Topic: Roots of complex numbers

Question: Which of the following is a cube root of z?

$$z = 64\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$$

Answer choices:

A $\quad 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

B $\quad 6\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$

C $\quad 4\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$

D $\quad 6\left(\cos\frac{11\pi}{2} + i\sin\frac{11\pi}{2}\right)$

Solution: C

We're looking for the third (or cube) roots of z, which means there will be 3 of them, given by $k = 0, 1, 2$. And since the complex number is given in radians, we'll plug $n = 3$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[3]{z} = \sqrt[3]{r}\left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i\sin\left(\frac{\theta + 2\pi k}{3}\right)\right]$$

With $r = 64$ and $\theta = 11\pi/8$ from the complex number, we get

$$\sqrt[3]{z} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{11\pi}{8} + 2\pi k}{3}\right)\right]$$

Now we'll find values for $k = 0, 1, 2$.

For $k = 0$:

$$\sqrt[3]{z}_{k=0} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8} + 2\pi(0)}{3}\right) + i\sin\left(\frac{\frac{11\pi}{8} + 2\pi(0)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=0} = 4\left(\cos\frac{11\pi}{24} + i\sin\frac{11\pi}{24}\right)$$

For $k = 1$:

$$\sqrt[3]{z}_{k=1} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8} + 2\pi(1)}{3}\right) + i\sin\left(\frac{\frac{11\pi}{8} + 2\pi(1)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{27\pi}{24} + i\sin\frac{27\pi}{24}\right)$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$$

For $k = 2$:

$$\sqrt[3]{z}_{k=2} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8} + 2\pi(2)}{3}\right) + i\sin\left(\frac{\frac{11\pi}{8} + 2\pi(2)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=2} = 4\left(\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right)$$

The roots are

$$\sqrt[3]{z}_{k=0} = 4\left(\cos\frac{11\pi}{24} + i\sin\frac{11\pi}{24}\right)$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$$

$$\sqrt[3]{z}_{k=2} = 4\left(\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right)$$

The matching root is from $k = 1$.

Topic: Roots of complex numbers

Question: How many of the seventh roots of z lie in the third quadrant of the complex plane?

$$z = 15\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$

Answer choices:

A One

B Two

C Three

D None

Solution: B

We're looking for the seventh roots of z, which means there will be 7 of them, given by $k = 0, 1, 2, 3, 4, 5, 6$. And since the complex number is given in radians, we'll plug $n = 7$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[7]{z} = \sqrt[7]{r}\left[\cos\left(\frac{\theta + 2\pi k}{7}\right) + i\sin\left(\frac{\theta + 2\pi k}{7}\right)\right]$$

With $r = 15$ and $\theta = \pi/10$ from the complex number, we get

$$\sqrt[7]{z} = \sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10} + 2\pi k}{7}\right) + i\sin\left(\frac{\frac{\pi}{10} + 2\pi k}{7}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4, 5, 6$.

For $k = 0$:

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10} + 2\pi(0)}{7}\right) + i\sin\left(\frac{\frac{\pi}{10} + 2\pi(0)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{15}\left(\cos\frac{\pi}{70} + i\sin\frac{\pi}{70}\right)$$

For $k = 1$:

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10}+2\pi(1)}{7}\right)+i\sin\left(\frac{\frac{\pi}{10}+2\pi(1)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left(\cos\frac{21\pi}{70}+i\sin\frac{21\pi}{70}\right)$$

For $k=2$:

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10}+2\pi(2)}{7}\right)+i\sin\left(\frac{\frac{\pi}{10}+2\pi(2)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left(\cos\frac{41\pi}{70}+i\sin\frac{41\pi}{70}\right)$$

