Identity matrices

We already know that multiply a matrix by a scalar of 0 will turn the matrix into a matrix with all 0 entries,

$$0\begin{bmatrix} 6 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 0(6) & 0(2) \\ 0(-1) & 0(-4) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and that multiplying a matrix by a scalar of 1 will keep the matrix the same as the original.

$$1\begin{bmatrix} 6 & 2 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 1(6) & 1(2) \\ 1(-1) & 1(-4) \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -1 & -4 \end{bmatrix}$$

But is there any matrix that you can multiply by another, without changing the value of that second matrix? The answer is yes, and we call it the **identity matrix**.

We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate $I_{2\times2}$ as just I_2, $I_{3\times3}$ as just I_3, etc.

We'll talk more later about why the identity matrix is always square. But for now, here's what an identity matrix looks like:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When you multiply the identity matrix by another matrix, you don't change the value of the other matrix. Let's see what happens when we multiply the identity matrix by another matrix.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}7&3&4\\1&6&1\\2&2&3\end{bmatrix}$$

$$=\begin{bmatrix}7(1)+1(0)+2(0)&3(1)+6(0)+2(0)&4(1)+1(0)+3(0)\\7(0)+1(1)+2(0)&3(0)+6(1)+2(0)&4(0)+1(1)+3(0)\\7(0)+1(0)+2(1)&3(0)+6(0)+2(1)&4(0)+1(0)+3(1)\end{bmatrix}$$

$$=\begin{bmatrix}7+0+0&3+0+0&4+0+0\\0+1+0&0+6+0&0+1+0\\0+0+2&0+0+2&0+0+3\end{bmatrix}$$

$$=\begin{bmatrix}7&3&4\\1&6&1\\2&2&3\end{bmatrix}$$

Notice how multiplying by the identity matrix I_3 didn't change the value of the second matrix.

Dimensions of the identity matrix

Let's prove to ourselves that the identity matrix will always be square. We'll start with some other matrix, like this 3×2:

$$A=\begin{bmatrix}4&-6\\1&1\\-2&9\end{bmatrix}$$

Because we know that the identity matrix won't change the value of A, we can set up this equation:

$$I \cdot \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix}$$

If we think about the dimensions of A in the context of this equation, we'll see why the identity matrix must be a square. The dimensions of A are 3×2, so let's substitute those into the equation to get a visual picture of the dimensions.

$$I \cdot 3 \times 2 = 3 \times 2$$

Then let's break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 2 = 3 \times 2$$

First, we know that in order to be able to multiply matrices at all, we need the same number of columns in the first matrix as we have rows in the second matrix. So we know the identity matrix must have 3 columns.

$$R \times \boxed{3 \cdot 3} \times 2 = 3 \times 2$$

We also know that the dimensions of the result matrix on the right, come from the rows of the first matrix and the columns of the second matrix.

$$\boxed{R} \times 3 \cdot 3 \times \boxed{2} = \boxed{3} \times \boxed{2}$$

So we know the identity matrix must have 3 rows.

$$3 \times 3 \cdot 3 \times 2 = 3 \times 2$$

Therefore, the identity matrix in this case turns out to be a square 3×3 matrix. And this works for a matrix with any dimensions. Here are some examples:

For a 2×4 matrix, the identity matrix has to be I_2:

$$I \cdot 2 \times 4 = 2 \times 4$$

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

$$\boxed{2} \times \boxed{2 \cdot 2} \times \boxed{4} = \boxed{2} \times \boxed{4}$$

$$\boxed{2 \times 2} \cdot 2 \times 4 = 2 \times 4$$

For a 3×1 matrix, the identity matrix has to be I_3:

$$I \cdot 3 \times 1 = 3 \times 1$$

$$R \times C \cdot 3 \times 1 = 3 \times 1$$

$$\boxed{3} \times \boxed{3 \cdot 3} \times \boxed{1} = \boxed{3} \times \boxed{1}$$

$$\boxed{3 \times 3} \cdot 3 \times 1 = 3 \times 1$$

Let's do an example problem.

Example

Choose the correct identity matrix for the given matrix, and then find the product of the identity matrix and the given matrix.

$$A = \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix}$$

We know that the dimensions of the identity matrix are always given by the number of rows in the other matrix, which means the identity matrix we need for matrix A will be I_3. We can prove it, too:

$$I \cdot 3 \times 2 = 3 \times 2$$

$$R \times C \cdot 3 \times 2 = 3 \times 2$$

$$\boxed{R} \times \boxed{3 \cdot 3} \times \boxed{2} = \boxed{3} \times \boxed{2}$$

$$\boxed{3 \times 3} \cdot 3 \times 2 = 3 \times 2$$

So the identity matrix is 3×3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the product of I_3 and matrix A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4(1) + 1(0) - 2(0) & -6(1) + 1(0) + 9(0) \\ 4(0) + 1(1) - 2(0) & -6(0) + 1(1) + 9(0) \\ 4(0) + 1(0) - 2(1) & -6(0) + 1(0) + 9(1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 0 - 0 & -6 + 0 + 0 \\ 0 + 1 - 0 & 0 + 1 + 0 \\ 0 + 0 - 2 & 0 + 0 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_3.

Properties of the identity matrix

When it comes to the identity matrix, it doesn't matter whether you multiply a matrix by the identity matrix, or multiply the identity matrix by a matrix; you'll get the original matrix either way. But the dimensions of the identity matrix may change, depending on whether it's the first or second matrix in the multiplication.

$IA = A$: I has the same number of columns as A has rows

$AI = A$: I has the same number of rows as A has columns