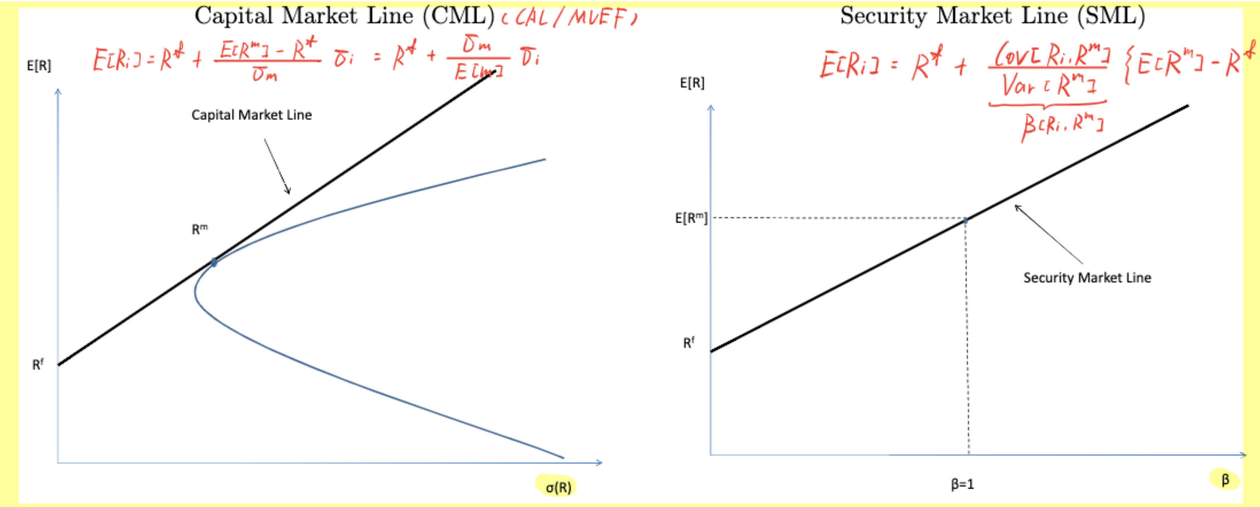
- Capital Asset Pricing Model (CAPM)
- (with 4 assumptions above) TFST holds, and the market portfolio is the (market value weighted) aggregate wealth portfolio including all traded risky assets, which is by assumption all wealth
- CAPM jargon
- *For investors' position*: MVEF / Capital Allocation Line CAL / Capital Market Line CML in CAPM

$$MVEF/CAL/CML : E[R_i] = R^f + \frac{E[R^m] - R^f}{\sigma_{R^m}}\sigma_i = R^f + \frac{\sigma_m}{E[m]}\sigma_i$$

and $\rho_{R_i,m} = -1$ on MVEF

- *For individual assets*: Tangent Portfolio -> SDF -> β Representation / Security Market Line SML

$$\rho_{R^m,m} = -1 \implies m = a + bR^m \implies \underbrace{E[R_i] = R^f + \beta_{[R_i,R^m]}\left\{E[R^m] - R^f\right\}}_{\text{Security Market Line (SML)/CAPM}}$$

- 

CAPM Implications

- Performance evaluation with the CAPM
- Alpha in CAPM

$$E[R_i^e] = E[R_i] - R^f = \alpha_i + \beta_{[R_i,R^m]}\left\{E[R^m] - R^f\right\} = \alpha_i + \beta_{[R_i,R^m]}E[R^{e,m}]$$

  - Positive α_i: underprice
- Using CAPM to value cash-flows: using SML:

$$p_x = \frac{E[x]}{E[R_x]} = \frac{E[x]}{R^f + \underbrace{\beta_{[R_x,R^m]}(E[R^m] - R^f)}_{RP(x,m)}}$$

the denominator $R^{m,x} = R^f + RP(x,m)$ can be viewed as a risk-adjusted discount rate

Mean-Variance Approach vs. SDF

- Mean-variance vs. SDF
- Drawbacks of M-V Approach:
  - Asymmetric information and disagreement
  - Market incompleteness (e.g. imperfect insurance of labour income, fundamental non-contractable states)
  - Quadratic preference implies an ARA increasing with wealth

    - Normality assumption cannot hold for multiple time periods and at odds with data
    - Hard to generalise to multiperiods
- Addition: using stock index as a proxy ignores non-traded assets and a many traded assets (e.g. bonds)

