Topic: Sketching a parametric curve and its orientation

Question: Consider the parametric equations $x = 3t^3 - 2t + 5$ and $y = 4 - 6t^2$ where $-3 \le t \le 4$. Which value of t in that interval yields the point with coordinates $(x, y) = (-15, -20)$?

Answer choices:

A $t = 2$

B $t = -3$

C $t = -2$

D $t = 4$

Solution: C

Let's evaluate $x = 3t^3 - 2t + 5$ and $y = 4 - 6t^2$ for each of the answer choices, and see which one corresponds to the point with $(x, y) = (-15, -20)$.

Answer choice A ($t = 2$):

$$x = 3(2^3) - 2(2) + 5$$

$$x = 3(8) - 4 + 5$$

$$x = 24 - 4 + 5$$

$$x = 20 + 5$$

$$x = 25 \neq -15$$

This tells us that answer choice A is incorrect.

Answer choice B ($t = -3$):

$$x = 3((-3)^3) - 2(-3) + 5$$

$$x = 3(-27) + 6 + 5$$

$$x = -81 + 6 + 5$$

$$x = -75 + 5$$

$$x = -70 \neq -15$$

Now we know that answer choice B is incorrect.

Answer choice C ($t = -2$):

$$x = 3((-2)^3) - 2(-2) + 5$$

$$x = 3(-8) + 4 + 5$$

$$x = -24 + 4 + 5$$

$$x = -20 + 5$$

$$x = -15$$

and

$$y = 4 - 6((-2)^2)$$

$$y = 4 - 6(4)$$

$$y = 4 - 24$$

$$y = -20$$

It looks as though answer choice C is correct. However, there's a possibility that there are two different values of t that yield the point with coordinates $(x, y) = (-15, -20)$. In that case, the parametric curve would have a point of self-intersection.

To check that answer choice D ($t = 4$) is incorrect, notice that since $y = 4 - 6t^2$, any two values of t that give us the same value of y have to be “negatives” of each other. In answer choice D, $t = 4$; in answer choice C, $t = -2$. Since 4 and -2 aren't “negatives” of each other, we see that answer choice D is incorrect.

Topic: Sketching a parametric curve and its orientation

Question: Consider the parametric equations $x = 3 + 2\sin t$ and $y = 2 - \cos t$ where t is between $-\pi/2$ and $\pi/2$. Which of the following describes the type of parametric curve that corresponds to these parametric equations?

Answer choices:

A The lower half of an ellipse traced out in the counterclockwise direction

B The upper half of a circle traced out in the clockwise direction

C The right half of an ellipse traced out in the clockwise direction

D The left half of a circle traced out in the counterclockwise direction

Solution: A

Let's solve the equations $x = 3 + 2\sin t$ and $y = 2 - \cos t$ for $\sin t$ and $\cos t$, respectively:

