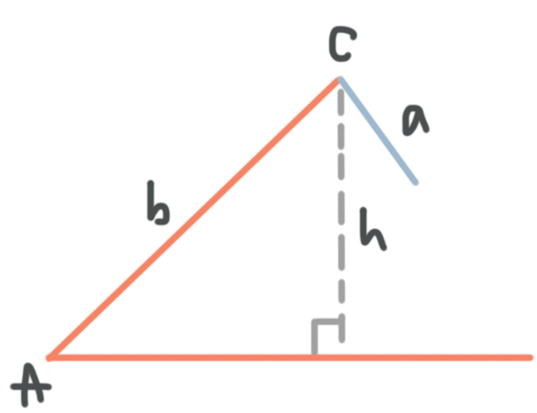
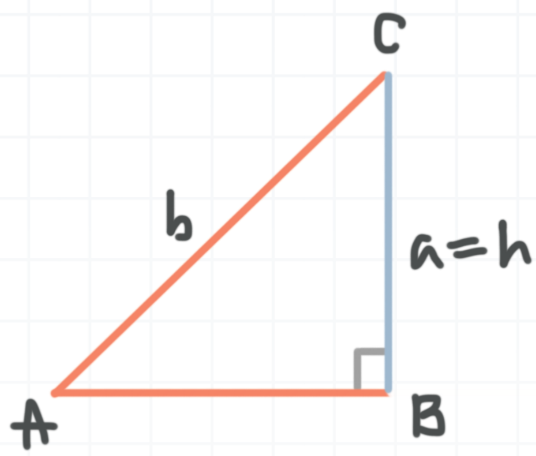
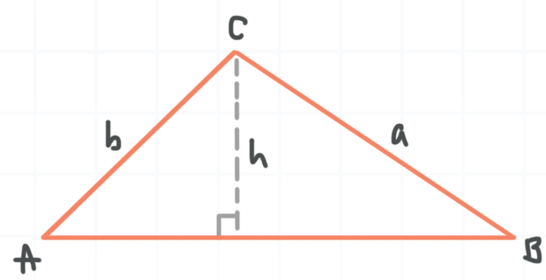
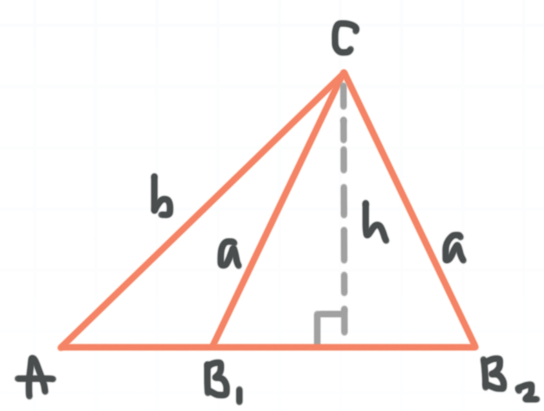
The ambiguous case of the law of sines

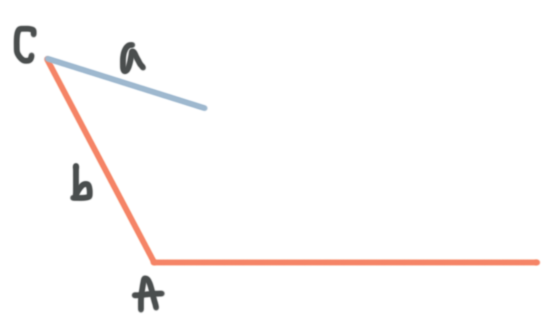
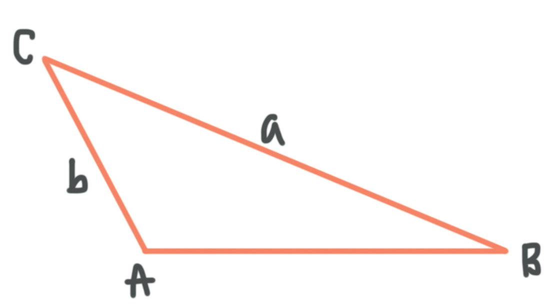
In the last lesson, we mentioned the ambiguous case of the law of sines, which we said occurred in an SSA triangle, for which we know the length of two sides of the triangle, and a non-included angle (one of the angles that's *not* between the two known side lengths).

In this SSA case, it's possible that exactly zero triangles are possible, that exactly one triangle is possible, or that exactly two triangles are possible. How many triangles we can get from the given information depends on the lengths of the two known sides and the measure of the one known angle.

Because we don't know initially how many triangles we'll have, this is called the **ambiguous case** of the law of sines. Below is a table summarizing all possible ways that we can get 0, 1, or 2 triangles. In every case in the table, we're given two sides a and b and the non-included angle A. By the definition of sine, the altitude is $h = b \sin A$.

Case	# of triangles	Sketch	Conditions
A is acute			
I	0		$a < h$, $h = b \sin A$

II	1		$a = h$
III	1		$a > b$
IV	2		$b > a > h$

A is obtuse			
V	0		$a \leq b$
VI	1		$a > b$

Notice in this table how Case II is a right triangle. As we've already seen, right triangles will always have exactly one solution, so it makes sense that there's exactly 1 triangle possible in a Case II situation.

To solve the ambiguous case by figuring out the number of possible triangles and then solving for every angle and side length for any possible triangle(s), here's the row from the table in the previous lesson showing the problem solving process:

Known information	How to solve
SSA Two sides and a non-included angle	**The ambiguous case**. If two triangles exist, use this same set of steps to find both triangles. 1. Use law of sines to find an angle 2. Use A+B+C=180º to find the remaining angle 3. Use law of sines to find the remaining side

Let's do an example of case VI, which is the case where there's exactly one triangle possible, and the triangle includes an obtuse angle.

