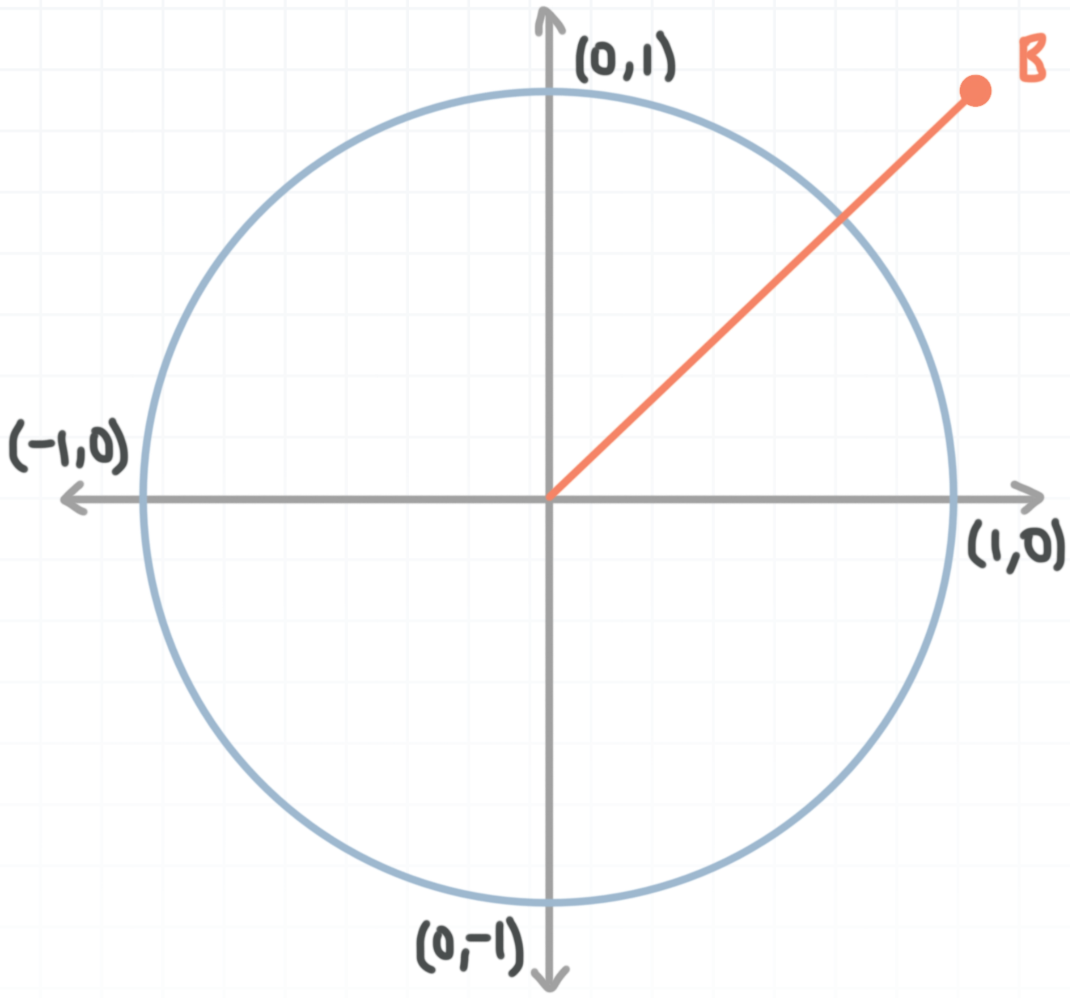
Points not on the unit circle

Up to now, we've been hyper-focused on points along the unit circle. But we also want to be able to deal with points that don't fall exactly on this perfect circle with radius $r = 1$.

