Sum-difference identities for sine and cosine

Previously, we introduced the reciprocal identities, quotient identities, Pythagorean identities, and the even-odd identities (which are sometimes called the negative-angle identities). Together, all of these make up the **fundamental identities**.

Now we want to look at some of the other trig identities that we'll use to solve trig equations, starting in this lesson with the sum-difference identities (also called the addition and subtraction identities) for sine and cosine.

The sum-difference identities for sine and cosine

When we want to find the sine or cosine of the sum or difference of two angles, we can use the sum-difference identities. For two angles θ and α, the sum-difference identities for sine are

$$\sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$$

and the sum-difference identities for cosine are

$$\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

These identities are especially helpful for finding sine and cosine of angles that aren't directly represented on the unit circle.

Remember that the unit circle only shows angles that are multiples of $\pi/6$, like $\pi/6$, $\pi/3$, $\pi/2$, $2\pi/3$, etc., and angles that are multiples of $\pi/4$, like $\pi/4$, $\pi/2$, $3\pi/4$, π, etc.

Because the unit circle only gives us this finite set of angles, up to now we haven't had a way to calculate sine and cosine of an angle like $\pi/12$. But if we realize that $\pi/12$ is equivalent to

$$\frac{\pi}{3}-\frac{\pi}{4}$$

$$\frac{\pi}{3}\left(\frac{4}{4}\right)-\frac{\pi}{4}\left(\frac{3}{3}\right)$$

$$\frac{4\pi}{12}-\frac{3\pi}{12}$$

$$\frac{\pi}{12}$$

And the unit circle *does* give us the values of $\pi/3$ and $\pi/4$. So even though we can't get $\pi/12$ from the unit circle directly, we can express it as the difference $(\pi/3)-(\pi/4)$. Then the sine and cosine of $\pi/12$ can be calculated as

$$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$\sin\frac{\pi}{12}=\left(\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right)-\left(\cos\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\sin\frac{\pi}{12}=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\sin\frac{\pi}{12}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$$

$$\sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

and

$$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$\cos\frac{\pi}{12}=\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right)+\left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\cos\frac{\pi}{12}=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\cos\frac{\pi}{12}=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$$

$$\cos\frac{\pi}{12}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

Let’s do another example with some more sine and cosine values.

Example

Find the exact values of $\cos(7\pi/12)$ and $\sin(7\pi/12)$.

From just the unit circle, we wouldn't know the values of sine and cosine at $7\pi/12$, but we can rewrite $7\pi/12$ as

$$\frac{7\pi}{12}=\frac{(4+3)\pi}{12}=\frac{4\pi}{12}+\frac{3\pi}{12}=\frac{\pi}{3}+\frac{\pi}{4}$$

Therefore, by the sum identity for the cosine function,

$$\cos\frac{7\pi}{12}=\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$$

$$\cos\frac{7\pi}{12}=\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right)-\left(\sin\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\cos\frac{7\pi}{12}=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\cos\frac{7\pi}{12}=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$$

$$\cos\frac{7\pi}{12}=\frac{\sqrt{2}-\sqrt{6}}{4}$$

By the sum identity for the sine function,

$$\sin\frac{7\pi}{12}=\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$$

$$\sin\frac{7\pi}{12}=\left(\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}\right)+\left(\cos\frac{\pi}{3}\right)\left(\sin\frac{\pi}{4}\right)$$

$$\sin\frac{7\pi}{12}=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\sin\frac{7\pi}{12}=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$$

$$\sin\frac{7\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

Sometimes we won't have both the sine and cosine of the two angles we're using in the sum difference identities. Let's do an example where we only know the sine of one angle, the cosine of the angle, and the quadrants of both angles.

Example

Suppose θ is an angle in the fourth quadrant whose cosine is $2\sqrt{5}/5$, and α is an angle in the second quadrant whose sine is $1/5$. Find the exact values of $\sin\theta$, $\cos\alpha$, $\cos(\theta-\alpha)$, and $\sin(\theta+\alpha)$.

