Precalculus Quizzes

krista king MATH

Topic: Converting polar coordinates to rectangular

Question: What are the rectangular coordinates (x, y) of the point that has polar coordinates $(r, \theta) = (4, \pi)$?

Answer choices:

A $(x, y) = (4,0)$

B $(x, y) = (0, -4)$

C $(x, y) = (0,4)$

D $(x, y) = (-4,0)$

Solution: D

Here, $r = 4$ and $\theta = \pi$. Since $r > 0$ and an angle of measure π is in the interval $[0,2\pi)$, the polar coordinates $(4,\pi)$ are the "basic" polar coordinates of the point in question. Thus the distance of this point from the pole is $r = 4$. Since (the terminal side of) an angle of measure π is on the negative horizontal axis, this point is on the negative horizontal axis (hence its y coordinate is 0) and it lies 4 units to the left of the pole (hence its x coordinate is -4), so its rectangular coordinates are $(-4,0)$.

Topic: Converting polar coordinates to rectangular

Question: Which of the following most closely approximates the rectangular coordinates (x, y) of the point whose polar coordinates are $(r, \theta) = (5, 11\pi/7)$?

Answer choices:

A $(x, y) = (-1.24, -2.38)$

B $(x, y) = (1.12, -4.88)$

C $(x, y) = (-4.87, 1.11)$

D $(x, y) = (1.24, -2.38)$

Solution: B

Here, $r = 5$ and $\theta = 11\pi/7$. Note that

$$\frac{3\pi}{2} = \frac{21\pi}{14} < \frac{22\pi}{14} = \frac{11\pi}{7} < \frac{14\pi}{7} = 2\pi$$

Thus an angle of measure $11\pi/7$ is not only in the interval $[0,2\pi)$ but in the fourth quadrant. Since r is positive, $(5,11\pi/7)$ are the “basic” polar coordinates of the point in question, so the point is in the fourth quadrant. Therefore, its x coordinate is positive and its y coordinate is negative.

Using the general equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

we get

$$x = 5\cos\frac{11\pi}{7}$$

and

$$y = 5\sin\frac{11\pi}{7}$$

With the help of a calculator, we find that $\cos(11\pi/7) \approx 0.223$ and $\sin(11\pi/7) \approx -0.975$, so $x \approx 5(0.223) \approx 1.12$ and $y \approx 5(-0.975) \approx -4.88$.

Topic: Converting polar coordinates to rectangular

Question: What are the rectangular coordinates (x, y) of the point that has polar coordinates $(r, \theta) = (21, 9\pi/8)$?

Answer choices:

A $$(x, y) = \left(-21\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right), -21\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)\right)$$

B $$(x, y) = \left(-21\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}, -21\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}\right)$$

C $$(x, y) = \left(-21\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}, -21\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}\right)$$

D $$(x, y) = \left(-21\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right), -21\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)\right)$$

Solution: C

Here, $r = 21$ and $\theta = 9\pi/8$. Note that

$$\pi = \frac{8\pi}{8} < \frac{9\pi}{8} = \frac{27\pi}{24} < \frac{36\pi}{24} = \frac{3\pi}{2}$$

Thus an angle of measure $9\pi/8$ is not only in the interval $[0,2\pi)$ but in the third quadrant. Since r is positive, $(21,9\pi/8)$ are the “basic” polar coordinates of the point in question, so the point is in the third quadrant. Therefore, its x and y coordinates are both negative.

To determine x and y, we'll use the equations

$$x = r\cos\theta = 21\cos\frac{9\pi}{8}$$

$$y = r\sin\theta = 21\sin\frac{9\pi}{8}$$

and the half-angle identities for cosine and sine. By the half-angle identity for cosine,

$$\cos^2\frac{9\pi}{8} = \frac{1}{2}\left\{1 + \cos\left[2\left(\frac{9\pi}{8}\right)\right]\right\} = \frac{1}{2}\left[1 + \cos\left(\frac{9\pi}{4}\right)\right]$$

By the half-angle identity for sine,

$$\sin^2\frac{9\pi}{8} = \frac{1}{2}\left\{1 - \cos\left[2\left(\frac{9\pi}{8}\right)\right]\right\} = \frac{1}{2}\left[1 - \cos\left(\frac{9\pi}{4}\right)\right]$$

Now

$$\frac{9\pi}{4} = \frac{8\pi + \pi}{4} = 2\pi + \frac{\pi}{4}$$

Thus an angle of measure $9\pi/4$ is coterminal with an angle of measure $\pi/4$.

Recall that

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Thus

$$\cos\frac{9\pi}{4} = \frac{1}{\sqrt{2}}$$

Substituting this result, we obtain

$$\cos^2\frac{9\pi}{8} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

and

$$\sin^2\frac{9\pi}{8} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

Since x is negative, $\cos(9\pi/8)$ is negative. Therefore,

$$x = 21\cos\frac{9\pi}{8} = 21\left(-\sqrt{\cos^2\frac{9\pi}{8}}\right)$$

$$x = -21\sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)}$$

$$x = -21\sqrt{\frac{1}{2}\left(\frac{\sqrt{2}(1) + 1(1)}{\sqrt{2}}\right)}$$

$$x = -21\sqrt{\frac{1}{2}\left(\frac{\sqrt{2} + 1}{\sqrt{2}}\right)}$$

$$x = -21\sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}$$

And since y is negative, $\sin(9\pi/8)$ is negative, so

$$y = 21\sin\frac{9\pi}{8} = 21\left(-\sqrt{\sin^2\frac{9\pi}{8}}\right)$$

$$y = -21\sqrt{\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)}$$

$$y = -21\sqrt{\frac{1}{2}\left(\frac{\sqrt{2}(1) - 1(1)}{\sqrt{2}}\right)}$$

$$y = -21\sqrt{\frac{1}{2}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)}$$

$$y = -21\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

Topic: Converting rectangular coordinates to polar

Question: Convert the rectangular coordinate point $(0,-14)$ into polar coordinates.

Answer choices:

A $\quad (r,\theta)=\left(-14,-\frac{\pi}{2}\right)$

B $\quad (r,\theta)=\left(14,\frac{3\pi}{2}\right)$

C $\quad (r,\theta)=\left(\sqrt{14},-\frac{\pi}{2}\right)$

D $\quad (r,\theta)=(14,\pi)$

Solution: B

To find r, we'll use the conversion formula

$$r^2 = x^2 + y^2$$

Plugging $(0, -14)$ into the formula gives

$$r^2 = (0)^2 + (-14)^2$$

$$r^2 = 196$$

$$r = 14$$

To find θ, we realize that since $x = 0$ and y is negative, the point in question is on the negative vertical axis, which must mean that $\theta = 3\pi/2$.

So the given point in polar coordinates is

$$\left(14, \frac{3\pi}{2}\right)$$

Topic: Converting rectangular coordinates to polar

Question: Which choice most closely represents the rectangular coordinate point $(-16,-22)$ in polar coordinates.

Answer choices:

A $(r,\theta)=(27.2,4.08)$

B $(r,\theta)=(20.7,54.1)$

C $(r,\theta)=(27.2,-1.20)$

D $(r,\theta)=(13.6,5.51)$

Solution: A

To find r, we'll use the conversion formula

$$r^2 = x^2 + y^2$$

Plugging $(-16, -22)$ into the formula gives

$$r^2 = (-16)^2 + (-22)^2$$

$$r^2 = 256 + 484$$

$$r^2 = 740$$

$$r \approx 27.2$$

Since both x and y are negative, this point is in the third quadrant, so

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

Using a calculator, we find that

$$\frac{y}{x} = \frac{-22}{-16} = \frac{22}{16} = 1.375$$

so

$$\tan^{-1}\left(\frac{y}{x}\right) \approx 0.942 \text{ radians}$$

Therefore,

$$\theta = \left(\tan^{-1}\left(\frac{y}{x}\right)\right) + \pi \approx (0.942 + \pi) \text{ radians}$$

$$\theta \approx 4.08 \text{ radians}$$

So the given point in polar coordinates is

$$(27.2, 4.08)$$

Topic: Converting rectangular coordinates to polar

Question: Convert the rectangular coordinate point $\left(17\sqrt{3}, -17\right)$ into polar coordinates.

Answer choices:

A $\quad (r, \theta) = \left(34, \frac{\pi}{6}\right)$

B $\quad (r, \theta) = \left(17, \frac{5\pi}{3}\right)$

C $\quad (r, \theta) = \left(17, \frac{\pi}{3}\right)$

D $\quad (r, \theta) = \left(34, \frac{11\pi}{6}\right)$

Solution: D

To find r, we'll use the conversion formula

$$r^2 = x^2 + y^2$$

Plugging $\left(17\sqrt{3}, -17\right)$ into the formula gives

$$r^2 = \left(17\sqrt{3}\right)^2 + (-17)^2$$

$$r^2 = 289(3) + 289$$

$$r^2 = 1{,}156$$

$$r = 34$$

Since x is positive and y is negative, this point is in the fourth quadrant, so

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2\pi$$

Now

$$\frac{y}{x} = \frac{-17}{17\sqrt{3}} = \frac{-1}{\sqrt{3}} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

Recall the following:

$$\sin\frac{\pi}{6} = \frac{1}{2} \text{ and } \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

By the odd and even identities for sine and cosine, respectively,

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} \text{ and } \cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Therefore,

$$\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{y}{x} = \frac{\sin\left(-\frac{\pi}{6}\right)}{\cos\left(-\frac{\pi}{6}\right)} = \tan\left(-\frac{\pi}{6}\right)$$

Note that

$$-\frac{\pi}{2} = -\frac{3\pi}{6} < -\frac{\pi}{6} < 0$$

That is, $-\pi/6$ is in the interval $(-\pi/2,0)$, so $-\pi/6$ is in the range of the inverse tangent function. Thus

$$\tan^{-1}\left(\frac{y}{x}\right) = -\frac{\pi}{6}$$

Using this result, we find that

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2\pi = -\frac{\pi}{6} + 2\pi = \frac{1(-\pi) + 6(2\pi)}{6} = \frac{-\pi + 12\pi}{6} = \frac{11\pi}{6}$$

So the given point in polar coordinates is

$$\left(34, \frac{11\pi}{6}\right)$$

Topic: Express the polar point multiple ways

Question: Convert the polar point to rectangular coordinates.

$$(r,\theta)=\left(-14,\frac{5\pi}{6}\right)$$

Answer choices:

A $\quad (x,y)=\left(7,-7\sqrt{3}\right)$

B $\quad (x,y)=\left(-\frac{7\sqrt{3}}{2},7\right)$

C $\quad (x,y)=\left(7\sqrt{3},-7\right)$

D $\quad (x,y)=\left(\frac{7\sqrt{3}}{2},-7\right)$

Solution: C

To convert a rectangular point to a polar point, we use the conversion formulas

$$x = r\cos\theta$$

$$y = r\cos\theta$$

Plugging the point into the conversion formulas give

$$x = -14\left(\cos\frac{5\pi}{6}\right)$$

$$y = -14\left(\sin\frac{5\pi}{6}\right)$$

We know that

$$\pi - \frac{5\pi}{6} = \frac{\pi}{6}$$

and by the reference angle theorem and the fact that an angle of measure $5\pi/6$ is in the second quadrant, we can say

$$\cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Substituting these results, we get

$$x = -14\left(-\frac{\sqrt{3}}{2}\right) = 7\sqrt{3}$$

$$y = -14\left(\frac{1}{2}\right) = -7$$

Therefore, the point in rectangular coordinates is

$$\left(7\sqrt{3}, -7\right)$$

Topic: Express the polar point multiple ways

Question: If one pair of polar coordinates of a point is $(-4.2, 41\pi/12)$, what is the pair of polar coordinates (r, θ) of that point if θ is in the interval $[-6\pi, -5\pi)$? Express θ in radians.

Answer choices:

A $\quad (r, \theta) = (27.2, 4.08)$

B $\quad (r, \theta) = (20.7, 54.1)$

C $\quad (r, \theta) = \left(4.2, -\dfrac{67\pi}{12}\right)$

D $\quad (r, \theta) = (13.6, 5.51)$

Solution: C

If (r, θ) is the pair of coordinates for this point in which θ is in the interval $[-6\pi, -5\pi)$, then θ must differ from $41\pi/12$ by $n\pi$ for some integer n. Note that

$$3\pi = \frac{36\pi}{12} < \frac{41\pi}{12} < \frac{48\pi}{12} = 4\pi$$

Now

$$-6\pi = 3\pi - 9\pi \text{ and } -5\pi = 4\pi - 9\pi$$

Therefore,

$$\theta = \frac{41\pi}{12} - 9\pi = \frac{1(41\pi) - 12(9\pi)}{12} = \frac{41\pi - 108\pi}{12} = -\frac{67\pi}{12}$$

Since 9 is an odd integer and the first coordinate in the given pair of polar coordinates of this point is -4.2 (hence negative), r must be positive (hence $r = 4.2$). Thus the indicated pair of polar coordinates of this point is $(4.2, -67\pi/12)$.

Topic: Express the polar point multiple ways

Question: Which of the following most closely approximates the polar coordinates (r,θ) of the point that has rectangular coordinates $(x,y)=(-9.6,-4.5)$ if θ is in the interval $[19\pi,20\pi)$? Express θ in radians.

Answer choices:

A $(r,\theta)=(13.9,57.0)$

B $(r,\theta)=(10.6,60.1)$

C $(r,\theta)=(-17.2,63.7)$

D $(r,\theta)=(-10.6,60.1)$

Solution: B

For the point P, both x and y are negative, so P is in the third quadrant. First, we'll find the basic polar coordinates of P.

Since the first of the two basic polar coordinates of any point other than the pole must be positive, and the second of its two basic polar coordinates must be an angle whose terminal side is in the same quadrant as the point itself, the basic polar coordinates of P are (s, α) for some positive number s and some angle α in the interval $[\pi, 3\pi/2)$.

In general, we know that

$$s = \sqrt{x^2 + y^2}$$

$$s = \sqrt{(-9.6)^2 + (-4.5)^2}$$

$$s = \sqrt{92.16 + 20.25}$$

$$s = \sqrt{112.41}$$

$$s \approx 10.6$$

Recall that the fact that α is in the third quadrant tells us that

$$\alpha = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

With the help of a calculator, we find that

$$\frac{y}{x} = \frac{-4.5}{-9.6} = \frac{4.5}{9.6} \approx 0.469$$

and so

$$\tan^{-1}\left(\frac{y}{x}\right) \approx \tan^{-1}(0.469) \approx 0.439 \text{ radians}$$

So

$$\alpha \approx (0.439 + \pi) \text{ radians} \approx 3.58 \text{ radians}$$

Since the interval $[\pi, 3\pi/2)$ is a subset of the interval $[\pi, 2\pi)$, α is in the interval $[\pi, 2\pi)$. Note that

$$19\pi = \pi + 18\pi \text{ and } 20\pi = 2\pi + 18\pi$$

Therefore,

$$\theta = \alpha + 18\pi \approx (3.58 + 18\pi) \approx 60.1 \text{ radians}$$

Since 18 is even, we know that r must be of the same sign as s (hence positive), and thus that $r = s$ (hence $r \approx 10.6$).

Topic: Converting equations from polar to rectangular

Question: What is the solution set of the polar equation in terms of rectangular coordinates?

$$5r\sin\theta + 10r\cos\theta = 15$$

Answer choices:

A The solution set is the set of all points on the line with slope -5 and y-intercept $(0,10)$.

B The solution set is the set of all points on the line with slope -2 and y-intercept $(0,3)$.

C The solution set is the set of all points on the line with slope -10 and y-intercept $(0,5)$.

D The solution set is the set of all points on the line with slope -4 and y-intercept $(0,-3)$.

Solution: B

Using the conversion equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

we find that the given polar equation, $5r\sin\theta + 10r\cos\theta = 15$, becomes

$$5y + 10x = 15$$

Dividing both sides by 5 gives

$$y + 2x = 3$$

Solving for y, we find that $y = -2x + 3$. This is the equation of the line with slope -2 and y-intercept $(0,3)$.

We'll now show that every point on the line $y = -2x + 3$ is a solution of the polar equation

$$5r\sin\theta + 10r\cos\theta = 15$$

Using the those same conversion equations and starting from the equation $y = -2x + 3$, we get

$$r\sin\theta = -2r\cos\theta + 3$$

$$r\sin\theta + 2r\cos\theta = 3$$

Multiplying both sides by 5 yields the given polar equation

$$5r\sin\theta + 10r\cos\theta = 15$$

Topic: Converting equations from polar to rectangular

Question: What is the solution set of the polar equation in terms of rectangular coordinates?

$$r = -6\sin\theta$$

Answer choices:

A The solution set is the set of all points on the circle $(x-3)^2 + y^2 = 9$.

B The solution set is the set of all points on the circle $(x+3)^2 + y^2 = 9$.

C The solution set is the set of all points on the circle $x^2 + (y-3)^2 = 9$.

D The solution set is the set of all points on the circle $x^2 + (y+3)^2 = 9$.

Solution: D

We have the conversion equation

$$y = r\sin\theta$$

$$\sin\theta = \frac{y}{r}$$

and we can substitute this result into the given polar equation, $r = -6\sin\theta$.

$$r = -6\sin\theta$$

$$r = -6\left(\frac{y}{r}\right)$$

Multiplying both sides by r yields

$$r^2 = -6y$$

Using the general equation $r^2 = x^2 + y^2$, we get

$$x^2 + y^2 = -6y$$

$$x^2 + y^2 + 6y = 0$$

Now we'll complete the square on the y part of the expression $x^2 + y^2 + 6y$ (that is, on $y^2 + 6y$). To do that, we need to add 9, because

$$y^2 + 6y + 9 = (y+3)^2$$

If we add 9, we also have to subtract 9, so

$$y^2 + 6y = (y^2 + 6y) + 9 - 9 = (y^2 + 6y + 9) - 9 = (y+3)^2 - 9$$

Substituting this expression for $y^2 + 6y$ in the equation $x^2 + y^2 + 6y = 0$, we get

$$x^2 + (y+3)^2 - 9 = 0$$

Adding 9 to both sides:

$$x^2 + (y+3)^2 = 9$$

$$(x-0)^2 + [y-(-3)]^2 = 3^2$$

This is the equation of the circle whose center has rectangular coordinates $(0, -3)$ and whose radius is 3.

It would be easy to check, by working backwards, that every point on that circle is indeed a solution of the given polar equation, $r = -6\sin\theta$.

Topic: Converting equations from polar to rectangular

Question: What is the solution set of the polar equation in terms of rectangular coordinates?

$$r^2 \sin(2\theta) = -8$$

Answer choices:

A The solution set is the set of all points on the hyperbola $y = -4/x$.

B The solution set is the set of all points on the hyperbola $y = 4/x$.

C The solution set is the set of all points on the parabola $y = x^2/4$.

D The solution set is the set of all points on the circle $x^2 + (y - 1/2)^2 = 4$.

Solution: A

By the double-angle formula for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Substituting this result into the given polar equation, $r^2\sin(2\theta) = -8$, we obtain

$$r^2(2\sin\theta\cos\theta) = -8$$

Using the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}$$

Substituting these results gives

$$r^2\left[2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)\right] = -8$$

If we cancel the r^2 in the numerator against the two factors of r in the denominator, we're left with the equation $2(yx) = -8$. Solving this equation for y yields

$$y = -\frac{4}{x}$$

This is the equation of a hyperbola. Because of the minus sign on the right-hand side of this equation, x and y must be of opposite sign, so this hyperbola resides in the second and fourth quadrants.

If we were to start from the equation $y = -4/x$ and work backwards, we would easily find that every point on this hyperbola is a solution of the given polar equation, $r^2 \sin(2\theta) = -8$.

Topic: Converting equations from rectangular to polar

Question: Which of the following equations do we get if we convert the rectangular equation $y = -6x + 5$ to polar coordinates?

Answer choices:

A $r(\sin\theta - 6\cos\theta) = 5$

B $\tan\theta = -\dfrac{5}{6}$

C $r(\sin\theta + 6\cos\theta) = 5$

D $\tan\theta + 6 = -\dfrac{5}{\cos\theta}$

Solution: C

Making the replacements $r\cos\theta$ for x, and $r\sin\theta$ for y, in the given rectangular equation, we get

$$r\sin\theta = -6r\cos\theta + 5$$

Adding $6r\cos\theta$ to both sides of this equation, we obtain

$$r\sin\theta + 6r\cos\theta = 5$$

Factoring out an r on the left-hand side gives

$$r(\sin\theta + 6\cos\theta) = 5$$

This shows that answer choice C is correct.

Since the four answer choices all have somewhat different forms, let's check to be sure that none of the other three answer choices is correct.

Answer choice A cannot be correct, because it agrees with answer choice C on everything but the sign of the term that includes $\cos\theta$.

We know that, in general,

$$\frac{y}{x} = \tan\theta$$

Thus the equation given in answer choice B is equivalent to

$$\frac{y}{x} = -\frac{5}{6}$$

This equation can be written as $y = -(5/6)x$, which is the equation of a line that has a slope of $-5/6$. The given rectangular equation, $y = -6x + 5$, is the equation of a line that has a slope of -6. Thus answer choice B cannot be correct.

Finally, suppose answer choice D is correct. Note that one of the points on the line given by the original (rectangular) equation $y = -6x + 5$ has rectangular coordinates $(x, y) = (1, -1)$, since

$$-6(1) + 5 = -6 + 5 = -1$$

One pair of polar coordinates of that point is

$$(r, \theta) = \left(\sqrt{2}, \frac{7\pi}{4}\right)$$

For that point, answer choice D yields

$$\tan\left(\frac{7\pi}{4}\right) + 6 = -\frac{5}{\cos\left(\frac{7\pi}{4}\right)}$$

Since $\tan(7\pi/4) = -1$ and $\cos(7\pi/4) = 1/\sqrt{2}$, that becomes

$$-1 + 6 = -\frac{5}{\left(\frac{1}{\sqrt{2}}\right)}$$

Simplifying, we get

$$5 = -5\sqrt{2}$$

which is absurd. Therefore, answer choice D is incorrect.

Topic: Converting equations from rectangular to polar

Question: Which of the following equations do we get if we convert the rectangular equation $x^2 + (y - 7)^2 = 49$ to polar coordinates?

Answer choices:

A $r = -14 \sin \theta$

B $r^2 = 14 \sin \theta$

C $r = 14 \cos \theta$

D $r = 14 \sin \theta$

Solution: D

Making the replacements $r\cos\theta$ for x, and $r\sin\theta$ for y, we have

$$(r\cos\theta)^2 + (r\sin\theta - 7)^2 = 49$$

Expanding the left-hand side (by performing the indicated squaring) yields

$$r^2\cos^2\theta + (r^2\sin^2\theta - 14r\sin\theta + 49) = 49$$

Regrouping some terms, we obtain

$$(r^2\cos^2\theta + r^2\sin^2\theta) - 14r\sin\theta + 49 = 49$$

We know that, in general,

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

Substituting this result, we find that

$$r^2 - 14r\sin\theta + 49 = 49$$

Subtracting 49 from both sides:

$$r^2 - 14r\sin\theta = 0$$

Factoring the left-hand side gives

$$r(r - 14\sin\theta) = 0$$

This implies that $r = 0$ or $r = 14\sin\theta$.

The only point for which $r = 0$ is the pole, and for any angle θ, $(0,\theta)$ is a pair of polar coordinates for the pole. Thus if we let $\theta = 0$, then the pole also satisfies the equation $r = 14\sin\theta$. Thus the given (rectangular) equation converts to the polar equation $r = 14\sin\theta$.

Topic: Converting equations from rectangular to polar

Question: Which of the following equations do we get if we convert the equation to polar coordinates?

$$x = \frac{1}{y+3}$$

Answer choices:

A $\quad r\cos^2\theta + 6r^2\sin\theta = -4$

B $\quad r^2\sin(2\theta) + 6r\cos\theta = 2$

C $\quad r^2\cos(2\theta) + 6r\sin\theta = -6$

D $\quad r\sin^2\theta - 6r\sin\theta = 2$

Solution: B

Replacing x with $r\cos\theta$, and y with $r\sin\theta$, we have

$$r\cos\theta = \frac{1}{r\sin\theta + 3}$$

Multiplying both sides of this equation by $r\sin\theta + 3$, we obtain

$$(r\cos\theta)(r\sin\theta + 3) = 1$$

Doing the indicated multiplication on the left-hand side:

$$r^2\cos\theta\sin\theta + 3r\cos\theta = 1$$

By the double-angle formula for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Thus

$$\cos\theta\sin\theta = \sin\theta\cos\theta = \left(\frac{1}{2}\right)(2\sin\theta\cos\theta) = \left(\frac{1}{2}\right)\sin(2\theta)$$

Substituting this result, we find that

$$r^2\left(\frac{1}{2}\right)\sin(2\theta) + 3r\cos\theta = 1$$

Multiplying both sides of this equation by 2:

$$r^2\sin(2\theta) + 6r\cos\theta = 2$$

This is answer choice B.

Let's check to be sure that none of the other answer choices is correct. Using our original equation,

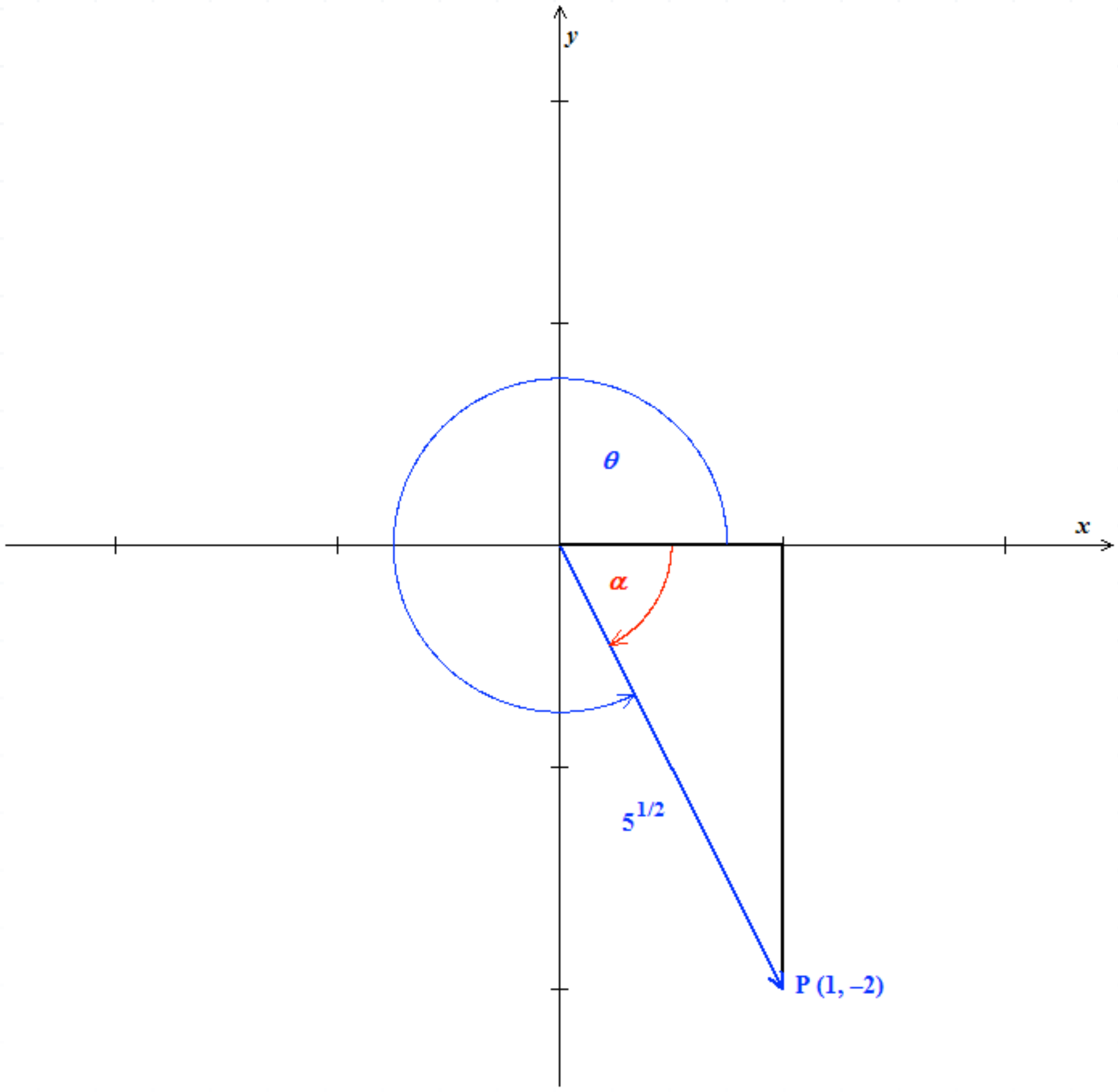
$$x = \frac{1}{y+3}$$

let's set x to 1 and solve for the corresponding value of y:

$$1 = \frac{1}{y+3}$$

Multiplying both sides by $y+3$, we get $1(y+3)=1$, which gives $y+3=1$, so $y=1-3=-2$. This point, with rectangular coordinates $(x, y)=(1,-2)$, is in the fourth quadrant.

We'll construct a right triangle, with one leg along the positive horizontal axis (and of length $x=1$), the other leg parallel to the negative vertical axis (and of length $|y|=|-2|=2$), and the acute interior angle α as shown in the figure.



Then the hypotenuse is of length

$$\sqrt{(1)^2 + (2)^2} = \sqrt{1 + 4} = \sqrt{5}$$

Since our point is in the fourth quadrant, we can use an angle θ in the interval $[3\pi/2, 2\pi)$, so

$$\theta = 2\pi - \alpha$$

and r (the distance of our point $(x, y) = (1, -2)$ from the pole) is equal to the length of the hypotenuse (namely, $\sqrt{5}$).

By our tried-and-true formulas for right triangles:

$$\cos\alpha = \frac{\text{length of leg adjacent to } \alpha}{\text{length of hypotenuse}} = \frac{1}{\sqrt{5}}$$

$$\sin\alpha = \frac{\text{length of leg opposite } \alpha}{\text{length of hypotenuse}} = \frac{2}{\sqrt{5}}$$

By the difference identity for cosine,

$$\cos(2\pi - \alpha) = \cos(2\pi)\cos\alpha + \sin(2\pi)\sin\alpha$$

$$\cos(2\pi - \alpha) = (1)\cos\alpha + (0)\sin\alpha$$

$$\cos(2\pi - \alpha) = \cos\alpha$$

$$\cos(2\pi - \alpha) = \frac{1}{\sqrt{5}}$$

And by the difference identity for sine,

$$\sin(2\pi - \alpha) = \sin(2\pi)\cos\alpha - \cos(2\pi)\sin\alpha$$

$$\sin(2\pi - \alpha) = (0)\cos\alpha - (1)\sin\alpha$$

$$\sin(2\pi - \alpha) = -\sin\alpha$$

$$\sin(2\pi - \alpha) = -\frac{2}{\sqrt{5}}$$

Therefore,

$$\cos\theta = \cos(2\pi - \alpha) = \cos\alpha = \frac{1}{\sqrt{5}}$$

and

$$\sin\theta = \sin(2\pi - \alpha) = -\sin\alpha = -\frac{2}{\sqrt{5}}$$

Let's plug these values of r, $\cos\theta$, and $\sin\theta$ for our chosen point into the equations given in the other three answer choices, and we'll see that none of those equations is satisfied.

The left-hand side of the equation in answer choice A is

$$r\cos^2\theta + 6r^2\sin\theta$$

Substituting the indicated values of r, $\cos\theta$, and $\sin\theta$, we get

$$r\cos^2\theta + 6r^2\sin\theta = \sqrt{5}\left(\frac{1}{\sqrt{5}}\right)^2 + 6(\sqrt{5})^2\left(-\frac{2}{\sqrt{5}}\right)$$

$$r\cos^2\theta + 6r^2\sin\theta = \sqrt{5}\left(\frac{1}{5}\right) + 6(5)\left(-\frac{2}{\sqrt{5}}\right)$$

$$r\cos^2\theta + 6r^2\sin\theta = \sqrt{5}\left(\frac{1}{5}\right) - 12\sqrt{5}$$

$$r\cos^2\theta + 6r^2\sin\theta = \sqrt{5}\left(\frac{1}{5} - 12\right)$$

This is not equal to the number on the right-hand side of the equation in answer choice A (namely, -4).

The left-hand side of the equation in answer choice C is

$$r^2\cos(2\theta) + 6r\sin\theta$$

By the double-angle formula for cosine,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{1}{\sqrt{5}}\right)^2 - \left(-\frac{2}{\sqrt{5}}\right)^2 = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$$

Substituting the indicated values of r, $\cos\theta$, and $\sin\theta$, the left-hand side of that equation gives

$$r^2\cos(2\theta) + 6r\sin\theta = \left(\sqrt{5}\right)^2\left(-\frac{3}{5}\right) + 6\sqrt{5}\left(-\frac{2}{\sqrt{5}}\right)$$

$$r^2\cos(2\theta) + 6r\sin\theta = 5\left(-\frac{3}{5}\right) - 12$$

$$r^2\cos(2\theta) + 6r\sin\theta = -3 - 12$$

$$r^2\cos(2\theta) + 6r\sin\theta = -15$$

This is not equal to the number on the right-hand side of the equation in answer choice C (namely, -6).

The left-hand side of the equation in answer choice D is

$$r\sin^2\theta - 6r\sin\theta$$

Substituting the indicated values of r, $\cos\theta$, and $\sin\theta$, the left-hand side of that equation gives

$$r\sin^2\theta - 6r\sin\theta = \sqrt{5}\left(-\frac{2}{\sqrt{5}}\right)^2 - 6\sqrt{5}\left(-\frac{2}{\sqrt{5}}\right)$$

$$r\sin^2\theta - 6r\sin\theta = \sqrt{5}\left(\frac{4}{5}\right) + 12$$

This is not equal to the number on the right-hand side of the equation in answer choice D (namely, 2).

We have shown that answer choices A, C, and D are incorrect. For the sake of completeness, let's show that the equation in answer choice B is satisfied by the values of r, $\cos\theta$, and $\sin\theta$ that correspond to our point $(x, y) = (1, -2)$.

The left-hand side of the equation in answer choice B is

$$r^2\sin(2\theta) + 6r\cos\theta$$

By the double-angle formula for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = -\frac{4}{\left(\sqrt{5}\right)^2} = -\frac{4}{5}$$

Therefore,

$$r^2\sin(2\theta) + 6r\cos\theta = \left(\sqrt{5}\right)^2\left(-\frac{4}{5}\right) + 6\sqrt{5}\left(\frac{1}{\sqrt{5}}\right)$$

$$r^2\sin(2\theta) + 6r\cos\theta = 5\left(-\frac{4}{5}\right) + 6$$

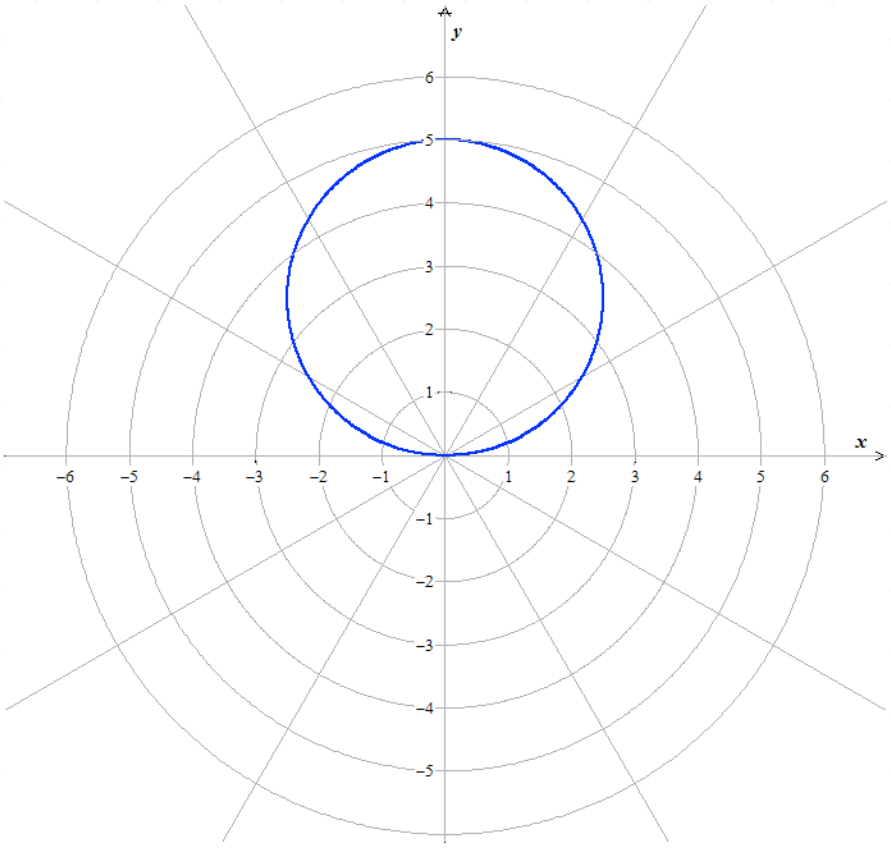
$$r^2\sin(2\theta) + 6r\cos\theta = -4 + 6$$

$$r^2\sin(2\theta) + 6r\cos\theta = 2$$

This is indeed the number on the right-hand side of the equation in answer choice B.

