Precalculus Workbook Solutions

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MATH

COMPLEX NUMBERS

■ 1. Simplify the imaginary number.

i^{437}

Solution:

We need to look for the largest number less than or equal to 437 that's divisible by 4. 437 isn't divisible by 4, so we try 436. 436 is the first number we come to that's divisible by 4, so we separate the exponent.

$$i^{437}$$

$$i^{436+1}$$

$$i^{436}i^1$$

Rewrite i^{436} as a power of 4.

$$(i^4)^{109}i^1$$

We know that i^4 is always 1, so

$$(1)^{109}i^1$$

$$1i^1$$

$$i^1$$

$$i$$

■ 2. Simplify the imaginary number.

$i^{2,314}$

Solution:

We need to look for the largest number less than or equal to $2,314$ that's divisible by 4. $2,314$ isn't divisible by 4, so we try $2,313$, then $2,312$. $2,312$ is the first number we come to that's divisible by 4, so we separate the exponent.

$$i^{2,314}$$

$$i^{2,312+2}$$

$$i^{2,312}i^2$$

Rewrite $i^{2,312}$ as a power of 4.

$$(i^4)^{578}i^2$$

We know that i^4 is always 1, so

$$(1)^{578}i^2$$

$$1i^2$$

$$i^2$$

-1

■ 3. Name the real and imaginary parts of the complex number.

$z = -5 + 17i$

Solution:

For a complex number in the form $z = a + bi$, a is always the real part and b is always the imaginary part. If b is negative, you have to remember to include the negative sign when you name the imaginary part. So in the complex number $z = -5 + 17i$, -5 is the real part, and 17 is the imaginary part.

■ 4. Name the real and imaginary parts of the complex number.

$z = \sqrt{7} - 4\pi i$

Solution:

For a complex number in the form $z = a + bi$, a is always the real part and b is always the imaginary part. If b is negative, you have to remember to include the negative sign when you name the imaginary part. So in the complex number $z = \sqrt{7} - 4\pi i$, $\sqrt{7}$ is the real part, and -4π is the imaginary part.

5. How can the numbers be classified?

$z = -3 + 9i$

$z = 0 - 15i$

$z = 6 + 0i$

Solution:

For the number $z = -3 + 9i$, both the real part and the imaginary part are non-zero. So, this is a **complex number**.

For the number $z = 0 - 15i$, the real part is 0 and the imaginary part is non-zero. Because the real-number part of the complex number is 0, we end up with

$$z = 0 - 15i$$

$$z = -15i$$

This is a **pure imaginary** number. Because every pure imaginary number is also a complex number, we can call this a complex number as well.

For the number $z = 6 + 0i$, the real part is non-zero and the imaginary part is 0. Because the imaginary part of the complex number is 0, we end up with

$$z = 6 + 0i$$

$$z = 6$$

This is a **real number**. Because every real number is also a complex number, we can call this a complex number as well.

■ 6. How can the numbers be classified?

$$z = 0 - \pi i$$

$$z = -\sqrt{5} + 0i$$

$$z = -11 + \frac{2}{3}i$$

Solution:

For the number $z = 0 - \pi i$, the real part is 0 and the imaginary part is non-zero. Because the real-number part of the complex number is 0, we end up with

$$z = 0 - \pi i$$

$$z = -\pi i$$

This is a **pure imaginary** number. Because every pure imaginary number is also a complex number, we can call this a complex number as well.

For the number $z = -\sqrt{5} + 0i$, the real part is non-zero and the imaginary part is 0. Because the imaginary part of a complex number is 0, we end up with

$$z = -\sqrt{5} + 0i$$

$$z = -\sqrt{5}$$

This is a **real number**. Because every real number is also a complex number, we can call this a complex number as well.

For the number $z = -11 + (2/3)i$, both the real part and the imaginary part are non-zero. So, this is a **complex number**.

COMPLEX NUMBER OPERATIONS

■ 1. Find the sum and difference of the complex numbers.

$$\frac{7}{5}-\frac{2}{3}i$$

$$\frac{7}{2}-\frac{8}{3}i$$

Solution:

The sum of the complex numbers is

$$\left(\frac{7}{5}-\frac{2}{3}i\right)+\left(\frac{7}{2}-\frac{8}{3}i\right)$$

$$\left(\frac{7}{5}+\frac{7}{2}\right)+\left(-\frac{2}{3}i-\frac{8}{3}i\right)$$

$$\left(\frac{14}{10}+\frac{35}{10}\right)+\left(-\frac{10}{3}i\right)$$

$$\frac{49}{10}-\frac{10}{3}i$$

The difference of the complex numbers is

$$\left(\frac{7}{5}-\frac{2}{3}i\right)-\left(\frac{7}{2}-\frac{8}{3}i\right)$$

$$\frac{7}{5} - \frac{2}{3}i - \frac{7}{2} + \frac{8}{3}i$$

$$\frac{7}{5} - \frac{7}{2} - \frac{2}{3}i + \frac{8}{3}i$$

$$\frac{14}{10} - \frac{35}{10} - \frac{2}{3}i + \frac{8}{3}i$$

$$-\frac{21}{10} + \frac{6}{3}i$$

$$-\frac{21}{10} + 2i$$

■ 2. Find the product of the complex numbers.

$-7i$

$-5 + 9i$

Solution:

Use the distributive property to find the product of the complex numbers.

$$-7i(-5 + 9i)$$

$$(-7i)(-5) + (-7i)(9i)$$

$$35i - 63i^2$$

Using $i^2 = -1$ in the last term, we get

$$35i - 63(-1)$$

$$35i + 63$$

$$63 + 35i$$

■ 3. Find the product of the complex numbers.

$5 - 2i$

$6 - 11i$

Solution:

Use FOIL to find the product of the complex numbers.

$$(5 - 2i)(6 - 11i)$$

$$(5)(6) + (5)(-11i) + (-2i)(6) + (-2i)(-11i)$$

$$30 - 55i - 12i + 22i^2$$

$$30 - 67i + 22i^2$$

Using $i^2 = -1$ in the last term, we get

$$30 - 67i + 22(-1)$$

$$30 - 67i - 22$$

$$8 - 67i$$

■ 4. Divide the complex number $-4 + 15i$ by the imaginary number $5i$.

Solution:

Set up the division.

$$\frac{-4 + 15i}{5i}$$

$$\frac{-4}{5i} + \frac{15i}{5i}$$

$$-\frac{4}{5}i^{-1} + 3$$

We know that i^{-1} is equal to $-i$.

$$-\frac{4}{5}(-i) + 3$$

$$\frac{4}{5}i + 3$$

$$3 + \frac{4}{5}i$$

■ 5. Find the complex conjugate of each complex number.

$9 - 9i$

$-3 + 13i$

$11 - 22i$

Solution:

For each of these, we keep the real part (9, -3, or 11) and change the sign of the imaginary part (from -9 to 9, from 13 to -13, or from -22 to 22). So the complex conjugates are:

The complex conjugate of $9 - 9i$ is $9 + 9i$.

The complex conjugate of $-3 + 13i$ is $-3 - 13i$.

The complex conjugate of $11 - 22i$ is $11 + 22i$.

■ 6. Express the fraction in the form $a + bi$ where a and b are real numbers.

$$\frac{-3 + 7i}{4 - 5i}$$

Solution:

Multiply by the conjugate of the denominator.

$$\left(\frac{-3+7i}{4-5i}\right)\left(\frac{4+5i}{4+5i}\right)$$

$$\frac{(-3+7i)(4+5i)}{(4-5i)(4+5i)}$$

Use FOIL to expand the numerator and denominator.

$$\frac{-12-15i+28i+35i^2}{16+20i-20i-25i^2}$$

$$\frac{-12+13i+35i^2}{16-25i^2}$$

Using $i^2=-1$ gives

$$\frac{-12+13i+35(-1)}{16-25(-1)}$$

$$\frac{-12+13i-35}{16+25}$$

$$\frac{-47+13i}{41}$$

$$-\frac{47}{41}+\frac{13}{41}i$$

GRAPHING COMPLEX NUMBERS

■ 1. Graph $-3+5i$, $2-4i$, and 5 in the complex plane.

Solution:

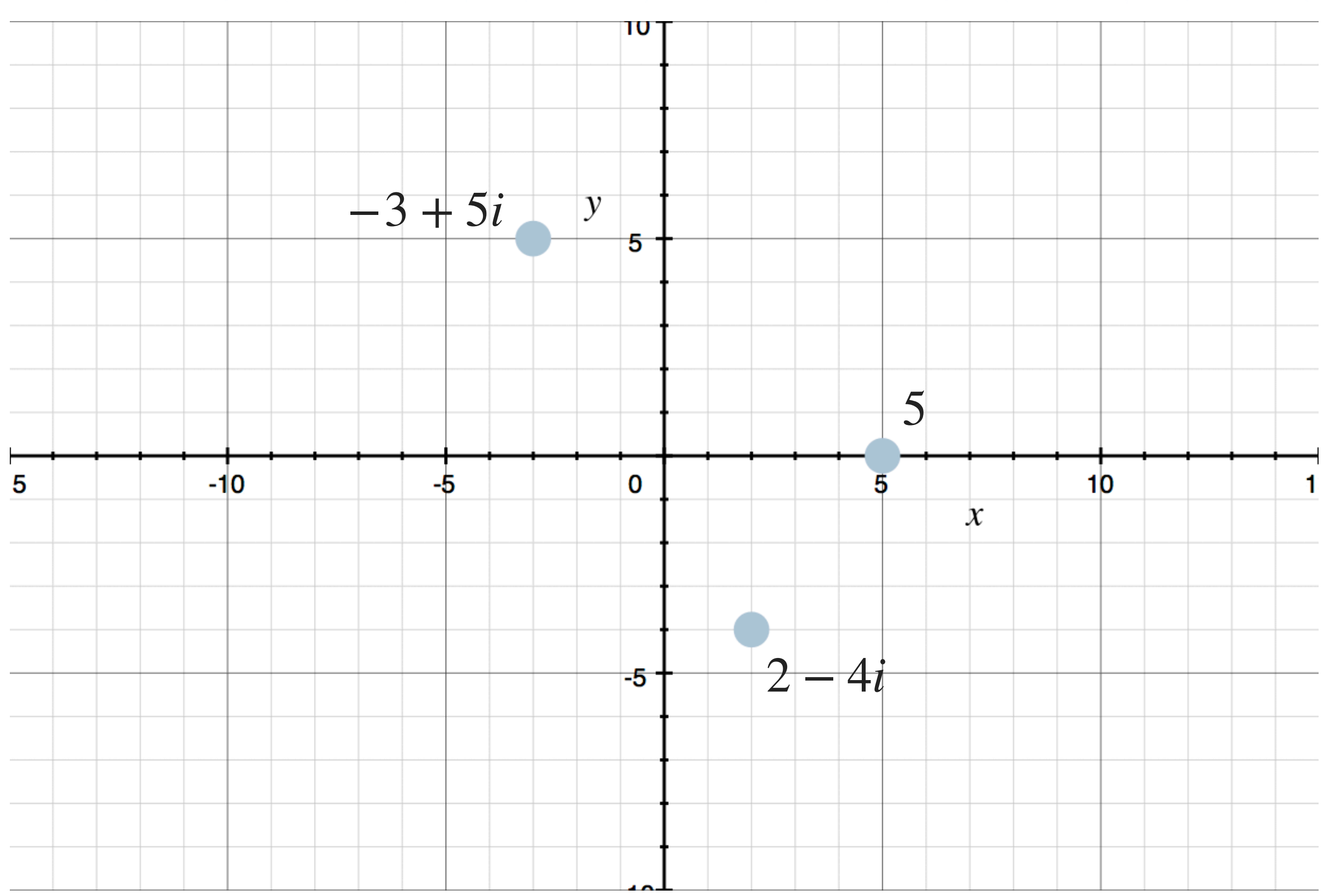
Let's break down each complex number.

$-3+5i$ has real part $a=-3$ and imaginary part $b=5$.

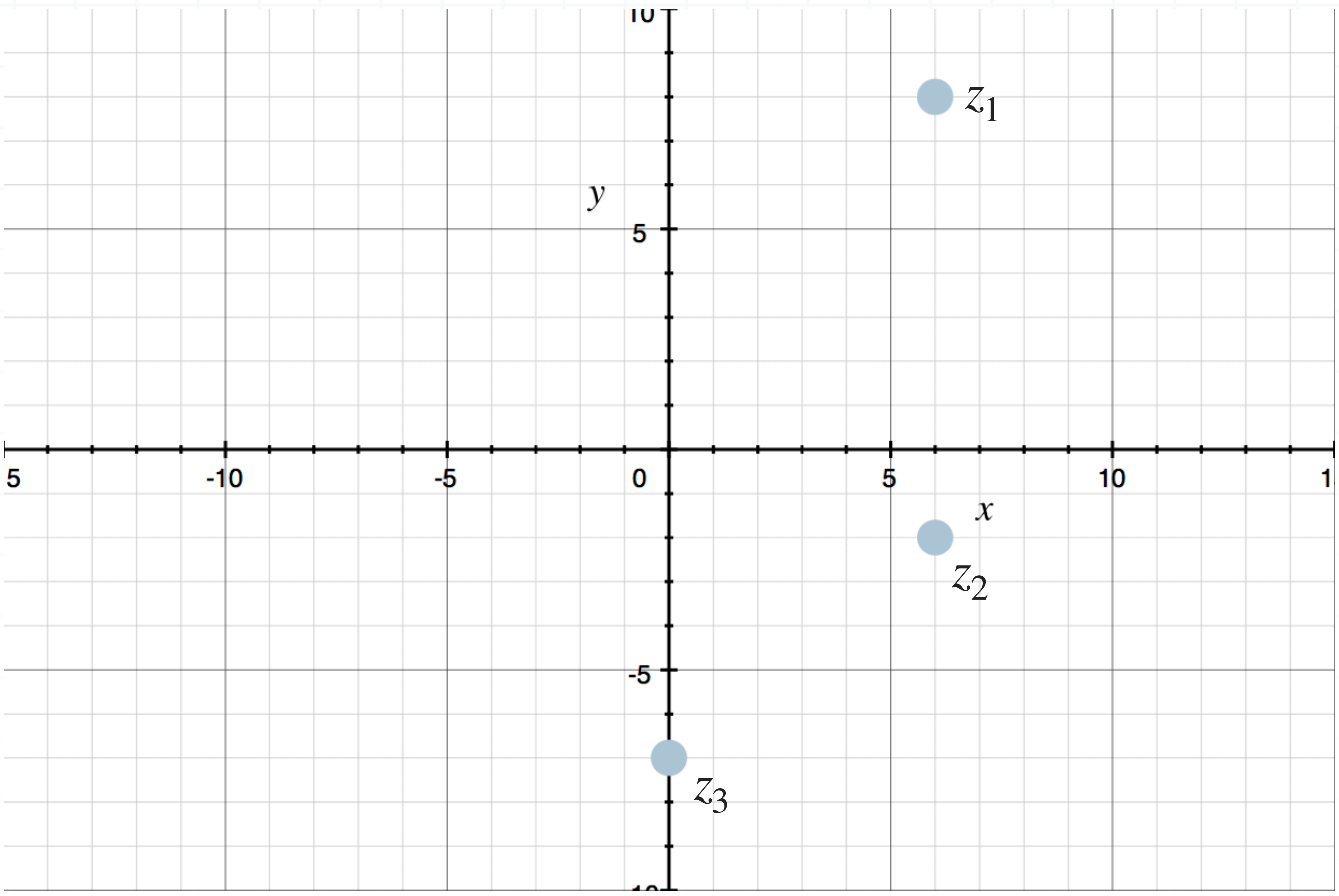
$2-4i$ has real part $a=2$ and imaginary part $b=-4$.

5 has real part $a=5$ and imaginary part $b=0$.

Now we can graph all of them together.



■ 2. Which three complex numbers are represented in the graph?



Solution:

The point z_1 is 6 units to the right of the vertical axis and 8 units above the horizontal axis, so it's the complex number $6 + 8i$.

The point z_2 is 6 units to the right of the vertical axis and 2 units below the horizontal axis, so it's the complex number $6 - 2i$.

The point z_3 is 0 units to the left or right of the vertical axis and 7 units below the horizontal axis, so it's the complex number $-7i$.

■ 3. Graph the sum of the complex numbers $5 - 4i$ and $-1 + 10i$.

Solution:

First, find the sum.

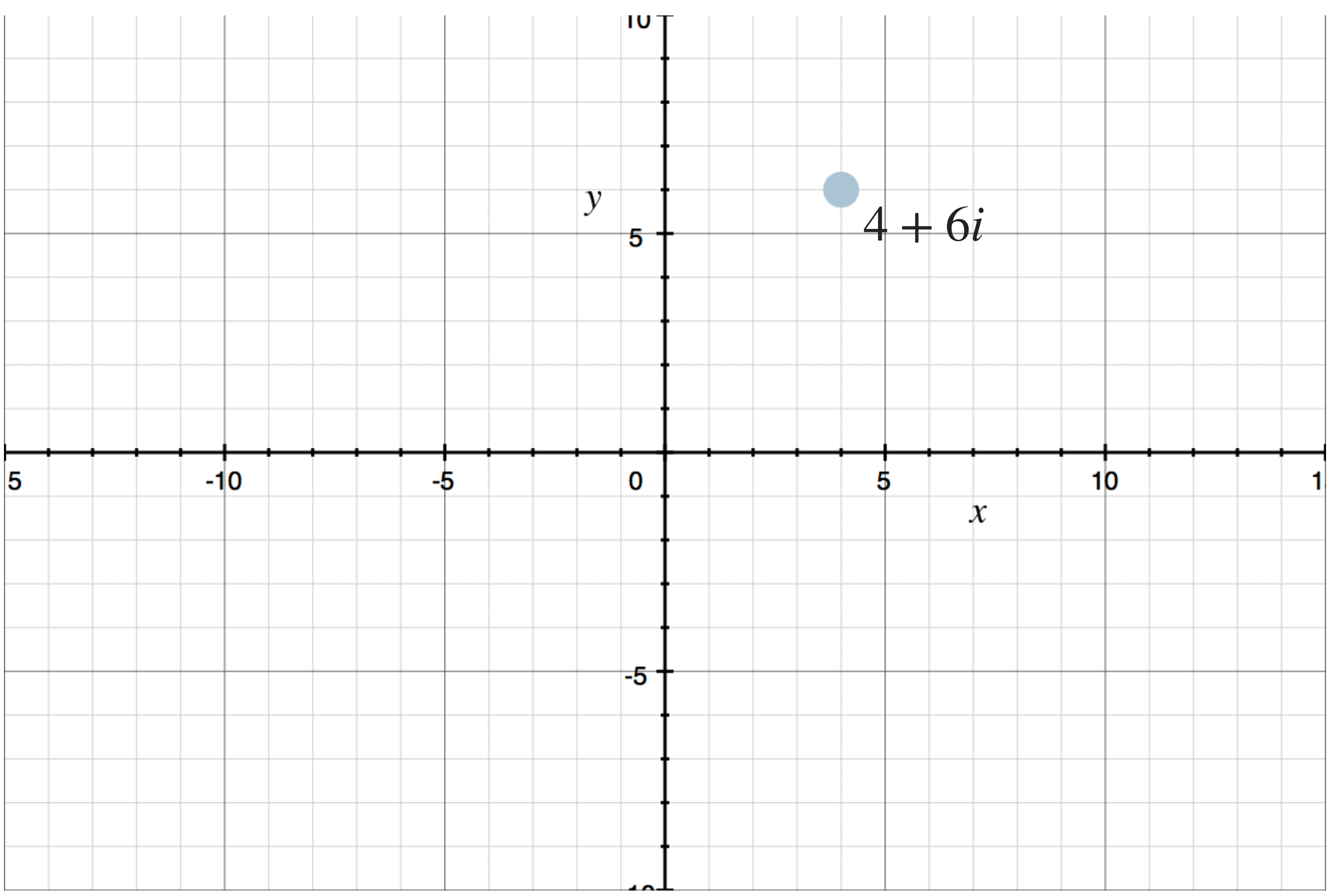
$$(5 - 4i) + (-1 + 10i)$$

$$(5 + (-1)) + (-4 + 10)i$$

$$(5 - 1) + (-4 + 10)i$$

$$4 + 6i$$

Now graph the complex number $4 + 6i$, which has a real part 4 and an imaginary part 6.



■ 4. Graph the difference of the complex numbers $8 - 7i$ and $13 - 4i$.

Solution:

First, find the difference.

