Product-to-sum identities

Sometimes we'll have the product of trig functions and we'll want to break up the product into a sum or difference. For instance, given the product $\sin\theta\cos\alpha$, we'll want to break up the product and be able to write this as

$$\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

which is the sum of two different sine functions. This is what the product-to-sum identities do; they change a product into a sum (or a difference).

Product-to-sum identities from the sum-difference identities

These product-to-sum identities can be built directly from the sum-difference identities we learned about earlier. As a reminder, here are the sum-difference identities:

$$\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta-\alpha)=\sin\theta\cos\alpha-\cos\theta\sin\alpha$$

$$\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha$$

$$\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha$$

If we add the two sine identities together, we'll add the left sides of the equations, $\sin(\theta+\alpha)$ and $\sin(\theta-\alpha)$, and the right sides of the equations, $\sin\theta\cos\alpha+\cos\theta\sin\alpha$ and $\sin\theta\cos\alpha-\cos\theta\sin\alpha$.

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$$\left[\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha\right]+\left[\sin(\theta-\alpha)=\sin\theta\cos\alpha-\cos\theta\sin\alpha\right]$$

$$\sin(\theta+\alpha)+\sin(\theta-\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha+\sin\theta\cos\alpha-\cos\theta\sin\alpha$$

$$\sin(\theta+\alpha)+\sin(\theta-\alpha)=\sin\theta\cos\alpha+\sin\theta\cos\alpha$$

$$\sin(\theta+\alpha)+\sin(\theta-\alpha)=2\sin\theta\cos\alpha$$

$$\sin\theta\cos\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

This is the first product-to-sum identity. We can find another one for sine by subtracting the identity for $\sin(\theta-\alpha)$ from the identity for $\sin(\theta+\alpha)$.

$$\left[\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha\right]-\left[\sin(\theta-\alpha)=\sin\theta\cos\alpha-\cos\theta\sin\alpha\right]$$

$$\sin(\theta+\alpha)-\sin(\theta-\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha-(\sin\theta\cos\alpha-\cos\theta\sin\alpha)$$

$$\sin(\theta+\alpha)-\sin(\theta-\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha-\sin\theta\cos\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta+\alpha)-\sin(\theta-\alpha)=\cos\theta\sin\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta+\alpha)-\sin(\theta-\alpha)=2\cos\theta\sin\alpha$$

$$\cos\theta\sin\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)-\sin(\theta-\alpha)\right]$$

In the same way we just found these first two product-to-sum identities, we can find two more product-to-sum identities by adding and subtracting the sum-difference identities for cosine. When we add the identities, we get

$$\left[\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha\right]+\left[\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha\right]$$

$$\cos(\theta+\alpha)+\cos(\theta-\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha+\cos\theta\cos\alpha+\sin\theta\sin\alpha$$

$$\cos(\theta+\alpha)+\cos(\theta-\alpha)=\cos\theta\cos\alpha+\cos\theta\cos\alpha$$

$$\cos(\theta+\alpha)+\cos(\theta-\alpha)=2\cos\theta\cos\alpha$$

$$\cos\theta\cos\alpha=\frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

and when we subtract the identities, we get

$$\left[\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha\right]-\left[\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha\right]$$

$$\cos(\theta+\alpha)-\cos(\theta-\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha-(\cos\theta\cos\alpha+\sin\theta\sin\alpha)$$

$$\cos(\theta+\alpha)-\cos(\theta-\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha-\cos\theta\cos\alpha-\sin\theta\sin\alpha$$

$$\cos(\theta+\alpha)-\cos(\theta-\alpha)=-\sin\theta\sin\alpha-\sin\theta\sin\alpha$$

$$\cos(\theta+\alpha)-\cos(\theta-\alpha)=-2\sin\theta\sin\alpha$$

$$\sin\theta\sin\alpha=-\frac{1}{2}\left[\cos(\theta+\alpha)-\cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha=\frac{1}{2}\left[-\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha=\frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

If we summarize what we've built, we get the four **product-to-sum identities**.

$$\sin\theta\cos\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

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$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha) + \cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha) - \cos(\theta+\alpha)\right]$$

Notice also that identities do something for us other than break up products into sums and differences.

The first identity takes the product of a sine and cosine function and changes it into the sum of two sine functions, thereby eliminating cosine from the expression completely.

The second identity takes the product of a sine and cosine function and changes it into the difference of two sine functions, again eliminating cosine completely.

And the fourth identity takes the product of two sines functions and changes it into the difference of two cosine functions, completely eliminating sine from the expression.

So these product-to-sum identities can also help us eliminate sine or cosine from the expression, or put the expression completely in terms of sine only or cosine only. And sometimes that'll be valuable to us.

