Intersection points of the polar curves

In this lesson, we're going to discuss how to find the points at which pairs of polar curves intersect. To do this, we'll use the polar equations of the two curves in the pair.

Example

Find the points of intersection of the graphs of the polar equations $r = 2$ and $r = -4\sin\theta$.

As you may recall, the graph of each of these polar equations is a circle.

Suppose we solve the two polar equations simultaneously, by equating the expressions for r in those equations:

$$r \text{ for first circle} = r \text{ for second circle}$$

$$2 = -4\sin\theta$$

Dividing both sides of the equation $2 = -4\sin\theta$ by -4 gives

$$-\frac{1}{2} = \sin\theta$$

Since

$$-1 < -\frac{1}{2} < 0$$

there are two angles θ in the interval $[0,2\pi)$ that satisfy this equation: one in the third quadrant and one in the fourth quadrant.

Now recall that

$$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

and (by the odd identity for sine)

$$\sin\left(-\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

An angle of measure $-\pi/6$ is in the fourth quadrant, but it isn't in the interval $[0,2\pi)$. However, an angle of measure

$$-\frac{\pi}{6}+2\pi\ \left(=\frac{11\pi}{6}\right)$$

is in the interval $[0,2\pi)$ as well as in the fourth quadrant, and it has a sine of $-1/2$.

By the sum identity for sine,

$$\sin\left(\frac{\pi}{6}+\pi\right)=\sin\left(\frac{\pi}{6}\right)\cos(\pi)+\cos\left(\frac{\pi}{6}\right)\sin(\pi)$$

$$\sin\left(\frac{\pi}{6}+\pi\right)=\sin\left(\frac{\pi}{6}\right)(-1)+\cos\left(\frac{\pi}{6}\right)(0)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

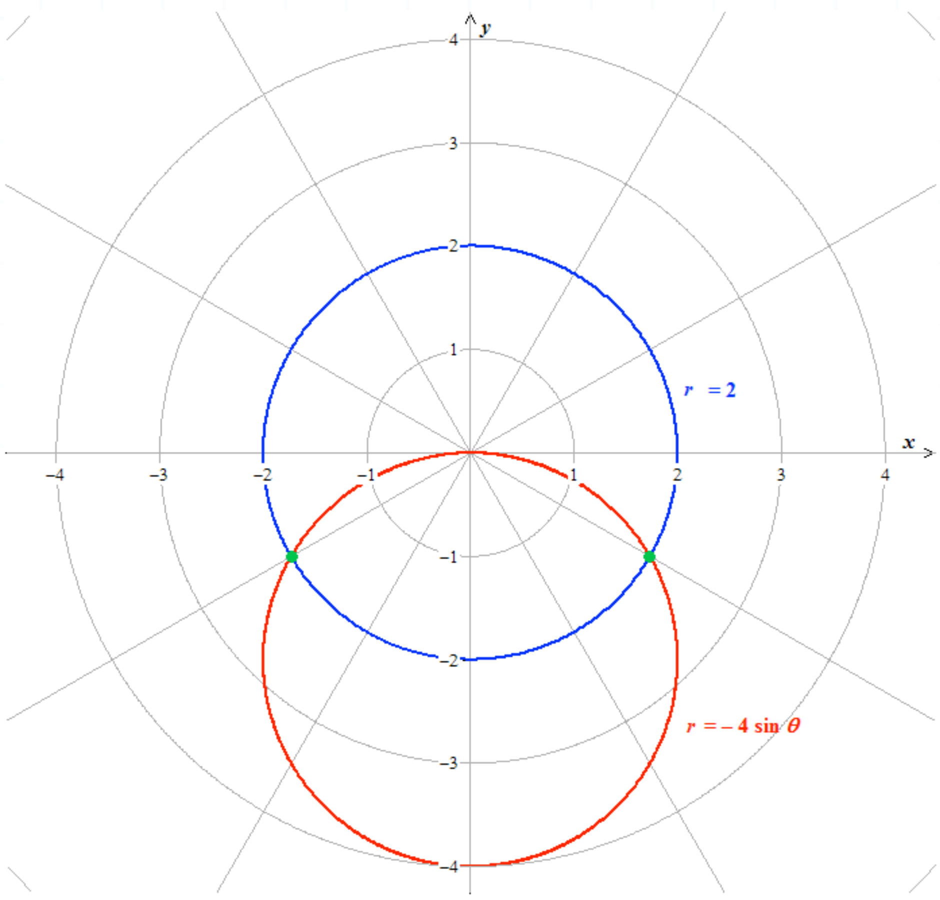
Thus the angle that's in the interval $[0,2\pi)$ as well in the third quadrant and has a sine of $-1/2$ is of measure