Week 8: Derivatives I: Pricing Forwards and Options

Lecture Slides

- 08 LS [Derivatives I_ Pricing Forwards and Options] - A
    - Definitions
    - Notation and Terminology
    - Payoffs
    - No-arbitrage bounds
    - Forwards pricing
        - Model-free Bounds on Option Prices
        - Put-Call Parity
    - Risk-neutral valuation
    - Binomial Trees
        - generalize
        - FAPE/No-Arbitrage ensures $p \in (0, 1)$
        - what happens to option prices if riskpremia change?
        - Two-step binomial tree
        - N-step binomial tree
    - Black & Scholes Formula
    - Intuition and Implications
        - Intuition for no-arbitrage approach
        - Power of Risk-Neutral
        - Replicating Strategies
    - Option Price from FAPE
    - Leveraged bets through options
    - Importance of underlying volatility
- 08 LS L Derivatives I_ Pricing Forwards and Options - A
    - What are derivatives?
    - FAPE applied to derivatives
        - convention for Forwards
    - Two simple applications of NA
        - Cost of replicating strategy must equal the derivative price
        - Return of fully hedged portfolio must be the risk-free return

- Uniqueness of derivatives
- Binomial trees as a tool
- Leverage
- Volatility
- PS4 2020
- Exercises Lecture Notes 8

Lecture Notes 08 LN Derivatives I_ Pricing Forwards and Options - A

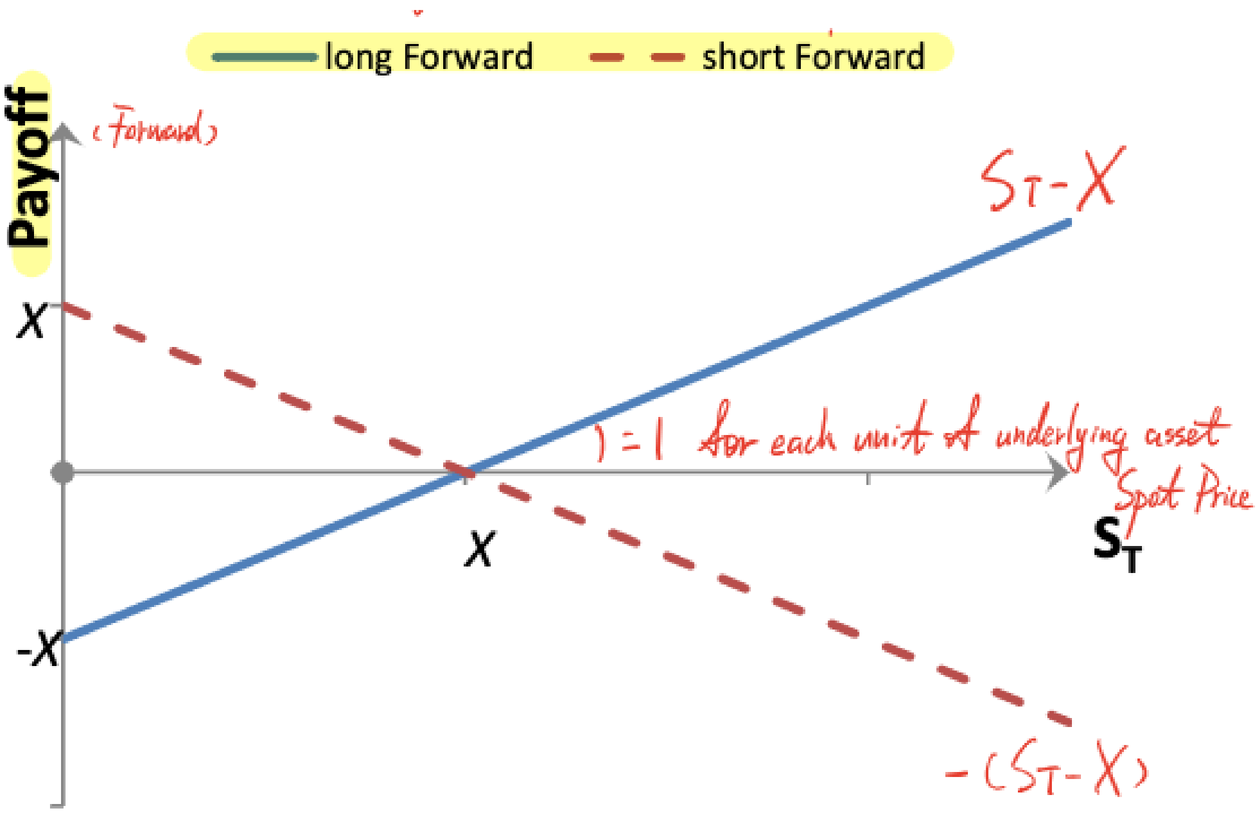
Introduction, Notations, Terminology

- Derivatives
- Notation
- Forwards: agreements (*commitments*) made today to buy (or sell) an asset at a future date for a known price $X = F_0$ (Convention: at inception: $Cost = 0$)
- Options: agreements made today to give the buyer of the option the *right* to buy (call option) or sell (put option) an asset at a future date T for a known strike price X
    - Call/Put price C, P, strike price X, spot price of underlying asset S_t
- Terminology and convention: in, at, out of the money (comparing S_t and X)

