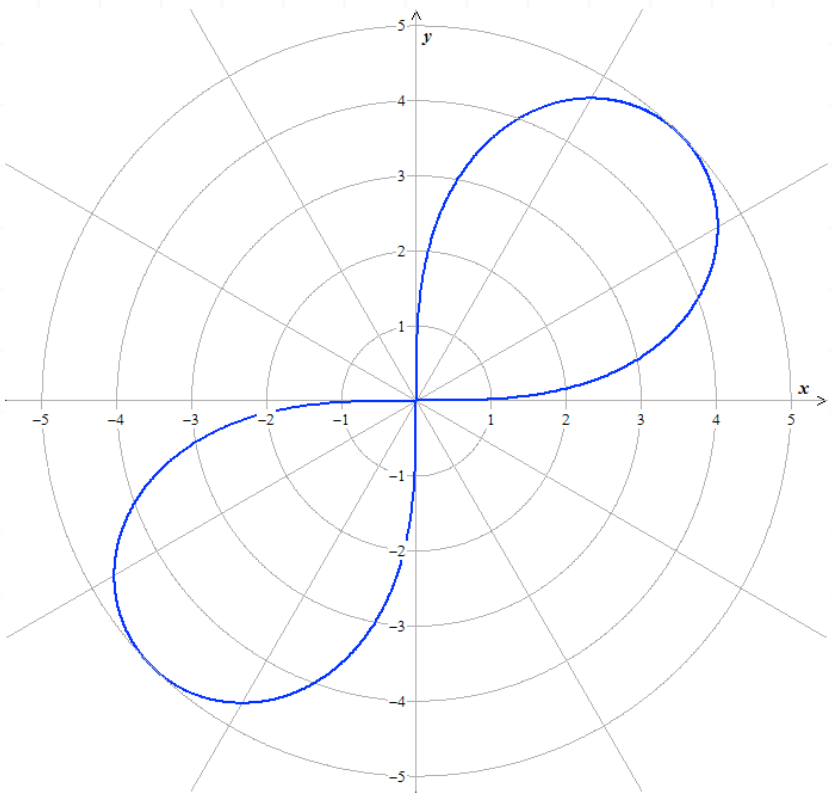
Topic: Graph the polar curve, lemniscate

Question: The following curve is the graph of one of the polar equations given below. Which polar equation is it?



Answer choices:

A $r^2 = 5\cos(2\theta)$

B $r^2 = 25\sin(2\theta)$

C $r^2 = 5\sin(2\theta)$

D $r^2 = 25\cos(2\theta)$

Solution: B

The given curve is symmetric with respect to the pole, but it isn't symmetric with respect to the horizontal axis or the vertical axis, so it's the graph of a "sine" lemniscate. Therefore, we can eliminate answer choices A and B, because they're the polar equations of "cosine" lemniscates, which are symmetric with respect to the horizontal axis and the vertical axis.

Also, the points on this curve that are furthest from the pole are at a distance of 5 units from it. Since $r^2 = 25\sin(2\theta)$ is the polar equation of a "sine" lemniscate with $a = 25$, we know that the points of that lemniscate that are furthest from the pole are a distance of

$$\sqrt{a} = \sqrt{25} = 5$$

units from it. This is indeed true of the given curve.

In the case of the "sine" lemniscate which is the graph of the polar equation given in answer choice C, that is, $r^2 = 5\sin(2\theta)$, we see that $a = 5$, so the points of that lemniscate that are furthest from the pole are at a distance of only

$$\sqrt{a} = \sqrt{5}$$

units from it, which is inconsistent with the given curve.

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Question: One of the following pairs of polar coordinates corresponds to a point of the lemniscate $r^2 = 8\cos(2\theta)$. Which pair of polar coordinates is it?