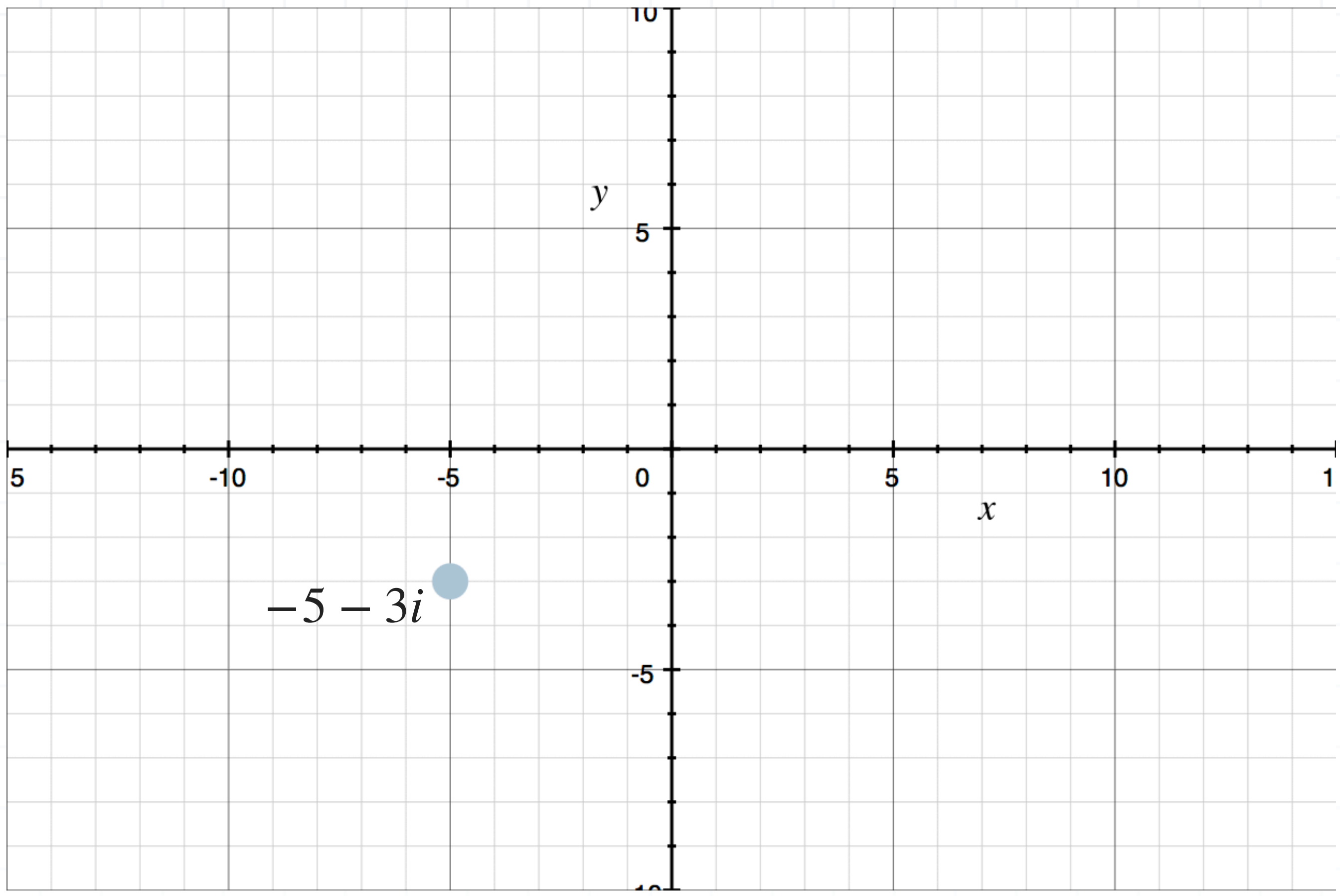
$$(8 - 7i) - (13 - 4i)$$

$$(8 - 13) + (-7 - (-4))i$$

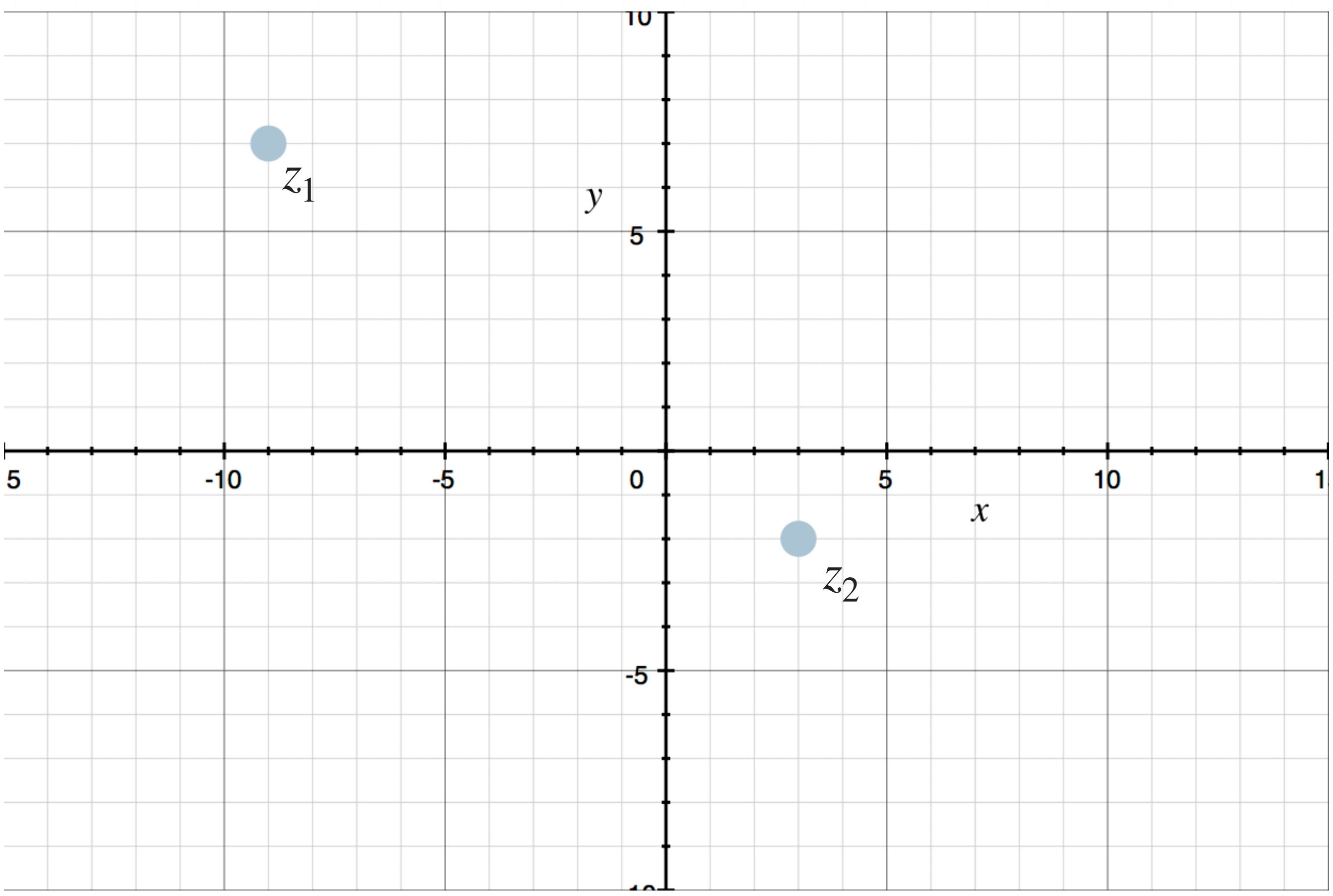
$$(8 - 13) + (-7 + 4)i$$

$$-5 - 3i$$

Now graph the complex number $-5 - 3i$, which has a real part -5 and an imaginary part -3.



■ 5. Graph the sum of the complex numbers z_1 and z_2.



Solution:

The point z_1 is 9 units to the left of the vertical axis and 7 units above the horizontal axis, which means that complex number is $z_1 = -9 + 7i$.

The point z_2 is 3 units to the right of the vertical axis and 2 units below the horizontal axis, which means that complex number is $z_2 = 3 - 2i$.

The sum of z_1 and z_2 is

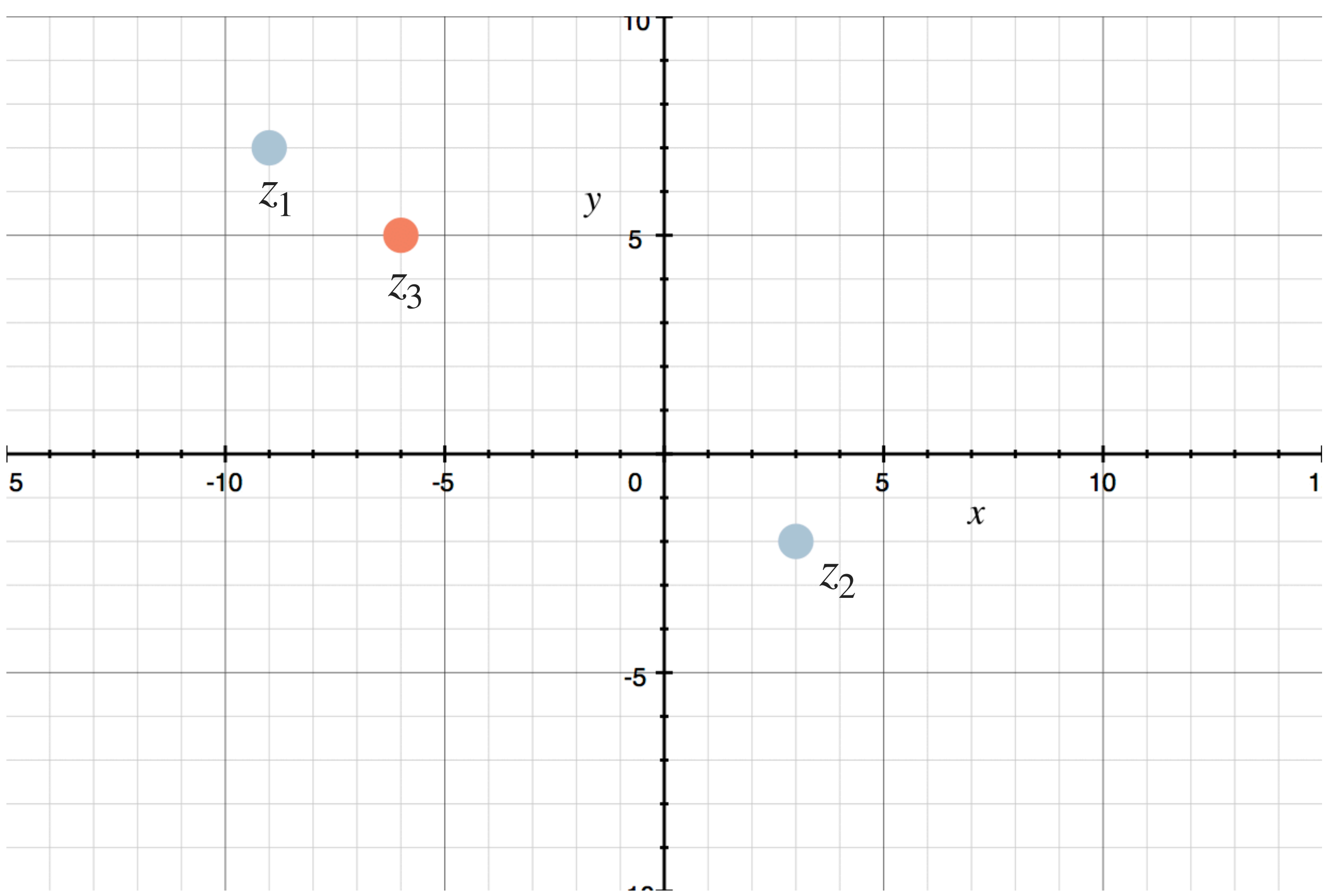
$$z_1 + z_2 = (-9 + 7i) + (3 - 2i)$$

$$z_1 + z_2 = (-9 + 3) + (7 + (-2))i$$

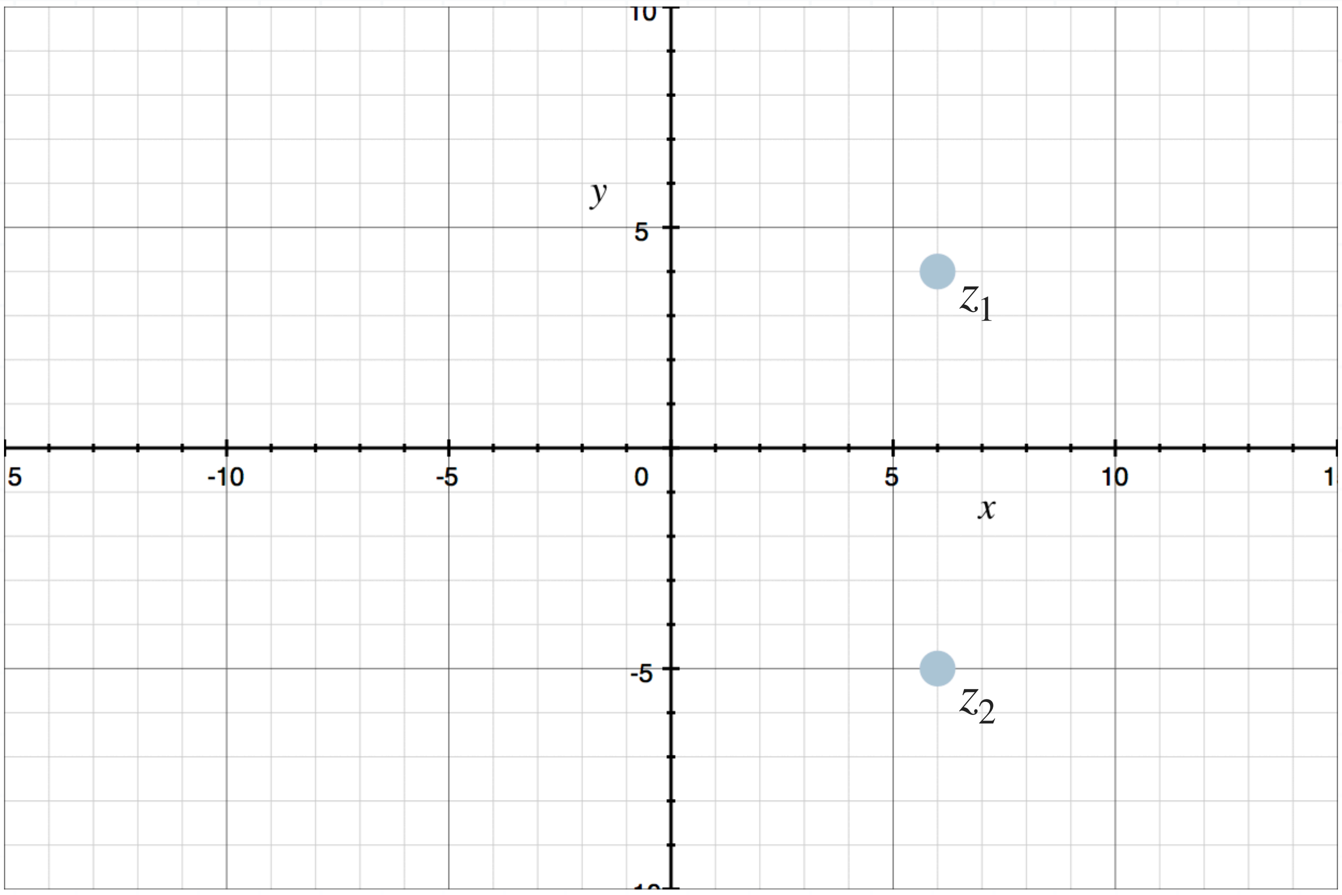
$$z_1 + z_2 = (-9 + 3) + (7 - 2)i$$

$$z_1 + z_2 = -6 + 5i$$

So if we plot the sum on the same set of axes, we get



■ 6. Graph the difference of the complex numbers z_1 and z_2.



Solution:

The point z_1 is 6 units to the right of the vertical axis and 4 units above the horizontal axis, which means that complex number is $z_1 = 6 + 4i$.

The point z_2 is 6 units to the right of the vertical axis and 5 units below the horizontal axis, which means that complex number is $z_2 = 6 - 5i$.

The difference of z_1 and z_2 is

$$z_1 - z_2 = (6 + 4i) - (6 - 5i)$$

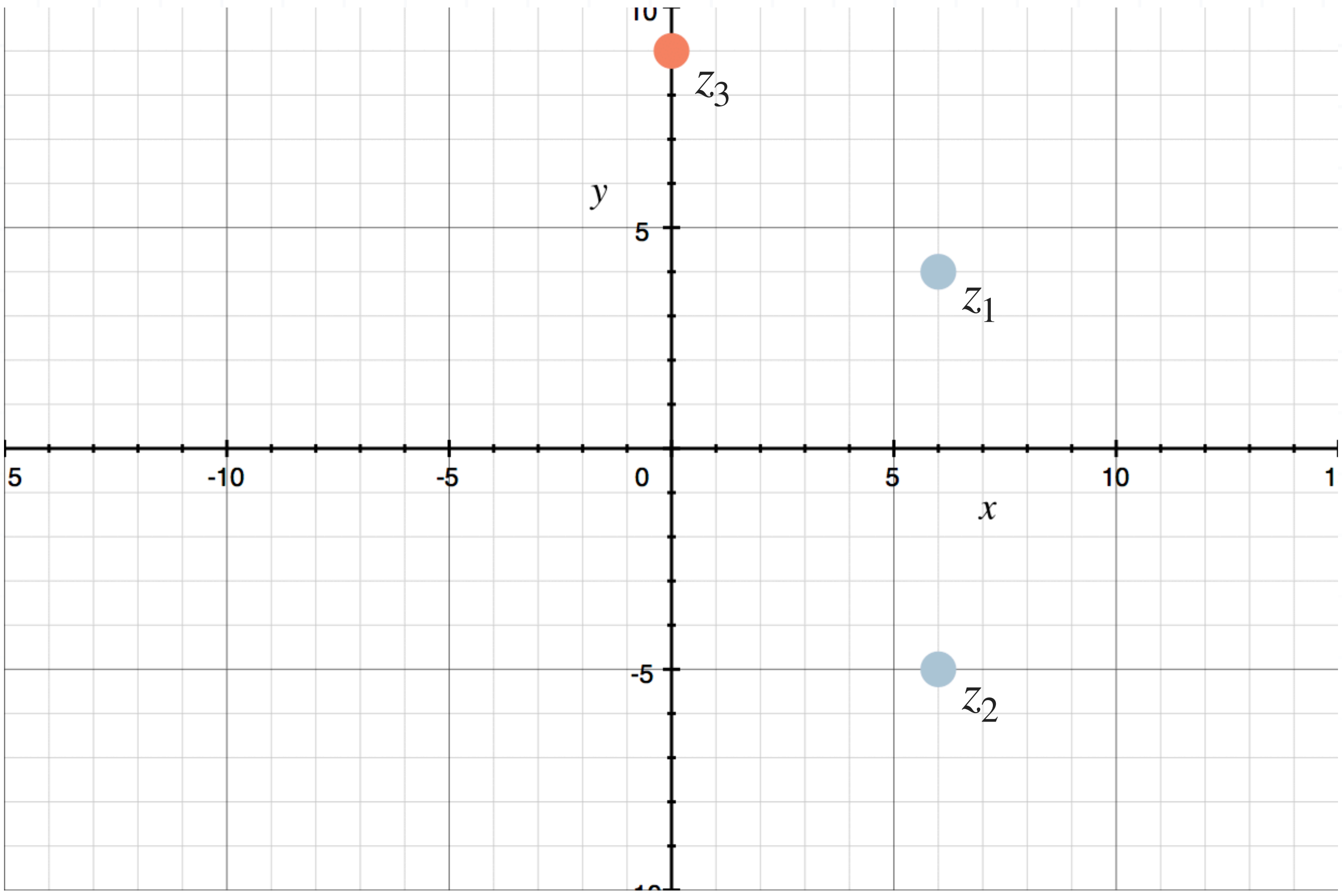
$$z_1 - z_2 = (6 - 6) + (4 - (-5))i$$

$$z_1 - z_2 = (6 - 6) + (4 + 5)i$$

$$z_1 - z_2 = 0 + 9i$$

$$z_1 - z_2 = 9i$$

So if we plot the difference on the same set of axes, we get



DISTANCES AND MIDPOINTS

■ 1. Find the distance between $s = 5 + 3i$ and $t = 1 - i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(5-1)^2 + (3-1)^2}$$

$$d = \sqrt{4^2 + 2^2}$$

$$d = \sqrt{16 + 4}$$

$$d = \sqrt{20}$$

$$d = 2\sqrt{5}$$

■ 2. Find the distance between $u = -5 - 3i$ and $v = 4 + 2i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(-5-4)^2 + (-3-2)^2}$$

$$d = \sqrt{(-9)^2 + (-5)^2}$$

$$d = \sqrt{81 + 25}$$

$$d = \sqrt{106}$$

■ 3. Find the distance between $w = 2 + 6i$ and $z = -2 - 6i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(2 - (-2))^2 + (6 - (-6))^2}$$

$$d = \sqrt{(2 + 2)^2 + (6 + 6)^2}$$

$$d = \sqrt{4^2 + 12^2}$$

$$d = \sqrt{16 + 144}$$

$$d = \sqrt{160}$$

$$d = 4\sqrt{10}$$

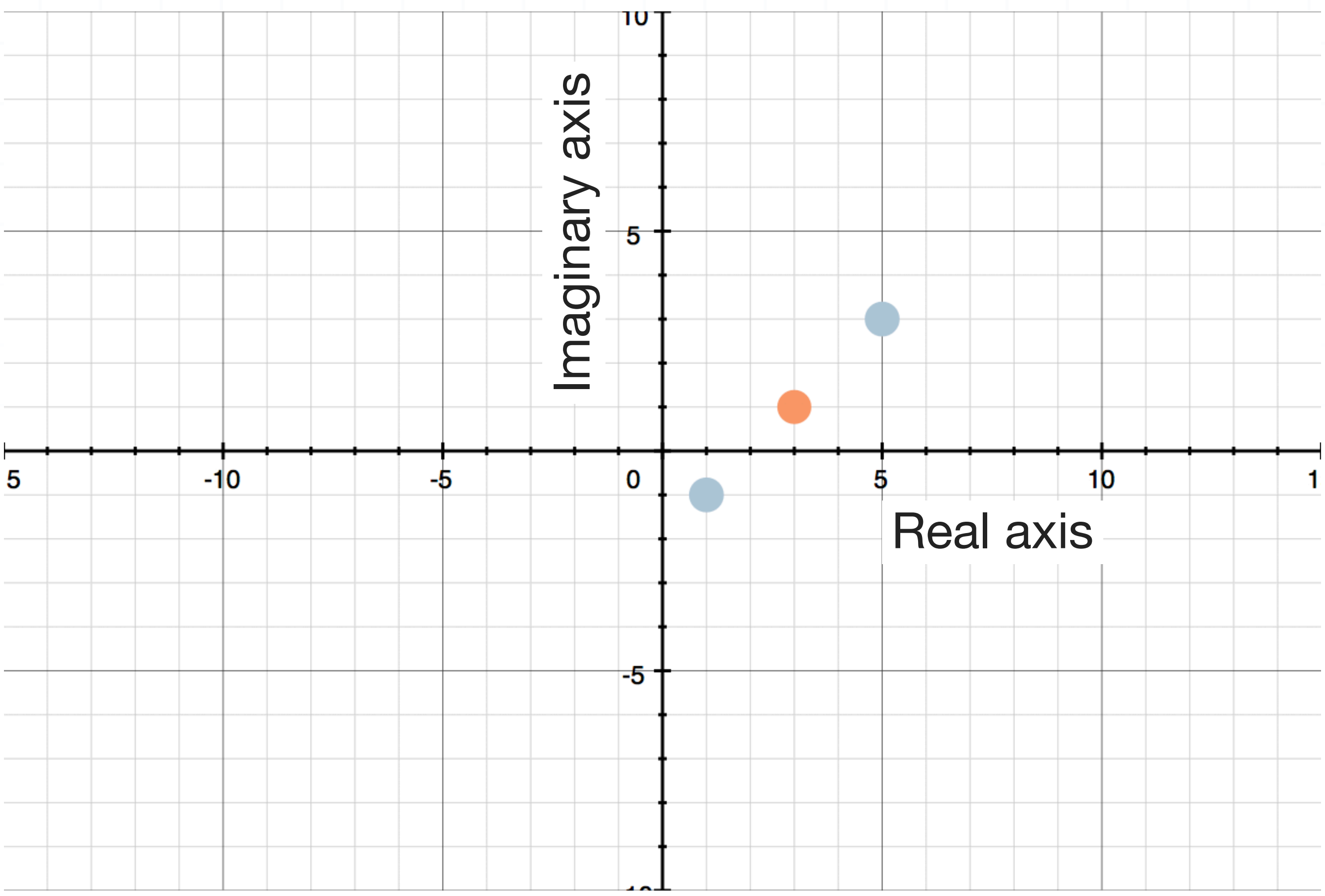
■ 4. Find the midpoint between $s = 5 + 3i$ and $t = 1 - i$.

Solution:

The distance between the real parts is $5 - 1 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 5 and 2 units from 2 is 3. The midpoint of the real parts is 3.

The distance between the imaginary parts is $3 - (-1) = 3 + 1 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 3 and 2 units from -1 is 1. The midpoint of the imaginary parts is 1.

The midpoint between $s = 5 + 3i$ and $t = 1 - i$ is $m = 3 + i$.