Payoffs

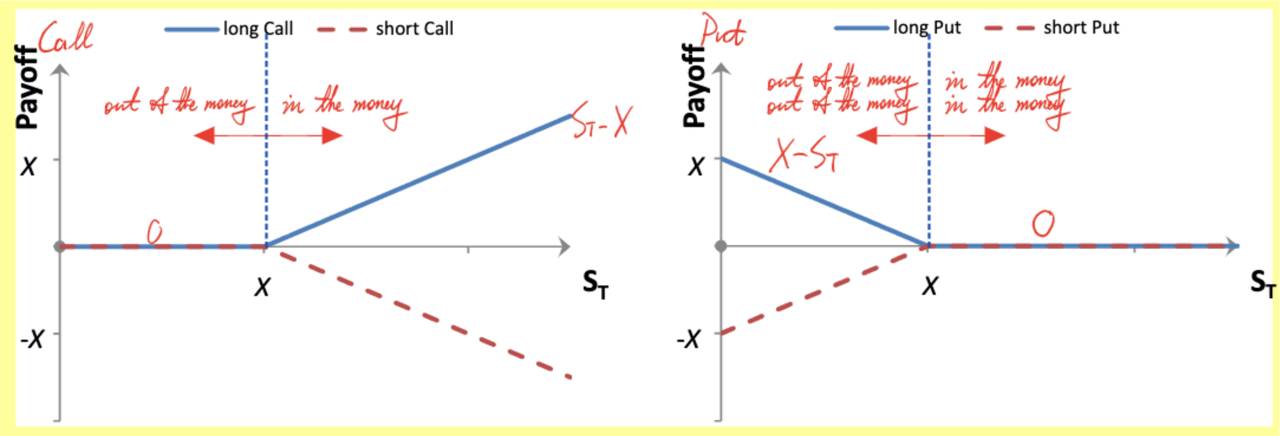
- Payoff of Forwards (= Profit of Forwards because initial price is 0 by convention)

$$Payoff = \begin{cases} S_T - X(or\ F_0) & \text{, for long Forwards} \\ X(or\ F_0) - S_T & \text{, for short Forwards} \end{cases}$$



- Payoff of Options

$$Payoff = \begin{cases} \max\{S_T - X, 0\} & \text{, for Long Call Options} \\ \max\{X - S_T, 0\} & \text{, for Long Put Options} \end{cases}$$



- Profit of Options

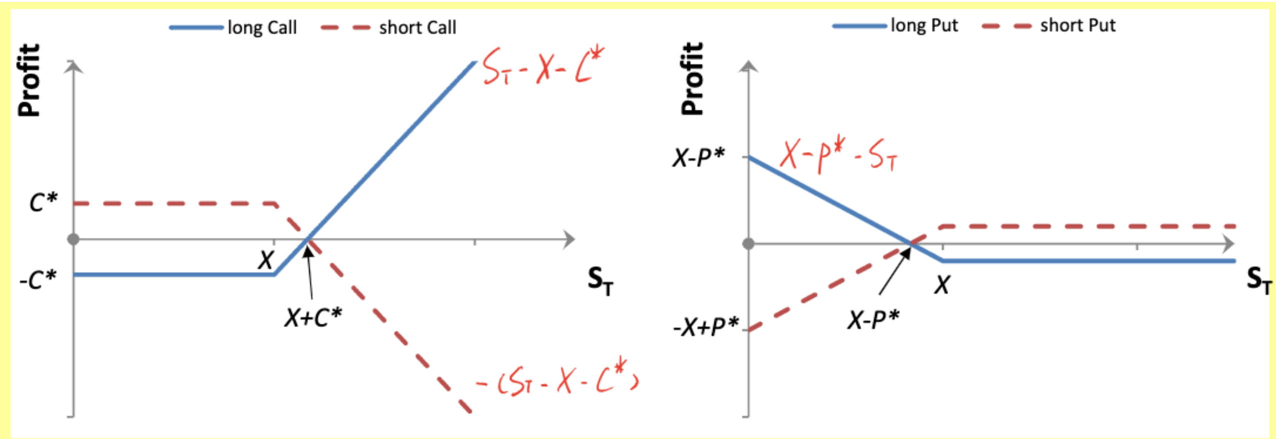
$$Profit = \begin{cases} \max\{S_T - X, 0\} - C^* & \text{, for Long Call Options} \\ \max\{X - S_T, 0\} - P^* & \text{, for Long Put Options} \end{cases}$$

- Call

$$Profit = \begin{cases} \max\{S_T - X, 0\} - C^* & \text{, for Long Call Options} \\ C^* - \max\{S_T - X, 0\} & \text{, for Short Call Options} \end{cases}$$

- Put

$$Profit = \begin{cases} \max\{X - S_T, 0\} - P^* & \text{, for Long Put Options} \\ P^* - \max\{X - S_T, 0\} & \text{, for Short Put Options} \end{cases}$$