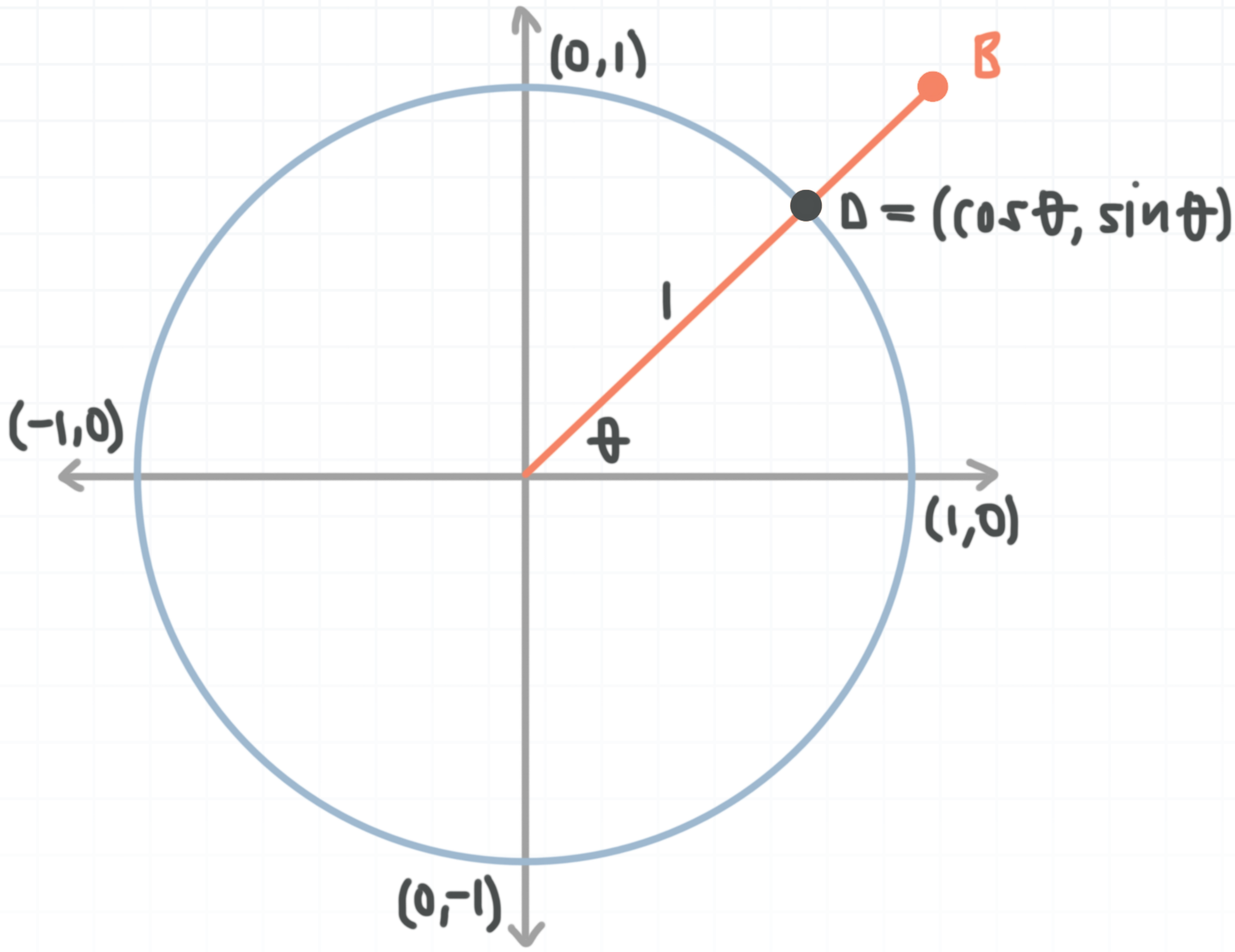
The good news is that we can still use the unit circle to help us find sine and cosine of an angle that's defined by a point off of the unit circle.