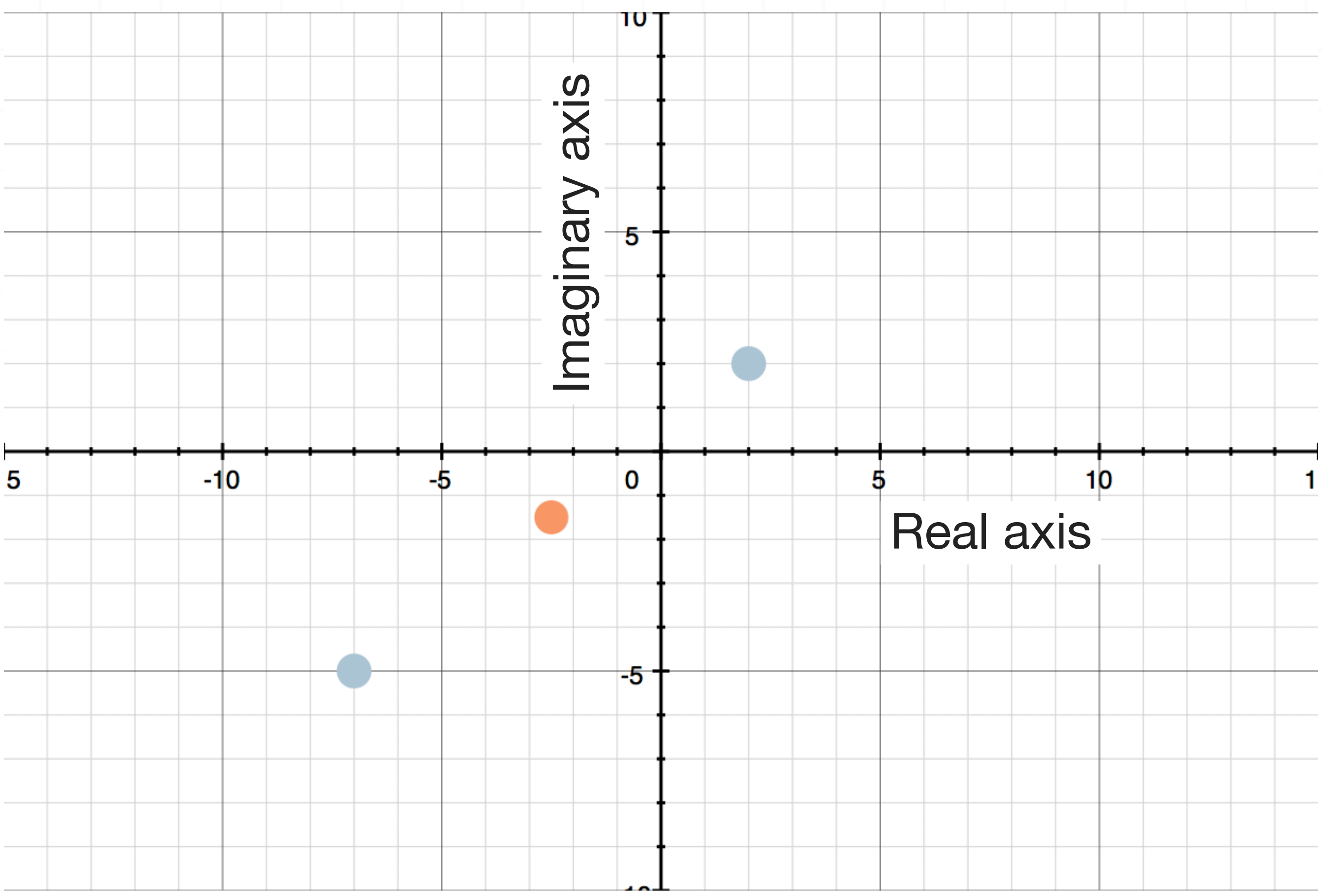
■ 5. Find the midpoint between $u = -7 - 5i$ and $z = 2 + 2i$.

Solution:

The distance between the real parts is $-7-2=-9$, and half of that distance is $-9/2=-4.5$. The value that's -4.5 units from -7 and -4.5 units from 2 is -2.5. The midpoint of the real parts is -2.5.

The distance between the imaginary parts is $-5-2=-7$, and half of that distance is $-7/2=-3.5$. The value that's -3.5 units from -7 and -3.5 units from 2 is -1.5. The midpoint of the imaginary parts is -1.5.

The midpoint between $u=-7-5i$ and $z=2+2i$ is $z=-2.5-1.5i$.



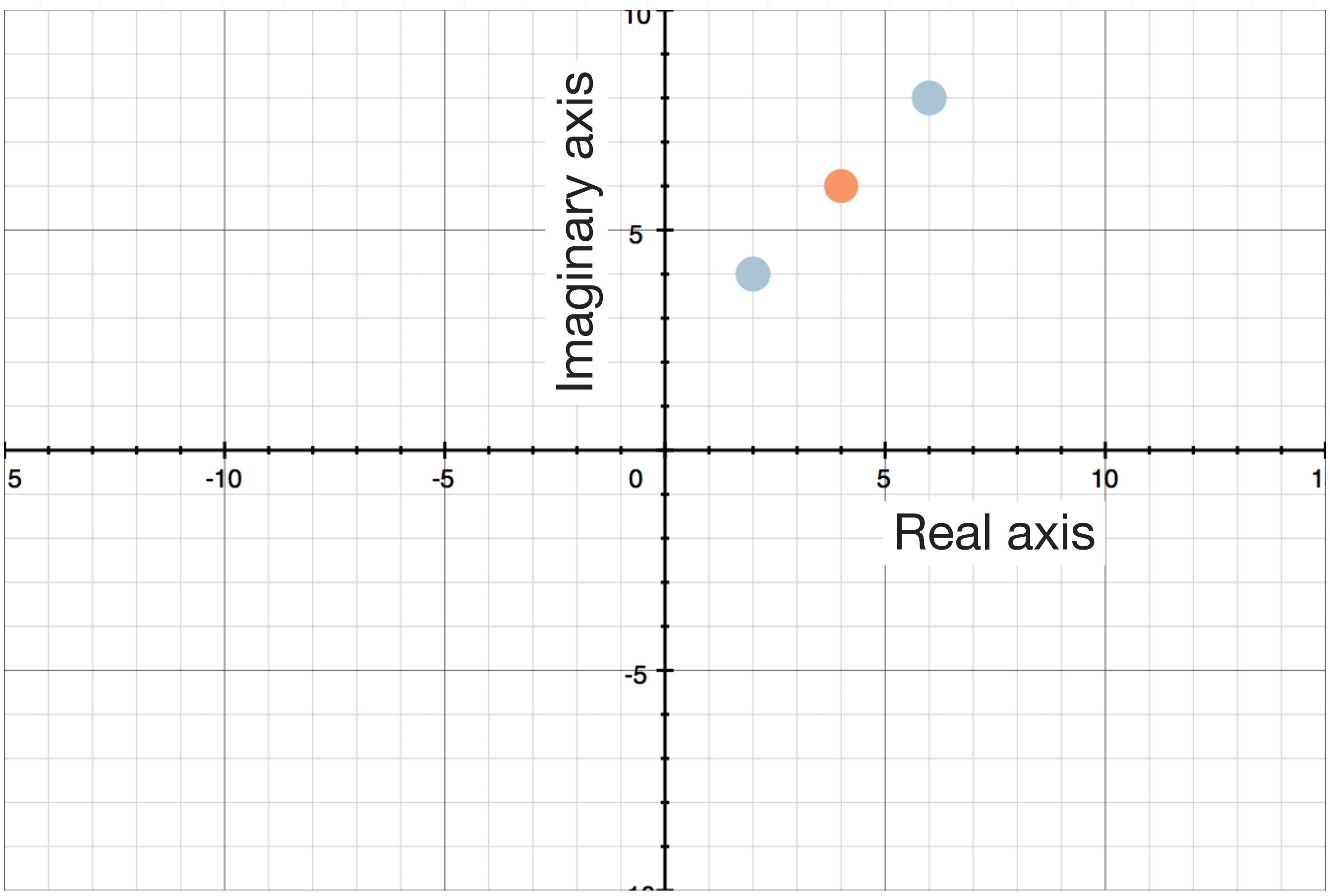
■ 6. Graph the midpoint between $w=6+8i$ and $z=2+4i$.

Solution:

The distance between the real parts is $6 - 2 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 6 and 2 units from 2 is 4. The midpoint of the real parts is 4.

The distance between the imaginary parts is $8 - 4 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 8 and 2 units from 4 is 6. The midpoint of the imaginary parts is 6.

The midpoint between $w = 6 + 8i$ and $z = 2 + 4i$ is $m = 4 + 6i$.



COMPLEX NUMBERS IN POLAR FORM

■ 1. If the complex number $6 - 2i$ is expressed in polar form, which quadrant contains the angle θ?

Solution:

If we set the complex number equal to its polar form, we get

$$6 - 2i = r(\cos\theta + i\sin\theta)$$

$$6 - 2i = r\cos\theta + ri\sin\theta$$

From this equation, we know that

$$6 = r\cos\theta$$

$$\cos\theta = \frac{6}{r}$$

The value of r is always positive, since r represents a distance, so $6/r$ has to be greater than 0, which means $\cos\theta$ has to be positive.

We also know from $6 - 2i = r\cos\theta + ri\sin\theta$ that

$$-2 = r\sin\theta$$

$$\sin\theta = -\frac{2}{r}$$

Because the value of r is always positive, $-2/r$ has to be less than 0, which means $\sin\theta$ has to be negative.

Angles with a positive cosine and negative sine are always in the fourth quadrant.

■ 2. Find r for the complex number.

$-9-3i$

Solution:

In the complex number, the real part is $a=-9$ and the imaginary part is $b=-3$, so the value of r will be

$$r=\sqrt{a^2+b^2}$$

$$r=\sqrt{(-9)^2+(-3)^2}$$

$$r=\sqrt{81+9}$$

$$r=\sqrt{90}$$

$$r=3\sqrt{10}$$

■ 3. What is the polar form of the complex number?

$5 + 12i$

Solution:

If we match up $5 + 12i$ with the standard form $a + bi$, we get $a = 5$ and $b = 12$, so

$$r = \sqrt{a^2 + b^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

The value of θ is

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{12}{5}$$

$$\arctan(\tan\theta) = \arctan\frac{12}{5}$$

$$\theta = \arctan\frac{12}{5}$$

$$\theta \approx 1.18$$

Then the complex number written in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z \approx 13\left[\cos(1.18) + i\sin(1.18)\right]$$

■ 4. Write the complex number in polar form.

$11i$

Solution:

The complex number $11i$ can be written as $0+11i$, so its real part is 0, which means the number is located on the imaginary axis. Because $a=0$ and $b=11$, the distance of $0+11i$ from the origin is

$$r=\sqrt{a^2+b^2}=\sqrt{0^2+11^2}=\sqrt{0+121}=\sqrt{121}=11$$

Since the imaginary part of $0+11i$ is 11, which is positive, $0+11i$ is located on the positive imaginary axis, so $\theta=\pi/2$. In polar form, we get

$$r(\cos\theta+i\sin\theta)$$

$$11\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

■ 5. What is the polar form of the complex number?

$$z=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$$

Solution:

In the complex number, the real part is $a = -\sqrt{3}/2$ and the imaginary part is $b = -1/2$, so the value of r will be

$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$r = \sqrt{\frac{3}{4} + \frac{1}{4}}$$

$$r = \sqrt{1}$$

$$r = 1$$

The value of θ is

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \left(-\frac{1}{2}\right)\left(-\frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\arctan(\tan\theta) = \arctan\frac{\sqrt{3}}{3}$$

$$\theta = \arctan\frac{\sqrt{3}}{3}$$

$\theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{7\pi}{6}$

Because the complex number is in quadrant III, we use $\theta = 7\pi/6$. Then the complex number written in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 1\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

■ 6. Write the complex number in polar form.

-5

Solution:

The complex number -5 can be written as $-5 + 0i$, so its imaginary part is 0, which means the number is located on the real axis. Because $a = -5$ and $b = 0$, the distance of $-5 + 0i$ from the origin is

$$r = \sqrt{a^2 + b^2} = \sqrt{(-5)^2 + 0^2} = \sqrt{25 + 0} = \sqrt{25} = 5$$

Since the real part of $-5 + 0i$ is -5, which is negative, $-5 + 0i$ is located on the negative real axis, so $\theta = \pi$. In polar form, we get

$$r(\cos\theta + i\sin\theta)$$

$$5(\cos\pi + i\sin\pi)$$

MULTIPLYING AND DIVIDING POLAR FORMS

■ 1. What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$z_1z_2 = \left(5 \cdot \sqrt{2}\right)\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right]$$

Simplify.

$$z_1z_2 = 5\sqrt{2}\left[\cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) + i\sin\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)\right]$$

$$z_1z_2 = 5\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

■ 2. What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = \sqrt{3}\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$$

$$z_2 = \frac{\sqrt{5}}{3}\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$z_1z_2 = \left(\sqrt{3}\cdot\frac{\sqrt{5}}{3}\right)\left[\cos\left(\frac{4\pi}{5} + \frac{11\pi}{8}\right) + i\sin\left(\frac{4\pi}{5} + \frac{11\pi}{8}\right)\right]$$

Simplify.

$$z_1z_2 = \frac{\sqrt{15}}{3}\left[\cos\left(\frac{32\pi}{40} + \frac{55\pi}{40}\right) + i\sin\left(\frac{32\pi}{40} + \frac{55\pi}{40}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{15}}{3}\left(\cos\frac{87\pi}{40} + i\sin\frac{87\pi}{40}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we subtract 2π from the angle, we get

$$z_1z_2=\frac{\sqrt{15}}{3}\left[\cos\left(\frac{87\pi}{40}-2\pi\right)+i\sin\left(\frac{87\pi}{40}-2\pi\right)\right]$$

$$z_1z_2=\frac{\sqrt{15}}{3}\left[\cos\left(\frac{87\pi}{40}-\frac{80\pi}{40}\right)+i\sin\left(\frac{87\pi}{40}-\frac{80\pi}{40}\right)\right]$$

$$z_1z_2=\frac{\sqrt{15}}{3}\left(\cos\frac{7\pi}{40}+i\sin\frac{7\pi}{40}\right)$$

■ 3. What is the quotient z_1/z_2 of the complex numbers in polar form?

$$z_1=12\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$$

$$z_2=15\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

Solution:

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{12}{15}\left[\cos\left(\frac{7\pi}{6} - \frac{\pi}{2}\right) + i\sin\left(\frac{7\pi}{6} - \frac{\pi}{2}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2} = \frac{4}{5}\left[\cos\left(\frac{7\pi}{6} - \frac{3\pi}{6}\right) + i\sin\left(\frac{7\pi}{6} - \frac{3\pi}{6}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{4}{5}\left(\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}\right)$$

$$\frac{z_1}{z_2} = \frac{4}{5}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

■ 4. What is the quotient z_1/z_2 of the complex numbers in polar form?

$$z_1 = \sqrt{7}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$z_2 = \frac{1}{\sqrt{2}}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Solution:

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{\sqrt{7}}{\frac{1}{\sqrt{2}}}\left[\cos\left(\frac{\pi}{12} - \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{12} - \frac{2\pi}{3}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2} = \sqrt{14}\left[\cos\left(\frac{\pi}{12} - \frac{8\pi}{12}\right) + i\sin\left(\frac{\pi}{12} - \frac{8\pi}{12}\right)\right]$$

$$\frac{z_1}{z_2} = \sqrt{14}\left(\cos\frac{-7\pi}{12} + i\sin\frac{-7\pi}{12}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we add 2π to the angle, we get

$$\frac{z_1}{z_2} = \sqrt{14}\left[\cos\left(\frac{-7\pi}{12} + 2\pi\right) + i\sin\left(\frac{-7\pi}{12} + 2\pi\right)\right]$$

$$\frac{z_1}{z_2} = \sqrt{14}\left[\cos\left(\frac{-7\pi}{12} + \frac{24\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12} + \frac{24\pi}{12}\right)\right]$$

$$z_1 z_2 = \sqrt{14}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

■ 5. What is the product $z_1 z_2$ of the complex numbers in polar form?

$$z_1 = \frac{\sqrt{15}}{4}\left(\cos\frac{7\pi}{2} + i\sin\frac{7\pi}{2}\right)$$

$$z_2 = \frac{1}{\sqrt{5}}\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$z_1 z_2 = \left(\frac{\sqrt{15}}{4}\cdot\frac{1}{\sqrt{5}}\right)\left[\cos\left(\frac{7\pi}{2} + \frac{6\pi}{5}\right) + i\sin\left(\frac{7\pi}{2} + \frac{6\pi}{5}\right)\right]$$

Simplify.

$$z_1 z_2 = \frac{\sqrt{15}}{4\sqrt{5}}\left[\cos\left(\frac{35\pi}{10} + \frac{12\pi}{10}\right) + i\sin\left(\frac{35\pi}{10} + \frac{12\pi}{10}\right)\right]$$

$$z_1 z_2 = \frac{\sqrt{3}}{4}\left(\cos\frac{47\pi}{10} + i\sin\frac{47\pi}{10}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we subtract $2 \cdot 2\pi = 4\pi$ from the angle, we get

$$z_1z_2 = \frac{\sqrt{3}}{4}\left[\cos\left(\frac{47\pi}{10} - 4\pi\right) + i\sin\left(\frac{47\pi}{10} - 4\pi\right)\right]$$

$$z_1z_2 = \frac{\sqrt{3}}{4}\left[\cos\left(\frac{47\pi}{10} - \frac{40\pi}{10}\right) + i\sin\left(\frac{47\pi}{10} - \frac{40\pi}{10}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{3}}{4}\left(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10}\right)$$

■ 6. Suppose that a complex number z is the product $z_1 \cdot z_2$ of the given complex numbers. If z is expressed in polar form, $r(\cos\theta + i\sin\theta)$, where is θ located?

$$z_1 = 3\sqrt{5}\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$$

$$z_2 = 6\left(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$z_1z_2 = \left(3\sqrt{5} \cdot 6\right)\left[\cos\left(\frac{2\pi}{5} + \frac{7\pi}{10}\right) + i\sin\left(\frac{2\pi}{5} + \frac{7\pi}{10}\right)\right]$$

Simplify.

$$z_1z_2 = 18\sqrt{5}\left[\cos\left(\frac{4\pi}{10} + \frac{7\pi}{10}\right) + i\sin\left(\frac{4\pi}{10} + \frac{7\pi}{10}\right)\right]$$

$$z_1z_2 = 18\sqrt{5}\left(\cos\frac{11\pi}{10} + i\sin\frac{11\pi}{10}\right)$$

The fraction $11/10$ is equal to 1.1, so the angle is 1.1π, which is in the third quadrant.

POWERS OF COMPLEX NUMBERS AND DE MOIVRE’S THEOREM

■ 1. Find z^5 in polar form.

$$z = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Solution:

Plug $r = 2$, $\theta = \pi/12$, and $n = 5$ into De Moivre's theorem.

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^5 = 2^5\left[\cos\left(5\cdot\frac{\pi}{12}\right) + i\sin\left(5\cdot\frac{\pi}{12}\right)\right]$$

Then simplify.

$$z^5 = 32\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

■ 2. Find z^7 in polar form.

$$z = \sqrt{5}\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$$

Solution:

Plug $r = \sqrt{5}$, $\theta = 2\pi/5$, and $n = 7$ into De Moivre's theorem.

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^7 = \left(\sqrt{5}\right)^7 \left[\cos\left(7 \cdot \frac{2\pi}{5}\right) + i\sin\left(7 \cdot \frac{2\pi}{5}\right)\right]$$

Then simplify.

$$z^7 = 125\sqrt{5}\left(\cos\frac{14\pi}{5} + i\sin\frac{14\pi}{5}\right)$$

We could leave the answer this way, but the angle $14\pi/5$ is larger than 2π, so we can reduce the angle to one that's coterminal with $14\pi/5$, but within the interval $[0,2\pi)$.

$$\frac{14\pi}{5} - 2\pi = \frac{14\pi}{5} - 2\pi\left(\frac{5}{5}\right) = \frac{14\pi}{5} - \frac{10\pi}{5} = \frac{4\pi}{5}$$

So the complex number z^4 in polar form can be written as

$$z^7 = 125\sqrt{5}\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$$

■ 3. Find z^6 in rectangular form $a + bi$.

$$z = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

Solution:

Plug $r = \sqrt{2}/2$, $\theta = \pi/8$, and $n = 6$ into De Moivre's theorem.

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^6 = \left(\frac{\sqrt{2}}{2}\right)^6 \left[\cos\left(6 \cdot \frac{\pi}{8}\right) + i\sin\left(6 \cdot \frac{\pi}{8}\right)\right]$$

Then simplify.

$$z^6 = \frac{1}{8}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$z^6 = \frac{1}{8}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$z^6 = \frac{1}{8}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{8}\left(\frac{\sqrt{2}}{2}i\right)$$

$$z^6 = -\frac{\sqrt{2}}{16} + \frac{\sqrt{2}}{16}i$$

■ 4. Find z^3 in rectangular form $a + bi$.

$$z = 2\sqrt{6}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Solution:

Plug $r = 2\sqrt{6}$, $\theta = 5\pi/3$, and $n = 3$ into De Moivre's theorem.

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^3 = \left(2\sqrt{6}\right)^3 \left[\cos\left(3 \cdot \frac{5\pi}{3}\right) + i\sin\left(3 \cdot \frac{5\pi}{3}\right)\right]$$

Then simplify.

$$z^3 = 8 \cdot 6\sqrt{6}(\cos 5\pi + i\sin 5\pi)$$

$$z^3 = 48\sqrt{6}(\cos 5\pi + i\sin 5\pi)$$

$$z^3 = 48\sqrt{6}(-1 + i(0))$$

$$z^3 = -48\sqrt{6}$$

■ 5. Find z^5 in polar form.

$z = -4 - 4i$

Solution:

First, convert $z = -4 - 4i$ to polar form by finding the modulus $|z|$ and the angle θ. The distance from the origin is

$$|z| = r = \sqrt{a^2 + b^2}$$

$$|z| = r = \sqrt{(-4)^2 + (-4)^2}$$

$$|z| = r = \sqrt{16 + 16}$$

$$|z| = r = \sqrt{32}$$

$$|z| = r = 4\sqrt{2}$$

and the angle is

$$\theta = \arctan \frac{b}{a} = \arctan \frac{-4}{-4} = \arctan(1) = \frac{\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

Because the complex number $z = -4 - 4i$ is in quadrant III, we use $\theta = \dfrac{5\pi}{4}$.

Then $z = -4 - 4i$ in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

To find z^5, plug $r = 4\sqrt{2}$, $\theta = 5\pi/4$, and $n = 5$ into De Moivre's theorem.

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^5 = \left(4\sqrt{2}\right)^5\left[\cos\left(5 \cdot \frac{5\pi}{4}\right) + i\sin\left(5 \cdot \frac{5\pi}{4}\right)\right]$$

Then simplify.

$$z^5 = 4{,}096\sqrt{2}\left(\cos\frac{25\pi}{4} + i\sin\frac{25\pi}{4}\right)$$

We could leave the answer this way, but the angle $25\pi/4$ is larger than 2π, so we can reduce the angle to one that's coterminal with $25\pi/4$, but within the interval $[0,2\pi)$.

$$\frac{25\pi}{4} - 3(2\pi) = \frac{25\pi}{4} - 6\pi\left(\frac{4}{4}\right) = \frac{25\pi}{4} - \frac{24\pi}{4} = \frac{\pi}{4}$$

So the complex number z^5 in polar form can be written as

$$z^5 = 4{,}096\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

■ 6. Find z^4 in rectangular form $a+bi$.

$z = \sqrt{6} - \sqrt{2}i$

Solution:

First, convert $z = \sqrt{6} - \sqrt{2}i$ to polar form by finding the modulus $|z|$ and the angle θ. The distance from the origin is

$$|z| = r = \sqrt{a^2 + b^2}$$

$$|z| = r = \sqrt{\left(\sqrt{6}\right)^2 + \left(-\sqrt{2}\right)^2}$$

$$|z| = r = \sqrt{6+2}$$

$$|z| = r = \sqrt{8}$$

$$|z| = r = 2\sqrt{2}$$

and the angle is

$$\theta = \arctan\frac{b}{a} = \arctan\frac{-\sqrt{2}}{\sqrt{6}} = \arctan\frac{-\sqrt{3}}{3} = -\frac{\pi}{6}$$

Then $z = \sqrt{6} - \sqrt{2}i$ in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

To find z^4, plug $r = 2\sqrt{2}$, $\theta = -\pi/6$, and $n = 4$ into De Moivre's theorem.