We can start to see how we're just adding $20\pi/70$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[7]{z}_{k=0}=\sqrt[7]{15}\left(\cos\frac{\pi}{70}+i\sin\frac{\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left(\cos\frac{21\pi}{70}+i\sin\frac{21\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left(\cos\frac{41\pi}{70}+i\sin\frac{41\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=3}=\sqrt[7]{15}\left(\cos\frac{61\pi}{70}+i\sin\frac{61\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=4}=\sqrt[7]{15}\left(\cos\frac{81\pi}{70}+i\sin\frac{81\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=5} = \sqrt[7]{15}\left(\cos\frac{101\pi}{70} + i\sin\frac{101\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=6} = \sqrt[7]{15}\left(\cos\frac{121\pi}{70} + i\sin\frac{121\pi}{70}\right)$$

If we find the decimal approximations of these angles, we get

For $k = 0$, $(1/70)\pi \approx 0.01\pi$

For $k = 1$, $(21/70)\pi \approx 0.3\pi$

For $k = 2$, $(41/70)\pi \approx 0.59\pi$

For $k = 3$, $(61/70)\pi \approx 0.87\pi$

For $k = 4$, $(81/70)\pi \approx 1.16\pi$

For $k = 5$, $(101/70)\pi \approx 1.44\pi$

For $k = 6$, $(121/70)\pi \approx 1.73\pi$

Anything in the third quadrant will fall in the interval $(1\pi, 1.5\pi)$, which in this case are the angles for $k = 4$ and $k = 5$, so two of the seventh roots fall in the third quadrant.

Topic: Roots of complex numbers

Question: Find the 4th root of the complex number that lies in the fourth quadrant of the complex plane.

$$z = 16\left(\cos 30^\circ + i\sin 30^\circ\right)$$

Answer choices:

A $2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right]$

B $2\left[\cos(297.5^\circ) + i\sin(297.5^\circ)\right]$

C $2\left[\cos(317.5^\circ) + i\sin(317.5^\circ)\right]$

D $2\left[\cos(337.5^\circ) + i\sin(337.5^\circ)\right]$

Solution: A

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k=0, 1, 2, 3$. And since the complex number is given in degrees, we'll plug $n=4$ into the formula for nth roots in degrees.

$$\sqrt[n]{z}=\sqrt[n]{r}\left[\cos\left(\frac{\theta+360^\circ k}{n}\right)+i\sin\left(\frac{\theta+360^\circ k}{n}\right)\right]$$

$$\sqrt[4]{z}=\sqrt[4]{r}\left[\cos\left(\frac{\theta+360^\circ k}{4}\right)+i\sin\left(\frac{\theta+360^\circ k}{4}\right)\right]$$

With $r=16$ and $\theta=30^\circ$ from the complex number, we get

$$\sqrt[4]{z}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ k}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ k}{4}\right)\right]$$

Now we'll find values for $k=0, 1, 2, 3$.

For $k=0$:

$$\sqrt[4]{z}_{k=0}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ(0)}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0}=2\left[\cos(7.5^\circ)+i\sin(7.5^\circ)\right]$$

For $k=1$:

$$\sqrt[4]{z}_{k=1}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ(1)}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1} = 2\left[\cos(97.5^\circ) + i\sin(97.5^\circ)\right]$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{16}\left[\cos\left(\frac{30^\circ + 360^\circ(2)}{4}\right) + i\sin\left(\frac{30^\circ + 360^\circ(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = 2\left[\cos(187.5^\circ) + i\sin(187.5^\circ)\right]$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{16}\left[\cos\left(\frac{30^\circ + 360^\circ(3)}{4}\right) + i\sin\left(\frac{30^\circ + 360^\circ(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3} = 2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right]$$

The roots are

$$\sqrt[4]{z}_{k=0} = 2\left[\cos(7.5^\circ) + i\sin(7.5^\circ)\right] \approx 1.982 + 0.262i$$

$$\sqrt[4]{z}_{k=1} = 2\left[\cos(97.5^\circ) + i\sin(97.5^\circ)\right] \approx -0.262 + 1.982i$$

$$\sqrt[4]{z}_{k=2} = 2\left[\cos(187.5^\circ) + i\sin(187.5^\circ)\right] \approx -1.982 - 0.262i$$

$$\sqrt[4]{z}_{k=3} = 2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right] \approx 0.262 - 1.982i$$

The root in the fourth quadrant will have a positive real part and a negative imaginary part, which is the root for $k = 3$.