Using a point off of the unit circle

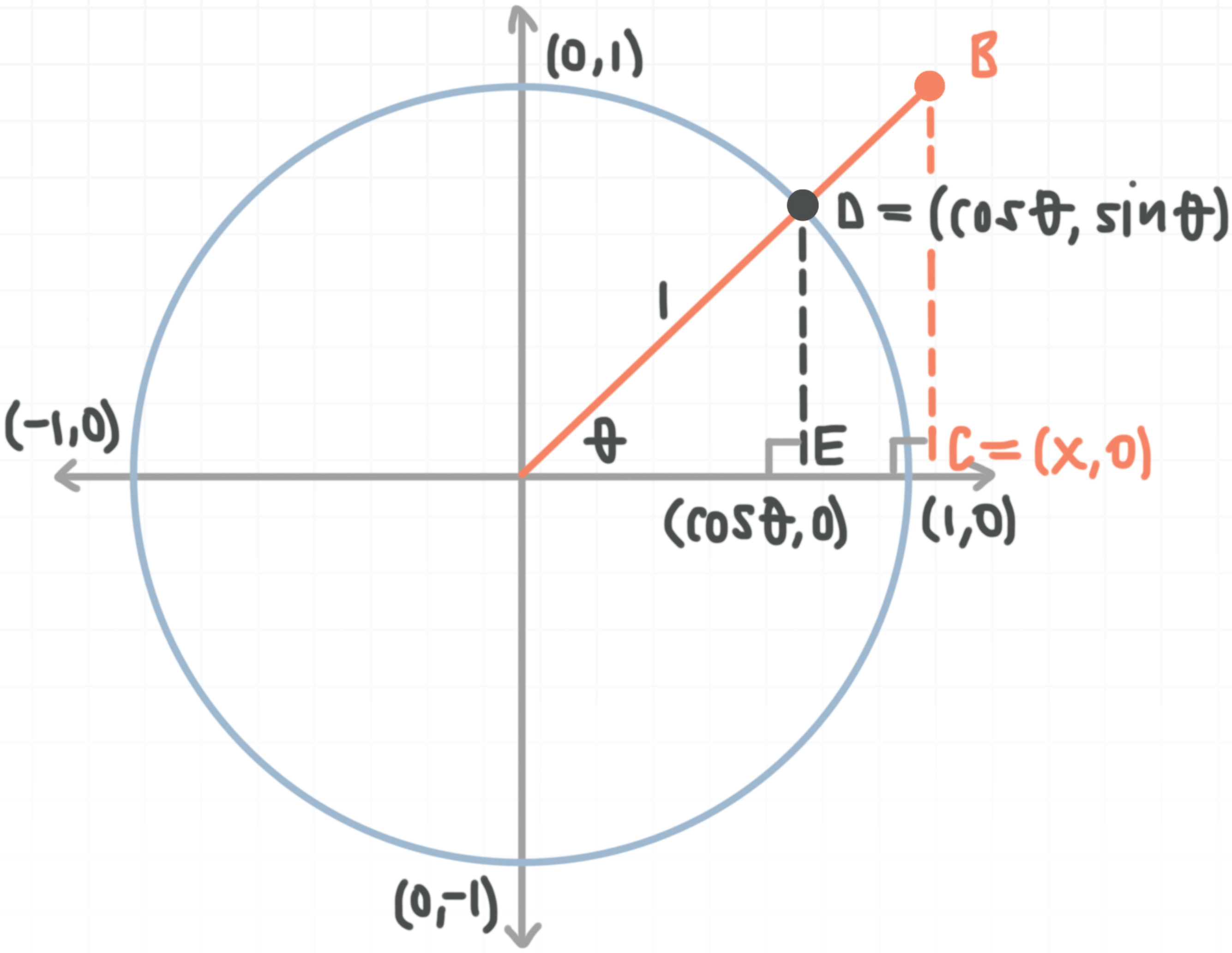
Let's plot a point B that's sitting somewhere outside the unit circle. We want to find the value of sine and cosine at the angle that B forms, but B isn't on the unit circle, which means we can't use the unit circle to find values of x and y.



