# Graph the polar curve, rose

In this lesson, you're going to see how to graph a polar curve that looks like a flower - and for that reason, such a curve is called a rose. We'll consider only certain curves of this type that have the center of the rose at the pole. The polar equation of every rose that we'll deal with is either of the form

$$r = a\cos(n\theta)$$

or of the form

$$r = a\sin(n\theta)$$

for some nonzero number $a$ and some integer $n$ with $|n| \geq 2$.

To see why we're stipulating that $|n| \geq 2$, let's consider what happens if we have an equation of either of these forms and $n$ is either $0$, $1$, or $-1$.

If $n = 0$, an equation of the form $r = a\cos(n\theta)$ reduces to

$$r = a\cos(0)$$

$$r = a(1)$$

$$r = a$$

This is just the equation of a circle of radius $|a|$ that has its center at the pole.

An equation of the form $r = a\sin(n\theta)$ with $n = 0$ reduces to

$$r = a\sin(0)$$

$$r = a(0)$$

$$r = 0$$

This is just the equation of a single point (namely, the equation of the pole).

Also, you have already learned that if $a \neq 0$, then a polar equation of the form $r = a\cos\theta$ or $r = a\sin\theta$ is the equation of a circle. This takes care of $n = 1$.

Now let's consider what happens when $n = -1$. By the even identity for cosine,

$$\cos(-\theta) = \cos\theta$$

so we get exactly the same circle as for the polar equation

$$r = a\cos(-\theta)$$

that we get for the polar equation

$$r = a\cos\theta$$

From this point on, we'll assume that $a \neq 0$ and that $n$ is an integer of absolute value $2$ or larger.

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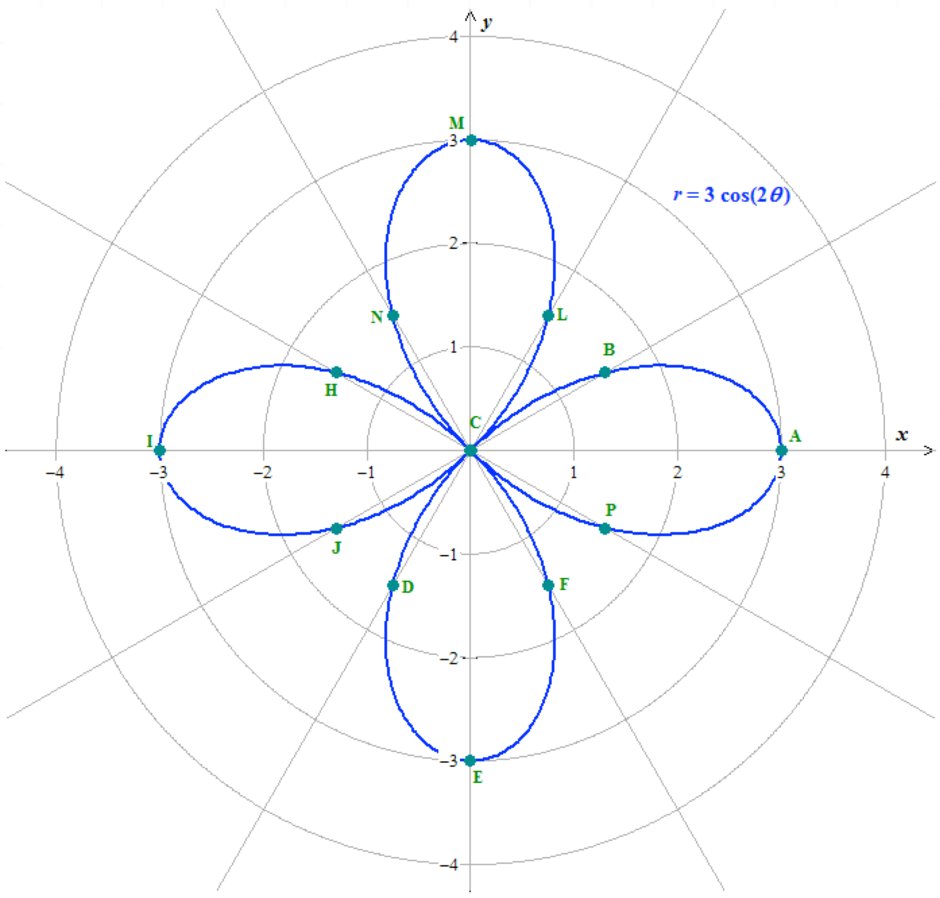
**Example**

Graph the polar equation $r = 3\cos(2\theta)$.

In the following table, the values of $\cos(2\theta)$ and $r = 3\cos(2\theta)$ for a number of angles $\theta$ in the interval $[0,2\pi)$ are shown. In the table, we also give one pair of polar coordinates, $(r,\theta)$, for points where the equation $r = 3\cos(2\theta)$ gives us a positive value of $r$, and we give two pairs of polar coordinates, $(r,\theta)$ and $(-r,\theta+\pi)$, for points where the equation $r = 3\cos(2\theta)$ gives us a negative value of $r$. Four of the points in the table (the ones that are labeled with the letters C, G, K, and O) correspond to the pole, so in the accompanying graph only the first of these labels (namely, C) is used for the pole.

| Point | $\theta$ | $\cos(2\theta)$ | $r = 3\cos(2\theta)$ | $(r,\theta)$ | $(-r,\theta+\pi)$ |
|---|---|---|---|---|---|
| A | $0$ | $1$ | $3$ | $(3,0)$ | |
| B | $\pi/6$ | $1/2$ | $3/2$ | $(3/2,\pi/6)$ | |
| C=pole | $\pi/4$ | $0$ | $0$ | $(0,\pi/4)$ | |
| D | $\pi/3$ | $-1/2$ | $-3/2$ | $(-3/2,\pi/3)$ | $(3/2,4\pi/3)$ |
| E | $\pi/2$ | $-1$ | $-3$ | $(-3,\pi/2)$ | $(3,3\pi/2)$ |
| F | $2\pi/3$ | $-1/2$ | $-3/2$ | $(-3/2,2\pi/3)$ | $(3/2,5\pi/3)$ |
| G=pole | $3\pi/4$ | $0$ | $0$ | $(0,3\pi/4)$ | |
| H | $5\pi/6$ | $1/2$ | $3/2$ | $(3/2,5\pi/6)$ | |
| I | $\pi$ | $1$ | $3$ | $(3,\pi)$ | |
| J | $7\pi/6$ | $1/2$ | $3/2$ | $(3/2,7\pi/6)$ | |

