Half-angle identities

In this lesson we're looking at the set of half-angle identities. They allow us to rewrite a trig function when the argument is $\theta/2$. They transform the argument from $\theta/2$ to just θ.

The good news is that we can build these identities directly from the double-angle identities we just learned.

Half-angle identities from the double-angle identities

To find the half-angle identity for sine, we start with the double-angle identity for cosine, $\cos 2\theta = 1 - 2\sin^2\theta$, and solve it for $\sin\theta$.

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$2\sin^2\theta + \cos 2\theta = 1$$

$$2\sin^2\theta = 1 - \cos 2\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1 - \cos 2\theta}{2}}$$

Now we'll make the substitution $2\theta = \alpha$. If we solve $2\theta = \alpha$ for θ, we also get $\theta = \alpha/2$. So we'll replace θ with $\alpha/2$, and replace 2θ with α, and we get

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$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$$

We build the half-angle identity for cosine in the same way, but we start with the $\cos 2\theta=2\cos^2\theta-1$ double-angle identity instead, solving it for $\cos\theta$.

$$\cos 2\theta=2\cos^2\theta-1$$

$$2\cos^2\theta=1+\cos 2\theta$$

$$\cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$\cos\theta=\pm\sqrt{\frac{1+\cos 2\theta}{2}}$$

Now we'll make the substitution $2\theta=\alpha$. If we solve $2\theta=\alpha$ for θ, we also get $\theta=\alpha/2$. So we'll replace θ with $\alpha/2$, and replace 2θ with α, and we get

$$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Then to get the half-angle identity for tangent, we'll use the half-angles we just got for sine and cosine.

$$\tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$\tan\frac{\alpha}{2}=\frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}}$$

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$$\tan\frac{\alpha}{2}=\frac{\pm\frac{\sqrt{1-\cos\alpha}}{\sqrt{2}}}{\pm\frac{\sqrt{1+\cos\alpha}}{\sqrt{2}}}$$

$$\tan\frac{\alpha}{2}=\pm\frac{\sqrt{1-\cos\alpha}}{\sqrt{2}}\cdot\pm\frac{\sqrt{2}}{\sqrt{1+\cos\alpha}}$$

$$\tan\frac{\alpha}{2}=\pm\frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}$$

There are also two other alternate forms of this half-angle tangent identity. So if we pull together all three of the half-angle identities we've built so far, along with the two alternate forms for the tangent identity, we can summarize the **half-angle identities** as

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\tan\frac{\theta}{2}=\pm\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

Notice here that three of these half-angle identities include a ± sign. The sign we choose will depend on the quadrant of the angle. For instance, using the half-angle cosine identity, cosine is positive in the first and fourth quadrants and negative in the second and third quadrants. So if the angle

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is in the first or fourth quadrant, we'll choose the positive value of the root on the right side of that cosine identity.

$$\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$$

But if the angle is in the second or third quadrant, we'll choose the negative value of the root on the right side of that cosine identity.

$$\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}$$

Let's do an example where we find the half-angle values of sine and cosine for an angle in the third quadrant.

Example

If $\pi<\theta<3\pi/2$ and $\cos\theta=-\sqrt{6}/7$, find $\cos(\theta/2)$ and $\sin(\theta/2)$.

If we substitute $\cos\theta=-\sqrt{6}/7$ into the half-angle identity for cosine, we get

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\left(-\frac{\sqrt{6}}{7}\right)}{2}}$$

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$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\frac{7-\sqrt{6}}{7}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{7-\sqrt{6}}{14}}$$

If we also substitute $\cos\theta = -\sqrt{6}/7$ into the half-angle identity for sine, we get

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\left(-\frac{\sqrt{6}}{7}\right)}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{\frac{7+\sqrt{6}}{7}}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{7+\sqrt{6}}{14}}$$

To figure out the quadrant of $\theta/2$, we'll start with the fact that we were told θ is in the third quadrant, $\pi < \theta < 3\pi/2$. We'll divide through the inequality by 2 to change θ into $\theta/2$.

$$\pi < \theta < \frac{3\pi}{2}$$

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$$\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$$

This inequality tells us that the angle $\theta/2$ falls between $\pi/2$ (which is along the positive direction of the y-axis) and $3\pi/4$ (which is halfway through the second quadrant). Therefore, the bounds $\theta/2 = [\pi/2, 3\pi/4]$ define space only in the second quadrant, so $\theta/2$ must be in the second quadrant.

For any angle in the second quadrant, cosine is negative and sine is positive.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{7-\sqrt{6}}{14}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{7+\sqrt{6}}{14}}$$

Let's do an example with an angle that's outside the interval $[0, 2\pi)$.

Example

If $-11\pi/2 < \theta < -5\pi$ and $\sin\theta = 1/3$, find $\cos(\theta/2)$ and $\sin(\theta/2)$.