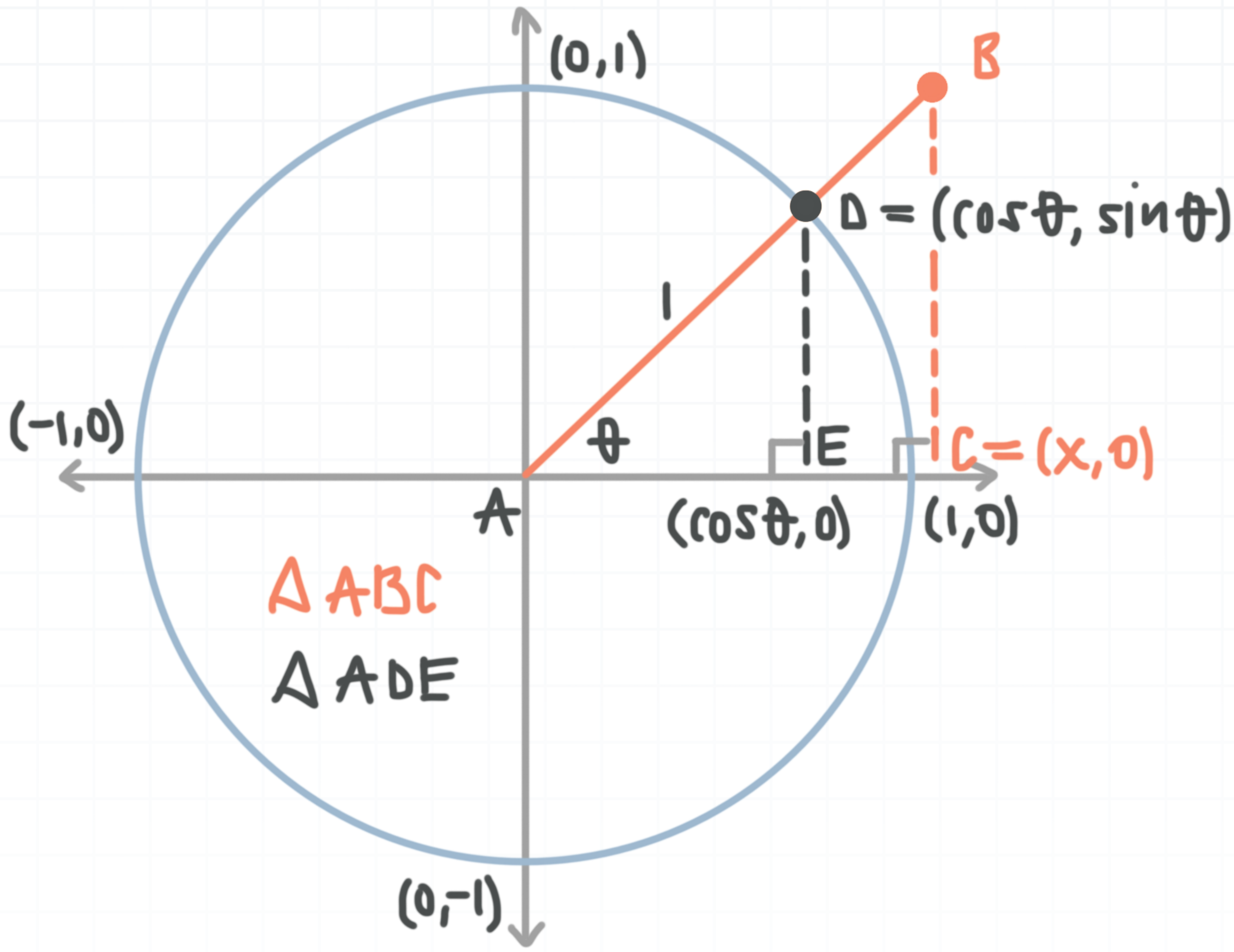
To figure this out, we'll label a point D that's on the terminal side of the angle θ created by B, and where D is on the unit circle. That means the coordinates of D are $(x, y) = (\cos\theta, \sin\theta)$, and because D is on the unit circle, $\overline{AD} = 1$.