| | | | | | |
|---|---|---|---|---|---|
| K=pole | $5\pi/4$ | $0$ | $0$ | $(0,5\pi/4)$ | |
| L | $4\pi/3$ | $-1/2$ | $-3/2$ | $(-3/2,4\pi/3)$ | $(3/2,7\pi/3)$ |
| M | $3\pi/2$ | $-1$ | $-3$ | $(-3,3\pi/2)$ | $(3,5\pi/2)$ |
| N | $5\pi/3$ | $-1/2$ | $-3/2$ | $(-3/2,5\pi/3)$ | $(3/2,8\pi/3)$ |
| O=pole | $7\pi/4$ | $0$ | $0$ | $(0,7\pi/4)$ | |
| P | $11\pi/6$ | $1/2$ | $3/2$ | $(3/2,11\pi/6)$ | |




Notice that for the rose $r = 3\cos(2\theta)$, the points which are furthest from the pole are at a distance of exactly $3$ units from it. More generally, it can easily be seen that for the rose $r = a\cos(n\theta)$, the points which are furthest from the pole are at a distance of exactly $|a|$ units from it.

Also, notice that the rose $r = 3\cos(2\theta)$ has four "petals." Let's investigate how many petals there are in the graph of the polar equation $r = a\cos(n\theta)$ in general. By the even identity for cosine,

$$\cos(-n\theta) = \cos(n\theta)$$

so it suffices to do this only for positive integers $n$.

The "tip" of each petal of a rose is at a distance of $|a|$ units from the pole, and every point which is at a distance of $|a|$ units from the pole is the tip of some petal of the rose. Therefore, the number of petals is equal to the numbers of points $(r, \theta)$ with $|r| = |a|$.

Our polar equation is

$$r = a\cos(n\theta)$$

so

$$|r| = |a| \iff |\cos(n\theta)| = 1$$

Now $|\cos(n\theta)| = 1$ if and only if either

$$\cos(n\theta) = 1$$

or

$$\cos(n\theta) = -1$$

Also, $|\cos(n\theta)| = 1$ if and only if $n\theta$ is an integer multiple of $\pi$.

In addition,

$$a\cos(n(\theta+2\pi)) = a\cos(n\theta+2n\pi) = a\cos(n\theta)$$

where the second equality follows from the fact that the angles $n\theta$ and $n\theta+2n\pi$ are coterminal and hence have the same value of the cosine function. Thus we need not consider angles outside the interval $[0,2\pi)$ (and we should not consider them, as we would be counting one or more petals twice).

If $\cos(n\theta)=1$, then $n\theta$ is an integer multiple of $2\pi$, so there is a nonnegative integer $k$ such that $n\theta=2k\pi$. This implies that

$$\theta = \frac{2k\pi}{n}$$

Since we want $\theta \in [0,2\pi)$, we have

$$0 \le \frac{2k\pi}{n} < 2\pi$$

Multiplying through by $n/(2\pi)$ gives

$$0 \le k < n$$

Since both $k$ and $n$ are integers, this means that

$$0 \le k \le n-1$$

What this tell us is that the tip of each petal for which $\cos(n\theta)=1$ is located at some point $(r,\theta)$ with $|r|=|a|$ and

$$\theta \in \left\{0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2(n-1)\pi}{n}\right\}$$

Now

$$2(n-1) = 2n-2$$

so

$$\theta \in \left\{0, \frac{2\pi}{n}, \frac{4\pi}{n}, \dots, \frac{(2n-2)\pi}{n}\right\}$$

If $\cos(n\theta) = -1$, then $n\theta$ is an odd integer multiple of $\pi$, so there is a nonnegative integer $k$ such that $n\theta = (2k+1)\pi$. Thus

$$\theta = \frac{(2k+1)\pi}{n}$$

We want $\theta \in [0,2\pi)$, so

$$0 \le \frac{(2k+1)\pi}{n} < 2\pi$$

Multiplying through by $n/\pi$, we get

$$0 \le \frac{2k+1}{n} < 2n$$

Now $k$ is a nonnegative integer, so $2k+1$ is an odd integer and $2k+1 \ge 1$, hence

$$1 \le 2k+1 < 2n$$

Also, $2k+1$ is an odd integer and $2n$ is an even integer, hence

$$1 \le 2k+1 \le 2n-1$$

Subtracting $1$ throughout, we have

$$0 \leq 2k \leq 2n - 2$$

That is,

$$0 \leq 2k \leq 2(n - 1)$$

Dividing through by $2$, we get

$$0 \leq k \leq n - 1$$

Thus the tip of each petal for which $\cos(n\theta) = -1$ is located at some point $(r, \theta)$ with $|r| = |a|$ and

$$\theta \in \left\{ \frac{\pi}{n}, \frac{3\pi}{n}, \ldots, \frac{[2(n-1)+1]\pi}{n} \right\}$$

Now

$$2(n - 1) + 1 = 2n - 2 + 1 = 2n - 1$$

so

$$\theta \in \left\{ \frac{\pi}{n}, \frac{3\pi}{n}, \ldots, \frac{(2n-1)\pi}{n} \right\}$$

Combining the results for $\cos(n\theta) = 1$ and $\cos(n\theta) = -1$, we find that the tip of each petal is located at some point $(r, \theta)$ with $|r| = |a|$ and some angle $\theta$ in the set

$$\left\{ 0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{(2n-1)\pi}{n} \right\}$$

Since $|\cos(n\theta)| = 1$ for every angle $\theta$ in this set, every element of the set corresponds to the tip of some petal of the rose.

This set of angles $\theta$ has exactly $2n$ elements, so you might think that the rose $r = a\cos(n\theta)$ has exactly $2n$ petals. Well, you would be right about that in the case where $n$ is an even integer, but such a rose has only $n$ petals if $n$ is an odd integer.