$$\frac{\pi}{6} + \pi = \frac{7\pi}{6}$$

We have found that there are two points of intersection of the graphs of the polar equations $r = 2$ and $r = -4\sin\theta$ and that those points have polar coordinates

$$(r, \theta) = \left(2, \frac{7\pi}{6}\right) \quad \text{and} \quad (r, \theta) = \left(2, \frac{11\pi}{6}\right)$$

If we look at graphs of the two curves, we find that that is indeed the case.



From that example, it looks as though the process of finding the points of intersection of two polar curves is rather easy and straightforward, doesn't it? Actually, however, it doesn't always work out quite that way.

Example

Find the points of intersection of the graphs of the polar equations $r = 2$ and $r = 2 - 4\cos\theta$.

As in the previous problem, the graph of the polar equation $r = 2$ is a circle. And as you may recall, the graph of the polar equation $r = 2 - 4\cos\theta$ is a limaçon with $a = 2$ and $b = 4$, hence $a < b$, which tell us that this limaçon has a loop.

Equating the expressions for r:

r for circle $=$ r for limaçon

$$2 = 2 - 4\cos\theta$$

Subtracting 2 from both sides of the equation $2 = 2 - 4\cos\theta$ gives

$$0 = -4\cos\theta$$

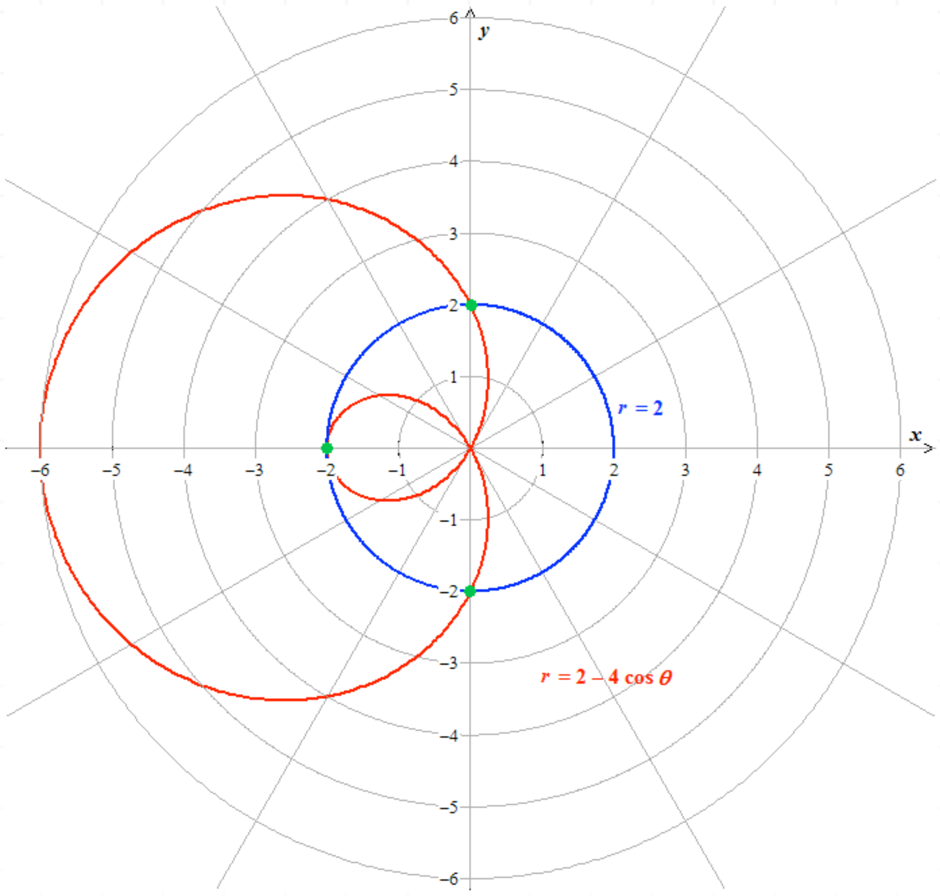
Dividing both sides by -4, we find that

$$0 = \cos\theta$$

The angles θ in the interval $[0,2\pi)$ for which $\cos\theta = 0$ are $\pi/2$ and $3\pi/2$. Thus we have found that there are two points of intersection and that they have polar coordinates

$$(r,\theta) = \left(2,\frac{\pi}{2}\right) \quad \text{and} \quad (r,\theta) = \left(2,\frac{3\pi}{2}\right)$$

Everything looks fine, right? Well, let's graph the two curves and see where they intersect.



As seen in the graph, the two points

$$(r, \theta) = \left(2, \frac{\pi}{2}\right) \quad \text{and} \quad (r, \theta) = \left(2, \frac{3\pi}{2}\right)$$

are indeed points of intersection of the polar curves in question, but there are actually three points of intersection. Why didn't we find all three of those points when we equated the expressions for r?