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^4 = \left(2\sqrt{2}\right)^4\left[\cos\left(4\cdot-\frac{\pi}{6}\right) + i\sin\left(4\cdot-\frac{\pi}{6}\right)\right]$$

Then simplify.

$$z^4 = 64\left[\cos\left(-\frac{4\pi}{6}\right) + i\sin\left(-\frac{4\pi}{6}\right)\right]$$

$$z^4 = 64\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$$

$$z^4 = 64\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$z^4 = 64\left(-\frac{1}{2}\right) + 64\left(-\frac{\sqrt{3}}{2}i\right)$$

$$z^4 = -32 - 32\sqrt{3}i$$

COMPLEX NUMBER EQUATIONS

■ 1. Find the solutions of the complex equation.

$z^2 = 49$

Solution:

Rewrite z^2 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^2 = r^2 \left[\cos(2\theta) + i\sin(2\theta)\right]$$

Rewrite 49 as the complex number $49 + 0i$. The modulus and angle of $49 + 0i$ are

$$r = \sqrt{49^2 + 0^2}$$

$$r = \sqrt{49^2}$$

$$r = 49$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{49} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 49\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 49\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 49\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^2 = 49$, we can start making substitutions.

$$z^2 = 49$$

$$r^2\left[\cos(2\theta) + i\sin(2\theta)\right] = 49$$

$$r^2\left[\cos(2\theta) + i\sin(2\theta)\right] = 49\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^2 = 49$$

$$2\theta = 360^\circ k$$

From these equations, we get

$r^2 = 49$, so $r = 7$

$2\theta = 360^\circ k$, so $\theta = 180^\circ k$

To $\theta = 180^\circ k$, if we plug in $k = 0,\ 1,\ \dots$, we get

For $k = 0$, $\theta = 180^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 180^\circ(1) = 180^\circ$

...

We could keep going for $k = 2, 3, 4, 5, \ldots$, but $k = 2$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ,\ 180^\circ$.

Plugging these two angles and $r = 7$ into the formula for polar form of a complex number, we'll get the solutions to $z^2 = 49$.

$$z_1 = 7\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 7\left[1 + i(0)\right] = 7$$

$$z_2 = 7\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 7\left[-1 + i(0)\right] = -7$$

■ 2. Find the solution of the complex equation that lies in the third quadrant.

$z^3 = 216$

Solution:

Rewrite z^3 as

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^3 = r^3\left[\cos(3\theta) + i\sin(3\theta)\right]$$

Rewrite 216 as the complex number $216 + 0i$. The modulus and angle of $216 + 0i$ are

$$r = \sqrt{216^2 + 0^2}$$

$$r = \sqrt{216^2}$$

$$r = 216$$

and

$$\theta = \arctan \frac{b}{a} = \arctan \frac{0}{216} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 216 \left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i \sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 216 \left[\cos(2\pi k) + i \sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 216 \left[\cos(360^\circ k) + i \sin(360^\circ k)\right]$$

Starting again with $z^3 = 216$, we can start making substitutions.

$$z^3 = 216$$

$$r^3 \left[\cos(3\theta) + i \sin(3\theta)\right] = 216$$

$$r^3 \left[\cos(3\theta) + i \sin(3\theta)\right] = 216 \left[\cos(360^\circ k) + i \sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^3 = 216$$

$$3\theta = 360^\circ k$$

From these equations, we get

$r^3 = 216$, so $r = 6$

$3\theta = 360^\circ k$, so $\theta = 120^\circ k$

To $\theta = 120^\circ k$, if we plug in $k = 0, 1, 2, \dots$, we get

For $k = 0$, $\theta = 120^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 120^\circ(1) = 120^\circ$

For $k = 2$, $\theta = 120^\circ(2) = 240^\circ$

...

We could keep going for $k = 3, 4, 5, 6, \dots$, but $k = 3$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 120^\circ, 240^\circ$.

Plugging these three angles and $r = 6$ into the formula for polar form of a complex number, we'll get the solutions to $z^3 = 216$.

$$z_1 = 6\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 6\left[1 + i(0)\right] = 6$$

$$z_2 = 6\left[\cos(120^\circ) + i\sin(120^\circ)\right] = 6\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -3 + 3\sqrt{3}i$$

$$z_3 = 6\left[\cos(240^\circ) + i\sin(240^\circ)\right] = 6\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -3 - 3\sqrt{3}i$$

Roots in the third quadrant will have a negative real part and a negative imaginary part. So, z_3 is the solution in the third quadrant.

■ 3. Find the solutions of the complex equation.

$z^4 = 256$

Solution:

Rewrite z^4 as

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^4 = r^4\left[\cos(4\theta) + i\sin(4\theta)\right]$$

Rewrite 256 as the complex number $256 + 0i$. The modulus and angle of $256 + 0i$ are

$$r = \sqrt{256^2 + 0^2}$$

$$r = \sqrt{256^2}$$

$$r = 256$$

and

$$\theta = \arctan \frac{b}{a} = \arctan \frac{0}{256} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 256 \left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i \sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 256 \left[\cos(2\pi k) + i \sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 256 \left[\cos(360^\circ k) + i \sin(360^\circ k)\right]$$

Starting again with $z^4 = 256$, we can start making substitutions.

$$z^4 = 256$$

$$r^4 \left[\cos(4\theta) + i \sin(4\theta)\right] = 256$$

$$r^4 \left[\cos(4\theta) + i \sin(4\theta)\right] = 256 \left[\cos(360^\circ k) + i \sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^4 = 256$$

$$4\theta = 360^\circ k$$

From these equations, we get

$r^4 = 256$, so $r = 4$

$4\theta = 360^\circ k$, so $\theta = 90^\circ k$

To $\theta = 90^\circ k$, if we plug in $k = 0, 1, 2, 3, \ldots,$ we get

For $k = 0$, $\theta = 90^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 90^\circ(1) = 90^\circ$

For $k = 2$, $\theta = 90^\circ(2) = 180^\circ$

For $k = 3$, $\theta = 90^\circ(3) = 270^\circ$

...

We could keep going for $k = 4, 5, 6, 7, \ldots,$ but $k = 4$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ$.

Plugging these four angles and $r = 4$ into the formula for polar form of a complex number, we'll get the solutions to $z^4 = 256$.

$$z_1 = 4\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 4\left[1 + i(0)\right] = 4$$

$$z_2 = 4\left[\cos(90^\circ) + i\sin(90^\circ)\right] = 4\left[0 + i(1)\right] = 4i$$

$$z_3 = 4\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 4\left[-1 + i(0)\right] = -4$$

$$z_4 = 4\left[\cos(270^\circ) + i\sin(270^\circ)\right] = 4\left[0 + i(-1)\right] = -4i$$

■ 4. Find the solutions of the complex equation.

$$z^6 = 729$$

Solution:

Rewrite z^6 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^6 = r^6 \left[\cos(6\theta) + i\sin(6\theta)\right]$$

Rewrite 729 as the complex number $729 + 0i$. The modulus and angle of $729 + 0i$ are

$$r = \sqrt{729^2 + 0^2}$$

$$r = \sqrt{729^2}$$

$$r = 729$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{729} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 729\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 729\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 729\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^6 = 729$, we can start making substitutions.

$$z^6 = 729$$

$$r^6\left[\cos(6\theta) + i\sin(6\theta)\right] = 729$$

$$r^6\left[\cos(6\theta) + i\sin(6\theta)\right] = 729\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^6 = 729$$

$$6\theta = 360^\circ k$$

From these equations, we get

$r^6 = 729$, so $r = 3$

$6\theta = 360^\circ k$, so $\theta = 60^\circ k$

To $\theta = 60^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, 5, \ldots$, we get

For $k = 0$, $\theta = 60^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 60^\circ(1) = 60^\circ$

For $k = 2$, $\theta = 60^\circ(2) = 120^\circ$

For $k = 3$, $\theta = 60^\circ(3) = 180^\circ$

For $k = 4$, $\theta = 60^\circ(4) = 240^\circ$

For $k = 5$, $\theta = 60^\circ(5) = 300^\circ$

...

We could keep going for $k = 6, 7, 8, 9, \dots$, but $k = 6$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$.

Plugging these six angles and $r = 3$ into the formula for polar form of a complex number, we'll get the solutions to $z^6 = 729$.

$$z_1 = 3\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 3\left[1 + i(0)\right] = 3$$

$$z_2 = 3\left[\cos(60^\circ) + i\sin(60^\circ)\right] = 3\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$z_3 = 3\left[\cos(120^\circ) + i\sin(120^\circ)\right] = 3\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$z_4 = 3\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 3\left[-1 + i(0)\right] = -3$$

$$z_5 = 3\left[\cos(240^\circ) + i\sin(240^\circ)\right] = 3\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

$$z_6 = 3\left[\cos(300^\circ) + i\sin(300^\circ)\right] = 3\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

■ 5. Find the solutions of the complex equation.

$$z^5 = 32$$

Solution:

Rewrite z^5 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^5 = r^5 \left[\cos(5\theta) + i\sin(5\theta)\right]$$

Rewrite 32 as the complex number $32 + 0i$. The modulus and angle of $32 + 0i$ are

$$r = \sqrt{32^2 + 0^2}$$

$$r = \sqrt{32^2}$$

$$r = 32$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{32} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 32\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 32\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 32\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^5 = 32$, we can start making substitutions.

$$z^5 = 32$$

$$r^5\left[\cos(5\theta) + i\sin(5\theta)\right] = 32$$

$$r^5\left[\cos(5\theta) + i\sin(5\theta)\right] = 32\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^5 = 32$$

$$5\theta = 360^\circ k$$

From these equations, we get

$r^5 = 32$, so $r = 2$

$5\theta = 360^\circ k$, so $\theta = 72^\circ k$

To $\theta = 72^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, \dots,$ we get

For $k = 0$, $\theta = 72^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 72^\circ(1) = 72^\circ$

For $k = 2$, $\theta = 72^\circ(2) = 144^\circ$

For $k = 3$, $\theta = 72^\circ(3) = 216^\circ$

For $k = 4$, $\theta = 72^\circ(4) = 288^\circ$

...

We could keep going for $k = 5, 6, 7, 8, 9, \dots$, but $k = 5$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 72^\circ, 144^\circ, 216^\circ, 288^\circ$.

Plugging these five angles and $r = 2$ into the formula for polar form of a complex number, we'll get the solutions to $z^5 = 32$. Because these angles are not on the unit circle, we will need to use decimal approximations.

$$z_1 = 2\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 2\left[1 + i(0)\right] = 2$$

$$z_2 \approx 2\left[\cos(72^\circ) + i\sin(72^\circ)\right] \approx 2\left[0.309 + 0.951i\right] \approx 0.618 + 1.902i$$

$$z_3 \approx 2\left[\cos(144^\circ) + i\sin(144^\circ)\right] \approx 2\left[-0.809 + 0.588i\right] \approx -1.618 + 1.176i$$

$$z_4 \approx 2\left[\cos(216^\circ) + i\sin(216^\circ)\right] \approx 2\left[-0.809 - 0.588i\right] \approx -1.618 - 1.176i$$

$$z_5 \approx 2\left[\cos(288^\circ) + i\sin(288^\circ)\right] \approx 2\left[0.309 - 0.951i\right] \approx 0.618 - 1.902i$$

■ 6. How many solutions of the complex equation lie in the second quadrant?

$$z^8 = 256$$

Solution:

Rewrite z^8 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^8 = r^8 \left[\cos(8\theta) + i\sin(8\theta)\right]$$

Rewrite 256 as the complex number $256 + 0i$. The modulus and angle of $256 + 0i$ are

$$r = \sqrt{256^2 + 0^2}$$

$$r = \sqrt{256^2}$$

$$r = 256$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{256} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 256\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 256\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 256\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^8 = 256$, we can start making substitutions.

$$z^8 = 256$$

$$r^8\left[\cos(8\theta)+i\sin(8\theta)\right]=256$$

$$r^8\left[\cos(8\theta)+i\sin(8\theta)\right]=256\left[\cos(360^\circ k)+i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^8=256$$

$$8\theta=360^\circ k$$

From these equations, we get

$r^8=256$, so $r=2$

$8\theta=360^\circ k$, so $\theta=45^\circ k$

To $\theta=45^\circ k$, if we plug in $k=0, 1, 2, 3, 4, 5, 6, 7, ...$, we get

For $k=0$, $\theta=45^\circ(0)=0^\circ$

For $k=1$, $\theta=45^\circ(1)=45^\circ$

For $k=2$, $\theta=45^\circ(2)=90^\circ$

For $k=3$, $\theta=45^\circ(3)=135^\circ$

For $k=4$, $\theta=45^\circ(4)=180^\circ$

For $k=5$, $\theta=45^\circ(5)=225^\circ$

For $k=6$, $\theta=45^\circ(6)=270^\circ$

For $k=7$, $\theta=45^\circ(7)=315^\circ$

...

We could keep going for $k = 8, 9, 10, 11, \dots$, but $k = 8$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ, 225^\circ, 270^\circ, 315^\circ$.

Plugging these eight angles and $r = 2$ into the formula for polar form of a complex number, we'll get the solutions to $z^8 = 256$.

$$z_1 = 2\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 2\left[1 + i(0)\right] = 2$$

$$z_2 = 2\left[\cos(45^\circ) + i\sin(45^\circ)\right] = 2\left[\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2}\right)\right] = \sqrt{2} + \sqrt{2}i$$

$$z_3 = 2\left[\cos(90^\circ) + i\sin(90^\circ)\right] = 2\left[0 + i(1)\right] = 2i$$

$$z_4 = 2\left[\cos(135^\circ) + i\sin(135^\circ)\right] = 2\left[-\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2}\right)\right] = -\sqrt{2} + \sqrt{2}i$$

$$z_5 = 2\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 2\left[-1 + i(0)\right] = -2$$

$$z_6 = 2\left[\cos(225^\circ) + i\sin(225^\circ)\right] = 2\left[-\frac{\sqrt{2}}{2} - i\left(\frac{\sqrt{2}}{2}\right)\right] = -\sqrt{2} - \sqrt{2}i$$

$$z_7 = 2\left[\cos(270^\circ) + i\sin(270^\circ)\right] = 2\left[0 + i(-1)\right] = -2i$$

$$z_8 = 2\left[\cos(315^\circ) + i\sin(315^\circ)\right] = 2\left[\frac{\sqrt{2}}{2} - i\left(\frac{\sqrt{2}}{2}\right)\right] = \sqrt{2} - \sqrt{2}i$$

Roots in the second quadrant will have a negative real part and a positive imaginary part. That's only z_4, so there's one solution in the second quadrant.

ROOTS OF COMPLEX NUMBERS

■ 1. Find the cube roots of the complex number.

$$z = 27\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Solution:

We're looking for the third (or cube) roots of z, which means there will be 3 of them, given by $k = 0,\ 1,\ 2$. And since the complex number is given in radians, we'll plug $n = 3$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[3]{z} = \sqrt[3]{r}\left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i\sin\left(\frac{\theta + 2\pi k}{3}\right)\right]$$

With $r = 27$ and $\theta = \pi/4$ from the complex number, we get

$$\sqrt[3]{z} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2\pi k}{3}\right)\right]$$

Now we'll find values for $k = 0,\ 1,\ 2$.

For $k = 0$:

$$\sqrt[3]{z}_{k=0} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(0)}{3}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\pi(0)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=0} = 3\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

For $k=1$:

$$\sqrt[3]{z}_{k=1} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(1)}{3}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\pi(1)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=1} = 3\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

For $k=2$:

$$\sqrt[3]{z}_{k=2} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(2)}{3}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\pi(2)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=2} = 3\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right)$$

The roots are

$$\sqrt[3]{z}_{k=0} = 3\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

$$\sqrt[3]{z}_{k=1} = 3\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

$$\sqrt[3]{z}_{k=2} = 3\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

■ 2. Find the 4th root of the complex number.

$z = 256\left(\cos 60^\circ + i\sin 60^\circ\right)$

Solution:

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k = 0,\ 1,\ 2,\ 3$. And since the complex number is given in degrees, we'll plug $n = 4$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[4]{z} = \sqrt[4]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{4}\right) + i\sin\left(\frac{\theta + 360^\circ k}{4}\right)\right]$$

With $r = 256$ and $\theta = 60^\circ$ from the complex number, we get

$$\sqrt[4]{z} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ k}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ k}{4}\right)\right]$$

Now we'll find values for $k = 0,\ 1,\ 2,\ 3$.

For $k = 0$:

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(0)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0} = 4\left[\cos(15^\circ) + i\sin(15^\circ)\right]$$

For $k = 1$:

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(1)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1} = 4\left[\cos(105^\circ) + i\sin(105^\circ)\right]$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(2)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = 4\left[\cos(195^\circ) + i\sin(195^\circ)\right]$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(3)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3} = 4\left[\cos(285^\circ) + i\sin(285^\circ)\right]$$

The roots are

$$\sqrt[4]{z}_{k=0} = 4\left[\cos(15^\circ) + i\sin(15^\circ)\right]$$

$$\sqrt[4]{z}_{k=1} = 4\left[\cos(105^\circ) + i\sin(105^\circ)\right]$$

$$\sqrt[4]{z}_{k=2} = 4\left[\cos(195^\circ) + i\sin(195^\circ)\right]$$

$$\sqrt[4]{z}_{k=3} = 4\left[\cos(285^\circ) + i\sin(285^\circ)\right]$$

■ 3. Find the 5th roots of the complex number that lies in the first quadrant of the complex plane.

$$z = 25\left(\cos 80^\circ + i\sin 80^\circ\right)$$

Solution:

We're looking for the 5th roots of z, which means there will be 5 of them, given by $k = 0,\ 1,\ 2,\ 3,\ 4$. And since the complex number is given in degrees, we'll plug $n = 5$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[5]{z} = \sqrt[5]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{5}\right) + i\sin\left(\frac{\theta + 360^\circ k}{5}\right)\right]$$

With $r = 25$ and $\theta = 80^\circ$ from the complex number, we get

$$\sqrt[5]{z} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ k}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ k}{5}\right)\right]$$

Now we'll find values for $k=0, 1, 2, 3, 4$.

For $k=0$:

$$\sqrt[5]{z}_{k=0}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(0)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(0)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=0}=\sqrt[5]{25}\left[\cos(16^\circ)+i\sin(16^\circ)\right]$$

For $k=1$:

$$\sqrt[5]{z}_{k=1}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(1)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(1)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=1}=\sqrt[5]{25}\left[\cos(88^\circ)+i\sin(88^\circ)\right]$$

For $k=2$:

$$\sqrt[5]{z}_{k=2}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(2)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(2)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=2}=\sqrt[5]{25}\left[\cos(160^\circ)+i\sin(160^\circ)\right]$$

For $k=3$:

$$\sqrt[5]{z}_{k=3}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(3)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(3)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=3}=\sqrt[5]{25}\left[\cos(232^\circ)+i\sin(232^\circ)\right]$$

For $k=4$:

$$\sqrt[5]{z}_{k=4} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ(4)}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ(4)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=4} = \sqrt[5]{25}\left[\cos(304^\circ) + i\sin(304^\circ)\right]$$

The roots are

$$\sqrt[5]{z}_{k=0} = \sqrt[5]{25}\left[\cos(16^\circ) + i\sin(16^\circ)\right]$$

$$\sqrt[5]{z}_{k=1} = \sqrt[5]{25}\left[\cos(88^\circ) + i\sin(88^\circ)\right]$$

$$\sqrt[5]{z}_{k=2} = \sqrt[5]{25}\left[\cos(160^\circ) + i\sin(160^\circ)\right]$$

$$\sqrt[5]{z}_{k=3} = \sqrt[5]{25}\left[\cos(232^\circ) + i\sin(232^\circ)\right]$$

$$\sqrt[5]{z}_{k=4} = \sqrt[5]{25}\left[\cos(304^\circ) + i\sin(304^\circ)\right]$$

Anything in the first quadrant will fall in the interval $(0^\circ,90^\circ)$. In this case, the angles for $k=0$ and $k=1$ are in the first quadrant.

■ 4. Find the 4th roots of the complex number.

$$z = 34\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$$

Solution:

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k = 0, 1, 2, 3$. And since the complex number is given in radians, we'll plug $n = 4$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[4]{z} = \sqrt[4]{r}\left[\cos\left(\frac{\theta + 2\pi k}{4}\right) + i\sin\left(\frac{\theta + 2\pi k}{4}\right)\right]$$

With $r = 34$ and $\theta = 3\pi/5$ from the complex number, we get

$$\sqrt[4]{z} = \sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5} + 2\pi k}{4}\right) + i\sin\left(\frac{\frac{3\pi}{5} + 2\pi k}{4}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3$.

For $k = 0$:

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5} + 2\pi(0)}{4}\right) + i\sin\left(\frac{\frac{3\pi}{5} + 2\pi(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{34}\left(\cos\frac{3\pi}{20} + i\sin\frac{3\pi}{20}\right)$$

For $k = 1$:

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5} + 2\pi(1)}{4}\right) + i\sin\left(\frac{\frac{3\pi}{5} + 2\pi(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{34}\left(\cos\frac{13\pi}{20} + i\sin\frac{13\pi}{20}\right)$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5} + 2\pi(2)}{4}\right) + i\sin\left(\frac{\frac{3\pi}{5} + 2\pi(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{34}\left(\cos\frac{23\pi}{20} + i\sin\frac{23\pi}{20}\right)$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5} + 2\pi(3)}{4}\right) + i\sin\left(\frac{\frac{3\pi}{5} + 2\pi(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{34}\left(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right)$$

The roots are

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{34}\left(\cos\frac{3\pi}{20} + i\sin\frac{3\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{34}\left(\cos\frac{13\pi}{20} + i\sin\frac{13\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{34}\left(\cos\frac{23\pi}{20} + i\sin\frac{23\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{34}\left(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right)$$

■ 5. Find the 6th roots of the complex number that lie in the second quadrant of the complex plane.

$$z = 11\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Solution:

We're looking for the 6th roots of z, which means there will be 6 of them, given by $k = 0, 1, 2, 3, 4, 5$. And since the complex number is given in radians, we'll plug $n = 6$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[6]{z} = \sqrt[6]{r}\left[\cos\left(\frac{\theta + 2\pi k}{6}\right) + i\sin\left(\frac{\theta + 2\pi k}{6}\right)\right]$$

With $r = 11$ and $\theta = 5\pi/6$ from the complex number, we get

$$\sqrt[6]{z} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi k}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi k}{6}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4, 5.$

For $k = 0$:

$$\sqrt[6]{z}_{k=0} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi(0)}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi(0)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=0} = \sqrt[6]{11}\left(\cos\frac{5\pi}{36} + i\sin\frac{5\pi}{36}\right)$$

For $k = 1$:

$$\sqrt[6]{z}_{k=1} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi(1)}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi(1)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=1} = \sqrt[6]{11}\left(\cos\frac{17\pi}{36} + i\sin\frac{17\pi}{36}\right)$$

For $k = 2$:

$$\sqrt[6]{z}_{k=2} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi(2)}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi(2)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=2} = \sqrt[6]{11}\left(\cos\frac{29\pi}{36} + i\sin\frac{29\pi}{36}\right)$$

We can start to see how we're just adding $12\pi/36$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[6]{z}_{k=0} = \sqrt[6]{11}\left(\cos\frac{5\pi}{36} + i\sin\frac{5\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=1} = \sqrt[6]{11}\left(\cos\frac{17\pi}{36} + i\sin\frac{17\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=2} = \sqrt[6]{11}\left(\cos\frac{29\pi}{36} + i\sin\frac{29\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=3} = \sqrt[6]{11}\left(\cos\frac{41\pi}{36} + i\sin\frac{41\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=4} = \sqrt[6]{11}\left(\cos\frac{53\pi}{36} + i\sin\frac{53\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=5} = \sqrt[6]{11}\left(\cos\frac{65\pi}{36} + i\sin\frac{65\pi}{36}\right)$$

If we find the decimal approximations of these angles, we get

For $k = 0$, $(5/36)\pi \approx 0.14\pi$

For $k = 1$, $(17/36)\pi \approx 0.47\pi$

For $k = 2$, $(29/36)\pi \approx 0.81\pi$

For $k = 3$, $(41/36)\pi \approx 1.14\pi$

For $k=4$, $(53/36)\pi \approx 1.47\pi$

For $k=5$, $(65/36)\pi \approx 1.81\pi$

Anything in the second quadrant will fall in the interval $(0.5\pi, 1.0\pi)$, which in this case is the angle for $k=2$.