Let's see why we get a different result for the two cases. To do this, we'll consider the value of $a\cos(n(\theta+\pi))$ for angles $\theta$ in the interval $[0,\pi)$. First, we have

$$\cos(n(\theta+\pi)) = \cos(n\theta + n\pi)$$

By the sum identity for cosine,

$$\cos(n\theta + n\pi) = \cos(n\theta)\cos(n\pi) - \sin(n\theta)\sin(n\pi)$$

$$\cos(n\theta + n\pi) = \cos(n\theta)\cos(n\pi) - \sin(n\theta)(0)$$

$$\cos(n\theta + n\pi) = \cos(n\theta)\cos(n\pi)$$

Recall that

$$\cos(n\pi) = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

Thus

$$\cos(n(\theta+\pi)) = \begin{cases} \cos(n\theta), & n \text{ even} \\ -\cos(n\theta), & n \text{ odd} \end{cases}$$

Now you might be wondering what this has to do with the number of petals in a rose. Well, the polar equation

$$r = a\cos(n\theta)$$

tells us that if $n$ is even and the tip of a petal of the rose has polar coordinates

$$(a\cos(n\theta), \theta) = (a, \theta)$$

then there is also a petal whose tip has polar coordinates

$$(a\cos(n(\theta + \pi)), \theta + \pi) = (a, \theta + \pi)$$

because

$$a\cos(n\theta) = a \Longrightarrow a\cos(n(\theta + \pi)) = a\cos(n\theta) = a$$

Since the first polar coordinate ($a$) is the same for the tips of these two petals and the second polar coordinate is different ($\theta$ for one of them, and $\theta$ for the other one), the tips of these two petals are located at different points of the rose, hence they correspond to different petals.

Similarly, if $n$ is even and the tip of some petal of the rose has polar coordinates

$$(a\cos(n\theta), \theta) = (-a, \theta)$$

for some $\theta \in [0,\pi)$, then there is also a petal whose tip has polar coordinates

$$(a\cos(n(\theta + \pi)), \theta + \pi) = (-a, \theta + \pi)$$

Thus the tips of these two petals are located at different points of the rose, hence they correspond to different petals.

Also, the angle $\theta$ for the first pair of petals is different from the angle $\theta$ for the second pair of petals. This is because for each $\theta$ in the interval $[0,\pi)[0,\pi)$, there is only one value of $a\cos(n\theta)$. Thus the petals in the second pair are not only different from each other but also different from the petals in the first pair.

Now recall that an angle $\theta$ in the interval $[0,2\pi)$ corresponds to the tip of some petal of the rose if and only if

$$\theta \in \left\{0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{(2n-1)\pi}{n}\right\}$$

Note that half of the angles in this set are in the interval $[0,\pi)$ and the other half are in the interval $[\pi,2\pi)$. Also, for every angle $\theta$ in this set which is in the interval $[0,\pi)$, the angle $\theta + \pi$ is also in this set.

Thus the rose $r = a\cos(n\theta)$ has a total of $2n$ petals if $n$ is even, and the tips of the petals of the rose are the points with polar coordinates $(a\cos(n\theta), \theta)$ for

$$\theta \in \left\{0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{(2n-1)\pi}{n}\right\}$$

If, on the other hand, $n$ is odd and the tip of a petal of the rose has polar coordinates

$$(a\cos(n\theta), \theta) = (a, \theta)$$

for some $\theta \in [0,\pi)$, then there is also a petal whose tip has polar coordinates

$$(a\cos(n(\theta+\pi)), \theta+\pi) = (-a, \theta+\pi)$$

because

$$a\cos(n\theta) = a \Longrightarrow a\cos(n(\theta+\pi)) = a(-\cos(n\theta)) = -\,a\cos(n\theta) = -\,a$$

However, $(a,\theta)$ and $(-a,\theta+\pi)$ are two pairs of polar coordinates for the same point. Thus these "two petals" are actually the same petal.

Similarly, if $n$ is odd and the tip of a petal of the rose has polar coordinates

$$(a\cos(n\theta), \theta) = (-a, \theta)$$

for some $\theta$ in the interval $[0,\pi)$, then there is also a petal whose tip has polar coordinates

$$(a\cos(n(\theta+\pi)), \theta+\pi) = (a, \theta+\pi)$$

and these "two petals" are the same petal.

Combining these two results, we see that if $n$ is odd, then the rose $r = a\cos(n\theta)$ has only $n$ petals, and the tips of the petals of the rose are the points with polar coordinates $(a\cos(n\theta), \theta)$ for

$$\theta \in \left\{0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{(n-1)\pi}{n}\right\}$$

These general properties of roses also show that if $n$ is even, the set of points of the rose $r = a\cos(n\theta)$ is identical to the set of points of the rose

$r = -a\cos(n\theta)$. To see this, consider a point of the rose $r = a\cos(n\theta)$, and let $b = a\cos(n\theta)$; that is, this point has polar coordinates $(b, \theta)$. Then

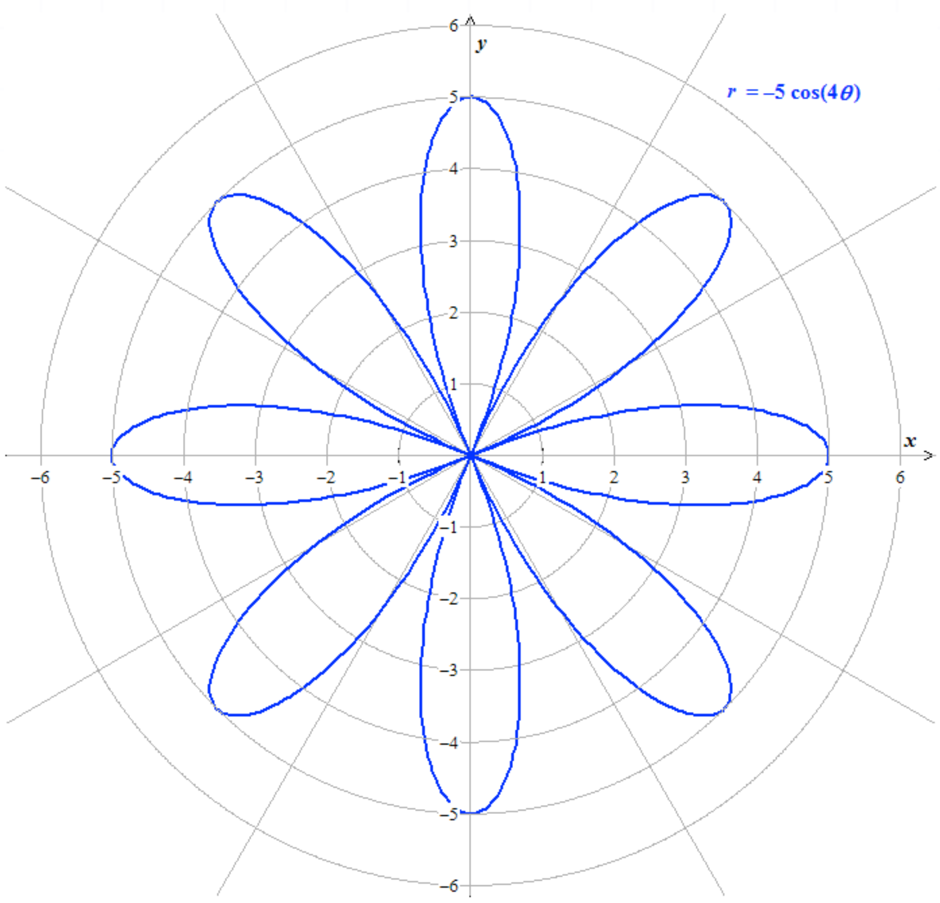
$$a\cos(n\theta) = b \implies -a\cos(n(\theta+\pi)) = -a\cos(n\theta) = -b$$

