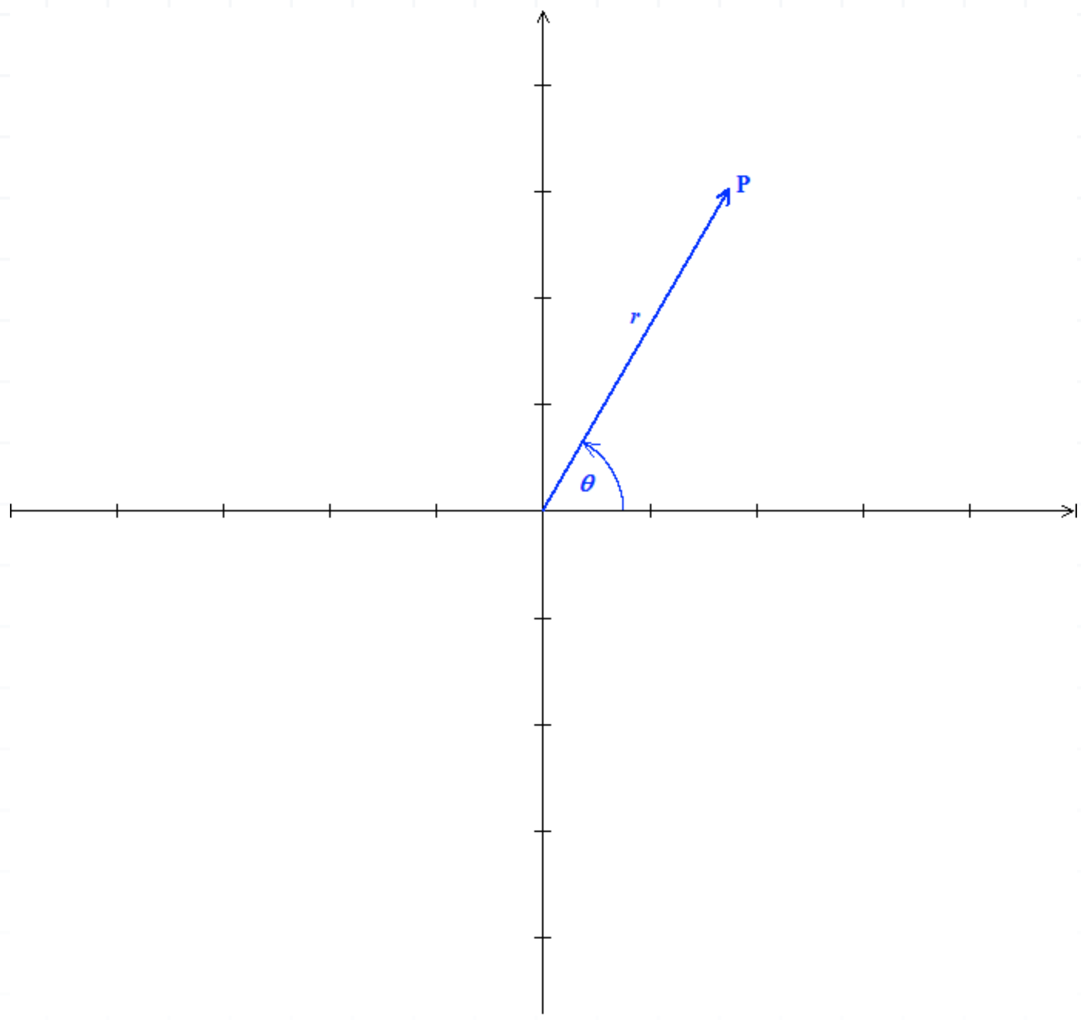
Converting polar coordinates to rectangular

