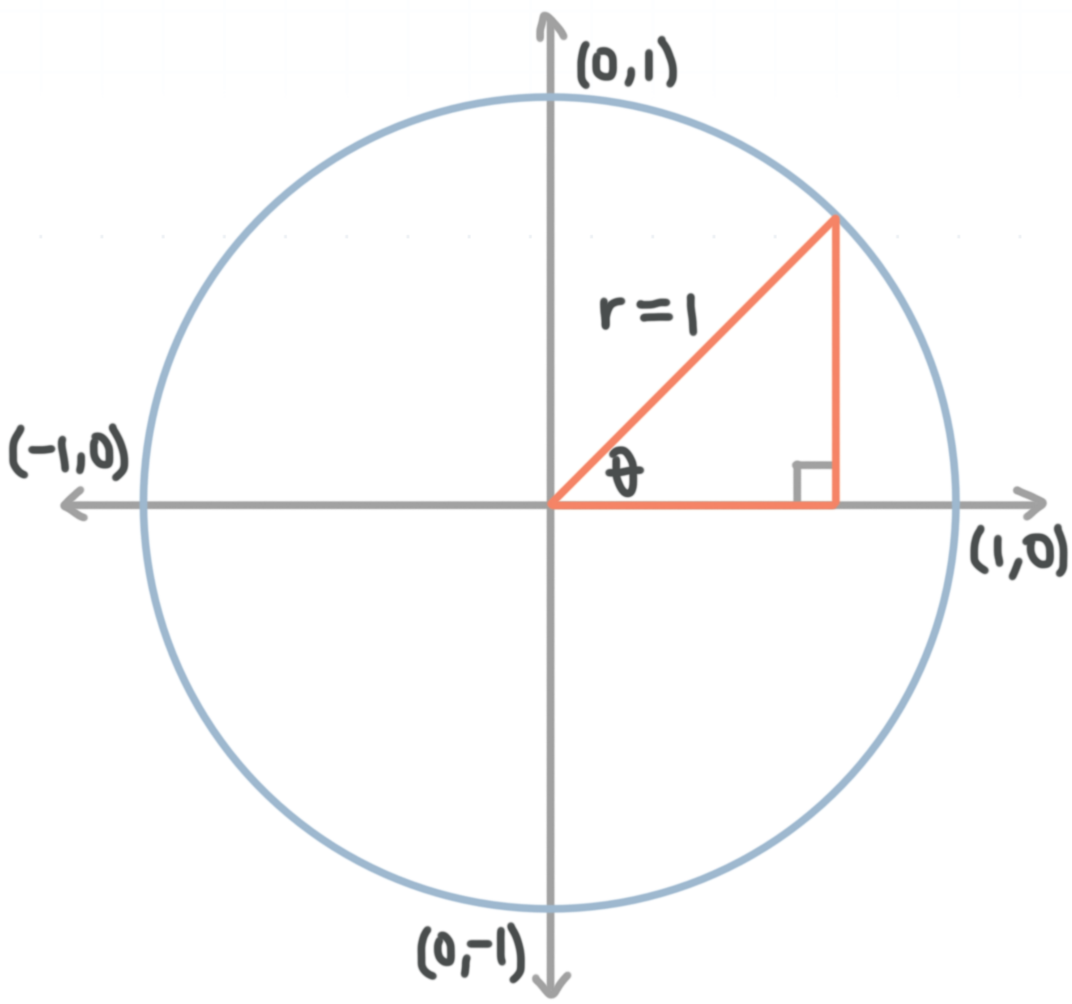
The quotient identities

We've now defined all six trig functions, and we've talked about the reciprocal relationships between them. The reciprocal identities show us that sine and cosecant are reciprocals, cosine and secant are reciprocals, and tangent and cotangent are reciprocals.

In this lesson, we want to build on those relationships in order to define two quotient identities.

The quotient identities

Let's look again at our right triangle in the first quadrant.



Notice that we've got the triangle intersecting the circle with radius 1. We know the radius is 1 because the circle passes through both $(1,0)$ along the x-axis and $(0,1)$ along the y-axis. And if the radius is 1, that means the hypotenuse of the triangle is also 1, since the hypotenuse forms a radius of the circle.

When we defined the six trig functions earlier, we said that

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

but in this particular circle, we already know the hypotenuse is $r = 1$. So we could rewrite this definition of sine as