Note that one pair of polar coordinates of the third point of intersection of the circle and the limaçon is $(r, \theta) = (2, \pi)$, so this point satisfies the equation of the circle ($r = 2$) if we use the polar coordinates $(r, \theta) = (2, \pi)$. However, if we substitute π for θ in the polar equation of the limaçon ($r = 2 - 4\cos\theta$), we get

$$r = 2 - 4(\cos\pi) = 2 - 4(-1) = 2 + 4 = 6 \neq 2$$

so the point doesn't satisfy the equation of the circle($r = 2$) if we use the polar coordinates $(r, \theta) = (2,\pi)$. From the graph, we see that the point with polar coordinates $(r, \theta) = (6,\pi)$ is a point of the limaçon, but that it isn't also a point of the circle.

Note that another pair of polar coordinates of the third point of intersection of the circle and the limaçon is $(r, \theta) = (-2,0)$. If we substitute 0 for θ in the polar equation of the limaçon, we get

$$r = 2 - 4\cos(0) = 2 - 4(1) = 2 - 4 = -2 \neq 2$$

so the point satisfies the equation of the limaçon if we use the polar coordinates $(r, \theta) = (-2,0)$, but it doesn't satisfy the equation of the circle $r = 2$ if we use those coordinates.

What we have found is an example of a pair of polar curves where one of the points of intersection satisfies the polar equations of those curves separately, but not with the same pair of polar coordinates, and therefore doesn't satisfy the two polar equations simultaneously.

Another situation in which we could have a point of intersection of two polar curves that doesn't emerge from solving the polar equations of the curves simultaneously is when the pole is a point of intersection. The reason for this is that the pole has polar coordinates $(0,\theta)$ for every angle θ. Thus the angle coordinate θ that satisfies the polar equation of one of the curves (and yields 0 for r) may be different from the angle coordinate θ that satisfies the polar equation of the other curve (and yields 0 for r).

Example

Find the points of intersection of the graphs of the polar equations $r = 3\sin\theta$ and $r = 1 + \sin\theta$.

As you may recall, $r = 3\sin\theta$ is the polar equation of a circle, and $r = 1 + \sin\theta$ is the polar equation of a cardioid.