Example

A triangle has one side with length 3 and another with length 5. The angle opposite the side with length 5 is 40°. Complete the triangle.

Let $a=3$ and $b=5$, and let $B=40^\circ$ be the angle opposite b. Substituting these values into the law of sines gives

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\frac{3}{\sin A} = \frac{5}{\sin 40^\circ} = \frac{c}{\sin C}$$

Use just the first two parts of this three-part equation in order to solve for A.

$$\frac{3}{\sin A} = \frac{5}{\sin 40^\circ}$$

$$3 = \frac{5}{\sin 40^\circ}(\sin A)$$

$$3 \sin 40^\circ = 5 \sin A$$

$$\sin A = \frac{3 \sin 40^\circ}{5}$$

$$\sin A \approx 0.386$$

Apply the inverse sine function to both sides to cancel the sine on the left and solve for A.

$$A \approx \arcsin(0.386)$$

$$A \approx 22.7^\circ$$

Since the sum of the interior angles of any triangle is 180°, the measure of the third interior angle C is approximately

$$C \approx 180^\circ - 40^\circ - 22.7^\circ$$

$$C \approx 117.3^\circ$$

Use the second and third parts of the three-part equation to solve for c.

$$\frac{5}{\sin 40^\circ} = \frac{c}{\sin C}$$

$$\frac{5}{\sin 40^\circ} \approx \frac{c}{\sin 117.3^\circ}$$

$$c \approx \frac{5}{\sin 40^\circ}(\sin 117.3^\circ)$$

$$c \approx 6.91$$

The side lengths of the triangle are $a = 3$, $b = 5$, and $c \approx 6.91$, and the angle measures are $A \approx 22.7^\circ$, $B = 40^\circ$, and $C = 117.3^\circ$.

Let's do an example of Case V, where no triangle is possible.

Example

Solve the triangle with side lengths 0.68 and 0.92, if angle $C = 118^\circ$ is opposite the side with length 0.68.

Because the side with length 0.68 is opposite the angle $C = 118^\circ$, we'll name that side as $c = 0.68$. We'll also name $b = 0.92$, then we'll plug everything we know into the law of sines.

$$\frac{a}{\sin A} = \frac{0.92}{\sin B} = \frac{0.68}{\sin 118^\circ}$$

Use the second and third parts of this three-part equation to solve for B.

$$\frac{0.92}{\sin B} = \frac{0.68}{\sin 118^\circ}$$

$$0.92 = \frac{0.68}{\sin 118^\circ}(\sin B)$$

$$0.92 \sin 118^\circ = 0.68 \sin B$$

$$\sin B = \frac{0.92 \sin 118^\circ}{0.68}$$

$$\sin B \approx 1.195$$

Remember that the sine of any angle must have a value on the interval $[-1,1]$. And when we're talking about a triangle, sine of an angle in the triangle needs to be a positive value on the interval $(0,1]$.

Because we're getting $\sin B \approx 1.195$, which is a value for sine greater than 1, a triangle with the given measurements is impossible.

Finally, let's do an example of Case IV, where two triangles are possible.

Example

Solve the triangle with side lengths 41 and 54, where the angle opposite $a = 41$ is 38°.

The problem names side a as $a = 41$. Since 38° is opposite that side, we'll say $A = 38^\circ$. And we'll name $b = 54$. Then we can substitute into the law of sines.

$$\frac{\sin 38^\circ}{41} = \frac{\sin B}{54} = \frac{\sin C}{c}$$

We'll use the first two parts of this three-part equation to solve for the angle B.

$$\frac{\sin 38^\circ}{41} = \frac{\sin B}{54}$$

$$\sin B = \frac{54 \sin 38^\circ}{41}$$

$$\sin B \approx 0.811$$

There are two possibilities here. We could find $\sin B \approx 0.811$ in both the first and second quadrants, since those are the quadrants where sine is positive. To find the value of angle B in the first quadrant, we'll apply the inverse sine function to both sides to cancel the sine on the left and solve for B.

$$B \approx \arcsin(0.811)$$

$$B \approx 54^\circ$$

Then to find the other possible value of B in the second quadrant, we'll subtract $B \approx 54^\circ$ from 180°.

$$B' \approx 180^\circ - 54^\circ$$

$$B' \approx 126^\circ$$

Then the remaining angle is either

$$C \approx 180^\circ - 38^\circ - 54^\circ$$

$$C \approx 88^\circ$$

or

$$C' \approx 180^\circ - 38^\circ - 126^\circ$$

$$C' \approx 16^\circ$$

Finally, we can find the measure of the remaining side. If we plug everything we know so far into the law of sines, we get either

$$\frac{\sin 38^\circ}{41} \approx \frac{\sin 54^\circ}{54} \approx \frac{\sin 88^\circ}{c}$$

$$0.0150 \approx 0.0150 \approx \frac{\sin 88^\circ}{c}$$

$$c \approx \frac{\sin 88^\circ}{0.0150}$$

$$c \approx 67$$

or

$$\frac{\sin 38^\circ}{41} \approx \frac{\sin 126^\circ}{54} \approx \frac{\sin 16^\circ}{c'}$$

$$0.0150 \approx 0.0150 \approx \frac{\sin 16^\circ}{c'}$$

$$c' = \frac{\sin 16^\circ}{0.0150}$$

$$c' \approx 18$$

If we summarize what we've found, we get the values for both triangles:

Triangle #1: $A = 38^\circ$, $B \approx 54^\circ$, $C \approx 88^\circ$, $a = 41$, $b = 54$, and $c \approx 67$

Triangle #2: $A = 38^\circ$, $B' \approx 126^\circ$, $C' \approx 16^\circ$, $a = 41$, $b = 54$, and $c \approx 18$