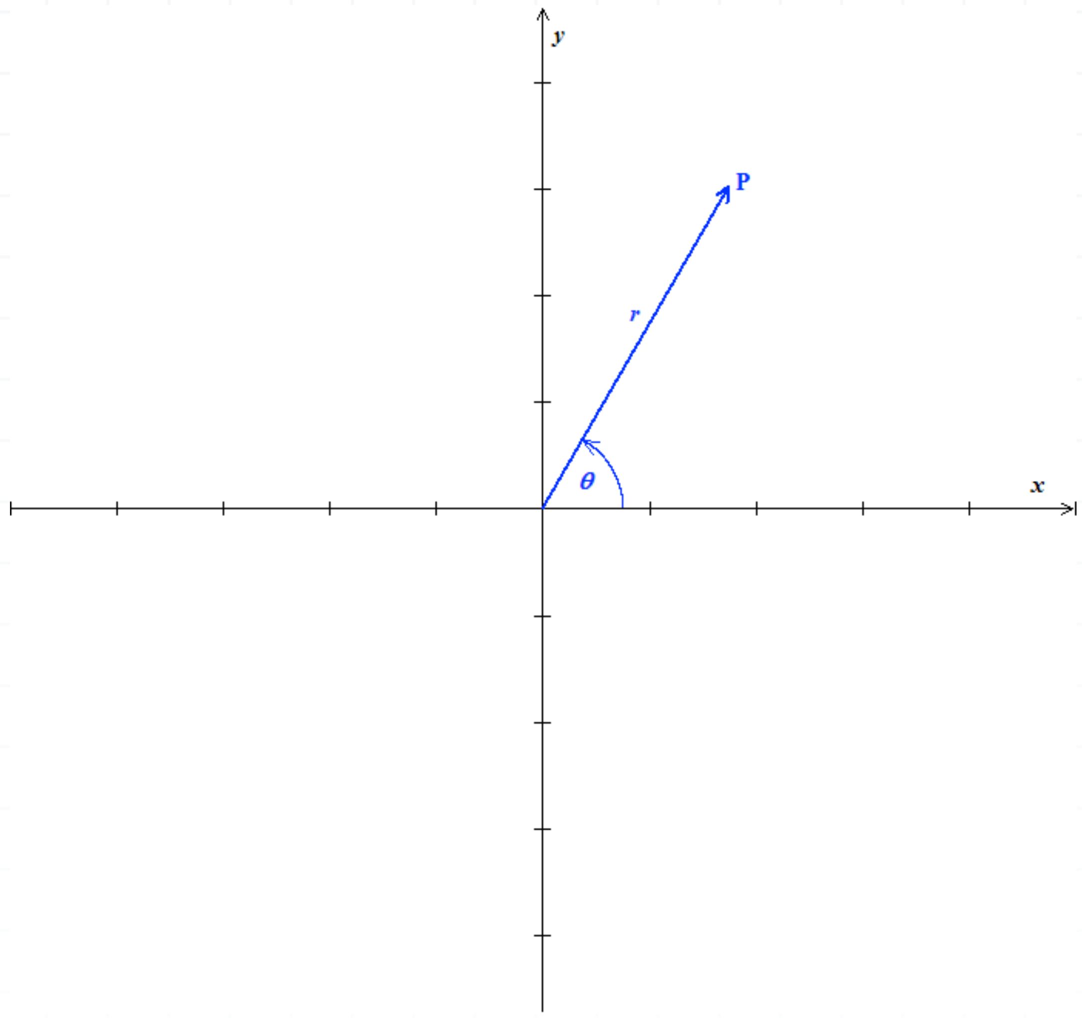
You know how to get the rectangular (Cartesian) coordinates (x, y) of a point in the plane (i.e., in two dimensions), and now you're going to learn how to get the polar coordinates (r, θ) of a point in the plane. One major difference between polar coordinates and rectangular coordinates is that every point in the plane has exactly one pair of rectangular coordinates, while every point in the plane has more than one (in fact, infinitely many) pairs of polar coordinates. In this lesson, you'll learn how to get (what we'll call) the “basic” pair of polar coordinates for a point in the plane; in a later lesson, we'll discuss how to get other pairs of polar coordinates.

So let P be a point in the plane. The basic r-coordinate of P is nonnegative, and r is equal to the distance of point P from the origin, though in polar coordinates the origin is known as the pole. The basic θ-coordinate is the angle θ in the interval $[0,2\pi)$ such that θ is in standard position and point P is located on the terminal side of θ. Thus r is the distance, along the terminal side of θ, from the pole to point P.