Thus $(-b, \theta+\pi)$ are the coordinates of a point of the rose $r = -a\cos(n\theta)$, but this point also has polar coordinates $(b, \theta)$.

## Example

Graph the polar equation $r = -5\cos(4\theta)$.

Notice that the rose $r = -5\cos(4\theta)$ has $8 = 2(4)$ petals.




If $n$ is odd, the set of points of the rose $r = -a\cos(n\theta)$ is the "reflection through the pole" of the set of points of the rose $r = a\cos(n\theta)$. To see this, consider a point of the rose $r = a\cos(n\theta)$, and let $b = a\cos(n\theta)$; that is, this point has polar coordinates $(b, \theta)$. Then

$$a\cos(n\theta) = b \implies -a\cos(n(\theta+\pi)) = -a(-\cos(n\theta)) = a\cos(n\theta) = b$$

Thus $(b, \theta+\pi)$ are the polar coordinates of a point of the rose $r = -a\cos(n\theta)$. This point is diametrically opposite the point that has polar coordinates $(b, \theta)$.

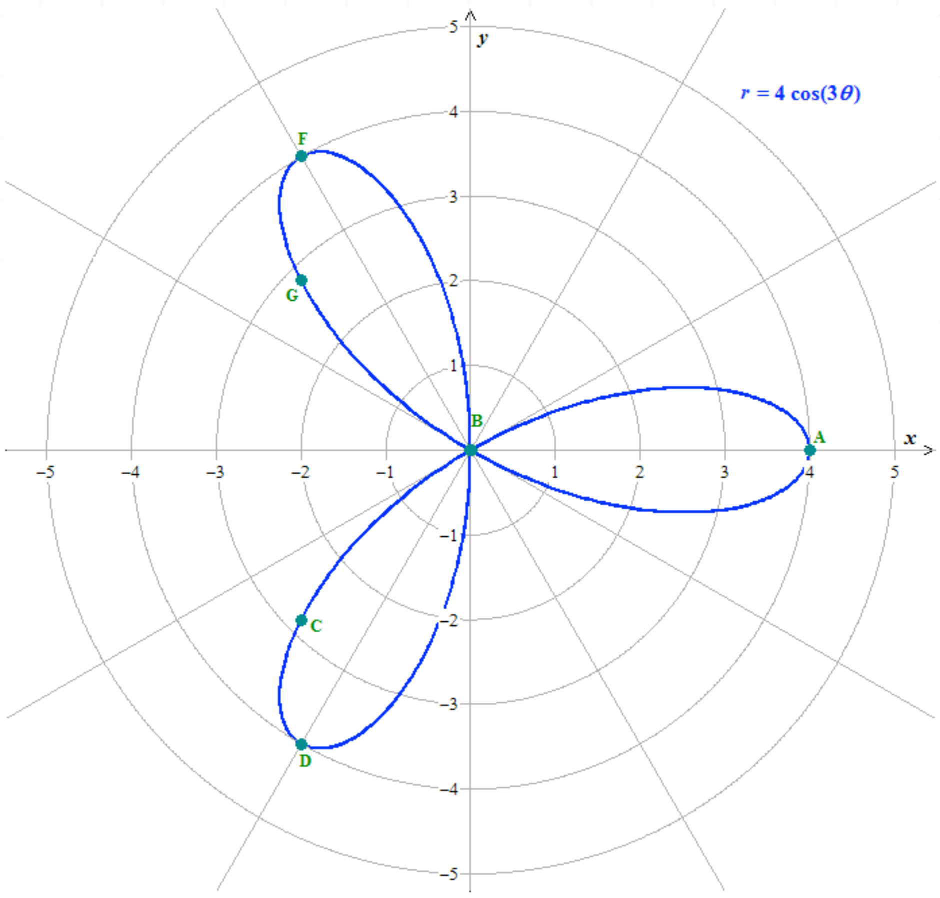
**Example**

Graph the polar equations $r = 4\cos(3\theta)$ and $r = -4\cos(3\theta)$.

In the following table, the values of $\cos(3\theta)$ and $r = 4\cos(3\theta)$ for a few angles $\theta$ in the interval $[0,\pi)$ are shown. In the table, we also give one pair of polar coordinates, $(r, \theta)$, for points where the equation $r = 4\cos(3\theta)$ gives us a positive value of $r$, and we give two pairs of polar coordinates, $(r, \theta)$ and $(-r, \theta+\pi)$, for points where the equation $r = 4\cos(3\theta)$ gives us a negative value of $r$.

| Point | $\theta$ | $\cos(3\theta)$ | $r = 4\cos(3\theta)$ | $(r, \theta)$ | $(-r, \theta+\pi)$ |
|---|---|---|---|---|---|
| A | $0$ | $1$ | $4$ | $(4,0)$ | |
| B=pole | $\pi/6$ | $0$ | $0$ | $(0,\pi/6)$ | |
| C | $\pi/4$ | $-\sqrt{2}/2$ | $-2\sqrt{2}$ | $\left(-2\sqrt{2}, \pi/4\right)$ | $\left(2\sqrt{2}, 5\pi/4\right)$ |
| D | $\pi/3$ | $-1$ | $-4$ | $(-4,\pi/3)$ | $(4,4\pi/3)$ |

