Graph the polar curve, lemniscate

In this lesson, we're going to discuss a polar curve that's known as a lemniscate. Such a curve is the graph of a polar equation of the form

$$r^2 = a\cos(2\theta)$$

or

$$r^2 = a\sin(2\theta)$$

where a is positive.

Since r^2 is nonnegative and a is positive, every point of a lemniscate of the form $r^2 = a\cos(2\theta)$ must satisfy

$$\cos(2\theta) \geq 0$$

and every point of a lemniscate of the form $r^2 = a\sin(2\theta)$ must satisfy

$$\sin(2\theta) \geq 0$$

Let's first consider lemniscates of the form $r^2 = a\cos(2\theta)$. The cosine of an angle is nonnegative if the angle is either in the first quadrant, in the fourth quadrant, on the positive horizontal axis, or on the (positive or negative) vertical axis. Thus if θ is an angle such that 2θ is in the interval $[0,2\pi)$ and $\cos(2\theta) \geq 0$, then

$$0 \leq 2\theta \leq \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} \leq 2\theta < 2\pi$$

Equivalently,

$$0 \le \theta \le \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4} \le \theta < \pi$$

If θ is in either of those two intervals and $\cos(2\theta) > 0$, then there are two values of r that satisfy the equation $r^2 = a\cos(2\theta)$:

$$r = \sqrt{a\cos(2\theta)}$$

and

$$r = -\sqrt{a\cos(2\theta)}$$

Thus there are two points that correspond to such an angle θ: one with polar coordinates

$$(r, \theta) = (\sqrt{a\cos(2\theta)}, \theta)$$

and the other with polar coordinates

$$(r, \theta) = (-\sqrt{a\cos(2\theta)}, \theta)$$

The latter point also has polar coordinates

$$(-r, \theta + \pi) = (\sqrt{a\cos(2\theta)}, \theta + \pi)$$

If $r = 0$, the "two points" coincide, and they do so at the pole (since the pole is the only point at which $r = 0$). Now if θ is in the interval $[0,\pi/4]$ or the interval $[3\pi/4, 2\pi)$, then

$$r = 0 \Longrightarrow \cos(2\theta) = 0 \Longrightarrow \theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{3\pi}{4}$$

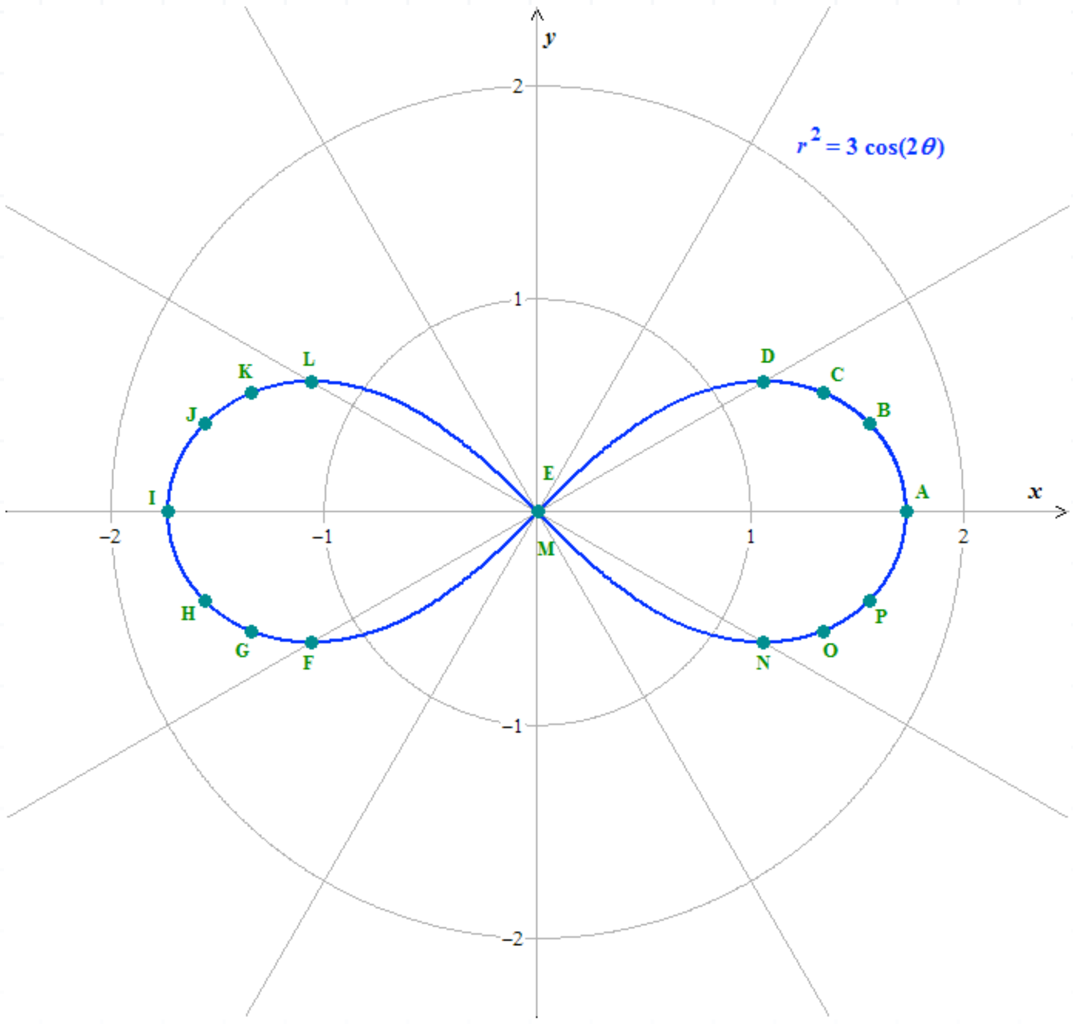
Thus for all θ in the intervals $[0,\pi/4)$ and $(3\pi/4,2\pi)$, there are two distinct points of the lemniscate with the same value of θ.