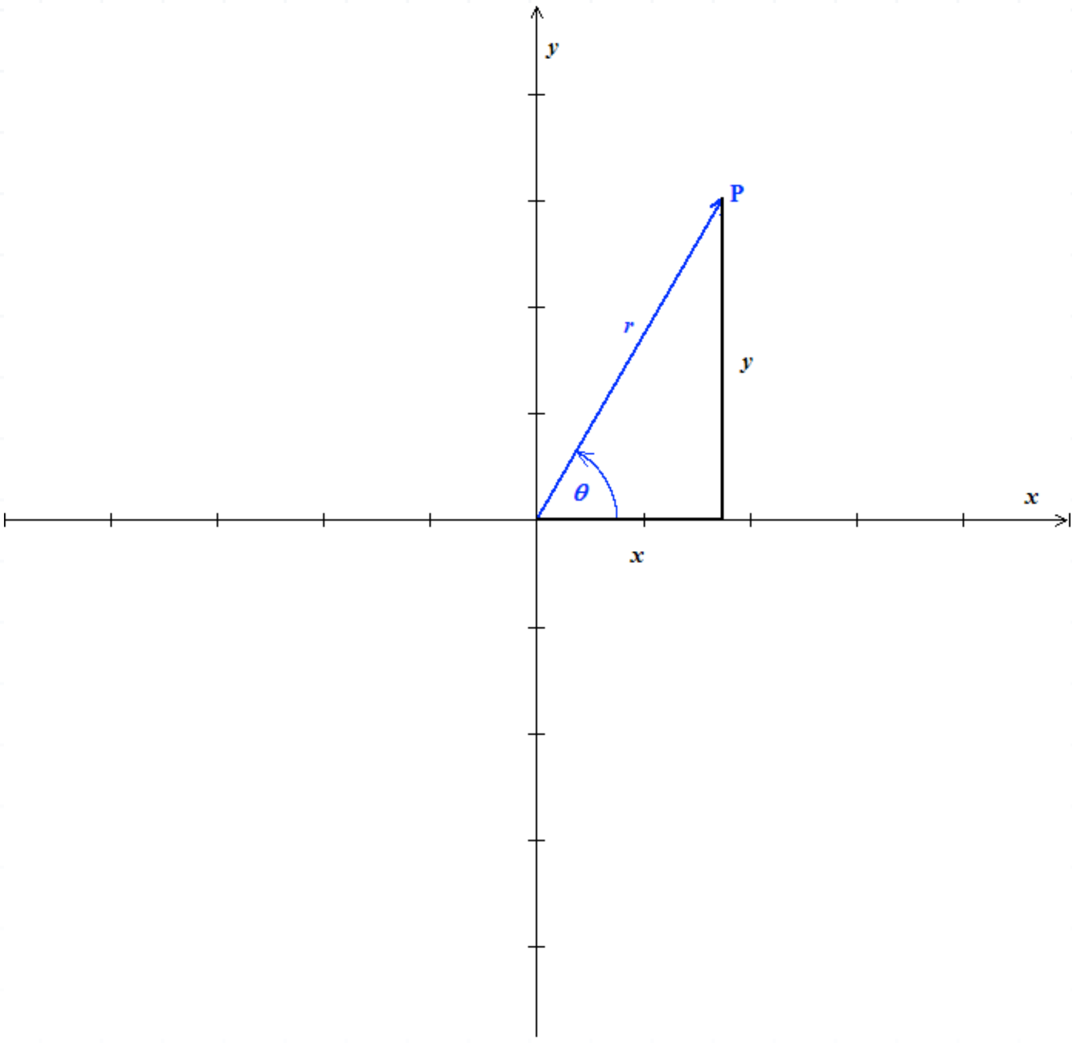
First, we'll exhibit the basic polar coordinates (r, θ) of a point P which is in the first quadrant.



At times, you may be given the polar coordinates of a point and will need to convert the polar coordinates of that point to its rectangular coordinates, so let's take that graph and label the horizontal and vertical axes with "x" and "y," respectively.



Next, we'll drop a perpendicular from point P to the horizontal axis. Since point P is in the first quadrant, both the x coordinate of P and the y coordinate of P are positive. Thus the x coordinate of P is equal to the length of the line segment that joins the pole to the foot of that perpendicular, and the y coordinate of P is equal to the length of that perpendicular.



Note that now we have a right triangle in which θ is one of the interior acute angles, one of the legs is on the positive horizontal axis, and the other leg is parallel to the positive vertical axis.

Since θ is acute, we know the following:

$$\cos\theta = \frac{\text{length of leg adjacent to } \theta}{\text{length of hypotenuse}} = \frac{x}{r}$$

$$\sin\theta = \frac{\text{length of leg opposite } \theta}{\text{length of hypotenuse}} = \frac{y}{r}$$

$$\tan\theta = \frac{\text{length of leg opposite } \theta}{\text{length of leg adjacent to } \theta} = \frac{y}{x}$$

Solving the first equation for x and the second equation for y, we obtain

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Example

Find the rectangular coordinates (x, y) of the point that has polar coordinates $(2,\pi/3)$.