$$\sin\theta = \frac{\text{opposite}}{1}$$

$$\sin\theta = \text{opposite}$$

In the same way, we said before that

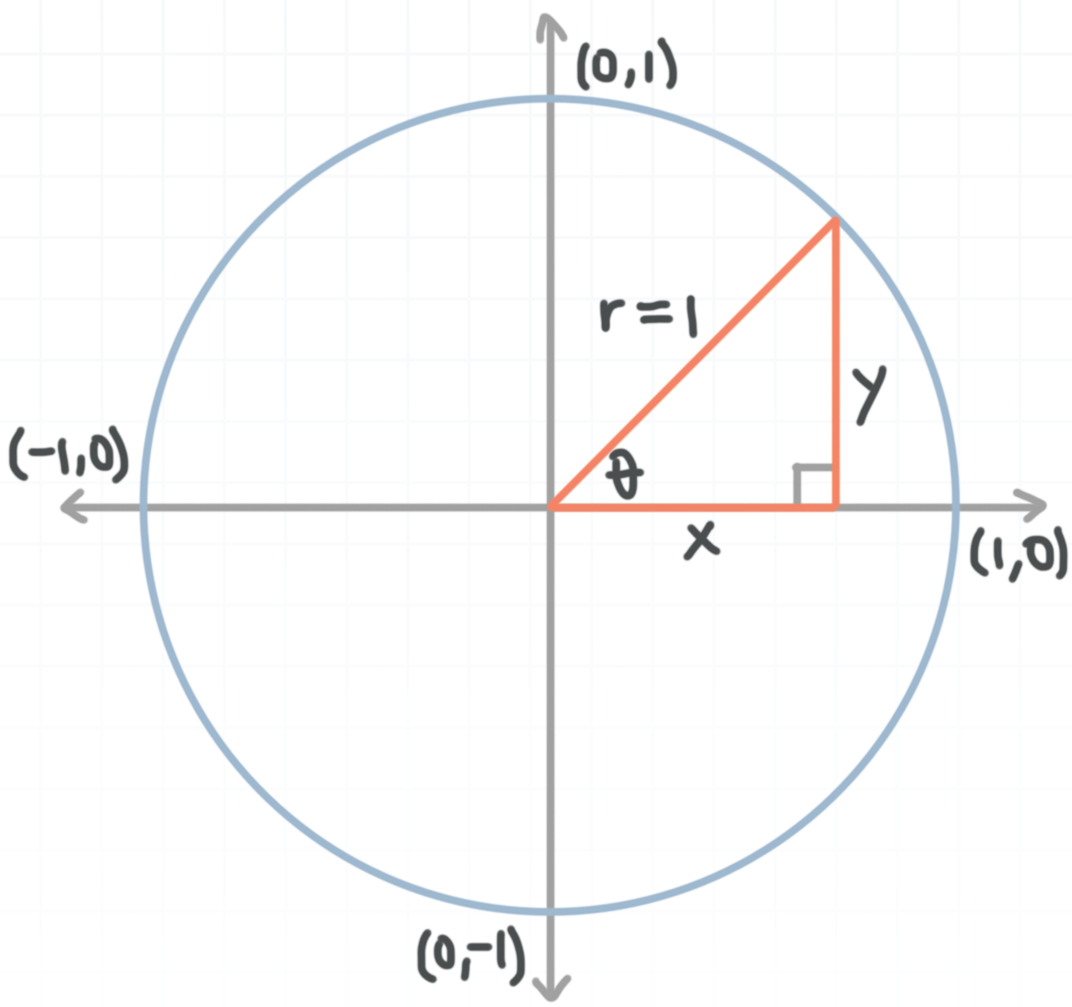
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

but in this particular circle, the hypotenuse is $r = 1$, so we can again rewrite this definition of cosine as

$$\cos\theta = \frac{\text{adjacent}}{1}$$

$$\cos\theta = \text{adjacent}$$

Then, if we define the horizontal leg of the triangle as having length x, and the vertical leg of the triangle as having length y,



then we get definitions for sine and cosine (when the radius of the circle is 1) of

$\sin\theta =$ opposite

$\sin\theta = y$

$y = \sin\theta$

and

$\cos\theta =$ adjacent

$\cos\theta = x$

$$x = \cos\theta$$

This is actually a really important point that we'll build on throughout Trigonometry. In a circle with radius $r = 1$, we realize now that we can always define the x value of the coordinate point along the circle with cosine of the angle, $x = \cos\theta$, and we can always define the y value of the coordinate point along the circle with sine of the angle, $y = \sin\theta$.

Now remember how we defined the six trig functions earlier in terms of x, y, and r:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

But we just concluded that $x = \cos\theta$ and $y = \sin\theta$. If we substitute these values, along with $r = 1$, into these six formulas, we get

$$\sin\theta = \frac{\sin\theta}{1} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{\cos\theta}{1} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

If we simplify these equations, we get

$$\sin\theta = \sin\theta \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \cos\theta \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

The top four equations aren't interesting to us. The equations for $\sin\theta$ and $\cos\theta$ tell us nothing, and the equations for $\csc\theta$ and $\sec\theta$ are the reciprocal identities that we already know. But the bottom two equations, the ones for $\tan\theta$ and $\cot\theta$, are new to us.

In fact, these are the two quotient identities that we wanted to introduce in this lesson. They tell us that for any right triangle, tangent of the angle is always equivalent to the quotient of sine and cosine of the same angle, and that cotangent of the angle is always equivalent to the quotient of cosine and sine of the same angle.

We'll use these two quotient identities all the time throughout trigonometry, so it's important that we have them memorized.

Let's use the quotient identities to find the tangent and cotangent of an angle.

Example

Find tangent and cotangent of the angle θ.

$$\sin\theta = \frac{1}{2}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

We can find tangent and cotangent of θ just by plugging these sine and cosine values into the quotient identities.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

We'll rationalize the denominator for the value of tangent.

$$\tan\theta = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{3}$$

So the values of tangent and cotangent for the same angle θ are

$$\tan\theta = \frac{\sqrt{3}}{3}$$

$$\cot\theta = \sqrt{3}$$

Lastly, remember how we saw that tangent and cotangent were reciprocals of one another when we learned about the reciprocal identities. We see that reciprocal relationship represented here as well in the quotient identities, since tangent is sine/cosine, and cotangent is the reciprocal of that, cosine/sine.

So in the last example, instead of calculating both tangent and cotangent of the angle using the quotient identities, we could instead have first calculated tangent using the quotient identity, and then found cotangent as tangent's reciprocal. Or we could have calculated cotangent using the quotient identity, and then found tangent as cotangent's reciprocal.

Both processes get us to the same, correct values for the tangent and cotangent of the angle.