Example

Graph the lemniscate $r^2 = 3\cos(2\theta)$.

In the following table, the values of $\cos(2\theta)$ and $r^2 = 3\cos(2\theta)$ are shown for a number of angles θ in the intervals $[0,\pi/4]$ and $[3\pi/4,\pi)$. In the table, we also give one pair of polar coordinates, (r,θ), for points where r is positive, and we give two pairs of polar coordinates, (r,θ) and $(-r,\theta+\pi)$, for points where r is negative.

Point	θ	$\cos(2\theta)$	$r^2=3\cos(2\theta)$	Polar coordinates (r,θ)	Polar coordinates $(-r,\theta+\pi)$ r negative
A	0	1	3	$(\sqrt{3},0)$	
B	$\frac{\pi}{12}$	$\frac{\sqrt{3}}{2}$	$\frac{3\sqrt{3}}{2}$	$\left(\sqrt{\frac{3\sqrt{3}}{2}},\frac{\pi}{12}\right)$	
C	$\frac{\pi}{8}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\left(\sqrt{\frac{3\sqrt{2}}{2}},\frac{\pi}{8}\right)$	
D	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\left(\sqrt{\frac{3}{2}},\frac{\pi}{6}\right)$	
$\text{E}=\text{pole}$	$\frac{\pi}{4}$	0	0	$\left(0,\frac{\pi}{4}\right)$	
F	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\left(-\sqrt{\frac{3}{2}},\frac{\pi}{6}\right)$	$\left(\sqrt{\frac{3}{2}},\frac{7\pi}{6}\right)$
G	$\frac{\pi}{8}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\left(-\sqrt{\frac{3\sqrt{2}}{2}},\frac{\pi}{8}\right)$	$\left(\sqrt{\frac{3\sqrt{2}}{2}},\frac{9\pi}{8}\right)$
H	$\frac{\pi}{12}$	$\frac{\sqrt{3}}{2}$	$\frac{3\sqrt{3}}{2}$	$\left(-\sqrt{\frac{3\sqrt{3}}{2}},\frac{\pi}{12}\right)$	$\left(\sqrt{\frac{3\sqrt{3}}{2}},\frac{13\pi}{12}\right)$
I	0	1	3	$(-\sqrt{3},0)$	$(\sqrt{3},\pi)$
J	$\frac{11\pi}{12}$	$\frac{\sqrt{3}}{2}$	$\frac{3\sqrt{3}}{2}$	$\left(\sqrt{\frac{3\sqrt{3}}{2}},\frac{11\pi}{12}\right)$	
K	$\frac{7\pi}{8}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\left(\sqrt{\frac{3\sqrt{2}}{2}},\frac{7\pi}{8}\right)$	
L	$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\left(\sqrt{\frac{3}{2}},\frac{5\pi}{6}\right)$	
$\text{M}=\text{pole}$	$\frac{3\pi}{4}$	0	0	$\left(0,\frac{3\pi}{4}\right)$	
N	$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\left(-\sqrt{\frac{3}{2}},\frac{5\pi}{6}\right)$	$\left(\sqrt{\frac{3}{2}},\frac{11\pi}{6}\right)$
O	$\frac{7\pi}{8}$	$\frac{\sqrt{2}}{2}$	$\frac{3\sqrt{2}}{2}$	$\left(-\sqrt{\frac{3\sqrt{2}}{2}},\frac{7\pi}{8}\right)$	$\left(\sqrt{\frac{3\sqrt{2}}{2}},\frac{15\pi}{8}\right)$
P	$\frac{11\pi}{12}$	$\frac{\sqrt{3}}{2}$	$\frac{3\sqrt{3}}{2}$	$\left(-\sqrt{\frac{3\sqrt{3}}{2}},\frac{11\pi}{12}\right)$	$\left(\sqrt{\frac{3\sqrt{3}}{2}},\frac{23\pi}{12}\right)$



