The Pythagorean identities

So far we've defined the six trig functions and introduced the reciprocal and quotient identities that relate the trig functions to one another.

In this lesson we'll look at the Pythagorean identities, which are another set of three identities that relate the trig functions to each other. We'll look at exactly how to prove each of them, but here they are for quick reference:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

We call them "Pythagorean identities" because they're derived from the Pythagorean theorem.

The Pythagorean theorem

Remember from Geometry that the Pythagorean theorem tells us that, for any right triangle, the sum of the squares of the side lengths is equal to the square of the length of the hypotenuse.

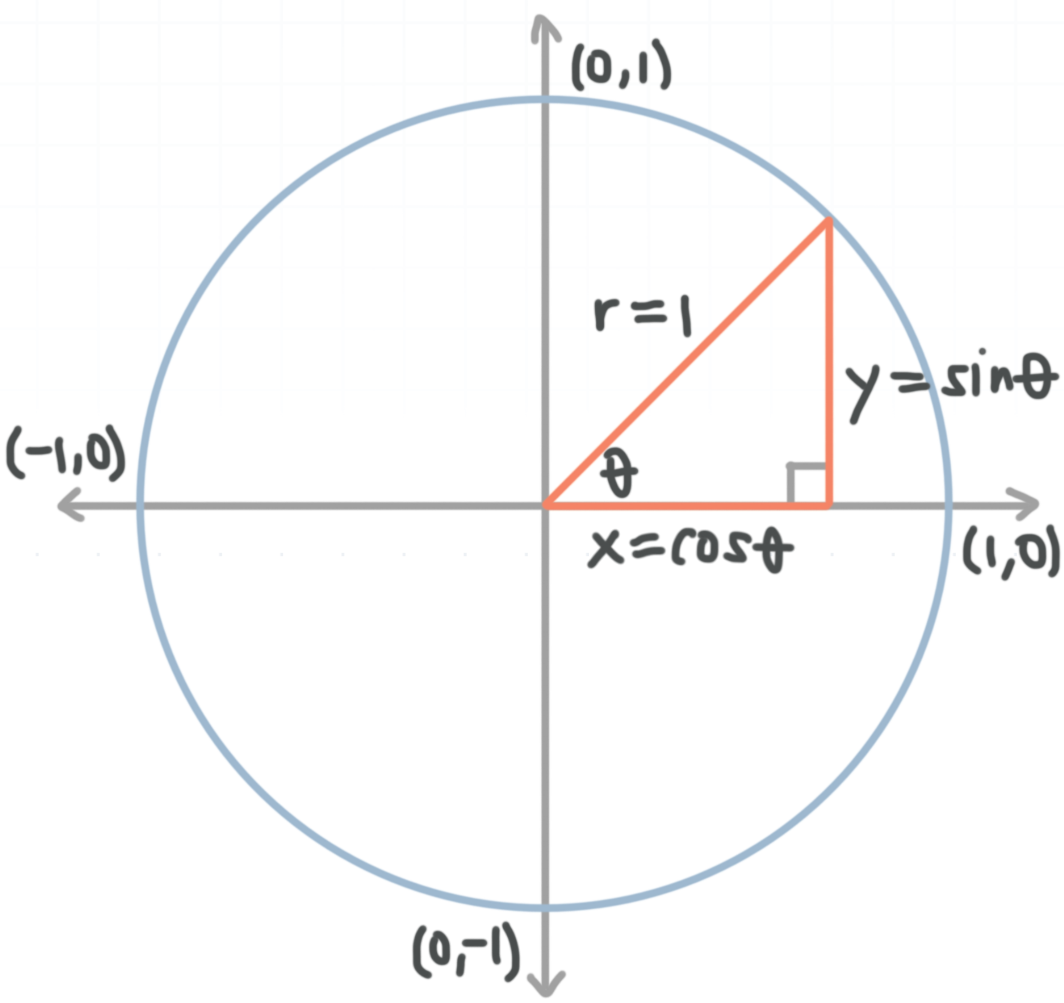
In other words, if we call the legs of the right triangle a and b, and the hypotenuse c, then the Pythagorean theorem says

$$a^2 + b^2 = c^2$$

What we want to do now is rewrite this theorem in terms of trig functions, instead of simple side lengths. Doing so will give us the three Pythagorean identities.