Topic: Graph the polar curve, circle

Question: The following curve is the graph of one of the polar equations given below. Which polar equation is it?



Answer choices:

A $r = -5\sin\theta$

B $r = (5/2)\cos\theta$

C $r = 5\sin\theta$

D $r = (5/2)\sin\theta$

Solution: C

The given curve is the circle with center at the point

$$(x, y) = (0,k) = \left(0,\frac{5}{2}\right)$$

and a radius of

$$c = |k| = \frac{5}{2}$$

Since k is positive, one pair of polar coordinates of the center of this circle is

$$(r, \theta) = \left(k, \frac{\pi}{2}\right) = \left(\frac{5}{2}, \frac{\pi}{2}\right)$$

Thus the given curve is the graph of the polar equation

$$r = a\sin\theta$$

for $a = 2k$. Since $k = 5/2$, we have $a = 2k = 2(5/2) = 5$. Therefore, the curve is the graph of the polar equation

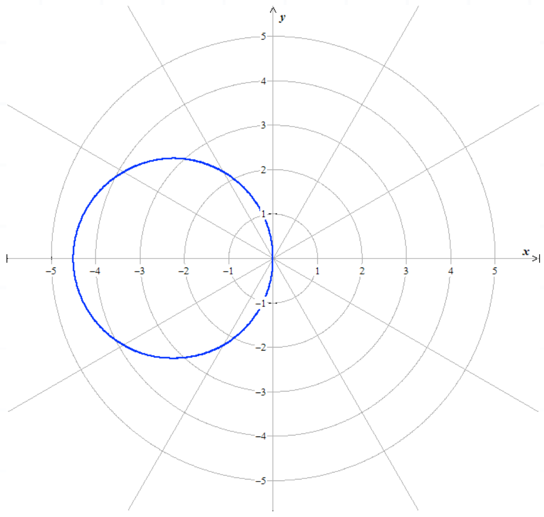
$$r = 5\sin\theta$$

Topic: Graph the polar curve, circle

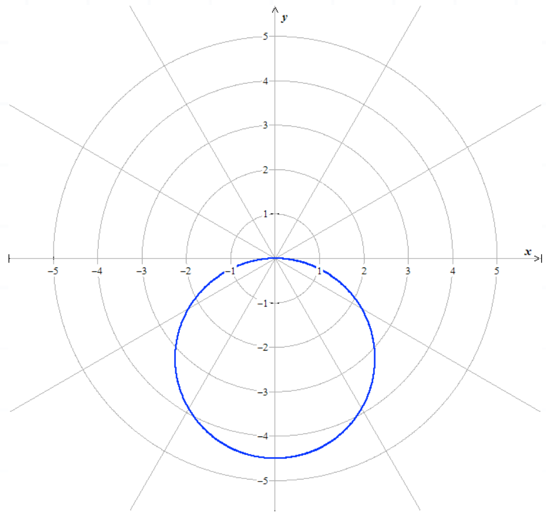
Question: Which of the following is the graph of the polar equation?

$$r = -9\cos\theta$$

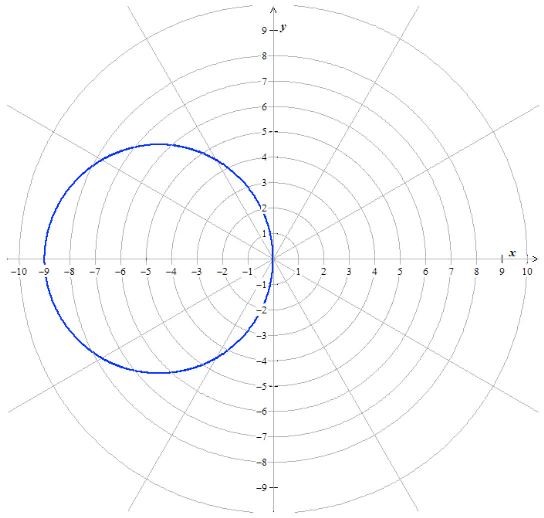
Answer choices:



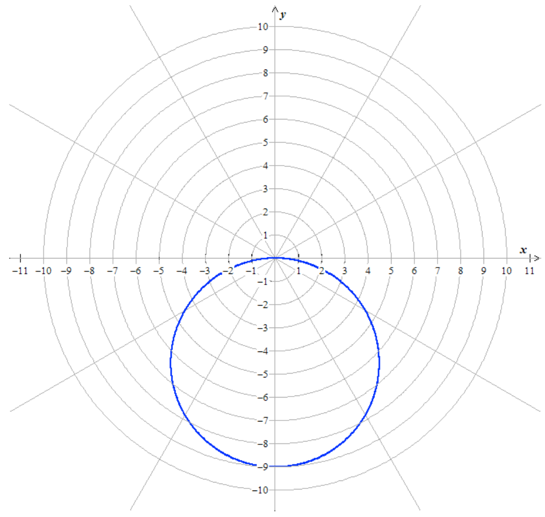
A



C



B



D

Solution: B

Since the coefficient of $\cos\theta$ is $a = -9$, we can set a equal to $2h$ (hence $h = a/2 = -9/2$), so

$$r = -9\cos\theta$$

is the polar equation of the circle that's centered at the point

$$(x, y) = (h, 0) = \left(\frac{a}{2}, 0\right) = \left(-\frac{9}{2}, 0\right)$$

and has a radius of

$$c = \frac{|a|}{2} = \frac{9}{2}$$

Since a is negative, one pair of polar coordinates of the center of this circle is

$$(r, \theta) = (-h, \pi) = \left(-\frac{a}{2}, \pi\right) = \left(\frac{9}{2}, \pi\right)$$

Inspection of the graphs given in the answer choices tells us that the graph of the polar equation $r = -9\cos\theta$ is the curve shown in answer choice B.

Topic: Graph the polar curve, circle

Question: Exactly one of the curves described below is the graph of the polar equation. Which curve is it?

$$r = 8\cos\left(\theta - \frac{\pi}{2}\right)$$

Answer choices:

A The circle with center $(x, y) = (0,4)$ and radius 4.

B The circle with center $(x, y) = (4,0)$ and radius 8.

C The circle with center $(x, y) = (-4,0)$ and radius 4.

D The circle with center $(x, y) = (0, -4)$ and radius 8.

Solution: A

By the difference identity for cosine,

$$\cos\left(\theta - \frac{\pi}{2}\right) = \cos\theta\cos\left(\frac{\pi}{2}\right) + \sin\theta\sin\left(\frac{\pi}{2}\right) = \cos\theta(0) + \sin\theta(1) = \sin\theta$$

Therefore, the polar equation

$$r = 8\cos\left(\theta - \frac{\pi}{2}\right)$$

is equivalent to the polar equation

$$r = 8\sin\theta$$

Since the coefficient of $\sin\theta$ is $a = 8$, we can set a equal to $2k$ (hence $k = a/2 = 4$), so

$$r = 8\cos\left(\theta - \frac{\pi}{2}\right)$$

is the polar equation of the circle with center at the point

$$(x, y) = (0,k) = \left(0,\frac{a}{2}\right) = (0,4)$$

and a radius of

$$c = \frac{|a|}{2} = 4$$

Topic: Graph the polar curve, rose

Question: How many petals are there in the graph of the rose?

$$r = 4\sin(6\theta)$$

Answer choices:

A 6 petals

B 4 petals

C 8 petals

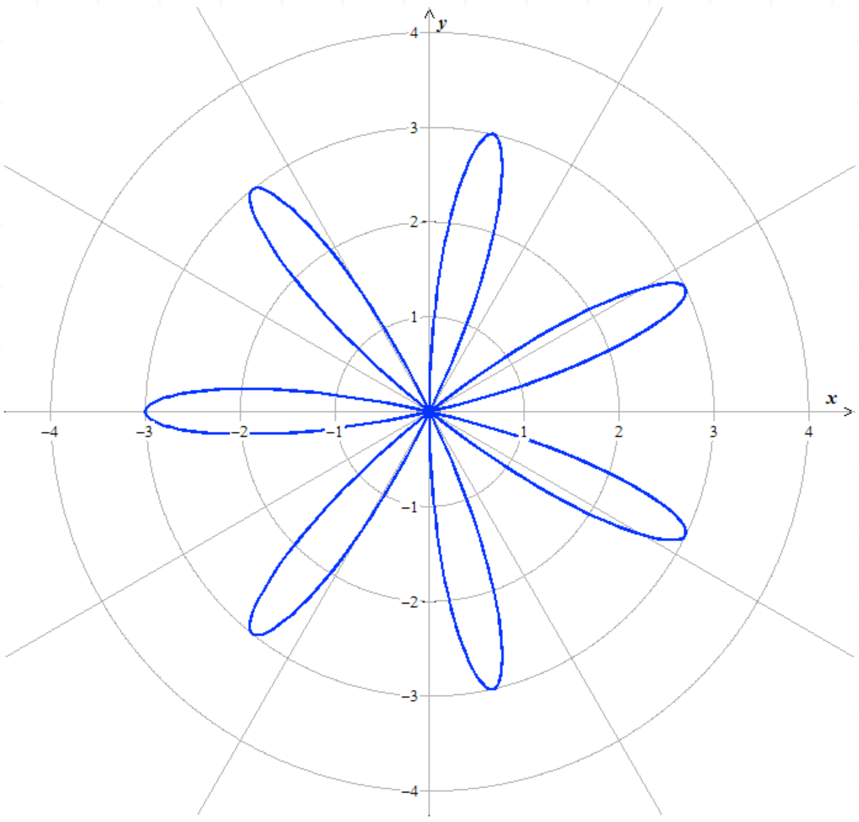
D 12 petals

Solution: D

The equation $r = 4\sin(6\theta)$ is in the form $r = a\sin(n\theta)$, with $a = 4$ and $n = 6$. Since n is even, this rose has $2n$ petals (i.e., 12 petals).

Topic: Graph the polar curve, rose

Question: The following curve is the graph of one of the polar equations given below. Which polar equation is it?



Answer choices:

A $\quad r = 3\cos(7\theta)$

B $\quad r = 7\sin(3\theta)$

C $\quad r = 3\sin(7\theta)$

D $\quad r = -3\cos(7\theta)$

Solution: D

This rose has 7 petals, which is an odd number, so it's the graph of a polar equation which is either of the form $r = a\cos(7\theta)$ or of the form $r = a\sin(7\theta)$ for some number a. Since the tip of one of the petals is on the horizontal axis, it's the graph of $r = a\cos(7\theta)$ for some number a. And since the tip of that petal is on the negative horizontal axis, and located at a point which is 3 units away from the pole, it's the graph of $r = -3\cos(7\theta)$.

Topic: Graph the polar curve, rose

Question: What is the measure of the smallest angle θ in the interval $[0,\pi)$ for which there is a petal of the rose $r = 8\sin(5\theta)$ whose tip has polar coordinates (r, θ) with $r = 8$, and what is the measure of the largest angle θ in the interval $[0,\pi)$ for which there is a petal of that rose whose tip has polar coordinates (r, θ) with $r = 8$?

Answer choices:

A Measure of smallest angle: $\theta = \pi/10$

Measure of largest angle: $\theta = 7\pi/10$

B Measure of smallest angle: $\theta = \pi/5$

Measure of largest angle: $\theta = 4\pi/5$

C Measure of smallest angle: $\theta = 0$

Measure of largest angle: $\theta = 4\pi/5$

D Measure of smallest angle: $\theta = \pi/10$

Measure of largest angle: $\theta = 9\pi/10$

Solution: D

The equation $r = 8\sin(5\theta)$ is of the form $r = a\sin(n\theta)$, with $a = 8$ and $n = 5$ (hence n is odd), so the tip of every petal of this rose has a pair of polar coordinates (r, θ) with $|r| = 8$ and some angle θ in the set

$$\left\{\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}\right\}$$

The angle of smallest measure in this set is $\theta = \pi/10$. Let's see whether $(8, \pi/10)$ is a pair of polar coordinates for one of the petals of this rose.

Well, for $\theta = \pi/10$,

$$r = 8\sin(5\theta) = 8\sin\left(5\left(\frac{\pi}{10}\right)\right) = 8\sin\left(\frac{\pi}{2}\right) = 8(1) = 8$$

so $(8, \pi/10)$ is indeed a pair of polar coordinates for the tip of a petal of this rose, hence $\theta = \pi/10$ is the measure of the smallest angle that we're looking for.

The angle of largest measure in the set given above is $\theta = 9\pi/10$. For the petal whose tip has $\theta = 9\pi/10$,

$$r = 8\sin(5\theta) = 8\sin\left(5\left(\frac{9\pi}{10}\right)\right) = 8\sin\left(\frac{9\pi}{2}\right) = 8\sin\left(4\pi + \frac{\pi}{2}\right)$$

Since 4π is a multiple of 2π, an angle of measure

$$4\pi + \frac{\pi}{2}$$

is coterminal with an angle of measure $\pi/2$, so

$$r = 8\sin\left(4\pi + \frac{\pi}{2}\right) = 8\sin\left(\frac{\pi}{2}\right) = 8(1) = 8$$

Therefore, $(8,9\pi/10)$ is a pair of polar coordinates for the tip of a petal of this rose, so $\theta = 9\pi/10$ is the measure of the largest angle that we're looking for.

Topic: Graph the polar curve, cardioid

Question: Which of the following is a pair of polar coordinates (r, θ) of some point of the cardioid?

$$r = 6(1 - \sin\theta)$$

Answer choices:

A $\quad (r, \theta) = \left(6 + 3\sqrt{3}, \frac{\pi}{3}\right)$

B $\quad (r, \theta) = \left(6 - 2\sqrt{2}, \frac{\pi}{6}\right)$

C $\quad (r, \theta) = \left(6 + 3\sqrt{3}, \frac{5\pi}{3}\right)$

D $\quad (r, \theta) = \left(6 + 2\sqrt{2}, -\frac{\pi}{4}\right)$

Solution: C

If we substitute the angle $\theta = 5\pi/3$ from answer choice C into the polar equation $r = 6(1 - \sin\theta)$ we get

$$r = 6\left[1 - \sin\left(\frac{5\pi}{3}\right)\right]$$

Now

$$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

so

$$r = 6\left[1 - \left(-\frac{\sqrt{3}}{2}\right)\right] = 6\left[1 + \frac{\sqrt{3}}{2}\right] = 6 + 3\sqrt{3}$$

Thus the point with polar coordinates

$$\left(6 + 3\sqrt{3}, \frac{5\pi}{3}\right)$$

is a point of the cardioid $r = 6(1 - \sin\theta)$.

Now we'll show that none of the other answer choices is correct.

If we substitute the angle $\theta = \pi/3$ from answer choice A into the polar equation $r = 6(1 - \sin\theta)$, we get

$$r = 6\left[1 - \sin\left(\frac{\pi}{3}\right)\right]$$

Now

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

so

$$r = 6\left(1 - \frac{\sqrt{3}}{2}\right) = 6 - 3\sqrt{3} \neq 6 + 3\sqrt{3}$$

Thus the point with polar coordinates

$$\left(6 + 3\sqrt{3}, \frac{\pi}{3}\right)$$

is not a point of the cardioid $r = 6(1 - \sin\theta)$.

For answer choice B, we'll substitute $\pi/6$ for θ, which gives

$$r = 6\left[1 - \sin\left(\frac{\pi}{6}\right)\right]$$

Now

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

so

$$r = 6\left(1 - \frac{1}{2}\right) = 6 - 3 = 3 \neq 6 - 2\sqrt{2}$$

Thus the point with polar coordinates

$$\left(6 - 2\sqrt{2}, \frac{\pi}{6}\right)$$

is not a point of the cardioid $r = 6(1 - \sin\theta)$.

For answer choice D, we'll substitute $-\pi/4$ for θ, which gives

$$r = 6\left[1 - \sin\left(-\frac{\pi}{4}\right)\right]$$

Now

$$\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

so

$$r = 6\left[1 - \left(-\frac{\sqrt{2}}{2}\right)\right] = 6\left(1 + \frac{\sqrt{2}}{2}\right) = 6 + 3\sqrt{2} \neq 6 + 2\sqrt{2}$$

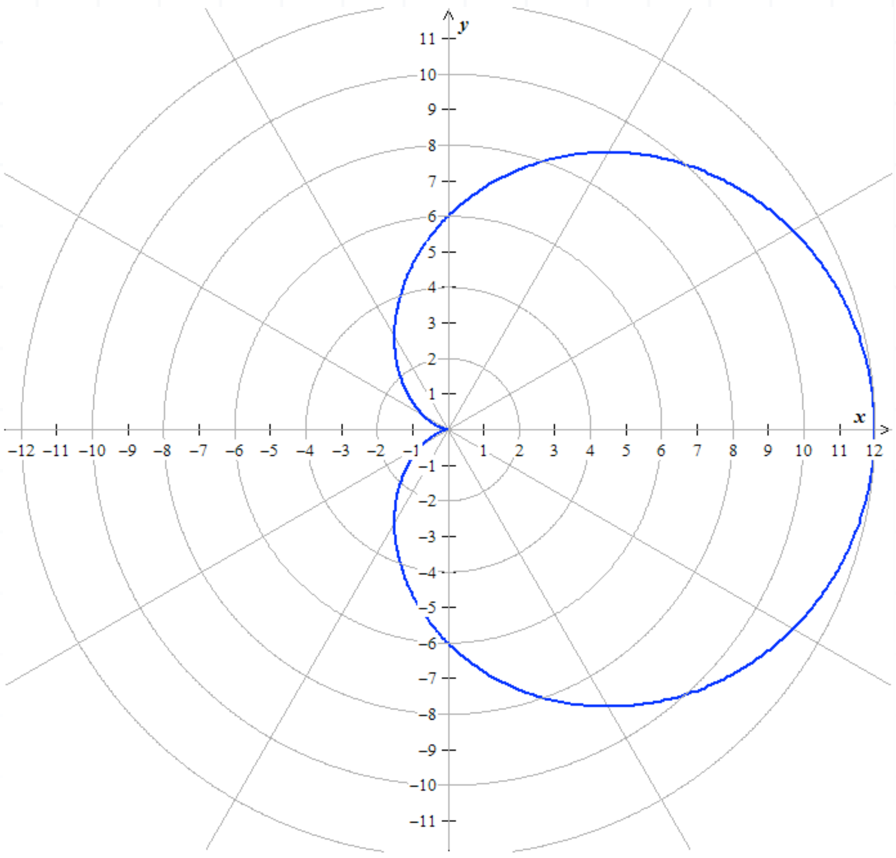
This shows that the point with polar coordinates

$$\left(6 + 2\sqrt{2}, -\frac{\pi}{4}\right)$$

is not a point of the cardioid $r = 6(1 - \sin\theta)$.

Topic: Graph the polar curve, cardioid

Question: The following curve is the graph of one of the polar equations given below. Which polar equation is it?



Answer choices:

A $r = 6(1 + \cos\theta)$

B $r = 12(1 + \cos\theta)$

C $r = 6(1 - \sin\theta)$

D $r = 12(1 + \sin\theta)$

Solution: A

This curve is a cardioid that's symmetric with respect to the horizontal axis, so it's a “cosine cardioid”. This means that we can eliminate answer choices C and D.

The point of the given curve which is furthest from the pole is at a distance of 12 units from the pole and has polar coordinates $(r, \theta) = (12,0)$. Now

$$\theta = 0 \Longrightarrow r = 6(1 + \cos\theta) = 6(1 + 1) = 6(2) = 12$$

and

$$\theta = 0 \Longrightarrow r = 12(1 + \cos\theta) = 12(1 + 1) = 12(2) = 24 \neq 12$$

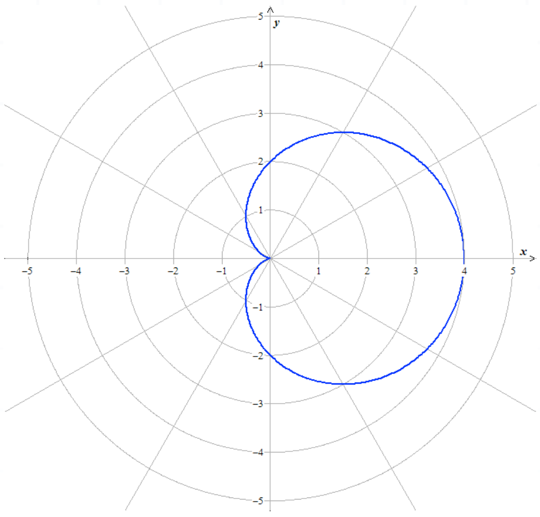
Thus answer choice A is correct.

Topic: Graph the polar curve, cardioid

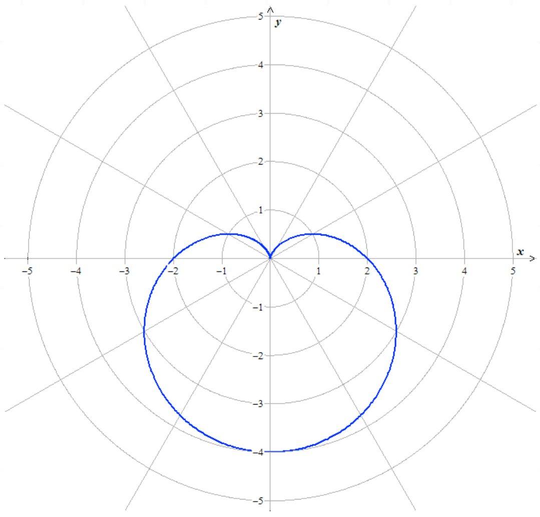
Question: Which of the following is the graph of the cardioid?

$$r = 4(1 + \sin\theta)$$

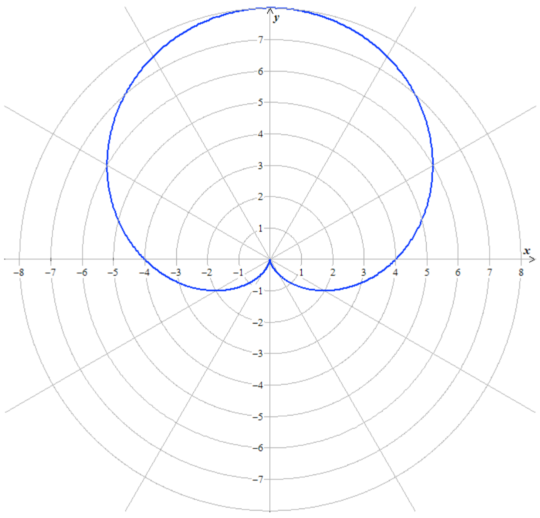
Answer choices:



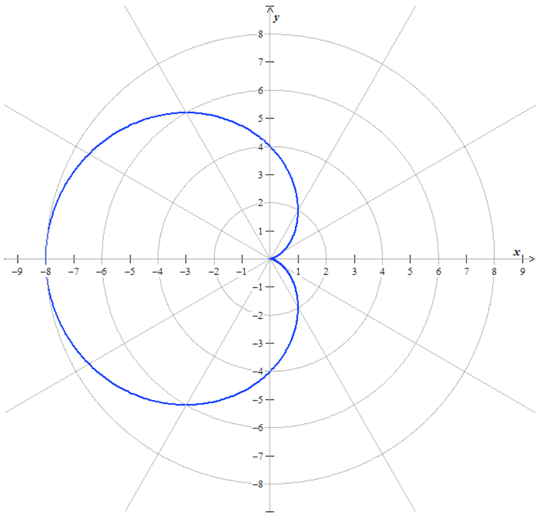
A



C



B



D

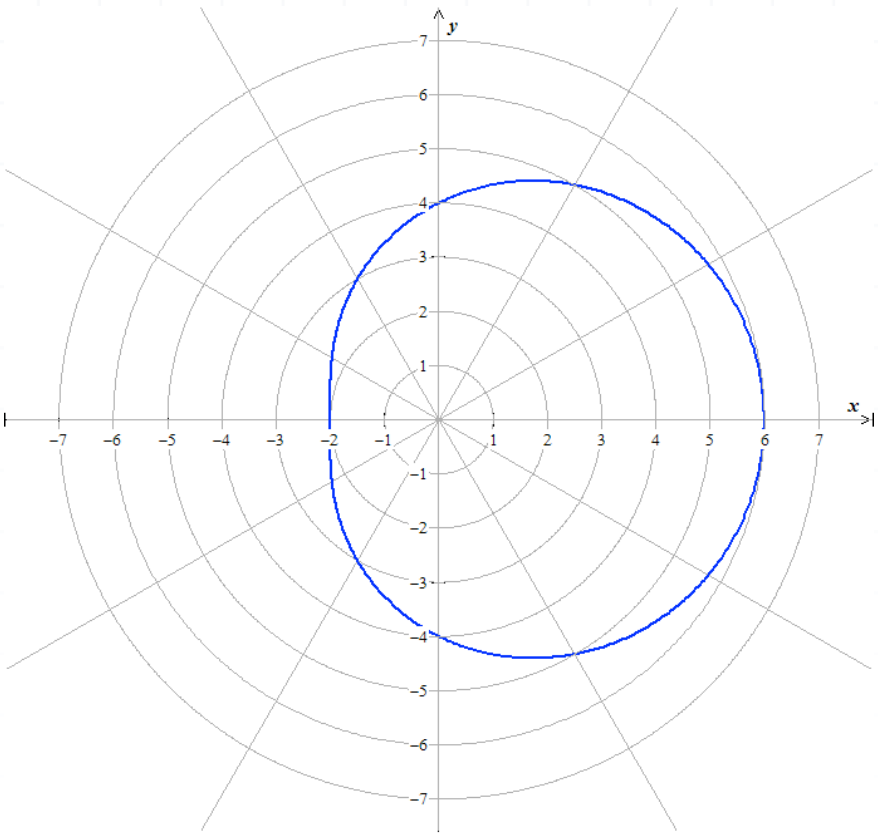
Solution: B

The equation $r = 4(1 + \sin\theta)$ is a “sine cardioid”, so its graph is symmetric with respect to the vertical axis. Therefore, we can eliminate answer choices A and D, because those curves are symmetric with respect to the horizontal axis.

The point of the cardioid $r = 4(1 + \sin\theta)$ which is furthest from the pole is $8 = 2(4)$ units away from the pole. The curve in answer choice C doesn't include a point which is 8 units away from the pole, so B is the correct answer.

Topic: Graph the polar curve, limacon

Question: The following curve is the graph of one of the polar equations given below. Which polar equation is it?



Answer choices:

A $\quad r = 2 - 4\cos\theta$

B $\quad r = 5 + \sin\theta$

C $\quad r = 1 - 4\sin\theta$

D $\quad r = 4 + 2\cos\theta$

Solution: D

This curve is symmetric with respect to the horizontal axis, so it's the graph of a “cosine” limacon (not a “sine” limacon). Therefore, we can eliminate answer choices B and C.

Also, the curve has no loop, so it's a limacon that satisfies either a polar equation $r = a + b\cos\theta$ or a polar equation $r = a - b\cos\theta$ for some positive numbers a and b with $a > b$. Thus we can eliminate answer choice A, since in that polar equation $a = 2$ and $b = 4$, so $a < b$.

The only answer choice that's left is D. We can verify that this is correct by determining the values of $r = 4 + 2\cos\theta$ for $\theta = 0$ and $\theta = \pi$:

$$\theta = 0 \Longrightarrow r = 4 + 2(\cos 0) = 4 + 2(1) = 6$$

$$\theta = \pi \Longrightarrow r = 4 + 2(\cos\pi) = 4 + 2(-1) = 2$$

Inspection of the given curve shows that the point with polar coordinates $(6,0)$ and the point with polar coordinates $(2,\pi)$ are both on it.

Topic: Graph the polar curve, limacon

Question: Which of the following are the angles θ_1, θ_2 in the interval $[0,2\pi)$ such that $\theta_1 < \theta_2$ and (θ_1, θ_2) is the subinterval of $[0,2\pi)$ on which the value of r in the polar equation $r = 1 + 2\sin\theta$ is negative?

Answer choices:

A $\theta_1 = \dfrac{\pi}{6}$ and $\theta_2 = \dfrac{5\pi}{6}$

B $\theta_1 = \dfrac{7\pi}{6}$ and $\theta_2 = -\dfrac{\pi}{6}$

C $\theta_1 = \dfrac{7\pi}{6}$ and $\theta_2 = \dfrac{11\pi}{6}$

D $\theta_1 = \dfrac{5\pi}{6}$ and $\theta_2 = \dfrac{7\pi}{6}$

Solution: C

Well, $r = 1 + 2\sin\theta$ is the polar equation of the limacon $r = a + b\cos\theta$ with $a = 1$ and $b = 2$. Since $a < b$, this limacon passes through the pole twice and has a loop. Thus the subinterval of $[0,2\pi)$ on which $r = 1 + 2\sin\theta$ is negative is (θ_1, θ_2), where θ_1 and θ_2 are the angles in the interval $[0,2\pi)$ at which $r = 0$ and $\theta_1 < \theta_2$. Now

$$r = 0 \Longrightarrow 1 + 2\sin\theta = 0 \Longrightarrow 2\sin\theta = -1 \Longrightarrow \sin\theta = -\frac{1}{2}$$

Recall that the sine function is negative in the third and fourth quadrants. Thus θ_1 (the smaller of the two angles at which $r = 0$) is in the third quadrant, and θ_2 is in the fourth quadrant.

What we need to do is determine the values of θ_1 and θ_2.

Let's first recall that

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

By the sum identity for sine,

$$\sin(\theta + \pi) = (\sin\theta)(\cos\pi) + (\cos\theta)(\sin\pi)$$

$$\sin(\theta + \pi) = \sin\theta(-1) + \cos\theta(0)$$

$$\sin(\theta + \pi) = -\sin\theta$$

Since the angle of measure $\pi/6$ is in the first quadrant, the angle of measure

$$\frac{\pi}{6}+\pi=\frac{7\pi}{6}$$

is in the third quadrant. Therefore,

$$\sin\left(\frac{7\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

Note that $7\pi/6$ is in the interval $[0,2\pi)$.

Also, by the odd identity for sine,

$$\sin(-\theta)=-\sin\theta$$

Thus

$$\sin\left(-\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$$

Since the angle of measure $\pi/6$ is in the first quadrant, the angle of measure $-\pi/6$ is in the fourth quadrant, but $-\pi/6$ isn't in the interval $[0,2\pi)$. However, the sine of any angle which differs in measure from $-\pi/6$ by an integer multiple of 2π is also equal to $-1/2$. One such angle is

$$-\frac{\pi}{6}+2\pi\left(=\frac{11\pi}{6}\right)$$

Note that $11\pi/6$ is in the interval $[0,2\pi)$.

What we have found is that

$$\theta_1=\frac{7\pi}{6} \text{ and } \theta_2=\frac{11\pi}{6}$$

Since the sine function is negative in the third and fourth quadrants, it is negative on the interval

$$\left(\frac{7\pi}{6},\frac{11\pi}{6}\right)$$

Moreover, the value of the sine function is less than $-1/2$ on that interval. That is,

$$\theta_1 < \theta < \theta_2 \Longrightarrow \sin\theta < \sin\theta_1 = -\frac{1}{2}$$

From this it follows that

$$r = 1 + 2\sin\theta < 1 + 2\left(-\frac{1}{2}\right) = 1 - 1 = 0$$

so the value of $r = 1 + 2\sin\theta$ is negative on the interval (θ_1, θ_2).

Note that for any angle θ in either the interval $[0,7\pi/6)$ or the interval $(11\pi/6,2\pi)$,

$$\sin\theta > \sin\theta_1 = -\frac{1}{2}$$

so the value of $r = 1 + 2\sin\theta$ in either of these intervals is

$$r = 1 + 2\sin\theta > 1 + 2\left(-\frac{1}{2}\right) = 1 - 1 = 0$$

What we have shown is that the angles θ in the interval $[0,2\pi)$ at which the value of $r = 1 + 2\sin\theta$ is negative are those in the interval

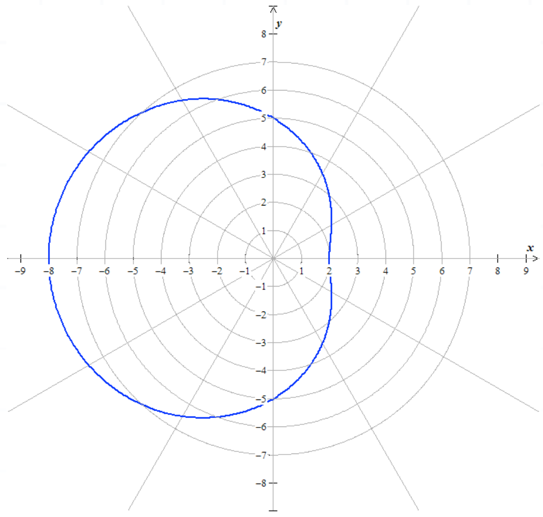
$$(\theta_1, \theta_2) = \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$$

Topic: Graph the polar curve, limacon

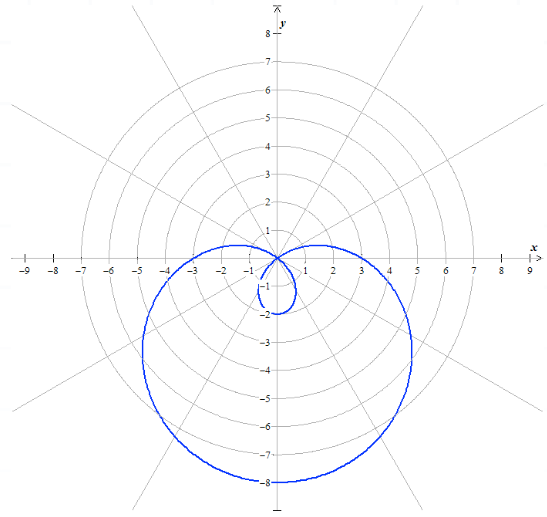
Question: Which of the following curves is the graph of the limacon?

$$r = 3 - 5\cos\theta$$

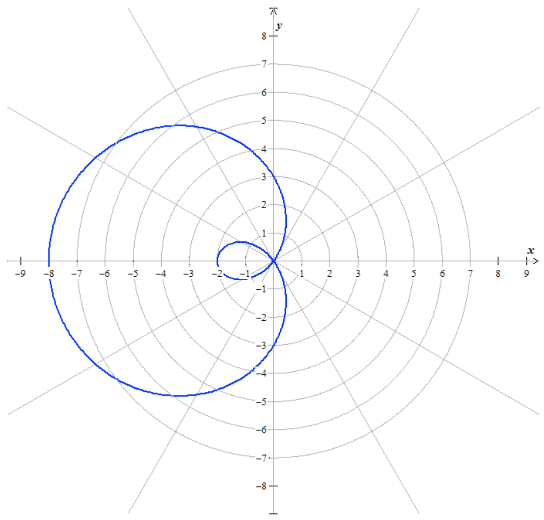
Answer choices:



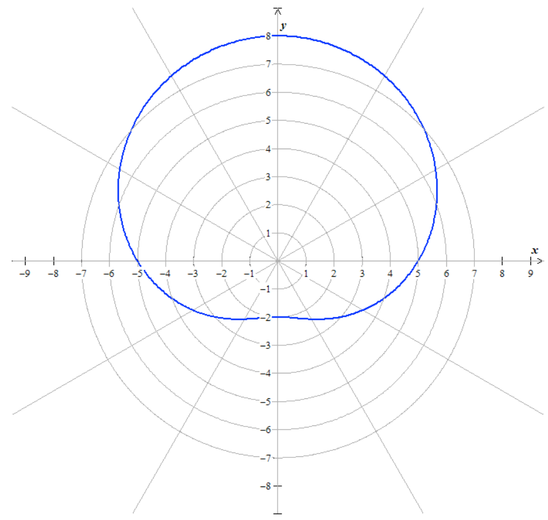
A



C



B



D

Solution: B

The equation $r = 3 - 5\cos\theta$ is a "cosine cardioid", so its graph is symmetric with respect to the horizontal axis. Therefore, we can eliminate answer choices C and D, because the curves given in those answer choices are symmetric with respect to the vertical axis.

Moreover, the equation $r = 3 - 5\cos\theta$ is in the form $a - b\cos\theta$ where $a = 3$ and $b = 5$. Thus $a < b$, so this limacon has a loop. This enables us to eliminate answer choice A, because the curve given in that answer choice has a depression, not a loop.

