Graph the polar region in the xy-plane

In the lessons on how to graph polar curves, we occasionally referred to the polar coordinate r in the equation of a polar curve as being a function of θ. In this lesson, we're going to graph these polar functions in the xy-plane, but with the values of the polar coordinate θ (instead of the x-coordinates of points) plotted along the horizontal axis and the corresponding values of the polar coordinate r (instead of the y-coordinates of points) plotted along the vertical axis. We'll refer to these graphs as rectangular graphs (to distinguish them from polar graphs, where we graph points in terms of distances from the pole and lines that pass through the pole).

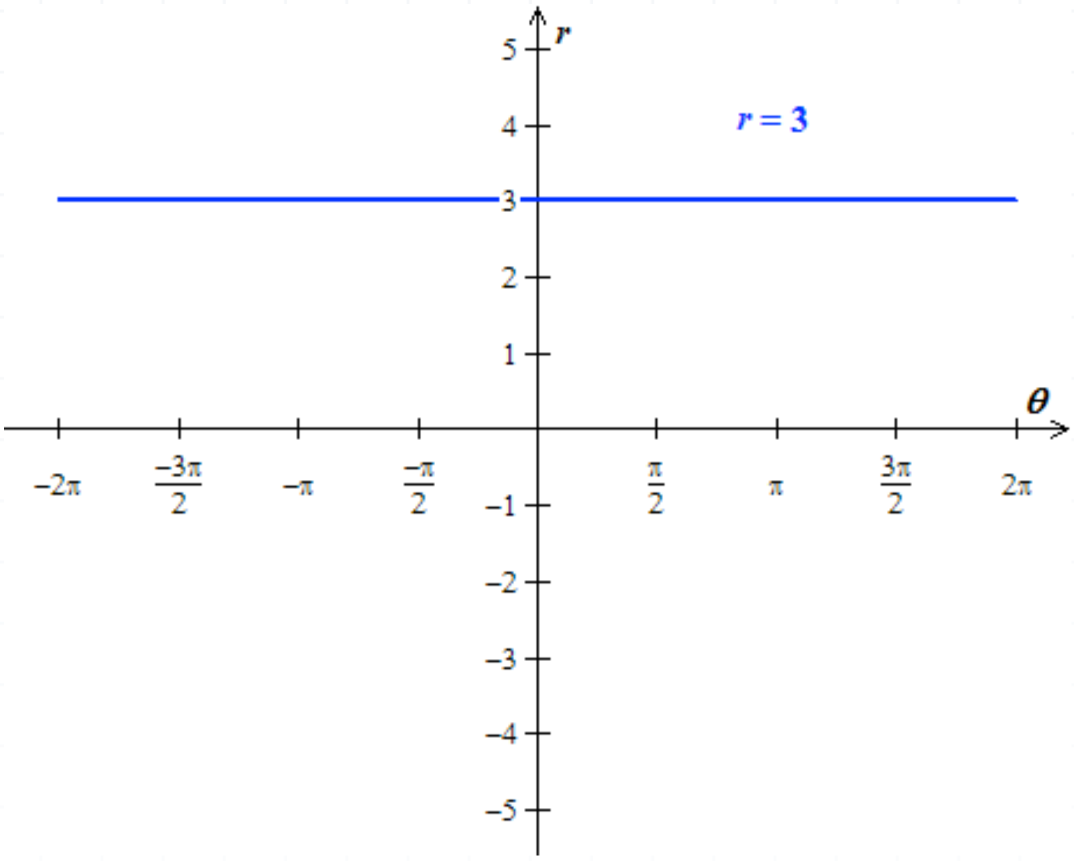
First, we'll look at a polar equation of the form $r = a$ where a is a nonzero constant.

Example

Sketch the rectangular graph of the polar equation $r = 3$.