The Pythagorean identity with sine and cosine

Remember in the last lesson that we put a right triangle in the first quadrant inside a circle with radius $r = 1$, and used that to define the side lengths of the triangle as $x = \cos\theta$ and $y = \sin\theta$.



If we start with the Pythagorean theorem, then we can substitute $a = x = \cos\theta$, $b = y = \sin\theta$, and $c = r = 1$.

$$a^2 + b^2 = c^2$$

$$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$$

$$\sin^2\theta + \cos^2\theta = 1$$

This is the first of the three Pythagorean identities. We can derive the other two from this first one.

The Pythagorean identity with cosecant and cotangent

To find the second Pythagorean identity, we can start with the first identity, $\sin^2\theta + \cos^2\theta = 1$, and divide through both sides of the equation by $\sin^2\theta$.

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2$$

The quotient identity for tangent lets us simplify the fraction on the left,

$$1 + (\cot\theta)^2 = \left(\frac{1}{\sin\theta}\right)^2$$

$$1 + \cot^2\theta = \left(\frac{1}{\sin\theta}\right)^2$$

and the reciprocal identity for cosecant lets us simplify the fraction on the right.

$$1 + \cot^2\theta = (\csc\theta)^2$$

$$1 + \cot^2\theta = \csc^2\theta$$

This is our second Pythagorean identity. Sometimes, we'll subtract $\cot^2\theta$ from both sides and see it written as

$$\csc^2\theta - \cot^2\theta = 1$$

The Pythagorean identity with secant and tangent

To find the third Pythagorean identity, we can start with the first identity, $\sin^2\theta + \cos^2\theta = 1$, and divide through both sides of the equation by $\cos^2\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

The quotient identity for tangent lets us simplify the fraction on the left,

$$(\tan\theta)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\tan^2\theta + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

and the reciprocal identity for secant lets us simplify the fraction on the right.

$$\tan^2\theta + 1 = (\sec\theta)^2$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

This is our third Pythagorean identity. Sometimes, we'll subtract $\tan^2\theta$ from both sides and see it written as

$$\sec^2\theta - \tan^2\theta = 1$$

Of course, just like any of the other trig identities we've already learned (reciprocal and quotient identities), and just like the trig identities we'll learn later, these Pythagorean identities relate the trig functions to one another. With the Pythagorean identities in particular, we can

- find sine given cosine, or vice versa,
- find cosecant given cotangent, or vice versa, and
- find secant given tangent, or vice versa.
- find cosecant or secant given sine and cosine

Let's do an example.

Example

Find $\cos\theta$.

$$\sin\theta = \frac{\sqrt{2}}{2}$$

The Pythagorean identity that relates sine and cosine will let us calculate cosine of the angle. We'll substitute the value of sine of the angle into the identity.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \cos^2\theta = 1$$

Simplify the left side of the equation, and then get $\cos^2\theta$ by itself.

$$\frac{2}{4} + \cos^2\theta = 1$$

$$\frac{1}{2} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{1}{2}$$

$$\cos^2\theta = \frac{1}{2}$$

Take the square root of both sides in order to solve for $\cos\theta$. When we add in a square root like this, we have to add in a $\pm$ on the right to indicate that we could get both a positive and negative solution.

$$\sqrt{\cos^2\theta} = \pm\sqrt{\frac{1}{2}}$$

$$\cos\theta = \pm\frac{\sqrt{1}}{\sqrt{2}}$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}}$$

Rationalize the denominator.

$$\cos\theta = \pm\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$\cos\theta = \pm\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}$$

$$\cos\theta = \pm\frac{\sqrt{2}}{2}$$

So, what is the answer we just found in the example problem actually telling us?