$$x = 3 + 2\sin t \implies \sin t = \frac{x-3}{2}$$

$$y = 2 - \cos t \implies \cos t = 2 - y$$

By the basic Pythagorean identity,

$$\sin^2 t + \cos^2 t = 1$$

Substituting $(x-3)/2$ and $2-y$ for $\sin t$ and $\cos t$, respectively, we get

$$\left(\frac{x-3}{2}\right)^2 + (2-y)^2 = 1$$

$$\frac{(x-3)^2}{4} + (2-y)^2 = 1$$

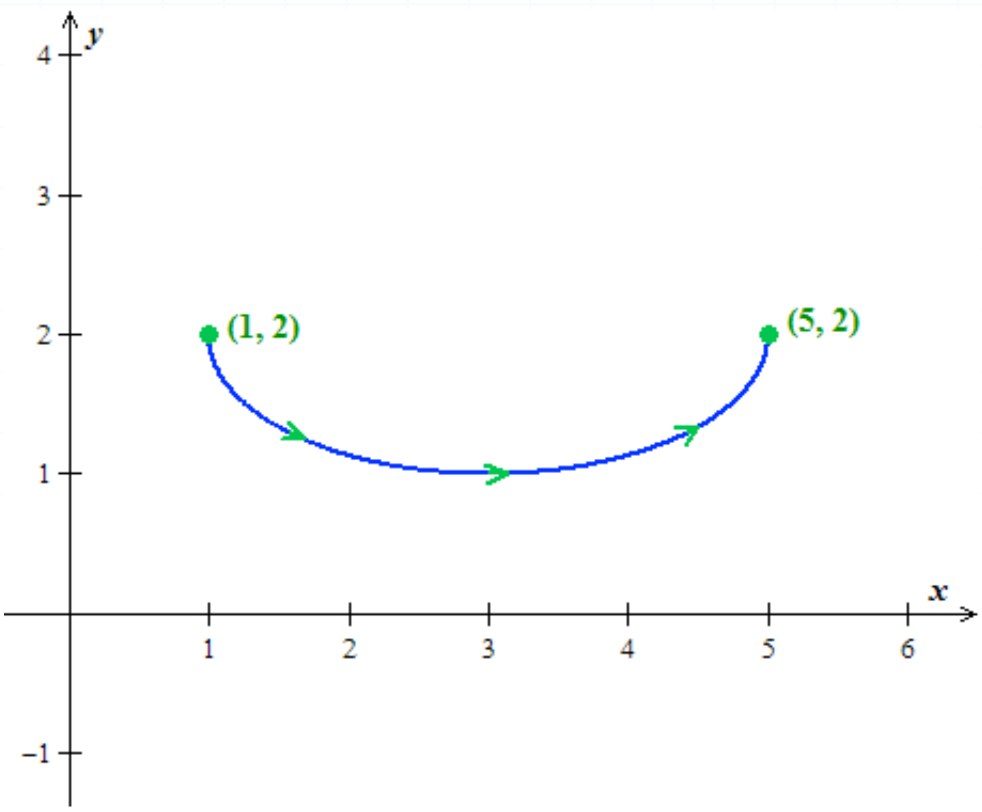
This is the equation of an ellipse.

To determine more specifically what the parametric curve consists of, let's tabulate values of x and y for several values of t in the interval $[-\pi/2,\pi/2]$.

t	$\sin t$	$\cos t$	$x = 3 + 2\sin t$	$y = 2 - \cos t$
$-\frac{\pi}{2}$	-1	0	1	2
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$3-\sqrt{2}$	$2-\frac{\sqrt{2}}{2}$

0	0	1	3	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$3+\sqrt{2}$	$2-\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1	0	5	2

Using those data, we can sketch the parametric curve.



What we see is that this parametric curve is the lower half of the ellipse

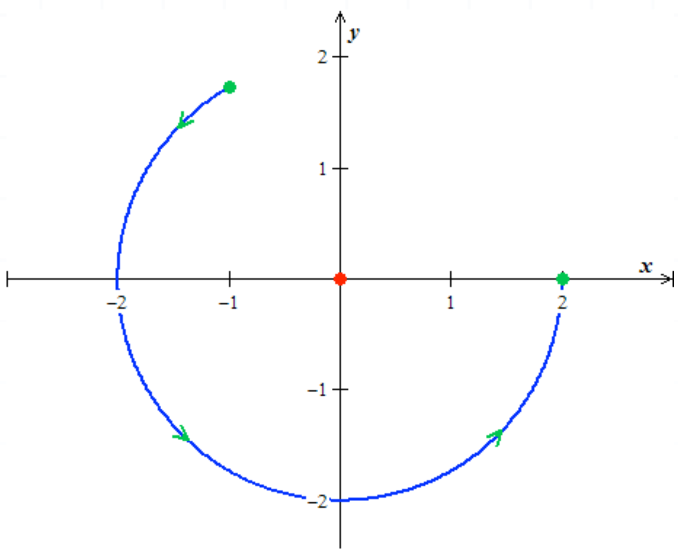
$$\frac{(x-3)^2}{4}+(2-y)^2=1$$

and that it's traced out in the counterclockwise direction.

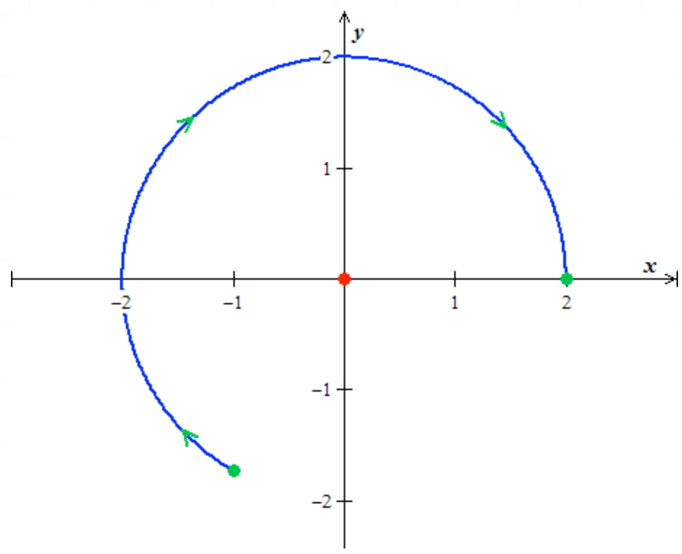
Topic: Sketching a parametric curve and its orientation

Question: One of the following curves is the parametric curve for the equations $x = 2\cos(3t)$ and $y = 2\sin(3t)$ where $-(\pi/6) \leq t \leq \pi/4$. Which one is it?