To check that the curve given in answer choice B is indeed the graph of the equation $r = 3 - 5\cos\theta$, we'll evaluate $r = 3 - 5\cos\theta$ at $\theta = 0$ and $\theta = \pi$:

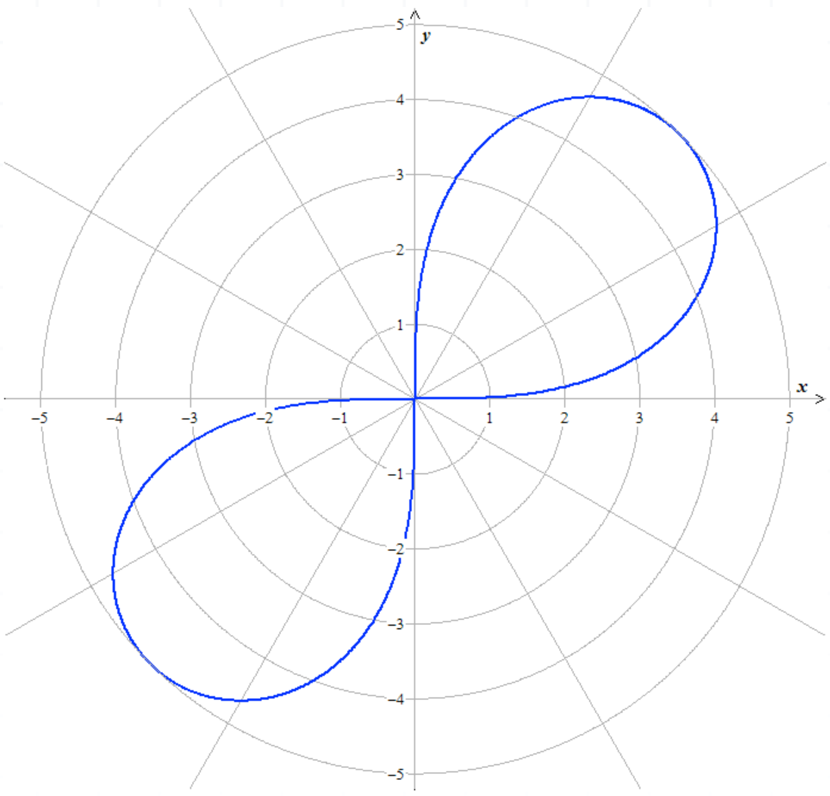
$$\theta = 0 \implies r = 3 - 5(\cos 0) = 3 - 5(1) = -2$$

$$\theta = \pi \implies r = 3 - 5(\cos\pi) = 3 - 5(-1) = 8$$

This would mean that the given curve includes the point with polar coordinates $(-2,0)$ and the point with polar coordinates $(8,\pi)$. Inspection of the curve tells us that the point with polar coordinates $(8,\pi)$ is definitely on it. Note that the point with polar coordinates $(-2,0)$ also has polar coordinates $(2,\pi)$. Clearly, this point is also on the given curve. (In fact, it's the leftmost point of the loop.)

Topic: Graph the polar curve, lemniscate

Question: The following curve is the graph of one of the polar equations given below. Which polar equation is it?



Answer choices:

A $\quad r^2 = 5\cos(2\theta)$

B $\quad r^2 = 25\sin(2\theta)$

C $\quad r^2 = 5\sin(2\theta)$

D $\quad r^2 = 25\cos(2\theta)$

Solution: B

The given curve is symmetric with respect to the pole, but it isn't symmetric with respect to the horizontal axis or the vertical axis, so it's the graph of a "sine" lemniscate. Therefore, we can eliminate answer choices A and B, because they're the polar equations of "cosine" lemniscates, which are symmetric with respect to the horizontal axis and the vertical axis.

Also, the points on this curve that are furthest from the pole are at a distance of 5 units from it. Since $r^2 = 25\sin(2\theta)$ is the polar equation of a "sine" lemniscate with $a = 25$, we know that the points of that lemniscate that are furthest from the pole are a distance of

$$\sqrt{a} = \sqrt{25} = 5$$

units from it. This is indeed true of the given curve.

In the case of the "sine" lemniscate which is the graph of the polar equation given in answer choice C, that is, $r^2 = 5\sin(2\theta)$, we see that $a = 5$, so the points of that lemniscate that are furthest from the pole are at a distance of only

$$\sqrt{a} = \sqrt{5}$$

units from it, which is inconsistent with the given curve.

Topic: Graph the polar curve, lemniscate

Question: One of the following pairs of polar coordinates corresponds to a point of the lemniscate $r^2 = 8\cos(2\theta)$. Which pair of polar coordinates is it?

Answer choices:

A $\left(4, \frac{\pi}{4}\right)$ and $\left(-4, \frac{5\pi}{4}\right)$

B $\left(4, \frac{2\pi}{3}\right)$ and $\left(-4, \frac{5\pi}{3}\right)$

C $\left(2, \frac{5\pi}{6}\right)$ and $\left(-2, \frac{11\pi}{6}\right)$

D $\left(\frac{4}{\sqrt{3}}, \frac{\pi}{8}\right)$ and $\left(-\frac{4}{\sqrt{3}}, \frac{9\pi}{8}\right)$

Solution: C

In each pair of polar coordinates, the first coordinate in the first pair is equal in absolute value (but opposite in sign) to the first coordinate in the second pair. Since the first coordinate is a value of r, the two pairs have the same value of r^2. Also, the angle coordinate in the second pair differs from the angle coordinate in the first pair by π. Thus in each answer choice, the two pairs of polar coordinates apply to the very same point.

What remains is to determine the answer choice that contains a pair of polar coordinates for some point of the lemniscate $r^2 = 8\cos(2\theta)$. Let's look at each answer choice in turn.

In answer choice A, we have $r^2 = 16$, so the angle coordinate θ in one of the pairs of polar coordinates would have to satisfy

$$16 = 8\cos(2\theta)$$

Dividing both sides of this equation by 8, we have

$$2 = \cos(2\theta)$$

This equation has no solution, because the cosine of any angle cannot be greater than 1.

In answer choice B, we also have $r^2 = 16$, and we just found that there is no angle θ that satisfies the equation

$$16 = 8\cos(2\theta)$$

In answer choice C, we have $r^2 = 4$, so the angle coordinate θ in one of the pairs of polar coordinates would have to satisfy

$$4 = 8\cos(2\theta)$$

Dividing both sides of this equation by 8 gives

$$\frac{1}{2} = \cos(2\theta)$$

Let's try the angle coordinate θ in the first pair of polar coordinates (i.e., $\theta = 5\pi/6$):

$$\cos(2\theta) = \cos\left(2\left(\frac{5\pi}{6}\right)\right) = \cos\left(\frac{5\pi}{3}\right)$$

Well,

$$\cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{6\pi - \pi}{3}\right) = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right)$$

By the even identity for cosine,

$$\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

so answer choice C is correct.

To be on the safe side, we'll check the pairs of polar coordinates given in answer choice D. In that case, $r^2 = 16/3$, so we need to see if the angle coordinate θ in either pair satisfies the equation

$$\frac{16}{3} = 8\cos(2\theta)$$

Dividing both sides by 8, we have

$$\frac{2}{3} = \cos(2\theta)$$

Let's try the angle coordinate θ in the first pair of polar coordinates (i.e., $\theta = \pi/8$):

$$\cos(2\theta) = \cos\left(2\left(\frac{\pi}{8}\right)\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \neq \frac{2}{3}$$

Clearly, that doesn't work. The angle coordinate θ in the second pair is $\theta = 9\pi/8$, so

$$\cos(2\theta) = \cos\left(2\left(\frac{9\pi}{8}\right)\right) = \cos\left(\frac{9\pi}{4}\right)$$

Well,

$$\cos\left(\frac{9\pi}{4}\right) = \cos\left(\frac{8\pi + \pi}{4}\right) = \cos\left(2\pi + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \neq \frac{2}{3}$$

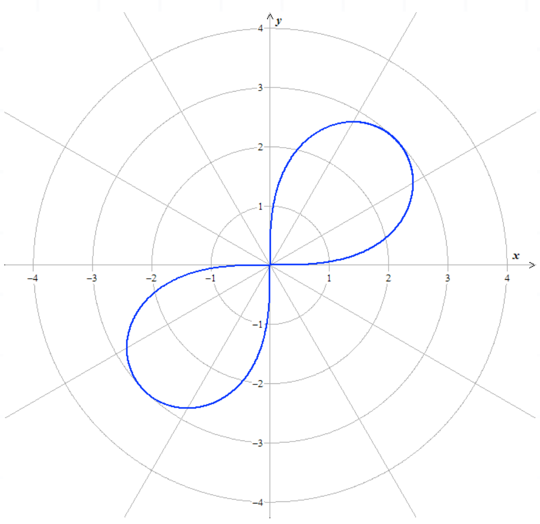
This shows that answer choice D is definitely incorrect.

Topic: Graph the polar curve, lemniscate

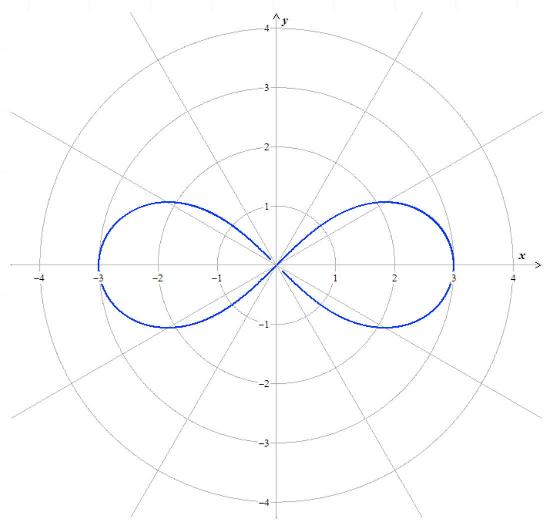
Question: Which of the following curves is the graph of the lemniscate?

$$r^2 = 9\sin(2\theta)$$

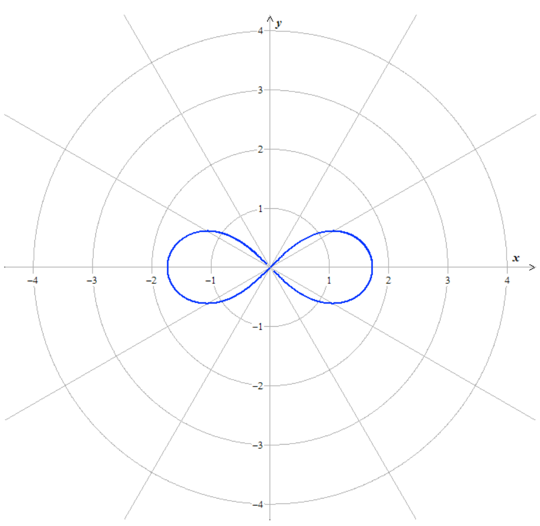
Answer choices:



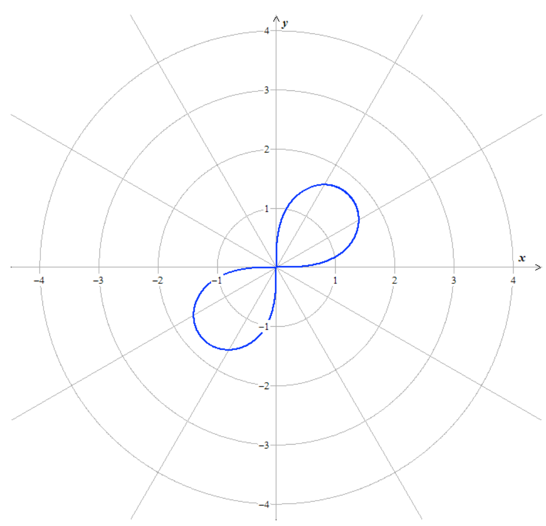
A



C



B



D

Solution: A

The equation $r^2 = 9\sin(2\theta)$ is that of a lemniscate with $a = 9$. Thus the points of this lemniscate that are furthest from the pole are at a distance of

$$\sqrt{a} = \sqrt{9} = 3$$

units from it. The only answer choices that show a curve with this property are answer choices A and C.

The curve shown in answer choice C is a “cosine” lemniscate, because it's symmetric with respect to both the horizontal axis and the vertical axis, whereas a “sine” lemniscate (the kind we're looking for) isn't symmetric with respect to either of those axes. Thus answer choice C is incorrect.

The curve shown in answer choice A is indeed correct, because the curve is symmetric with respect to the pole (but it isn't symmetric with respect to the horizontal axis or the vertical axis), hence it is a “sine” lemniscate. Also, the points on that curve that are furthest from the pole are at a distance of 3 units from it.

Topic: Intersection points of the polar curves

Question: Which of the following is a set of pairs of polar coordinates of all the points of intersection of the graphs of the polar equations $r = 2\cos\theta$ and $r = 3 - 4\cos\theta$?

Answer choices:

A $\left\{\left(\sqrt{3}, \frac{5\pi}{6}\right), (0,0), \left(\sqrt{3}, \frac{7\pi}{6}\right)\right\}$

B $\left\{\left(5, \frac{2\pi}{3}\right), (0,0), \left(5, \frac{4\pi}{3}\right)\right\}$

C $\left\{\left(1, \frac{\pi}{3}\right), (0,0), \left(1, \frac{5\pi}{3}\right)\right\}$

D $\left\{\left(3 - 2\sqrt{3}, \frac{\pi}{6}\right), (0,0), \left(3 - 2\sqrt{3}, \frac{11\pi}{6}\right)\right\}$

Solution: C

The graph of the polar equation $r = 2\cos\theta$ is a circle, and the graph of the polar equation $r = 3 - 4\cos\theta$ is a limacon (one with $a = 3$ and $b = 4$, so $a < b$ and this limacon has a loop).

Equating the expressions for r in the two polar equations gives

$$2\cos\theta = 3 - 4\cos\theta$$

$$6\cos\theta = 3$$

Dividing both sides by 6 gives

$$\cos\theta = \frac{1}{2}$$

The angles θ in the interval $[0,2\pi)$ that have a cosine of $1/2$ are $\pi/3$ and $5\pi/3$. For these two angles, the polar equation of the circle ($r = 2\cos\theta$) gives us the following values of r:

$$\theta = \frac{\pi}{3} \Longrightarrow r = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

$$\theta = \frac{5\pi}{3} \Longrightarrow r = 2\cos\left(\frac{5\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

The polar equation of the limacon $r = 3 - 4\cos\theta$ gives these same values of r for those two angles:

$$\theta = \frac{\pi}{3} \Longrightarrow r = 3 - 4\cos\left(\frac{\pi}{3}\right) = 3 - 4\left(\frac{1}{2}\right) = 3 - 2 = 1$$

$$\theta = \frac{5\pi}{3} \Longrightarrow r = 3 - 4\cos\left(\frac{5\pi}{3}\right) = 3 - 4\left(\frac{1}{2}\right) = 3 - 2 = 1$$

Thus the points with polar coordinates

$$(r, \theta) = \left(1, \frac{\pi}{3}\right) \quad \text{and} \quad (r, \theta) = \left(1, \frac{5\pi}{3}\right)$$

are points of intersection of the circle and the limacon. As always, we want to check to see if the pole (the origin) is a point of intersection.

We know a polar curve will pass through the origin when $r = 0$, so we'll set $r = 0$ in each curve. The circle $r = 2\cos\theta$ goes through the origin at

$$0 = 2\cos\theta$$

$$\cos\theta = 0$$

We know the cosine function is equal to 0 at $\pi/2$ and $3\pi/2$, so the circle $r = 2\cos\theta$ definitely passes through the origin. The limacon $r = 3 - 4\cos\theta$ goes through the origin at

$$0 = 3 - 4\cos\theta$$

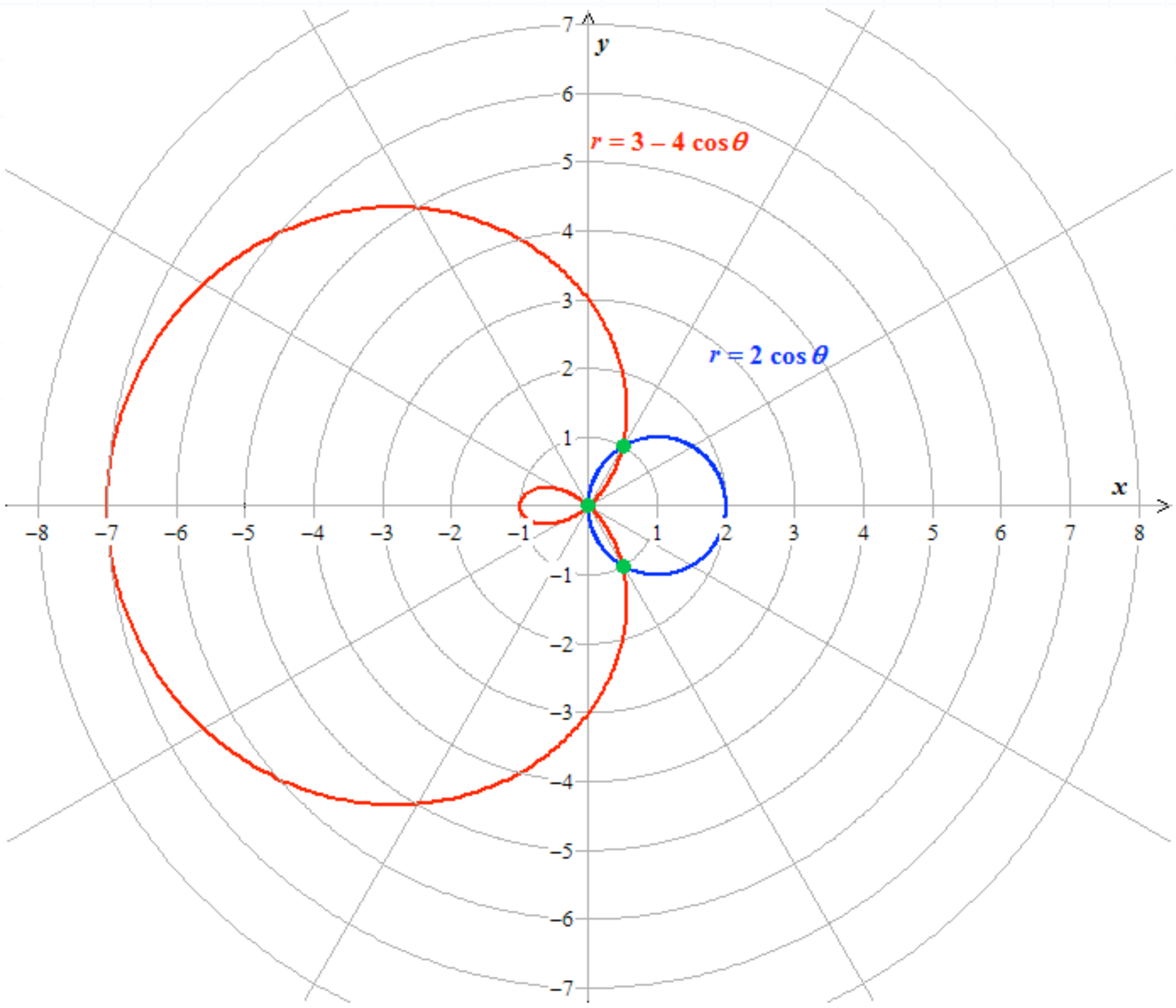
$$4\cos\theta = 3$$

$$\cos\theta = \frac{3}{4}$$

Because the values of the cosine function oscillate back and forth between 0 and 1, we know the value of the cosine function will be equal to $3/4$ at some point. We said that the limacon $r = 3 - 4\cos\theta$ will pass through the origin when $\cos\theta = 3/4$, and since we know that $\cos\theta$ will equal $3/4$ at some point, we know that the limacon also passes through the origin.

Since both curves pass through the origin, we can say that $(0,0)$ is also an intersection point.

We have found three points of intersection. To be sure we haven't missed any points of intersection, we'll graph the two polar equations.



From the graph, we see that there are indeed only three points of intersection.

Topic: Intersection points of the polar curves

Question: At how many points does the graph of the polar equation $r = 5\sin(3\theta)$ intersect the graph of the polar equation $r = 4$?

Answer choices:

A One

B Three

C Six

D Eight

Solution: C

The graph of the polar equation $r = 5\sin(3\theta)$ is a three-petal rose, and the graph of the polar equation $r = 4$ is the circle of radius 4 that has its center at the pole.

The points of the rose that are furthest from the pole are the tips of the petals, each of which is at a distance of 5 units from the pole. Also, every petal of the rose intersects the pole, hence every petal contains a point which is at a distance of 0 units from the pole.

Each "edge" of every petal of the rose $5\sin(3\theta)$ is a continuous curve, so for every real number x such that $0 < x < 5$, each edge of every petal of this rose contains a point which is at a distance of x units from the pole. This assures us that each edge of every petal of this rose contains a point which is at a distance of 4 units from the pole, hence a point which is on the circle $r = 4$.

Since the rose $r = 5\sin(3\theta)$ has three petals, and each petal has two edges, the number of points of intersection of the rose and the circle is $3 \times 2 = 6$.

To locate the points of intersection, we can equate the expressions for r in the polar equations of the two curves:

$$5\sin(3\theta) = 4$$

Dividing both sides of the equation $5\sin(3\theta) = 4$ by 5 gives

$$\sin(3\theta) = \frac{4}{5}$$

Since

$$0 < \frac{4}{5} < 1$$

there are points that satisfy this equation, and (because the value of $\sin(3\theta)$ is positive) all the angles 3θ that have a sine of $4/5$ must be in either the first quadrant or the fourth quadrant.

One such point has an angle coordinate θ such that 3θ is in the interval $(0,\pi/2)$, namely,

$$3\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

where $\sin^{-1}$ denotes the inverse sine function. Thus one point of intersection of the rose and the circle is

$$(r,\theta) = \left(4, \frac{\sin^{-1}\left(\frac{4}{5}\right)}{3}\right)$$

By the reference angle theorem, $\sin(\pi - 3\theta) = \sin(3\theta)$, so there is an angle θ such that 3θ is in the interval $(\pi/2,\pi)$ and $\sin(3\theta) = 4/5$:

$$3\theta = \pi - \sin^{-1}\left(\frac{4}{5}\right)$$

Thus another point of intersection is

$$(r,\theta) = \left(4, \frac{\pi - \sin^{-1}\left(\frac{4}{5}\right)}{3}\right)$$

The other four points of intersection correspond to angles θ with the following values of 3θ:

$$3\theta = \sin^{-1}\left(\frac{4}{5}\right) + 2\pi$$

$$3\theta = \left(\pi - \sin^{-1}\left(\frac{4}{5}\right)\right) + 2\pi$$

$$3\theta = \sin^{-1}\left(\frac{4}{5}\right) + 4\pi$$

$$3\theta = \left(\pi - \sin^{-1}\left(\frac{4}{5}\right)\right) + 4\pi$$

Thus these four points are

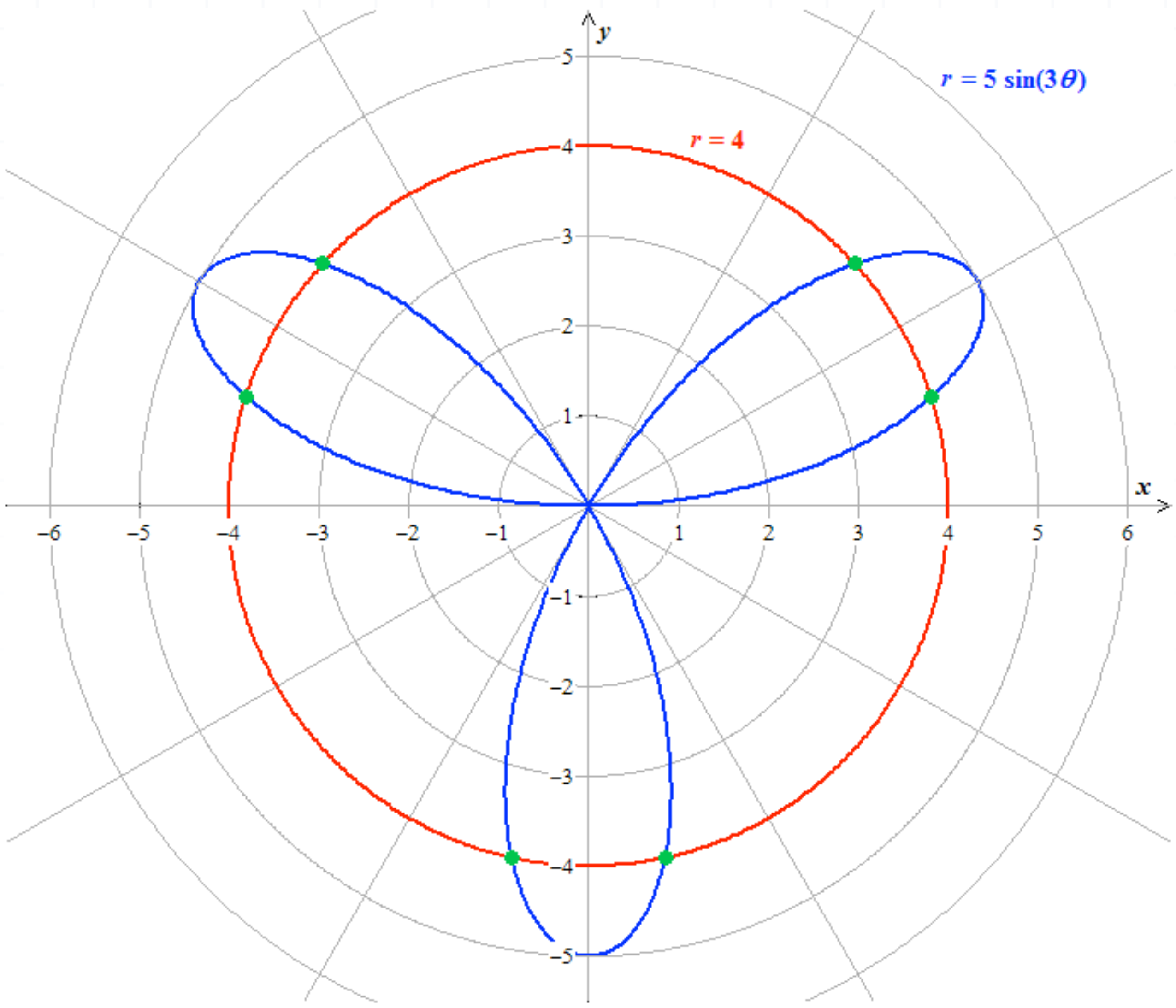
$$(r, \theta) = \left(4, \frac{\sin^{-1}\left(\frac{4}{5}\right) + 2\pi}{3}\right)$$

$$(r, \theta) = \left(4, \frac{3\pi - \sin^{-1}\left(\frac{4}{5}\right)}{3}\right)$$

$$(r, \theta) = \left(4, \frac{\sin^{-1}\left(\frac{4}{5}\right) + 4\pi}{3}\right)$$

$$(r,\theta)=\left(4,\frac{5\pi-\sin^{-1}\left(\frac{4}{5}\right)}{3}\right)$$

Now let's look at the graphs of these polar equations to confirm the points of intersection we've found and make sure that there are no others.



Topic: Intersection points of the polar curves

Question: At how many points do the graphs of the polar equations and intersect each other?

$$r = 2\sin(2\theta)$$

$$r^2 = 9\sin(2\theta)$$

Answer choices:

A Zero

B One

C Two

D Three

Solution: B

The graph of the polar equation $r = 2\sin(2\theta)$ is a four-petal rose, and the graph of the polar equation $r^2 = 9\sin(2\theta)$ is a lemniscate. As we know, every lemniscate passes through the pole, as does every rose, so the pole is a point of intersection of these two curves. Thus what we need to show is that they have no other points of intersection.

In this case, we can't directly equate the expressions for r, because the polar equation of the rose gives us an expression for r but the polar equation of the lemniscate gives us an expression for r^2. However, we can square both sides of the polar equation for the rose to get an expression for r^2:

$$r = 2\sin(2\theta) \Longrightarrow r^2 = 4\sin^2(2\theta)$$

Now we can equate the two expressions for r^2:

$$4\sin^2(2\theta) = 9\sin(2\theta)$$

$$4\sin^2(2\theta) - 9\sin(2\theta) = 0$$

Factoring out $\sin(2\theta)$ on the left-hand side, we obtain

$$\sin(2\theta)\left[4\sin(2\theta) - 9\right] = 0$$

Thus every solution of this equation would have to satisfy either $\sin(2\theta) = 0$ or $4\sin(2\theta) - 9 = 0$.

First, we'll consider the equation $\sin(2\theta) = 0$. Evaluating the expression for r^2 for the lemniscate, we obtain

$$\sin(2\theta) = 0 \implies r^2 = 9\sin(2\theta) = 9(0) = 0 \implies r = 0$$

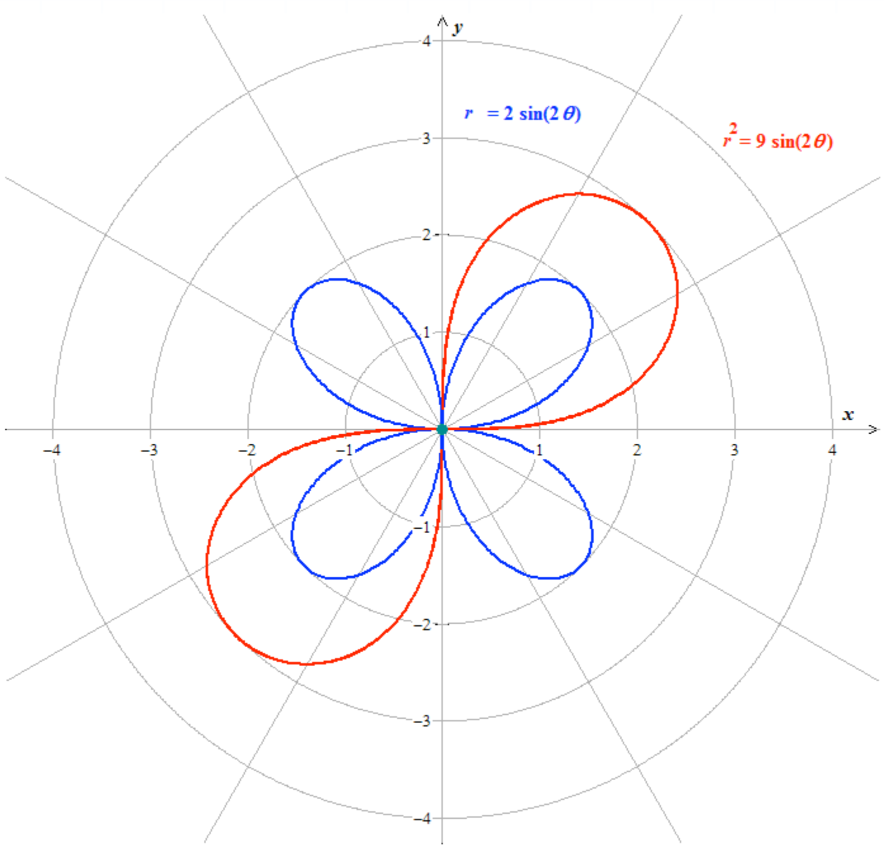
And evaluating the expression for r for the rose, we find that

$$\sin(2\theta) = 0 \implies r = 2\sin(2\theta) = 2(0) = 0$$

In both cases, we find that $r = 0$, so this is just the pole, which we have already stated to be a point of intersection of the two curves.

Next, we'll consider the equation $4\sin(2\theta) - 9 = 0$; equivalently, $\sin(2\theta) = 9/4$. Note that this equation has no solutions, because $9/4 > 1$, hence $\sin(2\theta)$ cannot be equal to $9/4$. (In fact, there is no angle whose sine is equal to $9/4$.)

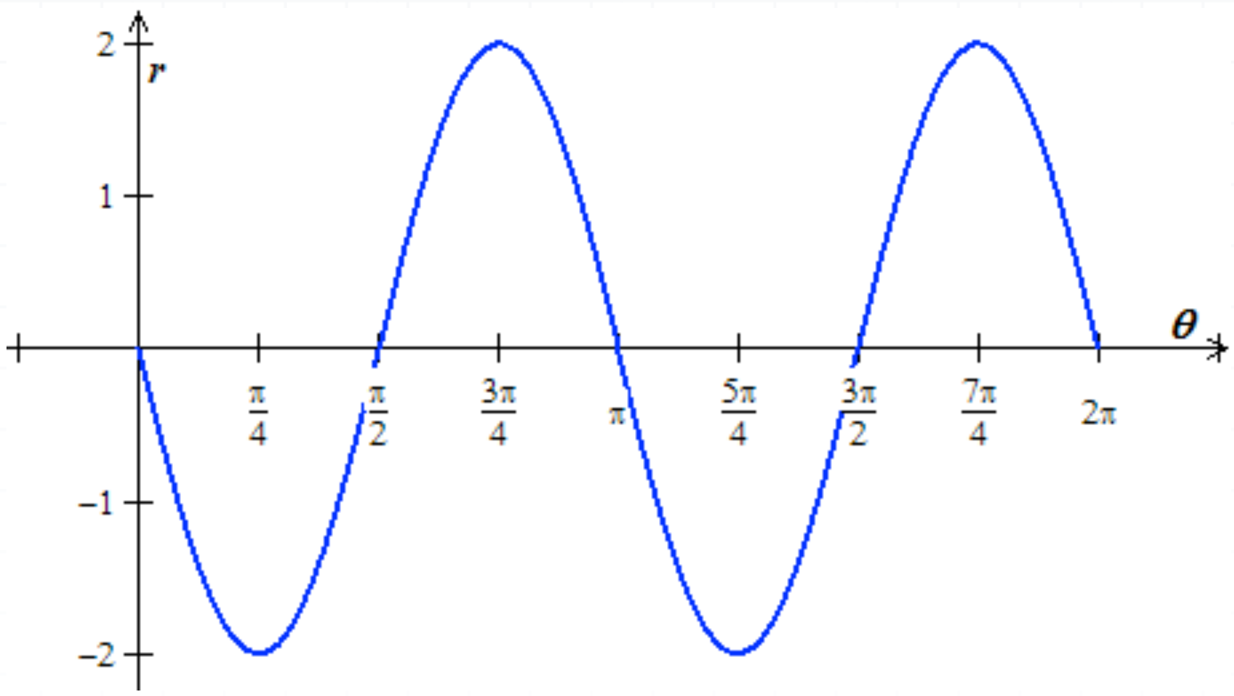
What we have found is that the graphs of the polar equations $r = 2\sin(2\theta)$ and $r^2 = 9\sin(2\theta)$ have just one point of intersection, namely the pole. To check this, we'll take a look at the graphs of those equations.



Sure enough, the pole is indeed the only point of intersection.

Topic: Graph the polar region in the xy-plane

Question: Which polar equation is it the graph of?



Answer choices:

A $r = 1 + 2\sin\theta$

B $r = \cos(2\theta)$

C $r = 2 + \cos\theta$

D $r = -2\sin(2\theta)$

Solution: D

Let's see if we can eliminate any of the answer choices.

First, notice that on the given rectangular graph the value of r at $\theta = 0$ is 0.

That tells us that answer choice A can't be correct, because if the given graph were the rectangular graph of the polar equation $r = 1 + 2\sin\theta$, then at $\theta = 0$ we would get

$$r = 1 + 2(\sin 0) = 1 + 2(0) = 1 + 0 = 1 \neq 0$$

The same thing applies to answer choice B, because there the value of r at $\theta = 0$ is

$$r = \cos(2(0)) = \cos(0) = 1 \neq 0$$

For the polar equation in answer choice C, we get the following at $\theta = 0$:

$$r = 2 + \cos(0) = 2 + 1 = 3 \neq 0$$

Thus the only answer choice that could be correct is D. Notice that if $r = -2\sin(2\theta)$, then at $\theta = 0$ we have

$$r = -2\sin(2(0)) = -2\sin(0) = -2(0) = 0$$

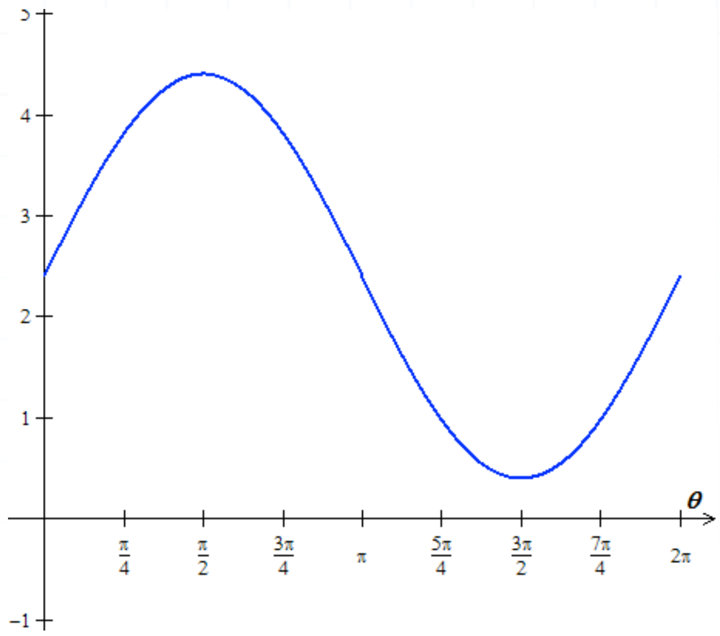
Another way to arrive at the correct answer would be to notice that the size and shape of the part of the given rectangular graph which corresponds to the interval $[\pi, 2\pi]$ are identical to the size and shape of the part which corresponds to the interval $[0, \pi]$, and that the graph on each of those two intervals looks like a complete sine curve. Therefore, it appears that the given rectangular graph is that of a sine function that has period π.

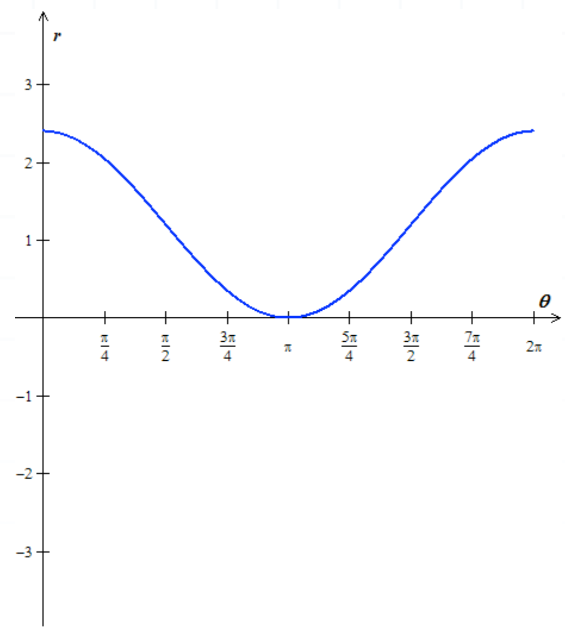
This is the case with the polar equation $r = -2\sin(2\theta)$, because the period of the function $-2\sin(2\theta)$ is $2\pi/b$ (where b is the coefficient of θ in the expression 2θ , which is 2), hence the period is $2\pi/2 = \pi$.

