The unit circle

Up to this point, we've talked a lot about the value of the trig functions as they relate to a circle with radius 1 that's centered at the origin. We've just been using the circle to define the trig functions, but now we really want to focus on defining the circle itself.

The unit circle

In fact, this special circle with center at the origin and radius 1 has a special name: it's called the **unit circle**. We call it the unit circle because its radius is 1 unit long. Of course, that means the unit circle intersects the positive x-axis at $(1,0)$, the positive y-axis at $(0,1)$, the negative x-axis at $(-1,0)$, and the negative y-axis at $(0,-1)$.

Remember that the standard equation of a circle centered at (h,k) with radius r is $(x-h)^2+(y-k)^2=r^2$. Because the unit circle is centered at the origin $(0,0)$ and has radius $r=1$, its equation is

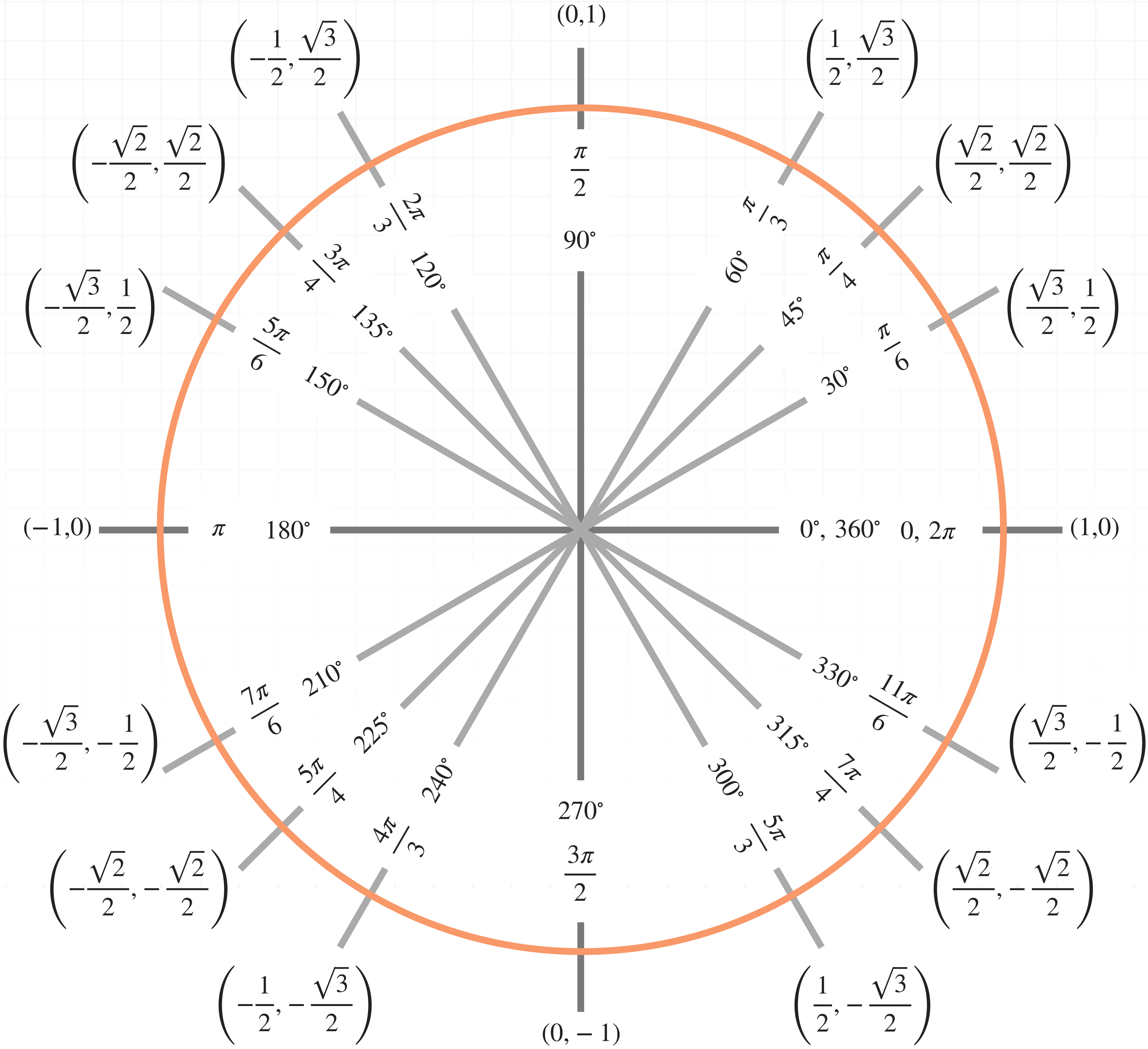
$$(x-0)^2+(y-0)^2=1^2$$

$$x^2+y^2=1$$

We already know the values of the six trig functions along the circle at the quadrantal angles along the major axes:

	sin	csc	cos	sec	tan	cot
0°=0	0	Undefined	1	1	0	Undefined
90°=π/2	1	1	0	Undefined	Undefined	0
180°=π	0	Undefined	-1	-1	0	Undefined
270°=3π/2	-1	-1	0	Undefined	Undefined	0
360°=2π	0	Undefined	1	1	0	Undefined

But there are lots of other special values along the circle. Because of how often we'll use the unit circle throughout Trigonometry and beyond into other more advanced math classes, the more familiar we can be with the values around this circle, the better.



