Topic: Write the equation as a parametric curve

Question: Which of the following is a parametric representation of the curve that satisfies the equation $x^2 - 2(y - 1) = 0$ and has $(2,3)$ and $(4,9)$ as its initial and terminal points, respectively?

Answer choices:

A $\quad x = t$ and $y = 2(t - 1)$ where $2 \le t \le 5$

B $\quad x = t$ and $y = \frac{t^2}{2} + 1$ where $2 \le t \le 4$

C $\quad x = \frac{t^2}{2}$ and $y = t + 1$ where $3 \le t \le 5$

D $\quad x = \frac{t^2}{2}$ and $y = t - 2$ where $3 \le t \le 4$

Solution: B

To see that answer choice B is correct, we'll first solve the given equation for y:

$$x^2 - 2(y-1) = 0$$

$$x^2 - 2y + 2 = 0$$

$$x^2 + 2 = 2y$$

$$\frac{x^2}{2} + 1 = y$$

Turning this equation around, we obtain

$$y = \frac{x^2}{2} + 1$$

Thus y is a function of x. If we let $x = t$, then

$$y = \frac{t^2}{2} + 1$$

For the smallest value of t given in answer choice B (i.e., $t = 2$),

$$x = t \Longrightarrow x = 2, \quad y = \frac{t^2}{2} + 1 \Longrightarrow y = \frac{(2)^2}{2} + 1 = \frac{4}{2} + 1 = 3$$

For the largest value of t given in answer choice B (i.e., $t = 4$),

$$x = t \Longrightarrow x = 4, \quad y = \frac{t^2}{2} + 1 \Longrightarrow y = \frac{(4)^2}{2} + 1 = \frac{16}{2} + 1 = 9$$

This shows that the initial point is $(2,3)$ and the terminal point is $(4,9)$.

Now we'll show that none of the other three answer choices is correct.

The parametric equation given for y in answer choice A is $y = 2(t - 1)$, and the smallest value given for t is 2. For $t = 2$, answer choice A yields

$$y = 2(2 - 1) = 2(1) = 2 \neq 3$$

so answer choice A doesn't give the correct y-coordinate for the initial point of the curve.

The parametric equation for x which is given in answer choices C and D is

$$x = \frac{t^2}{2}$$

and the smallest value of t given for both of them is 3. For $t = 3$, answer choices C and D yield

$$x = \frac{(3)^2}{2} = \frac{9}{2} \neq 2$$

Thus neither answer choice C nor answer choice D gives the correct x-coordinate for the initial point of the curve.

Topic: Write the equation as a parametric curve

Question: The given equation is a closed curve. Express the left half of this curve in parametric form, with $(-4,4)$ as the initial point.

$$\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=1$$

Answer choices:

A $x=-4+4\cos t$ and $y=2+2\sin t$ where $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$

B $x=-4-4\cos t$ and $y=2-4\sin t$ where $0\le t\le\pi$

C $x=-4+4\cos t$ and $y=-2-2\sin t$ where $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$

D $x=-4+4\cos t$ and $y=2+4\sin t$ where $0\le t\le\pi$

Solution: A

Notice that

$$\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=1$$

is the equation of an ellipse that's centered at the point $(-4,2)$. First, we'll check that the prescribed initial point, $(-4,4)$, is on this curve:

$$\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=\frac{(-4+4)^2}{16}+\frac{(4-2)^2}{4}=0+\frac{4}{4}=1$$

Since the x-coordinate of $(-4,4)$ is equal to the x-coordinate of the center of the ellipse, we see that $(-4,4)$ is on the boundary between the left and right halves of the ellipse.

Using $16=4^2$ and $4=2^2$, we can rewrite the equation of the ellipse as

$$\left(\frac{x+4}{4}\right)^2+\left(\frac{y-2}{2}\right)^2=1$$

We know that $\sin^2 t+\cos^2 t=1$ for any t. Because of the “natural” relationship between x and the cosine function, and the “natural” relationship between y and the sine function, let's let

$$\frac{x+4}{4}=\cos t, \quad \frac{y-2}{2}=\sin t$$

Multiplying both sides of the first equation by 4, and both sides of the second equation by 2, gives

$$x+4=4\cos t, \quad y-2=2\sin t$$

Solving the first equation for x and the second equation for y, we get the parametric equations

$$x = -4 + 4\cos t, \quad y = 2 + 2\sin t$$

Since we want only the left half of the curve, the largest value of t must exceed the smallest value of t by π, so using $\pi/2$ and $3\pi/2$ as the smallest and largest values of t makes sense.