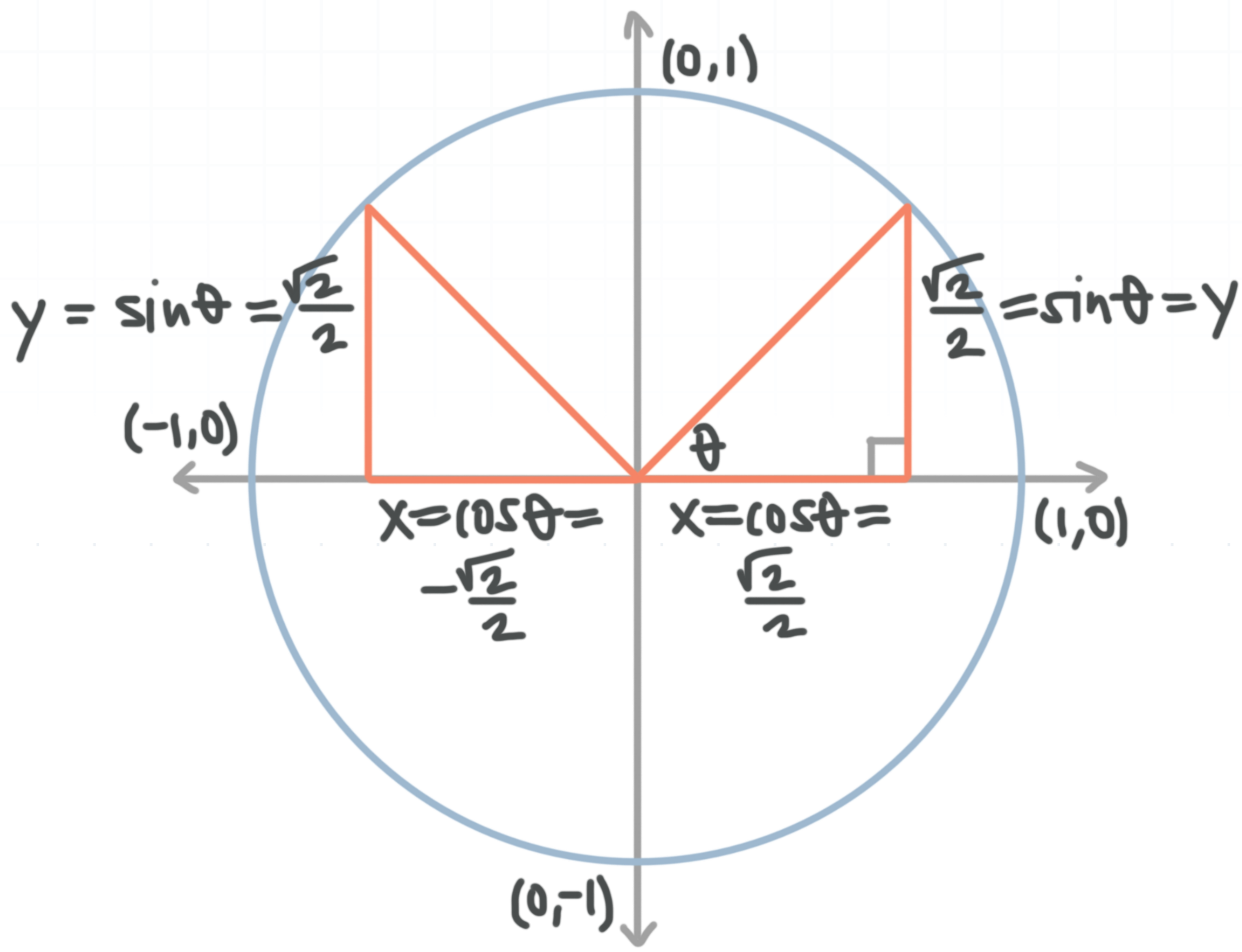
It's saying that for a circle with radius 1, sine of the angle (or the length of the vertical leg of the triangle), will be

$$y = \sin\theta = \frac{\sqrt{2}}{2}$$

when cosine of the angle (or the length of the horizontal leg of the triangle), is

$$x = \cos\theta = \pm\frac{\sqrt{2}}{2}$$

We can actually see these values in action if we sketch out two triangles in the coordinate plane.



The triangle in the first quadrant shows us the length of the horizontal leg as $x = \cos\theta = \sqrt{2}/2$, and the length of the vertical leg as $y = \sin\theta = \sqrt{2}/2$. And the triangle in the second quadrant shows us the length of the horizontal leg as $x = \cos\theta = -\sqrt{2}/2$, and the length of the vertical leg as $y = \sin\theta = \sqrt{2}/2$.