Sum-difference identities for tangent

In the same way that we had sum-difference identities for the sine and cosine functions, we also have a sum identity and difference identity for the tangent function:

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

Because $\tan\theta=\sin\theta/\cos\theta$, we can actually build these sum-difference identities for tangent by dividing the sine identity by the cosine identity. Here's how we build the sum identity for tangent,

$$\tan(\theta+\alpha)=\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}$$

$$\tan(\theta+\alpha)=\frac{\sin\theta\cos\alpha+\cos\theta\sin\alpha}{\cos\theta\cos\alpha-\sin\theta\sin\alpha}$$

$$\tan(\theta+\alpha)=\frac{\sin\theta\cos\alpha+\cos\theta\sin\alpha}{\cos\theta\cos\alpha-\sin\theta\sin\alpha}\cdot\frac{\frac{1}{\cos\theta\cos\alpha}}{\frac{1}{\cos\theta\cos\alpha}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sin\theta\cos\alpha+\cos\theta\sin\alpha}{\cos\theta\cos\alpha}}{\frac{\cos\theta\cos\alpha-\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sin\theta\cos\alpha}{\cos\theta\cos\alpha}+\frac{\cos\theta\sin\alpha}{\cos\theta\cos\alpha}}{\frac{\cos\theta\cos\alpha}{\cos\theta\cos\alpha}-\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sin\theta}{\cos\theta}+\frac{\sin\alpha}{\cos\alpha}}{1-\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

and here's how we build the difference identity for tangent.

$$\tan(\theta-\alpha)=\frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)}$$

$$\tan(\theta-\alpha)=\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\cos\theta\cos\alpha+\sin\theta\sin\alpha}$$

$$\tan(\theta-\alpha)=\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\cos\theta\cos\alpha+\sin\theta\sin\alpha}\cdot\frac{\frac{1}{\cos\theta\cos\alpha}}{\frac{1}{\cos\theta\cos\alpha}}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\cos\theta\cos\alpha}}{\frac{\cos\theta\cos\alpha+\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sin\theta\cos\alpha}{\cos\theta\cos\alpha}-\frac{\cos\theta\sin\alpha}{\cos\theta\cos\alpha}}{\frac{\cos\theta\cos\alpha}{\cos\theta\cos\alpha}+\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sin\theta}{\cos\theta}-\frac{\sin\alpha}{\cos\alpha}}{1+\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}}$$

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

Alternatively, to determine the difference identity for tangent, we can use the even-odd identity $\tan(-\theta) = -\tan(\theta)$. Substitute $-\alpha$ for α into the sum identity for tangent.

$$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

$$\tan(\theta + (-\alpha)) = \frac{\tan\theta + \tan(-\alpha)}{1 - \tan\theta\tan(-\alpha)}$$

$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 - \tan\theta(-\tan\alpha)}$$

$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

Let's do an example where we know the cosines of two different angles, and the quadrants of each angle, and need to use that information to find the tangent of both the sum and difference.

Example

Find the exact values of $\tan(\theta + \alpha)$ and $\tan(\theta - \alpha)$ if θ is an angle in the third quadrant whose cosine is $-2\sqrt{7}/7$ and α is an angle in the first quadrant whose cosine is $4/7$.

Before we can find the values of $\tan(\theta + \alpha)$ and $\tan(\theta - \alpha)$, we need to find the values of $\sin\theta$ and $\sin\alpha$, because right now we only have the cosines. To find the sines from the cosines, we'll rewrite the Pythagorean identity with sine and cosine,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

and then substitute $\cos\theta = -2\sqrt{7}/7$.