Let's do an example where we use a product-to-sum identity to break up the product of two sine functions.

Example

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Express $\sin(7\theta)\sin(11\theta)$ as the sum or difference of trig functions.

Because we have the product of sine functions, we'll use the product-to-sum identity for the product of sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

$$\sin(7\theta)\sin(11\theta) = \frac{1}{2}\left[\cos(7\theta-11\theta)-\cos(7\theta+11\theta)\right]$$

$$\sin(7\theta)\sin(11\theta) = \frac{1}{2}\left[\cos(-4\theta)-\cos(18\theta)\right]$$

We could leave the expression this way. But we can also further simplify the first cosine expression using the even identity for cosine, which tells us that

$$\cos(-4\theta) = \cos(4\theta)$$

Therefore, we'll simplify the equation to

$$\sin(7\theta)\sin(11\theta) = \frac{1}{2}\left[\cos(4\theta)-\cos(18\theta)\right]$$

Let's look at another example where we use the product-to-sum identities to calculate the values of a set of expressions.

Example

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Find the exact values of $(\sin \pi/8)(\cos \pi/8)$, $\sin^2(\pi/8)$, and $\cos^2(\pi/8)$.

To compute $(\sin \pi/8)(\cos \pi/8)$, we can use the product-to-sum identity for the product of a sine and cosine function.

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left[\sin\left(\frac{\pi}{8}+\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{8}-\frac{\pi}{8}\right)\right]$$

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left(\sin\frac{\pi}{4}+\sin 0\right)$$

Pulling the values of sine on the right side from the unit circle, we get

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2}+0\right)$$

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = \frac{\sqrt{2}}{4}$$

To find $\sin^2(\pi/8)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

$$\left(\sin\frac{\pi}{8}\right)\left(\sin\frac{\pi}{8}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{8}-\frac{\pi}{8}\right)-\cos\left(\frac{\pi}{8}+\frac{\pi}{8}\right)\right]$$

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$$\sin^2\frac{\pi}{8}=\frac{1}{2}\left(\cos 0-\cos\frac{\pi}{4}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\frac{\pi}{8}=\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)$$

$$\sin^2\frac{\pi}{8}=\frac{1}{2}-\frac{\sqrt{2}}{4}$$

$$\sin^2\frac{\pi}{8}=\frac{2}{4}-\frac{\sqrt{2}}{4}$$

$$\sin^2\frac{\pi}{8}=\frac{2-\sqrt{2}}{4}$$

To get the value of $\cos^2(\pi/8)$, we'll use the product-to-sum identity for the product of two cosine functions.

$$\cos\theta\cos\alpha=\frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

$$\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right)=\frac{1}{2}\left[\cos\left(\frac{\pi}{8}+\frac{\pi}{8}\right)+\cos\left(\frac{\pi}{8}-\frac{\pi}{8}\right)\right]$$

$$\cos^2\frac{\pi}{8}=\frac{1}{2}\left(\cos\frac{\pi}{4}+\cos 0\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

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$$\cos^2\frac{\pi}{8}=\frac{1}{2}\left(\frac{\sqrt{2}}{2}+1\right)$$

$$\cos^2\frac{\pi}{8}=\frac{\sqrt{2}}{4}+\frac{1}{2}$$

$$\cos^2\frac{\pi}{8}=\frac{\sqrt{2}}{4}+\frac{2}{4}$$

$$\cos^2\frac{\pi}{8}=\frac{2+\sqrt{2}}{4}$$

Let's do one more example where we use the product-to-sum identity to break up the product of a cosine and sine function.

Example

Find the exact value of $\cos(17\pi/12)\sin(\pi/12)$.

We'll use the product-to-sum identity for the product of sine and cosine.

$$\cos\theta\sin\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)-\sin(\theta-\alpha)\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[\sin\left(\frac{17\pi}{12}+\frac{\pi}{12}\right)-\sin\left(\frac{17\pi}{12}-\frac{\pi}{12}\right)\right]$$

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$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[\sin\left(\frac{18\pi}{12}\right)-\sin\left(\frac{16\pi}{12}\right)\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[\sin\left(\frac{3\pi}{2}\right)-\sin\left(\frac{4\pi}{3}\right)\right]$$

Pulling the values of sine on the right side from the unit circle, we get

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left[-1-\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{1}{2}\left(-1+\frac{\sqrt{3}}{2}\right)$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=-\frac{1}{2}+\frac{\sqrt{3}}{4}$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=-\frac{2}{4}+\frac{\sqrt{3}}{4}$$

$$\left(\cos\frac{17\pi}{12}\right)\left(\sin\frac{\pi}{12}\right)=\frac{-2+\sqrt{3}}{4}$$

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