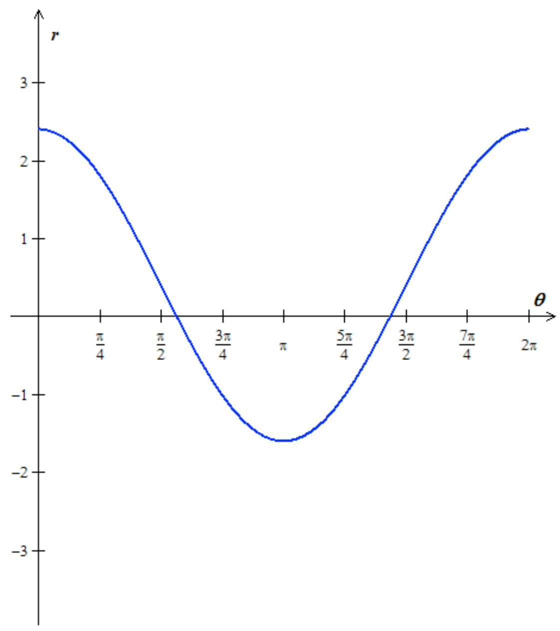
Topic: Graph the polar region in the xy-plane

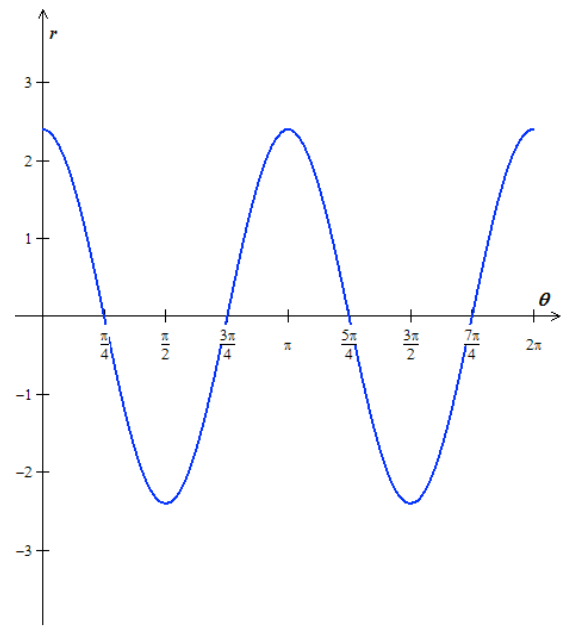
Question: Which of the following is the rectangular graph of the polar equation $r = 1.2(1 + \cos\theta)$ on the interval $[0, 2\pi]$?

Answer choices:

A


B


C


D


Solution: B

From the given polar equation, $r = 1.2(1 + \cos\theta)$, we find that the value of r at $\theta = 0$ is

$$r = 1.2(1 + \cos(0)) = 1.2(1 + 1) = 1.2(2) = 2.4$$

All four of these graphs look as though they would yield $r \approx 2.4$ at $\theta = 0$, so we'll have to consider some other value(s) of θ to eliminate the wrong answer choices.

In the rectangular graph given in answer choice A, the value of r at $\theta = \pi/2$ is greater than 4. From the given polar equation, $r = 1.2(1 + \cos\theta)$, what we find is that the value of r at $\theta = \pi/2$ is

$$r = 1.2\left[1 + \cos\left(\frac{\pi}{2}\right)\right] = 1.2(1 + 0) = 1.2(1) = 1.2$$

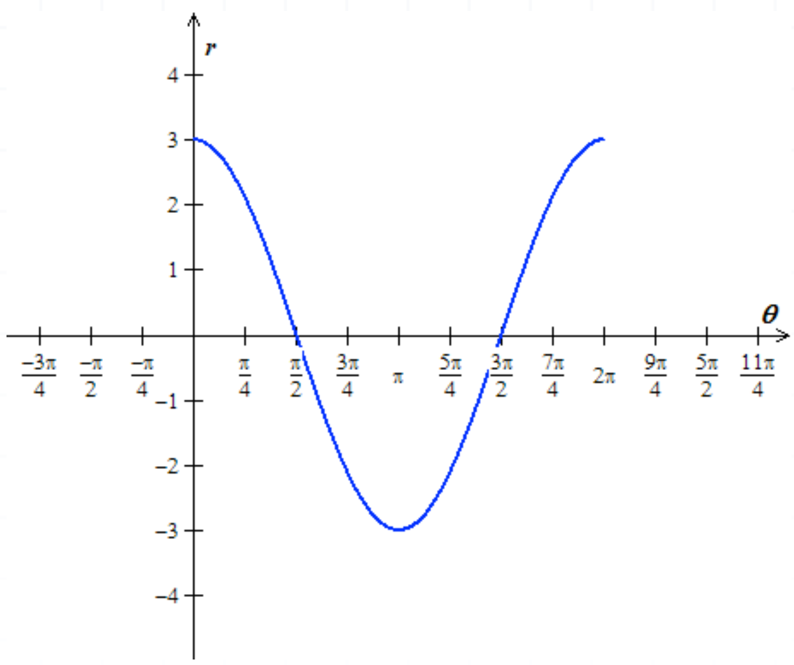
which is well below 4. Thus we can eliminate answer choice A.

In the rectangular graph given in answer choice C, the value of r at $\theta = \pi/2$ is positive but well below 1.2, so we can eliminate answer choice C.

In the rectangular graph given in answer choice D, the value of r at $\theta = \pi/2$ is negative, hence it isn't equal to 1.2, so we can eliminate answer choice D.

Topic: Graph the polar region in the xy-plane

Question: The following graph is the rectangular graph of a certain polar equation on the interval $[0,2\pi]$. Which type of curve would you get if you were to convert this rectangular graph to a polar graph?



Answer choices:

A Cardioid

B Circle

C Limacon

D Rose

Solution: B

Inspection of the given rectangular graph tells us that it's just the graph of the basic cosine function (over one complete period of length 2π) magnified by a factor of 3. Thus it's the rectangular graph of the polar equation $r = 3\cos\theta$ on the interval $[0,2\pi]$. The curve we get when we draw the polar graph of this equation is the circle whose center is on the horizontal axis and located $3/2 = 1.5$ units to the right of the pole.

Topic: Sketching a parametric curve and its orientation

Question: Consider the parametric equations $x = 3t^3 - 2t + 5$ and $y = 4 - 6t^2$ where $-3 \leq t \leq 4$. Which value of t in that interval yields the point with coordinates $(x, y) = (-15, -20)$?

Answer choices:

A $t = 2$

B $t = -3$

C $t = -2$

D $t = 4$

Solution: C

Let's evaluate $x = 3t^3 - 2t + 5$ and $y = 4 - 6t^2$ for each of the answer choices, and see which one corresponds to the point with $(x, y) = (-15, -20)$.

Answer choice A ($t = 2$):

$$x = 3(2^3) - 2(2) + 5$$

$$x = 3(8) - 4 + 5$$

$$x = 24 - 4 + 5$$

$$x = 20 + 5$$

$$x = 25 \neq -15$$

This tells us that answer choice A is incorrect.

Answer choice B ($t = -3$):

$$x = 3((-3)^3) - 2(-3) + 5$$

$$x = 3(-27) + 6 + 5$$

$$x = -81 + 6 + 5$$

$$x = -75 + 5$$

$$x = -70 \neq -15$$

Now we know that answer choice B is incorrect.

Answer choice C ($t = -2$):

$$x = 3((-2)^3) - 2(-2) + 5$$

$$x = 3(-8) + 4 + 5$$

$$x = -24 + 4 + 5$$

$$x = -20 + 5$$

$$x = -15$$

and

$$y = 4 - 6((-2)^2)$$

$$y = 4 - 6(4)$$

$$y = 4 - 24$$

$$y = -20$$

It looks as though answer choice C is correct. However, there's a possibility that there are two different values of t that yield the point with coordinates $(x, y) = (-15, -20)$. In that case, the parametric curve would have a point of self-intersection.

To check that answer choice D ($t = 4$) is incorrect, notice that since $y = 4 - 6t^2$, any two values of t that give us the same value of y have to be "negatives" of each other. In answer choice D, $t = 4$; in answer choice C, $t = -2$. Since 4 and -2 aren't "negatives" of each other, we see that answer choice D is incorrect.

Topic: Sketching a parametric curve and its orientation

Question: Consider the parametric equations $x = 3 + 2\sin t$ and $y = 2 - \cos t$ where t is between $-\pi/2$ and $\pi/2$. Which of the following describes the type of parametric curve that corresponds to these parametric equations?

Answer choices:

A The lower half of an ellipse traced out in the counterclockwise direction

B The upper half of a circle traced out in the clockwise direction

C The right half of an ellipse traced out in the clockwise direction

D The left half of a circle traced out in the counterclockwise direction

Solution: A

Let's solve the equations $x = 3 + 2\sin t$ and $y = 2 - \cos t$ for $\sin t$ and $\cos t$, respectively:

$$x = 3 + 2\sin t \implies \sin t = \frac{x-3}{2}$$

$$y = 2 - \cos t \implies \cos t = 2 - y$$

By the basic Pythagorean identity,

$$\sin^2 t + \cos^2 t = 1$$

Substituting $(x-3)/2$ and $2-y$ for $\sin t$ and $\cos t$, respectively, we get

$$\left(\frac{x-3}{2}\right)^2 + (2-y)^2 = 1$$

$$\frac{(x-3)^2}{4} + (2-y)^2 = 1$$

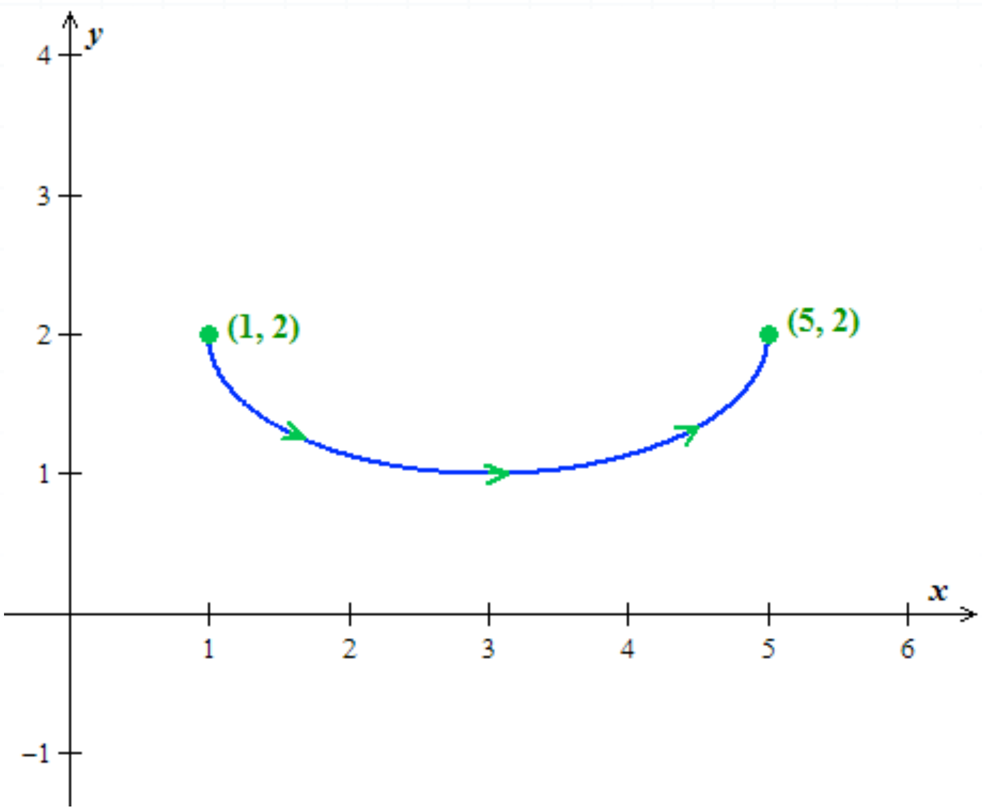
This is the equation of an ellipse.

To determine more specifically what the parametric curve consists of, let's tabulate values of x and y for several values of t in the interval $[-\pi/2,\pi/2]$.

t	$\sin t$	$\cos t$	$x = 3 + 2\sin t$	$y = 2 - \cos t$
$-\frac{\pi}{2}$	-1	0	1	2
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$3-\sqrt{2}$	$2-\frac{\sqrt{2}}{2}$

0	0	1	3	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$3+\sqrt{2}$	$2-\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1	0	5	2

Using those data, we can sketch the parametric curve.



What we see is that this parametric curve is the lower half of the ellipse

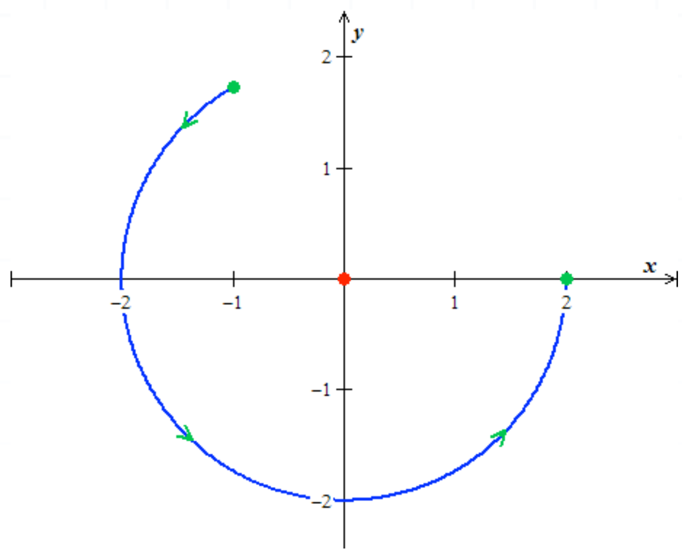
$$\frac{(x-3)^2}{4}+(2-y)^2=1$$

and that it's traced out in the counterclockwise direction.

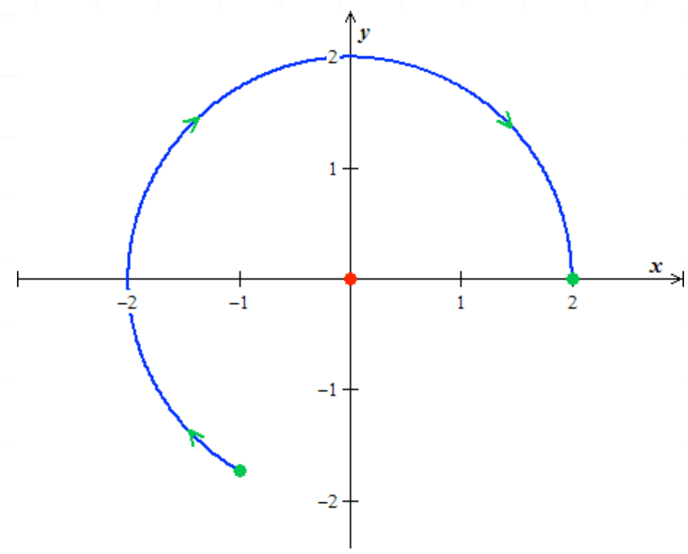
Topic: Sketching a parametric curve and its orientation

Question: One of the following curves is the parametric curve for the equations $x = 2\cos(3t)$ and $y = 2\sin(3t)$ where $-(\pi/6) \le t \le \pi/4$. Which one is it?

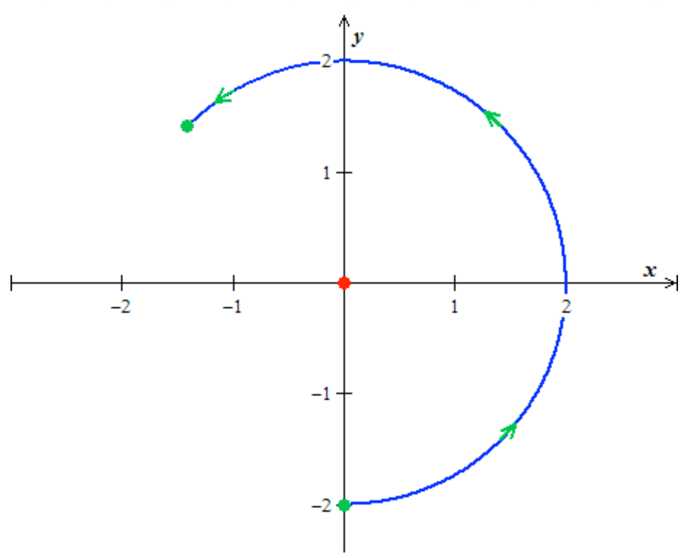
Answer choices:



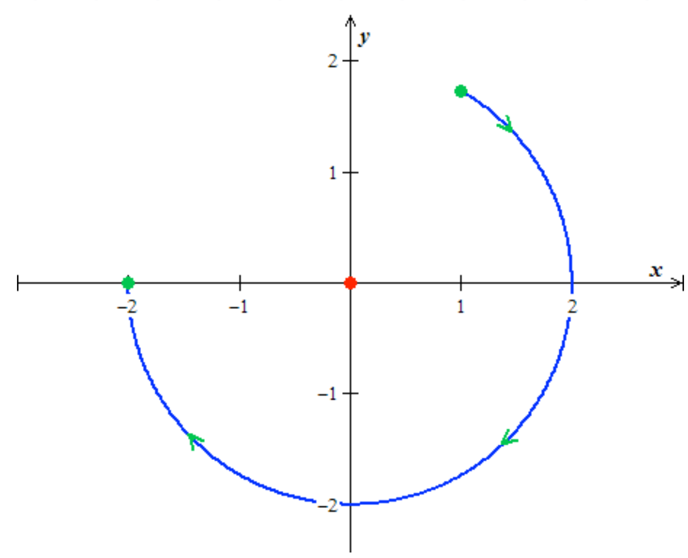
A



B



C



D

Solution: C

Let's evaluate $x = 2\cos(3t)$ and $y = 2\sin(3t)$ at the endpoints of the given interval for t (i.e., at $t = -\pi/6$ and $t = \pi/4$).

$$t = -\frac{\pi}{6} \Longrightarrow x = 2\cos\left(3\left(-\frac{\pi}{6}\right)\right) = 2\cos\left(-\frac{\pi}{2}\right) = 2(0) = 0$$

$$t = -\frac{\pi}{6} \Longrightarrow y = 2\sin\left(3\left(-\frac{\pi}{6}\right)\right) = 2\sin\left(-\frac{\pi}{2}\right) = 2(-1) = -2$$

$$t = \frac{\pi}{4} \Longrightarrow x = 2\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\cos\left(\frac{3\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$$

$$t = \frac{\pi}{4} \Longrightarrow y = 2\sin\left(3\left(\frac{\pi}{4}\right)\right) = 2\sin\left(\frac{3\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

Inspection of the given curves reveals that the only one with endpoints at $(x, y) = (0, -2)$ and $(x, y) = \left(-\sqrt{2}, \sqrt{2}\right)$ is the curve in answer choice C. The initial point (the point for $t = -\pi/6$) is $(0, -2)$, and the terminal point (the point for $t = \pi/4$) is $\left(-\sqrt{2}, \sqrt{2}\right)$. The arrows shown in the curve in answer choice C point away from the initial point and toward the terminal point, as they should. Thus C is indeed the correct answer.

Topic: Write the equation as a parametric curve

Question: Which of the following is a parametric representation of the curve that satisfies the equation $x^2 - 2(y - 1) = 0$ and has $(2,3)$ and $(4,9)$ as its initial and terminal points, respectively?

Answer choices:

A $\quad x = t$ and $y = 2(t - 1)$ where $2 \le t \le 5$

B $\quad x = t$ and $y = \frac{t^2}{2} + 1$ where $2 \le t \le 4$

C $\quad x = \frac{t^2}{2}$ and $y = t + 1$ where $3 \le t \le 5$

D $\quad x = \frac{t^2}{2}$ and $y = t - 2$ where $3 \le t \le 4$

Solution: B

To see that answer choice B is correct, we'll first solve the given equation for y:

$$x^2 - 2(y - 1) = 0$$

$$x^2 - 2y + 2 = 0$$

$$x^2 + 2 = 2y$$

$$\frac{x^2}{2} + 1 = y$$

Turning this equation around, we obtain

$$y = \frac{x^2}{2} + 1$$

Thus y is a function of x. If we let $x = t$, then

$$y = \frac{t^2}{2} + 1$$

For the smallest value of t given in answer choice B (i.e., $t = 2$),

$$x = t \Longrightarrow x = 2, \quad y = \frac{t^2}{2} + 1 \Longrightarrow y = \frac{(2)^2}{2} + 1 = \frac{4}{2} + 1 = 3$$

For the largest value of t given in answer choice B (i.e., $t = 4$),

$$x = t \Longrightarrow x = 4, \quad y = \frac{t^2}{2} + 1 \Longrightarrow y = \frac{(4)^2}{2} + 1 = \frac{16}{2} + 1 = 9$$

This shows that the initial point is $(2,3)$ and the terminal point is $(4,9)$.

Now we'll show that none of the other three answer choices is correct.

The parametric equation given for y in answer choice A is $y = 2(t - 1)$, and the smallest value given for t is 2. For $t = 2$, answer choice A yields

$$y = 2(2 - 1) = 2(1) = 2 \neq 3$$

so answer choice A doesn't give the correct y-coordinate for the initial point of the curve.

The parametric equation for x which is given in answer choices C and D is

$$x = \frac{t^2}{2}$$

and the smallest value of t given for both of them is 3. For $t = 3$, answer choices C and D yield

$$x = \frac{(3)^2}{2} = \frac{9}{2} \neq 2$$

Thus neither answer choice C nor answer choice D gives the correct x-coordinate for the initial point of the curve.

Topic: Write the equation as a parametric curve

Question: The given equation is a closed curve. Express the left half of this curve in parametric form, with $(-4,4)$ as the initial point.

$$\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=1$$

Answer choices:

A $\quad x=-4+4\cos t$ and $y=2+2\sin t$ where $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$

B $\quad x=-4-4\cos t$ and $y=2-4\sin t$ where $0\le t\le\pi$

C $\quad x=-4+4\cos t$ and $y=-2-2\sin t$ where $\frac{\pi}{2}\le t\le\frac{3\pi}{2}$

D $\quad x=-4+4\cos t$ and $y=2+4\sin t$ where $0\le t\le\pi$

Solution: A

Notice that

$$\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=1$$

is the equation of an ellipse that's centered at the point $(-4,2)$. First, we'll check that the prescribed initial point, $(-4,4)$, is on this curve:

$$\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=\frac{(-4+4)^2}{16}+\frac{(4-2)^2}{4}=0+\frac{4}{4}=1$$

Since the x-coordinate of $(-4,4)$ is equal to the x-coordinate of the center of the ellipse, we see that $(-4,4)$ is on the boundary between the left and right halves of the ellipse.

Using $16=4^2$ and $4=2^2$, we can rewrite the equation of the ellipse as

$$\left(\frac{x+4}{4}\right)^2+\left(\frac{y-2}{2}\right)^2=1$$

We know that $\sin^2 t+\cos^2 t=1$ for any t. Because of the "natural" relationship between x and the cosine function, and the "natural" relationship between y and the sine function, let's let

$$\frac{x+4}{4}=\cos t, \quad \frac{y-2}{2}=\sin t$$

Multiplying both sides of the first equation by 4, and both sides of the second equation by 2, gives

$$x+4=4\cos t, \quad y-2=2\sin t$$

Solving the first equation for x and the second equation for y, we get the parametric equations

$$x = -4 + 4\cos t, \quad y = 2 + 2\sin t$$

Since we want only the left half of the curve, the largest value of t must exceed the smallest value of t by π, so using $\pi/2$ and $3\pi/2$ as the smallest and largest values of t makes sense.

Let's check that we get $(-4,4)$ as the initial point of the curve when $t = \pi/2$. Well,

$$t = \frac{\pi}{2} \implies x = -4 + 4\cos\left(\frac{\pi}{2}\right) = -4 + 4(0) = -4$$

and

$$t = \frac{\pi}{2} \implies y = 2 + 2\sin\left(\frac{\pi}{2}\right) = 2 + 2(1) = 4$$

To be sure that we get the left half of the ellipse (and only the left half) with the parametric equations and t-values given in answer choice A, let's check the coordinates of the points that correspond to $t = \pi$ and $3\pi/2$.

$$t = \pi \implies x = -4 + 4\cos(\pi) = -4 + 4(-1) = -8$$

and

$$t = \pi \implies y = 2 + 2\sin(\pi) = 2 + 2(0) = 2$$

Clearly, the point $(-8,2)$ is on the left half of the ellipse, because its x-coordinate is less than that of the center of the ellipse.

$$t = \frac{3\pi}{2} \implies x = -4 + 4\cos\left(\frac{3\pi}{2}\right) = -4 + 4(0) = -4$$

and

$$t = \frac{3\pi}{2} \implies y = 2 + 2\sin\left(\frac{3\pi}{2}\right) = 2 + 2(-1) = 0$$

The point $(-4,0)$ is also on the left half of the ellipse. In fact, it's on the boundary between the left and right halves of the ellipse, because it lies directly below the initial point of the curve, $(-4,2)$, so it is indeed the terminal point of the curve. These results show that answer choice A is correct.

In answer choice B, the parametric equation for x is given as $x = -4 - 4\cos t$, and the smallest value of t is given as 0. Thus for the x-coordinate of the initial point, we would get

$$x = -4 - 4\cos(0) = -4 - 4(1) = -8 \neq -4$$

This shows that answer choice B is incorrect.

In answer choice C, the parametric equation for y is given as $y = -2 - 2\sin t$ and the smallest value of t is given as $\pi/2$. Thus for the y-coordinate of the initial point, we would get

$$y = -2 - 2\sin\left(\frac{\pi}{2}\right) = -2 - 2(1) = -4 \neq 4$$

This rules out answer choice C.

In answer choice D, the parametric equation given for x is $x = -4 + 4\cos t$, and the smallest value of t is given as 0, so for the x-coordinate of the initial point, we would get

$$x = -4 + 4\cos(0) = -4 + 4(1) = 0 \neq -4$$

Therefore, answer choice D is incorrect.

Topic: Write the equation as a parametric curve

Question: What type of curve has the following properties:

1. The curve satisfies the equation $y^2 - 10y - x + 22 = 0$.

2. The initial point of the curve is $(-3,5)$.

3. The y-coordinate of every point on the curve other than the initial point is greater than 5.

Answer choices:

A The lower half of a hyperbola

B A circle with center at $(-3,10)$

C The right half of an ellipse

D The upper half of a parabola that opens to the right

Solution: D

First, let's move the x to the right-hand side of the given equation:

$$y^2 - 10y + 22 = x$$

Turning this equation around, we get

$$x = y^2 - 10y + 22$$

Next, we'll take $y^2 - 10y$ and complete the square on it:

$$y^2 - 10y = \left(y^2 - 10y + 25\right) - 25$$

Now $y^2 - 10y + 25 = (y - 5)^2$, so our equation becomes

$$x = \left[(y - 5)^2 - 25\right] + 22$$

$$x = (y - 5)^2 - 3$$

This is the equation of a parabola that opens to the right and has its vertex at the point $(-3,5)$.

Note that for every real number y, there is a unique real number x that satisfies the equation $x = (y - 5)^2 - 3$, so x is a function of y. This tells us that we could represent this curve by using the parametric equations $y = t$ and $x = (t - 5)^2 - 3$ together with appropriate values of t.

Since we want y to be greater than 5 for all points on the curve other than the initial point, this curve is the upper half of the parabola, so we could represent the curve in parametric form as follows:

$x = (t - 5)^2 - 3$ and $y = t$ where $t \geq 5$

This curve has no terminal point; the larger the value of t, the larger the values of x and y.

Topic: Eccentricity and directrix of the conic section

Question: The total of the eccentricities of the following functions is equal to 6. Which type of curve is represented by r_4?

$$r_1 = \frac{16}{4 - 5\cos\theta_1}$$

$$r_2 = \frac{12}{2 - 7\cos\theta_2}$$

$$r_3 = \frac{24}{8 - 3\cos\theta_3}$$

$$r_4 = \frac{15}{6 - m\cos\theta_4}$$

Answer choices:

A	A circle

B	An ellipse

C	A hyperbola

D	A parabola

Solution: B

Find the eccentricity of each function.

For the eccentricity of r_1:

$$r_1 = \frac{16}{4 - 5\cos\theta_1}$$

$$r_1 = \frac{\frac{16}{4}}{\frac{4}{4} - \frac{5}{4}\cos\theta_1}$$

$$r_1 = \frac{4}{1 - \frac{5}{4}\cos\theta_1}$$

$$e_1 = \frac{5}{4}$$

For the eccentricity of r_2:

$$r_2 = \frac{12}{2 - 7\cos\theta_2}$$

$$r_2 = \frac{\frac{12}{2}}{\frac{2}{2} - \frac{7}{2}\cos\theta_2}$$

$$r_2 = \frac{6}{1 - \frac{7}{2}\cos\theta_2}$$

$$e_2 = \frac{7}{2}$$

For the eccentricity of r_3:

$$r_3 = \frac{24}{8 - 3\cos\theta_3}$$

$$r_3 = \frac{\frac{24}{8}}{\frac{8}{8} - \frac{3}{8}\cos\theta_3}$$

$$r_3 = \frac{3}{1 - \frac{3}{8}\cos\theta_3}$$

$$e_3 = \frac{3}{8}$$

For the eccentricity of r_4:

$$r_4 = \frac{15}{6 - m\cos\theta_4}$$

$$r_4 = \frac{\frac{15}{6}}{\frac{6}{6} - \frac{m}{6}\cos\theta_4}$$

$$r_4 = \frac{\frac{5}{2}}{1 - \frac{m}{6}\cos\theta_4}$$

$$e_4 = \frac{m}{6}$$

The sum of these eccentricities is therefore

$$\frac{5}{4} + \frac{7}{2} + \frac{3}{8} + \frac{m}{6} = 6$$

Which means that m is

$$\frac{30}{24} + \frac{84}{24} + \frac{9}{24} + \frac{4m}{24} = 6$$

$$30 + 84 + 9 + 4m = 144$$

$$4m = 21$$

$$m = \frac{21}{4}$$

So e_4 is

$$e_4 = \frac{\frac{21}{4}}{6}$$

$$e_4 = \frac{21}{24}$$

$$e_4 = \frac{7}{8}$$

Since $0 < e_4 < 1$, then r_4 represents an ellipse.

Topic: Eccentricity and directrix of the conic section

Question: Which polar curves have the same directrix?

$$r_1 = \frac{6}{1 - 7\cos\theta_1}$$

$$r_2 = \frac{12}{9 - 5\cos\theta_2}$$

$$r_3 = \frac{12}{5 - 14\cos\theta_3}$$

$$r_4 = \frac{5}{7 - \cos\theta_4}$$

Answer choices:

A The directrices of r_1 and r_4 are parallel.

B The directrices of r_1 and r_3 are parallel.

C The directrices of r_2 and r_3 are parallel.

D The directrices of r_2 and r_4 are parallel.

Solution: B

The directrix of

$$r_1 = \frac{6}{1 - 7\cos\theta_1}$$

is

$$d = \frac{6}{7}$$

The directrix of

$$r_3 = \frac{12}{5 - 14\cos\theta_3}$$

$$r_3 = \frac{\frac{12}{5}}{1 - \frac{14}{5}\cos\theta_3}$$

is

$$d = \frac{\frac{12}{5}}{\frac{14}{5}} = \frac{6}{7}$$

Therefore, r_1 and r_3 have the same directrix.

Topic: Eccentricity and directrix of the conic section

Question: The following polar functions are given, where e_1 and d_1 are the eccentricity and directrix of the function r_1, and e_2 and d_2 are the eccentricity and directrix of the function r_2. If $\theta_1 = \theta_2$, $e_1 = 3e_2$ and $3d_1 = d_2$, then which statement is true about the positions of the graphs of the given functions.

$$r_1 = \frac{a}{b - c\cos\theta_1}$$

$$r_2 = \frac{c}{b - a\cos\theta_2}$$

Answer choices:

A The graphs of r_1 and r_2 are the same.

B The graphs of r_1 and r_2 don't overlap.

C The graph of r_1 is 3 units above the graph of r_2.

D The graph of r_1 is 3 units below the graph of r_2.

Solution: A

We'll rewrite r_1.

$$r_1 = \frac{a}{b - c\cos\theta_1}$$

$$r_1 = \frac{\frac{a}{b}}{\frac{b}{b} - \frac{c}{b}\cos\theta_1}$$

$$r_1 = \frac{\frac{a}{b}}{1 - \frac{c}{b}\cos\theta_1}$$

We'll rewrite r_2.

$$r_2 = \frac{c}{b - a\cos\theta_2}$$

$$r_2 = \frac{\frac{c}{b}}{\frac{b}{b} - \frac{a}{b}\cos\theta_2}$$

$$r_2 = \frac{\frac{c}{b}}{1 - \frac{a}{b}\cos\theta_2}$$

Now we can say

$$e_1 d_1 = \frac{a}{b}$$

and

$$e_2 d_2 = \frac{c}{b}$$

Divide these equations side-by-side, and substitute $e_1 = 3e_2$ and $3d_1 = d_2$.

$$\frac{e_1d_1}{e_2d_2} = \frac{a}{c}$$

$$\frac{(3e_2)\,d_1}{e_2\,(3d_1)} = \frac{a}{c}$$

$$\frac{1}{1} = \frac{a}{c}$$

$$a = c$$

Replacing $a = c$ in the given functions gives

$$r_1 = \frac{a}{b - a\cos\theta_1}$$

$$r_2 = \frac{a}{b - a\cos\theta_2}$$

Since $\theta_1 = \theta_2$, the functions are the same.

Topic: Parabolas

Question: Find the vertex, focus, and directrix of the parabola.

$$2y^2 - 5x + 3y - 7 = 0$$

Answer choices:

A $V\left(-\frac{13}{8}, -\frac{3}{4}\right)$ $F\left(-1, -\frac{3}{4}\right)$ Directrix at $x = -\frac{9}{4}$

B $V\left(\frac{13}{8}, \frac{3}{4}\right)$ $F\left(-1, -\frac{3}{4}\right)$ Directrix at $x = \frac{9}{4}$

C $V\left(-\frac{13}{8}, -\frac{3}{4}\right)$ $F\left(-\frac{9}{4}, -\frac{3}{4}\right)$ Directrix at $x = -\frac{9}{4}$

D $V\left(-\frac{3}{4}, -\frac{13}{8}\right)$ $F\left(-1, -\frac{3}{4}\right)$ Directrix at $x = \frac{9}{4}$

Solution: A

First, reduce the equation of the parabola into standard form.

$$2y^2 - 5x + 3y - 7 = 0$$

$$2y^2 + 3y = 5x + 7$$

$$y^2 + \frac{3}{2}y = \frac{5}{2}x + \frac{7}{2}$$

$$y^2 + \frac{3}{2}y + \frac{9}{16} = \frac{5}{2}x + \frac{7}{2} + \frac{9}{16}$$

$$\left(y + \frac{3}{4}\right)^2 = \frac{5}{2}x + \frac{65}{16}$$

$$\left(y + \frac{3}{4}\right)^2 = \frac{5}{2}\left(x + \frac{13}{8}\right)$$

$$\left(y + \frac{3}{4}\right)^2 = 4\frac{5}{8}\left(x + \frac{13}{8}\right)$$

The parabola has a horizontal axis at $y = -3/4$ and opens to the right. Its vertex is at $(-13/8, -3/4)$. Since $a = 5/8$, its focus is at

$$\left(-\frac{13}{8} + \frac{5}{8}, -\frac{3}{4}\right)$$

$$\left(-1, -\frac{3}{4}\right)$$

The equation of the directrix is

$$x = -\frac{13}{8} - \frac{5}{8}$$

$$x = -\frac{9}{4}$$

Topic: Parabolas

Question: Which pair of equations represents a parabola with a directrix at $x = -5$?

Answer choices:

A $\quad x = t^2 + 1$ and $y = 5t - 9$

B $\quad x = t^2 - 1$ and $y = 2t - 3$

C $\quad x = 4t^2 - 1$ and $y = 4t - 1$

D $\quad x = t^2 - 1$ and $y = 4t - 1$

Solution: D

Choosing $x = t^2 - 1$ and $y = 4t - 1$ yields the following equations:

$$y + 1 = 4t$$

$$(y + 1)^2 = (4t)^2$$

$$(y + 1)^2 = 16t^2$$

Solve $x = t^2 - 1$ for t^2.

$$x + 1 = t^2$$

Replace $x + 1 = t^2$ in $(y + 1)^2 = 16t^2$.

$$(y + 1)^2 = 16(x + 1)$$

Therefore, the equation of the directrix is

$$x = -1 - 4$$

$$x = -5$$

Topic: Parabolas

Question: Which pair of parametric equations have their vertices at the same point?