$$\sin^2\theta = 1 - \left(-\frac{2\sqrt{7}}{7}\right)^2$$

$$\sin^2\theta = 1 - \frac{4(7)}{49}$$

$$\sin^2\theta = 1 - \frac{4}{7}$$

$$\sin^2\theta = \frac{3}{7}$$

$$\sin\theta = \pm\sqrt{\frac{3}{7}}$$

Since θ is in the third quadrant, we know that $\sin\theta$ is negative. So we can ignore the positive value and say

$$\sin\theta = -\sqrt{\frac{3}{7}} = -\frac{\sqrt{3}}{\sqrt{7}} = -\frac{\sqrt{3}\sqrt{7}}{7} = -\frac{\sqrt{21}}{7}$$

Now we need to find the value of $\sin\alpha$. Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha = 1 - \cos^2\alpha$$

Substituting $\cos\alpha = 4/7$, we get

$$\sin^2\alpha = 1 - \left(\frac{4}{7}\right)^2$$

$$\sin^2\alpha = 1 - \frac{16}{49}$$

$$\sin^2\alpha = \frac{33}{49}$$

$$\sin\alpha = \pm\sqrt{\frac{33}{49}}$$

Since α is in the first quadrant, we know that $\sin\alpha$ is positive. So we can ignore the negative value and say

$$\sin\alpha = \sqrt{\frac{33}{49}} = \frac{\sqrt{33}}{\sqrt{49}} = \frac{\sqrt{33}}{7}$$

Now we can find the values of $\tan\theta$ and $\tan\alpha$. Using the definition of the tangent function, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{21}}{7}}{-\frac{2\sqrt{7}}{7}} = \frac{\sqrt{21}}{7}\cdot\frac{7}{2\sqrt{7}} = \frac{\sqrt{21}}{2\sqrt{7}} = \frac{\sqrt{7}\sqrt{3}}{2\sqrt{7}} = \frac{\sqrt{3}}{2}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{\sqrt{33}}{7}}{\frac{4}{7}} = \frac{\sqrt{33}}{7}\cdot\frac{7}{4} = \frac{\sqrt{33}}{4}$$

By the sum identity for the tangent function,

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{33}}{4}}{1-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{33}}{4}\right)}$$

$$\tan(\theta+\alpha)=\frac{\frac{2\sqrt{3}+\sqrt{33}}{4}}{1-\frac{\sqrt{99}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\frac{2\sqrt{3}+\sqrt{3}\sqrt{11}}{4}}{1-\frac{3\sqrt{11}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\frac{\sqrt{3}(2+\sqrt{11})}{4}}{\frac{8-3\sqrt{11}}{8}}$$

$$\tan(\theta+\alpha)=\frac{\sqrt{3}(2+\sqrt{11})}{4}\cdot\frac{8}{8-3\sqrt{11}}$$

$$\tan(\theta+\alpha)=\frac{8\sqrt{3}(2+\sqrt{11})}{4(8-3\sqrt{11})}$$

$$\tan(\theta+\alpha)=\frac{2\sqrt{3}(2+\sqrt{11})}{8-3\sqrt{11}}$$

By the difference identity for the tangent function,

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sqrt{3}}{2}-\frac{\sqrt{33}}{4}}{1+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{33}}{4}\right)}$$

$$\tan(\theta-\alpha)=\frac{\frac{2\sqrt{3}-\sqrt{33}}{4}}{1+\frac{\sqrt{99}}{8}}$$

$$\tan(\theta-\alpha)=\frac{\frac{2\sqrt{3}-\sqrt{3}\sqrt{11}}{4}}{1+\frac{3\sqrt{11}}{8}}$$

$$\tan(\theta-\alpha)=\frac{\frac{\sqrt{3}(2-\sqrt{11})}{4}}{\frac{8+3\sqrt{11}}{8}}$$

$$\tan(\theta-\alpha)=\frac{\sqrt{3}(2-\sqrt{11})}{4}\cdot\frac{8}{8+3\sqrt{11}}$$

$$\tan(\theta-\alpha)=\frac{8\sqrt{3}(2-\sqrt{11})}{4(8+3\sqrt{11})}$$

$$\tan(\theta-\alpha)=\frac{2\sqrt{3}(2-\sqrt{11})}{8+3\sqrt{11}}$$