Equating the expressions for r in the two polar equations:

r for circle $=$ r for cardioid

$$3\sin\theta = 1 + \sin\theta$$

Subtracting $\sin\theta$ from both sides of the equation $3\sin\theta = 1 + \sin\theta$, we get

$$2\sin\theta = 1$$

Dividing both sides by 2 gives

$$\sin\theta = \frac{1}{2}$$

The angles θ in the interval $[0,2\pi)$ that have a sine of $1/2$ are

$$\frac{\pi}{6} \quad \text{and} \quad \pi - \frac{\pi}{6}\left(=\frac{5\pi}{6}\right)$$

For these two angles, the polar equation of the circle ($r = 3\sin\theta$) gives us the following values of r:

$$\theta = \frac{\pi}{6} \Longrightarrow r = 3\sin\left(\frac{\pi}{6}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

$$\theta = \frac{5\pi}{6} \Longrightarrow r = 3\sin\left(\frac{5\pi}{6}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

The polar equation of the cardioid ($r = 1 + \sin\theta$) gives these same values of r for those two angles:

$$\theta = \frac{\pi}{6} \Longrightarrow r = 1 + \sin\left(\frac{\pi}{6}\right) = 1 + \left(\frac{1}{2}\right) = \frac{3}{2}$$

$$\theta = \frac{5\pi}{6} \Longrightarrow r = 1 + \sin\left(\frac{5\pi}{6}\right) = 1 + \left(\frac{1}{2}\right) = \frac{3}{2}$$

Thus the points with polar coordinates

$$(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{6}\right) \quad \text{and} \quad (r, \theta) = \left(\frac{3}{2}, \frac{5\pi}{6}\right)$$

are points of intersection of the circle and the cardioid.

The pole is also a point of intersection.

For the circle ($r = 3\sin\theta$), we get $r = 0$ for angles θ that have a sine of 0. The angles θ in the interval $[0,2\pi)$ that have a sine of 0 are 0 and π. Thus the pairs of polar coordinates that satisfy the equation of the circle and have $r = 0$ are

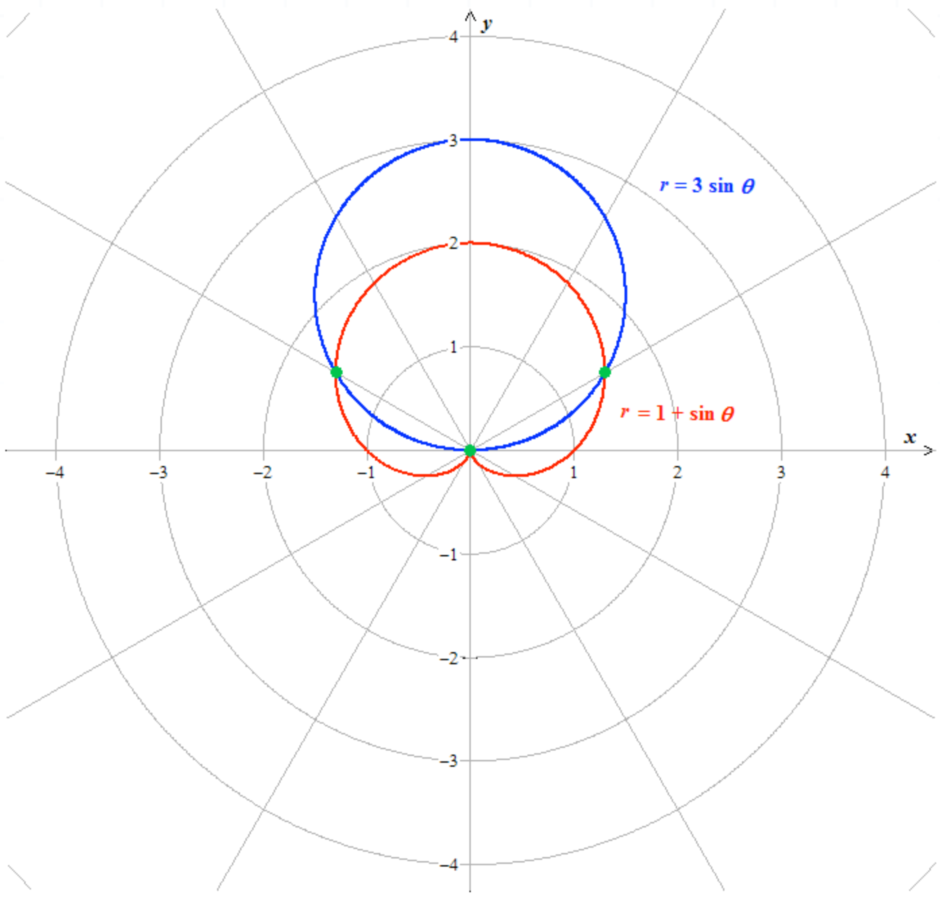
$$(r, \theta) = (0,0) \quad \text{and} \quad (r, \theta) = (0,\pi)$$