Answer choices:

A $\left(4, \frac{\pi}{4}\right)$ and $\left(-4, \frac{5\pi}{4}\right)$

B $\left(4, \frac{2\pi}{3}\right)$ and $\left(-4, \frac{5\pi}{3}\right)$

C $\left(2, \frac{5\pi}{6}\right)$ and $\left(-2, \frac{11\pi}{6}\right)$

D $\left(\frac{4}{\sqrt{3}}, \frac{\pi}{8}\right)$ and $\left(-\frac{4}{\sqrt{3}}, \frac{9\pi}{8}\right)$

Solution: C

In each pair of polar coordinates, the first coordinate in the first pair is equal in absolute value (but opposite in sign) to the first coordinate in the second pair. Since the first coordinate is a value of r, the two pairs have the same value of r^2. Also, the angle coordinate in the second pair differs from the angle coordinate in the first pair by π. Thus in each answer choice, the two pairs of polar coordinates apply to the very same point.

What remains is to determine the answer choice that contains a pair of polar coordinates for some point of the lemniscate $r^2 = 8\cos(2\theta)$. Let's look at each answer choice in turn.

In answer choice A, we have $r^2 = 16$, so the angle coordinate θ in one of the pairs of polar coordinates would have to satisfy

$$16 = 8\cos(2\theta)$$

Dividing both sides of this equation by 8, we have

$$2 = \cos(2\theta)$$

This equation has no solution, because the cosine of any angle cannot be greater than 1.

In answer choice B, we also have $r^2 = 16$, and we just found that there is no angle θ that satisfies the equation

$$16 = 8\cos(2\theta)$$

In answer choice C, we have $r^2 = 4$, so the angle coordinate θ in one of the pairs of polar coordinates would have to satisfy

$$4 = 8\cos(2\theta)$$

Dividing both sides of this equation by 8 gives

$$\frac{1}{2} = \cos(2\theta)$$

Let's try the angle coordinate θ in the first pair of polar coordinates (i.e., $\theta = 5\pi/6$):

$$\cos(2\theta) = \cos\left(2\left(\frac{5\pi}{6}\right)\right) = \cos\left(\frac{5\pi}{3}\right)$$

Well,

$$\cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{6\pi - \pi}{3}\right) = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right)$$

By the even identity for cosine,

$$\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

so answer choice C is correct.

To be on the safe side, we'll check the pairs of polar coordinates given in answer choice D. In that case, $r^2 = 16/3$, so we need to see if the angle coordinate θ in either pair satisfies the equation

$$\frac{16}{3} = 8\cos(2\theta)$$

Dividing both sides by 8, we have

$$\frac{2}{3} = \cos(2\theta)$$

Let's try the angle coordinate θ in the first pair of polar coordinates (i.e., $\theta = \pi/8$):

$$\cos(2\theta) = \cos\left(2\left(\frac{\pi}{8}\right)\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \neq \frac{2}{3}$$

Clearly, that doesn't work. The angle coordinate θ in the second pair is $\theta = 9\pi/8$, so

$$\cos(2\theta) = \cos\left(2\left(\frac{9\pi}{8}\right)\right) = \cos\left(\frac{9\pi}{4}\right)$$

Well,

$$\cos\left(\frac{9\pi}{4}\right) = \cos\left(\frac{8\pi + \pi}{4}\right) = \cos\left(2\pi + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \neq \frac{2}{3}$$

This shows that answer choice D is definitely incorrect.

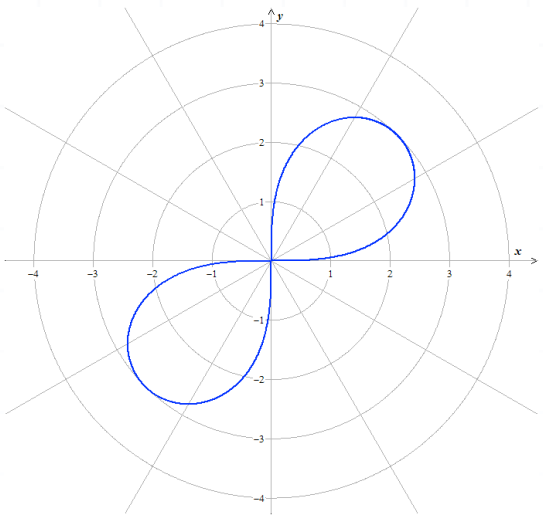
Topic: Graph the polar curve, lemniscate

Question: Which of the following curves is the graph of the lemniscate?

