Topic: Complex number equations

Question: Find the solution of the complex equation that lies in the third quadrant.

$$z^3 = 125$$

Answer choices:

A $\quad z = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$

B $\quad z = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

C $\quad z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

D $\quad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution: B

Rewrite z^3 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^3 = r^3 \left[\cos(3\theta) + i\sin(3\theta)\right]$$

Rewrite 125 as the complex number $125 + 0i$. The modulus and angle of $125 + 0i$ are

$$r = \sqrt{125^2 + 0^2}$$

$$r = \sqrt{125^2}$$

$$r = 125$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{125} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 125\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 125\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 125\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^3 = 125$, we can start making substitutions.

$$z^3 = 125$$

$$r^3 \left[\cos(3\theta) + i\sin(3\theta)\right] = 125$$

$$r^3 \left[\cos(3\theta) + i\sin(3\theta)\right] = 125 \left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^3 = 125$$

$$3\theta = 360^\circ k$$

From these equations, we get

$r^3 = 125$, so $r = 5$

$3\theta = 360^\circ k$, so $\theta = 120^\circ k$

To $\theta = 120^\circ k$, if we plug in $k = 0,\ 1,\ 2,\ \ldots$, we get

For $k = 0$, $\theta = 120^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 120^\circ(1) = 120^\circ$

For $k = 2$, $\theta = 120^\circ(2) = 240^\circ$

...

We could keep going for $k = 3,\ 4,\ 5,\ 6,\ \ldots$, but $k = 3$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ,\ 120^\circ,\ 240^\circ$.

Plugging these three angles and $r = 5$ into the formula for polar form of a complex number, we'll get the solutions to $z^3 = 125$.

$$z_1 = 5\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 5\left[1 + i(0)\right] = 5$$

$$z_2 = 5\left[\cos(120^\circ) + i\sin(120^\circ)\right] = 5\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

$$z_3 = 5\left[\cos(240^\circ) + i\sin(240^\circ)\right] = 5\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$$

Topic: Complex number equations

Question: Find the solutions of the complex equation.

$$z^2 = 81$$

Answer choices:

A $z = 3$ and $z = -3$

B $z = 3i$ and $z = -3i$

C $z = 9$ and $z = -9$

D $z = 9i$ and $z = -9i$

Solution: C

Rewrite z^2 as

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^2 = r^2\left[\cos(2\theta) + i\sin(2\theta)\right]$$

Rewrite 81 as the complex number $81 + 0i$. The modulus and angle of $81 + 0i$ are

$$r = \sqrt{81^2 + 0^2}$$

$$r = \sqrt{81^2}$$

$$r = 81$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{81} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 81\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 81\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 81\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^2 = 81$, we can start making substitutions.

$$z^2 = 81$$

$$r^2 \left[\cos(2\theta) + i\sin(2\theta)\right] = 81$$

$$r^2 \left[\cos(2\theta) + i\sin(2\theta)\right] = 81 \left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^2 = 81$$

$$2\theta = 360^\circ k$$

From these equations, we get

$r^2 = 81$, so $r = 9$

$2\theta = 360^\circ k$, so $\theta = 180^\circ k$

To $\theta = 180^\circ k$, if we plug in $k = 0, 1, \dots$, we get

For $k = 0$, $\theta = 180^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 180^\circ(1) = 180^\circ$

...

We could keep going for $k = 2, 3, 4, 5, \dots$, but $k = 2$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 180^\circ$.

Plugging these two angles and $r = 9$ into the formula for polar form of a complex number, we'll get the solutions to $z^2 = 81$.

$$z_1 = 9\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 9\left[1 + i(0)\right] = 9$$

$$z_2 = 9\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 9\left[-1 + i(0)\right] = -9$$

Topic: Complex number equations

Question: How many solutions of the complex equation lie in the fourth quadrant?

$$z^6 = 64$$

Answer choices:

A 1

B 2

C 3

D 4

Solution: A

Rewrite z^6 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^6 = r^6 \left[\cos(6\theta) + i\sin(6\theta)\right]$$

Rewrite 64 as the complex number $64 + 0i$. The modulus and angle of $64 + 0i$ are

$$r = \sqrt{64^2 + 0^2}$$

$$r = \sqrt{64^2}$$

$$r = 64$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{64} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 64\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \ldots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \ldots)\right]$$

$$z = 64\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 64\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^6 = 64$, we can start making substitutions.

$$z^6 = 64$$

$$r^6 \left[\cos(6\theta) + i\sin(6\theta)\right] = 64$$

$$r^6 \left[\cos(6\theta) + i\sin(6\theta)\right] = 64\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^6 = 64$$

$$6\theta = 360^\circ k$$

From these equations, we get

$r^6 = 64$, so $r = 2$

$6\theta = 360^\circ k$, so $\theta = 60^\circ k$

To $\theta = 60^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, 5, \ldots$, we get

For $k = 0$, $\theta = 60^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 60^\circ(1) = 60^\circ$

For $k = 2$, $\theta = 60^\circ(2) = 120^\circ$

For $k = 3$, $\theta = 60^\circ(3) = 180^\circ$

For $k = 4$, $\theta = 60^\circ(4) = 240^\circ$

For $k = 5$, $\theta = 60^\circ(5) = 300^\circ$

...

We could keep going for $k = 6, 7, 8, 9, \dots$, but $k = 6$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$.

Plugging these six angles and $r = 2$ into the formula for polar form of a complex number, we'll get the solutions to $z^6 = 64$.

$$z_1 = 2\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 2\left[1 + i(0)\right] = 2$$

$$z_2 = 2\left[\cos(60^\circ) + i\sin(60^\circ)\right] = 2\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = 1 + \sqrt{3}i$$

$$z_3 = 2\left[\cos(120^\circ) + i\sin(120^\circ)\right] = 2\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -1 + \sqrt{3}i$$

$$z_4 = 2\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 2\left[-1 + i(0)\right] = -2$$

$$z_5 = 2\left[\cos(240^\circ) + i\sin(240^\circ)\right] = 2\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -1 - \sqrt{3}i$$

$$z_6 = 2\left[\cos(300^\circ) + i\sin(300^\circ)\right] = 2\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = 1 - \sqrt{3}i$$

Roots in the fourth quadrant will have a positive real part and a negative imaginary part. That's only z_6, so there's one solution in the fourth quadrant.