■ 6. Find the 7th roots of the complex number.

$z = 20\left(\cos 120^\circ + i \sin 120^\circ\right)$

Solution:

We're looking for the 7th1 roots of z, which means there will be 7 of them, given by $k = 0, 1, 2, 3, 4, 5, 6$. And since the complex number is given in degrees, we'll plug $n=7$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i \sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[7]{z} = \sqrt[7]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{7}\right) + i \sin\left(\frac{\theta + 360^\circ k}{7}\right)\right]$$

With $r=20$ and $\theta = 120^\circ$ from the complex number, we get

$$\sqrt[7]{z} = \sqrt[7]{20}\left[\cos\left(\frac{120^\circ + 360^\circ k}{7}\right) + i \sin\left(\frac{120^\circ + 360^\circ k}{7}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4, 5, 6$.

For $k = 0$:

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{20}\left[\cos\left(\frac{120^\circ + 360^\circ(0)}{7}\right) + i\sin\left(\frac{120^\circ + 360^\circ(0)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{20}\left[\cos\left(\frac{120}{7}\right)^\circ + i\sin\left(\frac{120}{7}\right)^\circ\right]$$

For $k = 1$:

$$\sqrt[7]{z}_{k=1} = \sqrt[7]{20}\left[\cos\left(\frac{120^\circ + 360^\circ(1)}{7}\right) + i\sin\left(\frac{120^\circ + 360^\circ(1)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=1} = \sqrt[7]{20}\left[\cos\left(\frac{480}{7}\right)^\circ + i\sin\left(\frac{480}{7}\right)^\circ\right]$$

For $k = 2$:

$$\sqrt[7]{z}_{k=2} = \sqrt[7]{20}\left[\cos\left(\frac{120^\circ + 360^\circ(2)}{7}\right) + i\sin\left(\frac{120^\circ + 360^\circ(2)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=2} = \sqrt[7]{20}\left[\cos\left(\frac{840}{7}\right)^\circ + i\sin\left(\frac{840}{7}\right)^\circ\right]$$

We can start to see how we're just adding $360/7$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[7]{z}_{k=0}=\sqrt[7]{20}\left[\cos\left(\frac{120}{7}\right)^{\circ}+i\sin\left(\frac{120}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{20}\left[\cos\left(\frac{480}{7}\right)^{\circ}+i\sin\left(\frac{480}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{20}\left[\cos\left(\frac{840}{7}\right)^{\circ}+i\sin\left(\frac{840}{7}\right)^{\circ}\right]=\sqrt[7]{20}\left[\cos(120)^{\circ}+i\sin(120)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=3}=\sqrt[7]{20}\left[\cos\left(\frac{1,200}{7}\right)^{\circ}+i\sin\left(\frac{1,200}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=4}=\sqrt[7]{20}\left[\cos\left(\frac{1,560}{7}\right)^{\circ}+i\sin\left(\frac{1,560}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=5}=\sqrt[7]{20}\left[\cos\left(\frac{1,920}{7}\right)^{\circ}+i\sin\left(\frac{1,920}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=6}=\sqrt[7]{20}\left[\cos\left(\frac{2,280}{7}\right)^{\circ}+i\sin\left(\frac{2,280}{7}\right)^{\circ}\right]$$

MATRIX DIMENSIONS AND ENTRIES

■ 1. Give the dimensions of the matrix.

$$D = \begin{bmatrix} 11 & 9 \\ -4 & 8 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix D has 2 rows and 2 columns, so D is a 2×2 matrix.

■ 2. Give the dimensions of the matrix.

$$A = \begin{bmatrix} 3 & 5 & -2 & 1 & 8 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix A has 1 row and 5 columns, so A is a 1×5 matrix.

■ 3. Given matrix J, find $J_{4,1}$.

$$J = \begin{bmatrix} 6 \\ 2 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

The value of $J_{4,1}$ is the entry in the fourth row, first column of matrix J, which is 1, so $J_{4,1} = 1$.

■ 4. Given matrix C, find $C_{1,2}$.

$$C = \begin{bmatrix} 3 & 12 \\ 1 & 4 \\ 9 & 5 \\ -3 & 2 \end{bmatrix}$$

Solution:

The value of $C_{1,2}$ is the entry in the first row, second column of matrix C, which is 12, so $C_{1,2} = 12$.

■ 5. Given matrix N, state the dimensions and find $N_{1,3}$.

$$N = \begin{bmatrix} 1 & 5 & 9 \\ 14 & -8 & 6 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix N has 2 rows and 3 columns, so N is a 2×3 matrix.

The value of $N_{1,3}$ is the entry in the first row, third column of matrix N, which is 9, so $N_{1,3} = 9$.

■ 6. Given matrix S, state the dimensions and find $S_{3,4}$.

$$S = \begin{bmatrix} 3 & 6 & -7 & 1 & 0 \\ 0 & 9 & 15 & 3 & 4 \\ 4 & 0 & 2 & 11 & 8 \\ -5 & 8 & 7 & 9 & 2 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix S has 4 rows and 5 columns, so S is a 4×5 matrix.

The value of $S_{3,4}$ is the entry in the third row, fourth column of matrix S, which is 11, so $S_{3,4} = 11$.

REPRESENTING SYSTEMS WITH MATRICES

■ 1. Represent the system with an augmented matrix called A.

$$-2x + 5y = 12$$

$$6x - 2y = 4$$

Solution:

The system contains the variables x and y along with a constant. Which means the augmented matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into an augmented matrix gives

$$A = \begin{bmatrix} -2 & 5 & 12 \\ 6 & -2 & 4 \end{bmatrix}$$

■ 2. Represent the system with an augmented matrix called D.

$$9y - 3x + 12 = 0$$

$$8 - 4x = 11y$$

Solution:

This system can be reorganized by putting each equation in order, with x and y on the left side, and the constant on the right side.

$$-3x + 9y = -12$$

$$4x + 11y = 8$$

The system contains the variables x and y along with a constant. Which means the augmented matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into an augmented matrix gives

$$D = \begin{bmatrix} -3 & 9 & -12 \\ 4 & 11 & 8 \end{bmatrix}$$

■ 3. Represent the system with an augmented matrix called H.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b = -2c + 1$$

Solution:

The second equation can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no d-term in the second equation, so we add in a 0 "filler" term.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b + 2c + 0d = 1$$

The system contains the variables a, b, c, and d, along with a constant. Which means the augmented matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into an augmented matrix gives

$$H = \begin{bmatrix} 4 & 7 & -5 & 13 & 6 \\ 3 & -8 & 2 & 0 & 1 \end{bmatrix}$$

■ 4. Represent the system with an augmented matrix called M.

$$-2x + 4y = 9 - 6z$$

$$7y + 2z - 3 = -3t - 9x$$

Solution:

Both equations can be reorganized by putting x, y, z, and t on the left side, and the constant on the right side. We also recognize that there is no t -term in the first equation, so we add in a 0 "filler" term.

$$-2x + 4y + 6z + 0t = 9$$

$$9x + 7y + 2z + 3t = 3$$

The system contains the variables x, y, z, and t, along with a constant. Which means the augmented matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into an augmented matrix gives

$$M = \begin{bmatrix} -2 & 4 & 6 & 0 & 9 \\ 9 & 7 & 2 & 3 & 3 \end{bmatrix}$$

■ 5. Represent the system with an augmented matrix called A.

$3x - 8y + z = 7$

$2z = 3y - 2x + 4$

$5y = 12 - 9x$

Solution:

The second and third equations can be reorganized by putting x, y, and z on the left side, and the constant on the right side. We also recognize that there is no z-term in the third equation, so we add in a 0 “filler” term.

$$3x - 8y + z = 7$$

$$2x - 3y + 2z = 4$$

$$9x + 5y + 0z = 12$$

The system contains the variables x, y, and z, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into an augmented matrix gives

$$A = \begin{bmatrix} 3 & -8 & 1 & 7 \\ 2 & -3 & 2 & 4 \\ 9 & 5 & 0 & 12 \end{bmatrix}$$

■ 6. Represent the system with an augmented matrix called K.

$-4b + 2c = 3 - 7a$

$9c = 4 - 2b$

$8a - 2c = 5b$

Solution:

All three of these equations can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no a-term in the second equation, and no constant in the third equation, so we add in 0 "filler" terms.

$$7a - 4b + 2c = 3$$

$$0a + 2b + 9c = 4$$

$$8a - 5b - 2c = 0$$

The system contains the variables a, b, and c, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into an augmented matrix gives

$$K = \begin{bmatrix} 7 & -4 & 2 & 3 \\ 0 & 2 & 9 & 4 \\ 8 & -5 & -2 & 0 \end{bmatrix}$$

SIMPLE ROW OPERATIONS

■ 1. Write the new matrix after $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 2 & 6 & -4 & 1 \\ 8 & 2 & 1 & -5 \end{bmatrix}$$

Solution:

The operation described by $R_1 \leftrightarrow R_2$ is switching row 1 with row 2. The matrix after $R_1 \leftrightarrow R_2$ is

$$\begin{bmatrix} 8 & 2 & 1 & -5 \\ 2 & 6 & -4 & 1 \end{bmatrix}$$

■ 2. Write the new matrix after $R_2 \leftrightarrow R_4$.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 6 & 1 & 5 & -4 \\ -7 & 7 & 0 & 3 \\ 9 & 2 & 8 & 3 \end{bmatrix}$$

Solution:

The operation described by $R_2 \leftrightarrow R_4$ is switching row 2 with row 4. Nothing will happen to rows 1 and 3. The matrix after $R_2 \leftrightarrow R_4$ is

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 9 & 2 & 8 & 3 \\ -7 & 7 & 0 & 3 \\ 6 & 1 & 5 & -4 \end{bmatrix}$$

■ 3. Write the new matrix after $R_1 \leftrightarrow 3R_2$.

$$\begin{bmatrix} 9 & 2 & -7 \\ 1 & 6 & 4 \end{bmatrix}$$

Solution:

The operation described by $R_1 \leftrightarrow 3R_2$ is multiplying row 2 by a constant of 3 and then switching those two rows. The matrix after $3R_2$ is

$$\begin{bmatrix} 9 & 2 & -7 \\ 3 & 18 & 12 \end{bmatrix}$$

The matrix after $R_1 \leftrightarrow 3R_2$ is

$$\begin{bmatrix} 3 & 18 & 12 \\ 9 & 2 & -7 \end{bmatrix}$$

■ 4. Write the new matrix after $3R_2 \leftrightarrow 3R_4$.

$$\begin{bmatrix} 0 & 11 & 6 \\ 7 & -3 & 9 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

Solution:

The operation described by $3R_2 \leftrightarrow 3R_4$ is multiplying row 2 by a constant of 3, multiplying row 4 by a constant of 3, and then switching those two rows. Nothing will happen to rows 1 and 3. The matrix after $3R_2$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 21 & -9 & 27 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

The matrix after $3R_4$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 21 & -9 & 27 \\ 8 & 8 & 1 \\ 18 & 6 & 12 \end{bmatrix}$$

The matrix after $3R_2 \leftrightarrow 3R_4$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 18 & 6 & 12 \\ 8 & 8 & 1 \\ 21 & -9 & 27 \end{bmatrix}$$

■ 5. Write the new matrix after $R_1+2R_2\to R_1$.

$$\begin{bmatrix}6&2&7\\1&-5&15\end{bmatrix}$$

Solution:

The operation described by $R_1+2R_2\to R_1$ is multiplying row 2 by a constant of 2, adding that resulting row to row 1, and using that result to replace row 1. $2R_2$ is

$$\begin{bmatrix}2(1)&2(-5)&2(15)\end{bmatrix}$$

$$[2\quad -10\quad 30]$$

The sum R_1+2R_2 is

$$[6+2\quad 2-10\quad 7+30]$$

$$[8\quad -8\quad 37]$$

The matrix after $R_1+2R_2\to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix}8&-8&37\\1&-5&15\end{bmatrix}$$

■ 6. Write the new matrix after $4R_2+R_3\to R_3$.

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 4 & 1 & 7 & -3 \end{bmatrix}$$

Solution:

The operation described by $4R_2 + R_3 \to R_3$ is multiplying row 2 by a constant of 4, adding that resulting row to row 3, and using that result to replace row 3. $4R_2$ is

$$\begin{bmatrix} 4(8) & 4(2) & 4(0) & 4(6) \end{bmatrix}$$

$$\begin{bmatrix} 32 & 8 & 0 & 24 \end{bmatrix}$$

The sum $4R_2 + R_3$ is

$$\begin{bmatrix} 32+4 & 8+1 & 0+7 & 24-3 \end{bmatrix}$$

$$\begin{bmatrix} 36 & 9 & 7 & 21 \end{bmatrix}$$

The matrix after $4R_2 + R_3 \to R_3$, which is replacing row 3 with this row we just found, is

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 36 & 9 & 7 & 21 \end{bmatrix}$$

GAUSS-JORDAN ELIMINATION AND REDUCED ROW-ECHELON FORM

■ 1. Use Gauss-Jordan elimination to solve the system.

$x + 2y = -2$

$3x + 2y = 6$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 1 & 2 & = & -2 \\ 3 & 2 & = & 6 \end{bmatrix}$$

After $3R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & = & -2 \\ 0 & 4 & = & -12 \end{bmatrix}$$

The first column is done. After $(1/4)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & = & -2 \\ 0 & 1 & = & -3 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & = & 4 \\ 0 & 1 & = & -3 \end{bmatrix}$$

The second column is done, so we get the solution set

$$x = 4$$

$$y = -3$$

■ 2. Use Gauss-Jordan elimination to solve the system.

$$2x + 4y = 22$$

$$3x + 3y = 15$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 2 & 4 & = & 22 \\ 3 & 3 & = & 15 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$ and $(1/3)R_2 \to R_2$the matrix is

$$\begin{bmatrix} 1 & 2 & = & 11 \\ 1 & 1 & = & 5 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & = & 11 \\ 0 & 1 & = & 6 \end{bmatrix}$$

The first column is done. After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & = & -1 \\ 0 & 1 & = & 6 \end{bmatrix}$$

The second column is done, so we get the solution set

$$x = -1$$

$$y = 6$$

■ 3. Use Gauss-Jordan elimination to solve the system.

$$x - 3y - 6z = 4$$

$$y + 2z = -2$$

$$-4x + 12y + 21z = -4$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 1 & -3 & -6 & = & 4 \\ 0 & 1 & 2 & = & -2 \\ -4 & 12 & 21 & = & -4 \end{bmatrix}$$

After $4R_1 + R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & -3 & -6 & = & 4 \\ 0 & 1 & 2 & = & -2 \\ 0 & 0 & -3 & = & 12 \end{bmatrix}$$

The first column is done. After $3R_2 + R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & -2 \\ 0 & 1 & 2 & = & -2 \\ 0 & 0 & -3 & = & 12 \end{bmatrix}$$

The second column is done. After $(-1/3)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & -2 \\ 0 & 1 & 2 & = & -2 \\ 0 & 0 & 1 & = & -4 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & -2 \\ 0 & 1 & 0 & = & 6 \\ 0 & 0 & 1 & = & -4 \end{bmatrix}$$

The third column is done, so we get the solution set

$$x = -2$$

$$y = 6$$

$$z = -4$$

■ 4. Use Gauss-Jordan elimination to solve the system.

$$2y + 4z = 4$$

$$x + 3y + 3z = 5$$

$$2x + 7y + 6z = 10$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 0 & 2 & 4 & = & 4 \\ 1 & 3 & 3 & = & 5 \\ 2 & 7 & 6 & = & 10 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 0 & 1 & 2 & = & 2 \\ 1 & 3 & 3 & = & 5 \\ 2 & 7 & 6 & = & 10 \end{bmatrix}$$

Because the first entry in the first row is 0, swap it with the second row to get

$$\begin{bmatrix} 1 & 3 & 3 & = & 5 \\ 0 & 1 & 2 & = & 2 \\ 2 & 7 & 6 & = & 10 \end{bmatrix}$$

After $R_3 - 2R_1 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 3 & 3 & = & 5 \\ 0 & 1 & 2 & = & 2 \\ 0 & 1 & 0 & = & 0 \end{bmatrix}$$

The first column is done. After $R_1 - 3R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -1 \\ 0 & 1 & 2 & = & 2 \\ 0 & 1 & 0 & = & 0 \end{bmatrix}$$

After $R_2 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -1 \\ 0 & 1 & 2 & = & 2 \\ 0 & 0 & 2 & = & 2 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -1 \\ 0 & 1 & 2 & = & 2 \\ 0 & 0 & 1 & = & 1 \end{bmatrix}$$

After $R_1 + 3R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 2 & = & 2 \\ 0 & 0 & 1 & = & 1 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 0 & = & 0 \\ 0 & 0 & 1 & = & 1 \end{bmatrix}$$

The third column is done, so we get the solution set

$$x = 2$$

$$y = 0$$

$$z = 1$$

■ 5. Use Gauss-Jordan elimination to solve the system.

$3x + 12y + 42z = -27$

$x + 2y + 8z = -5$

$2x + 5y + 16z = -6$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 3 & 12 & 42 & = & -27 \\ 1 & 2 & 8 & = & -5 \\ 2 & 5 & 16 & = & -6 \end{bmatrix}$$

After $(1/3)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & = & -9 \\ 1 & 2 & 8 & = & -5 \\ 2 & 5 & 16 & = & -6 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & = & -9 \\ 0 & 2 & 6 & = & -4 \\ 2 & 5 & 16 & = & -6 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & = & -9 \\ 0 & 2 & 6 & = & -4 \\ 0 & 3 & 12 & = & -12 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & = & -9 \\ 0 & 1 & 3 & = & -2 \\ 0 & 3 & 12 & = & -12 \end{bmatrix}$$

After $R_1 - 4R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & = & -1 \\ 0 & 1 & 3 & = & -2 \\ 0 & 3 & 12 & = & -12 \end{bmatrix}$$

After $R_3 - 3R_2 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & = & -1 \\ 0 & 1 & 3 & = & -2 \\ 0 & 0 & 3 & = & -6 \end{bmatrix}$$

The second column is done. After $(1/3)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & = & -1 \\ 0 & 1 & 3 & = & -2 \\ 0 & 0 & 1 & = & -2 \end{bmatrix}$$

After $R_1 - 2R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 3 \\ 0 & 1 & 3 & = & -2 \\ 0 & 0 & 1 & = & -2 \end{bmatrix}$$

After $R_2 - 3R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 3 \\ 0 & 1 & 0 & = & 4 \\ 0 & 0 & 1 & = & -2 \end{bmatrix}$$

The third column is done, so we get the solution set

$$x = 3$$

$$y = 4$$

$$z = -2$$

■ 6. Use Gauss-Jordan elimination to solve the system.

$$4x + 8y + 4z = 20$$

$$4x + 6y = 4$$

$$3x + 3y - z = 1$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 4 & 8 & 4 & = & 20 \\ 4 & 6 & 0 & = & 4 \\ 3 & 3 & -1 & = & 1 \end{bmatrix}$$

After $(1/4)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & = & 5 \\ 4 & 6 & 0 & = & 4 \\ 3 & 3 & -1 & = & 1 \end{bmatrix}$$

After $4R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & = & 5 \\ 0 & 2 & 4 & = & 16 \\ 3 & 3 & -1 & = & 1 \end{bmatrix}$$

After $3R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & = & 5 \\ 0 & 2 & 4 & = & 16 \\ 0 & 3 & 4 & = & 14 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & = & 5 \\ 0 & 1 & 2 & = & 8 \\ 0 & 3 & 4 & = & 14 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -11 \\ 0 & 1 & 2 & = & 8 \\ 0 & 3 & 4 & = & 14 \end{bmatrix}$$

After $3R_2 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -11 \\ 0 & 1 & 2 & = & 8 \\ 0 & 0 & 2 & = & 10 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -11 \\ 0 & 1 & 2 & = & 8 \\ 0 & 0 & 1 & = & 5 \end{bmatrix}$$

After $R_1 + 3R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 4 \\ 0 & 1 & 2 & = & 8 \\ 0 & 0 & 1 & = & 5 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 4 \\ 0 & 1 & 0 & = & -2 \\ 0 & 0 & 1 & = & 5 \end{bmatrix}$$

The third column is done, so we get the solution set

$x = 4$

$y = -2$

$z = 5$

MATRIX ADDITION AND SUBTRACTION

■ 1. Add the matrices.

$$\begin{vmatrix} 7 & 6 \\ 17 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 8 \\ -2 & 5 \end{vmatrix}$$

Solution:

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{vmatrix} 7 & 6 \\ 17 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 8 \\ -2 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 7+0 & 6+8 \\ 17+(-2) & 9+5 \end{vmatrix}$$

$$\begin{vmatrix} 7 & 14 \\ 15 & 14 \end{vmatrix}$$

■ 2. Add the matrices.

$$\begin{vmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{vmatrix}$$

Solution:

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{vmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 8+6 & 3+7 \\ -4+2 & 7+(-3) \\ 6+9 & 0+11 \\ 1+7 & 13+(-2) \end{vmatrix}$$

$$\begin{vmatrix} 14 & 10 \\ -2 & 4 \\ 15 & 11 \\ 8 & 11 \end{vmatrix}$$

■ 3. Subtract the matrices.

$$\begin{vmatrix} 7 & 9 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 8 \\ 12 & -3 \end{vmatrix}$$

Solution:

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{vmatrix} 7 & 9 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 8 \\ 12 & -3 \end{vmatrix}$$

$$\begin{vmatrix} 7-3 & 9-8 \\ 4-12 & -1-(-3) \end{vmatrix}$$

$$\begin{vmatrix} 4 & 1 \\ -8 & 2 \end{vmatrix}$$

■ 4. Subtract the matrices.

$$\begin{vmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{vmatrix} - \begin{vmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{vmatrix}$$

Solution:

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{vmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{vmatrix} - \begin{vmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{vmatrix}$$

$$\begin{vmatrix} 8-6 & 11-11 & 2-7 & 9-(-4) \\ 6-5 & 3-8 & 16-1 & 8-15 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & -5 & 13 \\ 1 & -5 & 15 & -7 \end{vmatrix}$$

■ 5. Solve for m.