Answer choices:

A	$x = t + 1, \ y = 2t^2 - 2$	and	$x = t^2 - 4, \ y = 2t + 1$
B	$x = t^2 - 4, \ y = 2t + 1$	and	$x = t^2 + 4, \ y = t + 1$
C	$x = t^2 + 4, \ y = t + 1$	and	$x = t^2 + 4, \ y = 2t + 1$
D	$x = t + 1, \ y = 2t^2 + 2$	and	$x = t^2 - 4, \ y = 2t + 1$

Solution: C

From the parametric equations $x = t^2 + 4$ and $y = t + 1$, we have:

$$y - 1 = t$$

$$(y - 1)^2 = t^2$$

and

$$x = t^2 + 4$$

$$x - 4 = t^2$$

Replace $x - 4 = t^2$ in $(y - 1)^2 = t^2$.

$$(y - 1)^2 = x - 4$$

Thus, the vertex of this parabola is at $(4,1)$.

For parametric equations $x = t^2 + 4$ and $y = 2t + 1$, we have:

$$y - 1 = 2t$$

$$(y - 1)^2 = 4t^2$$

and

$$x = t^2 + 4$$

$$x - 4 = t^2$$

Replace $x - 4 = t^2$ in $(y - 1)^2 = 4t^2$.

$$(y - 1)^2 = 4(x - 4)$$

Thus, the vertex of this parabola is at $(4,1)$.

Topic: Equation of a parabolic conic section

Question: A parametric curve is defined by the equations $x = 4\sqrt{t}$ and $y = 24t$ within the interval $[0,\infty)$. Which statement describes the graph of this function?

Answer choices:

A The equations represent a half-parabola originating from the point $(6,0)$, and extending up into the first quadrant.

B The equations represent a half-parabola originating from the point $(0,6)$, and extending up into the second quadrant.

C The equations represent a half-parabola originating from the point $(0,0)$, and extending up into the second quadrant.

D The equations represent a half-parabola originating from the point $(0,0)$, and extending up into the first quadrant.

Solution: D

Square $x = 4\sqrt{t}$, and solve the result for t.

$$x^2 = \left(4\sqrt{t}\right)^2$$

$$x^2 = 16t$$

$$t = \frac{1}{16}x^2$$

Now, replace $t = (1/16)x^2$ in $y = 24t$.

$$y = 24\left(\frac{1}{16}x^2\right)$$

$$y = \frac{3}{2}x^2$$

Thus, the equations represent a half-parabola originating from the point $(0,0)$, and extending up above the x-axis and to the right of the y-axis.

Topic: Equation of a parabolic conic section

Question: For which values of m and n do the parametric equations $x = t - m$ and $y = 4t^2 - n$ represent the parabola $y = 4x^2 - 40x + 103$?

Answer choices:

A $m = 5$ and $n = 3$

B $m = -5$ and $n = -3$

C $m = -5$ and $n = 3$

D $m = 5$ and $n = -3$

Solution: B

Choosing $m = -5$ and $n = -3$ transforms the given parametric equations to the following forms:

$$x = t + 5$$

$$y = 4t^2 + 3$$

Eliminate 5 from the right side of $x = t + 5$, and then square the result.

$$x - 5 = t$$

$$(x - 5)^2 = t^2$$

Replace $(x - 5)^2 = t^2$ in $y = 4t^2 + 3$, and expand.

$$y = 4(x - 5)^2 + 3$$

$$y = 4\left(x^2 - 10x + 25\right) + 3$$

$$y = 4x^2 - 40x + 103$$

Topic: Equation of a parabolic conic section

Question: Which of the following parametric functions represents a full parabola?

Answer choices:

A $\sqrt[3]{x^2} = 3\sqrt[3]{t}$ and $y = 81t - 6$

B $x = 6\sqrt[4]{t}$ and $y = 4t - 5$

C $x = 12\sqrt{t}$ and $y = 2t - 3$

D $x = 4\sqrt{t} - 1$ and $y = 5t$

Solution: A

Assuming answer choice A is correct, solve $\sqrt[3]{x^2} = 3\sqrt[3]{t}$ for t.

$$\sqrt[3]{x^2} = 3\sqrt[3]{t}$$

$$\left(\sqrt[3]{x^2}\right)^3 = \left(3\sqrt[3]{t}\right)^3$$

$$x^2 = 27t$$

$$t = \frac{x^2}{27}$$

Replace $t = x^2/27$ in $y = 81t - 6$.

$$y = 81t - 6$$

$$y = 81\left(\frac{x^2}{27}\right) - 6$$

$$y = 3x^2 - 6$$

This function represents a graph within the domain $(-\infty, \infty)$.

Topic: Polar equation of a parabolic conic section

Question: Convert the parabola into polar coordinates.

$$y = \frac{1}{2}x^2 - x$$

Answer choices:

A $\quad r^2 = 2\left(1 + \tan^2\theta\right)(1 - \tan\theta)^2$

B $\quad r^2 = 2\left(1 - \tan^2\theta\right)(1 + \tan\theta)^2$

C $\quad r^2 = 4\left(1 + \tan^2\theta\right)(1 + \tan\theta)^2$

D $\quad r^2 = 4\left(1 - \tan^2\theta\right)(1 + \tan\theta)^2$

Solution: C

The equation of the parabola is

$$y = \frac{1}{2}x^2 - x$$

$$2y = x^2 - 2x$$

$$x^2 = 2x + 2y$$

$$x = \frac{2x + 2y}{x}$$

$$x = \frac{2x}{x} + \frac{2y}{x}$$

$$x = 2 + 2\frac{y}{x}$$

Replace $x = r\cos\theta$ and $\tan\theta = y/x$, and simplify.

$$r\cos\theta = 2 + 2\tan\theta$$

$$r = 2\left(\frac{1}{\cos\theta}\right)(1 + \tan\theta)$$

$$r^2 = 4\left(\frac{1}{\cos^2\theta}\right)(1 + \tan\theta)^2$$

Knowing that

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$

we can substitute and get

$$r^2 = 4\left(1 + \tan^2\theta\right)(1 + \tan\theta)^2$$

Topic: Polar equation of a parabolic conic section

Question: Convert the polar equation into a parabola in rectangular coordinates.

$$r\sin^2\theta - 2\cos\theta - 2 = 0$$

Answer choices:

A $\quad x = \frac{1}{4}y^2 - 1$

B $\quad x = \frac{1}{4}y^2 - 4$

C $\quad x = \frac{1}{2}y^2 - 1$

D $\quad x = \frac{1}{2}y^2 + 1$

Solution: A

Transform the given equation as follows:

$$r\sin^2\theta - 2\cos\theta - 2 = 0$$

$$r\sin^2\theta = 2\cos\theta + 2$$

$$r\sin^2\theta = 2(\cos\theta + 1)$$

$$r = \frac{2(\cos\theta + 1)}{\sin^2\theta}$$

We know that $\sin^2\theta = 1 - \cos^2\theta$.

$$r = \frac{2(\cos\theta + 1)}{1 - \cos^2\theta}$$

Factor the denominator, then cancel common factors.

$$r = \frac{2(\cos\theta + 1)}{(1 - \cos\theta)(1 + \cos\theta)}$$

$$r = \frac{2}{(1 - \cos\theta)}$$

$$r(1 - \cos\theta) = 2$$

$$r - r\cos\theta = 2$$

Use the conversion equation $x = r\cos\theta$ to substitute.

$$r - x = 2$$

$$r = x + 2$$

$$r^2 = (x+2)^2$$

Use the conversion equation $x^2 + y^2 = r^2$ to substitute.

$$x^2 + y^2 = x^2 + 4x + 4$$

$$y^2 = 4x + 4$$

$$y^2 - 4 = 4x$$

$$x = \frac{1}{4}y^2 - 1$$

Topic: Polar equation of a parabolic conic section

Question: Which equation can be equivalent to the given function?

$$rf(\theta) + r - 8 = 0$$

Answer choices:

A $\quad r = \dfrac{8}{1 + \sin\theta}$ with directrix $d = 8$.

B $\quad r = \dfrac{8}{1 + \tan\theta}$ with directrix $d = 8$.

C $\quad r = \dfrac{1}{8 + \sin\theta}$ with directrix $d = 1$.

D $\quad r = \dfrac{8}{8 + \sin\theta}$ with directrix $d = 8$.

Solution: A

Transform the given function as follows:

$$rf(\theta) + r - 8 = 0$$

$$r\left(f(\theta) + 1\right) - 8 = 0$$

$$r\left(f(\theta) + 1\right) = 8$$

$$r = \frac{8}{1 + f(\theta)}$$

Replacing $f(\theta)$ by $\sin\theta$ results in

$$r = \frac{8}{1 + \sin\theta}$$

where its directrix is $d = 8$.

Topic: Ellipses

Question: Which statement describes the graph of the parametric functions?

$$x = \sin t$$

$$y = 5\cos t - 7$$

Answer choices:

A The equations represent an ellipse centered at $(0, -7)$, with a semimajor axis with a length of 7 along the y-axis, and with a semiminor axis with a length of 1 along with the line $y = -7$.

B The equations represent an ellipse centered at $(0, -7)$, with a semimajor axis with a length of 5 along the y-axis, and with a semiminor axis with a length of 1 along with the line $y = -5$.

C The equations represent an ellipse centered at $(0, -7)$, with a semimajor axis with a length of 5 along the y-axis, and with a semiminor axis with a length of 1 along with the line $y = -7$.

D The equations represent an ellipse centered at $(0,7)$, with a semimajor axis with a length of 5 along the y-axis, and with a semiminor axis with a length of 1 along with the line $y = -7$.

Solution: C

Solve $y = 5\cos t - 7$ for $\cos t$, and square the result.

$$y = 5\cos t - 7$$

$$y + 7 = 5\cos t$$

$$\frac{y+7}{5} = \cos t$$

$$\frac{(y+7)^2}{25} = \cos^2 t$$

Square $x = \sin t$ and add this to the result.

$$x^2 + \frac{(y+7)^2}{25} = \sin^2 t + \cos^2 t$$

$$x^2 + \frac{(y+7)^2}{25} = 1$$

Thus the equations represent an ellipse with the following properties:

- Centered at $(0, -7)$
- Semimajor axis with a length of 5 along the y-axis
- Semiminor axis with a length of 1 along with the line $y = -7$

Topic: Ellipses

Question: The following ellipses are defined by parametric equations. The graph of which ellipse is closer to a circle than the other graphs?

Ellipse E:	$x = 2\sin t$	and	$y = 3\cos t - 1$
Ellipse F:	$x = 3\sin t$	and	$y = 5\cos t - 2$
Ellipse G:	$x = 2\sin t$	and	$y = \cos t - 1$
Ellipse H:	$x = \sin t$	and	$y = 4\cos t - 3$

Answer choices:

A Ellipse E is closer to a circle because its eccentricity, $e = \sqrt{5}/3 = 0.75$, is less than the eccentricities of the other ellipses.

B Ellipse F is closer to a circle because its eccentricity, $e = 6/7 = 0.88$, is less than the eccentricities of the other ellipses.

C Ellipse G is closer to a circle because its eccentricity, $e = \sqrt{5}/3 = 0.76$, is less than the eccentricities of the other ellipses.

D Ellipse G is closer to a circle because its eccentricity, $e = \sqrt{15}/4 = 0.97$, is greater than the eccentricities of the other ellipses.

Solution: A

Find a and b for each ellipse. Then calculate $e = c/a$.

For ellipse E, given by $x = 2\sin t$ and $y = 3\cos t - 1$, the eccentricity is

$$x^2 = 4\sin^2 t \text{ and } (y+1)^2 = 9\cos^2 t$$

$$\frac{x^2}{4} = \sin^2 t \text{ and } \frac{(y+1)^2}{9} = \cos^2 t$$

$$\frac{x^2}{4} + \frac{(y+1)^2}{9} = 1$$

$$e = \frac{\sqrt{5}}{3} = 0.75$$

For ellipse F, given by $x = 3\sin t$ and $y = 5\cos t - 2$, the eccentricity is

$$x^2 = 9\sin^2 t \text{ and } (y+2)^2 = 25\cos^2 t$$

$$\frac{x^2}{9} = \sin^2 t \text{ and } \frac{(y+2)^2}{25} = \cos^2 t$$

$$\frac{x^2}{9} + \frac{(y+2)^2}{25} = 1$$

$$e = \frac{4}{5} = 0.80$$

For ellipse G, given by $x = 2\sin t$ and $y = \cos t - 1$, the eccentricity is

$$\frac{x^2}{4} = \sin^2 t \text{ and } (y+1)^2 = \cos^2 t$$

$$\frac{x^2}{4} + (y+1)^2 = 1$$

$$e = \frac{\sqrt{3}}{2} = 0.87$$

For ellipse H, given by $x = \sin t$ and $y = 4\cos t - 3$, the eccentricity is

$x^2 = \sin^2 t$ and $\frac{(y+3)^2}{16} = \cos^2 t$

$$x^2 + \frac{(y+3)^2}{16} = 1$$

$$e = \frac{\sqrt{15}}{4} = 0.97$$

Topic: Ellipses

Question: Which ellipses have foci with the same x-coordinates?

Answer choices:

A Ellipses $x = \sin t$, $y = 3\cos t - 4$ and $x = 4\sin t$, $y = 5\cos t - 3$

B Ellipses $x = 3\sin t$, $y = \cos t + 9$ and $x = -4\sin t$, $y = -5\cos t + 1$

C Ellipses $x = -3\sin t + 9$, $y = 4\cos t$ and $x = \sin t - 6$, $y = -5\cos t$

D Ellipses $x = 5\sin t$, $y = 3\cos t - 2$ and $x = 5\sin t$, $y = 3\cos t - 4$

Solution: D

Choose the ellipse given by $x = 5\sin t$, $y = 3\cos t - 2$:

$$\frac{x^2}{25} = \sin^2 t$$

$$\frac{(y+2)^2}{9} = \cos^2 t$$

Therefore

$$\frac{x^2}{25} + \frac{(y+2)^2}{9} = 1$$

The x-coordinates of its foci are $c = -4$ and $c = 4$.

Choose the ellipse given by $x = 5\sin t$, $y = 3\cos t - 4$:

$$\frac{x^2}{25} = \sin^2 t$$

$$\frac{(y+4)^2}{9} = \cos^2 t$$

Therefore

$$\frac{x^2}{25} + \frac{(y+4)^2}{9} = 1$$

The x-coordinates of its foci are $c = -4$ and $c = 4$.

Topic: Equation of an elliptical conic section

Question: Which two parametric equations represent an ellipse centered at $(2, -1)$.

$$x = 2 - 3\sin t$$

$$x = t$$

$$x = 2 + 5\sin t$$

$$y = \cos t - 3$$

$$y = 4\cos t - 1$$

$$y = t^2 + 2$$

Answer choices:

A $x = 2 - 3\sin t$ and $y = 4\cos t - 1$

B $x = 2 + 5\sin t$ and $y = t^2 + 2$

C $x = t$ and $x = 2 + 5\sin t$

D $x = t$ and $y = t^2 + 2$

Solution: A

Choose the equations $x = 2 - 3\sin t$ and $y = 4\cos t - 1$, and solve them for $\sin^2 t$ and $\cos^2 t$ respectively.

$$x = 2 - 3\sin t$$

$$x - 2 = -3\sin t$$

$$\frac{(x-2)^2}{9} = \sin^2 t$$

and

$$y = 4\cos t - 1$$

$$\frac{y+1}{4} = \cos t$$

$$\frac{(y+1)^2}{16} = \cos^2 t$$

Because we know that $\sin^2 t + \cos^2 t = 1$, we can say

$$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{16} = 1$$

Then this is the equation of an ellipse centered at $(2, -1)$.

Topic: Equation of an elliptical conic section

Question: Which type of conic section is represented by the parametric functions?

$$x = 11\sin t - 3$$

$$y = 3\cos t - 11$$

Answer choices:

A Circle

B Ellipse

C Hyperbola

D Parabola

Solution: B

Solve the equations $x = 11\sin t - 3$ and $y = 3\cos t - 11$ for $\sin^2 t$ and $\cos^2 t$ respectively.

$$x = 11\sin t - 3$$

$$x + 3 = 11\sin t$$

$$\frac{x+3}{11} = \sin t$$

$$\frac{(x+3)^2}{121} = \sin^2 t$$

and

$$y = 3\cos t - 11$$

$$y + 11 = 3\cos t$$

$$\frac{y+11}{3} = \cos t$$

$$\frac{(y+11)^2}{9} = \cos^2 t$$

Then we can say

$$\frac{(x+3)^2}{121} + \frac{(y+11)^2}{9} = \sin^2 t + \cos^2 t$$

$$\frac{(x+3)^2}{121} + \frac{(y+11)^2}{9} = 1$$

So, the given equations represent an ellipse.

Topic: Equation of an elliptical conic section

Question: Any point on an ellipse satisfies the parametric equations. Which single equation defines the given ellipse?

$$x = \frac{m}{(m-1)\sec t} - 1$$

$$y = \frac{n}{(n-1)\csc t} - 1$$

Answer choices:

A $\quad \dfrac{(x+1)^2}{\frac{m+1}{m-1}} + \dfrac{(y+1)^2}{\frac{n+1}{n-1}} = 1$

B $\quad \dfrac{(x+1)^2}{\frac{m^2}{(m-1)^2}} + \dfrac{(y+1)^2}{\frac{n^2}{(n-1)^2}} = 1$

C $\quad \dfrac{(x+1)^2}{\frac{m-1}{m}} + \dfrac{(y+1)^2}{\frac{n-1}{n}} = 1$

D $\quad \dfrac{(x+1)^2}{\frac{m+1}{m}} + \dfrac{(y+1)^2}{\frac{n+1}{n}} = 1$

Solution: B

Solve the given equations for $\sin t$ and $\cos t$. Then square both sides of each equation.

$$x = \frac{m}{(m-1)\sec t} - 1$$

$$x + 1 = \frac{m\cos t}{(m-1)}$$

$$\frac{(x+1)^2}{\frac{m^2}{(m-1)^2}} = \cos^2 t$$

and

$$y = \frac{n}{(n-1)\csc t} - 1$$

$$y + 1 = \frac{n\sin t}{n-1}$$

$$\frac{(y+1)^2}{\frac{n^2}{(n-1)^2}} = \sin^2 t$$

Then we can say

$$\frac{(x+1)^2}{\frac{m^2}{(m-1)^2}} + \frac{(y+1)^2}{\frac{n^2}{(n-1)^2}} = 1$$

Topic: Polar equation of an elliptical conic section

Question: Which equation represents the polar equation of the ellipse?

$$x^2 + 4m^2y^2 - 4m^2 = 0$$

Answer choices:

A $\quad r = \dfrac{4m}{\sqrt{\cos^2\theta + 2m^2\sin^2\theta}}$

B $\quad r = \dfrac{4m}{\sqrt{\cos^2\theta + 4m\sin^2\theta}}$

C $\quad r = \dfrac{2m}{\sqrt{\cos^2\theta + 4m^2\sin^2\theta}}$

D $\quad r = \dfrac{2m}{\sqrt{\cos^2\theta + 4m\sin^2\theta}}$

Solution: C

For the equation we've been given

$$x^2 + 4m^2y^2 - 4m^2 = 0$$

we'll use the conversion equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

to substitute and get

$$(r\cos\theta)^2 + 4m^2\left(r\sin\theta\right)^2 - 4m^2 = 0$$

$$r^2\cos^2\theta + 4m^2r^2\sin^2\theta - 4m^2 = 0$$

$$r^2\left(\cos^2\theta + 4m^2\sin^2\theta\right) = 4m^2$$

$$r^2 = \frac{4m^2}{\cos^2\theta + 4m^2\sin^2\theta}$$

$$r = \sqrt{\frac{4m^2}{\cos^2\theta + 4m^2\sin^2\theta}}$$

$$r = \frac{2m}{\sqrt{\cos^2\theta + 4m^2\sin^2\theta}}$$

Topic: Polar equation of an elliptical conic section

Question: The polar equation represents which of the following ellipses?

$$r = \frac{5\cos\theta - 2\sin\theta \pm \sqrt{5(21\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta)}}{5\cos^2\theta + \sin^2\theta}$$

Answer choices:

A $\quad 5x^2 + y^2 + 10x - 2y - 16 = 0$

B $\quad 5x^2 + y^2 - 10x - 2y + 16 = 0$

C $\quad 5x^2 + y^2 - 10x + 4y + 16 = 0$

D $\quad 5x^2 + y^2 - 10x + 4y - 16 = 0$

Solution: D

Choose

$$5x^2 + y^2 - 10x + 4y - 16 = 0$$

from answer choice D. Use the conversion equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

to substitute into the equation.

$$5(r\cos\theta)^2 + (r\sin\theta)^2 - 10(r\cos\theta) + 4(r\sin\theta) - 16 = 0$$

$$5r^2\cos^2\theta + r^2\sin^2\theta - 10r\cos\theta + 4r\sin\theta - 16 = 0$$

$$(5\cos^2\theta + \sin^2\theta)r^2 + 2(-5\cos\theta + 2\sin\theta)r - 16 = 0$$

Now this is a quadratic equation, and we can use the quadratic formula to find its roots. With $a = (5\cos^2\theta + \sin^2\theta)$, $b = 2(-5\cos\theta + 2\sin\theta)$, and $c = -16$, we plug into the quadratic formula and get

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-2(-5\cos\theta + 2\sin\theta) \pm \sqrt{(2(-5\cos\theta + 2\sin\theta))^2 - 4(5\cos^2\theta + \sin^2\theta)(-16)}}{2(5\cos^2\theta + \sin^2\theta)}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{4(-5\cos\theta + 2\sin\theta)(-5\cos\theta + 2\sin\theta) + 64(5\cos^2\theta + \sin^2\theta)}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{4(25\cos^2\theta - 20\sin\theta\cos\theta + 4\sin^2\theta) + 320\cos^2\theta + 64\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{100\cos^2\theta - 80\sin\theta\cos\theta + 16\sin^2\theta + 320\cos^2\theta + 64\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{420\cos^2\theta - 80\sin\theta\cos\theta + 80\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{420\cos^2\theta - 40(2\sin\theta\cos\theta) + 80\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

Using the trig identity $\sin(2x) = 2\sin x\cos x$, we get

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{420\cos^2\theta - 40\sin 2\theta + 80\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{20(21\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta)}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm 2\sqrt{5(21\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta)}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{5\cos\theta - 2\sin\theta \pm \sqrt{5(21\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta)}}{5\cos^2\theta + \sin^2\theta}$$

Topic: Polar equation of an elliptical conic section

Question: Name the center of each ellipse.

$$r^2 = \frac{m^4n^4}{m^4\sin^2\theta_1 + n^4\cos^2\theta_1} \text{ and } r^2 = \frac{m^6n^6}{m^6\sin^2\theta_2 + n^6\cos^2\theta_2}$$

Answer choices:

A $(0,0)$ and $(1,0)$

B $(0,0)$ and $(0,0)$

C $(0,0)$ and $(0,1)$

D $(0,0)$ and $(-1,0)$

Solution: B

Rewrite the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ as

$\cos\theta_1 = \frac{x}{r}$ and $\sin\theta_1 = \frac{y}{r}$

and then make substitutions into the first equation.

$$r^2 = \frac{m^4n^4}{m^4\sin^2\theta_1 + n^4\cos^2\theta_1}$$

$$r^2 = \frac{m^4n^4}{m^4\left(\frac{y}{r}\right)^2 + n^4\left(\frac{x}{r}\right)^2}$$

$$r^2 = \frac{m^4n^4}{\frac{m^4y^2 + n^4x^2}{r^2}}$$

$$r^2\left(\frac{m^4y^2 + n^4x^2}{r^2}\right) = m^4n^4$$

$$m^4y^2 + n^4x^2 = m^4n^4$$

$$\frac{m^4y^2}{m^4n^4} + \frac{m^4x^2}{m^4n^4} = \frac{m^4n^4}{m^4n^4}$$

$$\frac{y^2}{n^4} + \frac{x^2}{n^4} = 1$$

$$\frac{y^2}{(n^2)^2} + \frac{x^2}{(n^2)^2} = 1$$

This is the equation of an ellipse centered at the origin with $a = n^2$ and $b = n^2$.

Now substitute into the second equation.

$$r^2 = \frac{m^6 n^6}{m^6 \sin^2 \theta_2 + n^6 \cos^2 \theta_2}$$

$$r^2 = \frac{m^6 n^6}{m^6 \left(\frac{y}{r}\right)^2 + n^6 \left(\frac{x}{r}\right)^2}$$

$$r^2 = \frac{m^6 n^6}{\frac{m^6 y^2 + n^6 x^2}{r^2}}$$

$$r^2 \left(\frac{m^6 y^2 + n^6 x^2}{r^2}\right) = m^6 n^6$$

$$m^6 y^2 + n^6 x^2 = m^6 n^6$$

$$\frac{m^6 y^2}{m^6 n^6} + \frac{m^6 x^2}{m^6 n^6} = \frac{m^6 n^6}{m^6 n^6}$$

$$\frac{y^2}{n^6} + \frac{x^2}{n^6} = 1$$

$$\frac{y^2}{(n^3)^2} + \frac{x^2}{(n^3)^2} = 1$$

This is the equation of an ellipse centered at the origin with $a = n^3$ and $b = n^3$.

Topic: Vertex, axis, focus, directrix of a hyperbola

Question: Which set of parametric equations defines the hyperbola with the given vertices?

$$\left(\sqrt{10},0\right)$$

$$\left(-\sqrt{10},0\right)$$

Answer choices:

A $x=2t+\frac{1}{5t}$ and $y=5t-\frac{1}{2t}$

B $x=5t+\frac{1}{2t}$ and $y=6t-\frac{1}{2t}$

C $x=5t+\frac{1}{2t}$ and $y=2t-\frac{1}{2t}$

D $x=5t+\frac{1}{2t}$ and $y=5t-\frac{1}{2t}$

Solution: D

Choose the equations from answer choice D,

$$x = 5t + \frac{1}{2t}$$

$$y = 5t - \frac{1}{2t}$$

Square both sides of each equation, and simplify.

$$x = 5t + \frac{1}{2t}$$

$$x^2 = \left(5t + \frac{1}{2t}\right)^2$$

$$x^2 = 25t^2 + \frac{1}{4t^2} + 5$$

and

$$y = 5t - \frac{1}{2t}$$

$$y^2 = \left(5t - \frac{1}{2t}\right)^2$$

$$y^2 = 25t^2 + \frac{1}{4t^2} - 5$$

Subtract the equation for y^2 from the equation for x^2.

$$x^2 - y^2 = \left(25t^2 + \frac{1}{4t^2} + 5\right) - \left(25t^2 + \frac{1}{4t^2} - 5\right)$$

$$x^2 - y^2 = 10$$

Therefore, the vertices of the hyperbola are at $\left(\sqrt{10},0\right)$ and $\left(-\sqrt{10},0\right)$.

Topic: Vertex, axis, focus, directrix of a hyperbola

Question: A hyperbola is defined by the given functions. Where are the foci of the hyperbola?

$$x = \frac{m}{\cos\theta} - 2 \text{ and } y = n\tan\theta - 3$$

Answer choices:

A $\left(-2+\sqrt{m^2+n^2}, -3\right)$ and $\left(-2-\sqrt{m^2+n^2}, -3\right)$

B $\left(3+\sqrt{m^2+n^2}, 2\right)$ and $\left(3-\sqrt{m^2+n^2}, 2\right)$

C $\left(-2+\sqrt{m^2-n^2}, -3\right)$ and $\left(-2-\sqrt{m^2-n^2}, -3\right)$

D $\left(4+\sqrt{m^2+n^2}, 3\right)$ and $\left(4-\sqrt{m^2+n^2}, 3\right)$

Solution: A

Rewrite each of the given equations.

$$x = \frac{m}{\cos\theta} - 2$$

$$x + 2 = \frac{m}{\cos\theta}$$

$$\frac{(x+2)^2}{m^2} = \frac{1}{\cos^2\theta}$$

$$\frac{(x+2)^2}{m^2} = 1 + \tan^2\theta$$

and

$$y = n\tan\theta - 3$$

$$y + 3 = n\tan\theta$$

$$\frac{y+3}{n} = \tan\theta$$

$$\frac{(y+3)^2}{n^2} = \tan^2\theta$$

Subtract this second equation from the first.

$$\frac{(x+2)^2}{m^2} - \frac{(y+3)^2}{n^2} = 1 + \tan^2\theta - \tan^2\theta$$

$$\frac{(x+2)^2}{m^2} - \frac{(y+3)^2}{n^2} = 1$$

Therefore, the foci are at

$$\left(-2+\sqrt{m^2+n^2},-3\right)$$

and

$$\left(-2-\sqrt{m^2+n^2},-3\right)$$

Topic: Vertex, axis, focus, directrix of a hyperbola

Question: Which hyperbola opens left and right and has its axes at $x = -4$ and $y = -3$s?

Answer choices:

A $\quad x = \dfrac{7}{\cos t} - 4 \qquad y = 5\tan t - 3$

B $\quad x = \dfrac{5}{\cos t} - 4 \qquad y = 7\tan t - 3$

C $\quad x = \dfrac{3}{\cos t} + 4 \qquad y = 4\tan t - 3$

D $\quad x = \dfrac{6}{\cos t} - 4 \qquad y = 6\tan t + 3$

Solution: A

Check answer choice A by rewriting both equations.

$$x = \frac{7}{\cos t} - 4$$

$$x + 4 = \frac{7}{\cos t}$$

$$\frac{x+4}{7} = \frac{1}{\cos t}$$

$$\frac{(x+4)^2}{7^2} = \frac{1}{\cos^2 t}$$

$$\frac{(x+4)^2}{7^2} = \sec^2 t$$

$$\frac{(x+4)^2}{7^2} = 1 + \tan^2 t$$

and

$$y = 5\tan t - 3$$

$$y + 3 = 5\tan t$$

$$\frac{y+3}{5} = \tan t$$

$$\frac{(y+3)^2}{5^2} = \tan^2 t$$

Subtract this second equation from the first equation we found.

$$\frac{(x+4)^2}{7^2}-\frac{(y+3)^2}{5^2}=1+\tan^2 t-\tan^2 t$$

$$\frac{(x+4)^2}{7^2}-\frac{(y+3)^2}{5^2}=1$$

The axes of this hyperbola are $x = -4$ and $y = -3$, which are the axes we're looking for, so answer choice A must be the correct choice.

Answer choice B also has its axes at $x = -4$ and $y = -3$, but it opens up and down.

$$\frac{(x+4)^2}{5^2}-\frac{(y+3)^2}{7^2}=1$$

Answer choice C has its axes at $x = 4$ and $y = -3$, and it opens left and right.

$$\frac{(x-4)^2}{3^2}-\frac{(y+3)^2}{4^2}=1$$

Answer choice D has its axes at $x = -4$ and $y = 3$, and it opens left and right.

$$\frac{(x+4)^2}{6^2}-\frac{(y-3)^2}{6^2}=1$$

Topic: Equation of a hyperbolic conic section

Question: Which function defines the same hyperbola?

$$x = \frac{4}{\cos t} + 1$$

$$y = 2\sin t\left(1 + \tan t \tan\frac{t}{2}\right) - 1$$

Answer choices:

A $\frac{(x-4)^2}{1^2} - \frac{(y+1)^2}{2^2} = 1$

B $\frac{(x-1)^2}{2^2} - \frac{(y+1)^2}{4^2} = 1$

C $\frac{(x-1)^2}{4^2} - \frac{(y+1)^2}{2^2} = 1$

D $\frac{(x+1)^2}{4^2} - \frac{(y-1)^2}{2^2} = 1$

Solution: C

Rewrite both equations.

$$x = \frac{4}{\cos t} + 1$$

$$x - 1 = \frac{4}{\cos t}$$

$$\frac{(x-1)^2}{4^2} = \frac{1}{\cos^2 t} = 1 + \tan^2 t$$

and

$$y = 2\sin t\left(1 + \tan t \tan \frac{t}{2}\right) - 1$$

$$y + 1 = 2\sin t\left(1 + \tan t \tan \frac{t}{2}\right)$$

$$\frac{y+1}{2} = \sin t\left(1 + \tan t \tan \frac{t}{2}\right)$$

$$\frac{y+1}{2} = \tan t$$

$$\frac{(y+1)^2}{2^2} = \tan^2 t$$

Subtract this second equation from the first.

$$\frac{(x-1)^2}{4^2} - \frac{(y+1)^2}{2^2} = 1 + \tan^2 t - \tan^2 t$$

$$\frac{(x-1)^2}{4^2} - \frac{(y+1)^2}{2^2} = 1$$

Topic: Equation of a hyperbolic conic section

Question: Which parametric functions represent the hyperbola?

$$3x^2 - y^2 + 6x + 4y - 10 = 0$$

Answer choices:

A $x = \sec t - 1$ and $y = 3\tan t - 2$

B $x = -1 + \sqrt{3}\sec t$ and $y = 2\tan t - 3$

C $x = \csc t - 1$ and $y = 2 + 3\tan t$

D $x = -1 + \sqrt{3}\sec t$ and $y = 2 + 3\tan t$

Solution: D

Complete the square with respect to both variables.

$$3x^2 - y^2 + 6x + 4y - 10 = 0$$

$$3\left(x^2 + 2x + 1\right) - \left(y^2 - 4y + 4\right) = 9$$

$$3(x + 1)^2 - (y - 2)^2 = 9$$

Divide out the coefficients.

$$\frac{(x + 1)^2}{3} - \frac{(y - 2)^2}{9} = \frac{9}{9}$$

$$\frac{(x + 1)^2}{3} - \frac{(y - 2)^2}{9} = 1$$

For a hyperbola in the form

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

we can parametrize the curve using the equations $x = h + a\sec t$ and $y = k + b\tan t$. In our case, we get

$$x = -1 + \sqrt{3}\sec t$$

and

$$y = 2 + 3\tan t$$

Topic: Equation of a hyperbolic conic section

Question: Which set of parametric equations represent the hyperbola?

$$4x^2 - y^2 - 8mx + 2ny + 4m^2 - n^2 = 16$$

Answer choices:

A $\quad x = \dfrac{2}{\cos t} - m$ and $y = 4\tan t - n$

B $\quad x = \dfrac{2}{\cos t} + m$ and $y = 4\tan t + n$

C $\quad x = \dfrac{4}{\cos t} + m$ and $y = 4\tan t + n$

D $\quad x = \dfrac{16}{\cos t} + m$ and $y = 4\tan t + n$

Solution: B

Choose the functions from answer choice B.

$$x = \frac{2}{\cos t} + m$$

$$y = 4\tan t + n$$

Rewrite both equations.

$$x = \frac{2}{\cos t} + m$$

$$x - m = \frac{2}{\cos t}$$

$$\frac{x - m}{2} = \frac{1}{\cos t}$$

$$\frac{(x - m)^2}{2^2} = \frac{1}{\cos^2 t}$$

$$\frac{(x - m)^2}{2^2} = 1 + \tan^2 t$$

and

$$y = 4\tan t + n$$

$$y - n = 4\tan t$$

$$\frac{y - n}{4} = \tan t$$

$$\frac{(y-n)^2}{4^2} = \tan^2 t$$

Subtract this second equation from the first.

$$\frac{(x-m)^2}{4} - \frac{(y^2-n)^2}{16} = 1 + \tan^2 t - \tan^2 t$$

$$\frac{(x-m)^2}{4} - \frac{(y^2-n)^2}{16} = 1$$

Expand the equation above, and simplify.

$$\frac{x^2 - 2mx + m^2}{4} - \frac{y^2 - 2ny + n^2}{16} = 1$$

$$4\left(x^2 - 2mx + m^2\right) - y^2 - 2ny + n^2 = 16$$

$$4x^2 - 8mx + 4m^2 - y^2 - 2ny + n^2 = 16$$

$$4x^2 - y^2 - 8mx + 2ny + 4m^2 - n^2 = 16$$

Topic: Polar equation of a hyperbolic conic section

Question: The polar functions of two hyperbolas are given. What is the ratio of x_1y_1 to x_2y_2?

$$r_1^2 = \frac{1}{4\sin 2\theta_1}$$

$$r_2^2 = \frac{1}{6\sin 2\theta_2}$$

Answer choices:

A $\quad \dfrac{x_1y_1}{x_2y_2} = \dfrac{2}{3}$

B $\quad \dfrac{x_1y_1}{x_2y_2} = \dfrac{3}{2}$

C $\quad \dfrac{x_1y_1}{x_2y_2} = \dfrac{3}{4}$

D $\quad \dfrac{x_1y_1}{x_2y_2} = \dfrac{4}{3}$

Solution: B

Rewrite both equations.

$$r_1^2 = \frac{1}{4\sin 2\theta_1}$$

$$r_1^2\left(4\sin 2\theta_1\right) = 1$$

$$r_1^2\left(8\sin\theta_1\cos\theta_1\right) = 1$$

$$8\left(r_1\cos\theta_1\right)\left(r_1\sin\theta_1\right) = 1$$

$$8x_1y_1 = 1$$

and

$$r_2^2 = \frac{1}{6\sin 2\theta_2}$$

$$r_2^2\left(6\sin 2\theta_2\right) = 1$$

$$r_2^2\left(12\sin\theta_2\cos\theta_2\right) = 1$$

$$12\left(r_2\cos\theta_2\right)\left(r_2\sin\theta_2\right) = 1$$

$$12x_2y_2 = 1$$

Now pair these two equations together in a ratio.

$$\frac{8x_1y_1}{12x_2y_2} = \frac{1}{1}$$

$$\frac{x_1y_1}{x_2y_2} = \frac{12}{8} = \frac{3}{2}$$

Topic: Polar equation of a hyperbolic conic section

Question: Which conic section is defined by the polar function?

$$r - 4r\cos\theta - 5 = 0$$

Answer choices:

A Circle

B Ellipse

C Parabola

D Hyperbola

Solution: D

Solve the equation for r.

$$r - 4r\cos\theta - 5 = 0$$

$$(1 - 4\cos\theta)\, r - 5 = 0$$

$$(1 - 4\cos\theta)\, r = 5$$

$$r = \frac{5}{1 - 4\cos\theta}$$

Therefore the conic section is a hyperbola.