Notice that on the graph, the two points of the lemniscate $r^2 = 3\cos(2\theta)$ which are furthest from the pole are at a distance of exactly $\sqrt{3}$ units from it. This is because the maximum value of $\sqrt{\cos(2\theta)})$ for θ in the set

$$\left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right)$$

is 1 (which is attained at the two points that correspond to $\theta = 0$). It can easily be seen that for any lemniscate of the form $r^2 = a\cos(2\theta)$, the two points which are furthest from the pole are at a distance of exactly $\sqrt{a}$ units from it.

Also, notice that the lemniscate $r^2 = 3\cos(2\theta)$ is symmetric with respect to both the horizontal axis and the vertical axis, and that (as derived earlier) it does indeed pass through the pole twice (once at the point that corresponds to $\theta = \pi/4$, and once at the point that corresponds to $\theta = 3\pi/4$). Both of these properties are true of all lemniscates of the form $r^2 = a\cos(2\theta)$.

Let's turn our attention now to lemniscates of the form $r^2 = a\sin(2\theta)$, where we must have $\sin(2\theta) \geq 0$. The sine of an angle is nonnegative if it's either in the first quadrant, in the second quadrant, on the positive vertical axis, or on the (positive or negative) horizontal axes. Thus if θ is an angle such that 2θ is in the interval $[0,2\pi)$ and $\sin(2\theta) \geq 0$, then

$$0 \leq 2\theta \leq \pi$$

Equivalently,

$$0 \leq \theta \leq \frac{\pi}{2}$$

If θ is in that interval and $\sin(2\theta) > 0$, then there are two values of rr that satisfy the equation $r^2 = a\sin(2\theta)$:

$$r = \sqrt{a\sin(2\theta)}$$

and

$$r = -\sqrt{a\sin(2\theta)}$$

Thus there are two points that correspond to such an angle θ: one with polar coordinates

$$(r, \theta) = (\sqrt{a\sin(2\theta)}, \theta)$$

and the other with polar coordinates

$$(r, \theta) = (-\sqrt{a \sin(2\theta)}, \theta)$$

The latter point also has polar coordinates

$$(-r, \theta + \pi) = (\sqrt{a \sin(2\theta)}, \theta + \pi)$$

If $r = 0$, the “two points” coincide, and they do so at the pole (since the pole is the only point at which $r = 0$). Now if θ is in the interval $[0,\pi/2]$, then