Next we'll connect B and D to the x-axis with vertical lines, and label new points, C and E, where we intersect the x-axis. Point C is at some generic $(x,0)$. Because E is directly below D, it has the same x-coordinate as D, which means it's sitting at $(\cos\theta,0)$.



Then we realize that we have two right triangles in this figure: triangle ABC and triangle ADE.



The full triangle and the triangle within the unit circle are **similar triangles**, which means they have equal angles,

$$\angle A = \angle A$$

$$\angle B = \angle D$$

$$\angle C = \angle E$$

and proportional side lengths.

$$\frac{a = \overline{BC}}{\overline{DE}} = \frac{b = \overline{AC}}{\overline{AE}} = \frac{c = \overline{AB}}{\overline{AD}}$$

For the small triangle ADE within the unit circle, we can define sine and cosine as

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\overline{DE}}{1} = \overline{DE}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\overline{AE}}{1} = \overline{AE}$$

Furthermore, for the large triangle ABC, we've defined B at (x, y), which means we can define sine and cosine as

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{c = \overline{AB}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{c = \overline{AB}}$$

And we can find the value of $c = \overline{AB}$ using the **distance formula** from Algebra,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which gives the distance between two points, (x_1, y_1) and (x_2, y_2). The side $c = \overline{AB}$ is defined between the origin $(0,0)$ and B at (x, y), so the distance formula gives

$$d = \sqrt{(x - 0)^2 + (y - 0)^2}$$

$$d = \sqrt{x^2 + y^2}$$

So for the large triangle ABC, the formulas for sine and cosine become

$$\sin\theta = \frac{y}{c} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos\theta = \frac{x}{c} = \frac{x}{\sqrt{x^2+y^2}}$$

These formulas hold for angles in all four quadrants, so let's look at some examples.

Example

Find sine and cosine of an angle whose terminal side contains the point $(3,4)$.

Substitute $(x, y) = (3,4)$ into the formulas for sine and cosine.

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{\sqrt{9+16}} = \frac{4}{\sqrt{25}} = \frac{4}{5}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{\sqrt{9+16}} = \frac{3}{\sqrt{25}} = \frac{3}{5}$$

So even though $(3,4)$ isn't on the unit circle, we're able to use formulas for sine and cosine to find the values of those trig functions at the angle created by $(3,4)$.

Let's try an example with an angle in the second quadrant, and this time we'll find the values of all six trig functions.

Example

Find all six trig functions of an angle whose terminal side contains the point $(-2,3)$.

Substitute $(x, y) = (-2,3)$ into the formulas for sine and cosine.

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{3}{\sqrt{(-2)^2+3^2}} = \frac{3}{\sqrt{4+9}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{-2}{\sqrt{(-2)^2+3^2}} = \frac{-2}{\sqrt{4+9}} = \frac{-2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$$

Use the quotient identity to find tangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3\sqrt{13}}{13}}{-\frac{2\sqrt{13}}{13}} = \frac{3\sqrt{13}}{13}\left(-\frac{13}{2\sqrt{13}}\right) = -\frac{39\sqrt{13}}{26\sqrt{13}} = -\frac{3}{2}$$

Then use the reciprocal identities to find cosecant, secant, and cotangent.

$$\csc\theta = \frac{13}{3\sqrt{13}} = \frac{\sqrt{13}}{3}$$

$$\sec\theta = -\frac{13}{2\sqrt{13}} = -\frac{\sqrt{13}}{2}$$

$$\cot\theta = -\frac{2}{3}$$

So even though $(-2,3)$ isn't on the unit circle, we're able to use formulas for sine and cosine to find the values of those trig functions at the angle created by $(-2,3)$, and then use our other trig identities to find the values of the other four trig functions at the same angle.

$$\sin\theta = \frac{3\sqrt{13}}{13} \qquad \csc\theta = \frac{\sqrt{13}}{3}$$

$$\cos\theta = -\frac{2\sqrt{13}}{13} \qquad \sec\theta = -\frac{\sqrt{13}}{2}$$

$$\tan\theta = -\frac{3}{2} \qquad \cot\theta = -\frac{2}{3}$$