Converting equations from rectangular to polar

Just as there may be times when you want to convert an equation from polar coordinates to rectangular coordinates, there may be times when you want to do the opposite: convert an equation in rectangular coordinates to polar coordinates. To do this, you'll replace every x with $r\cos\theta$, and every y with $r\sin\theta$, and then simplify (if any simplification is possible or desired).

One equation that may come in handy is the Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

For example, if we get

$$r^2\cos^2\theta + r^2\sin^2\theta$$

when we replace every x with $r\cos\theta$, and every y with $r\sin\theta$, then that becomes just r^2, because

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2(1) = r^2$$

Example

Convert the rectangular equation $x^2 + y^2 = 64$ to polar coordinates and describe its solution geometrically.

Replacing x with $r\cos\theta$, and y with $r\sin\theta$, we have

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 64$$

Expanding the left-hand side (by carrying out the indicated squaring), we get

$$r^2\cos^2\theta + r^2\sin^2\theta = 64$$

Using the fact that $r^2\sin^2\theta + r^2\cos^2\theta = r^2$, we obtain

$$r^2 = 64$$

The solution set of this equation is the set of points that have $r^2 = 64$ (equivalently, the union of all points with $r = \sqrt{64} = 8$ and the set of all points with $r = -\sqrt{64} = -8$).

We'll now show that the set of all points with $r = 8$ is identical to the set of all points with $r = -8$.

Clearly, every point that has a pair of polar coordinates (r, θ) where r positive and θ is any angle (i.e., any point with $r = \sqrt{64} = 8$) satisfies the polar equation $r = 8$. Recall that the r coordinate of a point is positive (and equal to 8) if and only if that point is at a distance of 8 units from the pole, hence the solution set of the equation $r = 8$ is the set of all points on the circle of radius 8 whose center is at the pole.

Now consider points with r negative (i.e., $r = -8$). Recall that every point with a pair of polar coordinates (r, θ) where $r = -8$ and θ is any angle also has a pair of polar coordinates $(s, \theta + \pi)$ where $s = -r = 8$. By the previous result, such a point is located on the circle of radius 8 whose center is at the pole.

Conversely, every point on that circle has a pair of polar coordinates $(s, \theta + \pi)$ where $s = 8$ and θ is any angle, hence that point also has a pair of polar coordinates (r, θ) where $r = -s = -8$. Thus every point with these characteristics satisfies the polar equation $r = -8$.

What we have found is that the set of all points that satisfy the equation $r^2 = 64$ is just the set of all points on the circle of radius 8 whose center is at the pole.

Another result that may help in getting the polar equivalent of a rectangular equation is

$$\frac{r \sin \theta}{r \cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

Example

Convert the rectangular equation $y = 25x$ to polar coordinates and describe its solution geometrically.

Making the usual replacements for x and y, we get

$$r \sin \theta = 25r \cos \theta$$

Dividing both sides by $r \cos \theta$ gives

$$\frac{r \sin \theta}{r \cos \theta} = 25$$

That is,

$$\tan\theta = 25$$

Since $\tan\theta$ is positive, the terminal side of θ is in either the first quadrant or the third quadrant. The solution of this equation is the union of the set of all points in the first quadrant whose tangent is equal to 25, the set of all points in the third quadrant whose tangent is equal to 25, and the set containing just the pole (since the pole has a pair of coordinates $(0,\theta)$ for any angle θ whose tangent is equal to 25).

There is only one angle θ whose terminal side is in the first quadrant and such that θ is in the interval $[0,\pi/2)$ and $\tan\theta = 25$, and there is only one angle θ whose terminal side is in the third quadrant and such that θ is in the interval $[\pi,3\pi/2)$ and $\tan\theta = 25$. Since $y/x = \tan\theta$, the equation $\tan\theta = 25$ is the equation of the line that passes through the pole and has a slope of $\tan\theta$ (i.e., a slope of 25).

Now let's take a look at a rectangular equation that's of neither of the two types discussed above.

Example

Convert the rectangular equation $y = -6x^2$ to polar coordinates and describe its solution geometrically.

