Task 3

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Contents

1 Problem Statement

The goal of Task 3 is to solve the 2-D Navier-Strokes equation for lid-driven cavity flow. The computational domains are (x,y)=[0,1][0,1] and t[0,0.1] and the equations are as;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial t} + \frac{\partial v^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial v^2}{\partial x^2} + \frac{\partial v^2}{\partial y^2}\right) \tag{3}$$

with Initial and boundary conditions as

$$u(x,1,t) = 1,$$
$$u(x,0,t) = u(0,y,t) = u(1,y,t) = 0$$
$$and u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$$

2 Solution : Numerical Scheme

As we have studied, FDM do not follow the conservation principle so here we will be using FVM for obtaining solution numerically. Also, the given problem (Lid-driven cavity flow) is a special case of the Navier-Strokes equation, which can be solved here by Artificial Compressibility. For that, we will consider a staggered grid system because of its advantages.

The ideal solution strategy for this is discretizing momentum equation (2) and [3] to get the value of u$_{i,j}^{n+1} and v_{i,j}^{n+1}$.

With the use of both velocity fields, the pressure field can be calculated.

2.1 Solving x-momentum and y-momentum equation

Rearranging equations (2) and (3) and solving via FVM, we get

1. x-momentum equation as;

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + \left(\left(\frac{u_{i+1,j} + u_{i,j}}{2}\right)^2 - \left(\frac{u_{i,j} + u_{i-1,j}}{2}\right)^2 \Delta y\right)$$

$+ \left(\left(\frac{(u_{i,j+1}+u_{i,j})(v_{i,j+1}+v_{i,j})}{2}\right) - \left(\frac{(u_{i,j-1}+u_{i,j})(v_{i,j-1}+v_{i,j})}{2}\right)\Delta x\right)$

- $\left(\frac{1}{Re}\right)\left(\left(\left(\frac{(u_{i+1,j}+u_{i-1,j})-2u_{i,j}}{\Delta x}\right)\Delta y\right) + \left(\left(\frac{(u_{i,j+1}+u_{i,j-1})-2u_{i,j}}{\Delta y}\right)\Delta x\right)\right)$
$+ \left(\frac{p_{i+1,j}-p_{i,j}}{\Delta x}\right)$ (4)

2. y-momentum equation as;

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} + \left(\left(\frac{v_{i1,j+1} + v_{i,j}}{2} \right)^2 - \left(\frac{v_{i,j} + v_{i,j-1}}{2} \right)^2 \Delta x \right)$$

$+ \left(\left(\frac{(u_{i+1,j}+u_{i,j})(v_{i+1,j}+v_{i,j})}{2} \right) - \left(\frac{(u_{i-1,j}+u_{i,j})(v_{i-1,j}+v_{i,j})}{2} \right) \Delta y \right)$

- $\left(\frac{1}{Re}\right) \left(\left(\left(\frac{(v_{i+1,j}+v_{i-1,j})-2v_{i,j}}{\Delta x} \right) \Delta y \right) + \left(\left(\frac{(v_{i,j+1}+v_{i,j-1})-2v_{i,j}}{\Delta y} \right) \Delta x \right) \right)$
$+ \left(\frac{p_{i,j+1}-p_{i,j}}{\Delta y} \right)$ (5)

i.e. simply we get

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t \left(P + D - C\right) \tag{6}$$

$$v_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t \left(P + D - C\right) \tag{7}$$

where, P is pressure gradient, D is diffusion and C is advection.

2.2 Solving conservation equation

For solving equation via Artificial Compressibility, we need to change (1) as;

$$\frac{1}{\delta} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8}$$

Thus its descretized version is as;

$$\frac{1}{\delta} \left(\frac{p_{i,j}^{n+1} - p_{i,j}^{n}}{\Delta t} \right) + (u_{i+1,j} - u_{i,j})\Delta x + (v_{i+1,j} - v_{i,j}\Delta y) = 0 \tag{9}$$

Thus we can calculate pressure at n+1 interval from using equations (6), (7) and (9), where we have to assume some value for δ.