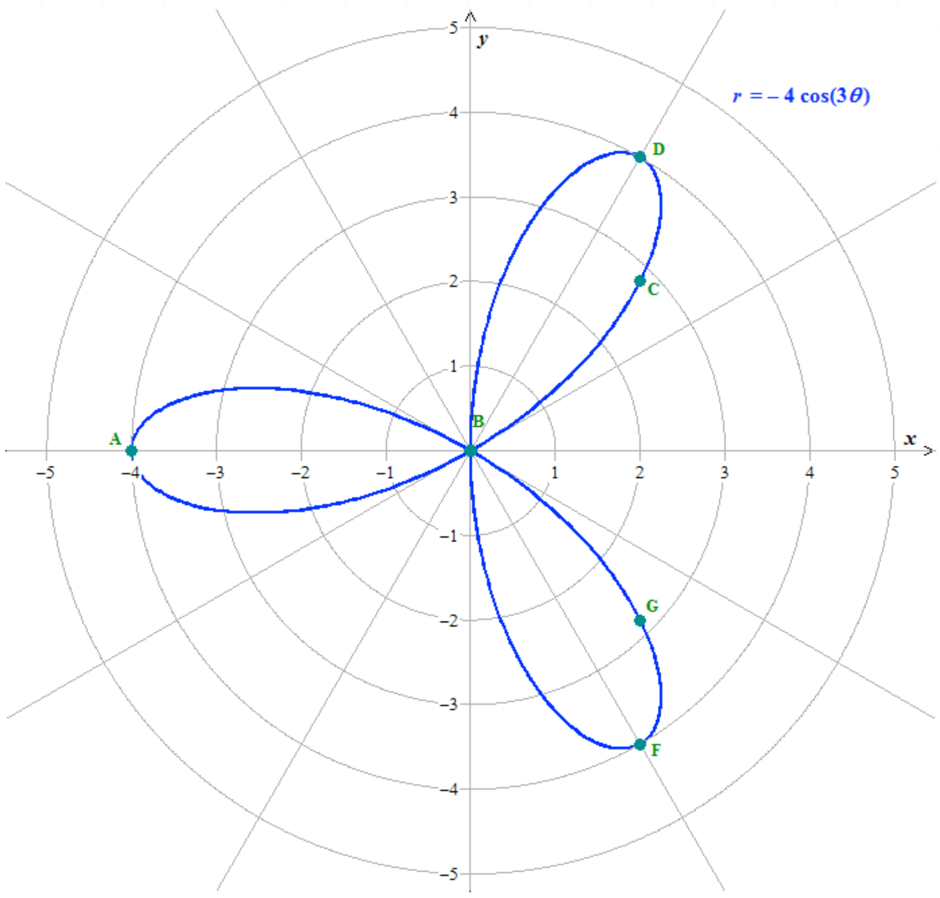
| | | | | |
|---|---|---|---|---|
| E=pole | $\pi/2$ | $0$ | $0$ | $(0,\pi/2)$ |
| F | $2\pi/3$ | $1$ | $4$ | $(4,2\pi/3)$ |
| G | $3\pi/4$ | $\sqrt{2}/2$ | $2\sqrt{2}$ | $\left(2\sqrt{2},3\pi/4\right)$ |
| H=pole | $5\pi/6$ | $0$ | $0$ | $(0,5\pi/6)$ |




Now we'll tabulate the values of $\cos(3\theta)$ and $r = -4\cos(3\theta)$ for those same angles $\theta$ in the interval $[0,\pi)$. Note that the points A, C, D, F, and G are labeled in such a way that they correspond to the points for the same values of $\theta$ from the previous table, hence they are the reflections of those points through the pole.

| Point | $\theta$ | $\cos(3\theta)$ | $r = 4\cos(3\theta)$ | $(r,\theta)$ | $(-r,\theta+\pi)$ |
|---|---|---|---|---|---|
| A | $0$ | $1$ | $-4$ | $(-4,0)$ | $(4,\pi)$ |

| | | | | | |
|---|---|---|---|---|---|
| B=pole | $\pi/6$ | $0$ | $0$ | $(0,\pi/6)$ | |
| C | $\pi/4$ | $-\sqrt{2}/2$ | $2\sqrt{2}$ | $\left(2\sqrt{2},\pi/4\right)$ | |
| D | $\pi/3$ | $-1$ | $4$ | $(4,\pi/3)$ | |
| E=pole | $\pi/2$ | $0$ | $0$ | $(0,\pi/2)$ | |
| F | $2\pi/3$ | $1$ | $-4$ | $(-4,2\pi/3)$ | $(4,5\pi/3)$ |
| G | $3\pi/4$ | $\sqrt{2}/2$ | $-2\sqrt{2}$ | $\left(-2\sqrt{2},3\pi/4\right)$ | $\left(2\sqrt{2},7\pi/4\right)$ |
| H=pole | $5\pi/6$ | $0$ | $0$ | $(0,5\pi/6)$ | |




Notice that the tip of one petal of the rose $r = 4\cos(3\theta)$ is on the positive horizontal axis, and the tip of the corresponding petal of the rose $r = -4\cos(3\theta)$ is on the negative horizontal axis. This property generalizes: If $n$ is odd and $a$ is positive, then the tip of one petal of the rose $r = a\cos(n\theta)$

is on the positive horizontal axis, and the tip of the corresponding petal of the rose $r = -a\cos(n\theta)$ is on the negative horizontal axis.

Now let's turn our attention to the rose which is the graph of the polar equation $r = a\sin(n\theta)$, and let's investigate the numbers and locations of the tips of the petals. Clearly, the tips of the petals are the points that have polar coordinates $(r, \theta)$ with $|r| = |a|$, and

$$|r| = |a| \iff |\sin(n\theta)| = 1$$

Now $|\sin(n\theta)| = 1$ if and only if either

$$\sin(n\theta) = 1$$

or

$$\sin(n\theta) = -1$$

Also, $|\sin(n\theta)| = 1$ if and only if $n\theta$ is an odd integer multiple of $\pi/2$.

In addition,

$$a\sin(n(\theta + 2\pi)) = a\sin(n\theta + 2n\pi) = a\sin(n\theta)$$

because the angles $n\theta$ and $n\theta + 2n\pi$ are coterminal and hence have the same value of the sine function. Thus, just as with the rose $r = a\cos(n\theta)$, we need not consider any angles $\theta$ outside the interval $[0,2\pi)$.

