Write the equation as a parametric curve

Now that you've seen how to take a pair of parametric equations (one for x as a function of the parameter t, and the other for y as a function of t), together with a range of values of t, and convert it to an equation (in x and y only) of the corresponding parametric curve, we'll discuss how to go in the opposite direction: taking an equation (in x and y only) of a curve, together with the coordinates (x, y) of the initial and terminal points of that curve, and converting it to parametric equations for x and y (as functions of t) together with the corresponding range of values of t.

If the given equation can be written in a form where y is a function of x, we can use $x = t$ as the parametric equation for x.

Example

Express in parametric form the curve given by the equation $x^3 - 6x^2 - y + 15 = 0$ from $(x, y) = (-2, -17)$ to $(x, y) = (3, -12)$.

We can rewrite the equation $x^3 - 6x^2 - y + 15 = 0$ by adding y to both sides:

$$x^3 - 6x^2 + 15 = y$$

Turning this equation around, we get

$$y = x^3 - 6x^2 + 15$$

For every real number x, this equation gives a unique value of y, so y is indeed a function of x.

If we use $x = t$ as our parametric equation for x, then we can substitute t for every x in the equation for y that we just found ($y = x^3 - 6x^2 + 15$) and get a parametric equation for y:

$$y = t^3 - 6t^2 + 15$$

When we derive a pair of parametric equations, we also need to specify the interval of values of the parameter t such that the smallest value of t in that interval corresponds to the initial point of the curve and the largest value of t corresponds to the terminal point.

In our example, the initial point is $(-2, -17)$ and the terminal point is $(3, -12)$. Since our parametric equation for x is $x = t$ (hence $x = -2$ at the initial point, and $x = 3$ at the terminal point), we see that $-2 \le t \le 3$. Thus we can express this curve in parametric form as follows:

$$x = t \text{ and } y = t^3 - 6t^2 + 15 \text{ where } -2 \le t \le 3$$

We can take the analogous approach if x can be expressed as a function of y.

Example

Express in parametric form the curve given by the equation $\sqrt{y} + x = 3$ from $(x, y) = (1,4)$ to $(x, y) = (-2,25)$.

We can rewrite the equation $\sqrt{y} + x = 3$ by subtracting $\sqrt{y}$ from both sides:

$$x = 3 - \sqrt{y}$$

For every nonnegative real number y, this equation gives a (unique) value of x, so x is indeed a function of y on the interval $[0,\infty)$.

Using $y = t$ as our parametric equation for y, and substituting t for y in the equation for x that we just obtained ($x = 3 - \sqrt{y}$), gives

$$x = 3 - \sqrt{t}$$

In this example, the initial point is $(1,4)$ and the terminal point is $(-2,25)$. If we use $y = t$ as our parametric equation for y, then (since $y = 4$ at the initial point, and $y = 25$ at the terminal point) we find that $4 \le t \le 25$. Thus we can express this curve in parametric form as follows:

$$x = 3 - \sqrt{t} \text{ and } y = t \text{ where } 4 \le t \le 25$$

Given an equation (in x and y alone) of a curve, the parametric equations for x and y in terms of the parameter t are by no means unique. All we need to do is come up with parametric equations for x and y that satisfy the given equation, and define the smallest and largest values of t so that they correspond to the given initial and terminal points of the curve, respectively.

In our first example, we could have used (say) $x = t + 3$ as our parametric equation for x. Then we would have gotten the following equation for y:

$$y = (t + 3)^3 - 6(t + 3)^2 + 15$$

Also, we would have needed to adjust the range of values of t accordingly. Recall that in that example, $x = -2$ for the initial point and $x = 3$ for the terminal point. Thus

$$x = -2 \Longrightarrow t + 3 = -2 \Longrightarrow t = -5, \quad x = 3 \Longrightarrow t + 3 = 3 \Longrightarrow t = 0$$

Therefore, we could have expressed the curve in that example in the following parametric form:

$$x = t + 3 \text{ and } y = (t + 3)^3 - 6(t + 3)^2 + 15 \text{ where } -5 \le t \le 0$$

Notice that this is more complicated than the form we originally found for this curve.