No-Arbitrage Bounds

- No-arbitrage bounds for derivatives

NA Bounds / Price for Forwards

- Forward Prices (at inception $X = F_0$)

$$X = F_0 = \underbrace{E_0^Q[S_T] = e^{r^f T} S_0}_{\text{Risk-Neutral Expectation of Asset Price}}$$

with *(static) replication strategy*: borrow $F_0 e^{-r^f T}$ and buy 1 underlying asset at price S_0:

$$\begin{cases} \text{Initial cashflow:} & F_0 e^{-r^f T} - S_0 = 0 \\ \text{Payoff at T:} & S_T - F_0 = \text{Payoff of the forward} \end{cases}$$

- or direct pricing with FAPE:

$$\underbrace{p_{f(F_t,T),t} = 0 \rightarrow \frac{E_t^Q[S_T - F_t]}{R_{T-t,t}^f} = 0 \rightarrow F_t = E_t^Q[S_T]}_{\text{Convention \& FAPE}} \underbrace{= R_{T-t}^f S_t = e^{r^f(T-t)} S_t}_{\text{R-N FAPE}}$$

- Interim Forward Price ($X \neq F_t$) （*好像不太对......*）

$$(F_t - X)e^{-r^f(T-t)}$$

where $F_t = S_t e^{r^f(T-t)}$ is the price of an incipient forward with $X = F_t$ at t

NA Bounds for Options

- No Arbitrage Bounds for Call Options

$$\max\left\{S_0 - Xe^{-r^f T}, 0\right\} \leq C_0 \leq S_0$$

- No Arbitrage Bounds for Put Options

$$\max\left\{Xe^{-r^f T} - S_0, 0\right\} \leq P_0 \leq Xe^{-r^f T}$$

Put-Call Parity

- *Put-Call Parity*

$$P_0 + S_0 = C_0 + Xe^{-r^f T}$$

  - Knowledge of $S_0 \iff P_0$

Risk-Neutral Valuation: Binomial Tress and Black & Scholes Formula

- Idea: Delta Hedging: hold 1 unit option and Δ unit of underlying asset to construct a risk-free portfolio (Dynamic replication of options), then we can discount the payoff with risk-free rate

1-Step Binomial Trees

- Option Pricing using 1-Step Binomial Tree Binomial trees:
  - *Risk-free portfolio*: 1 unit of option + Δ unit of underlying asset

- ensure $\Delta S_0 u + f_u = \Delta S_0 d + f_d \implies \Delta = \frac{f_d - f_u}{S_0(u-d)}$
- Pricing:

$$\begin{aligned} f_0 &= -\Delta S_0 + e^{-r^f T}(\Delta S_0 u + f_u) \text{ with } \Delta = \frac{f_d - f_u}{S_0(u-d)} \\ &= e^{-r^f T}\Big[(1-p)f_d + p f_u\Big] \text{ with } p = \frac{e^{r^f T} - d}{u-d} \end{aligned}$$

- where: $p = \frac{e^{r^f T} - d}{u-d} \in (0,1)$ are valid risk-neutral probabilities
- f_u, f_d are the payoffs/values of the option if the price of underlying asset goes up/down

Multi-Step Binomial Trees

- Option Pricing using 2-Step Binomial Tree Two-period binomial tree

$$\begin{aligned} f_0 &= e^{-r^f \frac{T}{2}}\left[(1-p)f_d + f_u\right] \\ &= e^{-r^f \frac{T}{2}}\Big\{(1-p)\underbrace{\Big[e^{-r^f \frac{T}{2}}\big((1-p)f_{dd} + p f_{ud}\big)\Big]}_{f_d} + p\underbrace{\Big[e^{-r^f \frac{T}{2}}\big((1-p)f_{ud} + p f_{uu}\big)\Big]}_{f_u}\Big\} \\ &= e^{-r^f T}\Big[(1-p)^2 f_{dd} + 2p(1-p)f_{ud} + p^2 f_{uu}\Big] \end{aligned}$$

  where $p = \frac{e^{r^f \frac{T}{2}} - d}{u-d}$ and $\frac{T}{2}$ is the period length

- Option Pricing using N-Step Binomial Tree N-period tree

$$f_0 = e^{-r^f T}\left[\sum_{n=0}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} f_{u^n d^{N-n}}\right]$$

  where

$$p = \frac{e^{r^f \frac{T}{N}} - d}{u-d}, u = e^{\sigma\sqrt{\frac{T}{N}}}, d = \frac{1}{u}, f_{u^n d^{N-n}} = \begin{cases} \max\{S_0 u^n d^{N-n} - X, 0\} & \text{, for a Call option} \\ \max\{X - S_0 u^n d^{N-n}, 0\} & \text{, for a Put option} \end{cases}$$

  and $\frac{T}{N}$ is the period length

  - If log returns have volatility σ per unit of time, then using:

  1. *Risk-neutral Probability Adjustment* $p = \frac{e^{r^f \frac{T}{N}} - d}{u-d}$ (due to changing compounding period)