For now, we'll assume that $n$ is positive. If $\sin(n\theta) = 1$, then there is a nonnegative integer $k$ such that

$$n\theta = \frac{\pi}{2} + 2k\pi$$

This implies that

$$\theta = \frac{\frac{\pi}{2} + 2k\pi}{n} = \frac{1(\pi) + 2(2k\pi)}{2n} = \frac{(4k+1)\pi}{2n}$$

Since we want $0 \in [0,2\pi)$, we have

$$0 \leq \frac{(4k+1)\pi}{2n} < 2\pi$$

Multiplying through by $2n/\pi$ gives

$$0 \leq 4k + 1 < 4n$$

Now $k$ is a nonnegative integer, so $4k + 1$ is an odd integer and $4k + 1 \geq 1$, hence

$$1 \leq 4k + 1 < 4n$$

Also, $4k + 1$ is an odd integer and $4n$ is an even integer, so

$$1 \leq 4k + 1 \leq 4n - 1$$

Subtracting $1$ throughout yields

$$0 \leq 4k \leq 4n - 2$$

Dividing through by $4$, we get

$$0 \leq k \leq n - \frac{1}{2}$$

Now $k$ is an integer, and $n - (1/2)$ isn't an integer, so

$$0 \leq k \leq n - 1$$

Thus the tip of each petal for which $\sin(n\theta) = 1$ is located at some point $(r, \theta)$ with $|r| = |a|$ and

$$\theta \in \left\{ \frac{\pi}{2n}, \frac{5\pi}{2n}, \dots, \frac{[4(n-1)+1]\pi}{2n} \right\}$$

Now

$$4(n-1) + 1 = 4n - 4 + 1 = 4n - 3$$

so

$$\theta \in \left\{ \frac{\pi}{2n}, \frac{5\pi}{2n}, \dots, \frac{(4n-3)\pi}{2n} \right\}$$

If $\sin(n\theta) = -1$, then there is a nonnegative integer $k$ such that

$$n\theta = \frac{3\pi}{2} + 2k\pi$$

This implies that

$$\theta = \frac{\frac{3\pi}{2} + 2k\pi}{n} = \frac{1(3\pi) + 2(2k\pi)}{2n} = \frac{(4k+3)\pi}{2n}$$

Since we want $0 \in [0,2\pi)$, we have

$$0 \le \frac{(4k+3)\pi}{2n} < 2\pi$$

Multiplying through by $2n/\pi$ gives

$$0 \le 4k + 3 < 4n$$

Now $k$ is a nonnegative integer, so $4k + 3$ is an odd integer and $4k + 3 \geq 3$, hence

$$3 \leq 4k + 3 < 4n$$

Also, $4k + 3$ is an odd integer and $4n$ is an even integer, so

$$3 \leq 4k + 3 \leq 4n - 1$$

Subtracting $3$ throughout yields

$$0 \leq 4k \leq 4n - 4$$

That is,

$$0 \leq 4k \leq 4(n - 1)$$

Dividing through by $4$, we obtain

$$0 \leq k \leq n - 1$$

Thus the tip of each petal for which $\sin(n\theta) = 1$ is located at some point $(r, \theta)$ with $|r| = |a|$ and

$$\theta \in \left\{ \frac{3\pi}{2n}, \frac{7\pi}{2n}, \ldots, \frac{[4(n-1)+3]\pi}{2n} \right\}$$

Now

$$4(n - 1) + 3 = 4n - 4 + 3 = 4n - 1$$

so

$$\theta \in \left\{ \frac{3\pi}{2n}, \frac{7\pi}{2n}, \ldots, \frac{(4n-1)\pi}{2n} \right\}$$

Combining the results for $\sin(n\theta) = 1$ and $\sin(n\theta) = -1$, we find that the tip of each petal is located at some point $(r, \theta)$ with $|r| = |a|$ and some angle $\theta$ in the set

$$\left\{ 0, \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \frac{7\pi}{2n}, \ldots, \frac{(4n-1)\pi}{2n} \right\}$$

Since $|\sin(n\theta)| = 1$ for every angle $\theta$ in this set, every element of the set corresponds to the tip of some petal of the rose.

This set of angles $\theta$ has exactly $2n$ elements. However, just as with the rose $r = a\cos(n\theta)$, the rose $r = a\sin(n\theta)$ has $2n$ petals if $n$ is even and only $n$ petals if $n$ is odd.

Let's see why we get a different result for the two cases. To do this, we'll consider the value of $a\sin(n(\theta + \pi))$ for angles $\theta$ in the interval $[0, \pi)$. First, we have

$$\sin(n(\theta + \pi)) = \sin(n\theta + n\pi)$$

By the sum identity for sine,

$$\sin(n\theta + n\pi) = \sin(n\theta)\cos(n\pi) + \cos(n\theta)\sin(n\pi)$$

$$\sin(n\theta + n\pi) = \sin(n\theta)\cos(n\pi) + \cos(n\theta)(0)$$

$$\sin(n\theta + n\pi) = \sin(n\theta)\cos(n\pi)$$

Recall that

$$\cos(n\pi) = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

Thus

$$\sin(n(\theta + \pi)) = \begin{cases} \sin(n\theta), & n \text{ even} \\ -\sin(n\theta), & n \text{ odd} \end{cases}$$

The polar equation

$$r = a\sin(n\theta)$$

tells us that if $n$ is even and the tip of a petal of the rose has polar coordinates

$$(a\sin(n\theta), \theta) = (a, \theta)$$

then there is also a petal whose tip has polar coordinates

$$(a\sin(n(\theta + \pi)), \theta + \pi) = (a, \theta + \pi)$$

because

$$a\sin(n\theta) = a \Longrightarrow a\sin(n(\theta + \pi)) = a\sin(n\theta) = a$$

Since the first polar coordinate ($a$) is the same for the tips of these two petals and the second polar coordinate is different ($\theta$ for one of them, and $\theta$ for the other one), the tips of these two petals are located at different points of the rose, hence they correspond to different petals.

Similarly, if $n$ is even and the tip of some petal of the rose has polar coordinates

$$(a\sin(n\theta), \theta) = (-a, \theta)$$

for some $\theta \in [0,\pi)$, then there is also a petal whose tip has polar coordinates

$$(a\sin(n(\theta + \pi)), \theta + \pi) = (-a, \theta + \pi)$$

Thus the tips of these two petals are located at different points of the rose, hence they correspond to different petals.

