Multiplying and dividing polar forms

Now that we know about complex, polar, and exponential forms, let's look at how to multiply and divide polar forms.

Multiplying polar forms

To multiply two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ that are given in polar form, we first pull out r_1 and r_2.

$$z_1 z_2 = \left[r_1(\cos\theta_1 + i\sin\theta_1)\right]\left[r_2(\cos\theta_2 + i\sin\theta_2)\right]$$

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

FOIL out the parentheses.

$$z_1 z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + (-1)\sin\theta_1\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$

Group together the real and imaginary parts. Then use the sum identities for sine and cosine to simplify.

$$z_1 z_2 = r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2)i\right]$$

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

When we look at this product, we can see that

1. its distance from the origin is the product of the original distances $r = r_1r_2$.

2. its angle is the sum of the original angles $\theta = \theta_1 + \theta_2$, and

Let's work through an example of multiplying complex numbers that are given in polar form.

Example

Find the product z_1z_2 of the complex numbers in polar form.

$$z_1 = 3\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$$

$$z_2 = \sqrt{5}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

Plugging these complex numbers into the formula for the product of complex numbers gives

$$z_1z_2 = r_1r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$z_1z_2 = (3 \cdot \sqrt{5})\left[\cos\left(\frac{3\pi}{8} + \frac{7\pi}{6}\right) + i\sin\left(\frac{3\pi}{8} + \frac{7\pi}{6}\right)\right]$$

$$z_1z_2 = 3\sqrt{5}\left[\cos\left(\frac{9\pi}{24} + \frac{28\pi}{24}\right) + i\sin\left(\frac{9\pi}{24} + \frac{28\pi}{24}\right)\right]$$

$$z_1 z_2 = 3\sqrt{5}\left(\cos\frac{37\pi}{24} + i\sin\frac{37\pi}{24}\right)$$

Dividing polar forms

To divide two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ that are given in polar form, we get

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

We could simplify this division using the complex conjugate of the denominator. But it's actually much easier to convert the complex numbers in the division to exponential form.

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$$

Then exponent rules let us simplify this to

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i\theta_1 - i\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1 - \theta_2)}$$

If, instead of converting to exponential form, we had gone forward with multiplying by the complex conjugate, we would have wound up with this formula:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

We can use either of these formulas (polar form or exponential form) to find the quotient of two complex numbers. Let's do an example.

Example

Find the quotient z_1/z_2 of the complex numbers in polar form.

$$z_1 = 13\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$$

$$z_2 = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Plugging these complex numbers into the formula for the quotient of complex numbers gives

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{13}{8}\left[\cos\left(\frac{8\pi}{5} - \frac{2\pi}{3}\right) + i\sin\left(\frac{8\pi}{5} - \frac{2\pi}{3}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{13}{8}\left[\cos\left(\frac{24\pi}{15} - \frac{10\pi}{15}\right) + i\sin\left(\frac{24\pi}{15} - \frac{10\pi}{15}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{13}{8}\left(\cos\frac{14\pi}{15} + i\sin\frac{14\pi}{15}\right)$$