Topic: Equation of a hyperbolic conic section

Question: Which function defines the same hyperbola?

$$x = \frac{4}{\cos t} + 1$$

$$y = 2\sin t\left(1 + \tan t \tan\frac{t}{2}\right) - 1$$

Answer choices:

A $\quad \dfrac{(x-4)^2}{1^2} - \dfrac{(y+1)^2}{2^2} = 1$

B $\quad \dfrac{(x-1)^2}{2^2} - \dfrac{(y+1)^2}{4^2} = 1$

C $\quad \dfrac{(x-1)^2}{4^2} - \dfrac{(y+1)^2}{2^2} = 1$

D $\quad \dfrac{(x+1)^2}{4^2} - \dfrac{(y-1)^2}{2^2} = 1$

Solution: C

Rewrite both equations.

$$x = \frac{4}{\cos t} + 1$$

$$x - 1 = \frac{4}{\cos t}$$

$$\frac{(x-1)^2}{4^2} = \frac{1}{\cos^2 t} = 1 + \tan^2 t$$

and

$$y = 2\sin t \left(1 + \tan t \tan \frac{t}{2}\right) - 1$$

$$y + 1 = 2\sin t \left(1 + \tan t \tan \frac{t}{2}\right)$$

$$\frac{y+1}{2} = \sin t \left(1 + \tan t \tan \frac{t}{2}\right)$$

$$\frac{y+1}{2} = \tan t$$

$$\frac{(y+1)^2}{2^2} = \tan^2 t$$

Subtract this second equation from the first.

$$\frac{(x-1)^2}{4^2} - \frac{(y+1)^2}{2^2} = 1 + \tan^2 t - \tan^2 t$$

$$\frac{(x-1)^2}{4^2}-\frac{(y+1)^2}{2^2}=1$$

Topic: Equation of a hyperbolic conic section

Question: Which parametric functions represent the hyperbola?

$$3x^2 - y^2 + 6x + 4y - 10 = 0$$

Answer choices:

A $x = \sec t - 1$ and $y = 3\tan t - 2$

B $x = -1 + \sqrt{3}\sec t$ and $y = 2\tan t - 3$

C $x = \csc t - 1$ and $y = 2 + 3\tan t$

D $x = -1 + \sqrt{3}\sec t$ and $y = 2 + 3\tan t$

Solution: D

Complete the square with respect to both variables.

$$3x^2 - y^2 + 6x + 4y - 10 = 0$$

$$3\left(x^2 + 2x + 1\right) - \left(y^2 - 4y + 4\right) = 9$$

$$3(x + 1)^2 - (y - 2)^2 = 9$$

Divide out the coefficients.

$$\frac{(x + 1)^2}{3} - \frac{(y - 2)^2}{9} = \frac{9}{9}$$

$$\frac{(x + 1)^2}{3} - \frac{(y - 2)^2}{9} = 1$$

For a hyperbola in the form

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

we can parametrize the curve using the equations $x = h + a\sec t$ and $y = k + b\tan t$. In our case, we get

$$x = -1 + \sqrt{3}\sec t$$

and

$$y = 2 + 3\tan t$$

Topic: Equation of a hyperbolic conic section

Question: Which set of parametric equations represent the hyperbola?

$$4x^2 - y^2 - 8mx + 2ny + 4m^2 - n^2 = 16$$

Answer choices:

A $\quad x = \dfrac{2}{\cos t} - m$ and $y = 4\tan t - n$

B $\quad x = \dfrac{2}{\cos t} + m$ and $y = 4\tan t + n$

C $\quad x = \dfrac{4}{\cos t} + m$ and $y = 4\tan t + n$

D $\quad x = \dfrac{16}{\cos t} + m$ and $y = 4\tan t + n$

Solution: B

Choose the functions from answer choice B.

$$x = \frac{2}{\cos t} + m$$

$$y = 4\tan t + n$$

Rewrite both equations.

$$x = \frac{2}{\cos t} + m$$

$$x - m = \frac{2}{\cos t}$$

$$\frac{x - m}{2} = \frac{1}{\cos t}$$

$$\frac{(x - m)^2}{2^2} = \frac{1}{\cos^2 t}$$

$$\frac{(x - m)^2}{2^2} = 1 + \tan^2 t$$

and

$$y = 4\tan t + n$$

$$y - n = 4\tan t$$

$$\frac{y - n}{4} = \tan t$$

$$\frac{(y-n)^2}{4^2} = \tan^2 t$$

Subtract this second equation from the first.

$$\frac{(x-m)^2}{4} - \frac{(y^2-n)^2}{16} = 1 + \tan^2 t - \tan^2 t$$

$$\frac{(x-m)^2}{4} - \frac{(y^2-n)^2}{16} = 1$$

Expand the equation above, and simplify.

$$\frac{x^2-2mx+m^2}{4} - \frac{y^2-2ny+n^2}{16} = 1$$

$$4\left(x^2-2mx+m^2\right) - y^2 - 2ny + n^2 = 16$$

$$4x^2 - 8mx + 4m^2 - y^2 - 2ny + n^2 = 16$$

$$4x^2 - y^2 - 8mx + 2ny + 4m^2 - n^2 = 16$$