Trig functions of inverse trig functions

Previously in an Algebra course, you may have studied different kinds of inverse relationships.

For example, exponents and radicals are inverses. So taking the square root of something is inverse to raising it to the power of 2, and vice versa. Those operations undo each other.

$$\sqrt{x^2} = (\sqrt{x})^2 = x$$

Similarly, exponential and log functions undo each other, because they're inverses. So taking the natural log of something is inverse to raising something to the base e, or vice versa.

$$e^{\ln x} = \ln(e^x) = x$$

Inverse operations with trig functions

Trig functions and inverse trig functions are the same way. So taking the sine of something is inverse to taking the inverse sine of something, which means those operations undo each other.

$\sin^{-1}(\sin x) = x$ for $x = [-\pi/2,\pi/2]$

$\sin(\sin^{-1} x) = x$ for $x = [-1,1]$

The same is true for all six of the trig functions.

$\cos^{-1}(\cos x) = x$ for $x = [0,\pi]$

$\cos(\cos^{-1} x) = x$	for $x = [-1,1]$
$\tan^{-1}(\tan x) = x$	for $x = (-\pi/2,\pi/2)$
$\tan(\tan^{-1} x) = x$	for $x = (-\infty, \infty)$
$\csc^{-1}(\csc x) = x$	for $x = [-\pi/2,0)$ or $x = (0,\pi/2]$
$\csc(\csc^{-1} x) = x$	for $x = (-\infty, -1]$ or $x = [1,\infty)$
$\sec^{-1}(\sec x) = x$	for $x = [0,\pi/2)$ or $x = (\pi/2,\pi]$
$\sec(\sec^{-1} x) = x$	for $x = (-\infty, -1]$ or $x = [1,\infty)$
$\cot^{-1}(\cot x) = x$	for $x = (0,\pi)$
$\cot(\cot^{-1} x) = x$	for $x = (-\infty, \infty)$

But we can also pair together trig functions and inverse trig functions that don't "match." For instance, we can calculate $\cos(\sin^{-1} x)$, $\cot(\cos^{-1} x)$, or $\csc(\tan^{-1} x)$, etc.

Below is a table of formulas showing how we calculate every trig function of every inverse trig function. Keep in mind that it only gives values for the trig function of an inverse trig function, not the other way around. In other words, the second row third column shows $\cos(\tan^{-1} x)$. There is no cell in the table that gives $\tan^{-1}(\cos x)$.

	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\csc^{-1} x$	$\sec^{-1} x$	$\cot^{-1} x$
sin of	x	$\sqrt{1-x^2}$	$\frac{x}{\sqrt{x^2+1}}$	$\frac{1}{x}$	$\sqrt{1-\frac{1}{x^2}}$	$-\frac{1}{x\sqrt{\frac{1}{x^2}+1}}$
cos of	$\sqrt{1-x^2}$	x	$\frac{1}{\sqrt{x^2+1}}$	$\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x}$	$\frac{1}{\sqrt{\frac{1}{x^2}+1}}$
tan of	$\frac{x}{\sqrt{1-x^2}}$	$\frac{\sqrt{1-x^2}}{x}$	x	$\frac{1}{x\sqrt{1-\frac{1}{x^2}}}$	$x\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x}$
csc of	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{\sqrt{x^2+1}}{x}$	x	$\frac{1}{\sqrt{1-\frac{1}{x^2}}}$	$x\sqrt{\frac{1}{x^2}+1}$
sec of	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\sqrt{x^2+1}$	$\frac{1}{\sqrt{1-\frac{1}{x^2}}}$	x	$\sqrt{\frac{1}{x^2}+1}$
cot of	$\frac{\sqrt{1-x^2}}{x}$	$\frac{x}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$x\sqrt{1-\frac{1}{x^2}}$	$\frac{1}{x\sqrt{1-\frac{1}{x^2}}}$	x

It would be really difficult to remember all these formulas, but they're fairly easy to build from what we know about right triangles.

For instance, consider $\sin(\cos^{-1} x)$, which we see in the first row and second column of the table. We can rewrite about $\cos^{-1} x$ as

$$\cos^{-1}\left(\frac{x}{1}\right)$$

Because this is the inverse cosine function, the input $x/1$ represents adjacent/hypotenuse, and the result of $\cos^{-1}(x/1)$ will be the angle within the triangle.

So if the adjacent side of the triangle is given by x, and the hypotenuse of the triangle is given by 1, then the opposite side of the triangle, by the Pythagorean Theorem, must be

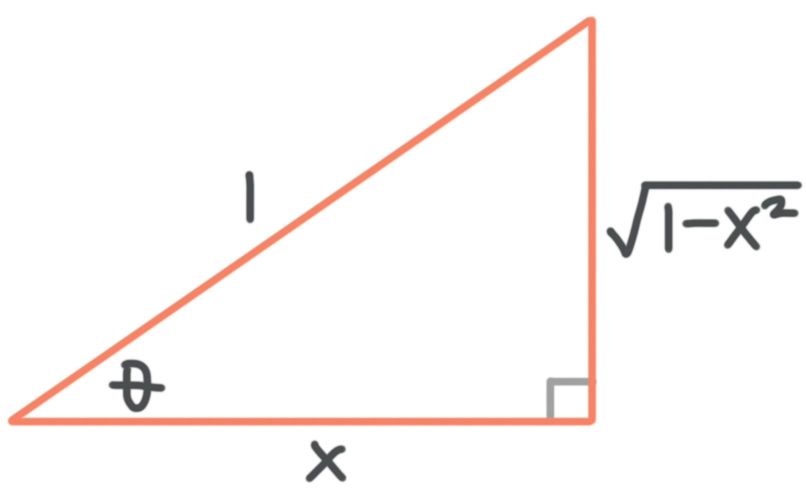
$$a^2 + b^2 = c^2$$

$$x^2 + b^2 = 1^2$$

$$b^2 = 1 - x^2$$

$$b = \sqrt{1 - x^2}$$

Now we know that the triangle we're describing has adjacent side x, opposite side $\sqrt{1 - x^2}$, and hypotenuse 1,



we can find the sine of the interior angle of that triangle. Because sine is equivalent to opposite/hypotenuse, we get

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Therefore, $\sin(\cos^{-1}x) = \sqrt{1-x^2}$.