For the cardioid ($r = 1 + \sin\theta$), we get $r = 0$ for angles θ such that $\sin\theta = -1$. The only angle θ in the interval $[0,2\pi)$ with a sine of -1 is $3\pi/2$. Thus the pair of polar coordinates that satisfies the equation of the cardioid and has $r = 0$ is

$$(r, \theta) = \left(0, \frac{3\pi}{2}\right)$$

All three pairs of polar coordinates that we have found with $r = 0$ correspond to the pole, but solving the equations of the two polar curves simultaneously doesn't yield the pole as a point of intersection of those curves, because the angle coordinates don't agree. However, the pole is definitely a point of intersection.

As shown in the graphs of the two curves, the three points of intersection we have found are their only points of intersection.



Let's next look at the intersection of a “cosine” cardioid and a “sine” cardioid.

Example

Find the points of intersection of the graphs of the polar equations $r=4(1-\cos\theta)$ and $r=4(1+\sin\theta)$.

Equating the expressions for r in the two polar equations:

r for "cosine" cardioid = r for "sine" cardioid

$$4(1-\cos\theta)=4(1+\sin\theta)$$

Dividing both sides of the equation $4(1-\cos\theta)=4(1+\sin\theta)$ by 4, we have

$$1-\cos\theta=1+\sin\theta$$

Subtracting 1 from both sides gives

$$-\cos\theta=\sin\theta$$

Dividing both sides of this equation by $\cos\theta$, we see that

$$-1=\frac{\sin\theta}{\cos\theta}$$

Therefore, $\tan\theta=-1$. The only angles θ in the interval $[0,2\pi)$ that have a tangent of -1 are $3\pi/4$ and $7\pi/4$.

Recall that

$$\cos\left(\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2}=\sin\left(\frac{7\pi}{4}\right)$$

and

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\left(\frac{7\pi}{4}\right)$$

For the angles $3\pi/4$ and $7\pi/4$, the polar equation of the "cosine" cardioid ($r = 4(1 - \cos\theta)$) gives us the following values of r:

$$\theta = \frac{3\pi}{4} \Longrightarrow r = 4\left[1 - \cos\left(\frac{3\pi}{4}\right)\right] = 4\left[1 - \left(-\frac{\sqrt{2}}{2}\right)\right] = 4\left(1 + \frac{\sqrt{2}}{2}\right) = 4 + 2\sqrt{2}$$

$$\theta = \frac{7\pi}{4} \Longrightarrow r = 4\left[1 - \cos\left(\frac{7\pi}{4}\right)\right] = 4\left(1 - \frac{\sqrt{2}}{2}\right) = 4 - 2\sqrt{2}$$

