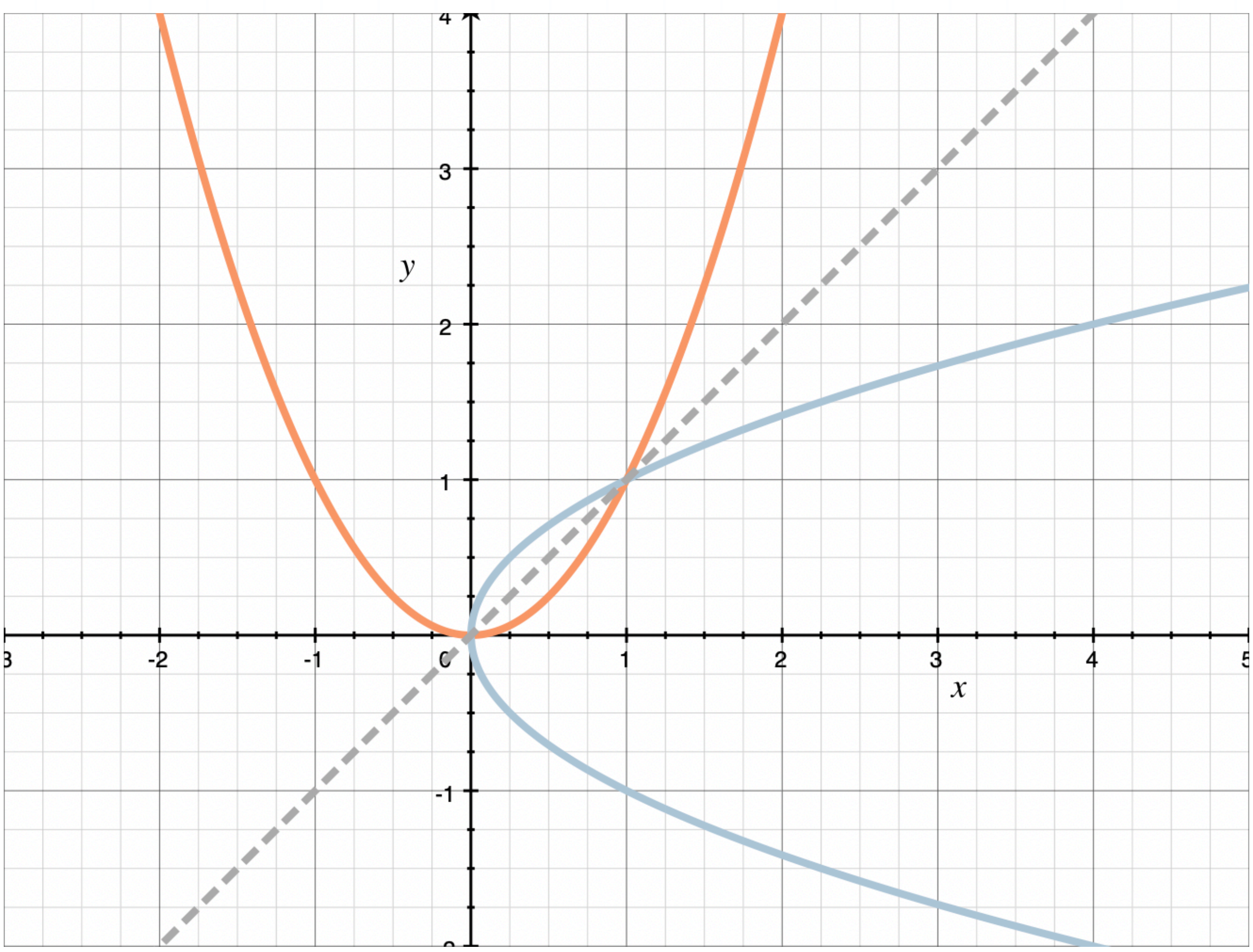
Inverse trig relations

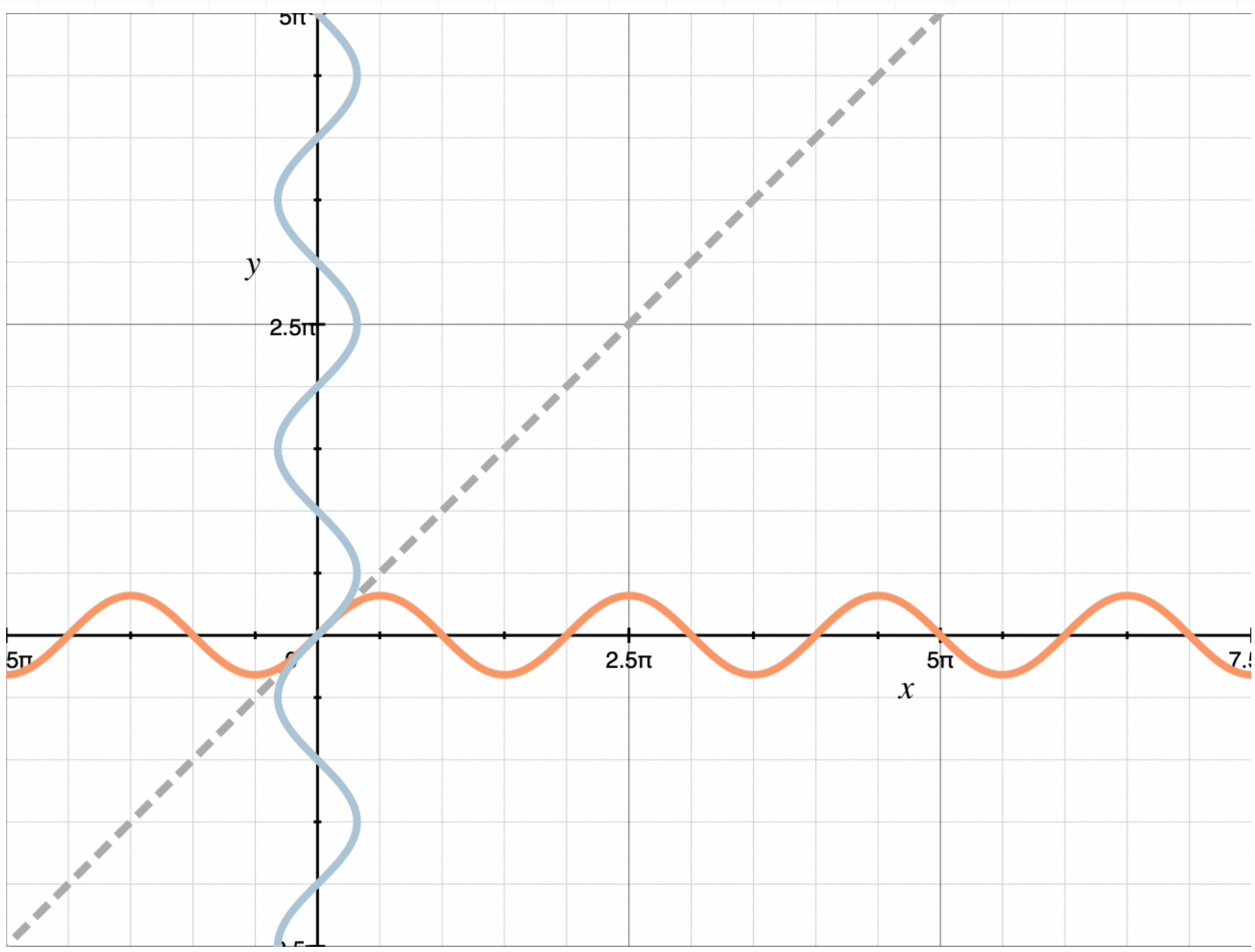
In Algebra, we learned that equations are inverses of one another when they have their x- and y-values swapped. For instance, the inverse of $y = x^2$ is $x = y^2$, because we've changed the places of the x and y variables.

Inverses as reflections

When we learned about inverses, we also learned that inverse curves will be reflections of each other over the line $y = x$. So if we sketch the graphs of $y = x^2$ in red and $x = y^2$ in blue, along with the dotted line $y = x$ in gray, we should see that $y = x^2$ and $x = y^2$ are perfect mirror images of one another across the line.



Inverse trig functions are just like these inverse functions from Algebra. To find a pair of inverse trig relations, we just swap the x- and y-values. So for example, the inverse of $y = \sin x$ is $x = \sin y$. We can see that these equations are inverses of each other if we sketch their graphs and the line $y = x$.



Notice how the blue curve $x = \sin y$ can't be a function. We learned about functions in Algebra, but as a review, a **function** will always pass the **Vertical Line Test**, which means that no perfectly vertical line will intersect the curve at more than one point.

