Even-odd identities

In this lesson, we're going to formalize all of the symmetry we learned in the last lesson into trig identities. We've already learned about the reciprocal, quotient, and Pythagorean identities, and now we're going to introduce the even-odd identities.

Even and odd functions

As a reminder from Algebra, every function can be classified as an even function, an odd function, or neither an even nor odd function.

In **even functions**, we replace x everywhere in the function with $-x$, and the function doesn't change. The function $f(x)=x^2$ would be an example, because we can replace x with $-x$ and get

$$f(x)=(-x)^2$$

$$f(x)=x^2$$

After we simplified, we got back to the original function, which means the function didn't change when we substituted $-x$. Therefore, it's an even function.

In **odd functions**, we replace x everywhere with $-x$ and end up with the original function multiplied by -1. The function $f(x)=x^3$ is an example, because we can replace x with $-x$ and get

$$f(x)=(-x)^3$$

$$f(x) = -x^3$$

$$f(x) = -(x^3)$$

After we simplified, we got back to the original function multiplied by -1. Therefore, it's an odd function.

In other words, the equation $f(-x) = f(x)$ holds true for even functions, and the equation $f(-x) = -f(x)$ holds true for odd functions.

The even-odd identities

Remember we said in the last lesson when we talked about symmetry that the value of x, and therefore the value of cosine, stayed the same when we reflected across the x-axis. That means we were essentially substituting $-\theta$ for θ, and coming out with the same cosine value. We could write that as

$$\cos(-\theta) = \cos\theta$$

Notice how this matches our rule for even functions, $f(-x) = f(x)$, which tells us that cosine is an even function.

We also said that the value of y, and therefore the value of sine, was multiplied by -1 when we reflected across the x-axis. That means we were essentially substituting $-\theta$ for θ, and coming out with the same sine value, but multiplied by -1. We could write that as

$$\sin(-\theta) = -\sin\theta$$

Notice how this matches our rule for odd functions, $f(-x) = -f(x)$, which tells us that sine is an odd function.

Then the quotient identity for tangent tells us that $\tan(-\theta)$ will be

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Then we use the reciprocal identities to find cosecant, secant, and cotangent at $(-\theta)$, and we get the full set of even-odd identities.

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Looking at these identities and comparing them to the even function $f(-x) = f(x)$ and the odd function $f(-x) = -f(x)$, we can say that

- cosine and secant are even functions
- sine, cosecant, tangent, and cotangent are odd functions

With these identities in hand, let's look at how we can use them to find values along the unit circle.

Example

Find the values of $\cos(-\pi/3)$ and $\sin(-\pi/3)$.

Since the cosine function is even,

$$\cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

Since the sine function is odd,

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

Let's do an example with an angle outside of one full rotation.

Example

Find the values of $\cos(-27\pi/4)$ and $\sin(-27\pi/4)$.

Since the cosine function is even and the sine function is odd, we can say

$$\cos\left(-\frac{27\pi}{4}\right) = \cos\frac{27\pi}{4}$$

$$\sin\left(-\frac{27\pi}{4}\right) = -\sin\frac{27\pi}{4}$$

Then to find the value of cosine of this positive angle, we'll get the coterminal angle for $27\pi/4$ by dividing it by 2π.

$$\frac{\frac{27\pi}{4}}{2\pi}$$

$$\frac{27\pi}{4}\cdot\frac{1}{2\pi}$$

$$\frac{27\pi}{8\pi}$$

$$3.375$$

So $27\pi/4$ is three full rotations, plus 0.375 of another rotation. Three full rotations is $3(2\pi)=6\pi$, so we'll rewrite the angle as

$$\frac{27\pi}{4}$$

$$\frac{24\pi}{4}+\frac{3\pi}{4}$$

$$6\pi+\frac{3\pi}{4}$$

Therefore, $27\pi/4$ is coterminal with $3\pi/4$, so

$$\cos\frac{27\pi}{4}=\cos\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$$

$$-\sin\frac{27\pi}{4}=-\sin\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$$

Finally, let's do an example where we use the even-odd identities to find the values of all six trig functions at a negative degree angle.

Example

Find the values of all six trig functions at -750°.

If we take out two full 360° rotations from 750°, we're left with 30°, which means -750° is coterminal with -30°.

$$-750^\circ + 2(360^\circ) = -750^\circ + 720^\circ = -30^\circ$$

The cosine and secant functions are even, so -30° is the same as 30°, and we can say

$$\cos(-750^\circ) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$$

$$\sec(-750^\circ) = \sec(30^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

The sine, cosecant, tangent, and cotangent functions are odd, so

$$\sin(-750^\circ) = \sin(-30^\circ) = -\sin(30^\circ) = -\frac{1}{2}$$

$$\csc(-750^\circ) = \csc(-30^\circ) = -\csc(30^\circ) = -\frac{2}{1} = -2$$

$$\tan(-750^\circ) = \tan(-30^\circ) = -\tan(30^\circ) = -\frac{\sin(30^\circ)}{\cos(30^\circ)} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right) = -\frac{\sqrt{3}}{3}$$

$$\cot(-750^\circ) = \cot(-30^\circ) = -\cot(30^\circ) = -\frac{\cos(30^\circ)}{\sin(30^\circ)} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$