In general, it's probably best to try to express a parametric curve (by means of a pair of parametric equations and a range of values of the parameter t) in a way that's as simple as possible.

Next, let's look at an example of a curve where the given equation (in x and y) cannot be put into a form where one of the variables (x or y) is expressed as a function of the other.

Example

$(x - 2)^2 + (y + 4)^2 = 10$ is the equation of a closed curve. Express this curve in parametric form, with $(2 + \sqrt{10}, -4)$ as the initial point, and define the range of values of t in such a way that we trace out the resulting parametric curve through one complete "turn" in the counterclockwise direction.

Notice that $(x - 2)^2 + (y + 4)^2 = 10$ is the equation of the circle that's centered at the point $(2, -4)$ and has a radius of $\sqrt{10}$.

As a first step, we'll divide both sides of the equation by 10:

$$\frac{(x-2)^2}{10}+\frac{(y+4)^2}{10}=1$$

Writing 10 as $(\sqrt{10})^2$, we get

$$\left(\frac{x-2}{\sqrt{10}}\right)^2+\left(\frac{y+4}{\sqrt{10}}\right)^2=1$$

We know that $\sin^2 t+\cos^2 t=1$ for any t. Because of the "natural" relationship between x and the cosine function, and the "natural" relationship between y and the sine function, let's let

$$\frac{x-2}{\sqrt{10}}=\cos t, \quad \frac{y+4}{\sqrt{10}}=\sin t$$

Multiplying both of these equations by $\sqrt{10}$ yields

$$x-2=\sqrt{10}\cos t, \quad y+4=\sqrt{10}\sin t$$

Solving the first equation for x and the second equation for y, we get the parametric equations

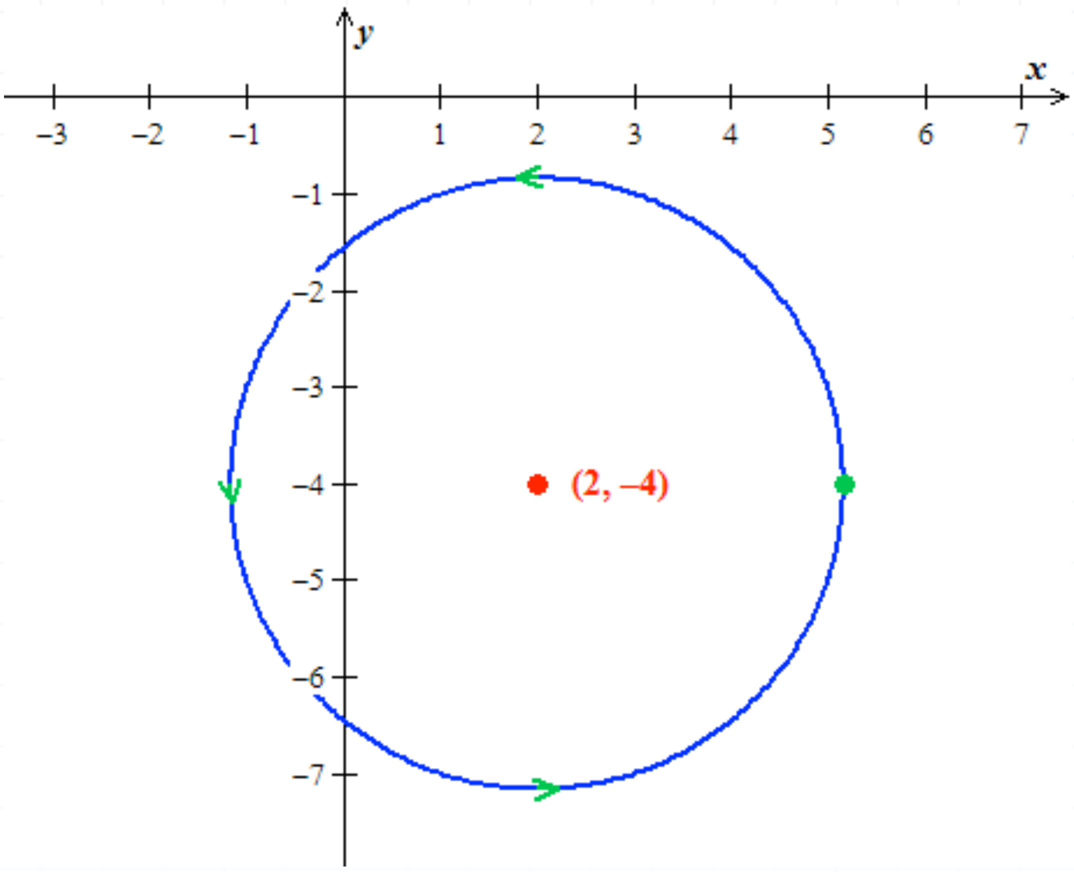
$$x=2+\sqrt{10}\cos t, \quad y=-4+\sqrt{10}\sin t$$