Let's do another example where we build another of these formulas.

Example

Find the value of $\sec(\cot^{-1}x)$.

Set $\theta = \cot^{-1}x$. Then we can say

$$\theta = \cot^{-1}\left(\frac{x}{1}\right)$$

$$\theta = \cot^{-1}\left(\frac{x = \text{adjacent}}{1 = \text{opposite}}\right)$$

Given a triangle with adjacent leg x and opposite leg 1, the hypotenuse must be

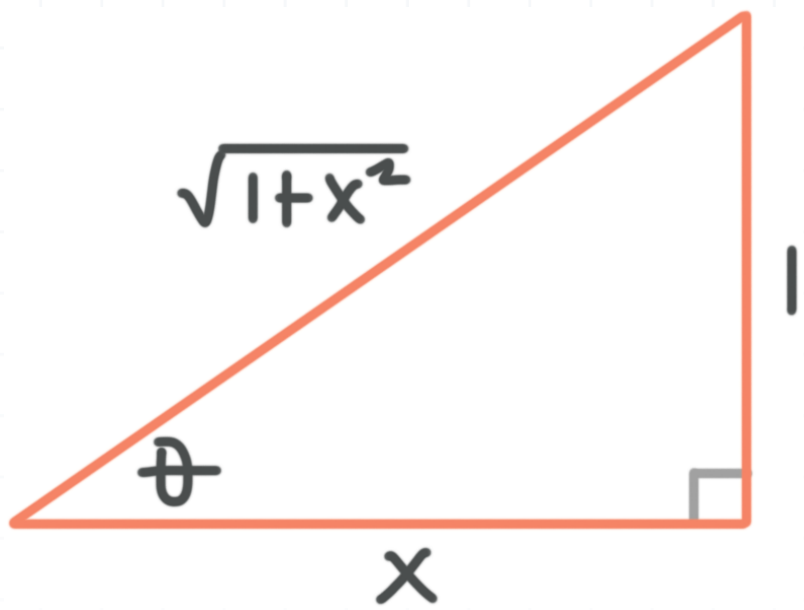
$$a^2 + b^2 = c^2$$

$$x^2 + 1^2 = c^2$$

$$c^2 = 1 + x^2$$

$$c = \sqrt{1+x^2}$$

Now we know that the triangle we're describing has adjacent leg x, opposite leg 1, and hypotenuse $\sqrt{1+x^2}$,



we can find the secant of the interior angle of that triangle. Because secant is equivalent to hypotenuse/adjacent, we get

$$\frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{1+x^2}}{x}$$

This expression represents $\sec(\cot^{-1} x)$, and now we just need to simplify it by putting the entire fraction under one square root. To do that, we can rewrite x as $\sqrt{x^2}$.

$$\sec(\cot^{-1} x) = \frac{\sqrt{1+x^2}}{\sqrt{x^2}}$$

When both the numerator and denominator of a fraction are under a root, we can simplify by taking the root of the whole fraction.

$$\sec(\cot^{-1} x) = \sqrt{\frac{1+x^2}{x^2}}$$

$$\sec(\cot^{-1} x) = \sqrt{\frac{1}{x^2} + \frac{x^2}{x^2}}$$

$$\sec(\cot^{-1} x) = \sqrt{\frac{1}{x^2} + 1}$$

Now let's work through an example where we evaluate one of these values at a particular point.

Example

Find the value of the expression.

$$\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$$

Let θ represent the angle in $[0,\pi]$ (because this is the range of the inverse cosine function) whose cosine is $-3/5$. Then we can say

$$\theta = \cos^{-1}\left(-\frac{3}{5}\right)$$

$$\cos\theta = -\frac{3}{5}$$

If θ is in $[0,\pi]$, it must be in the first or second quadrant. The value of $\cos\theta$ is positive in the first quadrant, and negative in the second quadrant, so because $\cos\theta$ is negative, θ can only be in the second quadrant.

If we imagine our right triangle in the second quadrant, because cosine is always equal to adjacent/hypotenuse, the adjacent side is -3 and the hypotenuse is 5. We can use the Pythagorean Theorem to find the length of the opposite side.

$$a^2 + b^2 = c^2$$

$$(-3)^2 + b^2 = 5^2$$

$$b^2 = 25 - 9$$

$$b^2 = 16$$

$$b = 4$$

Because tangent is equivalent to opposite/adjacent, tangent must be

$$\frac{\text{opposite}}{\text{adjacent}} = \frac{4}{-3} = -\frac{4}{3}$$

so

$$\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right) = \tan\theta = -\frac{4}{3}$$

This value also matches what we would have found using the formula from the table for $\tan(\cos^{-1} x)$,

$$\frac{\sqrt{1-x^2}}{x}$$

Let's do one more example.

Example

Find the value of the expression.

$$\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$$

The inverse property $\sin^{-1}(\sin x) = x$ applies for every x in $[-\pi/2,\pi/2]$. The value $x = 4\pi/3$ does not lie in $[-\pi/2,\pi/2]$. To evaluate this expression, we first need to find $\sin(4\pi/3)$.

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

The angle in $[-\pi/2,\pi/2]$ whose sine is $-\sqrt{3}/2$ is $-\pi/3$.

Therefore,

$$\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$