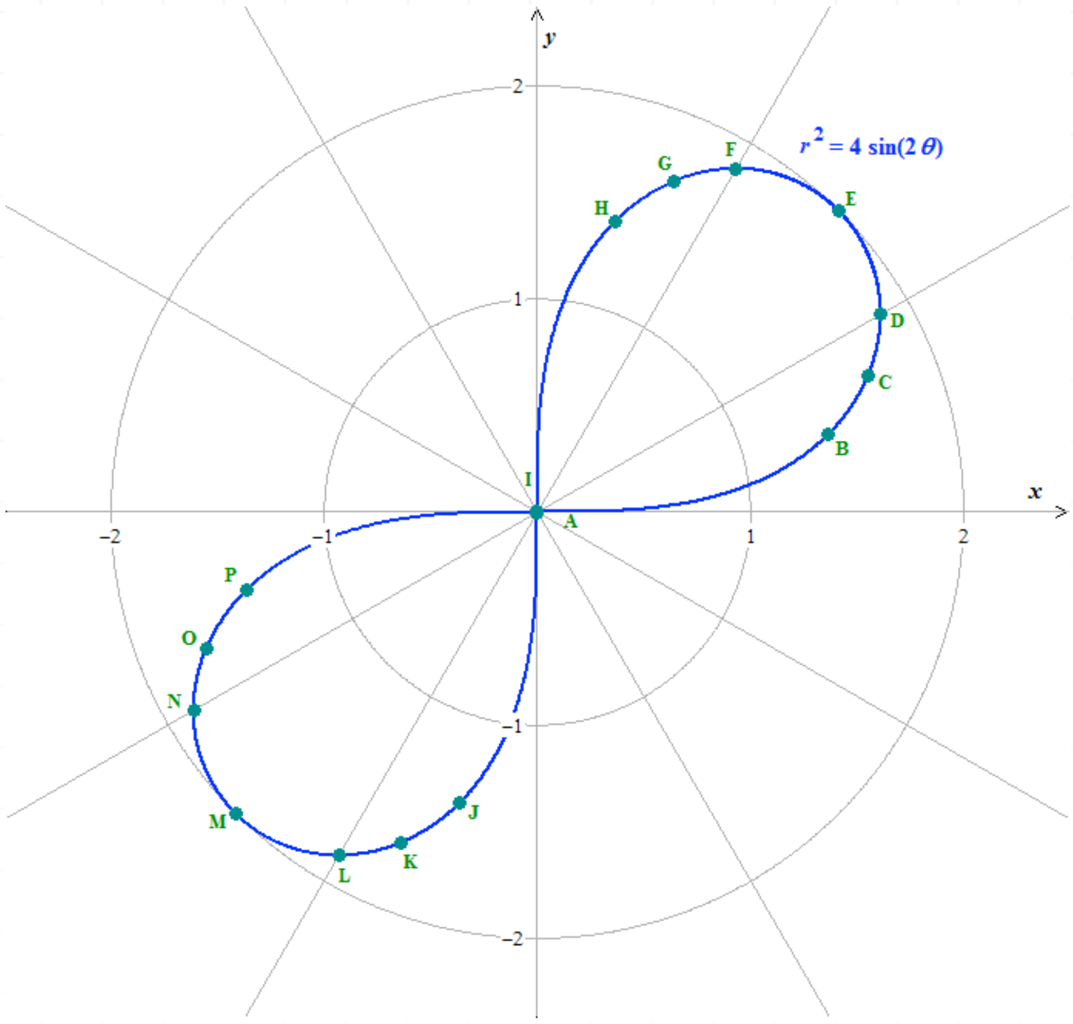
$$r = 0 \Longrightarrow \sin(2\theta) = 0 \Longrightarrow \theta = 0 \quad \text{or} \quad \theta = \frac{\pi}{2}$$

Thus for all θ in the interval $(0,\pi/2)$, there are two distinct points of the lemniscate with the same value of θ.

Example

Graph the lemniscate $r^2 = 4\sin(2\theta)$.