Topic: Polar equation of a hyperbolic conic section

Question: What are the eccentricity and directrix of the hyperbola?

$$3r - 5r\cos\theta - 9 = 0$$

Answer choices:

A $e = \frac{5}{3}$ and $d = \frac{27}{5}$

B $e = \frac{3}{5}$ and $d = \frac{27}{5}$

C $e = \frac{5}{3}$ and $d = \frac{9}{5}$

D $e = \frac{3}{5}$ and $d = \frac{9}{5}$

Solution: C

Transform the equation to standard polar form.

$$3r - 5r\cos\theta - 9 = 0$$

$$(3 - 5\cos\theta)r - 9 = 0$$

$$(3 - 5\cos\theta)r = 9$$

$$r = \frac{9}{3 - 5\cos\theta}$$

$$r = \frac{3}{1 - \frac{5}{3}\cos\theta}$$

In this form, we can see that the eccentricity and directrix are

$$e = \frac{5}{3}$$

$$d = 3 \div \frac{5}{3} = \frac{9}{5}$$

Topic: Hyperbolic identities

Question: Which of the following hyperbolic trigonometric identities is false?

Answer choices:

A $\text{csch}(x) = \dfrac{1}{\sinh(x)}$

B $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

C $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

D $\cosh^2(x) + \sinh^2(x) = 1$

Solution: D

The following list includes the basic hyperbolic identities.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \coth(x) = \frac{1}{\tanh(x)}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\tanh^2(x) + \operatorname{sech}^2(\mathrm{x}) = 1$$

$$\coth^2(\mathrm{x}) - \operatorname{csch}^2(\mathrm{x}) = 1$$

Answer choices A, B and C are all known identities, but answer choice D is not. It's similar to the identity $\cosh^2(x) - \sinh^2(x) = 1$, but the sign is wrong, so answer choice D is the correct answer.

Topic: Hyperbolic identities

Question: Which of the following hyperbolic trigonometric identities is true?

Answer choices:

A $\cosh(x) = \dfrac{e^x - e^{-x}}{2}$

B $\text{sech}(x) = \dfrac{1}{\cosh(x)}$

C $\text{coth}(x) = \dfrac{1}{\text{csch}(x)}$

D $\text{coth}^2(x) + \text{csch}^2(x) = 1$

Solution: B

The following list includes the basic hyperbolic identities

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \coth(x) = \frac{1}{\tanh(x)}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\tanh^2(x) + \operatorname{sech}^2(\mathrm{x}) = 1$$

$$\coth^2(\mathrm{x}) - \operatorname{csch}^2(\mathrm{x}) = 1$$

Let's look at our answer choices.

Answer choice A, $\cosh(x) = \dfrac{e^x - e^{-x}}{2}$ should be $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

Answer choice B, sech$(x) = \dfrac{1}{\cosh(x)}$ is correct

Answer choice C, coth$(x) = \dfrac{1}{\text{csch}(x)}$ should be coth$(x) = \dfrac{1}{\tanh(x)}$

Answer choice D, coth$^2(x) +$ csch$^2(x) = 1$ should be coth$^2(x) -$ csch$^2(x) = 1$

So answer choice B is the correct choice.

Topic: Hyperbolic identities

Question: Is the identity true or false?

$$\cosh^2(x) - \sinh^2(x) = \tanh^2(x) + \text{sech}^2(x)$$

Answer choices:

A True

B False

Solution: A

The following list includes the basic hyperbolic identities.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \coth(x) = \frac{1}{\tanh(x)}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\tanh^2(x) + \operatorname{sech}^2(x) = 1$$

$$\coth^2(x) - \operatorname{csch}^2(x) = 1$$

Given

$$\cosh^2(x) - \sinh^2(x) = \tanh^2(x) + \operatorname{sech}^2(x)$$

we can use the identity $\cosh^2(x) - \sinh^2(x) = 1$ to substitute into the given equation, and we get

$$1 = \tanh^2(x) + \operatorname{sech}^2(x)$$

Then, we'll use the identity $\tanh^2(x) + \operatorname{sech}^2(x) = 1$ to substitute into the right side of the equation and we get

$$1 = 1$$

Therefore, we've proven that the given equation is true.

Topic: Complex numbers

Question: Simplify the imaginary number.

$$i^{1,343}$$

Answer choices:

A i

B -1

C $-i$

D 1

Solution: C

We need to look for the largest number less than or equal to $1{,}343$ that's divisible by 4. $1{,}343$ isn't divisible by 4, so we try $1{,}342$, then $1{,}341$, then $1{,}340$. $1{,}340$ is the first number we come to that's divisible by 4, so we separate the exponent.

$$i^{1,343}$$

$$i^{1,340+3}$$

$$i^{1,340}i^3$$

Rewrite $1{,}340$ as a power of 4.

$$(i^4)^{335}i^3$$

We know that i^4 is always 1, so

$$(1)^{335}i^3$$

$$1i^3$$

$$i^3$$

We know that i^3 is equal to $-i$, so

$$i^{1,343} = -i$$

Topic: Complex numbers

Question: Name the imaginary part of the complex number.

$$z = 2 - 11i$$

Answer choices:

A $\quad 2$

B $\quad 11$

C $\quad -11$

D $\quad -11i$

Solution: C

For a complex number in the form $z = a + bi$, a is always the real part and b is always the imaginary part. If b is negative, you have to remember to include the negative sign when you name the imaginary part. So in the complex number $z = 2 - 11i$, 2 is the real part, and -11 is the imaginary part.

Topic: Complex numbers

Question: How can the number be classified?

$$z = 0 - 4i$$

Answer choices:

A Complex number

B Real number

C Pure imaginary number

D Both A and C

Solution: D

Every real number and every imaginary number is also a complex number. Because the number simplifies as

$$z = 0 - 4i$$

$$z = -4i$$

and the real part disappears, it can be classified as a pure imaginary number. But as a pure imaginary number, it's also automatically a complex number.

Topic: Complex number operations

Question: What are the sum and difference of the complex numbers?

$$2\frac{5}{6}-\frac{1}{3}i$$

$$-4\frac{1}{6}+\frac{1}{2}i$$

Answer choices:

A The sum is $-(11/6)+(1/6)i$

The difference is $(7/2)+(-1/3)i$

B The sum is $-(4/3)+(1/6)i$

The difference is $7+(-5/6)i$

C The sum is $-(7/6)+(1/3)i$

The difference is $5+(-2/3)i$

D The sum is $-1+(1/3)i$

The difference is $(20/3)+(-1/6)i$

Solution: B

The sum of the complex numbers is

$$\left(\frac{17}{6}-\frac{1}{3}i\right)+\left(-\frac{25}{6}+\frac{1}{2}i\right)$$

$$\left(\frac{17}{6}-\frac{25}{6}\right)+\left(-\frac{1}{3}i+\frac{1}{2}i\right)$$

$$\left(\frac{17}{6}-\frac{25}{6}\right)+\left(-\frac{1}{3}+\frac{1}{2}\right)i$$

$$-\frac{8}{6}+\frac{1}{6}i$$

$$-\frac{4}{3}+\frac{1}{6}i$$

The difference of the complex numbers is

$$\left(\frac{17}{6}-\frac{1}{3}i\right)-\left(-\frac{25}{6}+\frac{1}{2}i\right)$$

$$\frac{17}{6}-\frac{1}{3}i+\frac{25}{6}-\frac{1}{2}i$$

$$\frac{17}{6}+\frac{25}{6}-\frac{1}{3}i-\frac{1}{2}i$$

$$\frac{42}{6}-\frac{5}{6}i$$

$$7-\frac{5}{6}i$$

Topic: Complex number operations

Question: What is the product of the complex numbers?

$-9-5i$

$7+13i$

Answer choices:

A $-63-65i$

B $48+i$

C $-128-82i$

D $2-152i$

Solution: D

Use FOIL to find the product of the complex numbers.

$$(-9-5i)(7+13i)$$

$$(-9)(7)+(-9)(13i)+(-5i)(7)+(-5i)(13i)$$

$$-63+(-9)(13)i+(-5)(7)i+(-5)(13)(i^2)$$

$$-63-117i-35i+(-65)(i^2)$$

Using $i^2=-1$ in the last term, we get

$$-63-117i-35i+(-65)(-1)$$

$$-63-117i-35i+65$$

$$(-63+65)+(-117i-35i)$$

$$2+(-117-35)i$$

$$2-152i$$

Topic: Complex number operations

Question: Express the fraction in the form $a + bi$ where a and b are real numbers.

$$\frac{5+2i}{1+3i}$$

Answer choices:

A $\quad -\frac{5}{4} + \frac{7}{4}i$

B $\quad \frac{3}{5} - \frac{9}{10}i$

C $\quad \frac{11}{10} - \frac{13}{10}i$

D $\quad \frac{7}{8} + \frac{1}{8}i$

Solution: C

Multiply by the conjugate of the denominator.

$$\left(\frac{5+2i}{1+3i}\right)\left(\frac{1-3i}{1-3i}\right)$$

$$\frac{(5+2i)(1-3i)}{(1+3i)(1-3i)}$$

Use FOIL to expand the numerator and denominator.

$$\frac{5-15i+2i-6i^2}{1-3i+3i-9i^2}$$

$$\frac{5-13i-6i^2}{1-9i^2}$$

Using $i^2=-1$ gives

$$\frac{5-13i-6(-1)}{1-9(-1)}$$

$$\frac{5-13i+6}{1+9}$$

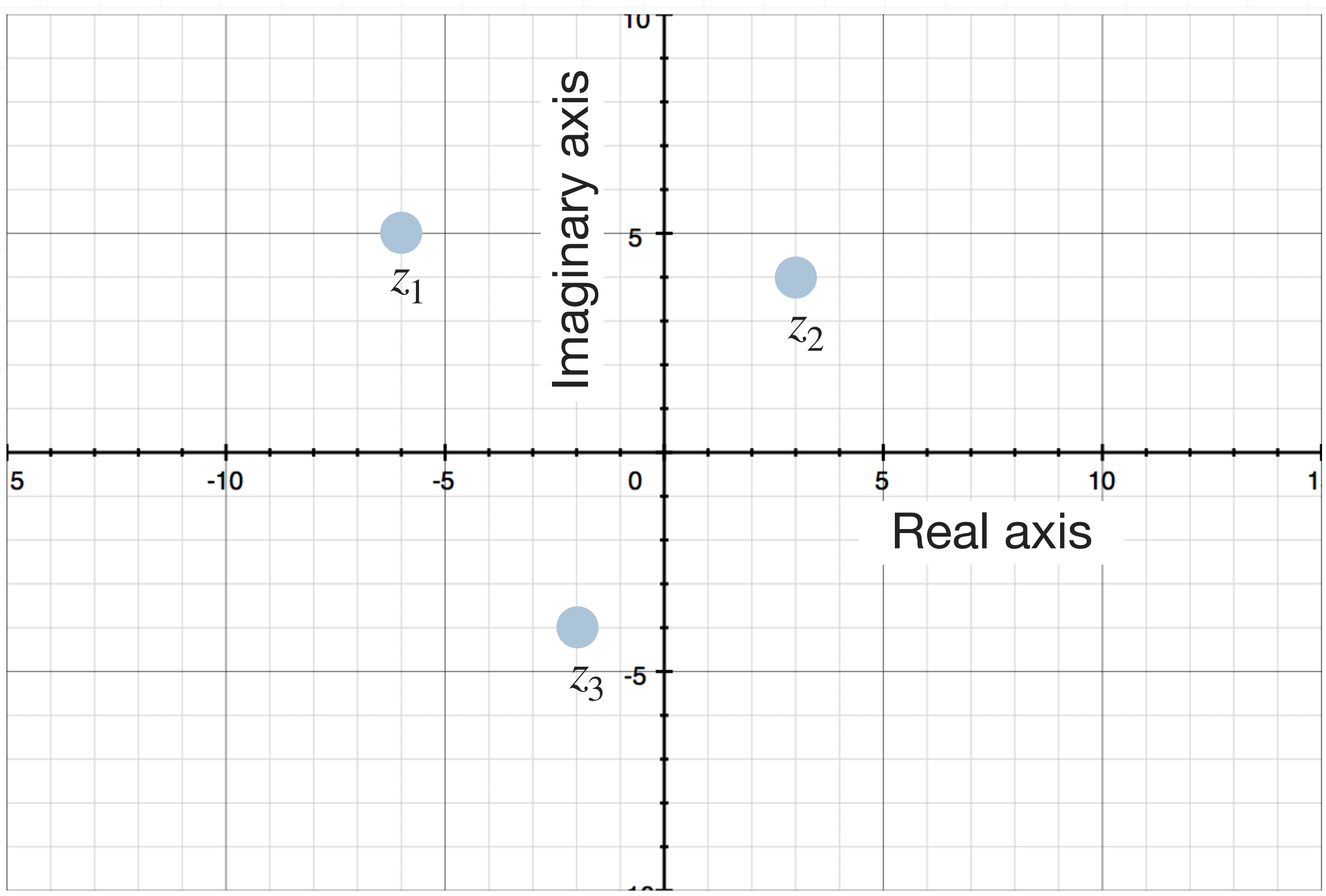
$$\frac{11-13i}{10}$$

Split the fraction.

$$\frac{11}{10}-\frac{13}{10}i$$

Topic: Graphing complex numbers

Question: Which three complex numbers are represented in the graph?



Answer choices:

A $\quad 3 - 4i$, $2 + 4i$, and $-5 + 6i$

B $\quad -2 - 4i$, $3 + 4i$, and $-6 + 5i$

C $\quad 5 + 6i$, $2 + 4i$, and $-3 - 4i$

D $\quad 3 + 4i$, $-6 - 5i$, and $-4 - 2i$

Solution: B

The point z_1 is 6 units to the left of the vertical axis and 5 units above the horizontal axis, so it's the complex number $-6+5i$.

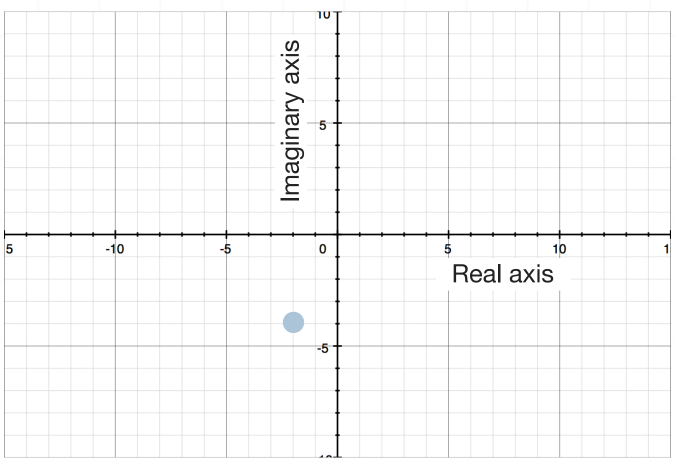
The point z_2 is 3 units to the right of the vertical axis and 4 units above the horizontal axis, so it's the complex number $3+4i$.

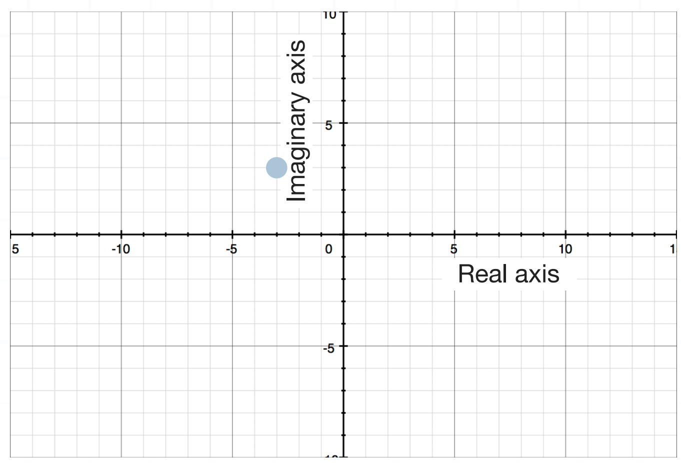
The point z_3 is 2 units to the left of the vertical axis and 4 units below the horizontal axis, so it's the complex number $-2-4i$.

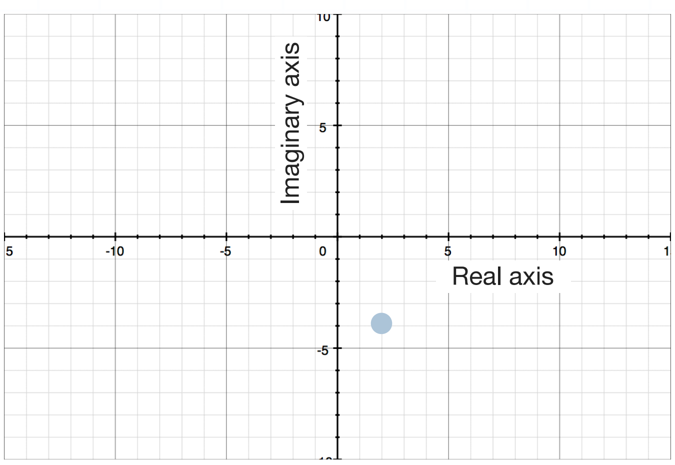
Topic: Graphing complex numbers

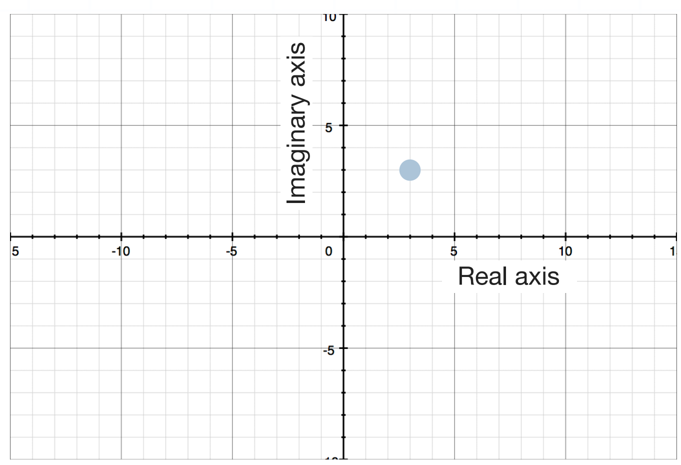
Question: Which graph shows the difference of $-8+5i$ and $-6+9i$ $((-8+5i)-(-6+9i))$?

Answer choices:

A


B


C


D


Solution: A

First, we'll compute the difference of the complex numbers.

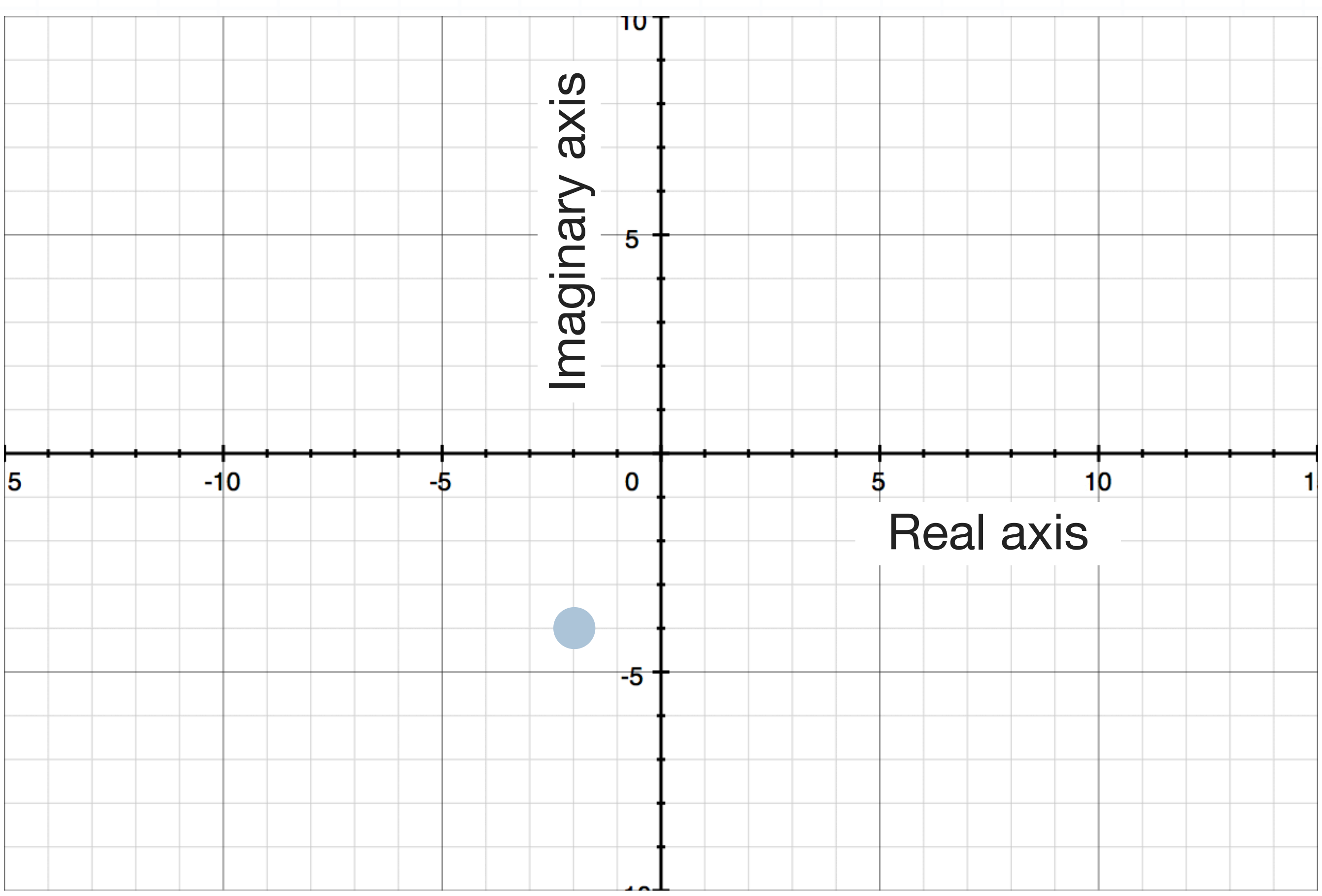
$$(-8+5i)-(-6+9i)$$

$$(-8-(-6))+(5-9)i$$

$$(-8+6)+(5-9)i$$

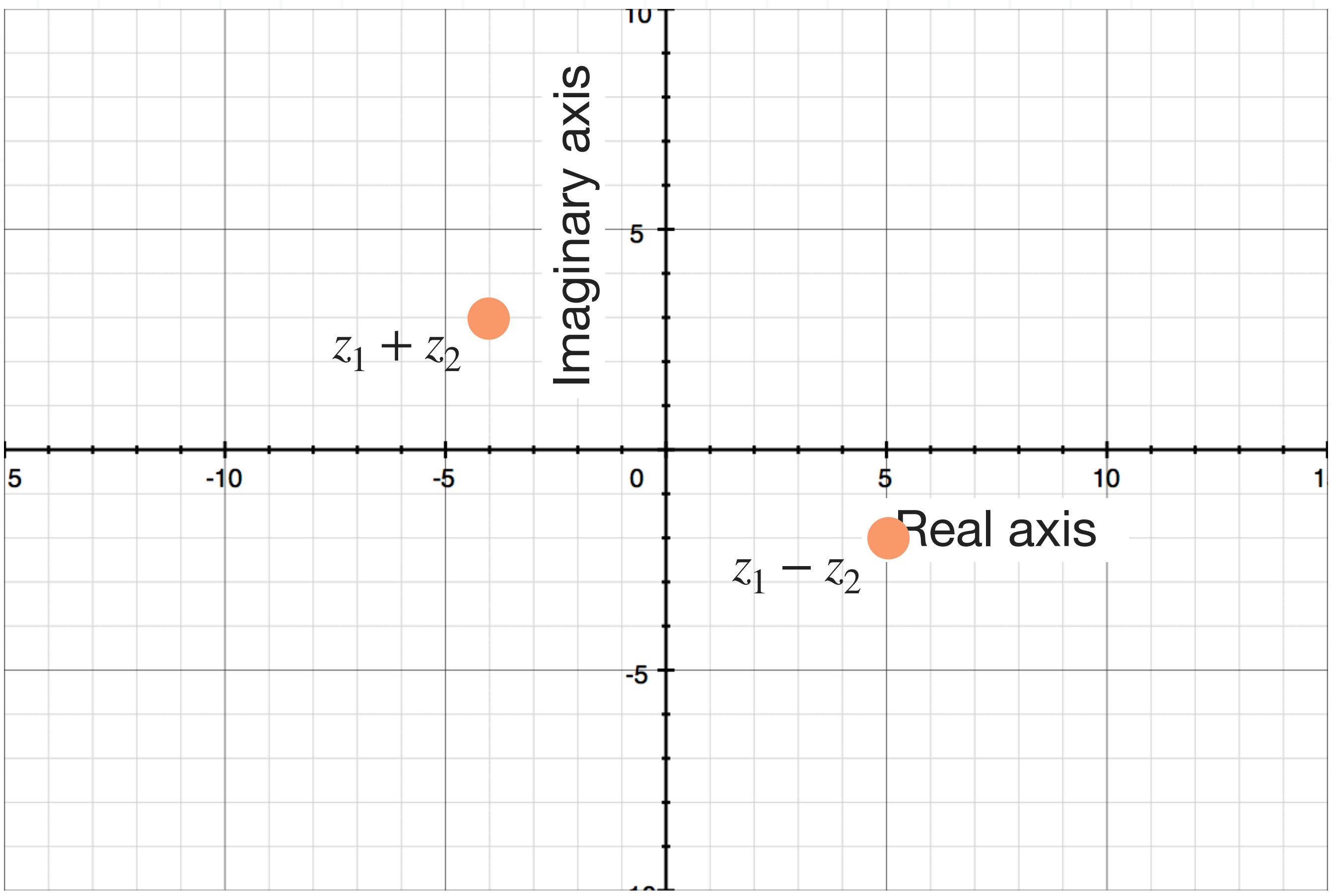
$$-2-4i$$

The real part of their difference is -2, and the imaginary part is -4. This means that the difference should be graphed 2 units to the left of the vertical axis and 4 units below the horizontal axis.



Topic: Graphing complex numbers

Question: The points on the graph are the sum $z_1 + z_2$ and difference $z_1 - z_2$ of two z_1 and z_2. Use a system of equations to find z_1 and z_2.



Answer choices:

A $\quad z_1 = (5/2) - (1/2)i$ and $z_2 = (1/2) + (1/2)i$

B $\quad z_1 = (3/2) + (7/2)i$ and $z_2 = -(5/2) - (3/2)i$

C $\quad z_1 = (1/2) + (1/2)i$ and $z_2 = -(9/2) + (5/2)i$

D $\quad z_1 = (7/2) - (3/2)i$ and $z_2 = (5/2) + (1/2)i$

Solution: C

The points on the graph are $z_1 + z_2 = -4 + 3i$ and $z_1 - z_2 = 5 - 2i$. We'll set up this system of equations:

$$z_1 + z_2 = -4 + 3i$$

$$z_1 - z_2 = 5 - 2i$$

Add the two equations together to eliminate z_2.

$$z_1 + z_2 = -4 + 3i$$

$$z_1 + z_2 + (z_1 - z_2) = -4 + 3i + (5 - 2i)$$

$$z_1 + z_2 + z_1 - z_2 = -4 + 3i + 5 - 2i$$

$$2z_1 = 1 + i$$

$$z_1 = \frac{1}{2} + \frac{1}{2}i$$

Substitute z_1 back into one of the other equations to find z_2.

$$z_1 + z_2 = -4 + 3i$$

$$\frac{1}{2} + \frac{1}{2}i + z_2 = -4 + 3i$$

$$z_2 = -4 + 3i - \frac{1}{2} - \frac{1}{2}i$$

$$z_2 = -\frac{9}{2} + \frac{5}{2}i$$

Topic: Distances and midpoints

Question: Use the distance formula to find the distance between the complex numbers $u = 3 + 4i$ and $z = 2 - 3i$.

Answer choices:

A $\quad d = \sqrt{15}$

B $\quad d = 5\sqrt{2}$

C $\quad d = \sqrt{26}$

D $\quad d = 4$

Solution: B

The distance formula is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The x-coordinates are the constants of the complex numbers and the y -coordinates are the coefficients of the imaginary numbers. Substitute the values into the distance formula and evaluate.

$$d = \sqrt{(3 - 2)^2 + (4 - (-3))^2}$$

$$d = \sqrt{(3 - 2)^2 + (4 + 3)^2}$$

$$d = \sqrt{1^2 + 7^2}$$

$$d = \sqrt{1 + 49}$$

$$d = \sqrt{50}$$

$$d = \sqrt{25 \cdot 2}$$

$$d = 5\sqrt{2}$$

Topic: Distances and midpoints

Question: Find the distance between the two complex numbers, $u = -3 - 3i$ and $z = -4 + 6i$, by graphing and using the Pythagorean theorem.

Answer choices:

A $c = \sqrt{10}$

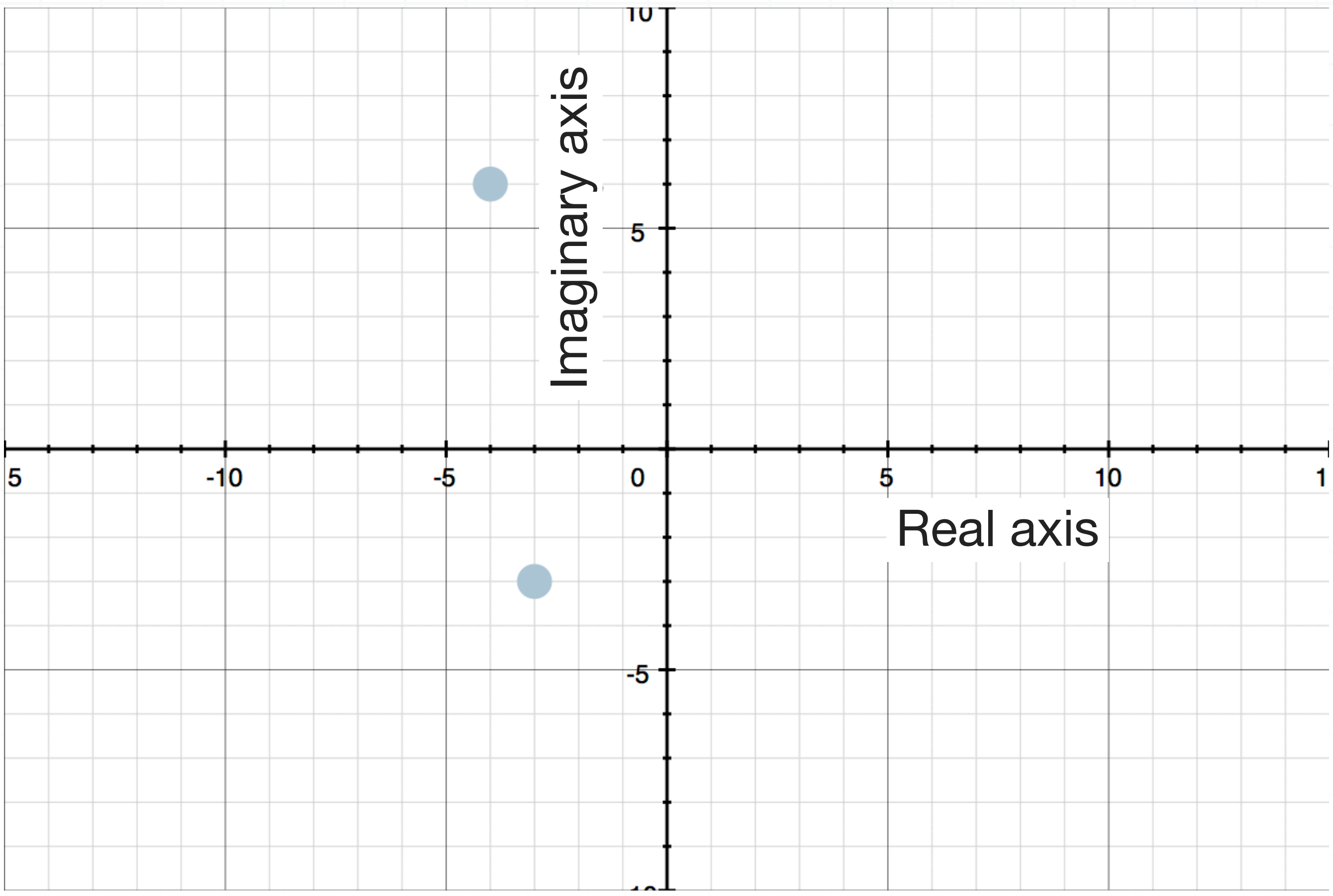
B $c = \sqrt{58}$

C $c = \sqrt{82}$

D $c = \sqrt{130}$

Solution: C

Graph $u = -3 - 3i$ and $z = -4 + 6i$ in the complex plane.



To find the distance between $u = -3 - 3i$ and $z = -4 + 6i$, start by finding the difference between the real parts and the imaginary parts.

The distance between the real parts is $-3 - (-4) = -3 + 4 = 1$, and the distance between the imaginary parts is $-3 - 6 = -9$. Then by the Pythagorean theorem, the distance between $u = -3 - 3i$ and $z = -4 + 6i$ is

$$1^2 + (-9)^2 = c^2$$

$$1 + 81 = c^2$$

$$82 = c^2$$

$$c = \sqrt{82}$$

Topic: Distances and midpoints

Question: Find the midpoint between $u = -3 - 3i$ and $z = -4 + 6i$.

Answer choices:

A $m = -3.5 + 1.5i$

B $m = -2 + i$

C $m = -1.5 + 1.5i$

D $m = -1 - 2i$

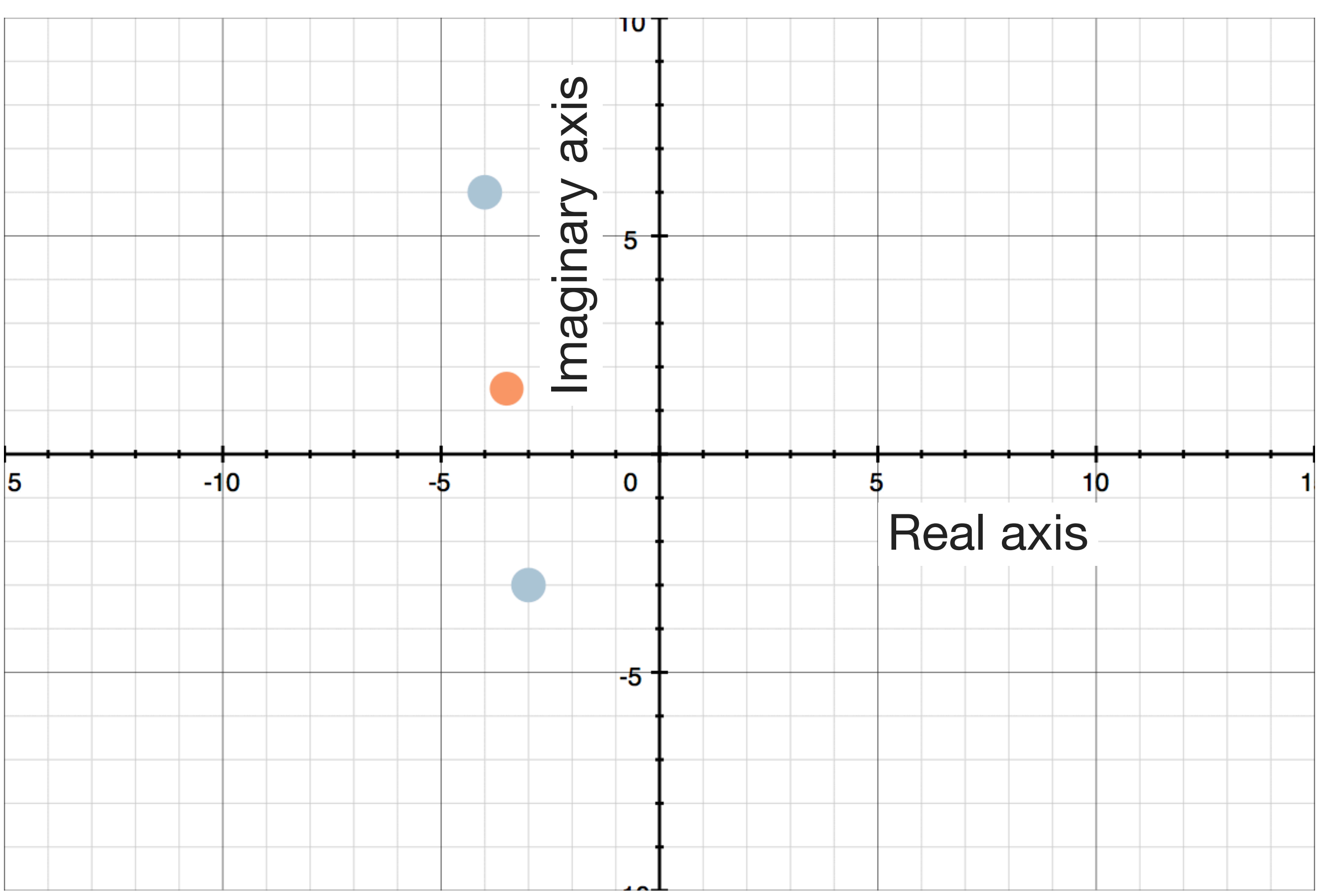
Solution: A

To find the midpoint between complex numbers, we find the midpoint of the real parts, and separately the midpoint of the imaginary parts.