Let's check that we get $(-4,4)$ as the initial point of the curve when $t = \pi/2$. Well,

$$t = \frac{\pi}{2} \Longrightarrow x = -4 + 4\cos\left(\frac{\pi}{2}\right) = -4 + 4(0) = -4$$

and

$$t = \frac{\pi}{2} \Longrightarrow y = 2 + 2\sin\left(\frac{\pi}{2}\right) = 2 + 2(1) = 4$$

To be sure that we get the left half of the ellipse (and only the left half) with the parametric equations and t-values given in answer choice A, let's check the coordinates of the points that correspond to $t = \pi$ and $3\pi/2$.

$$t = \pi \Longrightarrow x = -4 + 4\cos(\pi) = -4 + 4(-1) = -8$$

and

$$t = \pi \Longrightarrow y = 2 + 2\sin(\pi) = 2 + 2(0) = 2$$

Clearly, the point $(-8,2)$ is on the left half of the ellipse, because its x-coordinate is less than that of the center of the ellipse.

$$t=\frac{3\pi}{2}\implies x=-4+4\cos\left(\frac{3\pi}{2}\right)=-4+4(0)=-4$$

and

$$t=\frac{3\pi}{2}\implies y=2+2\sin\left(\frac{3\pi}{2}\right)=2+2(-1)=0$$

The point $(-4,0)$ is also on the left half of the ellipse. In fact, it's on the boundary between the left and right halves of the ellipse, because it lies directly below the initial point of the curve, $(-4,2)$, so it is indeed the terminal point of the curve. These results show that answer choice A is correct.

In answer choice B, the parametric equation for x is given as $x=-4-4\cos t$, and the smallest value of t is given as 0. Thus for the x-coordinate of the initial point, we would get

$$x=-4-4\cos(0)=-4-4(1)=-8\neq-4$$

This shows that answer choice B is incorrect.

In answer choice C, the parametric equation for y is given as $y=-2-2\sin t$ and the smallest value of t is given as $\pi/2$. Thus for the y-coordinate of the initial point, we would get

$$y=-2-2\sin\left(\frac{\pi}{2}\right)=-2-2(1)=-4\neq4$$

This rules out answer choice C.

In answer choice D, the parametric equation given for x is $x = -4 + 4\cos t$, and the smallest value of t is given as 0, so for the x-coordinate of the initial point, we would get

$$x = -4 + 4\cos(0) = -4 + 4(1) = 0 \neq -4$$

Therefore, answer choice D is incorrect.

Topic: Write the equation as a parametric curve

Question: What type of curve has the following properties:

1. The curve satisfies the equation $y^2 - 10y - x + 22 = 0$.

2. The initial point of the curve is $(-3,5)$.

3. The y-coordinate of every point on the curve other than the initial point is greater than 5.

Answer choices:

A The lower half of a hyperbola

B A circle with center at $(-3,10)$

C The right half of an ellipse

D The upper half of a parabola that opens to the right

Solution: D

First, let's move the x to the right-hand side of the given equation:

$$y^2 - 10y + 22 = x$$

Turning this equation around, we get

$$x = y^2 - 10y + 22$$

Next, we'll take $y^2 - 10y$ and complete the square on it:

$$y^2 - 10y = \left(y^2 - 10y + 25\right) - 25$$

Now $y^2 - 10y + 25 = (y - 5)^2$, so our equation becomes

$$x = \left[(y - 5)^2 - 25\right] + 22$$

$$x = (y - 5)^2 - 3$$

This is the equation of a parabola that opens to the right and has its vertex at the point $(-3,5)$.

Note that for every real number y, there is a unique real number x that satisfies the equation $x = (y - 5)^2 - 3$, so x is a function of y. This tells us that we could represent this curve by using the parametric equations $y = t$ and $x = (t - 5)^2 - 3$ together with appropriate values of t.

Since we want y to be greater than 5 for all points on the curve other than the initial point, this curve is the upper half of the parabola, so we could represent the curve in parametric form as follows:

$x = (t - 5)^2 - 3$ and $y = t$ where $t \geq 5$

This curve has no terminal point; the larger the value of t, the larger the values of x and y.