Now let's determine a pair of values of t that we could use (with this pair of parametric equations) as the smallest and largest values of t. Well, the circle is to be traced through one complete turn, so the initial and terminal points must correspond to values of t that differ by 2π. Since the curve is traced out in the counterclockwise direction, let's try using 0 and 2π. Then we get the following coordinates for some key points of the circle:

t	$x = 2 + \sqrt{10}\cos t$	$y = -4 + \sqrt{10}\sin t$
0	$2 + \sqrt{10}$	-4
$\frac{\pi}{2}$	2	$-4 + \sqrt{10}$
π	$2 - \sqrt{10}$	-4
$\frac{3\pi}{2}$	2	$-4 - \sqrt{10}$
2π	$2 + \sqrt{10}$	-4

As you can see, the coordinates of the initial point (the point for $t = 0$) are indeed $(2 + \sqrt{10}, -4)$. Note that the same is true of the terminal point (the point for $t = 2\pi$), which is identical to the initial point, because tracing out one complete “turn” of a circle means that we end up back at our initial point.

Also, the curve is indeed traced out in the counterclockwise direction as t increases from 0 to 2π.



Therefore, our parametric curve could be expressed as

$$x = 2 + \sqrt{10}\cos t \text{ and } y = -4 + \sqrt{10}\sin t \text{ where } 0 \le t \le 2\pi$$