Also, the angle $\theta$ for the first pair of petals is different from the angle $\theta$ for the second pair of petals. This is because for each $\theta$ in the interval $[0,\pi)$, there is only one value of $a\sin(n\theta)$. Thus the petals in the second pair are not only different from each other but also different from the petals in the first pair.

Now recall that an angle $\theta$ in the interval $[0,2\pi)$ corresponds to the tip of some petal of the rose if and only if

$$\theta \in \left\{\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{n}, \frac{5\pi}{n}, \ldots, \frac{(4n-1)\pi}{2n}\right\}$$

Note that half of the angles in this set are in the interval $[0,\pi)$ and the other half are in the interval $[\pi,2\pi)$. Also, for every angle $\theta$ in this set which is in the interval $[0,\pi)$, the angle $\theta + \pi$ is also in this set.

Thus the rose $r = a\sin(n\theta)$ has a total of $2n$ petals if $n$ is even, and the tips of the petals of the rose are the points with polar coordinates $(a\sin(n\theta), \theta)$ for

$$\theta \in \left\{\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \frac{7\pi}{2n}, \ldots, \frac{(4n-1)\pi}{2n}\right\}$$

If $n$ is odd and the tip of a petal of the rose has polar coordinates

$$(a\sin(n\theta),\theta)=(a,\theta)$$

for some $\theta\in[0,\pi)$, then there is also a petal whose tip has polar coordinates

$$(a\sin(n(\theta+\pi)),\theta+\pi)=(-a,\theta+\pi)$$

because

$$a\sin(n\theta)=a\Longrightarrow a\sin(n(\theta+\pi))=a(-\sin(n\theta))=-a\sin(n\theta)=-a$$

However, $(a,\theta)$ and $(-a,\theta+\pi)$ are two pairs of polar coordinates for the same point. Thus these "two petals" are actually the same petal.

Similarly, if $n$ is odd and the tip of a petal of the rose has polar coordinates

$$(a\sin(n\theta),\theta)=(-a,\theta)$$

for some $\theta$ in the interval $[0,\pi)$, then there is also a petal whose tip has polar coordinates

$$(a\sin(n(\theta+\pi)),\theta+\pi)=(a,\theta+\pi)$$

and these "two petals" are the same petal.

Combining these two results, we see that if $n$ is odd, then the rose $r=a\sin(n\theta)$ has only $n$ petals, and the tips of the petals of the rose are the points with polar coordinates $(a\sin(n\theta),\theta)$ for

$$\theta\in\left\{\frac{\pi}{2n},\frac{3\pi}{2n},\frac{5\pi}{2n},\frac{7\pi}{2n},\ldots,\frac{(2n-1)\pi}{2n}\right\}$$

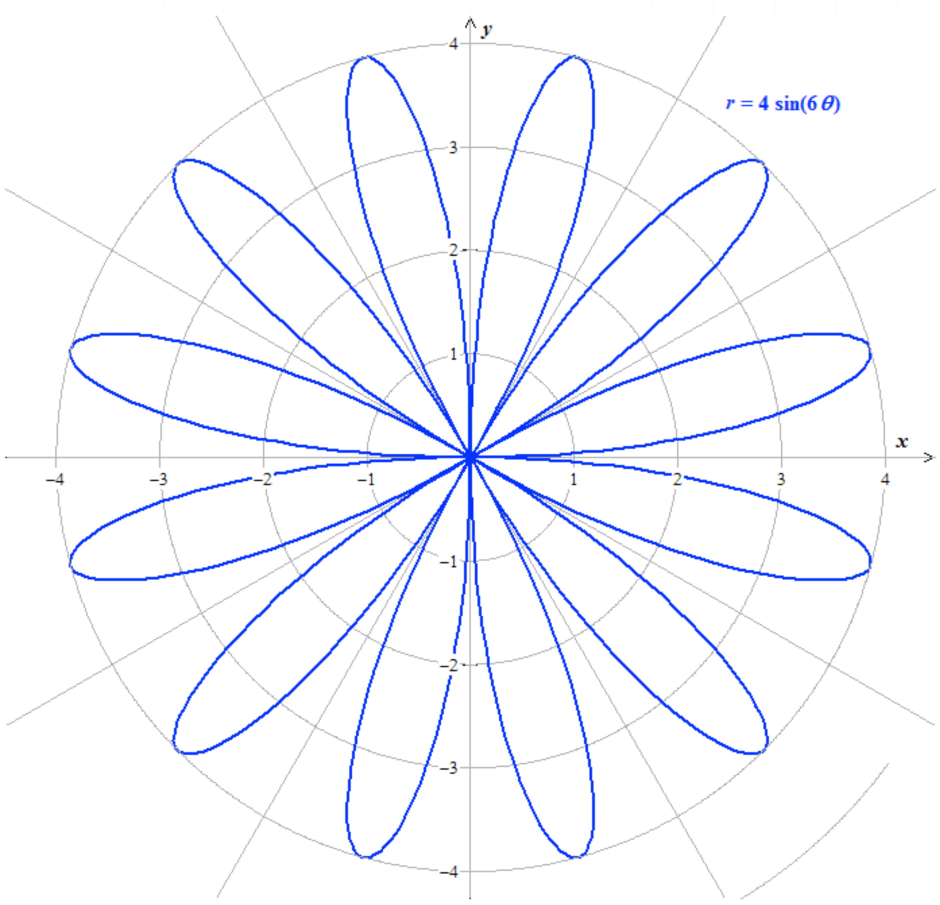
These general properties of roses also show that if $n$ is even, the set of points of the rose $r = a\sin(n\theta)$ is identical to the set of points of the rose $r = -a\sin(n\theta)$. To see this, consider a point of the rose $r = a\sin(n\theta)$, and let $b = a\sin(n\theta)$. Then

$$a\sin(n\theta) = b \implies -a\sin(n(\theta+\pi)) = -a\sin(n\theta) = -b$$

Thus $(-b, \theta+\pi)$ are the coordinates of a point of the rose $r = -a\sin(n\theta)$, but this point also has polar coordinates $(b, \theta)$.