The polar equation of the "sine" cardioid ($r = 4(1 + \sin\theta)$) gives these same values of r for those two angles:

$$\theta = \frac{3\pi}{4} \Longrightarrow r = 4\left[1 + \sin\left(\frac{3\pi}{4}\right)\right] = 4\left(1 + \frac{\sqrt{2}}{2}\right) = 4 + 2\sqrt{2}$$

$$\theta = \frac{7\pi}{4} \Longrightarrow r = 4\left[1 + \sin\left(\frac{7\pi}{4}\right)\right] = 4\left(1 - \frac{\sqrt{2}}{2}\right) = 4 - 2\sqrt{2}$$

We have found that the points with polar coordinates

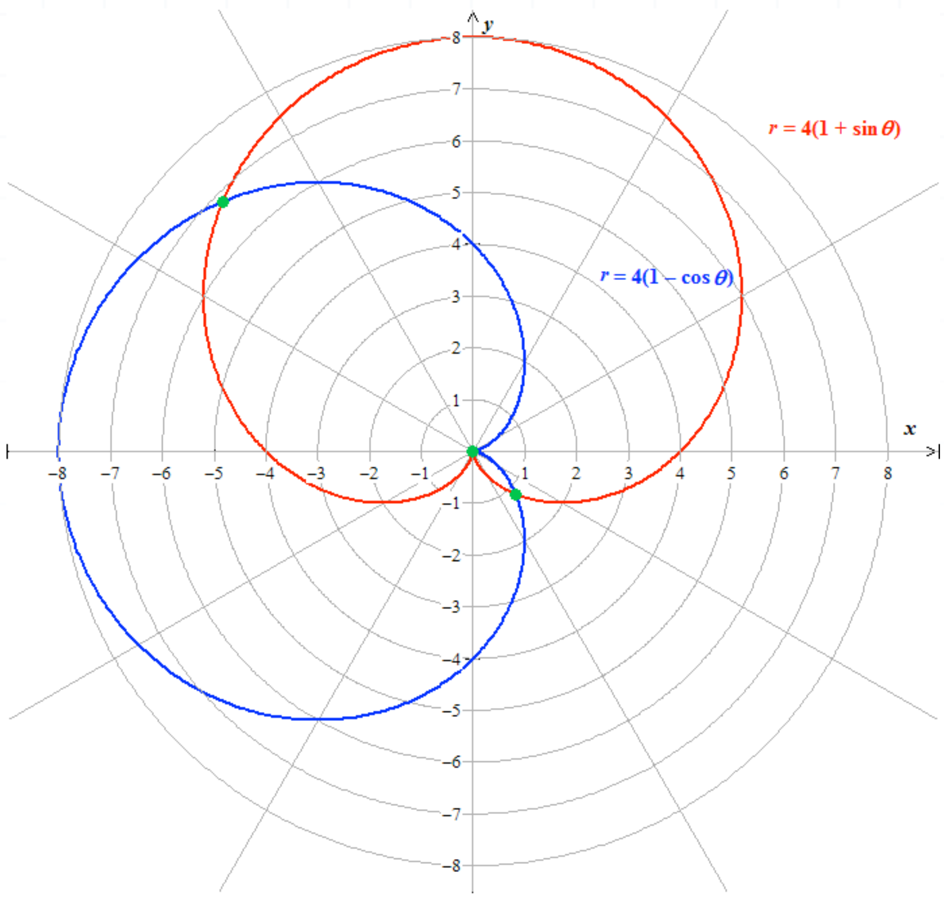
$$(r, \theta) = \left(4 + 2\sqrt{2}, \frac{3\pi}{4}\right) \quad \text{and} \quad (r, \theta) = \left(4 - 2\sqrt{2}, \frac{7\pi}{4}\right)$$

are points of intersection of the two given cardioids.

