Complex numbers

We know from algebra about the Cartesian (or rectangular) coordinate system, and we've learned also about the polar coordinate system.

The reason we have different number systems like this is because they allow us to solve problems that we wouldn't otherwise be able to using just one system alone. Different number systems can also make it easier to solve a particular kind of problem.

Imaginary numbers

Now we want to introduce the **complex number system**, which is based on the **imaginary unit** called i. This imaginary number is defined as

$i^2 = -1$ or $i = \sqrt{-1}$

Without imaginary numbers, we had no way to find the value of x in something like $x^2 = -16$, because we would take the square root of both sides to get

$$x = \pm\sqrt{-16}$$

and we'd be stuck, since the square root of a negative number isn't defined in the real number system. But if we use imaginary numbers, we're able to simplify this value of x as

$$x = \pm\sqrt{-16}$$

$$x = \pm\sqrt{16(-1)}$$

$$x = \pm\sqrt{16}\sqrt{-1}$$

$$x = \pm 4i$$

Simplifying powers of imaginary numbers

A natural extension of the definition of the imaginary unit i is that the powers of i follow a predictable, cyclical pattern. We already know that any non-zero value raised to the power of 0 is 1, so $i^0 = 1$. Putting that together with the definition of i, we have this:

$$i^0 = 1 \qquad i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

$$i^1 = \sqrt{-1} \qquad i^5 = i \cdot i^4 = \sqrt{-1}(1) = \sqrt{-1}$$

$$i^2 = \sqrt{-1}\sqrt{-1} = -1 \qquad i^6 = i^2 \cdot i^4 = (-1)(1) = -1$$

$$i^3 = i \cdot i^2 = -\sqrt{-1} = -i \qquad i^7 = i^3 \cdot i^4 = -i \cdot 1 = -i$$

This $\sqrt{-1}$, -1, $-i$, 1 pattern repeats over an over again, no matter how large you make the exponent on the imaginary number. Because the pattern is predictable, we can simplify any power of i just by pulling out the largest power that's divisible by 4. For instance,

$$i^{202}$$

$$i^{200} \cdot i^2$$

$$(i^4)^{50} \cdot i^2$$

$$(1)^{50} \cdot i^2$$

$$1 \cdot i^2$$

$$i^2$$

$$-1$$

Complex numbers

We've defined imaginary numbers, and we already know about real numbers from previous math classes. The interesting thing is that **complex numbers** actually include all of the real numbers *and* all of the imaginary numbers. In other words, all real numbers are complex numbers, and all imaginary numbers are complex numbers. Let's talk about why.

The standard form of a complex number is $z = a + bi$, where a and b are real numbers. If a and b are real numbers, then you can see that a complex number is always the sum of the real number a, and the imaginary number bi.

The **real part** of the complex number is a, and we describe that as Re(z). The **imaginary part** of the complex number is b, and we describe that as Im(z). These are all examples of complex numbers:

$$z = 4 + i$$

$$z = -2 + 3i$$

$$z = \pi - 6i$$

$$z = \sqrt{2} - ei$$

In each of these complex numbers, both the real part and the imaginary part are non-zero. If the real-number part of a complex number is 0, then we end up with

$$z = a + bi$$

$$z = 0 + bi$$

$$z = bi$$

Since we're left with only the imaginary part, we call $z = bi$ a **pure imaginary** number. But if instead the imaginary part of a complex number is 0, then we end up with

$$z = a + bi$$

$$z = a + 0i$$

$$z = a$$

Since we're left with only the real part, we call $z = a$ just a **real number**. But notice that in both cases we started with $z = a + bi$. What that tells you is that

1. all real numbers and all imaginary numbers are complex numbers,

2. in the special case of $a = 0$, the complex number is a pure imaginary number, and

3. in the special case of $b = 0$, the complex number is a real number.