Three sets of information in the unit circle

Notice that the circle really includes three sets of information:

1. Angles in degrees

2. Angles in radians

3. Coordinate points

The degree angles show us that the special points along the circle are at 30° increments: 0°, 30°, 60°, 90°, $\ldots$, and 45° increments: 0°, 45°, 90°, 135°, $\ldots$.

The radian angles show us that the special points along the circle are at $\pi/6$ increments: 0, $\pi/6$, $\pi/3$, $\pi/2$, $\ldots$, and $\pi/4$ increments: 0, $\pi/4$, $\pi/2$, $3\pi/4$, $\ldots$.

The coordinate points also have a pattern. Of course, we already know the points at the quadrantal angles, but all of the other points are fractions with a 2 in the denominator. And if we look in the first quadrant we can see that the numerators of the x-values are $\sqrt{3}$, $\sqrt{2}$, $\sqrt{1}=1$, and that the numerators of the y-values are the opposite: $\sqrt{1}=1$, $\sqrt{2}$, $\sqrt{3}$. The other three quadrants follow this same pattern.

Remembering these patterns can help us know these values without having to actually memorize the full circle.

Finding the values of all six trig functions

Remember that the sine function of an angle is represented by the y-value of the coordinate point, and that the cosine function of an angle is represented by the x-value. Which means we can think of each coordinate point along this circle as $(x, y) = (\cos\theta, \sin\theta)$.

For example, at the angle $\theta = 30^\circ = \pi/6$, the coordinate point along the unit circle is

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

Which means we know right away that, at $\theta = 30^\circ = \pi/6$,

$$\sin 30^\circ = \sin \frac{\pi}{6} = \frac{1}{2}$$

$$\cos 30^\circ = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Of course, now that we have sine and cosine, we can use the quotient identity to find tangent of $\theta = 30^\circ = \pi/6$,

$$\tan 30^\circ = \tan \frac{\pi}{6} = \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

and then we can use the reciprocal identities to find cosecant, secant, and cotangent of $\theta = 30^\circ = \pi/6$.

$$\csc 30^\circ = \csc \frac{\pi}{6} = \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = \frac{2}{1} = 2$$

$$\sec 30^\circ = \sec \frac{\pi}{6} = \frac{1}{\cos \frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 30^\circ = \cot \frac{\pi}{6} = \frac{1}{\tan \frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Let's do an example at another angle in a different quadrant.

Example

Use the unit circle to find the values of the six trig functions at $\theta = 5\pi/4$.

Looking at the unit circle, we know that sine of $\theta = 5\pi/4$ is the y-value of the coordinate point at that angle, and that cosine of $\theta = 5\pi/4$ is the x-value of the coordinate point at that angle.

$$\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

Use the quotient identity to find tangent.

$$\tan\frac{5\pi}{4} = \frac{\sin\frac{5\pi}{4}}{\cos\frac{5\pi}{4}} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

Then use the reciprocal identities to find cosecant as the reciprocal of sine, secant as the reciprocal of cosine, and cotangent as the reciprocal of tangent.

$$\csc\frac{5\pi}{4} = \frac{1}{\sin\frac{5\pi}{4}} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\sec\frac{5\pi}{4} = \frac{1}{\cos\frac{5\pi}{4}} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\cot\frac{5\pi}{4} = \frac{1}{\tan\frac{5\pi}{4}} = \frac{1}{1} = 1$$

Let's do another example with an angle at a $\pi/6$-increment.

Example

Find the values of all six trig functions at $\theta = \pi/3$.

From the unit circle, we know that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

The quotient identity for tangent gives

$$\tan\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \sqrt{3}$$

and the reciprocal identities for cosecant, secant, and cotangent give

$$\csc\frac{\pi}{3}=\frac{1}{\sin\frac{\pi}{3}}=\frac{1}{\frac{\sqrt{3}}{2}}=1\cdot\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

$$\sec\frac{\pi}{3}=\frac{1}{\cos\frac{\pi}{3}}=\frac{1}{\frac{1}{2}}=2$$

$$\cot\frac{\pi}{3}=\frac{1}{\tan\frac{\pi}{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$