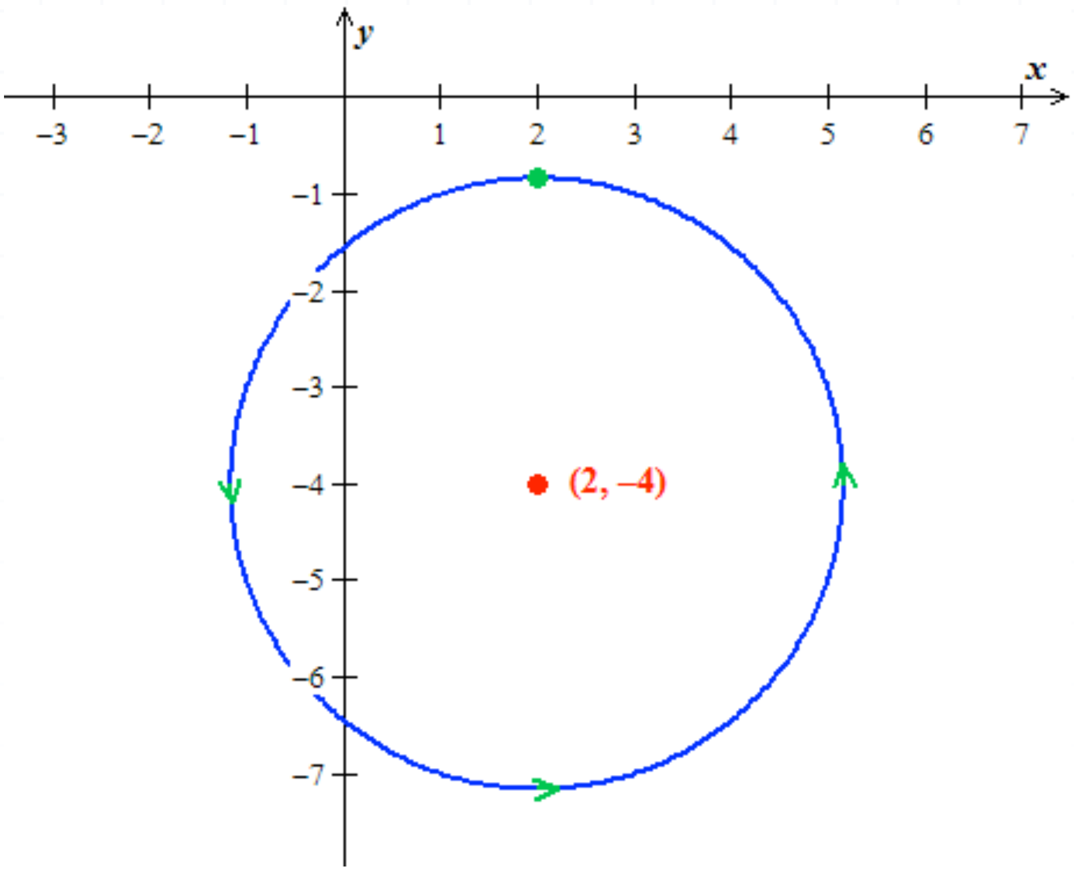
If we wanted to trace out this curve in the counterclockwise direction but use $(2, -4 + \sqrt{10})$ as our initial point, one way to do this would be to express the curve in parametric form as follows:

$$x = 2 + \sqrt{10}\cos t \text{ and } y = -4 + \sqrt{10}\sin t \text{ where } \frac{\pi}{2} \le t \le \frac{5\pi}{2}$$

In this case, our table takes the following form:

t	$x = 2 + \sqrt{10}\cos t$	$y = -4 + \sqrt{10}\sin t$
$\frac{\pi}{2}$	2	$-4 + \sqrt{10}$
π	$2 - \sqrt{10}$	-4

$\frac{3\pi}{2}$	2	$-4-\sqrt{10}$
2π	$2+\sqrt{10}$	-4
$\frac{5\pi}{2}$	2	$-4+\sqrt{10}$



Another way to do this would be to keep the range of values of t as $0 \leq t \leq 2\pi$ but use the parametric equations

$$x = 2 - \sqrt{10}\sin t$$

$$y = -4 + \sqrt{10}\cos t$$

To understand why this works, consider the following identities:

$$\cos\left(t+\frac{\pi}{2}\right) = -\sin t, \quad \sin\left(t+\frac{\pi}{2}\right) = \cos t$$

These identities can be derived from the sum identities for cosine and sine:

$$\cos\left(t+\frac{\pi}{2}\right)=(\cos t)\left(\cos\frac{\pi}{2}\right)-(\sin t)\left(\sin\frac{\pi}{2}\right)=(\cos t)(0)-(\sin t)(1)=-\sin t$$

$$\sin\left(t+\frac{\pi}{2}\right)=(\sin t)\left(\cos\frac{\pi}{2}\right)+(\cos t)\left(\sin\frac{\pi}{2}\right)=(\sin t)(0)+(\cos t)(1)=\cos t$$

Now our table takes the following form:

t	$x=2-\sqrt{10}\sin t$	$y=-4+\sqrt{10}\cos t$
0	2	$-4+\sqrt{10}$
$\frac{\pi}{2}$	$2-\sqrt{10}$	-4
π	2	$-4-\sqrt{10}$
$\frac{3\pi}{2}$	$2+\sqrt{10}$	-4
2π	2	$-4+\sqrt{10}$

Notice that we get exactly the same points (and in the same order) as in the previous table.