We'll start by using $\sin\theta = 1/3$ and the Pythagorean identity with sine and cosine to find the corresponding value of $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(\frac{1}{3}\right)^2$$

$$\cos^2\theta = 1 - \frac{1}{9}$$

$$\cos^2\theta = \frac{8}{9}$$

$$\cos\theta = \pm\sqrt{\frac{8}{9}}$$

We were told that $-11\pi/2 < \theta < -5\pi$. To figure out the quadrant in which θ lies, we'll find coterminal angles for both $-11\pi/2$ and -5π by adding 6π to both angles.

$$-\frac{11\pi}{2} + 6\pi < \theta < -5\pi + 6\pi$$

$$-\frac{11\pi}{2} + \frac{12\pi}{2} < \theta < \pi$$

$$\frac{\pi}{2} < \theta < \pi$$

So θ is in the second quadrant. The cosine of every angle in the second quadrant is negative, so

$$\cos\theta = -\sqrt{\frac{8}{9}} = -\frac{\sqrt{8}}{\sqrt{9}} = -\frac{2\sqrt{2}}{3}$$

Then, by the half-angle identity for cosine, we get

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$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\left(-\frac{2\sqrt{2}}{3}\right)}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{\frac{3-2\sqrt{2}}{3}}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{3-2\sqrt{2}}{6}}$$

By the half-angle identity for sine, we get

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\left(-\frac{2\sqrt{2}}{3}\right)}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{\frac{3+2\sqrt{2}}{3}}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{3+2\sqrt{2}}{6}}$$

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To find the quadrant of the angle $\theta/2$, we'll divide through the inequality we were given by 2.

$$-\frac{11\pi}{2} < \theta < -5\pi$$

$$-\frac{11\pi}{4} < \frac{\theta}{2} < -\frac{5\pi}{2}$$

To see where the angles $-11\pi/4$ and $-5\pi/4$ lie, we'll add 4π to both angles to find coterminal angles.

$$-\frac{11\pi}{4} + 4\pi < \frac{\theta}{2} < -\frac{5\pi}{2} + 4\pi$$

$$-\frac{11\pi}{4} + \frac{16\pi}{4} < \frac{\theta}{2} < -\frac{5\pi}{2} + \frac{8\pi}{2}$$

$$\frac{5\pi}{4} < \frac{\theta}{2} < \frac{3\pi}{2}$$

The angle $5\pi/4$ is halfway through the third quadrant, and the angle $3\pi/2$ is along the negative side of the y-axis, so $\theta/2$ has to be in the third quadrant. Both the cosine function and the sine function are negative for all angles in the third quadrant, so

$$\cos\frac{\theta}{2} = -\sqrt{\frac{3-2\sqrt{2}}{6}}$$

$$\sin\frac{\theta}{2} = -\sqrt{\frac{3+2\sqrt{2}}{6}}$$

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Even if we're given the value of $\tan\theta$ for some angle θ and the interval in which it lies, we can find $\cos(\theta/2)$ and $\sin(\theta/2)$.

Example

Find $\sin(\theta/2)$ and $\cos(\theta/2)$ for the angle θ such that $\tan\theta = 21$ and θ lies in the interval $(10\pi, 10\pi + (\pi/2))$.

We'll start with the Pythagorean identity with secant and tangent.

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\sec^2\theta = 1 + (21^2)$$

$$\sec^2\theta = 1 + 441$$

$$\sec^2\theta = 442$$

Using the reciprocal identity for cosine, we get

$$\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{442}$$

We've been told that

$$10\pi < \theta < 10\pi + \frac{\pi}{2}$$

Since 10π is an integer multiple of 2π, that angle lies along the positive side of the x-axis. Which means that $10\pi + (\pi/2)$ must be along the positive side

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of the y-axis. These two angles therefore bound the entire first quadrant, which means θ is in the first quadrant, so $\cos\theta$ is positive.

$$\cos\theta = \sqrt{\frac{1}{442}} = \frac{\sqrt{1}}{\sqrt{442}} = \frac{1}{\sqrt{442}} = \frac{\sqrt{442}}{\sqrt{442}\sqrt{442}} = \frac{\sqrt{442}}{442}$$

Then by the half-angle identity for cosine,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\frac{\sqrt{442}}{442}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\frac{442+\sqrt{442}}{442}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{442+\sqrt{442}}{884}}$$

And by the half-angle identity for sine,

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\frac{\sqrt{442}}{442}}{2}}$$

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$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{\frac{442-\sqrt{442}}{442}}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{442-\sqrt{442}}{884}}$$

If we divide through the inequality we were given by 2, we get

$$10\pi<\theta<10\pi+\frac{\pi}{2}$$

$$5\pi<\frac{\theta}{2}<5\pi+\frac{\pi}{4}$$

The angle 5π is an integer multiple of π, so it lies on the negative x-axis, which means the angle $5\pi+(\pi/4)$ lies halfway through the third quadrant, and therefore that $\theta/2$ must lie in the third quadrant. So both $\cos\theta$ and $\sin\theta$ are negative.

$$\cos\frac{\theta}{2}=-\sqrt{\frac{442+\sqrt{442}}{884}}$$

$$\sin\frac{\theta}{2}=-\sqrt{\frac{442-\sqrt{442}}{884}}$$