Making the usual replacements for x and y, we get

$$r\sin\theta = -6(r\cos\theta)^2$$

Here, we could divide both sides by $(r\cos\theta)^2$, which would give us

$$\frac{r\sin\theta}{(r\cos\theta)^2} = -6$$

$$\frac{r\sin\theta}{(r\cos\theta)(r\cos\theta)} = -6$$

Using the definition of $\tan\theta$ (as the ratio of sine to cosine), we obtain

$$\frac{r\sin\theta}{(r\cos\theta)(r\cos\theta)} = \left(\frac{r\sin\theta}{r\cos\theta}\right)\left(\frac{1}{r\cos\theta}\right) = \tan\theta\left(\frac{1}{r\cos\theta}\right) = \frac{\tan\theta}{r\cos\theta}$$

Therefore, our equation becomes

$$\frac{\tan\theta}{r\cos\theta} = -6$$

Equivalently, we get the following equation if we multiply both sides by r:

$$\frac{\tan\theta}{\cos\theta} = -6r$$

You're probably wondering how this polar equation would help you to envision the solution set - and with good reason! This is a case where the equation in rectangular coordinates is much more “transparent” (than the polar equation) in revealing the nature of the solution set. You may recognize the rectangular equation $y = -6x^2$ as the equation of a parabola that opens downwards and has its vertex at the origin.

This example points up the fact that when we wish to express an equation, it's best to express it in the coordinates that most readily “reveal” the nature of the solution. It's probably safe to say that polar coordinates aren't a good choice when it comes to expressing the equation of a parabola.

If you ever take a course in calculus, you'll find that the choice of coordinates can make a big difference when it comes to solving certain kinds of problems, especially those that involve finding the area of a two-dimensional region or the volume of a (three-dimensional) solid.

Let's look at still other examples of converting an equation from rectangular coordinates to polar coordinates.

Example

Convert the rectangular equation $(x+9)^2+(y-13)^2=64$ to polar coordinates and describe its solution geometrically.

Replacing x with $r\cos\theta$, and y with $r\sin\theta$, we have

$$(r\cos\theta+9)^2+(r\sin\theta-13)^2=64$$

Expanding this equation (by doing the indicated squaring), we get

$$(r^2\cos^2\theta+18r\cos\theta+81)+(r^2\sin^2\theta-26r\sin\theta+169)=64$$

Rearranging (by combining like terms etc.), we get

$$(r^2\cos^2\theta+r^2\sin^2\theta)+(18r\cos\theta-26r\sin\theta)+(81+169)=64$$

Equivalently,

$$r^2+18r\cos\theta-26r\sin\theta+250=64$$

Subtracting 250 from both sides, we obtain

$$r^2 + 18r\cos\theta - 26r\sin\theta = -186$$

Here again, the polar equation doesn't appear to be very "revealing," whereas the original (rectangular) equation is in the form

$$(x-h)^2 + (y-k)^2 = r^2$$

with $h = -9$, $k = 13$, and $r = 8$. You may recognize this as the equation of the circle of radius 8 whose center has rectangular coordinates $(-9,13)$. Thus it looks as though rectangular coordinates are preferable in a case like this.

Example

Convert the rectangular equation to polar coordinates and describe its solution geometrically.

$$\frac{(x+5)^2}{9} + \frac{(y-7)^2}{4} = 1$$

Replacing x with $r\cos\theta$ and y with $r\sin\theta$, we get

$$\frac{(r\cos\theta+5)^2}{9} + \frac{(r\sin\theta-7)^2}{4} = 1$$

We can clear the fractions by multiplying both sides of this equation by 36, which yields

$$4(r\cos\theta+5)^2 + 9(r\sin\theta-7)^2 = 36$$

Expanding the left-hand side (by carrying out the indicated squaring and multiplication) gives

$$4(r^2\cos^2\theta + 10r\cos\theta + 25) + 9(r^2\sin^2\theta - 14r\sin\theta + 49) = 36$$

$$(4r^2\cos^2\theta + 40r\cos\theta + 100) + (9r^2\sin^2\theta - 126r\sin\theta + 441) = 36$$

Here, the coefficient of $r^2\sin^2\theta$ is different from the coefficient of $r^2\cos^2\theta$, so we can't combine even those two terms and get something simple.

As you can see, we again end up with a polar equation that isn't very instructive as to the nature of the solution set. You may recognize the original (rectangular) equation as that of a certain ellipse, but in polar coordinates it looks rather daunting.

By now, you're probably thinking that polar coordinates are of very little use, and that the kinds of polar equations which "reveal" the nature of their solution set may be rather limited - and may comprise little more than polar equations that correspond to rectangular equations of the form $y = cx$ or $x^2 + y^2 = c^2$ for some nonzero constant c. Well, take heart, because you're eventually going to learn that there are polar equations which are not only simple and revealing but also have a solution set that (when shown on a graph) is visually very appealing. Stay tuned!