Now suppose we want to trace out this circle in the clockwise direction, and we want the initial (and therefore the terminal) point to be $(2+\sqrt{10},-4)$. We can still use 0 and 2π for the values of t that correspond to the initial and terminal points, respectively, but if we do that, we have to use a different pair of parametric equations for x and y. In this case, we could use

$$x = 2 + \sqrt{10}\cos(2\pi - t)$$

$$y = -4 + \sqrt{10}\sin(2\pi - t)$$

To see this, recall the difference identities for cosine and sine:

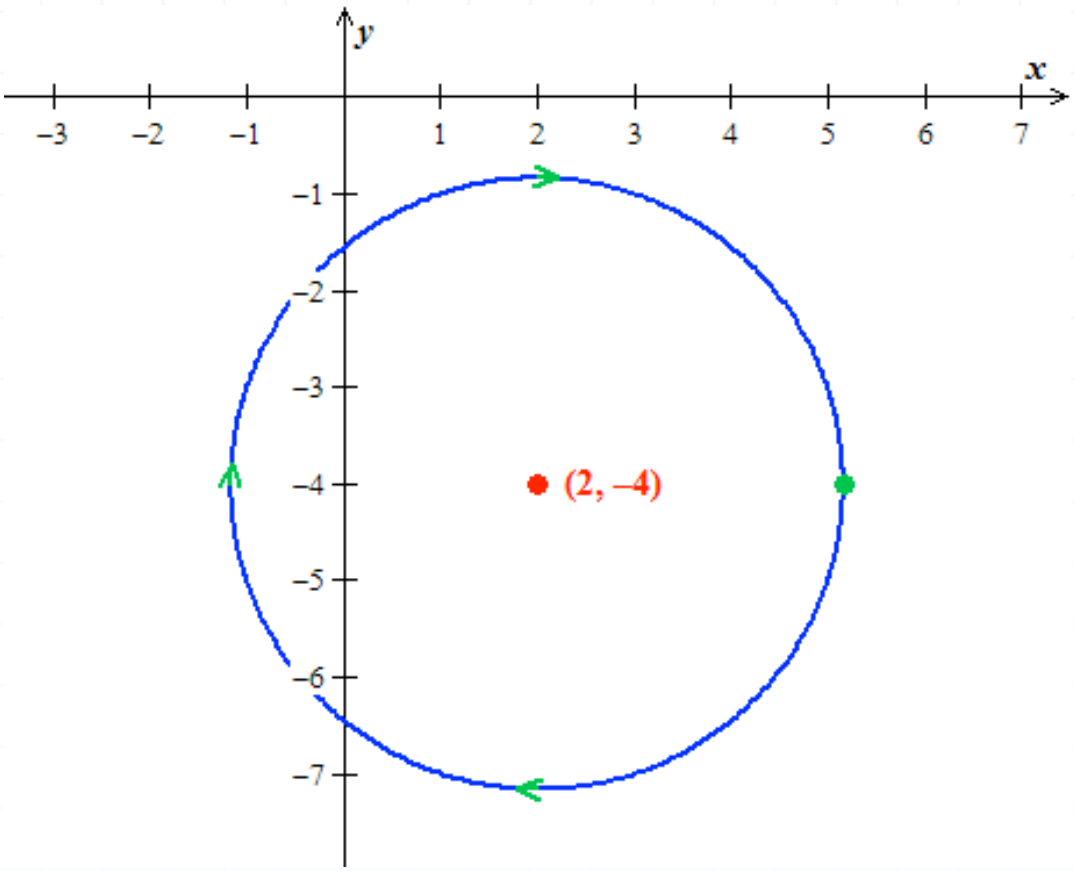
$$\cos(2\pi - t) = (\cos 2\pi)(\cos t) + (\sin 2\pi)(\sin t) = (1)(\cos t) + (0)(\sin t) = \cos t$$

$$\sin(2\pi - t) = (\sin 2\pi)(\cos t) - (\cos 2\pi)(\sin t) = (0)(\cos t) - (1)(\sin t) = -\sin t$$

Therefore, we get the following coordinates for those key points of the circle:

	$x = 2 + \sqrt{10}\cos(2\pi - t)$	$y = -4 + \sqrt{10}\sin(2\pi - t)$
t	$= 2 + \sqrt{10}\cos t$	$= -4 - \sqrt{10}\sin t$
0	$2 + \sqrt{10}$	-4
$\frac{\pi}{2}$	2	$-4 - \sqrt{10}$
π	$2 - \sqrt{10}$	-4
$\frac{3\pi}{2}$	2	$-4 + \sqrt{10}$
2π	$2 + \sqrt{10}$	-4

The curve is indeed traced out in the clockwise direction, and its initial point is $(2 + \sqrt{10}, -4)$.



What we have found is that we could express this curve in parametric form as follows:

$$x = 2 + \sqrt{10}\cos t \text{ and } y = -4 - \sqrt{10}\sin t \text{ where } 0 \le t \le 2\pi$$

Next, let's look at an equation that gives us an entirely different kind of curve.

Example

Express in parametric form the curve given by the equation $(x-3)^2 - (y-4)^2 = 4$ from $(x, y) = (7, 4 - 2\sqrt{3})$ to $(x, y) = (7, 4 + 2\sqrt{3})$.

Let's start by dividing both sides of the given equation by 4:

$$\frac{(x-3)^2}{4} - \frac{(y-4)^2}{4} = 1$$

Since $4 = 2^2$, we can rewrite this as

$$\left(\frac{x-3}{2}\right)^2 - \left(\frac{y-4}{2}\right)^2 = 1$$

This is an equation of a hyperbola. Since the minus sign is in front of the "y part" of it, one branch of this hyperbola opens to the right and the other branch opens to the left.

Recall that one of the Pythagorean identities is

$$\sec^2 t - \tan^2 t = 1$$

and notice that our equation

$$\left(\frac{x-3}{2}\right)^2 - \left(\frac{y-4}{2}\right)^2 = 1$$

has that same form. Thus let's let

$$\frac{x-3}{2} = \sec t$$

$$\frac{y-4}{2} = \tan t$$

Multiplying both of these equations by 2 yields

$$x - 3 = 2\sec t$$

$$y - 4 = 2\tan t$$

Solving the first equation for x and the second equation for y, we get

$$x = 3 + 2\sec t$$

$$y = 4 + 2\tan t$$

What we still need to do is come up with smallest and largest values of the parameter t such that the former corresponds to the initial point of our curve (the point $(7,4 - 2\sqrt{3})$) and the latter corresponds to the terminal point (the point $(7,4 + 2\sqrt{3})$).

Well, here we go:

$$x = 7 \Longrightarrow 3 + 2\sec t = 7 \Longrightarrow 2\sec t = 4 \Longrightarrow \sec t = 2$$

$$y = 4 - 2\sqrt{3} \Longrightarrow 4 + 2\tan t = 4 - 2\sqrt{3} \Longrightarrow 2\tan t = -2\sqrt{3} \Longrightarrow \tan t = -\sqrt{3}$$

$$y = 4 + 2\sqrt{3} \Longrightarrow 4 + 2\tan t = 4 + 2\sqrt{3} \Longrightarrow 2\tan t = 2\sqrt{3} \Longrightarrow \tan t = \sqrt{3}$$

For the initial point, we can use $t = -\pi/3$, because

$$\sec\left(-\frac{\pi}{3}\right) = \frac{1}{\cos\left(-\frac{\pi}{3}\right)} = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{1}{2}} = 2$$

and

$$\tan\left(-\frac{\pi}{3}\right) = \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} = \frac{-\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

For the terminal point, we can use $t = \pi/3$, because

$$\sec\left(\frac{\pi}{3}\right)=\frac{1}{\cos\left(\frac{\pi}{3}\right)}=\frac{1}{\frac{1}{2}}=2$$