3 Stability Solution

From (4) and (5) we can say that u and v are functions of x,y and t.

$$u(x, y, t) = v(x, y, t) = G(k)e^{i,k,x}e^{i,l,x}$$

For stability of numerical scheme $|G(k)| \leq 1$ Consider here x-momentum equation only,

$$u_{i,j}^{n} = G(k)e^{ik\Delta x}e^{il\Delta y} \tag{10}$$

$$u_{i,j}^{n+1} = G(k+1)e^{ik\Delta x}e^{il\Delta y} \tag{11}$$

Thus from (4), (10) and (11) we can write, (Assuming eqvidistant grid with $\Delta x = \Delta y = \Delta h$)

$$\frac{G(k+1)e^{ik\Delta x}e^{il\Delta y} - G(k)e^{ik\Delta x}e^{il\Delta y}}{\Delta t} = \frac{G(k)}{h^2 Re}e^{ik\Delta x}e^{il\Delta y}(e^{ik\Delta x}+e^{-ik\Delta x}+e^{il\Delta y}+e^{-il\Delta y}-4)$$

-u$_x G(k)e^{ik\Delta x}e^{il\Delta y}(\frac{e^{ik\Delta x}-e^{-ik\Delta x}}{2h}) - u_y G(k)e^{ik\Delta x}e^{il\Delta y}(\frac{e^{il\Delta y}-e^{-il\Delta y}}{2h}) + p_x$

$\Longrightarrow$

$$\frac{G-1}{\Delta t} = \frac{1}{h^2 Re}(e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta y} + e^{-il\Delta y} - 4)$$

-u$_x G(k)e^{ik\Delta x}e^{il\Delta y}(\frac{e^{ik\Delta x}-e^{-ik\Delta x}}{2h}) - u_y G(k)e^{ik\Delta x}e^{il\Delta y}(\frac{e^{il\Delta y}-e^{-il\Delta y}}{2h}) + p_x$

We know, $cosx = \frac{e_{ix}+e_{-ix}}{2}$ and $sinx = \frac{e_{ix}-e_{-ix}}{2}$

Thus we get, $\Longrightarrow$

$$G(= 1 + \frac{2\Delta t}{h^2 Re}(cos(kh) + cos(lh) - 2) - i\frac{\Delta t}{h}(u_x sin(kh) + u_y sin(lh)) \qquad (12)$$

let $z = \frac{2\Delta t}{h^2 Re}(cos(kh) + cos(lh) - 2)$ and $b = \Delta t(u_x sin(kh) + u_y sin(lh))$

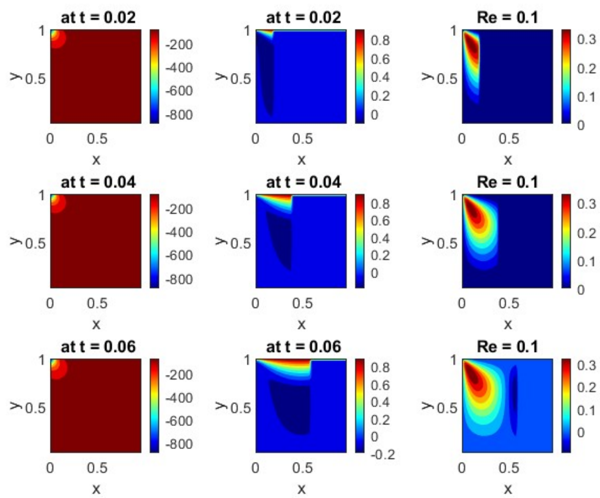
Now calculate $|G|^2$ and solve for Δt :

$\Longrightarrow$ For positive values,

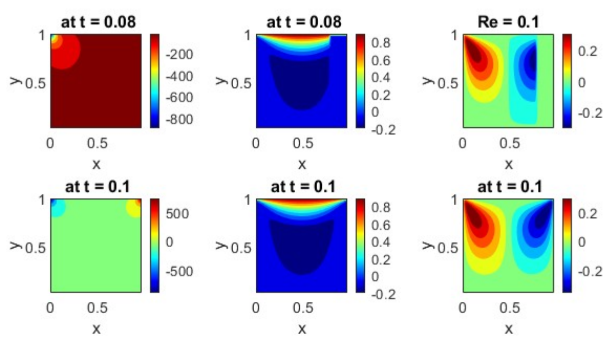
$$\Delta t \leq \frac{h^2 Re}{4} \qquad (13)$$