  2. *Magnitude of u/d (Step Size) Adjustment* $u = e^{\sigma\sqrt{\frac{T}{N}}}, d = \frac{1}{u}$ (to confine volatility): as $N \to \infty$, the asset price distribution approaches a Lognormal Distribution with volatility σ

  - Delta has the same expression $\Delta = \frac{f_d - f_u}{S_t(u-d)}$, but its magnitude changes as u, d change

Caveats

- Note that in binomial trees, $f_{u^m d^n}$ are payoffs of options in the final node / value of options in intermediate nodes
- IR compounding periods need to adjust flexibly according to specifications
- probabilities calculated using replication strategies will be consistent with risk-neutral probabilities if FAPE holds

- always use CCIR here
- always draw those trees when using them

Black & Scholes Formula

- Black & Scholes Model
- Black & Scholes Formula (limit of the binomial tree model $N \to \infty$ with $u = e^{\sigma\sqrt{\frac{T}{N}}}$)

$$\begin{cases} C_0 = S_0 N(d_1) - X e^{-r^f T} N(d_2) & \text{, for Call Options} \\ P_0 = X e^{-r^f T} N(-d_2) - S_0 N(-d_1) & \text{, for Put Options} \end{cases}$$

where $N(.) = \Phi(.)$ is the CDF of $\mathcal{N}(0,1)$ and

$$\begin{cases} d_1 = \frac{\ln(\frac{S_0}{X}) + \left(r^f + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 = \frac{\ln(\frac{S_0}{X}) + \left(r^f - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{cases}$$

Intuition for No-Arbitrage Approach

- Intuition for no-arbitrage approach
- How can option prices do not depend on actual probability of spot price going up/down?
    - All about *relative pricing*: RP, SDF are included in the absolute spot price of underlying assets
- Directly apply R-N FAPE to options

Leveraged Bets through Options

- Leveraged bets through options
- Tradeoff:
    - Upside: options allow the investor to earn returns that much higher if the asset price goes to the desirable direction
    - Downside: a modest movement towards the opposite direct cause $r = -100\%$

Role of Underlying Volatility

- Importance of underlying volatility: *higher volatility of the underlying asset always push up price for options*: one side of the payoff distribution is cut off (the undesirable tail always has $r = -100\%$) -- an increase in the dispersion of the future asset price enhance its payoff on the upside without the offsetting effect from the opposite direction.

Increase In	Call Price (C_0)	Put Pice (P_0)
Risk-Free Rate ($r_t^f \uparrow$)	+	-
Spot Price ($S_0 \uparrow$)	+	-
Strike Price ($X \uparrow$)	-	+

Increase In	Call Price (C_0)	Put Pice (P_0)
Volatility ($\sigma \uparrow$)	+	+
Time to Maturity ($T \uparrow$)	Uncertain	Uncertain

Key Points of Replicating Forwards and Options

- Replication of forwards is static (parallel shift of spot payoff); Replication of options is dynamic. Thus,
    - we can easily see the replication strategy for forwards, but not for options
    - volatility matters for options but not forwards
- The aim of replication is to retrieve a risk-free portfolio, so
    - the expected return is risk-free rate
    - risk-free rate is the appropriate discount rate for all cash flows
    - so we only need to know risk-free rate (and volatility for options), but not the actual expected change

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Week 9: Derivative II: Applications

Lecture Slides

Lecture Notes Derivative Pricing: Applications

Derivatives with Interim Cash Flows

- Assets with Interim Cash Flows

Risk-Free Intuition

- Risk-Neutral Intuition for intermediate payoffs
- With interim payment, we need less return from the derivative itself
    - Known cashflow occurring at CCIR q per period, then we substitute S_0e^{-qT} where we have S_0
    - Know discrete payment with total present value I, then we substitute $S_0 - I$ where we have S_0

Forwards with Interim Cashflows

- Forwards
- *Known discrete cashflow* with present value I
    - Strategy: borrow $Xe^{r_fT} + I$ and buy 1 underlying asset
    - Initial cashflow: $(X + Ie^{r_fT})e^{-r_fT} - S_0$

- Forward price:

$$F = (S_0 - I)e^{r^f T}$$

- *Known yield* (CCIR) q:
  - Forward price:

$$F_0 = S_0 e^{(r^f - q)T}$$

Options with Interim Cashflows

- Options
- Only focus on the case with a *known yield* (CCIR) q
- Put-Call Parity
  - Portfolios:
    - 1 call + cash of PV of the strike $Xe^{-r^f T}$
    - 1 put + e^{-qT} units of the asset
  - Parity:

$$P_0 + S_0 e^{-qT} = C_0 + Xe^{-r^f T}$$

- *Binomial Tree and Black & Scholes*
  - 1-Step Tree
    - Change the Δ to ensure risk-free:

$$\Delta = e^{-qT} \frac{f_u - f_d}{S_0(u - d)}$$

    - Risk-neutral probabilities will be changed accordingly:

$$p = \frac{e^{(r^f - q)T} - d}{u - d}$$

  - Multi-Step Trees and Black & Scholes formula
    - we just replace $S_0 \rightarrow S_0 e^{-qT}$:
      - Risk-neutral Probability Adjustment:

$$p = \frac{e^{(r^f - q)\frac{T}{N}} - d}{u - d}$$

      - Same Step Size Adjustment:

$$u = e^{\sigma\sqrt{\frac{T}{N}}}, d = \frac{1}{u}$$

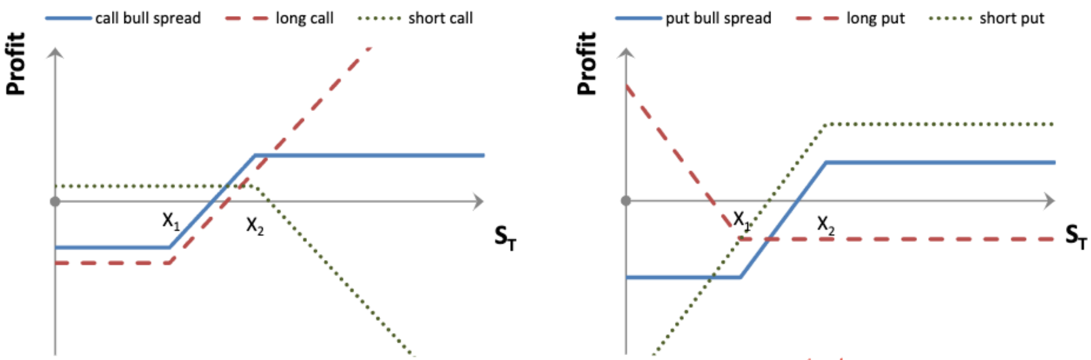
- Expression for Delta changes here:

$$\Delta = e^{-q\frac{T}{N}} \frac{f_u - f_d}{S_t(u-d)}$$

Option Trading Strategies

- Key No-arbitrage implication: the profit line as a function of S_T cannot be either non-negative or non-positive (because the profit line represents the payoff of a 0-cost strategy)

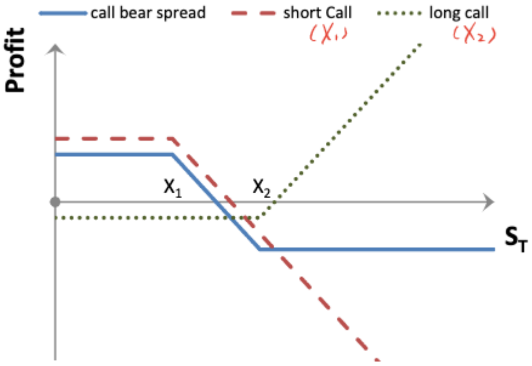
Spreads

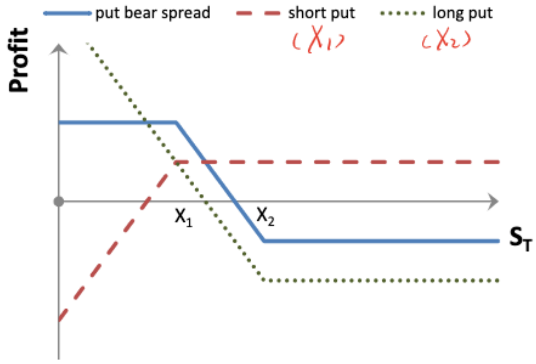
- Bull and Bear spreads
    - Bull spread:
        - Bet on higher price, but limit up/down side and cost less than buying the asset
        - *With call*: Buy 1 call with low strike price X_1 and sell 1 call with high strike price X_2
            - Cost (in future value): $C = e^{r_f T}\Big[C_0(X_1) - C_0(X_2)\Big]$
            - No-arbitrage: $X_2 - X_1 > C$
        - *With put*: Buy 1 put with low strike price X_1 and sell 1 put with high strike price X_2
        - 