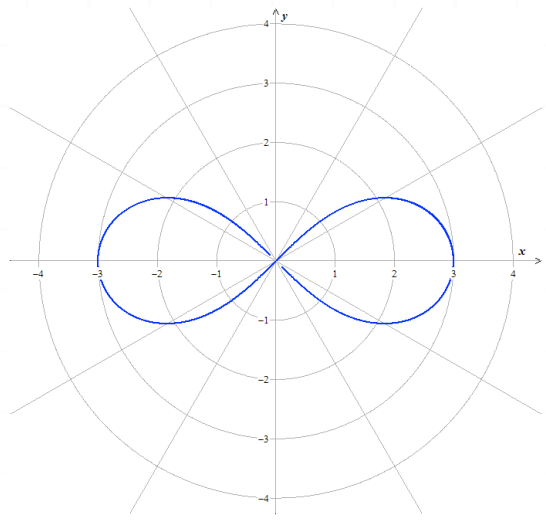
$$r^2 = 9\sin(2\theta)$$

Answer choices:

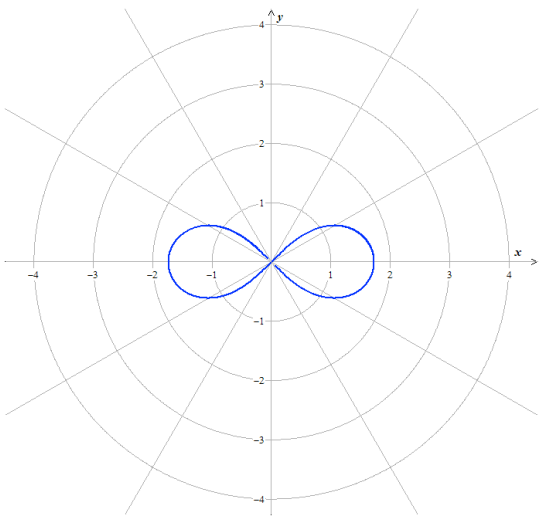
A



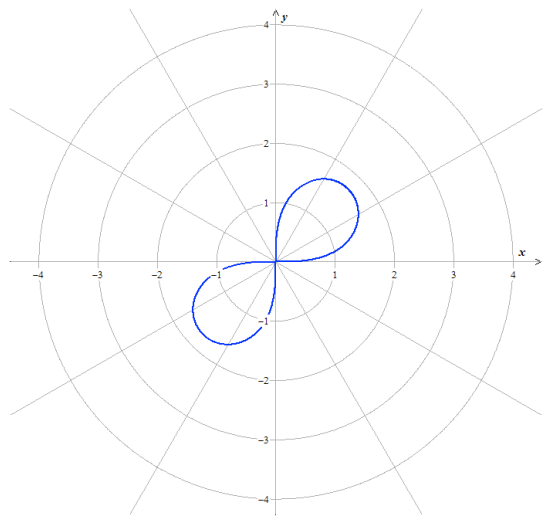
C



B



D



Solution: A

The equation $r^2 = 9\sin(2\theta)$ is that of a lemniscate with $a = 9$. Thus the points of this lemniscate that are furthest from the pole are at a distance of

$$\sqrt{a} = \sqrt{9} = 3$$

units from it. The only answer choices that show a curve with this property are answer choices A and C.

The curve shown in answer choice C is a “cosine” lemniscate, because it's symmetric with respect to both the horizontal axis and the vertical axis, whereas a “sine” lemniscate (the kind we're looking for) isn't symmetric with respect to either of those axes. Thus answer choice C is incorrect.

The curve shown in answer choice A is indeed correct, because the curve is symmetric with respect to the pole (but it isn't symmetric with respect to the horizontal axis or the vertical axis), hence it is a “sine” lemniscate. Also, the points on that curve that are furthest from the pole are at a distance of 3 units from it.