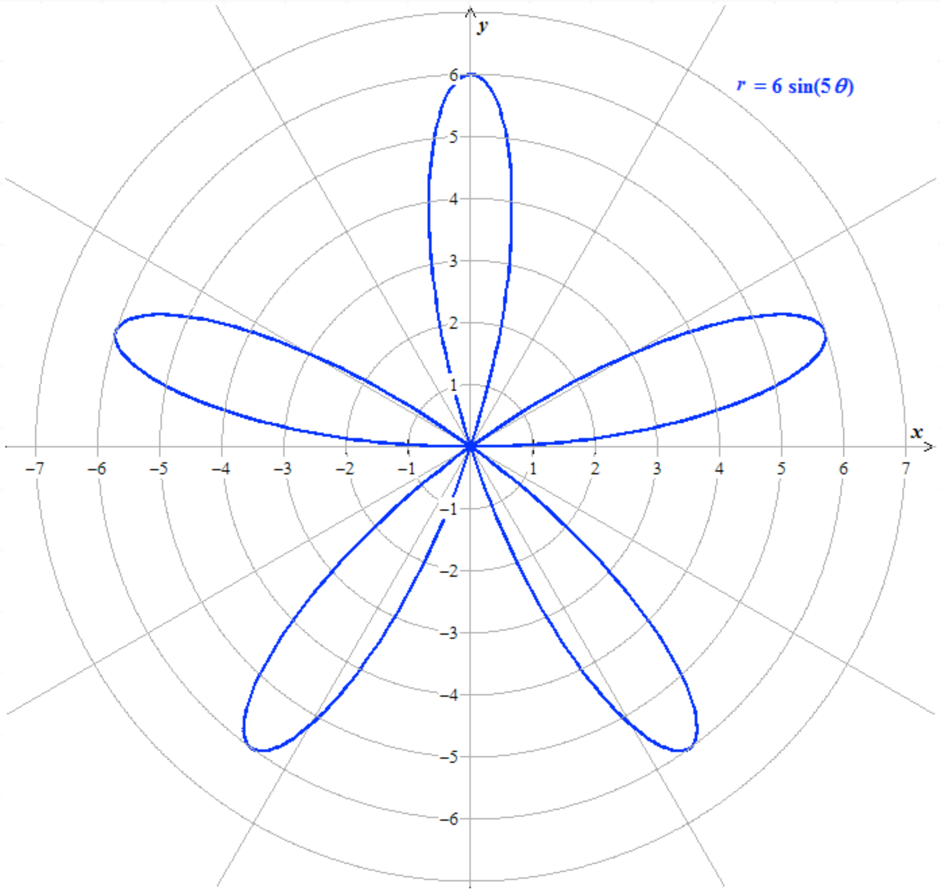
## Example

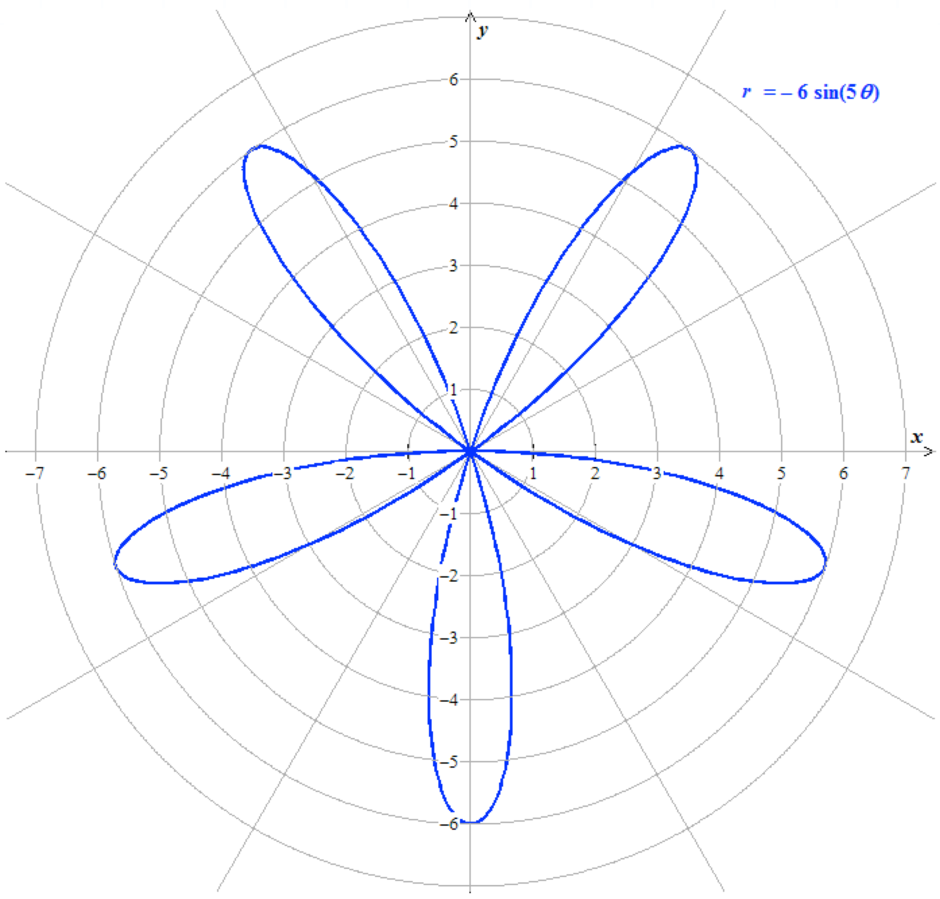
Graph the polar equation $4\sin(6\theta)$.




## Example

Graph the polar equations $r = 6\sin(5\theta)$ and $r = -6\sin(5\theta)$.



Notice that the tip of one petal of the rose $r = 6\sin(5\theta)$ is on the positive vertical axis, and the tip of the corresponding petal of the rose $r = -6\sin(5\theta)$ is on the negative vertical axis. This property generalizes: If $n$ is odd and $a$ is positive, then the tip of one petal of the rose $r = a\sin(n\theta)$ is on the positive vertical axis, and the tip of the corresponding petal of the rose $r = -a\sin(n\theta)$ is on the negative vertical axis. Thus these roses are rotated through an angle of measure $\pi/2$ with respect to the roses $r = a\cos(n\theta)$ and $r = -a\cos(n\theta)$, respectively.

The only case that we haven't yet addressed is that of a polar equation of the form $r = a\sin(n\theta)$ where $n$ is negative. Well, let $n = -m$. (Note that $m$ is positive.) By the odd identity for sine,

$$\sin(n\theta) = \sin(-m\theta) = -\sin(m\theta)$$

Thus the rose $r = a\sin(n\theta)$ is identical to the rose $r = -a\sin(m\theta)$.