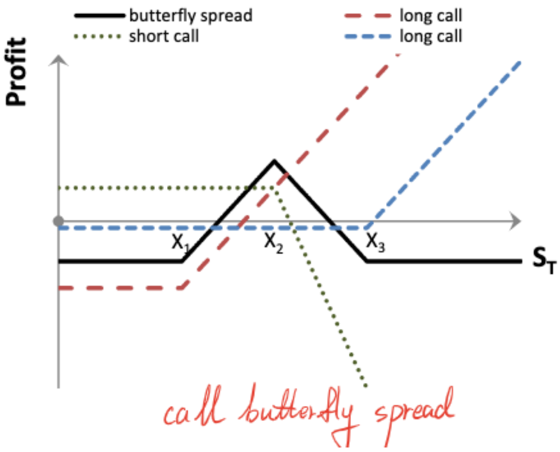
call bull spread   put bull spread

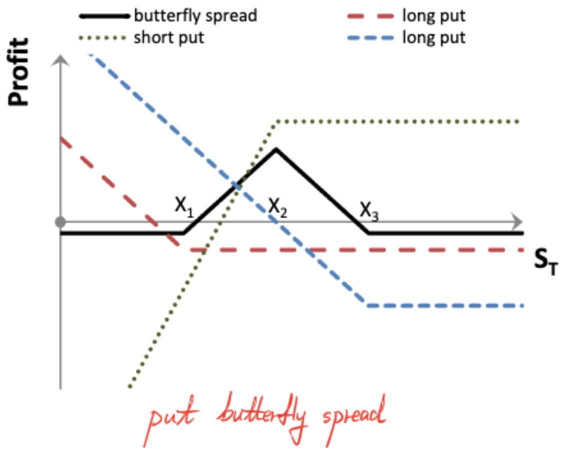
	Underlying asset price S_T		
	none exercised	X_1 exercised	both exercised
	$S_T \le X_1$	$X_1 < S_T < X_2$	$S_T \ge X_2$
long call strike price X_1	0	$S_T - X_1$	$S_T - X_1$
short call strike price X_2	0	0	$X_2 - S_T$
Combined payoff	0	$S_T - X_1$	$X_2 - X_1$
Combined profit	$-C$	$S_T - X_1 - C$	$X_2 - X_1 - C$

- Bear spread:

- Inverse of bull spread
- *With call*: Buy 1 call with high strike price X_2 and sell 1 call with low strike price X_1
- *With put*: Buy 1 put with high strike price X_2 and sell 1 put with low strike price X_1
- 



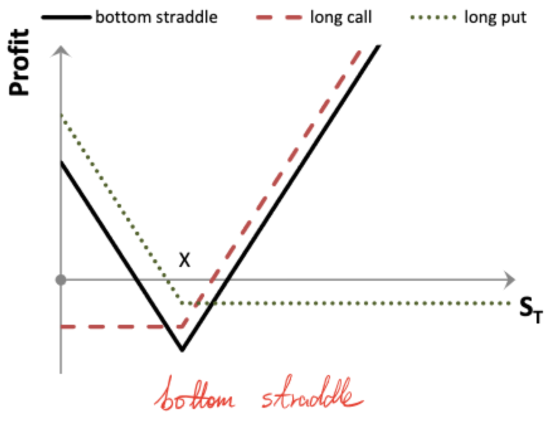
- Butterfly spreads
  - Bet on price staying close to a given price
  - *With call*: Buy 1 call with low strike price X_1, buy 1 call with high strike price X_3, and sell 2 call with intermediate strike price X_2
    - Cost (in future value): $C = e^{r_f T}\Big[C_0(X_1) - 2C_0(X_2) + C_0(X_3)\Big]$
    - Assume $X_2 \geq \frac{X_1+X_3}{2}$, no-arbitrage implies: $C_0(X_2) \leq \frac{C_0(X_1)+C_0(X_3)}{2}$
  - *With put*: Buy 1 put with low strike price X_1, buy 1 put with high strike price X_3, and sell 2 put with intermediate strike price X_2
    - Assume $X_2 \geq \frac{X_1+X_3}{2}$, no-arbitrage implies: $P_0(X_2) \leq \frac{P_0(X_1)+P_0(X_3)}{2}$
  - 

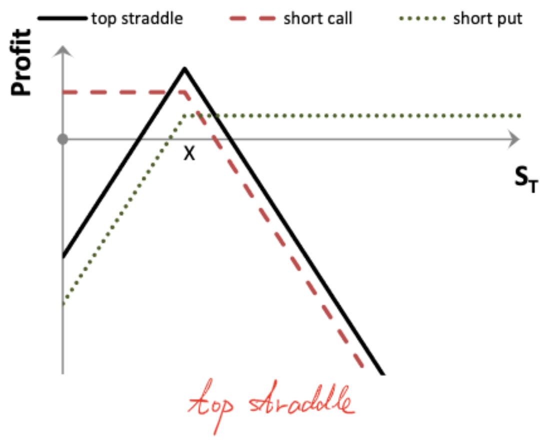


Combinations

- Straddles
  - *Bottom straddle*
    - Bet on price stays out of a range (volatility trade), no limit on the upside
    - Buy 1 call and 1 put with the same strike price X
  - *Top straddle*
    - Bet on price stays in a range (volatility trade), no limit on the downside