$$\begin{vmatrix} 6 & 5 \\ 9 & -9 \end{vmatrix} + \begin{vmatrix} 3 & 7 \\ 1 & 6 \end{vmatrix} = m + \begin{vmatrix} 7 & 12 \\ -3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 8 \\ 4 & -7 \end{vmatrix}$$

Solution:

Let's start with the matrix addition on the left side of the equation and the matrix subtraction on the right side of the equation.

$$\begin{vmatrix} 6 & 5 \\ 9 & -9 \end{vmatrix} + \begin{vmatrix} 3 & 7 \\ 1 & 6 \end{vmatrix} = m + \begin{vmatrix} 7 & 12 \\ -3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 8 \\ 4 & -7 \end{vmatrix}$$

$$\begin{vmatrix} 6+3 & 5+7 \\ 9+1 & -9+6 \end{vmatrix} = m + \begin{vmatrix} 7-1 & 12-8 \\ -3-4 & -1-(-7) \end{vmatrix}$$

$$\begin{vmatrix} 9 & 12 \\ 10 & -3 \end{vmatrix} = m + \begin{vmatrix} 6 & 4 \\ -7 & 6 \end{vmatrix}$$

To isolate m, we'll subtract the matrix on the right from both sides in order to move it to the left.

$$\begin{vmatrix} 9 & 12 \\ 10 & -3 \end{vmatrix} - \begin{vmatrix} 6 & 4 \\ -7 & 6 \end{vmatrix} = m$$

$$\begin{vmatrix} 9-6 & 12-4 \\ 10-(-7) & -3-6 \end{vmatrix} = m$$

$$\begin{vmatrix} 3 & 8 \\ 17 & -9 \end{vmatrix} = m$$

The conclusion is that the value of m that makes the equation true is this matrix:

$$m = \begin{vmatrix} 3 & 8 \\ 17 & -9 \end{vmatrix}$$

■ 6. Solve for n.

$$\begin{vmatrix} 4 & 12 \\ 9 & 8 \end{vmatrix} - \begin{vmatrix} 0 & 3 \\ 9 & 9 \end{vmatrix} = n - \begin{vmatrix} 6 & 3 \\ 5 & 11 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -18 & 1 \end{vmatrix}$$

Solution:

Let's start with the matrix subtraction on the left side of the equation and the matrix addition on the right side of the equation.

$$\begin{vmatrix} 4 & 12 \\ 9 & 8 \end{vmatrix} - \begin{vmatrix} 0 & 3 \\ 9 & 9 \end{vmatrix} = n - \begin{vmatrix} 6 & 3 \\ 5 & 11 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -18 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 4-0 & 12-3 \\ 9-9 & 8-9 \end{vmatrix} = n - \begin{vmatrix} 6+7 & 3+(-4) \\ 5+(-18) & 11+1 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 9 \\ 0 & -1 \end{vmatrix} = n - \begin{vmatrix} 13 & -1 \\ -13 & 12 \end{vmatrix}$$

To isolate n, we'll add the matrix on the right to both sides in order to move it to the left.

$$\begin{vmatrix} 4 & 9 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 13 & -1 \\ -13 & 12 \end{vmatrix} = n$$

$$\begin{vmatrix} 4+13 & 9+(-1) \\ 0+(-13) & -1+12 \end{vmatrix} = n$$

$$\begin{vmatrix} 17 & 8 \\ -13 & 11 \end{vmatrix} = n$$

The conclusion is that the value of n that makes the equation true is this matrix:

$$n = \begin{vmatrix} 17 & 8 \\ -13 & 11 \end{vmatrix}$$

SCALAR MULTIPLICATION AND ZERO MATRICES

■ 1. Use scalar multiplication to simplify the expression.

$$\frac{1}{4}\begin{bmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{bmatrix}$$

Solution:

The scalar $1/4$ is being multiplied by the matrix. Distribute the scalar across every entry in the matrix.

$$\frac{1}{4}\begin{bmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4}(12) & \frac{1}{4}(8) & \frac{1}{4}(3) \\ \frac{1}{4}(2) & \frac{1}{4}(-16) & \frac{1}{4}(0) \\ \frac{1}{4}(1) & \frac{1}{4}(5) & \frac{1}{4}(7) \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & \frac{3}{4} \\ \frac{1}{2} & -4 & 0 \\ \frac{1}{4} & \frac{5}{4} & \frac{7}{4} \end{bmatrix}$$

■ 2. Solve for y.

$$4\begin{vmatrix} 2 & 9 \\ -5 & 0 \end{vmatrix} + y = 5\begin{vmatrix} 1 & -3 \\ 6 & 8 \end{vmatrix}$$

Solution:

Apply the scalars to the matrices.

$$\begin{vmatrix} 4(2) & 4(9) \\ 4(-5) & 4(0) \end{vmatrix} + y = \begin{vmatrix} 5(1) & 5(-3) \\ 5(6) & 5(8) \end{vmatrix}$$

$$\begin{vmatrix} 8 & 36 \\ -20 & 0 \end{vmatrix} + y = \begin{vmatrix} 5 & -15 \\ 30 & 40 \end{vmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate y.

$$y = \begin{vmatrix} 5 & -15 \\ 30 & 40 \end{vmatrix} - \begin{vmatrix} 8 & 36 \\ -20 & 0 \end{vmatrix}$$

$$y = \begin{vmatrix} 5-8 & -15-36 \\ 30-(-20) & 40-0 \end{vmatrix}$$

$$y = \begin{vmatrix} -3 & -51 \\ 50 & 40 \end{vmatrix}$$

■ 3. Solve for n.

$$-2\begin{bmatrix} 6 & 5 \\ 0 & 11 \end{bmatrix} = n - 4\begin{bmatrix} 2 & 4 \\ -1 & 9 \end{bmatrix}$$

Solution:

Apply the scalars to the matrices.

$$\begin{bmatrix} -2(6) & -2(5) \\ -2(0) & -2(11) \end{bmatrix} = n - \begin{bmatrix} 4(2) & 4(4) \\ 4(-1) & 4(9) \end{bmatrix}$$

$$\begin{bmatrix} -12 & -10 \\ 0 & -22 \end{bmatrix} = n - \begin{bmatrix} 8 & 16 \\ -4 & 36 \end{bmatrix}$$

Add the matrix on the right to both sides of the equation in order to isolate n.

$$\begin{bmatrix} -12 & -10 \\ 0 & -22 \end{bmatrix} + \begin{bmatrix} 8 & 16 \\ -4 & 36 \end{bmatrix} = n$$

$$\begin{bmatrix} -12+8 & -10+16 \\ 0+(-4) & -22+36 \end{bmatrix} = n$$

$$\begin{bmatrix} -4 & 6 \\ -4 & 14 \end{bmatrix} = n$$

$$n = \begin{bmatrix} -4 & 6 \\ -4 & 14 \end{bmatrix}$$

■ 4. Add the zero matrix to the given matrix.

$$\begin{vmatrix} 8 & 17 \\ -6 & 0 \end{vmatrix}$$

Solution:

Adding the zero matrix to any other matrix doesn’t change the value of the matrix, so

$$\begin{vmatrix} 8 & 17 \\ -6 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 8 & 17 \\ -6 & 0 \end{vmatrix}$$

■ 5. Find the opposite matrix.

$$\begin{vmatrix} 6 & 8 & 0 \\ 2 & -3 & 11 \\ 4 & 12 & 9 \end{vmatrix}$$

Solution:

To get the opposite of a matrix, multiply it by a scalar of -1. Then the opposite of the given matrix is

$$(-1)\begin{vmatrix} (-1)6 & (-1)8 & (-1)0 \\ (-1)2 & (-1)(-3) & (-1)11 \\ (-1)4 & (-1)12 & (-1)9 \end{vmatrix}$$

$$\begin{vmatrix} -6 & -8 & 0 \\ -2 & 3 & -11 \\ -4 & -12 & -9 \end{vmatrix}$$

■ 6. Multiply the matrix by a scalar of 0.

$$\begin{vmatrix} 14 & -1 & 7 & 5 \\ 3 & 7 & 18 & -4 \end{vmatrix}$$

Solution:

Multiplying any matrix by a scalar of 0 results in a zero matrix.

$$(0)\begin{vmatrix} 14(0) & -1(0) & 7(0) & 5(0) \\ 3(0) & 7(0) & 18(0) & -4(0) \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

MATRIX MULTIPLICATION

■ 1. If matrix A is 3×3 and matrix B is 3×4, say whether AB or BA is defined, and give the dimensions of any product that's defined.

Solution:

Line up the dimensions for the products AB and BA, and compare the middle terms, which represent the columns from the first matrix and the rows from the second matrix.

$$AB: \quad 3 \times \boxed{3 \qquad 3} \times 4$$

$$BA: \quad 3 \times \boxed{4 \qquad 3} \times 3$$

The middle numbers match for AB, so that product is defined. For BA, the middle numbers don't match, so that product isn't defined.

The dimensions of AB are given by the outside numbers, which are the rows from the first matrix and the columns from the second matrix.

$$AB: \quad \boxed{3} \times 3 \qquad 3 \times \boxed{4}$$

So the dimensions of AB will be 3×4.

■ 2. Find the product of matrices A and B.

$$A = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2(-2) + 6(5) & 2(0) + 6(-4) \\ -3(-2) + 1(5) & -3(0) + 1(-4) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 26 & -24 \\ 11 & -4 \end{bmatrix}$$

■ 3. Find the product of matrices A and B.

$$A = \begin{bmatrix} 5 & -1 \\ 0 & 11 \\ 7 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 1 & 8 \\ -3 & 0 & 4 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 5 & -1 \\ 0 & 11 \\ 7 & -2 \end{bmatrix} \cdot \begin{bmatrix} 6 & 1 & 8 \\ -3 & 0 & 4 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 5(6) + (-1)(-3) & 5(1) + (-1)(0) & 5(8) + (-1)(4) \\ 0(6) + 11(-3) & 0(1) + 11(0) & 0(8) + 11(4) \\ 7(6) + (-2)(-3) & 7(1) + (-2)(0) & 7(8) + (-2)(4) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 33 & 5 & 36 \\ -33 & 0 & 44 \\ 48 & 7 & 48 \end{bmatrix}$$

■ 4. Find the product of matrices A and B.

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 8 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 3 & -2 \\ 1 & 8 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 3(5) + (-2)(4) & 3(2) + (-2)(8) \\ 1(5) + 8(4) & 1(2) + 8(8) \\ 0(5) + 3(4) & 0(2) + 3(8) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 7 & -10 \\ 37 & 66 \\ 12 & 24 \end{bmatrix}$$

■ 5. Use the distributive property to find $A(B + C)$.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 1 \\ 3 & -1 \end{bmatrix}$$

Solution:

Applying the distributive property to the initial expression, we get

$$A(B + C) = AB + AC$$

Use matrix multiplication to find $AB + AC$.

$$AB+AC=\begin{bmatrix}2 & 0\\4 & -2\end{bmatrix}\cdot\begin{bmatrix}3 & 1\\5 & 4\end{bmatrix}+\begin{bmatrix}2 & 0\\4 & -2\end{bmatrix}\begin{bmatrix}6 & 1\\3 & -1\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}2(3)+(0)(5) & 2(1)+0(4)\\4(3)+(-2)(5) & 4(1)+(-2)(4)\end{bmatrix}$$

$$+\begin{bmatrix}2(6)+0(3) & 2(1)+0(-1)\\4(6)+(-2)(3) & 4(1)+(-2)(-1)\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}6 & 2\\2 & -4\end{bmatrix}+\begin{bmatrix}12 & 2\\18 & 6\end{bmatrix}$$

Now use matrix addition.

$$AB+AC=\begin{bmatrix}6+12 & 2+2\\2+18 & -4+6\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}18 & 4\\20 & 2\end{bmatrix}$$

So the value of the original expression is

$$A(B+C)=\begin{bmatrix}18 & 4\\20 & 2\end{bmatrix}$$

■ 6. Find the product of matrices A and B.

$$A=\begin{bmatrix}0 & 0\\0 & 0\\0 & 0\\0 & 0\end{bmatrix}$$

$$B = \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

IDENTITY MATRICES

■ 1. Write the identity matrix I_4.

Solution:

We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate I_{2x2} as just I_2, or I_{3x3} as just I_3, etc. So, I_4 is the 4×4 identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 2. If we want to find the product IA, where I is the identity matrix and A is a 4×2, then what are the dimensions of I?

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 4 \times 2 = 4 \times 2$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 4 \times 2 = 4 \times 2$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times \boxed{4 \cdot 4} \times 2 = 4 \times 2$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$\boxed{4} \times 4 \cdot 4 \times 2 = \boxed{4} \times 2$$

Therefore, the identity matrix in this case is I_4.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 3. If we want to find the product IA, where I is the identity matrix and A is a 3×4, then what are the dimensions of I?

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 3 \times 4 = 3 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 4 = 3 \times 4$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times \boxed{3 \cdot 3} \times 4 = 3 \times 4$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$\boxed{3} \times 3 \cdot 3 \times 4 = \boxed{3} \times 4$$

Therefore, the identity matrix in this case is I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■ 4. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 3 \times 2 = 3 \times 2$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 2 = 3 \times 2$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times \boxed{3 \cdot 3} \times 2 = 3 \times 2$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$\boxed{3} \times 3 \cdot 3 \times 2 = \boxed{3} \times 2$$

Therefore, the identity matrix in this case is I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product IA is

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(2)+0(-2)+0(3) & 1(8)+0(7)+0(5) \\ 0(2)+1(-2)+0(3) & 0(8)+1(7)+0(5) \\ 0(2)+0(-2)+1(3) & 0(8)+0(7)+1(5) \end{bmatrix}$$

$$IA = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_3.

■ 5. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 2 \times 4 = 2 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times \boxed{2 \cdot 2} \times 4 = 2 \times 4$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$\boxed{2} \times 2 \cdot 2 \times 4 = \boxed{2} \times 4$$

Therefore, the identity matrix in this case is I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The product IA is

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(7)+0(5) & 1(1)+0(5) & 1(3)+0(2) & 1(-2)+0(9) \\ 0(7)+1(5) & 0(1)+1(5) & 0(3)+1(2) & 0(-2)+1(9) \end{bmatrix}$$

$$IA = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_2.

■ 6. If A is a 2×4 matrix what are the dimensions of the identity matrix that make the equation true?

$A \cdot I = A$

Solution:

Set up the equation $A \cdot I = A$, then substitute the dimensions for A into the equation.

$$A \cdot I = A$$

$$2 \times 4 \cdot I = 2 \times 4$$

Break up the dimensions of I as $R \times C$.

$$2 \times 4 \cdot R \times C = 2 \times 4$$

The number of rows in the second matrix must be equal to the number of columns from the first matrix.

$$2 \times \boxed{4 \cdot 4} \times C = 2 \times 4$$

The dimensions of the product come from the rows of the first matrix and the columns of the second matrix, so

$$2 \times 4 \cdot 4 \times \boxed{4} = 2 \times \boxed{4}$$

So the identity matrix is I_4, the 4×4 identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

TRANSFORMATIONS

■ 1. Find the resulting vector $\overrightarrow{b}$ after $\overrightarrow{a}=(1,6)$ undergoes a transformation by matrix M.

$$M=\begin{bmatrix}-7 & 1\\ 0 & -2\end{bmatrix}$$

Solution:

To apply a transformation matrix to vector $\overrightarrow{a}$, we'll multiply the matrix by the vector.

$$\overrightarrow{b}=M\overrightarrow{a}=\begin{bmatrix}-7 & 1\\ 0 & -2\end{bmatrix}\cdot\begin{bmatrix}1\\ 6\end{bmatrix}$$

$$\overrightarrow{b}=M\overrightarrow{a}=\begin{bmatrix}-7(1)+1(6)\\ 0(1)-2(6)\end{bmatrix}$$

$$\overrightarrow{b}=M\overrightarrow{a}=\begin{bmatrix}-7+6\\ 0-12\end{bmatrix}$$

$$\overrightarrow{b}=M\overrightarrow{a}=\begin{bmatrix}-1\\ -12\end{bmatrix}$$

■ 2. Sketch triangle $\triangle ABC$ with vertices $(2,3)$, $(-3,-1)$, and $(1,-4)$, and the transformation of $\triangle ABC$ after it's transformed by matrix L.

$$L = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Solution:

Put the vertices of $\triangle ABC$ into a matrix.

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & -1 & -4 \end{bmatrix}$$

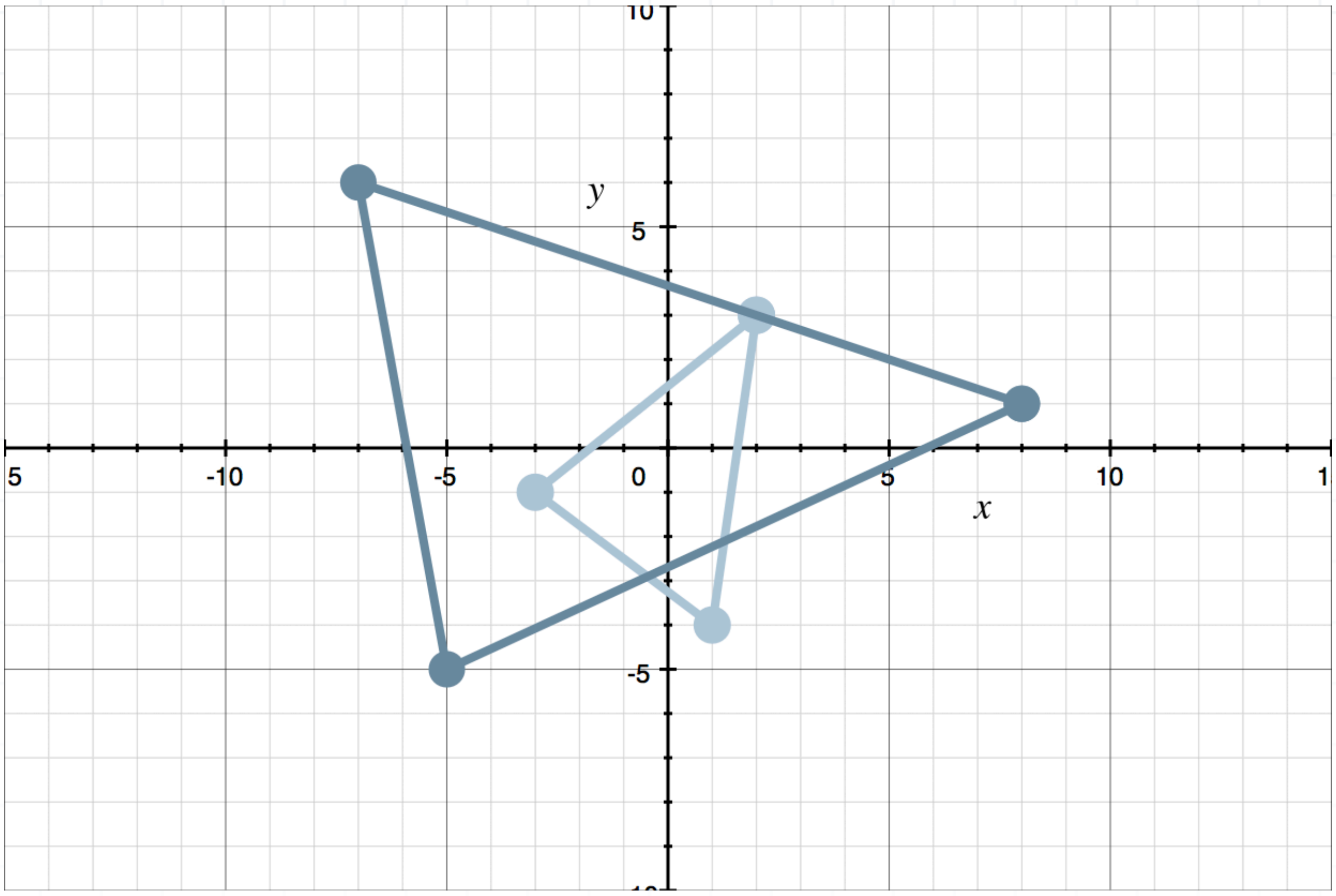
Apply the transformation of L to the vertex matrix.

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 1 \\ 3 & -1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1(2) + 2(3) & 1(-3) + 2(-1) & 1(1) + 2(-4) \\ 2(2) - 1(3) & 2(-3) - 1(-1) & 2(1) - 1(-4) \end{bmatrix}$$

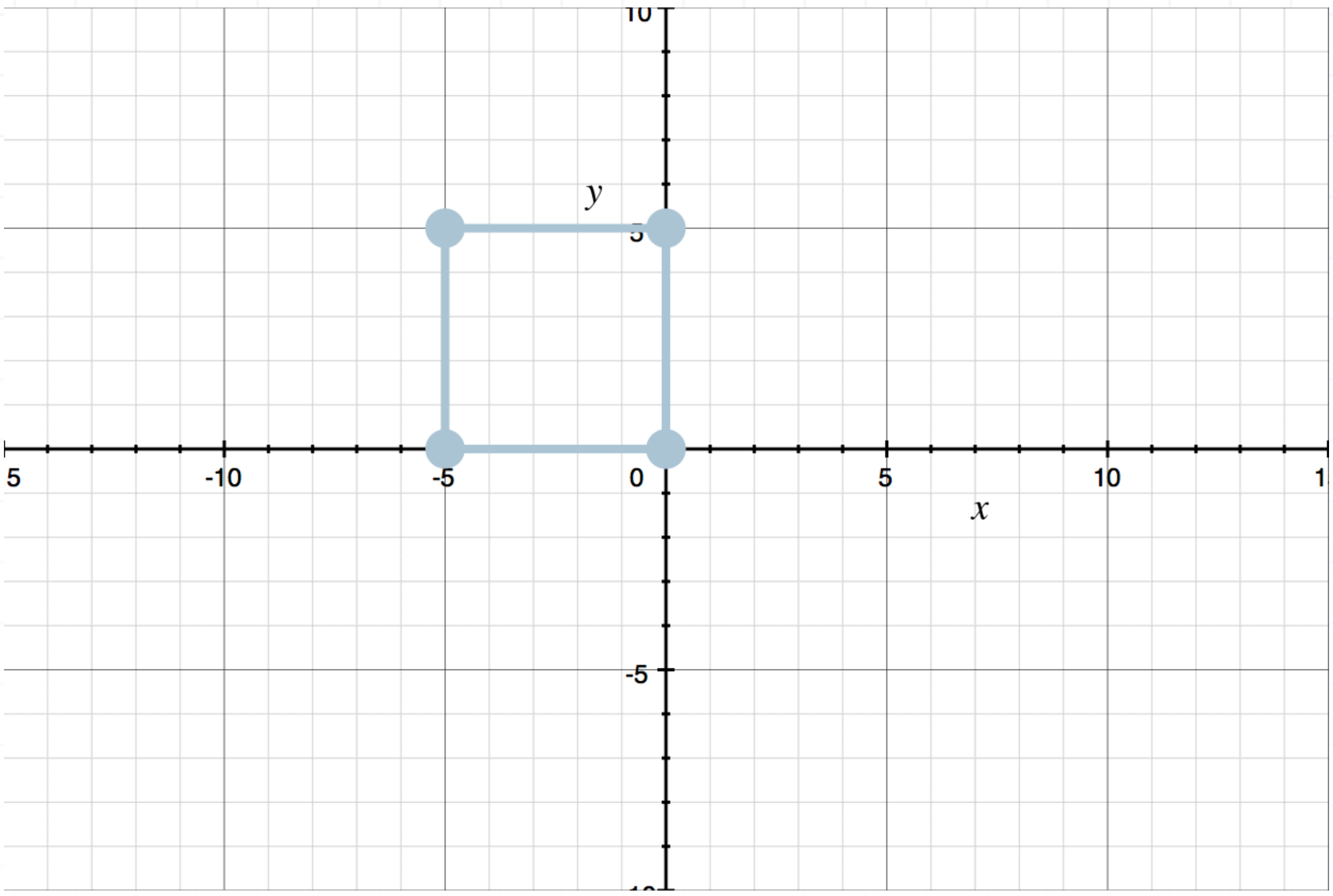
$$\begin{bmatrix} 8 & -5 & -7 \\ 1 & -5 & 6 \end{bmatrix}$$

The original triangle $\triangle ABC$ is sketched in light blue, and its transformation after L is in dark blue.