The distance between the real parts of $u = -3 - 3i$ and $z = -4 + 6i$ is $-3 - (-4) = -3 + 4 = 1$. Half of that distance is $1/2 = 0.5$, so we look for the value that's 0.5 units from -3 and 0.5 units from -4, so the midpoint between those real parts must be -3.5.

The distance between the imaginary parts of $u = -3 - 3i$ and $z = -4 + 6i$ is $-3 - 6 = -9$. Half of that distance is $-9/2 = -4.5$, so we look for the value that's -4.5 units from -3 and -4.5 units from -6, so the midpoint between those imaginary parts must be 1.5.

So the midpoint between $u = -3 - 3i$ and $z = -4 + 6i$ is $m = -3.5 + 1.5i$. If we graph all three of these in the complex plane, we get



Topic: Complex numbers in polar form

Question: If the complex number $-3 - 7i$ is expressed in polar form, which quadrant contains the angle θ?

Answer choices:

A In the first quadrant

B On the negative vertical axis

C In the third quadrant

D On the positive horizontal axis

Solution: C

If we set the complex number equal to its polar form, we get

$$-3-7i=r(\cos\theta+i\sin\theta)$$

$$-3-7i=r\cos\theta+ri\sin\theta$$

From this equation, we know that

$$-3=r\cos\theta$$

$$\cos\theta=-\frac{3}{r}$$

The value of r is always positive, since r represents a distance, so $-3/r$ has to be less than 0, which means $\cos\theta$ has to be negative.

We also know from $-3-7i=r\cos\theta+ri\sin\theta$ that

$$-7=r\sin\theta$$

$$\sin\theta=-\frac{7}{r}$$

Because the value of r is always positive, $-7/r$ has to be less than 0, which means $\sin\theta$ has to be negative.

The values of $\cos\theta$ and $\sin\theta$ are negative in the third quadrant.

Topic: Complex numbers in polar form

Question: What is the polar form of the complex number?

$$-4 + 6i$$

Answer choices:

A $\quad 2\sqrt{5}\left[\cos(0.98) + i\sin(0.98)\right]$

B $\quad 2\sqrt{13}\left[\cos(2.16) + i\sin(2.16)\right]$

C $\quad 2\sqrt{13}\left[\cos(5.30) + i\sin(5.30)\right]$

D $\quad 2\sqrt{5}\left[\cos(3.14) + i\sin(3.14)\right]$

Solution: B

If we write the complex number $-4+6i$ as $a+bi$, we get $a=-4$ and $b=6$, so

$$r=\sqrt{a^2+b^2}=\sqrt{(-4)^2+6^2}=\sqrt{16+36}=\sqrt{52}=\sqrt{4(13)}=2\sqrt{13}$$

and

$$\tan\theta=\frac{b}{a}=\frac{6}{-4}=-\frac{3}{2}$$

Using the trigonometric identity $\sec^2\theta=1+\tan^2\theta$, we get

$$\sec^2\theta=1+\left(-\frac{3}{2}\right)^2=1+\frac{9}{4}=\frac{4(1)+1(9)}{4}=\frac{13}{4}$$

$$\cos^2\theta=(\cos\theta)^2=\left(\frac{1}{\sec\theta}\right)^2=\frac{1}{\sec^2\theta}=\frac{1}{\left(\frac{13}{4}\right)}=\frac{4}{13}$$

So

$$\cos\theta=\pm\sqrt{\frac{4}{13}}=\pm\frac{2}{\sqrt{13}}$$

The real part of z is $a=-4$, and the imaginary part is $b=6$, which puts the complex number in the second quadrant. Since the cosine of every angle in the second quadrant is negative, we get

$$\cos\theta=-\frac{2}{\sqrt{13}}$$

$$\arccos(\cos\theta) = \arccos\left(-\frac{2}{\sqrt{13}}\right)$$

$$\theta = \arccos\left(-\frac{2}{\sqrt{13}}\right)$$

$\theta \approx 2.16$ radians

Substituting the values of r and θ into the polar form for a complex number, we get

$$r(\cos\theta + i\sin\theta)$$

$$2\sqrt{13}\left[\cos(2.16) + i\sin(2.16)\right]$$

Topic: Complex numbers in polar form

Question: Write the complex number in polar form.

$$-14i$$

Answer choices:

A $\quad -14\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

B $\quad -14\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

C $\quad 14\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

D $\quad 14\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

Solution: D

The complex number $-14i$ can be written as $0-14i$, so its real part is 0, which means the number is located on the imaginary axis. Because $a=0$ and $b=-14$, the distance of $0-14i$ from the origin is

$$r=\sqrt{a^2+b^2}=\sqrt{0^2+(-14)^2}=\sqrt{0+196}=\sqrt{196}=14$$

Since the imaginary part of $0-14i$ is -14, which is negative, $0-14i$ is located on the negative imaginary axis, so $\theta=3\pi/2$. In polar form, we get

$$r(\cos\theta+i\sin\theta)$$

$$14\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$$

Topic: Multiplying and dividing polar forms

Question: What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = \sqrt{2}\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$$

$$z_2 = \frac{7}{3\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Answer choices:

A $\quad \dfrac{7}{\sqrt{6}}\left(\cos\dfrac{21\pi}{20} + i\sin\dfrac{21\pi}{20}\right)$

B $\quad \dfrac{7}{3}\left(\cos\dfrac{13\pi}{10} + i\sin\dfrac{13\pi}{10}\right)$

C $\quad \dfrac{3\sqrt{2}}{7}\left(\cos\dfrac{23\pi}{20} + i\sin\dfrac{23\pi}{20}\right)$

D $\quad \dfrac{7}{3}\left(\cos\dfrac{3\pi}{20} + i\sin\dfrac{3\pi}{20}\right)$

Solution: D

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$$

$$z_1z_2 = \left(\sqrt{2}\cdot\frac{7}{3\sqrt{2}}\right)\left[\cos\left(\frac{7\pi}{5}+\frac{3\pi}{4}\right)+i\sin\left(\frac{7\pi}{5}+\frac{3\pi}{4}\right)\right]$$

Simplify.

$$z_1z_2 = \frac{7}{3}\left[\cos\left(\frac{28\pi}{20}+\frac{15\pi}{20}\right)+i\sin\left(\frac{28\pi}{20}+\frac{15\pi}{20}\right)\right]$$

$$z_1z_2 = \frac{7}{3}\left(\cos\frac{43\pi}{20}+i\sin\frac{43\pi}{20}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we subtract 2π from the angle, we get

$$z_1z_2 = \frac{7}{3}\left[\cos\left(\frac{43\pi}{20}-\frac{40\pi}{20}\right)+i\sin\left(\frac{43\pi}{20}-\frac{40\pi}{20}\right)\right]$$

$$z_1z_2 = \frac{7}{3}\left(\cos\frac{3\pi}{20}+i\sin\frac{3\pi}{20}\right)$$

Topic: Multiplying and dividing polar forms

Question: What is the quotient z_1/z_2 of the complex numbers in polar form?

$$z_1 = 16\left(\cos\frac{9\pi}{13} + i\sin\frac{9\pi}{13}\right)$$

$$z_2 = \frac{5}{\sqrt{3}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Answer choices:

A $\quad \dfrac{16\sqrt{3}}{5}\left(\cos\dfrac{41\pi}{78} + i\sin\dfrac{41\pi}{78}\right)$

B $\quad \dfrac{80}{\sqrt{3}}\left(\cos\dfrac{17\pi}{39} + i\sin\dfrac{17\pi}{39}\right)$

C $\quad \dfrac{16}{5\sqrt{3}}\left(\cos\dfrac{9\pi}{78} + i\sin\dfrac{9\pi}{78}\right)$

D $\quad \dfrac{16\sqrt{3}}{5}\left(\cos\dfrac{17\pi}{39} + i\sin\dfrac{17\pi}{39}\right)$

Solution: A

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{16}{\frac{5}{\sqrt{3}}}\left[\cos\left(\frac{9\pi}{13} - \frac{\pi}{6}\right) + i\sin\left(\frac{9\pi}{13} - \frac{\pi}{6}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2} = 16 \cdot \frac{\sqrt{3}}{5}\left[\cos\left(\frac{54\pi}{78} - \frac{13\pi}{78}\right) + i\sin\left(\frac{54\pi}{78} - \frac{13\pi}{78}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{16\sqrt{3}}{5}\left(\cos\frac{41\pi}{78} + i\sin\frac{41\pi}{78}\right)$$

Topic: Multiplying and dividing polar forms

Question: Suppose that a complex number z is the quotient z_1/z_2 of the given complex numbers. If z is expressed in polar form, $r(\cos\theta + i\sin\theta)$, where is θ located?

$$z_1 = 4\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right)$$

$$z_2 = \frac{17}{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Answer choices:

A In the first quadrant

B On the negative horizontal axis

C In the second quadrant

D On the positive vertical axis

Solution: C

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

$$\frac{z_1}{z_2}=\frac{4}{\frac{17}{3}}\left[\cos\left(\frac{13\pi}{9}-\frac{5\pi}{6}\right)+i\sin\left(\frac{13\pi}{9}-\frac{5\pi}{6}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2}=4\cdot\frac{3}{17}\left[\cos\left(\frac{78\pi}{54}-\frac{45\pi}{54}\right)+i\sin\left(\frac{78\pi}{54}-\frac{45\pi}{54}\right)\right]$$

$$\frac{z_1}{z_2}=\frac{12}{17}\left(\cos\frac{33\pi}{54}+i\sin\frac{33\pi}{54}\right)$$

The fraction $33/54$ is approximately equal to 0.6, so the angle is about 0.6π, which is in the second quadrant.

Topic: Powers of complex numbers and De Moivre's theorem

Question: Find z^5 in rectangular form $a+bi$?

$$z=3\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

Answer choices:

A $\quad 75\sqrt{2}+75\sqrt{2}i$

B $\quad 972-972i$

C $\quad 243\sqrt{2}+243\sqrt{2}i$

D $\quad 75-225i$

Solution: B

Plug $r = 3\sqrt{2}$, $\theta = 3\pi/4$, and $n = 5$ into De Moivre's theorem.

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^5 = \left(3\sqrt{2}\right)^5 \left[\cos\left(5 \cdot \frac{3\pi}{4}\right) + i\sin\left(5 \cdot \frac{3\pi}{4}\right)\right]$$

Then simplify.

$$z^5 = 3^5\left(\sqrt{2}\right)^5 \left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right)$$

$$z^5 = 243\left(4\sqrt{2}\right)\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$z^5 = 243\left(2\sqrt{2}\right)\left(\sqrt{2} - \sqrt{2}i\right)$$

$$z^5 = 486\sqrt{2}\left(\sqrt{2} - \sqrt{2}i\right)$$

$$z^5 = 486\sqrt{2}\sqrt{2} - 486\sqrt{2}\sqrt{2}i$$

$$z^5 = 486(2) - 486(2)i$$

$$z^5 = 972 - 972i$$

Topic: Powers of complex numbers and De Moivre's theorem

Question: Find z^4 in polar form?

$$z = \sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Answer choices:

A $\quad 6\left(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)$

B $\quad 18\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

C $\quad 9\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

D $\quad 3\left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right)$

Solution: C

Plug $r = \sqrt{3}$, $\theta = 2\pi/3$, and $n = 4$ into De Moivre's theorem.

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^4 = \left(\sqrt{3}\right)^4 \left[\cos\left(4 \cdot \frac{2\pi}{3}\right) + i\sin\left(4 \cdot \frac{2\pi}{3}\right)\right]$$

Then simplify.

$$z^4 = 9\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$$

We could leave the answer this way, but the angle $8\pi/3$ is larger than 2π, so we can reduce the angle to one that's coterminal with $8\pi/3$, but within the interval $[0,2\pi)$.

$$\frac{8\pi}{3} - 2\pi = \frac{8\pi}{3} - 2\pi\left(\frac{3}{3}\right) = \frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$$

So the complex number z^4 in polar form can be written as

$$z^4 = 9\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Topic: Powers of complex numbers and De Moivre's theorem

Question: In the complex number z, a is a positive real number and k is a nonnegative integer. Where in the complex plane is z^6 located?

$$z = a\left[\cos\left(\frac{(2k+3)\pi}{2}\right) + i\sin\left(\frac{(2k+3)\pi}{2}\right)\right]$$

Answer choices:

A In the first quadrant

B On the positive vertical axis

C In the third quadrant

D On the negative horizontal axis

Solution: D

From the given complex number, we have

$$r = a$$

$$\theta = \frac{(2k+3)\pi}{2}$$

Because we're looking for z^6, we know $n = 6$, and we plug everything into De Moivre's theorem.

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^6 = a^6 \left[\cos\left(6 \cdot \frac{(2k+3)\pi}{2}\right) + i\sin\left(6 \cdot \frac{2k+3)\pi}{2}\right)\right]$$

Then simplify.

$$z^6 = a^6 \left[\cos\left(\frac{(12k+18)\pi}{2}\right) + i\sin\left(\frac{(12k+18)\pi}{2}\right)\right]$$

$$z^6 = a^6 \left[\cos(6k+9)\pi + i\sin(6k+9)\pi\right]$$

If we test a couple of k-values, we get

For $k = 0$:

$$z^6 = a^6 \left[\cos(6(0)+9)\pi + i\sin(6(0)+9)\pi\right]$$

$$z^6 = a^6(\cos 9\pi + i\sin 9\pi)$$

$$z^6 = a^6(-1+0i)$$

For $k = 1$:

$$z^6 = a^6\left[\cos(6(1)+9)\pi + i\sin(6(1)+9)\pi\right]$$

$$z^6 = a^6(\cos 15\pi + i\sin 15\pi)$$

$$z^6 = a^6(-1+0i)$$

For $k = 2$:

$$z^6 = a^6\left[\cos(6(2)+9)\pi + i\sin(6(2)+9)\pi\right]$$

$$z^6 = a^6(\cos 21\pi + i\sin 21\pi)$$

$$z^6 = a^6(-1+0i)$$

We could keep going, but we realize that we're getting the same value each time, which is $z^6 = a^6(-1+0i)$.

The problem told us that a is a positive real number, which means that when we distribute it across the $(-1+0i)$, we'll still have a negative real part and a zero imaginary part. In the case when the real part is negative, and the imaginary part is 0, that always put us somewhere on the negative half of the horizontal (real) axis.

Topic: Complex number equations

Question: Find the solution of the complex equation that lies in the third quadrant.

$$z^3 = 125$$

Answer choices:

A $\quad z = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$

B $\quad z = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

C $\quad z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

D $\quad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution: B

Rewrite z^3 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^3 = r^3 \left[\cos(3\theta) + i\sin(3\theta)\right]$$

Rewrite 125 as the complex number $125 + 0i$. The modulus and angle of $125 + 0i$ are

$$r = \sqrt{125^2 + 0^2}$$

$$r = \sqrt{125^2}$$

$$r = 125$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{125} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 125\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 125\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 125\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^3 = 125$, we can start making substitutions.

$$z^3 = 125$$

$$r^3 \left[\cos(3\theta) + i\sin(3\theta)\right] = 125$$

$$r^3 \left[\cos(3\theta) + i\sin(3\theta)\right] = 125 \left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^3 = 125$$

$$3\theta = 360^\circ k$$

From these equations, we get

$r^3 = 125$, so $r = 5$

$3\theta = 360^\circ k$, so $\theta = 120^\circ k$

To $\theta = 120^\circ k$, if we plug in $k = 0,\ 1,\ 2,\ \dots$, we get

For $k = 0$, $\theta = 120^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 120^\circ(1) = 120^\circ$

For $k = 2$, $\theta = 120^\circ(2) = 240^\circ$

...

We could keep going for $k = 3,\ 4,\ 5,\ 6,\ \dots$, but $k = 3$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ,\ 120^\circ,\ 240^\circ$.

Plugging these three angles and $r = 5$ into the formula for polar form of a complex number, we'll get the solutions to $z^3 = 125$.

$$z_1 = 5\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 5\left[1 + i(0)\right] = 5$$

$$z_2 = 5\left[\cos(120^\circ) + i\sin(120^\circ)\right] = 5\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

$$z_3 = 5\left[\cos(240^\circ) + i\sin(240^\circ)\right] = 5\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$$

Topic: Complex number equations

Question: Find the solutions of the complex equation.

$$z^2 = 81$$

Answer choices:

A $z = 3$ and $z = -3$

B $z = 3i$ and $z = -3i$

C $z = 9$ and $z = -9$

D $z = 9i$ and $z = -9i$

Solution: C

Rewrite z^2 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^2 = r^2 \left[\cos(2\theta) + i\sin(2\theta)\right]$$

Rewrite 81 as the complex number $81 + 0i$. The modulus and angle of $81 + 0i$ are

$$r = \sqrt{81^2 + 0^2}$$

$$r = \sqrt{81^2}$$

$$r = 81$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{81} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 81\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 81\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 81\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^2 = 81$, we can start making substitutions.

$$z^2 = 81$$

$$r^2 \left[\cos(2\theta) + i\sin(2\theta)\right] = 81$$

$$r^2 \left[\cos(2\theta) + i\sin(2\theta)\right] = 81 \left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^2 = 81$$

$$2\theta = 360^\circ k$$

From these equations, we get

$r^2 = 81$, so $r = 9$

$2\theta = 360^\circ k$, so $\theta = 180^\circ k$

To $\theta = 180^\circ k$, if we plug in $k = 0,\ 1,\ \ldots$, we get

For $k = 0$, $\theta = 180^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 180^\circ(1) = 180^\circ$

...

We could keep going for $k = 2,\ 3,\ 4,\ 5,\ \ldots$, but $k = 2$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ,\ 180^\circ$.

Plugging these two angles and $r = 9$ into the formula for polar form of a complex number, we'll get the solutions to $z^2 = 81$.

$$z_1 = 9\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 9\left[1 + i(0)\right] = 9$$

$$z_2 = 9\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 9\left[-1 + i(0)\right] = -9$$

Topic: Complex number equations

Question: How many solutions of the complex equation lie in the fourth quadrant?

$$z^6 = 64$$

Answer choices:

A $\quad 1$

B $\quad 2$

C $\quad 3$

D $\quad 4$

Solution: A

Rewrite z^6 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^6 = r^6 \left[\cos(6\theta) + i\sin(6\theta)\right]$$

Rewrite 64 as the complex number $64 + 0i$. The modulus and angle of $64 + 0i$ are

$$r = \sqrt{64^2 + 0^2}$$

$$r = \sqrt{64^2}$$

$$r = 64$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{64} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 64\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 64\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 64\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^6 = 64$, we can start making substitutions.

$$z^6 = 64$$

$$r^6 \left[\cos(6\theta) + i\sin(6\theta)\right] = 64$$

$$r^6 \left[\cos(6\theta) + i\sin(6\theta)\right] = 64 \left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^6 = 64$$

$$6\theta = 360^\circ k$$

From these equations, we get

$r^6 = 64$, so $r = 2$

$6\theta = 360^\circ k$, so $\theta = 60^\circ k$

To $\theta = 60^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, 5, \dots$, we get

For $k = 0$, $\theta = 60^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 60^\circ(1) = 60^\circ$

For $k = 2$, $\theta = 60^\circ(2) = 120^\circ$

For $k = 3$, $\theta = 60^\circ(3) = 180^\circ$

For $k = 4$, $\theta = 60^\circ(4) = 240^\circ$

For $k = 5$, $\theta = 60^\circ(5) = 300^\circ$

...

We could keep going for $k = 6, 7, 8, 9, \ldots$, but $k = 6$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$.

Plugging these six angles and $r = 2$ into the formula for polar form of a complex number, we'll get the solutions to $z^6 = 64$.

$$z_1 = 2\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 2\left[1 + i(0)\right] = 2$$

$$z_2 = 2\left[\cos(60^\circ) + i\sin(60^\circ)\right] = 2\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = 1 + \sqrt{3}i$$

$$z_3 = 2\left[\cos(120^\circ) + i\sin(120^\circ)\right] = 2\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -1 + \sqrt{3}i$$

$$z_4 = 2\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 2\left[-1 + i(0)\right] = -2$$

$$z_5 = 2\left[\cos(240^\circ) + i\sin(240^\circ)\right] = 2\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -1 - \sqrt{3}i$$

$$z_6 = 2\left[\cos(300^\circ) + i\sin(300^\circ)\right] = 2\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = 1 - \sqrt{3}i$$

Roots in the fourth quadrant will have a positive real part and a negative imaginary part. That's only z_6, so there's one solution in the fourth quadrant.

Topic: Roots of complex numbers

Question: Which of the following is a cube root of z?

$$z = 64\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$$

Answer choices:

A $\quad 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

B $\quad 6\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$

C $\quad 4\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$

D $\quad 6\left(\cos\frac{11\pi}{2} + i\sin\frac{11\pi}{2}\right)$

Solution: C

We're looking for the third (or cube) roots of z, which means there will be 3 of them, given by $k = 0, 1, 2$. And since the complex number is given in radians, we'll plug $n = 3$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[3]{z} = \sqrt[3]{r}\left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i\sin\left(\frac{\theta + 2\pi k}{3}\right)\right]$$

With $r = 64$ and $\theta = 11\pi/8$ from the complex number, we get

$$\sqrt[3]{z} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{11\pi}{8} + 2\pi k}{3}\right)\right]$$

Now we'll find values for $k = 0, 1, 2$.

For $k = 0$:

$$\sqrt[3]{z}_{k=0} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8} + 2\pi(0)}{3}\right) + i\sin\left(\frac{\frac{11\pi}{8} + 2\pi(0)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=0} = 4\left(\cos\frac{11\pi}{24} + i\sin\frac{11\pi}{24}\right)$$

For $k = 1$:

$$\sqrt[3]{z}_{k=1} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8} + 2\pi(1)}{3}\right) + i\sin\left(\frac{\frac{11\pi}{8} + 2\pi(1)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{27\pi}{24} + i\sin\frac{27\pi}{24}\right)$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$$

For $k = 2$:

$$\sqrt[3]{z}_{k=2} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8} + 2\pi(2)}{3}\right) + i\sin\left(\frac{\frac{11\pi}{8} + 2\pi(2)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=2} = 4\left(\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right)$$

The roots are

$$\sqrt[3]{z}_{k=0} = 4\left(\cos\frac{11\pi}{24} + i\sin\frac{11\pi}{24}\right)$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$$

$$\sqrt[3]{z}_{k=2} = 4\left(\cos\frac{43\pi}{24} + i\sin\frac{43\pi}{24}\right)$$

The matching root is from $k = 1$.

Topic: Roots of complex numbers

Question: How many of the seventh roots of z lie in the third quadrant of the complex plane?

$$z = 15\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$

Answer choices:

A One

B Two

C Three

D None

Solution: B

We're looking for the seventh roots of z, which means there will be 7 of them, given by $k = 0, 1, 2, 3, 4, 5, 6$. And since the complex number is given in radians, we'll plug $n = 7$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[7]{z} = \sqrt[7]{r}\left[\cos\left(\frac{\theta + 2\pi k}{7}\right) + i\sin\left(\frac{\theta + 2\pi k}{7}\right)\right]$$

With $r = 15$ and $\theta = \pi/10$ from the complex number, we get

$$\sqrt[7]{z} = \sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10} + 2\pi k}{7}\right) + i\sin\left(\frac{\frac{\pi}{10} + 2\pi k}{7}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4, 5, 6$.

For $k = 0$:

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10} + 2\pi(0)}{7}\right) + i\sin\left(\frac{\frac{\pi}{10} + 2\pi(0)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{15}\left(\cos\frac{\pi}{70} + i\sin\frac{\pi}{70}\right)$$

For $k = 1$:

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10}+2\pi(1)}{7}\right)+i\sin\left(\frac{\frac{\pi}{10}+2\pi(1)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left(\cos\frac{21\pi}{70}+i\sin\frac{21\pi}{70}\right)$$

For $k=2$:

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10}+2\pi(2)}{7}\right)+i\sin\left(\frac{\frac{\pi}{10}+2\pi(2)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left(\cos\frac{41\pi}{70}+i\sin\frac{41\pi}{70}\right)$$

We can start to see how we're just adding $20\pi/70$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[7]{z}_{k=0}=\sqrt[7]{15}\left(\cos\frac{\pi}{70}+i\sin\frac{\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left(\cos\frac{21\pi}{70}+i\sin\frac{21\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left(\cos\frac{41\pi}{70}+i\sin\frac{41\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=3}=\sqrt[7]{15}\left(\cos\frac{61\pi}{70}+i\sin\frac{61\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=4}=\sqrt[7]{15}\left(\cos\frac{81\pi}{70}+i\sin\frac{81\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=5} = \sqrt[7]{15}\left(\cos\frac{101\pi}{70} + i\sin\frac{101\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=6} = \sqrt[7]{15}\left(\cos\frac{121\pi}{70} + i\sin\frac{121\pi}{70}\right)$$

If we find the decimal approximations of these angles, we get

For $k = 0$, $(1/70)\pi \approx 0.01\pi$

For $k = 1$, $(21/70)\pi \approx 0.3\pi$

For $k = 2$, $(41/70)\pi \approx 0.59\pi$

For $k = 3$, $(61/70)\pi \approx 0.87\pi$

For $k = 4$, $(81/70)\pi \approx 1.16\pi$

For $k = 5$, $(101/70)\pi \approx 1.44\pi$

For $k = 6$, $(121/70)\pi \approx 1.73\pi$

Anything in the third quadrant will fall in the interval $(1\pi, 1.5\pi)$, which in this case are the angles for $k = 4$ and $k = 5$, so two of the seventh roots fall in the third quadrant.

Topic: Roots of complex numbers

Question: Find the 4th root of the complex number that lies in the fourth quadrant of the complex plane.

$$z = 16\left(\cos 30^\circ + i\sin 30^\circ\right)$$

Answer choices:

A $\quad 2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right]$

B $\quad 2\left[\cos(297.5^\circ) + i\sin(297.5^\circ)\right]$

C $\quad 2\left[\cos(317.5^\circ) + i\sin(317.5^\circ)\right]$

D $\quad 2\left[\cos(337.5^\circ) + i\sin(337.5^\circ)\right]$

Solution: A

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k=0, 1, 2, 3$. And since the complex number is given in degrees, we'll plug $n=4$ into the formula for nth roots in degrees.

$$\sqrt[n]{z}=\sqrt[n]{r}\left[\cos\left(\frac{\theta+360^\circ k}{n}\right)+i\sin\left(\frac{\theta+360^\circ k}{n}\right)\right]$$

$$\sqrt[4]{z}=\sqrt[4]{r}\left[\cos\left(\frac{\theta+360^\circ k}{4}\right)+i\sin\left(\frac{\theta+360^\circ k}{4}\right)\right]$$

With $r=16$ and $\theta=30^\circ$ from the complex number, we get

$$\sqrt[4]{z}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ k}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ k}{4}\right)\right]$$

Now we'll find values for $k=0, 1, 2, 3$.

For $k=0$:

$$\sqrt[4]{z}_{k=0}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ(0)}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0}=2\left[\cos(7.5^\circ)+i\sin(7.5^\circ)\right]$$

For $k=1$:

$$\sqrt[4]{z}_{k=1}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ(1)}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1} = 2\left[\cos(97.5^\circ) + i\sin(97.5^\circ)\right]$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{16}\left[\cos\left(\frac{30^\circ + 360^\circ(2)}{4}\right) + i\sin\left(\frac{30^\circ + 360^\circ(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = 2\left[\cos(187.5^\circ) + i\sin(187.5^\circ)\right]$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{16}\left[\cos\left(\frac{30^\circ + 360^\circ(3)}{4}\right) + i\sin\left(\frac{30^\circ + 360^\circ(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3} = 2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right]$$

The roots are

$$\sqrt[4]{z}_{k=0} = 2\left[\cos(7.5^\circ) + i\sin(7.5^\circ)\right] \approx 1.982 + 0.262i$$

$$\sqrt[4]{z}_{k=1} = 2\left[\cos(97.5^\circ) + i\sin(97.5^\circ)\right] \approx -0.262 + 1.982i$$

$$\sqrt[4]{z}_{k=2} = 2\left[\cos(187.5^\circ) + i\sin(187.5^\circ)\right] \approx -1.982 - 0.262i$$

$$\sqrt[4]{z}_{k=3} = 2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right] \approx 0.262 - 1.982i$$

The root in the fourth quadrant will have a positive real part and a negative imaginary part, which is the root for $k = 3$.

Topic: Matrix dimensions and entries

Question: Give the dimensions of the matrix.

$$K = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 5 & 6 & -2 \end{bmatrix}$$

Answer choices:

A The dimensions are 4×2

B The dimensions are 1×8

C The dimensions are 2×4

D The dimensions are 3×3

Solution: C

We always give the dimensions of a matrix as rows $\times$ columns. Matrix K has 2 rows and 4 columns, so K is a 2×4 matrix.

Topic: Matrix dimensions and entries

Question: Given matrix B, find $B_{2,1}$.

$$B = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

Answer choices:

A -1

B 0

C -1

D 3

Solution: B

The value of $B_{2,1}$ is the entry in the second row, first column of matrix B, which is 0, so $B_{2,1}=0$.

Topic: Matrix dimensions and entries

Question: Give the dimensions of matrix M and find $M_{3,2}$.

$$M = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -1 & 2 \\ 9 & 4 & 6 \end{bmatrix}$$

Answer choices:

A The dimensions are 3×3 and $M_{3,2} = 4$

B The dimensions are 2×3 and $M_{3,2} = 2$

C The dimensions are 3×3 and $M_{3,2} = 2$

D The dimensions are 3×1 and $M_{3,2} = 4$

Solution: A

We always give the dimensions of a matrix as rows $\times$ columns. Matrix M has 3 rows and 3 columns, so M is a 3×3 matrix.

The value of $M_{3,2}$ is the entry in the third row, second column of matrix M, which is 4, so $M_{3,2} = 4$.

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called B.

$$4x + 2y = 8$$

$$-2x + 7y = 11$$

Answer choices:

A $\quad B = \begin{bmatrix} 8 & 2 & | & 4 \\ 11 & -2 & | & 7 \end{bmatrix}$

B $\quad B = \begin{bmatrix} 4 & 2 & | & 8 \\ 7 & -2 & | & 11 \end{bmatrix}$

C $\quad B = \begin{bmatrix} 2 & 4 & | & 8 \\ -2 & 7 & | & 11 \end{bmatrix}$

D $\quad B = \begin{bmatrix} 4 & 2 & | & 8 \\ -2 & 7 & | & 11 \end{bmatrix}$

Solution: D

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

This problem is straightforward because the system is set up correctly with all variables in both equations.

$$4x + 2y = 8$$

$$-2x + 7y = 11$$

The system contains the variables x and y along with a constant. Which means the augmented matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows.

Plugging the coefficients and constants into an augmented matrix gives

$$B = \begin{bmatrix} 4 & 2 & | & 8 \\ -2 & 7 & | & 11 \end{bmatrix}$$

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called G.

$$a - 3b + 9c + 6d = 4$$

$$8a + 6c = 9d + 15$$

Answer choices:

A $G = \begin{bmatrix} 1 & -3 & 9 & 6 & 4 \\ 8 & 0 & 6 & -9 & 15 \end{bmatrix}$

B $G = \begin{bmatrix} 1 & 9 & 6 & 4 \\ 8 & 6 & -9 & 15 \end{bmatrix}$

C $G = \begin{bmatrix} 1 & 3 & 9 & 6 & 4 \\ 8 & 0 & 6 & 9 & 15 \end{bmatrix}$

D $G = \begin{bmatrix} 1 & -3 & 9 & 6 & 4 \\ 15 & 6 & 0 & 5 & 8 \end{bmatrix}$

Solution: A

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

To do this, the second equation can be reorganized by putting a, c, and d on the left side, and the constant on the right side. We also recognize that there is no b-term in the second equation, so we add in a 0 "filler" term.

$$a - 3b + 9c + 6d = 4$$

$$8a + 0b + 6c - 9d = 15$$

The system contains the variables a, b, c, and d, along with a constant. Which means the augmented matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows.

Plugging the coefficients and constants into an augmented matrix gives

$$G = \begin{bmatrix} 1 & -3 & 9 & 6 & 4 \\ 8 & 0 & 6 & -9 & 15 \end{bmatrix}$$

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called N.

$$6a + 4b - c = 9$$

$$5b = -6a + 7c - 6$$

$$3c = 14 - 2a$$

Answer choices:

A $\quad N = \begin{bmatrix} 6 & 4 & -1 & 9 \\ 5 & -6 & 7 & -6 \\ 3 & 14 & -2 & 0 \end{bmatrix}$

B $\quad N = \begin{bmatrix} 6 & 4 & -1 & 9 \\ -6 & 5 & 7 & -6 \\ -2 & 3 & -14 & 0 \end{bmatrix}$

C $\quad N = \begin{bmatrix} 6 & 4 & -1 & 9 \\ 6 & 5 & -7 & -6 \\ 2 & 0 & 3 & 14 \end{bmatrix}$

D $\quad N = \begin{bmatrix} -2 & 3 & 0 & -14 \\ 6 & 4 & 1 & 9 \\ 6 & 5 & 7 & 6 \end{bmatrix}$

Solution: C

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

To do this, the second two equations can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no b-term in the third equation, so we add in a 0 "filler" term.

$$6a + 4b - c = 9$$

$$6a + 5b - 7c = -6$$

$$2a + 0b + 3c = 14$$

The system contains the variables a, b, and c, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into an augmented matrix gives

$$N = \begin{bmatrix} 6 & 4 & -1 & 9 \\ 6 & 5 & -7 & -6 \\ 2 & 0 & 3 & 14 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 1 & -2 & 5 \\ 6 & 7 & 0 \\ 7 & 4 & 9 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 1 & -2 & 5 \\ 7 & 4 & 9 \\ 6 & 7 & 0 \end{bmatrix}$

B $\begin{bmatrix} 7 & 4 & 9 \\ 6 & 7 & 0 \\ 1 & -2 & 5 \end{bmatrix}$

C $\begin{bmatrix} 6 & 7 & 0 \\ 1 & -2 & 5 \\ 7 & 4 & 9 \end{bmatrix}$

D $\begin{bmatrix} 7 & 4 & 9 \\ 1 & -2 & 5 \\ 6 & 7 & 0 \end{bmatrix}$

Solution: C

The operation described by $R_1 \leftrightarrow R_2$ is switching row 1 with row 2. Nothing will happen to row 3. The matrix after $R_1 \leftrightarrow R_2$ is

$$\begin{bmatrix} 6 & 7 & 0 \\ 1 & -2 & 5 \\ 7 & 4 & 9 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $2R_2 \leftrightarrow 4R_3$.