Because we could draw many vertical lines near $x = 0$ that would cross the blue curve at more than one point (in fact, at infinitely many points), the blue curve $x = \sin y$ is not a function. For this reason, we call these **inverse trig relations**, instead of inverse trig functions.

Working backwards to solve inverse equations

So we understand the general idea of an inverse, and what inverse trig equations look like, but now we want to talk about the meaning behind trig functions and their inverses.

We know that a trig function $y = \sin\theta$ is an equation that lets us input a particular angle θ, and get back the corresponding value y. This is the kind of relationship we're used to dealing with so far: we choose an angle θ, and then find the value of a trig function at that angle. So the question we've been answering is "What value y do we get for a given angle θ?"

If we take the inverse of $y = \sin\theta$, we're really switching y and θ, and we get $\theta = \sin y$. With this equation, we're answering the opposite question: "What angle θ do we get for a given y-value?"

So if we think back to the unit circle, we can see these opposite questions in action. Previously, we'd take something like $\sin(\pi/2)$, and say that the value of $\sin(\pi/2)$ is 1. When we have the inverse instead, we're looking at the equation

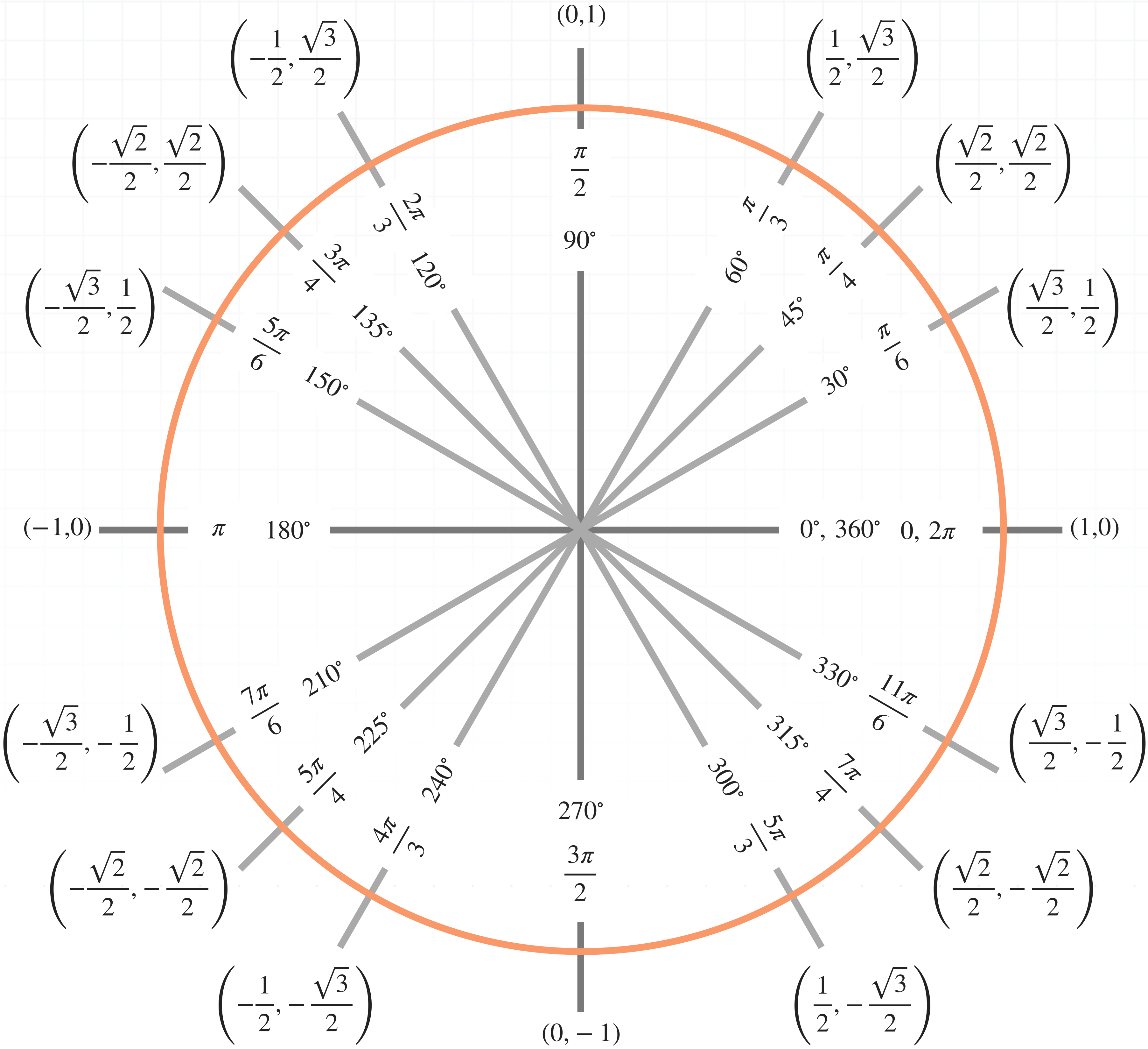
$$\sin y = 1$$

To solve this equation, we have to think about where the sine function is equal to 1. In the unit circle, sine of $\pi/2$ gives 1, and so does sine of all angles coterminal with $\pi/2$. So if we know that the result of the sine function is 1, then the solution to the equation above is

$$y = \frac{\pi}{2} + 2n\pi$$

where n is any integer.

Let's do a few examples in which we've already been given the value from the coordinate point on the unit circle, and we need to find the angle that's giving us that value. For reference in the next two examples, here is the unit circle again:



Example

In both radians and degrees, use the unit circle to find the set of angles whose sine is $1/2$.

On the unit circle, we know that the y-value in the coordinate point is the value that gives us the sine of the angle. Therefore, because we're told that sine of the angle is $1/2$, we need to find the angles in the unit circle where the corresponding coordinate point has a y-value equal to $1/2$.

Those angles are $\pi/6$ and $5\pi/6$. To give the full set of radian angles, we have to give all of the angles that are coterminal with these two:

$$\theta = \frac{\pi}{6} + 2n\pi \text{ and } \theta = \frac{5\pi}{6} + 2n\pi$$

We know that $\pi/6 = 30^\circ$ and $5\pi/6 = 150^\circ$, so the set of angles in degrees will be:

$$\theta = 30^\circ + n(360^\circ) \text{ and } \theta = 150^\circ + n(360^\circ)$$

Let's do an example with cosine.

Example

In both radians and degrees, use the unit circle to find the set of angles whose cosine is -1.

On the unit circle, we know that the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that cosine of the angle is -1, we need to find the angles in the unit circle where the corresponding coordinate point has an x-value equal to -1.

The only angle that does this is the angle π. To give the full set of radian angles, we have to give all of the angles that are coterminal π:

$$\theta = \pi + 2n\pi$$

We know that $\pi = 180^\circ$, so the set of angles in degrees will be:

$$\theta = 180^\circ + n(360^\circ)$$

How we express inverse relations

We know that $x = \sin y$ is the inverse sine relation, but we usually like to express equations for y in terms of x, which means we want to be able to solve $x = \sin y$ for y.

The way we do that is by applying the inverse sine to both sides of the equation. We indicate inverse sine as either $\sin^{-1}$ or as $\arcsin$. These mean the same thing, they're just written different way. When we take inverse sine of both sides, we get

$$\arcsin(x) = \arcsin(\sin y)$$

On the right side, the $\arcsin$ will cancel out the $\sin$, leaving us with just y.

$$\arcsin(x) = y$$

$$y = \arcsin x$$

This equation can also be written as $y = \sin^{-1} x$. The statement we're making with both equation is "y is the angle whose sine is x."

We also have to be careful when we use the notation $\sin^{-1}$. In Algebra, we would make a negative exponent positive by moving the term to the denominator. For instance, x^{-2} could be rewritten as $1/(x^2)$. But the -1 in $\sin^{-1}$ isn't a negative exponent, it's simply notation to indicate "inverse sine." So

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

If we extend this to the other trig functions, we get the following table:

The inverse of	is written as	and is equivalent to
$y = \sin x$	$y = \sin^{-1} x$ or $y = \arcsin x$	$x = \sin y$
$y = \cos x$	$y = \cos^{-1} x$ or $y = \arccos x$	$x = \cos y$
$y = \tan x$	$y = \tan^{-1} x$ or $y = \arctan x$	$x = \tan y$
$y = \csc x$	$y = \csc^{-1} x$ or $y = \text{arccsc} x$	$x = \csc y$
$y = \sec x$	$y = \sec^{-1} x$ or $y = \text{arcsec} x$	$x = \sec y$
$y = \cot x$	$y = \cot^{-1} x$ or $y = \text{arccot} x$	$x = \cot y$