How about the pole? Well, for the "cosine" cardioid, the angles θ for which the given polar equation ($r = 4(1 - \cos\theta)$) yields $r = 0$ are those for which

$\cos \theta = 1$. There is just one such angle in the interval $[0,2\pi)$, namely $\theta = 0$. For the "sine" cardioid, the angles θ for which the given polar equation ($r = 4(1 + \sin \theta)$) yields $r = 0$ are those for which $\sin \theta = -1$. There is just one such angle in the interval $[0,2\pi)$, namely $\theta = 3\pi/2$. Since each of the two given polar equations has (at least) one angle θ that yields $r = 0$, the pole is indeed a point of intersection of the graphs of two given cardioids.

The graphs of these two curves show that the three points of intersection we have found are indeed their only points of intersection.



Finally, let's look at the intersection of a rose and a lemniscate.

Example

Find the points of intersection of the graphs of the polar equations $r = 3\cos(2\theta)$ and $r^2 = 9\cos(2\theta)$.

The graph of the polar equation $r = 3\cos(2\theta)$ is a four-petal rose, and the graph of the polar equation $r^2 = 9\cos(2\theta)$ is a lemniscate.

In this case, we can't directly equate the expressions for r, because the polar equation of the lemniscate gives us an expression for r^2 but the polar equation of the rose gives us an expression for r. However, we can square both sides of the polar equation of the rose to get an expression for r^2. (Such an approach could introduce extraneous solutions, so we'll check out possible solutions later to see if they're "genuine.")

$$r = 3\cos(2\theta) \Longrightarrow r^2 = 9\cos^2(2\theta)$$

Now we can equate the two expressions for r^2:

r^2 for rose $= r^2$ for lemniscate

$$9\cos^2(2\theta) = 9\cos(2\theta)$$

Dividing both sides of the equation $9\cos(2\theta) = 9\cos^2(\theta)$ by 9 gives

$$\cos^2(2\theta) = \cos(2\theta)$$

Subtracting $\cos(2\theta)$ from both sides of this equation, we obtain

$$\cos^2(2\theta) - \cos(2\theta) = 0$$

Factoring out $\cos(2\theta)$ on the left-hand side:

$$\cos(2\theta)[\cos(2\theta) - 1] = 0$$

Note that every solution to this equation would have to satisfy either $\cos(2\theta) = 0$ or $\cos(2\theta) - 1 = 0$, and that the latter equation is equivalent to $\cos(2\theta) = 1$.

Let's solve those two equations separately.

First, we'll deal with the equation $\cos(2\theta) = 0$. Evaluating the expression for r^2 for the lemniscate, we obtain

$$\cos(2\theta) = 0 \Longrightarrow r^2 = 9\cos(2\theta) = 9(0) = 0 \Longrightarrow r = 0$$

And evaluating the expression for r for the rose, we find that

$$\cos(2\theta) = 0 \Longrightarrow r = 3\cos(2\theta) = 3(0) = 0 \Longrightarrow r = 0$$

In both cases, we find that $r = 0$, so the pole is a point of intersection of the two curves. This should come as no surprise, because every lemniscate passes through the origin and so does every rose.

Now we'll deal with the equation $\cos(2\theta) = 1$. Evaluating the expression for r^2 for the lemniscate, we get

$$\cos(2\theta) = 1 \Longrightarrow r^2 = 9\cos(2\theta) = 9(1) = 9 \Longrightarrow r = \pm 3$$

Evaluating the expression for r for the rose, we find that

$$\cos(2\theta) = 1 \Longrightarrow r = 3\cos(2\theta) = 3(1) = 3$$

Thus the points of intersection that arise from the equation $\cos(2\theta) = 1$ have $r = 3$, so we'll have to eliminate “solutions” with $r = -3$.

Recall that $\cos(2\theta) = 1$ if and only if there is some integer n such that $2\theta = 2n\pi$. Thus

$$\cos(2\theta) = 1 \iff \theta = n\pi$$

If 2θ is in the interval $[0,2\pi)$, then $0 \leq \theta \leq \pi$, so either $\theta = 0$ or $\theta = \pi$.

