Topic: Powers of complex numbers and De Moivre's theorem

Question: Find z^5 in rectangular form $a+bi$?

$$z = 3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Answer choices:

A $\quad 75\sqrt{2} + 75\sqrt{2}i$

B $\quad 972 - 972i$

C $\quad 243\sqrt{2} + 243\sqrt{2}i$

D $\quad 75 - 225i$

Solution: B

Plug $r = 3\sqrt{2}$, $\theta = 3\pi/4$, and $n = 5$ into De Moivre's theorem.

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^5 = \left(3\sqrt{2}\right)^5\left[\cos\left(5\cdot\frac{3\pi}{4}\right) + i\sin\left(5\cdot\frac{3\pi}{4}\right)\right]$$

Then simplify.

$$z^5 = 3^5\left(\sqrt{2}\right)^5\left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right)$$

$$z^5 = 243\left(4\sqrt{2}\right)\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$z^5 = 243\left(2\sqrt{2}\right)\left(\sqrt{2} - \sqrt{2}i\right)$$

$$z^5 = 486\sqrt{2}\left(\sqrt{2} - \sqrt{2}i\right)$$

$$z^5 = 486\sqrt{2}\sqrt{2} - 486\sqrt{2}\sqrt{2}i$$

$$z^5 = 486(2) - 486(2)i$$

$$z^5 = 972 - 972i$$

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Question: Find z^4 in polar form?

$$z = \sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Answer choices:

A $\quad 6\left(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)$

B $\quad 18\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

C $\quad 9\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

D $\quad 3\left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right)$

Solution: C

Plug $r = \sqrt{3}$, $\theta = 2\pi/3$, and $n = 4$ into De Moivre's theorem.

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^4 = \left(\sqrt{3}\right)^4 \left[\cos\left(4 \cdot \frac{2\pi}{3}\right) + i\sin\left(4 \cdot \frac{2\pi}{3}\right)\right]$$

Then simplify.

$$z^4 = 9\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$$

We could leave the answer this way, but the angle $8\pi/3$ is larger than 2π, so we can reduce the angle to one that's coterminal with $8\pi/3$, but within the interval $[0,2\pi)$.

$$\frac{8\pi}{3} - 2\pi = \frac{8\pi}{3} - 2\pi\left(\frac{3}{3}\right) = \frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$$

So the complex number z^4 in polar form can be written as

$$z^4 = 9\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

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Question: In the complex number z, a is a positive real number and k is a nonnegative integer. Where in the complex plane is z^6 located?

$$z = a\left[\cos\left(\frac{(2k+3)\pi}{2}\right) + i\sin\left(\frac{(2k+3)\pi}{2}\right)\right]$$

Answer choices:

A In the first quadrant

B On the positive vertical axis

C In the third quadrant

D On the negative horizontal axis

Solution: D

From the given complex number, we have

$$r = a$$

$$\theta = \frac{(2k+3)\pi}{2}$$

Because we're looking for z^6, we know $n = 6$, and we plug everything into De Moivre's theorem.

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^6 = a^6\left[\cos\left(6 \cdot \frac{(2k+3)\pi}{2}\right) + i\sin\left(6 \cdot \frac{2k+3)\pi}{2}\right)\right]$$

Then simplify.

$$z^6 = a^6\left[\cos\left(\frac{(12k+18)\pi}{2}\right) + i\sin\left(\frac{(12k+18)\pi}{2}\right)\right]$$

$$z^6 = a^6\left[\cos(6k+9)\pi + i\sin(6k+9)\pi\right]$$

If we test a couple of k-values, we get

For $k = 0$:

$$z^6 = a^6\left[\cos(6(0)+9)\pi + i\sin(6(0)+9)\pi\right]$$

$$z^6 = a^6(\cos 9\pi + i\sin 9\pi)$$

$$z^6 = a^6(-1 + 0i)$$

For $k = 1$:

$$z^6 = a^6\left[\cos(6(1) + 9)\pi + i\sin(6(1) + 9)\pi\right]$$

$$z^6 = a^6(\cos 15\pi + i\sin 15\pi)$$

$$z^6 = a^6(-1 + 0i)$$

For $k = 2$:

$$z^6 = a^6\left[\cos(6(2) + 9)\pi + i\sin(6(2) + 9)\pi\right]$$

$$z^6 = a^6(\cos 21\pi + i\sin 21\pi)$$

$$z^6 = a^6(-1 + 0i)$$

We could keep going, but we realize that we're getting the same value each time, which is $z^6 = a^6(-1 + 0i)$.

The problem told us that a is a positive real number, which means that when we distribute it across the $(-1 + 0i)$, we'll still have a negative real part and a zero imaginary part. In the case when the real part is negative, and the imaginary part is 0, that always put us somewhere on the negative half of the horizontal (real) axis.