■ 3. Sketch the transformation of the square in the graph after it's transformed by matrix Z.

$$Z = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$



Solution:

Put the vertices of the square into a matrix.

$$\begin{bmatrix} 0 & -5 & -5 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

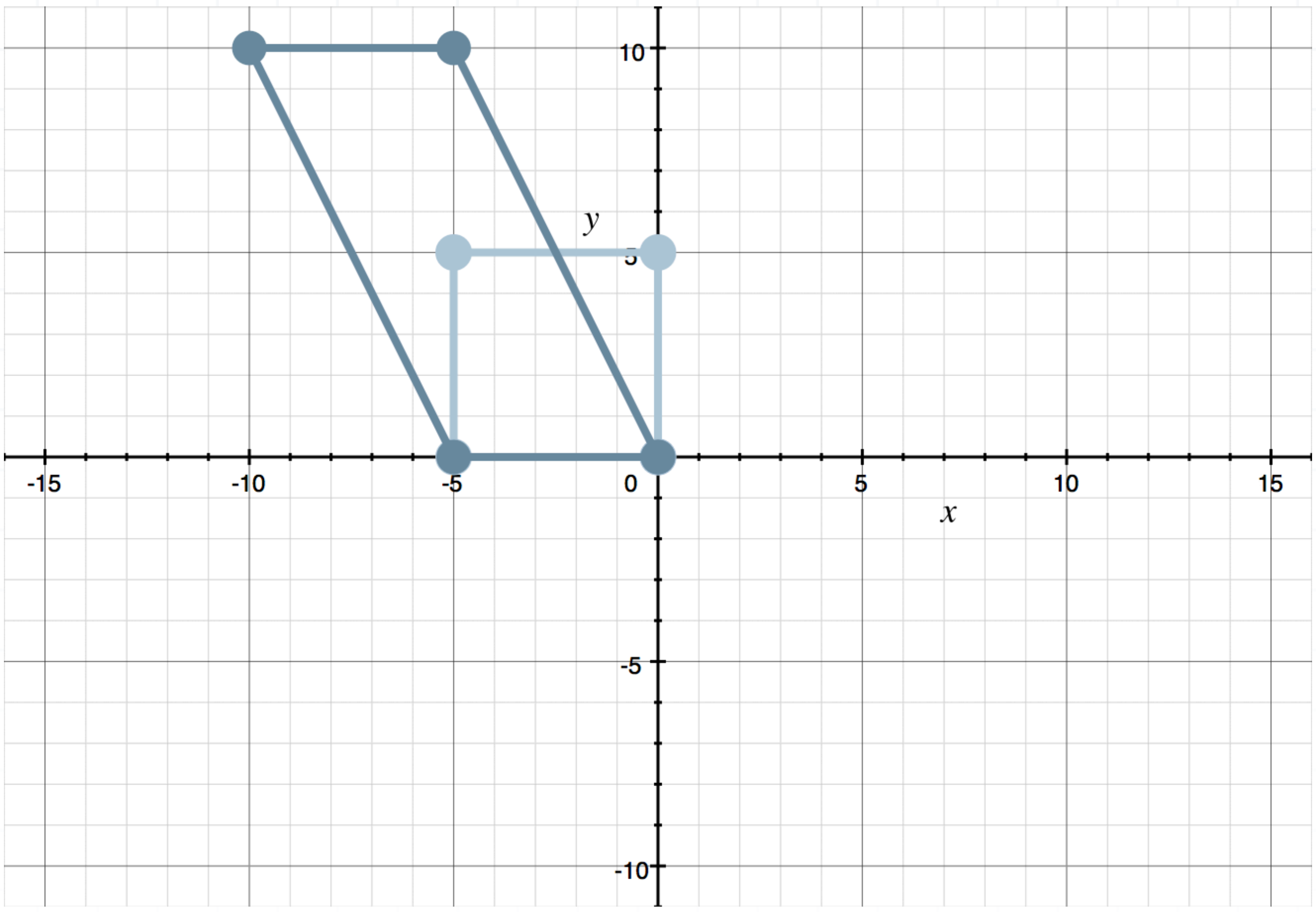
Apply the transformation of Z to the vertex matrix.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & -5 & -5 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1(0)-1(0) & 1(-5)-1(0) & 1(-5)-1(5) & 1(0)-1(5) \\ 0(0)+2(0) & 0(-5)+2(0) & 0(-5)+2(5) & 0(0)+2(5) \end{bmatrix}$$

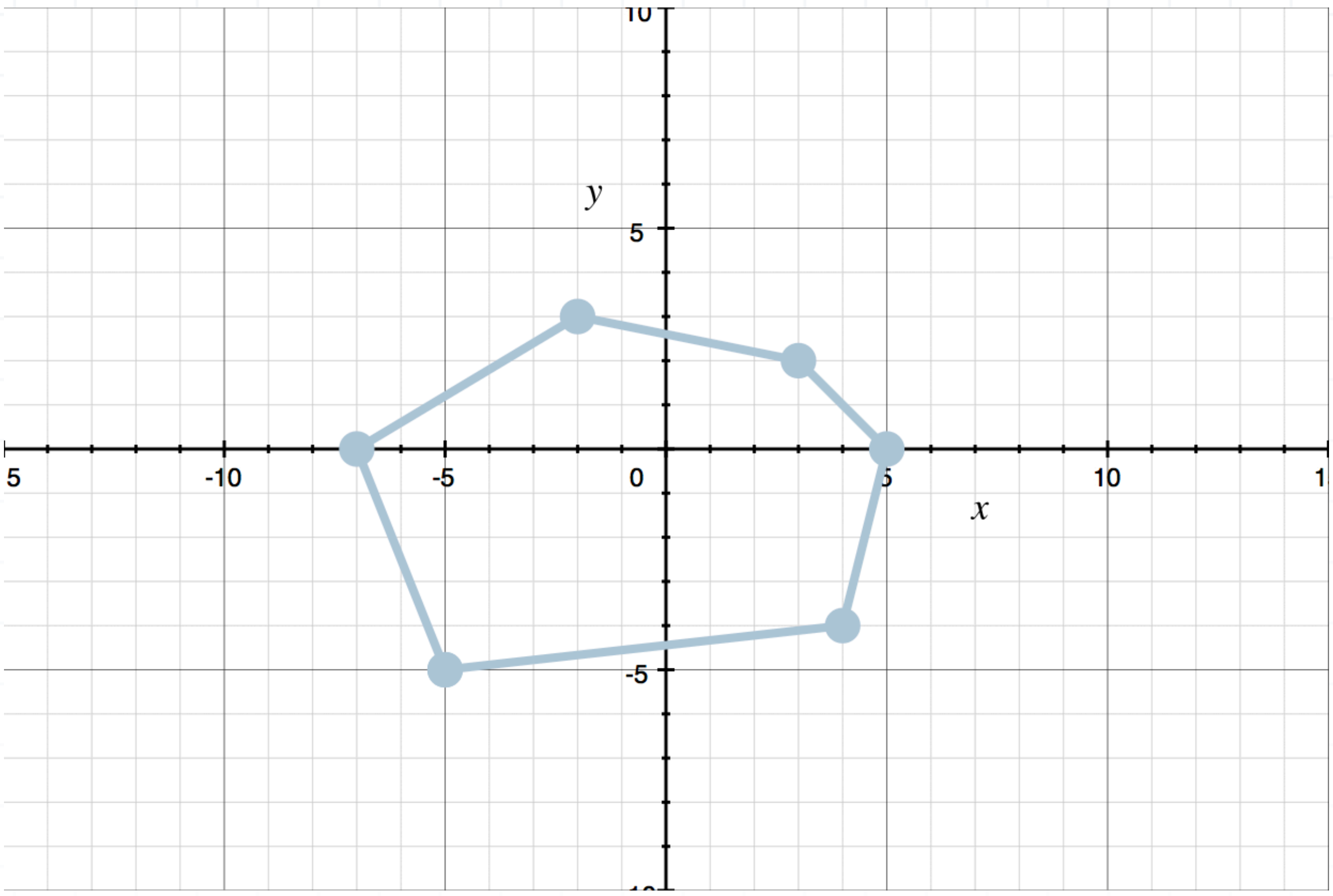
$$\begin{bmatrix} 0 & -5 & -10 & -5 \\ 0 & 0 & 10 & 10 \end{bmatrix}$$

The original square is sketched in light blue, and its transformation after Z is in dark blue.



■ 4. Sketch the transformation of the hexagon after it's transformed by matrix Y.

$$Y = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$



Solution:

Put the vertices of the hexagon into a matrix.

$$\begin{bmatrix} -5 & 4 & 5 & 3 & -2 & -7 & -5 \\ -5 & -4 & 0 & 2 & 3 & 0 & -5 \end{bmatrix}$$

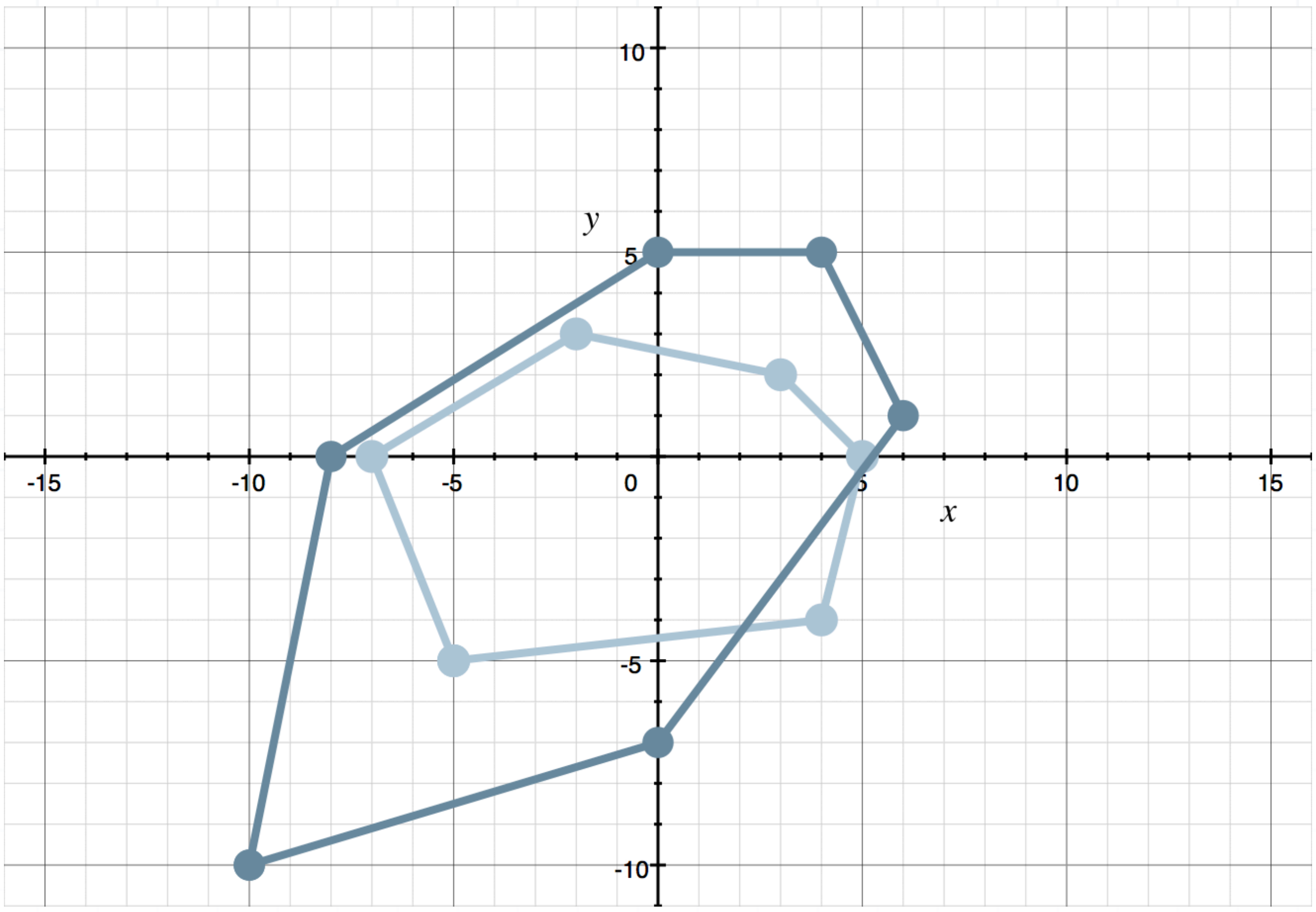
Apply the transformation of Z to the vertex matrix.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -5 & 4 & 5 & 3 & -2 & -7 \\ -5 & -4 & 0 & 2 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -5(0)-5(2) & 4(0)-4(2) & 5(0)+0(2) & 3(0)+2(2) & -2(0)+3(2) & -7(0)+0(2) \\ -5(1)-5(1) & 4(1)-4(1) & 5(1)+0(1) & 3(1)+2(1) & -2(1)+3(1) & -7(1)+0(1) \end{bmatrix}$$

$$\begin{bmatrix} -10 & -8 & 0 & 4 & 6 & 0 \\ -10 & 0 & 5 & 5 & 1 & -7 \end{bmatrix}$$

The original hexagon is sketched in light blue, and its transformation after Y is in dark blue.



■ 5. What happens to the unit vector $\overrightarrow{a} = (1,0)$ after the transformation given by matrix K.

$$K = \begin{bmatrix} 3 & -5 \\ -1 & 0 \end{bmatrix}$$

Solution:

Where $\overrightarrow{a} = (1,0)$ lands is given by the first column of the transformation matrix. So $\overrightarrow{a}$ will land on $(3, -1)$ after the transformation by K.

■ 6. What happens to the unit vector $\overrightarrow{b}=(0,1)$ after the transformation given by matrix K.

$$K=\begin{bmatrix}3 & -5\\ -1 & 0\end{bmatrix}$$

Solution:

Where $\overrightarrow{b}=(0,1)$ lands is given by the second column of the transformation matrix. So $\overrightarrow{b}$ will land on $(-5,0)$ after the transformation by K.

MATRIX INVERSES, AND INVERTIBLE AND SINGULAR MATRICES

■ 1. Find the determinant of the matrix.

$$B = \begin{bmatrix} -3 & 8 \\ 0 & -2 \end{bmatrix}$$

Solution:

The determinant is

$$B = \begin{vmatrix} -3 & 8 \\ 0 & -2 \end{vmatrix}$$

$$B = (-3)(-2) - (8)(0)$$

$$B = 6 - 0$$

$$B = 6$$

■ 2. Find the determinant of the matrix.

$$B = \begin{bmatrix} 1 & -6 \\ 5 & 5 \end{bmatrix}$$

Solution:

The determinant is

$$B = \begin{vmatrix} 1 & -6 \\ 5 & 5 \end{vmatrix}$$

$$B = (1)(5) - (-6)(5)$$

$$B = 5 + 30$$

$$B = 35$$

■ 3. Find the inverse of matrix G.

$$G = \begin{bmatrix} -3 & 8 \\ 0 & -2 \end{bmatrix}$$

Solution:

Plug into the formula for the inverse matrix.

$$G^{-1} = \frac{1}{|G|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$G^{-1} = \frac{1}{\begin{vmatrix} -3 & 8 \\ 0 & -2 \end{vmatrix}} \begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \frac{1}{(-3)(-2) - (8)(0)} \begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \frac{1}{6}\begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{4}{3} \\ 0 & -\frac{1}{2} \end{bmatrix}$$

■ 4. Find the inverse of matrix N.

$$N = \begin{bmatrix} 11 & -4 \\ 5 & -3 \end{bmatrix}$$

Solution:

Plug into the formula for the inverse matrix.

$$N^{-1} = \frac{1}{|N|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$N^{-1} = \frac{1}{\begin{vmatrix} 11 & -4 \\ 5 & -3 \end{vmatrix}}\begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \frac{1}{(11)(-3)-(-4)(5)}\begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \frac{1}{-33+20}\begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = -\frac{1}{13}\begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \begin{bmatrix} \frac{3}{13} & -\frac{4}{13} \\ \frac{5}{13} & -\frac{11}{13} \end{bmatrix}$$

■ 5. Is the matrix invertible or singular?

$$Z = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$$

Solution:

Find the determinant of the matrix.

$$\begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix}$$

$$(4)(-1) - (2)(-2)$$

$$-4 + 4$$

$$0$$

Because the determinant is 0, Z is a singular matrix that has no inverse.

■ 6. Is the matrix invertible or singular?

$$Y = \begin{bmatrix} 0 & 6 \\ 2 & -1 \end{bmatrix}$$

Solution:

Find the determinant of the matrix.

$$\begin{vmatrix} 0 & 6 \\ 2 & -1 \end{vmatrix}$$

$$(0)(-1) - (6)(2)$$

$$0 - 12$$

$$-12$$

Because the determinant is non-zero, Y is an invertible matrix with a defined inverse.

SOLVING SYSTEMS WITH INVERSE MATRICES

■ 1. Use an inverse matrix to solve the system.

$-4x + 3y = -14$

$7x - 4y = 32$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} -4 & 3 \\ 7 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -14 \\ 32 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(-4)(-4) - (3)(7)} \begin{bmatrix} -4 & -3 \\ -7 & -4 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{5} \begin{bmatrix} -4 & -3 \\ -7 & -4 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{7}{5} & \frac{4}{5} \end{bmatrix}$$

The solution to the system is

$$\vec{a} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{7}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} -14 \\ 32 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{4}{5}(-14) + \frac{3}{5}(32) \\ \frac{7}{5}(-14) + \frac{4}{5}(32) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{56}{5} + \frac{96}{5} \\ -\frac{98}{5} + \frac{128}{5} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{40}{5} \\ \frac{30}{5} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

■ 2. Use an inverse matrix to solve the system.

$$6x - 11y = 2$$

$$-10x + 7y = -26$$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} 6 & -11 \\ -10 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(6)(7) - (-11)(-10)} \begin{bmatrix} 7 & 11 \\ 10 & 6 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{68} \begin{bmatrix} 7 & 11 \\ 10 & 6 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{7}{68} & -\frac{11}{68} \\ -\frac{10}{68} & -\frac{6}{68} \end{bmatrix}$$

The solution to the system is

$$\vec{a} = \begin{bmatrix} -\frac{7}{68} & -\frac{11}{8} \\ -\frac{10}{68} & -\frac{6}{68} \end{bmatrix} \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{7}{68}(2) - \frac{11}{68}(-26) \\ -\frac{10}{68}(2) - \frac{6}{68}(-26) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{14}{68} + \frac{286}{68} \\ -\frac{20}{68} + \frac{156}{68} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{272}{68} \\ \frac{136}{68} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

■ 3. Use an inverse matrix to solve the system.

$13y - 6x = -81$

$7x + 17 = -22y$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} -6 & 13 \\ 7 & 22 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -81 \\ -17 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(-6)(22) - (13)(7)} \begin{bmatrix} 22 & -13 \\ -7 & -6 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{223} \begin{bmatrix} 22 & -13 \\ -7 & -6 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{22}{223} & \frac{13}{223} \\ \frac{7}{223} & \frac{6}{223} \end{bmatrix}$$

The solution to the system is

$$\vec{a} = \begin{bmatrix} -\frac{22}{223} & \frac{13}{223} \\ \frac{7}{223} & \frac{6}{223} \end{bmatrix} \begin{bmatrix} -81 \\ -17 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{22}{223}(-81) + \frac{13}{223}(-17) \\ \frac{7}{223}(-81) + \frac{6}{223}(-17) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{1,782}{223} - \frac{221}{223} \\ -\frac{567}{223} - \frac{102}{223} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{1,561}{223} \\ -\frac{669}{223} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

■ 4. Sketch a graph of vectors to visually find the solution to the system.