If $\theta = 0$, the polar equation for the rose yields

$$r = 3\cos(2(0)) = 3\cos(0) = 3(1) = 3$$

By the polar equation for the lemniscate,

$$r^2 = 9\cos(2(0)) = 9\cos(0) = 9(1) = 9$$

Thus one point of intersection of the two curves that arises as a solution of the equation $\cos(2\theta) = 1$ is

$$(r, \theta) = (3,0)$$

If $\theta = \pi$, the polar equation for the rose gives

$$r = 3\cos(2\pi) = 3(1) = 3$$

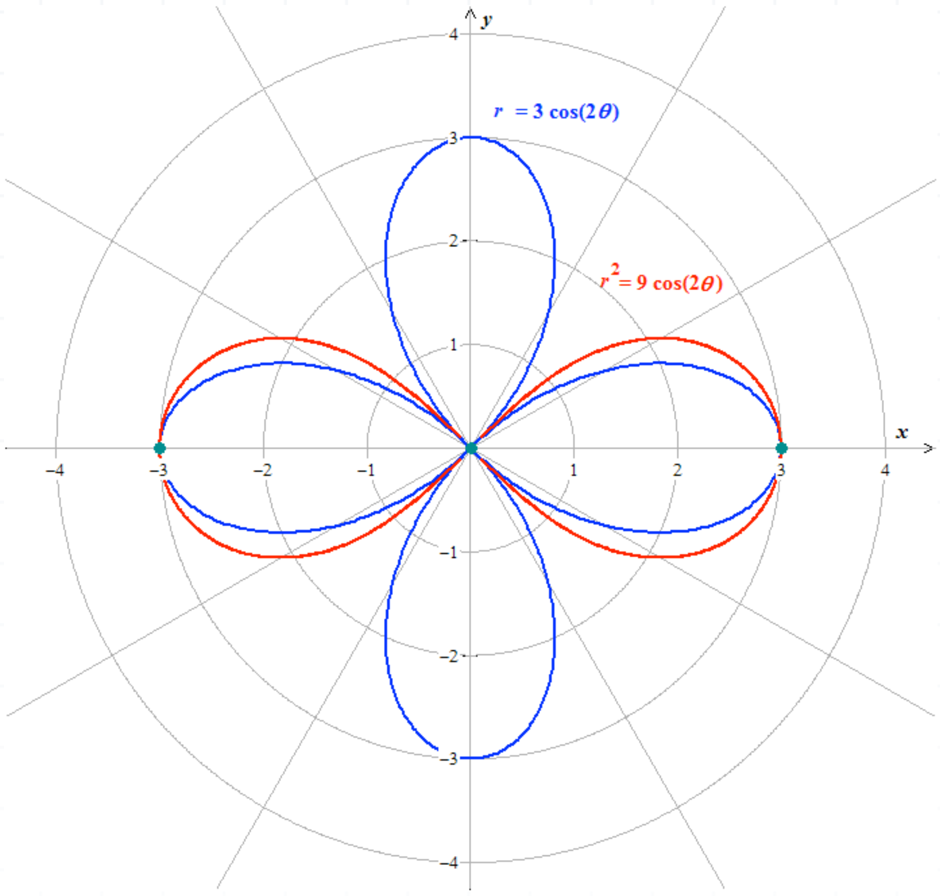
By the polar equation for the lemniscate,

$$r^2 = 9\cos(2\pi) = 9\cos(1) = 9$$

Thus another point of intersection of the two curves that arises as a solution of the equation $\cos(2\theta) = 1$ is

$$(r, \theta) = (3,\pi)$$

To see that the three points we have found are indeed the only points of intersection of the two curves, we'll graph them.


y
x
$r = 3\cos(2\theta)$
$r^2 = 9\cos(2\theta)$
4
3
2
1
-1
-2
-3
-4
-4
-3
-2
-1
1
2
3
4