We're starting with two angles, θ and α. We have the cosine for θ and we have the sine for α. So we need to start by finding the sine for θ and the cosine for α.

Remember that we can always find cosine of an angle when we know the sine and the quadrant, and we can always find the sine of an angle when we know the cosine and the quadrant.

To do that, we'll rewrite the Pythagorean identity for sine and cosine as

$$\sin^2\theta+\cos^2\theta=1$$

$$\sin^2\theta=1-\cos^2\theta$$

Let's start by working on sine of the angle θ. From the information we were given, we'll substitute $\cos\theta=2\sqrt{5}/5$ to get

$$\sin^2\theta=1-\left(\frac{2\sqrt{5}}{5}\right)^2$$

$$\sin^2\theta=1-\frac{4(5)}{25}$$

$$\sin^2\theta=1-\frac{4}{5}$$

$$\sin^2\theta=\frac{1}{5}$$

$$\sin\theta=\pm\sqrt{\frac{1}{5}}$$

Since θ is in the fourth quadrant, we know that $\sin\theta$ is negative. So we can ignore the positive value and say

$$\sin\theta=-\sqrt{\frac{1}{5}}=-\frac{\sqrt{1}}{\sqrt{5}}=-\frac{1}{\sqrt{5}}=-\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)=-\frac{\sqrt{5}}{5}$$

Now we'll work on cosine of the angle α. Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substituting $\sin\alpha = 1/5$, we get

$$\cos^2\alpha = 1 - \left(\frac{1}{5}\right)^2$$

$$\cos^2\alpha = 1 - \frac{1}{25}$$

$$\cos^2\alpha = \frac{24}{25}$$

$$\cos\alpha = \pm\sqrt{\frac{24}{25}}$$

Since α is in the second quadrant, we know that $\cos\alpha$ is negative. So we can ignore the positive value and say that

$$\cos\alpha = -\sqrt{\frac{24}{25}} = -\frac{\sqrt{24}}{\sqrt{25}} = -\frac{2\sqrt{6}}{5}$$

Now we have both sine and cosine for both θ and α:

$$\sin\theta = -\frac{\sqrt{5}}{5}$$

$$\cos\theta = \frac{2\sqrt{5}}{5}$$

$$\sin\alpha = \frac{1}{5}$$

$$\cos\alpha = -\frac{2\sqrt{6}}{5}$$

Which means of the values we were asked to find, $\sin\theta$, $\cos\alpha$, $\cos(\theta-\alpha)$, and $\sin(\theta+\alpha)$, we've already found $\sin\theta$ and $\cos\alpha$. Now we just need to find $\cos(\theta-\alpha)$ and $\sin(\theta+\alpha)$, which we can get from the sum-difference identities.

By the difference identity for the cosine function,

$$\cos(\theta-\alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$\cos(\theta-\alpha) = \left(\frac{2\sqrt{5}}{5}\right)\left(-\frac{2\sqrt{6}}{5}\right) + \left(-\frac{\sqrt{5}}{5}\right)\left(\frac{1}{5}\right)$$

$$\cos(\theta-\alpha) = -\frac{4\sqrt{30}}{25} - \frac{\sqrt{5}}{25}$$

$$\cos(\theta-\alpha) = -\frac{4\sqrt{30}+\sqrt{5}}{25}$$

By the sum identity for the sine function,

$$\sin(\theta+\alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$\sin(\theta+\alpha) = \left(-\frac{\sqrt{5}}{5}\right)\left(-\frac{2\sqrt{6}}{5}\right) + \left(\frac{2\sqrt{5}}{5}\right)\left(\frac{1}{5}\right)$$

$$\sin(\theta+\alpha) = \frac{2\sqrt{30}}{25} + \frac{2\sqrt{5}}{25}$$

$$\sin(\theta+\alpha)=\frac{2\sqrt{30}+2\sqrt{5}}{25}$$