Point	θ	$\sin(2\theta)$	$r^2 = 4\sin(2\theta)$	Polar coordinates (r, θ)	Polar coordinates $(-r, \theta+\pi)$ r negative
A = pole	0	0	0	$(0, 0)$	
B	$\frac{\pi}{12}$	$\frac{1}{2}$	2	$\left(\sqrt{2}, \frac{\pi}{12}\right)$	
C	$\frac{\pi}{8}$	$\frac{\sqrt{2}}{2}$	$2\sqrt{2}$	$\left(\sqrt{2\sqrt{2}}, \frac{\pi}{8}\right)$	
D	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$2\sqrt{3}$	$\left(\sqrt{2\sqrt{3}}, \frac{\pi}{6}\right)$	
E	$\frac{\pi}{4}$	1	4	$\left(2, \frac{\pi}{4}\right)$	
F	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$2\sqrt{3}$	$\left(\sqrt{2\sqrt{3}}, \frac{\pi}{3}\right)$	
G	$\frac{3\pi}{8}$	$\frac{\sqrt{2}}{2}$	$2\sqrt{2}$	$\left(\sqrt{2\sqrt{2}}, \frac{3\pi}{8}\right)$	
H	$\frac{5\pi}{12}$	$\frac{1}{2}$	2	$\left(2, \frac{5\pi}{12}\right)$	
I = pole	$\frac{\pi}{2}$	0	0	$\left(0, \frac{\pi}{2}\right)$	
J	$\frac{5\pi}{12}$	$\frac{1}{2}$	2	$\left(-2, \frac{5\pi}{12}\right)$	$\left(2, \frac{17\pi}{12}\right)$
K	$\frac{3\pi}{8}$	$\frac{\sqrt{2}}{2}$	$2\sqrt{2}$	$\left(-\sqrt{2\sqrt{2}}, \frac{3\pi}{8}\right)$	$\left(\sqrt{2\sqrt{2}}, \frac{11\pi}{8}\right)$
L	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$2\sqrt{3}$	$\left(-\sqrt{2\sqrt{3}}, \frac{\pi}{3}\right)$	$\left(\sqrt{2\sqrt{3}}, \frac{4\pi}{3}\right)$
M	$\frac{\pi}{4}$	1	4	$\left(-2, \frac{\pi}{4}\right)$	$\left(2, \frac{5\pi}{4}\right)$
N	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$2\sqrt{3}$	$\left(-\sqrt{2\sqrt{3}}, \frac{\pi}{6}\right)$	$\left(\sqrt{2\sqrt{3}}, \frac{7\pi}{6}\right)$
O	$\frac{\pi}{8}$	$\frac{\sqrt{2}}{2}$	$2\sqrt{2}$	$\left(-\sqrt{2\sqrt{2}}, \frac{\pi}{8}\right)$	$\left(\sqrt{2\sqrt{2}}, \frac{9\pi}{8}\right)$
P	$\frac{\pi}{12}$	$\frac{1}{2}$	2	$\left(-\sqrt{2}, \frac{\pi}{12}\right)$	$\left(\sqrt{2}, \frac{13\pi}{12}\right)$



Notice that on the graph, the two points of the lemniscate $r^2 = 4\sin(2\theta)$ which are furthest from the pole are at a distance of exactly $\sqrt{4} = 2$ units from it. This is because the maximum value of $\sqrt{\sin(2\theta)}$ for θ in the interval $[0,\pi/2]$ is 1 (which is attained at the two points that correspond to $\theta = \pi/4$). It can easily be seen that for any lemniscate of the form $r^2 = a\sin(2\theta)$, the two points which are furthest from the pole are at a distance of exactly $\sqrt{a}$ units from it.

Also, notice that the lemniscate $r^2 = 4\sin(2\theta)$ is symmetric with respect to the pole (but is not symmetric with respect to either the horizontal axis or the vertical axis), and that (as derived earlier) it does indeed pass through the pole twice (once at the point that corresponds to $\theta = 0$, and once at the point that corresponds to $\theta = \pi/2$). Both of these properties are true of all lemniscates of the form $r^2 = a\sin(2\theta)$.

The only key feature that makes a difference from one lemniscate of the form $r^2 = a\cos(2\theta)$ to another is in their values of a, and the same thing is true of any two lemniscates of the form $r^2 = a\sin(2\theta)$.