4 Comparison of plots

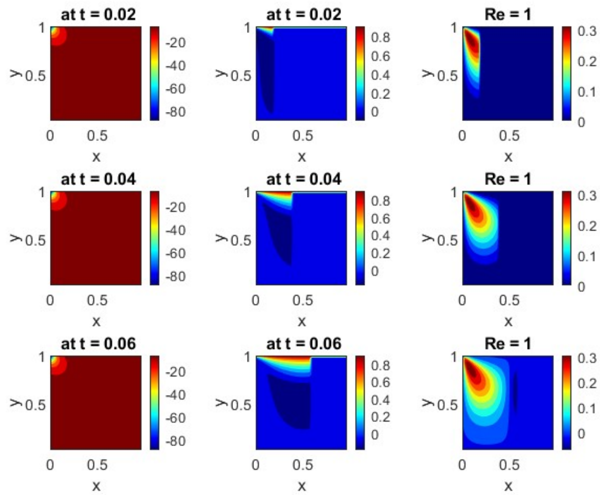
As Reynold's Number increases, the velocity value also increases resulting in increase in vorticity. Also, boundary layer thickness at wall decreases which can be validaed from plots.



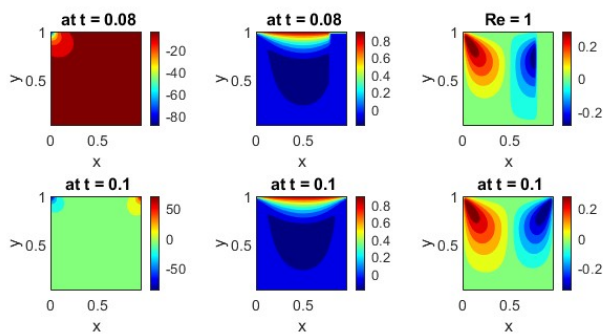
First 3 time steps for Re = 0.1



Last 2 time steps for Re = 0.1



First 3 time steps for Re = 1



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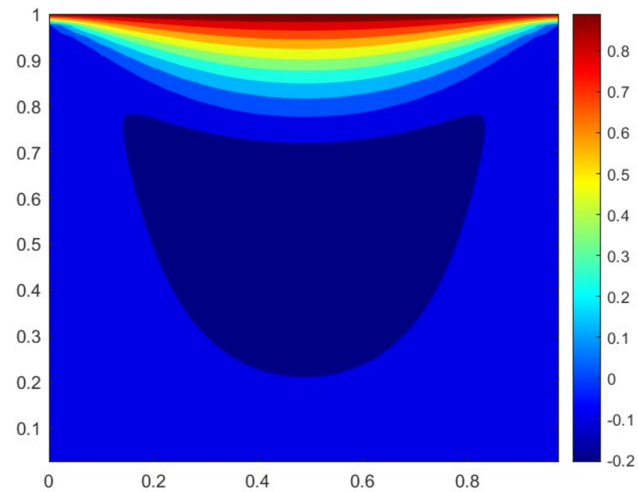
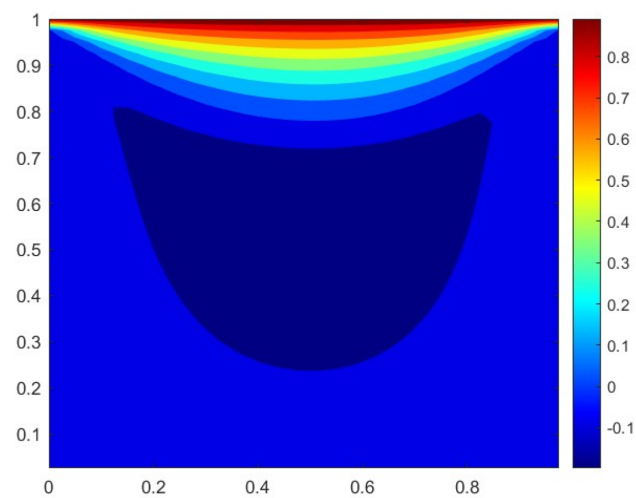


Velocity profile Re = 0.1



Velocity profile Re = 1