We're given that $r = 2$ and $\theta = \pi/3$. Note that an angle of measure $\pi/3$ is not only in the interval $[0,2\pi)$ but in the first quadrant. Since r is positive, $(2,\pi/3)$ are the "basic" polar coordinates of the point in question, so the point is in the first quadrant.

Now we'll use the equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

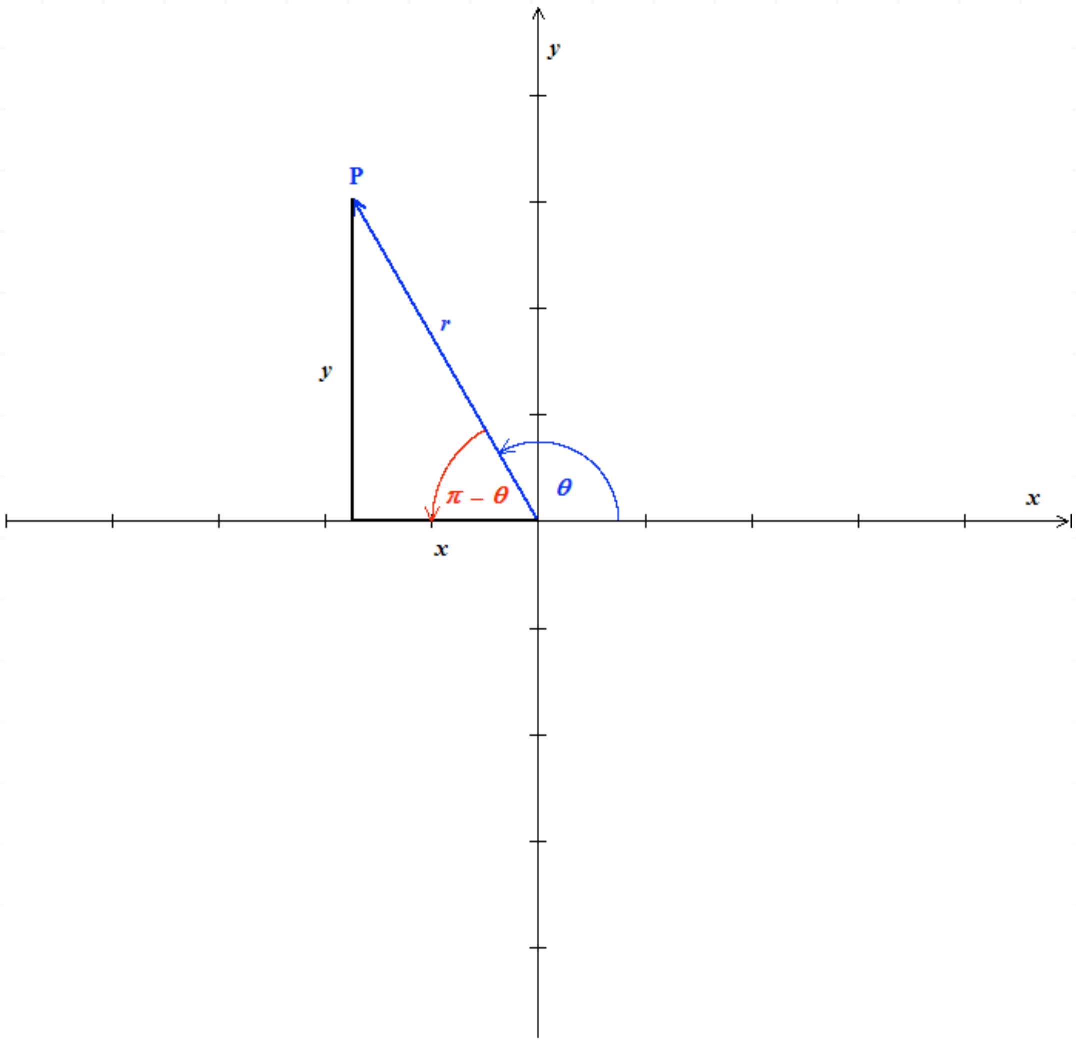
Substituting the data, we obtain

$$x = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

$$y = 2\left(\sin\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

Thus the rectangular coordinates of this point are $\left(1,\sqrt{3}\right)$.

Next, we'll exhibit the basic polar coordinates (r, θ) of a point P which is in the second quadrant.



In this case, we have a right triangle in which $\pi - \theta$ is one of the interior acute angles, one of the legs is on the negative horizontal axis, and the other leg is parallel to the positive vertical axis.