        - Short 1 call and 1 put with the same strike price X





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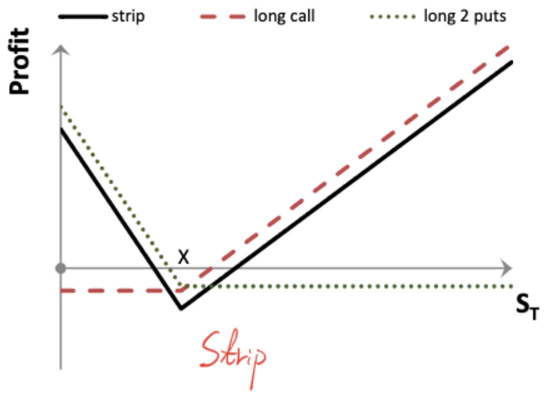
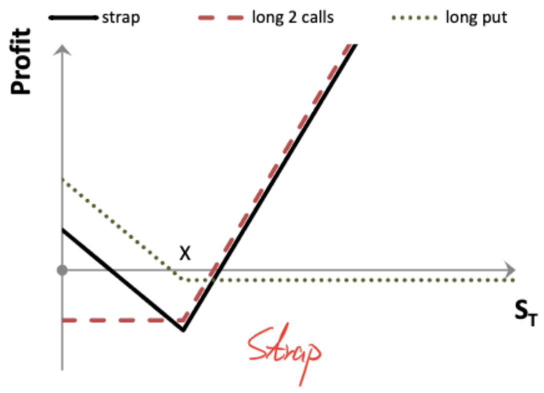
- Strips and Straps
    - Volatility + Skewness trade
    - *Strip*
        - Buy 1 call and short 2 put with the same strike price X
    - *Strap*
        - Buy 2 call and short 1 put with the same strike price X





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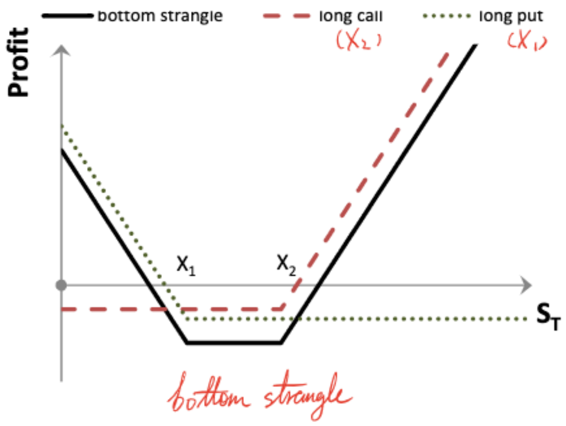
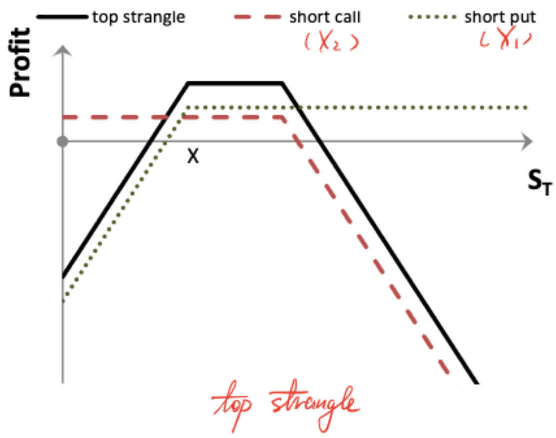
- Strangle
    - Similar to straddle or butterfly spreads, but increases the price range necessary for a gain/loss
    - *Bottom strangle*
        - Buy 1 call with high strike price X_2 and 1 put with low strike price X_1
    - *Top strangle*
        - Short 1 call with high strike price X_2 and 1 put with low strike price X_1

- 



Applications

Completing the Market with Options

- Completing the market with options
- If an asset has different payoffs in different states, we can always use options on it to complete the market
- But this cannot help with states that are not verifiable

Pricing the Equity of A Leveraged Firm

- Pricing the equity of a firm
- A firm has debt D, and value of assets S_t
- Due to limited liability, equity holders' payoff will be:

$$\max\{S_T - D, 0\}$$

- This is equivalent to holding a call option on the firm's asset with strike price D and maturity T.
- We can price the value of equities (equals to C_0) with put-call parity:

$$C_0 = P_0 + S_0 - De^{-r^f T}$$

  - Owning a firm is equal to: 1. own the assets S_0, 2. own a put option bought from the debt holders, 3. borrow the present value of debt
  - Value of debt: $De^{-r^f T} - P_0 = S_0 - C_0$
  - Interest rate charged: $r^d = \frac{1}{T}\ln(\frac{D}{S_0 - C_0})$
  - Agency problem; Gamble for resurrection

Real Options

- Real options
- Right (not obligation) to do something always has a positive price

Bubbles and Heterogeneity

- Bubbles and heterogeneity
- With short-selling constraints, if pessimism/optimism changes over time, price will be the combination of "fundamental value" + an option value captures the options of reselling to someone even more optimistic in the future

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Week 10: FAPE Review: Intuition, Implication, Generality and Extensions

Lecture Slides

- 10 LS Review - A (Review)
- 10 LS L Review - A (Past paper)