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -2 & 3 & 7 & 9 \\ 5 & -2 & 0 & 1 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 6 & 1 & 5 & -8 \\ 20 & -8 & 0 & 4 \\ -4 & 6 & 14 & 18 \end{bmatrix}$

B $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 5 & -2 & 0 & 1 \end{bmatrix}$

C $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -2 & 3 & 7 & 9 \\ 20 & -8 & 0 & 4 \end{bmatrix}$

D $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 20 & -8 & 0 & 4 \end{bmatrix}$

Solution: A

The operation described by $2R_2 \leftrightarrow 4R_3$ is multiplying row 2 by a constant of 2, multiplying row 3 by a constant of 4, and then switching those two rows. Nothing will happen to row 1. The matrix after $2R_2$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 5 & -2 & 0 & 1 \end{bmatrix}$$

The matrix after $4R_3$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 20 & -8 & 0 & 4 \end{bmatrix}$$

The matrix after $2R_2 \leftrightarrow 4R_3$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ 20 & -8 & 0 & 4 \\ -4 & 6 & 14 & 18 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $3R_1 + R_3 \to R_1$.

$$\begin{bmatrix} 7 & 8 & -2 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 21 & 24 & -6 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$

B $\begin{bmatrix} 7 & 8 & -2 & 0 \\ 5 & 1 & 6 & 13 \\ 25 & 17 & -3 & 9 \end{bmatrix}$

C $\begin{bmatrix} 7 & 8 & -2 & 0 \\ 25 & 17 & -3 & 9 \\ 4 & -7 & 3 & 9 \end{bmatrix}$

D $\begin{bmatrix} 25 & 17 & -3 & 9 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$

Solution: D

The operation described by $3R_1 + R_3 \to R_1$ is multiplying row 1 by a constant of 3, adding that resulting row to row 3, and using that result to replace row 1. The matrix after $3R_1$ is

$$\begin{bmatrix} 21 & 24 & -6 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

The sum $3R_1 + R_3$ is

$$\begin{bmatrix} 25 & 17 & -3 & 9 \end{bmatrix}$$

The matrix after $3R_1 + R_3 \to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix} 25 & 17 & -3 & 9 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

Topic: Gauss-Jordan elimination and reduced row-echelon form

Question: Use Gauss-Jordan elimination to solve the system.

$$x + 3y = 13$$

$$2x + 4y = 16$$

Answer choices:

A $x = 5,\ y = -2$

B $x = 3,\ y = -1$

C $x = -1,\ y = 3$

D $x = -2,\ y = 5$

Solution: D

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & = & 13 \\ 2 & 4 & = & 16 \end{bmatrix}$$

The first row already has a leading 1. After $2R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 3 & = & 13 \\ 0 & 2 & = & 10 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 3 & = & 13 \\ 0 & 1 & = & 5 \end{bmatrix}$$

After $R_1 - 3R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & = & -2 \\ 0 & 1 & = & 5 \end{bmatrix}$$

The second column is done, and we get the solution set

$$x = -2$$

$$y = 5$$

Topic: Gauss-Jordan elimination and reduced row-echelon form

Question: Use Gauss-Jordan elimination to solve the system.

$$x + 4z = 11$$

$$x - y + 4z = 6$$

$$2x + 9z = 25$$

Answer choices:

A $x = -1,\ y = 5,\ z = 3$

B $x = 11,\ y = 6,\ z = 25$

C $x = 1,\ y = 0,\ z = 12$

D $x = -3,\ y = 8,\ z = 3$

Solution: A

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 4 & = & 11 \\ 1 & -1 & 4 & = & 6 \\ 2 & 0 & 9 & = & 25 \end{bmatrix}$$

The first row already has a leading 1. After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & = & 11 \\ 0 & 1 & 0 & = & 5 \\ 2 & 0 & 9 & = & 25 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & = & 11 \\ 0 & 1 & 0 & = & 5 \\ 0 & 0 & -1 & = & -3 \end{bmatrix}$$

The first and second columns are done. After $(-1)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & = & 11 \\ 0 & 1 & 0 & = & 5 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

After $R_1 - 4R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & -1 \\ 0 & 1 & 0 & = & 5 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

The third column is done, and we get the solution set

$$x = -1$$

$$y = 5$$

$$z = 3$$

Topic: Gauss-Jordan elimination and reduced row-echelon form

Question: Use Gauss-Jordan elimination to solve the system.

$$2x + 4y + 10z = 30$$

$$x + y + 3z = 10$$

$$2x + y + 2z = 9$$

Answer choices:

A $x = 7,\ y = -3,\ z = 5$

B $x = -4,\ y = 1,\ z = 0$

C $x = 2,\ y = -1,\ z = 3$

D $x = 30,\ y = 10,\ z = 9$

Solution: C

The augmented matrix is

$$\begin{bmatrix} 2 & 4 & 10 & = & 30 \\ 1 & 1 & 3 & = & 10 \\ 2 & 1 & 2 & = & 9 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & = & 15 \\ 1 & 1 & 3 & = & 10 \\ 2 & 1 & 2 & = & 9 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & = & 15 \\ 0 & 1 & 2 & = & 5 \\ 2 & 1 & 2 & = & 9 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & = & 15 \\ 0 & 1 & 2 & = & 5 \\ 0 & 3 & 8 & = & 21 \end{bmatrix}$$

The first column is done. After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & = & 5 \\ 0 & 1 & 2 & = & 5 \\ 0 & 3 & 8 & = & 21 \end{bmatrix}$$

After $R_3 - 3R_2 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & = & 5 \\ 0 & 1 & 2 & = & 5 \\ 0 & 0 & 2 & = & 6 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & = & 5 \\ 0 & 1 & 2 & = & 5 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

After $R_1 - R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 2 & = & 5 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 0 & = & -1 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

The third column is done, and we get the solution set

$$x = 2$$

$$y = -1$$

$$z = 3$$

Topic: Matrix addition and subtraction

Question: Add the matrices.

$$\begin{bmatrix} 4 & -3 & 6 \\ 8 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ 11 & 4 & -9 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 4 & -3 & 6 \\ 19 & 6 & -8 \end{bmatrix}$

B $\begin{bmatrix} 7 & -3 & 7 \\ 19 & 6 & -8 \end{bmatrix}$

C $\begin{bmatrix} 7 & -3 & 7 \\ 8 & 2 & 1 \end{bmatrix}$

D $\begin{bmatrix} 7 & 3 & 7 \\ 19 & 6 & 8 \end{bmatrix}$

Solution: B

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{bmatrix}4 & -3 & 6\\ 8 & 2 & 1\end{bmatrix}+\begin{bmatrix}3 & 0 & 1\\ 11 & 4 & -9\end{bmatrix}$$

$$\begin{bmatrix}4+3 & -3+0 & 6+1\\ 8+11 & 2+4 & 1+(-9)\end{bmatrix}$$

$$\begin{bmatrix}7 & -3 & 7\\ 19 & 6 & -8\end{bmatrix}$$

Topic: Matrix addition and subtraction

Question: Subtract the matrices.

$$\begin{bmatrix} 8 & 1 & 3 \\ 6 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 12 & 5 \\ 5 & 1 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 14 & 13 & 8 \\ 1 & 5 & 5 \\ 2 & 6 & 7 \end{bmatrix}$

B $\begin{bmatrix} -2 & 11 & 2 \\ -1 & 5 & -5 \\ -2 & 6 & -7 \end{bmatrix}$

C $\begin{bmatrix} 14 & 13 & 8 \\ 11 & 7 & 5 \\ -2 & 8 & 11 \end{bmatrix}$

D $\begin{bmatrix} 2 & -11 & -2 \\ 1 & -5 & 5 \\ 2 & -6 & 7 \end{bmatrix}$

Solution: D

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{bmatrix}8 & 1 & 3\\6 & -4 & 5\\0 & 1 & 9\end{bmatrix}-\begin{bmatrix}6 & 12 & 5\\5 & 1 & 0\\-2 & 7 & 2\end{bmatrix}$$

$$\begin{bmatrix}8-6 & 1-12 & 3-5\\6-5 & -4-1 & 5-0\\0-(-2) & 1-7 & 9-2\end{bmatrix}$$

$$\begin{bmatrix}2 & -11 & -2\\1 & -5 & 5\\2 & -6 & 7\end{bmatrix}$$

Topic: Matrix addition and subtraction

Question: Solve for x.

$$\begin{bmatrix} 8 & 2 \\ 7 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = x + \begin{bmatrix} 5 & 7 \\ -5 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 6 & -4 \end{bmatrix}$$

Answer choices:

A $\quad x = \begin{bmatrix} 13 & 6 \\ 5 & 13 \end{bmatrix}$

B $\quad x = \begin{bmatrix} -13 & -6 \\ -5 & -13 \end{bmatrix}$

C $\quad x = \begin{bmatrix} -1 & -8 \\ 3 & 3 \end{bmatrix}$

D $\quad x = \begin{bmatrix} 1 & 8 \\ -3 & -3 \end{bmatrix}$

Solution: C

Let's start with the matrix subtraction on the left side of the equation and the matrix addition on the right side of the equation.

$$\begin{bmatrix} 8 & 2 \\ 7 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = x + \begin{bmatrix} 5 & 7 \\ -5 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 6 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 8-2 & 2-3 \\ 7-3 & 9-1 \end{bmatrix} = x + \begin{bmatrix} 5+2 & 7+0 \\ -5+6 & 9+(-4) \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 \\ 4 & 8 \end{bmatrix} = x + \begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix}$$

To isolate x, we'll subtract the matrix on the right from both sides in order to move it to the left.

$$\begin{bmatrix} 6 & -1 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix} = x$$

$$\begin{bmatrix} 6-7 & -1-7 \\ 4-1 & 8-5 \end{bmatrix} = x$$

$$\begin{bmatrix} -1 & -8 \\ 3 & 3 \end{bmatrix} = x$$

The conclusion is that the value of x that makes the equation true is this matrix:

$$x = \begin{bmatrix} -1 & -8 \\ 3 & 3 \end{bmatrix}$$

Topic: Scalar multiplication and zero matrices

Question: Use scalar multiplication to simplify the expression.

$$4\begin{bmatrix} 5 & 2 & 1 \\ -2 & 4 & 7 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 9 & 6 & 5 \\ 2 & 8 & 11 \end{bmatrix}$

B $\begin{bmatrix} 20 & 8 & 4 \\ -2 & 4 & 7 \end{bmatrix}$

C $\begin{bmatrix} 5 & 2 & 1 \\ -8 & 16 & 28 \end{bmatrix}$

D $\begin{bmatrix} 20 & 8 & 4 \\ -8 & 16 & 28 \end{bmatrix}$

Solution: D

In this problem, 4 is the scalar. We distribute the scalar across every entry in the matrix, and the result of the scalar multiplication is

$$4\begin{bmatrix}5 & 2 & 1\\ -2 & 4 & 7\end{bmatrix}$$

$$\begin{bmatrix}4(5) & 4(2) & 4(1)\\ 4(-2) & 4(4) & 4(7)\end{bmatrix}$$

$$\begin{bmatrix}20 & 8 & 4\\ -8 & 16 & 28\end{bmatrix}$$

Topic: Scalar multiplication and zero matrices

Question: Solve for x.

$$3\begin{bmatrix}7 & 1\\8 & 3\end{bmatrix}+x=-4\begin{bmatrix}0 & -5\\-2 & 3\end{bmatrix}$$

Answer choices:

A $\quad x=\begin{bmatrix}-21 & 17\\-16 & -21\end{bmatrix}$

B $\quad x=\begin{bmatrix}21 & 23\\32 & -3\end{bmatrix}$

C $\quad x=\begin{bmatrix}21 & -17\\16 & 21\end{bmatrix}$

D $\quad x=\begin{bmatrix}-21 & -23\\-32 & 3\end{bmatrix}$

Solution: A

Apply the scalars to the matrices.

$$\begin{bmatrix}3(7) & 3(1)\\ 3(8) & 3(3)\end{bmatrix} + x = \begin{bmatrix}-4(0) & -4(-5)\\ -4(-2) & -4(3)\end{bmatrix}$$

$$\begin{bmatrix}21 & 3\\ 24 & 9\end{bmatrix} + x = \begin{bmatrix}0 & 20\\ 8 & -12\end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate x.

$$x = \begin{bmatrix}0 & 20\\ 8 & -12\end{bmatrix} - \begin{bmatrix}21 & 3\\ 24 & 9\end{bmatrix}$$

$$x = \begin{bmatrix}0-21 & 20-3\\ 8-24 & -12-9\end{bmatrix}$$

$$x = \begin{bmatrix}-21 & 17\\ -16 & -21\end{bmatrix}$$

Topic: Scalar multiplication and zero matrices

Question: Choose the O_{4x2} matrix.

Answer choices:

A $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

B $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

C $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution: B

We always name the zero matrix with a capital O. And optionally, you can add a subscript with the dimensions of the zero matrix. Since the values in a zero matrix are all zeros, just having the dimensions of the zero matrix tells you what the entire matrix looks like. So O_{4x2} is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Topic: Matrix multiplication

Question: Find the product of matrices A and B.

$$A = \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 9 & 1 \\ 6 & -1 \end{bmatrix}$$

Answer choices:

A $\quad A \cdot B = \begin{bmatrix} 43 & 3 \\ -5 & 6 \end{bmatrix}$

B $\quad A \cdot B = \begin{bmatrix} 0 & -14 \\ 23 & 4 \end{bmatrix}$

C $\quad A \cdot B = \begin{bmatrix} 57 & 3 \\ -12 & 2 \end{bmatrix}$

D $\quad A \cdot B = \begin{bmatrix} 9 & 6 \\ 23 & -8 \end{bmatrix}$

Solution: C

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 9 & 1 \\ 6 & -1 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 5(9) + 2(6) & 5(1) + 2(-1) \\ 0(9) + (-2)(6) & 0(1) + (-2)(-1) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 57 & 3 \\ -12 & 2 \end{bmatrix}$$

Topic: Matrix multiplication

Question: Find the product of matrices A and B.

$$A = \begin{bmatrix} 7 & 2 & -4 \\ -5 & 10 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 1 \\ 7 & 2 \\ -2 & 6 \end{bmatrix}$$

Answer choices:

A $\quad A \cdot B = \begin{bmatrix} 71 & -13 \\ 29 & 33 \end{bmatrix}$

B $\quad A \cdot B = \begin{bmatrix} 45 & -30 \\ -16 & 52 \end{bmatrix}$

C $\quad A \cdot B = \begin{bmatrix} -41 & 56 \\ 29 & -16 \end{bmatrix}$

D $\quad A \cdot B = \begin{bmatrix} 43 & 33 \\ 82 & 19 \end{bmatrix}$

Solution: A

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 7 & 2 & -4 \\ -5 & 10 & 3 \end{bmatrix} \cdot \begin{bmatrix} 7 & 1 \\ 7 & 2 \\ -2 & 6 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 7(7) + 2(7) + (-4)(-2) & 7(1) + 2(2) + (-4)(6) \\ (-5)(7) + 10(7) + 3(-2) & (-5)(1) + 10(2) + 3(6) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 71 & -13 \\ 29 & 33 \end{bmatrix}$$

Topic: Matrix multiplication

Question: Use the distributive property to find $A(B+C)$.

$$A=\begin{bmatrix}3 & -1\\ 1 & 4\end{bmatrix}$$

$$B=\begin{bmatrix}5 & 2\\ -2 & 3\end{bmatrix}$$

$$C=\begin{bmatrix}2 & 0\\ 6 & 2\end{bmatrix}$$

Answer choices:

A $\quad A(B+C)=\begin{bmatrix}-2 & 15\\ 3 & 32\end{bmatrix}$

B $\quad A(B+C)=\begin{bmatrix}17 & 1\\ 23 & 22\end{bmatrix}$

C $\quad A(B+C)=\begin{bmatrix}3 & -14\\ 27 & 1\end{bmatrix}$

D $\quad A(B+C)=\begin{bmatrix}8 & 9\\ -14 & 17\end{bmatrix}$

Solution: B

Applying the distributive property to the initial expression, we get

$$A(B+C)=AB+AC$$

Now use matrix multiplication.

$$AB+AC=\begin{bmatrix}3 & -1\\1 & 4\end{bmatrix}\cdot\begin{bmatrix}5 & 2\\-2 & 3\end{bmatrix}+\begin{bmatrix}3 & -1\\1 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\6 & 2\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}3(5)+(-1)(-2) & 3(2)+(-1)(3)\\1(5)+4(-2) & 1(2)+4(3)\end{bmatrix}+\begin{bmatrix}3(2)+(-1)(6) & 3(0)+(-1)(2)\\1(2)+4(6) & 1(0)+4(2)\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}17 & 3\\-3 & 14\end{bmatrix}+\begin{bmatrix}0 & -2\\26 & 8\end{bmatrix}$$

Adding the matrices gives

$$AB+AC=\begin{bmatrix}17+0 & 3+(-2)\\-3+26 & 14+8\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}17 & 1\\23 & 22\end{bmatrix}$$

So the value of the original expression is

$$A(B+C)=\begin{bmatrix}17 & 1\\23 & 22\end{bmatrix}$$

Topic: Identity matrices

Question: Which identity matrix is I_3?

Answer choices:

A $I_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B $I_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D $I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Solution: C

We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate I_{2x2} as just I_2, or I_{3x3} as just I_3, etc. So, I_3 is the 3×3 identity matrix.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topic: Identity matrices

Question: Which identity matrix can be multiplied by A (in other words, $I \cdot A$), if A is a 2×4 matrix?

Answer choices:

A $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B $\quad I_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C $\quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D $\quad I_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Solution: A

Start by setting up the equation $I \cdot A = A$. Next, substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 2 \times 4 = 2 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns.

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

In order to be able to multiply matrices, we need the same number of columns in the first matrix as we have rows in the second matrix. So the identity matrix must have 2 columns.

$$R \times \boxed{2 \cdot 2} \times 4 = 2 \times 4$$

And the dimensions of the resulting matrix come from the rows of the first matrix and the columns of the second matrix. So the identity matrix must have 2 rows.

$$\boxed{2} \times 2 \cdot 2 \times 4 = \boxed{2} \times 4$$

Therefore, the identity matrix we need is I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Topic: Identity matrices

Question: If we want to find IA, which identity matrix should we use, and what is the product?

$$A = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

Answer choices:

A Use $I_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} 2 & 8 & 4 \\ -3 & 7 & 1 \end{bmatrix}$

B Use $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$

C Use $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$

D Use $I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} 2 & 8 & 4 \\ -3 & 7 & 1 \end{bmatrix}$

Solution: B

A is a 2×3 matrix, and we need to find IA. We also know that IA will be 2×3. So we'll set up an equation of dimensions.

$$I \cdot A = A$$

$$I \cdot 2 \times 3 = 2 \times 3$$

$$R \times C \cdot 2 \times 3 = 2 \times 3$$

For matrix multiplication to be valid, we need the same number of columns in the first matrix as we have rows in the second matrix.

$$R \times \boxed{2 \cdot 2} \times 3 = 2 \times 3$$

The dimensions of the result are given by the rows from the first matrix, and columns from the second matrix.

$$\boxed{2} \times 2 \cdot 2 \times 3 = \boxed{2} \times 3$$

So the identity matrix is 2×2, which means it's I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the product of I_2 and matrix A is

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(-3) + 0(2) & 1(7) + 0(8) & 1(1) + 0(4) \\ 0(-3) + 1(2) & 0(7) + 1(8) & 0(1) + 1(4) \end{bmatrix}$$

$$IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

Topic: Transformations

Question: Find the resulting vector $\overrightarrow{b}$ after $\overrightarrow{a} = (-4,2)$ undergoes a transformation by matrix Q.

$$Q = \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix}$$

Answer choices:

A $\quad \overrightarrow{b} = \begin{bmatrix} 42 \\ -12 \end{bmatrix}$

B $\quad \overrightarrow{b} = \begin{bmatrix} -42 \\ 12 \end{bmatrix}$

C $\quad \overrightarrow{b} = \begin{bmatrix} -42 \\ -12 \end{bmatrix}$

D $\quad \overrightarrow{b} = \begin{bmatrix} 42 \\ 12 \end{bmatrix}$

Solution: C

To apply a transformation matrix to vector $\overrightarrow{a}$, we'll multiply the matrix by the vector.

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} 11(-4) + 1(2) \\ 0(-4) - 6(2) \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} -44 + 2 \\ 0 - 12 \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} -42 \\ -12 \end{bmatrix}$$

Topic: Transformations

Question: What are the vertices of the transformation of the polygon with vertices $(-2,1)$, $(1,3)$, $(2,-2)$, and $(-3,-1)$ after it's transformed by matrix P.

$$P = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$$

Answer choices:

A $(2,2)$, $(-3,7)$, $(-3,2)$, and $(4,-3)$

B $(2,2)$, $(-3,7)$, $(-4,-4)$, and $(6,-6)$

C $(4,1)$, $(-2,10)$, $(-3,2)$, and $(4,-3)$

D $(4,1)$, $(-2,10)$, $(-4,-4)$, and $(6,-6)$

Solution: D

Put the vertices of the polygon into a matrix.

$$\begin{bmatrix} -2 & 1 & 2 & -3 \\ 1 & 3 & -2 & -1 \end{bmatrix}$$

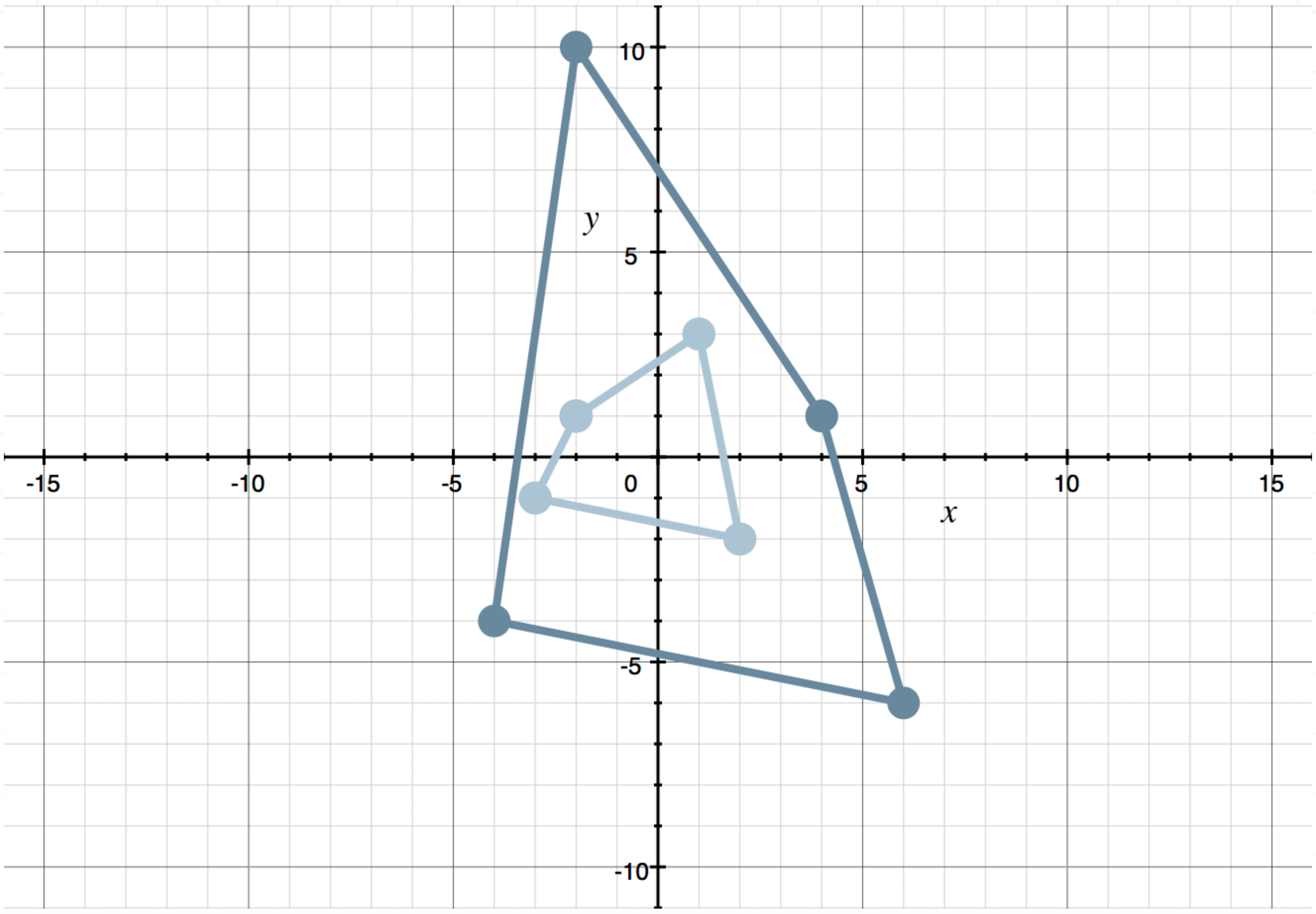
Apply the transformation of Z to the vertex matrix.

$$\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 & -3 \\ 1 & 3 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2(-2)+0(1) & -2(1)+0(3) & -2(2)+0(-2) & -2(-3)+0(-1) \\ 1(-2)+3(1) & 1(1)+3(3) & 1(2)+3(-2) & 1(-3)+3(-1) \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & -4 & 6 \\ 1 & 10 & -4 & -6 \end{bmatrix}$$

The original polygon is sketched in light blue, and its transformation after P is in dark blue.


10
y
5
-15
-10
-5
0
5
10
15
x
-5
-10

Topic: Transformations

Question: What are the vertices of the transformation of the triangle with vertices $(-3,0)$, $(1,2)$, and $(1,-2)$ after it's transformed by matrix S.

$$S = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$$

Answer choices:

A $(0,-6)$, $(-2,4)$, and $(2,0)$

B $(0,-4)$, $(-1,3)$, and $(2,2)$

C $(1,-3)$, $(-1,6)$, and $(3,1)$

D $(2,-1)$, $(0,2)$, and $(1,4)$

Solution: A

Put the vertices of the triangle into a matrix.

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

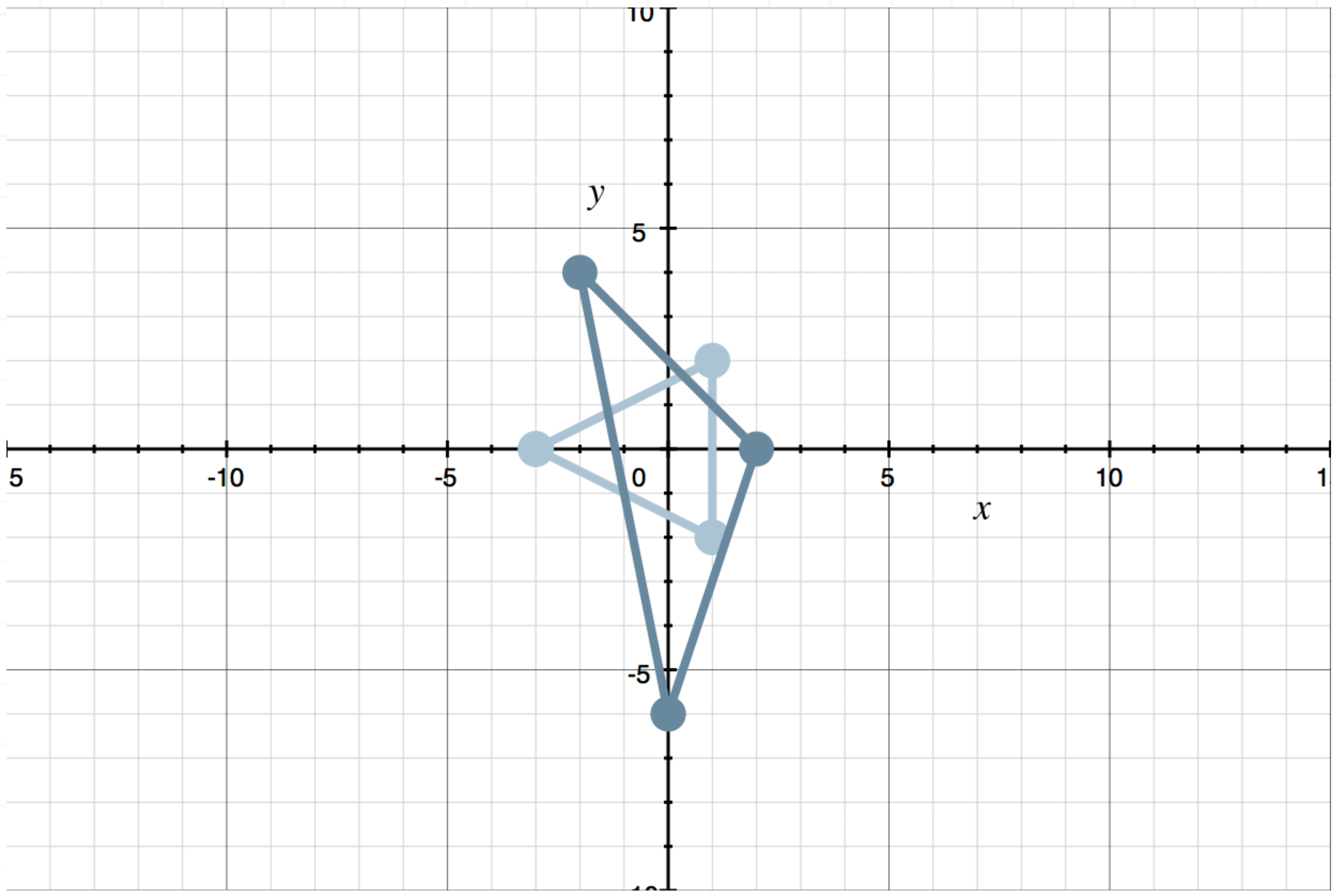
Apply the transformation of S to the vertex matrix.

$$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0(-3) - 1(0) & 0(1) - 1(2) & 0(1) - 1(-2) \\ 2(-3) + 1(0) & 2(1) + 1(2) & 2(1) + 1(-2) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 2 \\ -6 & 4 & 0 \end{bmatrix}$$

The original triangle is sketched in light blue, and its transformation after S is in dark blue.


y
x
10
5
0
-5
10
5
-10
-5
1

Topic: Matrix inverses, and invertible and singular matrices

Question: Are the matrices inverses of one another?

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

Answer choices:

A Yes

B No

C There's not enough information to know

Solution: A

To find the inverse of matrix A, plug it into the formula for the inverse matrix.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix}}\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix}$$

Find the determinant in the denominator of the fraction.

$$A^{-1} = \frac{1}{(2)(1)-(5)(3)}\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2-15}\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{13}\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix}$$

Then distribute the scalar across the matrix.

$$A^{-1} = \begin{bmatrix} -\frac{1}{13}(1) & -\frac{1}{13}(-5) \\ -\frac{1}{13}(-3) & -\frac{1}{13}(2) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

Because the value we found matches matrix B, it means that matrices A and B are inverses of one another.

Topic: Matrix inverses, and invertible and singular matrices

Question: Find the inverse of matrix M.

$$M = \begin{bmatrix} 0 & -2 \\ -4 & 5 \end{bmatrix}$$

Answer choices:

A $\quad M^{-1} = \begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{5}{8} \end{bmatrix}$

B $\quad M^{-1} = \begin{bmatrix} -\frac{5}{8} & -\frac{1}{4} \\ -\frac{1}{2} & 0 \end{bmatrix}$

C $\quad M^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{2} & -\frac{5}{8} \end{bmatrix}$

D $\quad M^{-1} = \begin{bmatrix} -\frac{5}{8} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}$

Solution: B

Plug the values from the matrix into the formula for the inverse matrix.

$$M^{-1}=\frac{1}{|M|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$M^{-1}=\frac{1}{\begin{vmatrix} 0 & -2 \\ -4 & 5 \end{vmatrix}}\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

Find the determinant in the denominator of the fraction.

$$M^{-1}=\frac{1}{(0)(5)-(-2)(-4)}\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

$$M^{-1}=\frac{1}{0-8}\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

$$M^{-1}=-\frac{1}{8}\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

Then distribute the scalar across the matrix.

$$M^{-1}=\begin{bmatrix} -\frac{1}{8}(5) & -\frac{1}{8}(2) \\ -\frac{1}{8}(4) & -\frac{1}{8}(0) \end{bmatrix}$$

$$M^{-1}=\begin{bmatrix} -\frac{5}{8} & -\frac{1}{4} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

Topic: Matrix inverses, and invertible and singular matrices

Question: Classify the matrix.

$$L = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$$

Answer choices:

A The matrix is invertible

B The matrix is singular

C The matrix is invertible and singular

D The matrix is neither invertible nor singular

Solution: A

A matrix is either invertible or singular, it can never be both. To determine whether a matrix in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible or singular, we need to look at the ratio of a to b, compared to the ratio of c to d.

For the given matrix L,

$$\frac{a}{b} = \frac{3}{7}$$

$$\frac{c}{d} = \frac{0}{-1} = 0$$

Because these ratios aren't equivalent, that means matrix L is invertible. If the ratios had been equivalent, the matrix would have been singular.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$3x + 12y = 51$$

$$-2x + 6y = -6$$

Answer choices:

A $x = -9$ and $y = -2$

B $x = -9$ and $y = 2$

C $x = 9$ and $y = -2$

D $x = 9$ and $y = 2$

Solution: D

Start by transferring the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} 3 & 12 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 51 \\ -6 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} 3 & 12 \\ -2 & 6 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(3)(6) - (12)(-2)} \begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix}$$

$$M^{-1} = \frac{1}{42} \begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{21} & \frac{1}{14} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\overrightarrow{a} = M^{-1}\overrightarrow{b}$$

$$\overrightarrow{a} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{21} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 51 \\ -6 \end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}\frac{1}{7}(51)-\frac{2}{7}(-6)\\ \frac{1}{21}(51)+\frac{1}{14}(-6)\end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}\frac{51}{7}+\frac{12}{7}\\ \frac{51}{21}-\frac{6}{14}\end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}\frac{63}{7}\\ \frac{17}{7}-\frac{3}{7}\end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}\frac{63}{7}\\ \frac{14}{7}\end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}9\\ 2\end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x=9$ and $y=2$.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$y - 5x = -15$$

$$3x + 8y = 95$$

Answer choices:

A $x = 5$ and $y = 10$

B $x = -5$ and $y = 10$

C $x = 5$ and $y = -10$

D $x = -5$ and $y = -10$

Solution: A

Start by transferring the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 \\ 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -15 \\ 95 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} -5 & 1 \\ 3 & 8 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(-5)(8) - (1)(3)} \begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{43} \begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{8}{43} & \frac{1}{43} \\ \frac{3}{43} & \frac{5}{43} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\overrightarrow{a} = M^{-1}\overrightarrow{b}$$

$$\overrightarrow{a} = \begin{bmatrix} -\frac{8}{43} & \frac{1}{43} \\ \frac{3}{43} & \frac{5}{43} \end{bmatrix} \begin{bmatrix} -15 \\ 95 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{8}{43}(-15) + \frac{1}{43}(95) \\ \frac{3}{43}(-15) + \frac{5}{43}(95) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{120}{43} + \frac{95}{43} \\ -\frac{45}{43} + \frac{475}{43} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{215}{43} \\ \frac{430}{43} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x = 5$ and $y = 10$.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$4x + 8y = -20$$

$$-12x - 3y = -66$$

Answer choices:

A $x = 7$ and $y = -6$

B $x = -7$ and $y = 6$

C $x = 7$ and $y = 6$

D $x = -7$ and $y = -6$

Solution: A

We could divide through both equations in the system to reduce them. The first equation $4x + 8y = -20$ becomes

$$\frac{4}{4}x + \frac{8}{4}y = -\frac{20}{4}$$

$$x + 2y = -5$$

And the equation $-12x - 3y = -66$ becomes

$$\frac{-12}{-3}x + \frac{-3}{-3}y = \frac{-66}{-3}$$

$$4x + y = 22$$

Then transfer the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 22 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(1)(1) - (2)(4)} \begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{7}\begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\overrightarrow{a} = M^{-1}\overrightarrow{b}$$

$$\overrightarrow{a} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}\begin{bmatrix} -5 \\ 22 \end{bmatrix}$$

$$\overrightarrow{a} = \begin{bmatrix} -\frac{1}{7}(-5) + \frac{2}{7}(22) \\ \frac{4}{7}(-5) - \frac{1}{7}(22) \end{bmatrix}$$

$$\overrightarrow{a} = \begin{bmatrix} \frac{5}{7} + \frac{44}{7} \\ -\frac{20}{7} - \frac{22}{7} \end{bmatrix}$$

$$\overrightarrow{a} = \begin{bmatrix} \frac{49}{7} \\ -\frac{42}{7} \end{bmatrix}$$

$$\overrightarrow{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x = 7$ and $y = -6$.

Topic: Solving systems with Cramer's rule

Question: Which expression would give the solution for y in this system?

$$3x - 2y = 21$$

$$-6x - 5y = 12$$

Answer choices:

A $\dfrac{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}{\begin{vmatrix} 21 & -2 \\ 12 & -5 \end{vmatrix}}$

B $\dfrac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & 6 \\ -2 & 1 \end{vmatrix}}$

C $\dfrac{\begin{vmatrix} 21 & -2 \\ 12 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$

D $\dfrac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$

Solution: D

Using the given system

$$3x - 2y = 21$$

$$-6x - 5y = 12$$

we can say

$$D = \begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}$$

and

$$D_y = \begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}$$

We can put those together to solve for the value of y.

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$$

Topic: Solving systems with Cramer's rule

Question: Which expression below would give the solution for x in this system?

$$3x + 3y = 9$$

$$2x - y = -9$$

Answer choices:

A $\dfrac{\begin{vmatrix} 3 & 9 \\ 2 & -9 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$

B $\dfrac{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$

C $\dfrac{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}$

D $\dfrac{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 9 \\ 2 & -9 \end{vmatrix}}$

Solution: B

Using the given system

$$3x + 3y = 9$$

$$2x - y = -9$$

we can say

$$D = \begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}$$

and

$$D_x = \begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}$$

We can put those together to solve for the value of x.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$$

Topic: Solving systems with Cramer's rule

Question: Which system below would give this value?

$$\frac{D_x}{D} = \frac{\begin{vmatrix} 1 & -5 \\ 15 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix}}$$

Answer choices:

A $\quad 3x - 5y = 1$ and $x + 2y = 15$

B $\quad x - 5y = 3$ and $15x - 2y = 1$

C $\quad 3x + y = -5$ and $x - 15y = 2$

D $\quad x - 2y = 1$ and $3x + 15y = 2$

Solution: A

One way to start this is to figure out the D for each answer choice and see which one(s) match the given expression.

For answer choice A we get

$$D = \begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix}$$

For answer choice B we get

$$D = \begin{vmatrix} 1 & -5 \\ 15 & -2 \end{vmatrix}$$

For answer choice C we get

$$D = \begin{vmatrix} 3 & 1 \\ 1 & -15 \end{vmatrix}$$

For answer choice D we get

$$D = \begin{vmatrix} 1 & -2 \\ 3 & 15 \end{vmatrix}$$

Only answer choice A matched the D in the given expression, so there's no need to check the D_x determinant.