In this case, the x coordinate of point P is negative, and the y coordinate is positive. Thus the length of the leg of that right triangle which is on the negative horizontal axis is $-x$ (because a length has to be nonnegative), and the length of the leg that's parallel to the positive vertical axis is y, so we have the following:

$$\cos(\pi - \theta) = \frac{\text{length of leg adjacent to } \pi - \theta}{\text{length of hypotenuse}} = \frac{-x}{r} = -\frac{x}{r}$$

$$\sin(\pi-\theta)=\frac{\text{length of leg opposite } \pi-\theta}{\text{length of hypotenuse}}=\frac{y}{r}$$

$$\tan(\pi-\theta)=\frac{\text{length of leg opposite } \pi-\theta}{\text{length of leg adjacent to } \pi-\theta}=\frac{y}{-x}=-\frac{y}{x}$$

By the difference identity for cosine,

$$\cos(\pi-\theta)=\cos(\pi)\cos(\theta)+\sin(\pi)\sin(\theta)$$

$$\cos(\pi-\theta)=(-1)\cos\theta+(0)\sin\theta$$

$$\cos(\pi-\theta)=-\cos\theta$$

And by the difference identity for sine,

$$\sin(\pi-\theta)=\sin(\pi)\cos(\theta)-\cos(\pi)\sin(\theta)$$

$$\sin(\pi-\theta)=(0)\cos\theta-(-1)\sin\theta$$

$$\sin(\pi-\theta)=\sin\theta$$

Therefore,

$$\tan(\pi-\theta)=\frac{\cos(\pi-\theta)}{\sin(\pi-\theta)}=\frac{-\cos\theta}{\sin\theta}=-\frac{\cos\theta}{\sin\theta}=-\tan\theta$$

Substituting these results, we obtain

$$-\cos\theta=-\frac{x}{r}, \qquad \sin\theta=\frac{y}{r}, \qquad -\tan\theta=-\frac{y}{x}$$

Equivalently,

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x}$$

Solving the first two equations for x and y, respectively, we again obtain

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Example

Find the rectangular coordinates of the point that has polar coordinates $(6,3\pi/4)$.

Here, $r = 6$ and $\theta = 3\pi/4$. Note that an angle of measure $3\pi/4$ is not only in the interval $[0,2\pi)$ but in the second quadrant. Since r is positive, $(6,3\pi/4)$ are the "basic" polar coordinates of the point in question. Thus the point is in the second quadrant.

Now we'll use the equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Since θ is in the second quadrant, the reference angle for $3\pi/4$ is

$$\pi - \frac{3\pi}{4} = \frac{\pi}{4}$$

By the reference angle theorem and the fact that an angle of measure $3\pi/4$ is in the second quadrant,

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{3\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Substituting the data and rationalizing the denominators, we obtain

$$x = 6\left(-\frac{1}{\sqrt{2}}\right) = -\frac{6}{\sqrt{2}} = \left(-\frac{6}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = -\frac{6\sqrt{2}}{2} = -3\sqrt{2}$$

and

$$y = 6\left(\frac{1}{\sqrt{2}}\right) = \frac{6}{\sqrt{2}} = \left(\frac{6}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Thus the rectangular coordinates of this point are $\left(-3\sqrt{2}, 3\sqrt{2}\right)$.

You can use similar techniques to convince yourself that if (r, θ) are the basic polar coordinates of a point P in either the third or fourth quadrant, then the same equations as before are satisfied:

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x}$$

Thus

$$x = r\cos\theta$$

$$y = r\sin\theta$$

All five of those equations are also satisfied for any point P (other than the pole) on the horizontal (x) axis.

For a point P (other than the pole) on the vertical (y) axis, the only one of the five equations that doesn't apply is

$$\tan\theta = \frac{y}{x}$$

This is because the x coordinate of such a point is 0, and so the ratio y/x is undefined (as is $\tan\theta$).

The rectangular coordinates of the pole are $(x, y) = (0,0)$, and its basic polar coordinate r is equal to 0, so it satisfies the equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

However, since $x = 0$ and $r = 0$, the ratios x/r, y/r, and y/x are undefined (as is $\tan\theta$).

We have found that the equations $x = r\cos\theta$ and $y = r\sin\theta$ are satisfied for every point P, where (r, θ) are the basic polar coordinates of P and (x, y) are its rectangular coordinates, so it is also universally true that

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta)$$

By the basic Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$, this yields $x^2 + y^2 = r^2$.