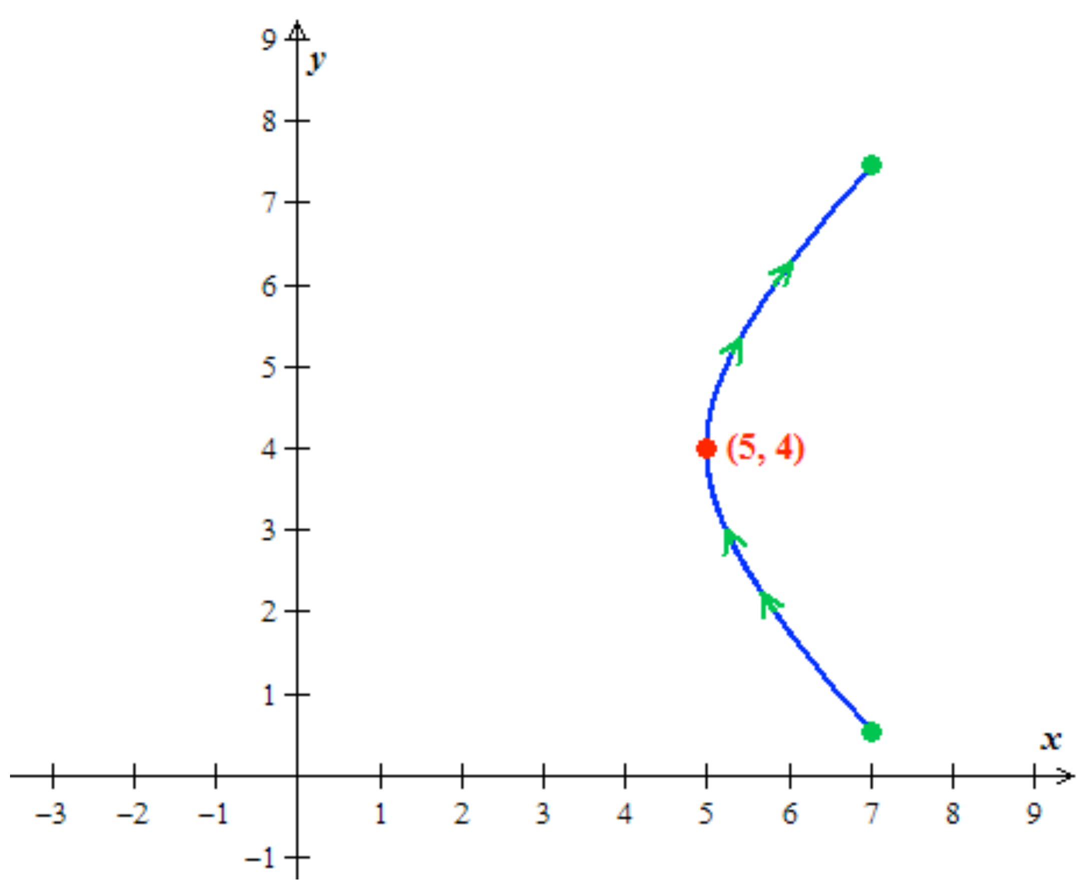
and

$$\tan\left(\frac{\pi}{3}\right)=\frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}$$

Thus this curve can be written in parametric form as

$x=3+2\sec t$ and $y=4+2\tan t$ where $-\frac{\pi}{3}\le t\le\frac{\pi}{3}$

This curve is part of the right branch of the hyperbola $(x-3)^2-(y-4)^2=4$. The vertex of this branch is at the point $(5,4)$, and the curve is traced out from a point on the lower half of that branch (namely, the point $(7,4-2\sqrt{3})$) to a point on the upper half of it (namely, the point $(7,4+2\sqrt{3})$).



The vertex of the left branch of this hyperbola is at the point $(1,4)$. You should convince yourself that you could express the part of the left branch of it from the point $(1,4-2\sqrt{3})$ to the point $(1,4+2\sqrt{3})$ in parametric form as follows:

$x = 3 - 2\sec t$ and $y = 4 + 2\tan t$ where $-\dfrac{\pi}{3} \le t \le \dfrac{\pi}{3}$