$3x = 3$

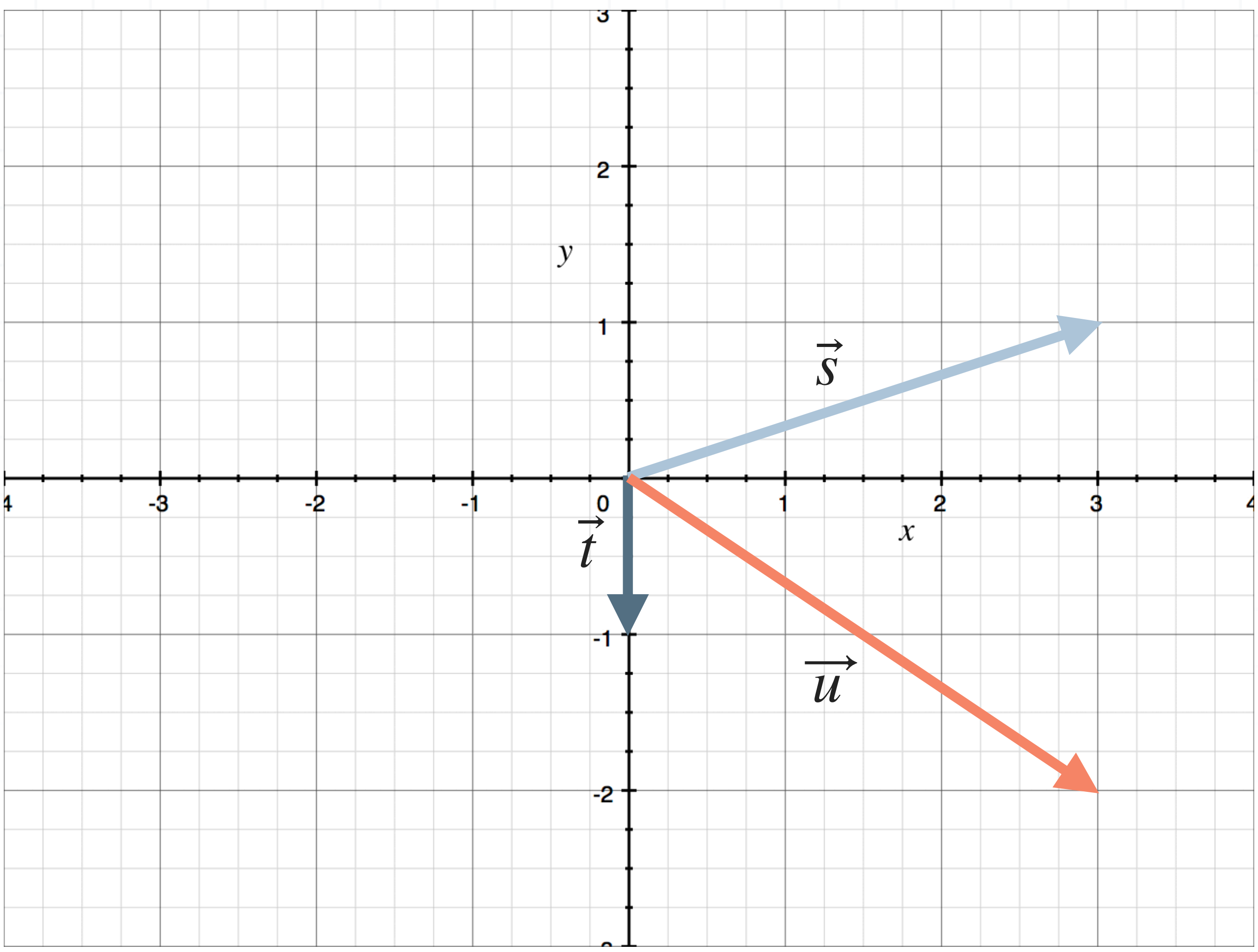
$x - y = -2$

Solution:

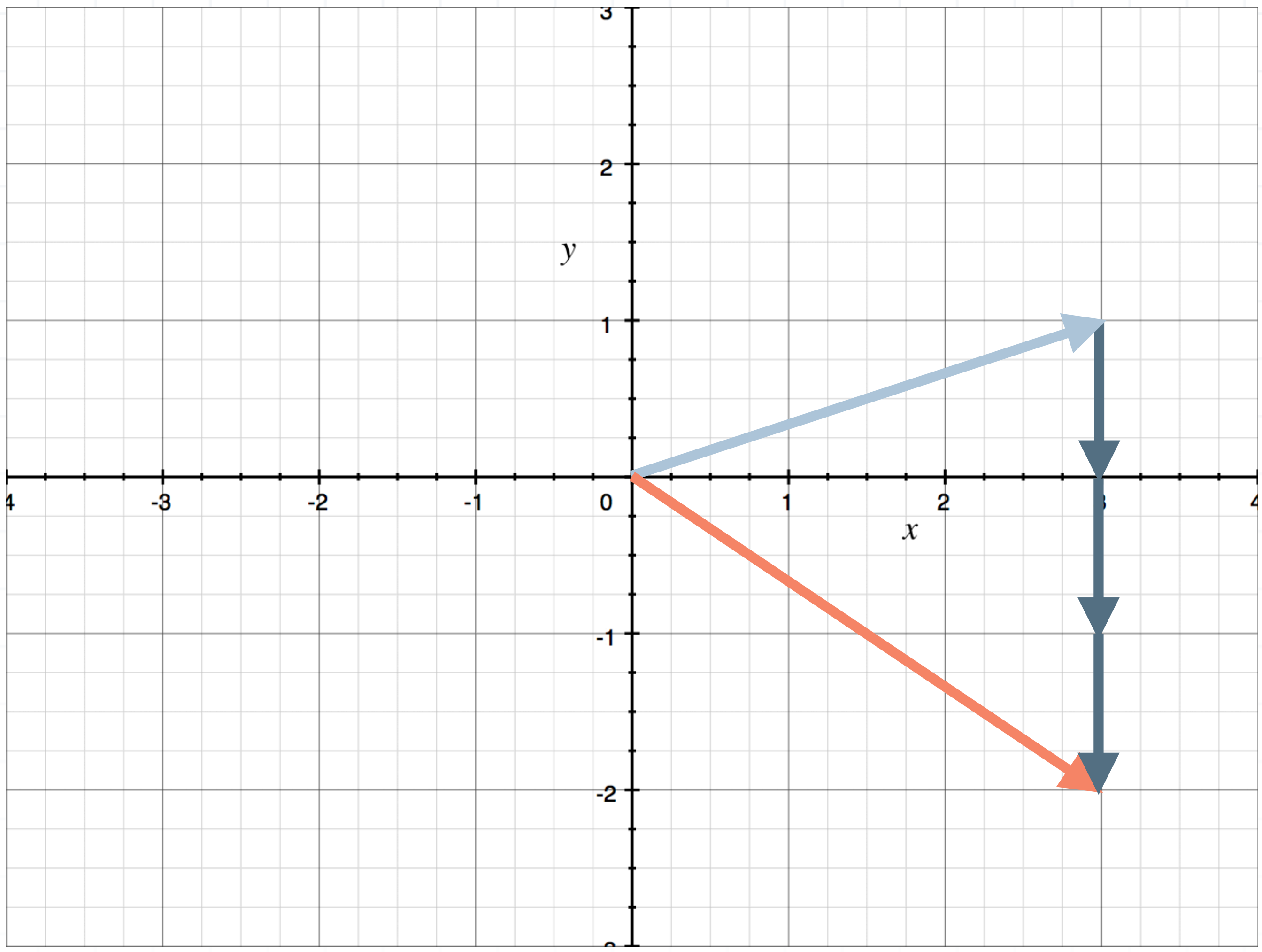
Put the system into a matrix equation.

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} y = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s} = (3,1)$ for x, $\vec{t} = (0, -1)$ for y, and the resulting vector $\vec{u} = (3, -2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and three $\vec{t}$s together will get us back to the terminal point of $\vec{u}$, so $x = 1$ and $y = 3$.



■ 5. Sketch a graph of vectors to visually find the solution to the system.

$$-y = -4$$

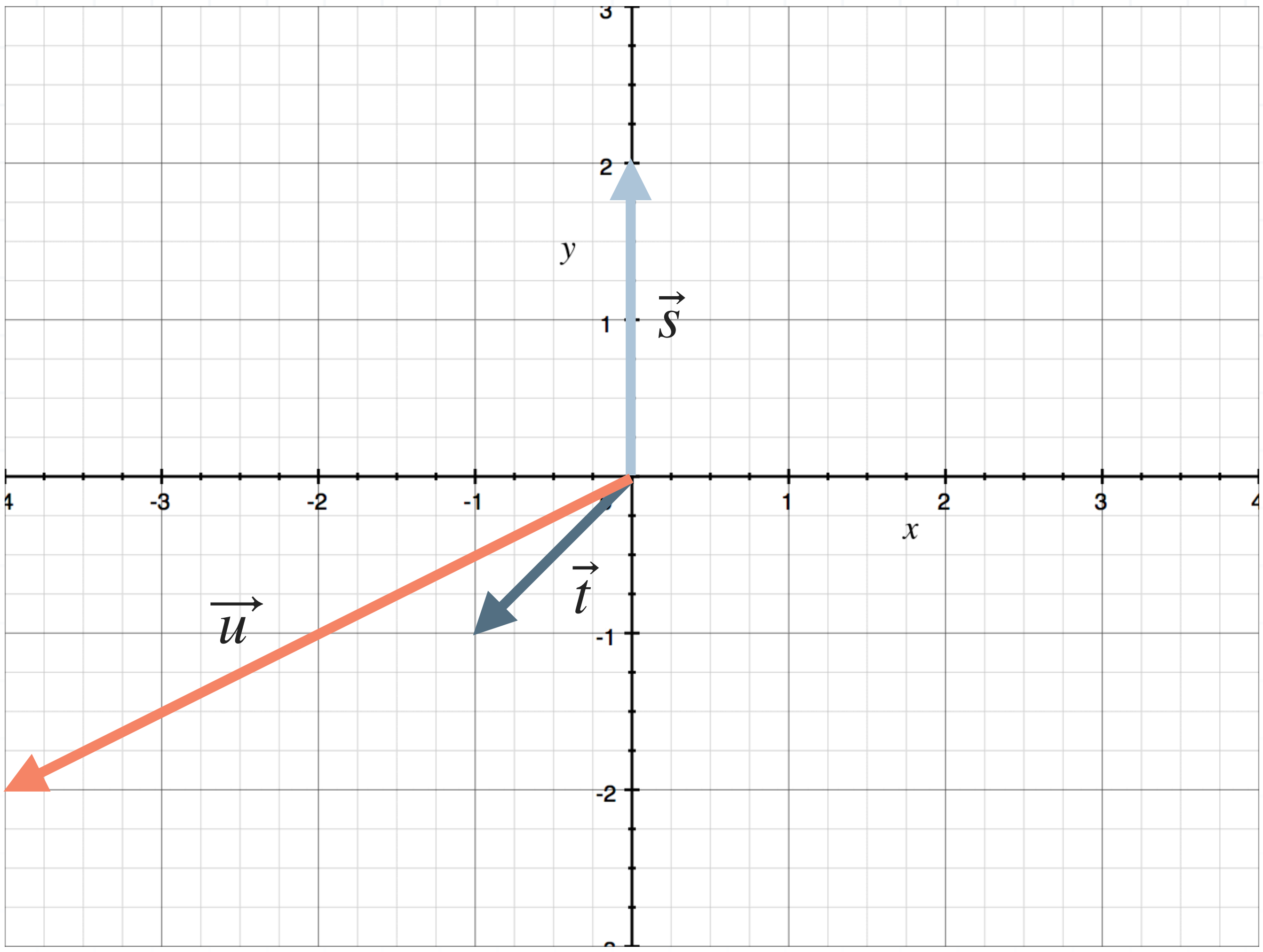
$$2x - y = -2$$

Solution:

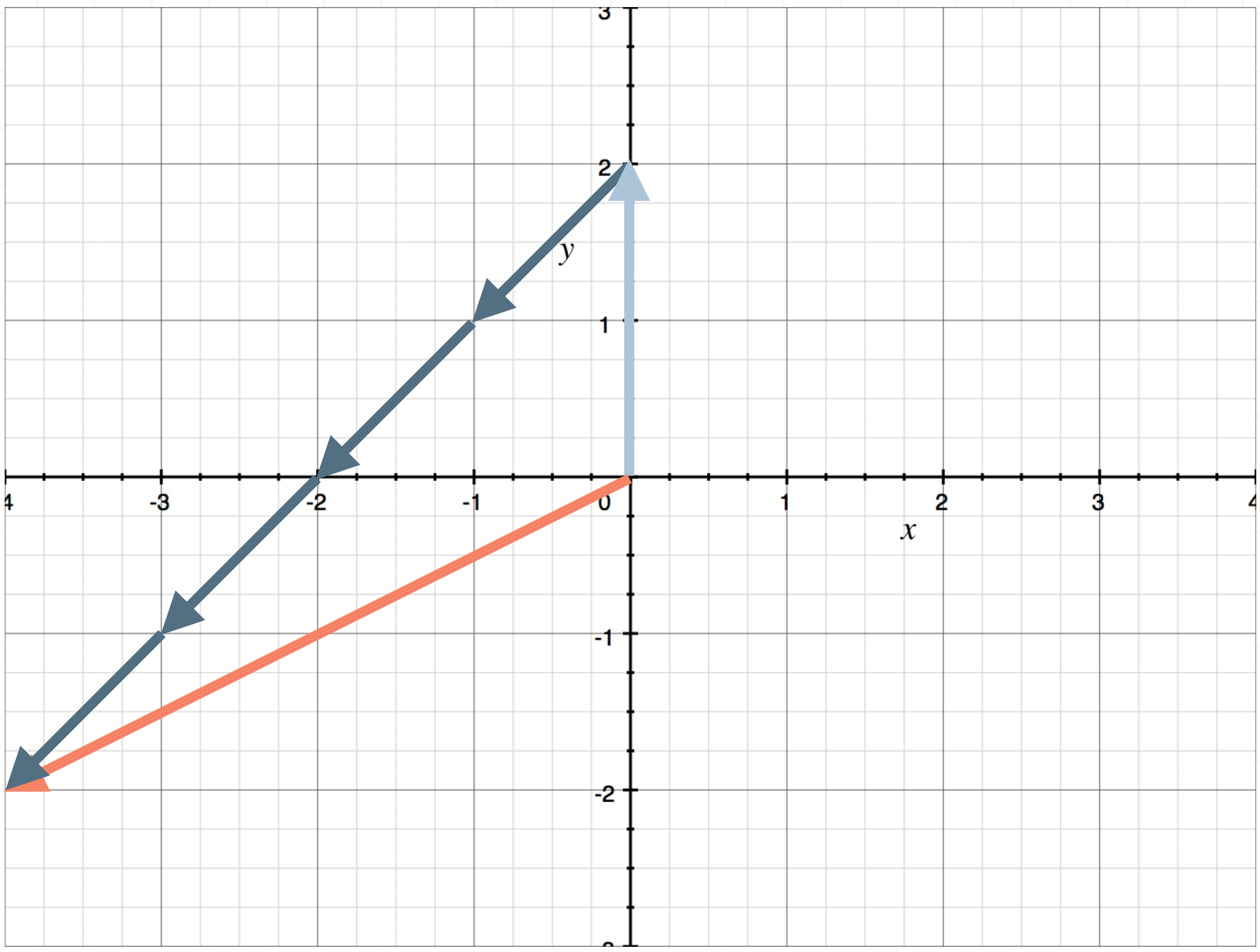
Put the system into a matrix equation.

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ -1 \end{bmatrix} y = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s} = (0,2)$ for x, $\vec{t} = (-1, -1)$ for y, and the resulting vector $\overrightarrow{u} = (-4, -2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and four $\vec{t}$s together will get us back to the terminal point of $\overrightarrow{u}$, so $x = 1$ and $y = 4$.



■ 6. Sketch a graph of vectors to visually find the solution to the system.

$x - y = 0$

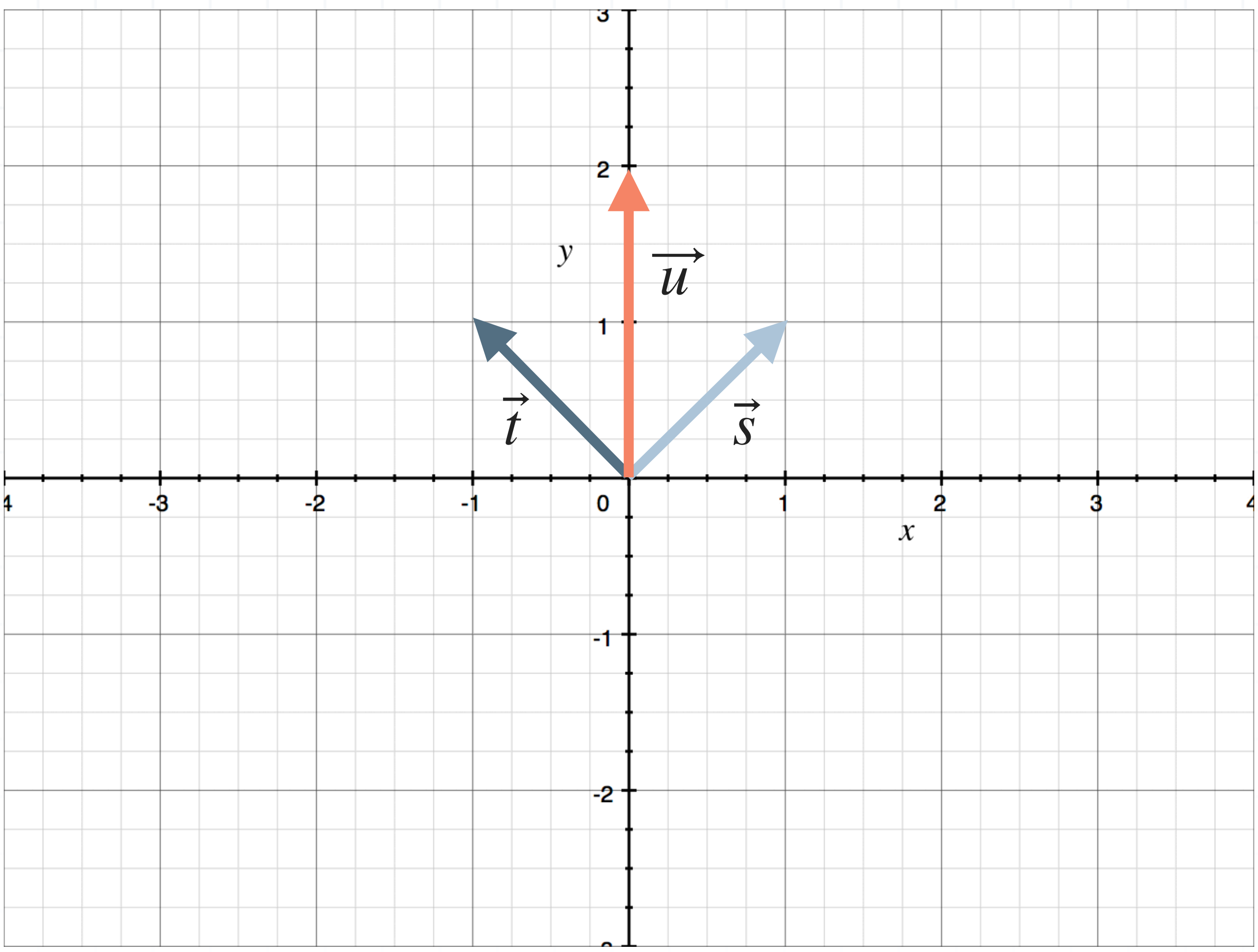
$x + y = 2$

Solution:

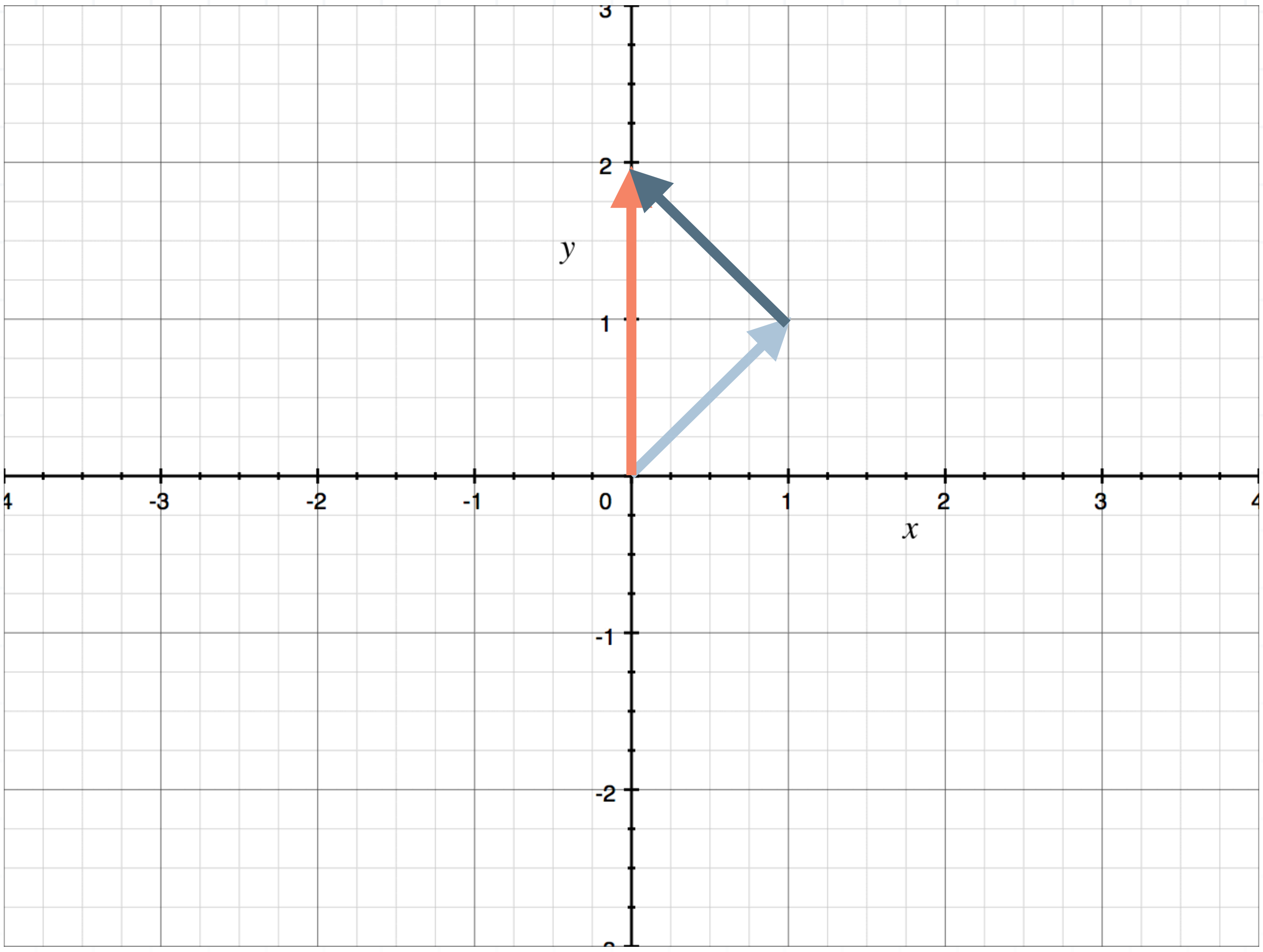
Put the system into a matrix equation.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s}=(1,1)$ for x, $\vec{t}=(-1,1)$ for y, and the resulting vector $\vec{u}=(0,2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and one $\vec{t}$ together will get us back to the terminal point of $\vec{u}$, so $x=1$ and $y=1$.



y
x
0
1
2
3
-1
-2
-3

SOLVING SYSTEMS WITH CRAMER'S RULE

■ 1. Use Cramer's rule to find the expression that would give the solution for x. You do not need to solve the system.

$2x - y = 5$

$x + 3y = 15$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} 5 & -1 \\ 15 & 3 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} 5 & -1 \\ 15 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}$$

2. Use Cramer's rule to find the expression that would give the solution for x. You do not need to solve the system.

$$ax + by = e$$

$$cx + dy = f$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

■ 3. Use Cramer's rule to find the expression that would give the solution for y. You do not need to solve the system.

$3x + 4y = 11$

$2x - 3y = -4$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} 3 & 11 \\ 2 & -4 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$\frac{\begin{vmatrix} 3 & 11 \\ 2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}}$$

■ 4. Use Cramer's rule to find the expression that would give the solution for y. You do not need to solve the system.

$ax + by = e$

$cx + dy = f$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

■ 5. Use Cramer's rule to solve for x.

$3x + 2y = 1$

$6x + 5y = 4$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}$$

Calculate the value of x.

$$x = \frac{1(5) - 2(4)}{3(5) - 2(6)}$$

$$x = \frac{5 - 8}{15 - 12}$$

$$x = \frac{-3}{3}$$

$$x = -1$$

■ 6. Use Cramer's rule to solve for y.

$3x + 2y = 1$

$6x + 5y = 4$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$\frac{\begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}$$

Calculate the value of y.

$$y = \frac{3(4) - 1(6)}{3(5) - 2(6)}$$

$$y = \frac{12 - 6}{15 - 12}$$

$$y = \frac{6}{3}$$

$$y = 2$$

■ 7. Use Cramer's rule to solve for x.

$3x + 5y = 6$

$9x + 10y = 14$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 5 \\ 9 & 10 \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} 6 & 5 \\ 14 & 10 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} 6 & 5 \\ 14 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 9 & 10 \end{vmatrix}}$$

Calculate the value of x.

$$x = \frac{6(10) - 5(14)}{3(10) - 5(9)}$$

$$x = \frac{60 - 70}{30 - 45}$$

$$x = \frac{-10}{-15}$$

$$x = \frac{2}{3}$$

■ 8. Use Cramer's rule to solve for y.

$3x + 5y = 6$

$9x + 10y = 14$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 5 \\ 9 & 10 \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} 3 & 6 \\ 9 & 14 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$\frac{\begin{vmatrix} 3 & 6 \\ 9 & 14 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 9 & 10 \end{vmatrix}}$$

Calculate the value of y.

$$y = \frac{3(14) - 6(9)}{3(10) - 5(9)}$$

$$y = \frac{42 - 54}{30 - 45}$$

$$y = \frac{-12}{-15}$$

$$y = \frac{4}{5}$$

W W W . K R I S T A K I N G M A T H . C O M