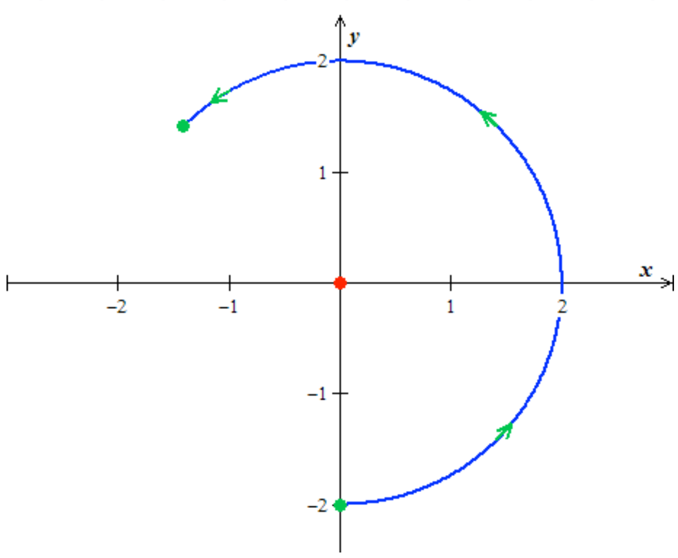
Answer choices:



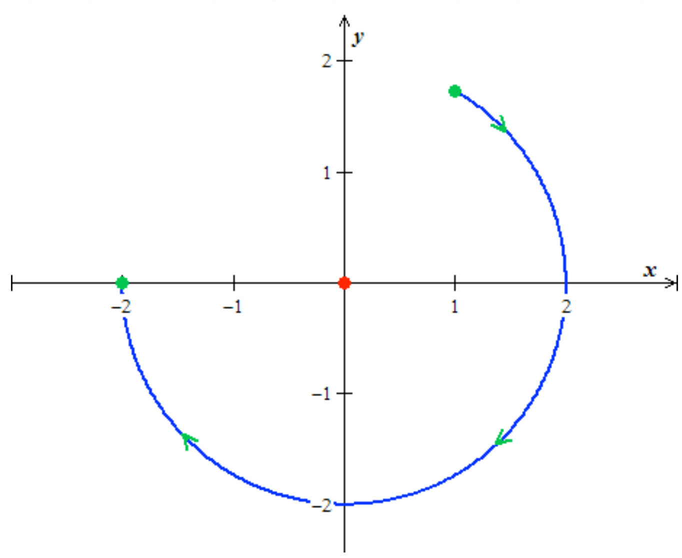
A



B



C



D

Solution: C

Let's evaluate $x = 2\cos(3t)$ and $y = 2\sin(3t)$ at the endpoints of the given interval for t (i.e., at $t = -\pi/6$ and $t = \pi/4$).

$$t = -\frac{\pi}{6} \Longrightarrow x = 2\cos\left(3\left(-\frac{\pi}{6}\right)\right) = 2\cos\left(-\frac{\pi}{2}\right) = 2(0) = 0$$

$$t = -\frac{\pi}{6} \Longrightarrow y = 2\sin\left(3\left(-\frac{\pi}{6}\right)\right) = 2\sin\left(-\frac{\pi}{2}\right) = 2(-1) = -2$$

$$t = \frac{\pi}{4} \Longrightarrow x = 2\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\cos\left(\frac{3\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$$

$$t = \frac{\pi}{4} \Longrightarrow y = 2\sin\left(3\left(\frac{\pi}{4}\right)\right) = 2\sin\left(\frac{3\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

Inspection of the given curves reveals that the only one with endpoints at $(x, y) = (0, -2)$ and $(x, y) = \left(-\sqrt{2}, \sqrt{2}\right)$ is the curve in answer choice C. The initial point (the point for $t = -\pi/6$) is $(0, -2)$, and the terminal point (the point for $t = \pi/4$) is $\left(-\sqrt{2}, \sqrt{2}\right)$. The arrows shown in the curve in answer choice C point away from the initial point and toward the terminal point, as they should. Thus C is indeed the correct answer.