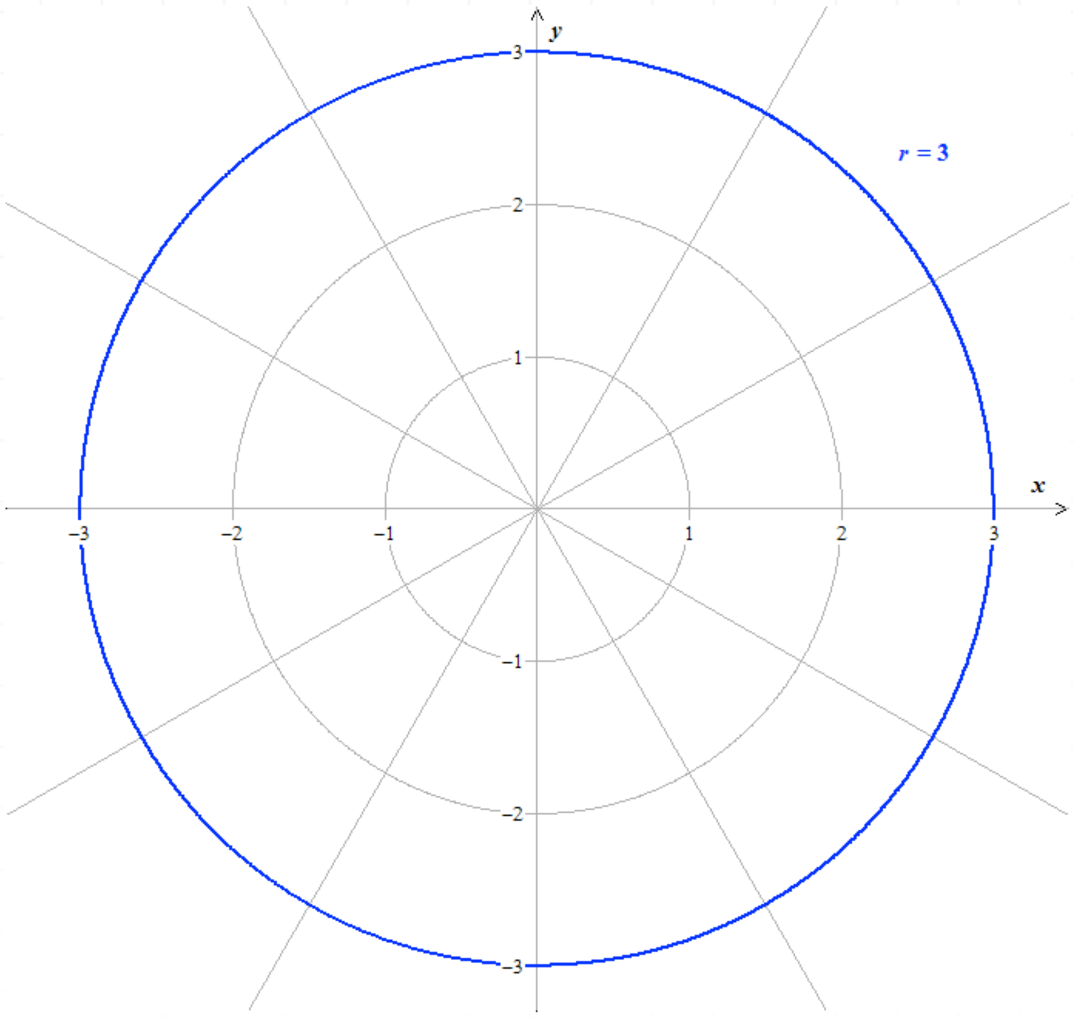
We graph this equation in exactly the same way that we graph the function $f(x) = 3$ (except that the variables on the axes are θ and r instead of x and y, respectively).

In this case, the value of r is constant (and equal to 3) for all angles θ, so the rectangular graph of the polar equation $r = 3$ is the horizontal line that's located 3 units above the horizontal axis.



We have graphed this equation on the interval $[-2\pi,2\pi]$. However, $r = 3$ for all angles θ, so the graph actually extends all the way to $\theta = -\infty$ on the left, and all the way to $\theta = +\infty$ on the right.

If you take this graph and convert it to a polar graph, you'll get a circle, since every point of the polar graph is at a distance of $r = 3$ units from the pole (regardless of the measure of angle θ).



In any interval of angle measure which is of length 2π, the graph of the polar equation is the entire circle. That is, if you continued the polar graph beyond such an interval, you would simply be retracing part/all of the circle that you've already drawn.

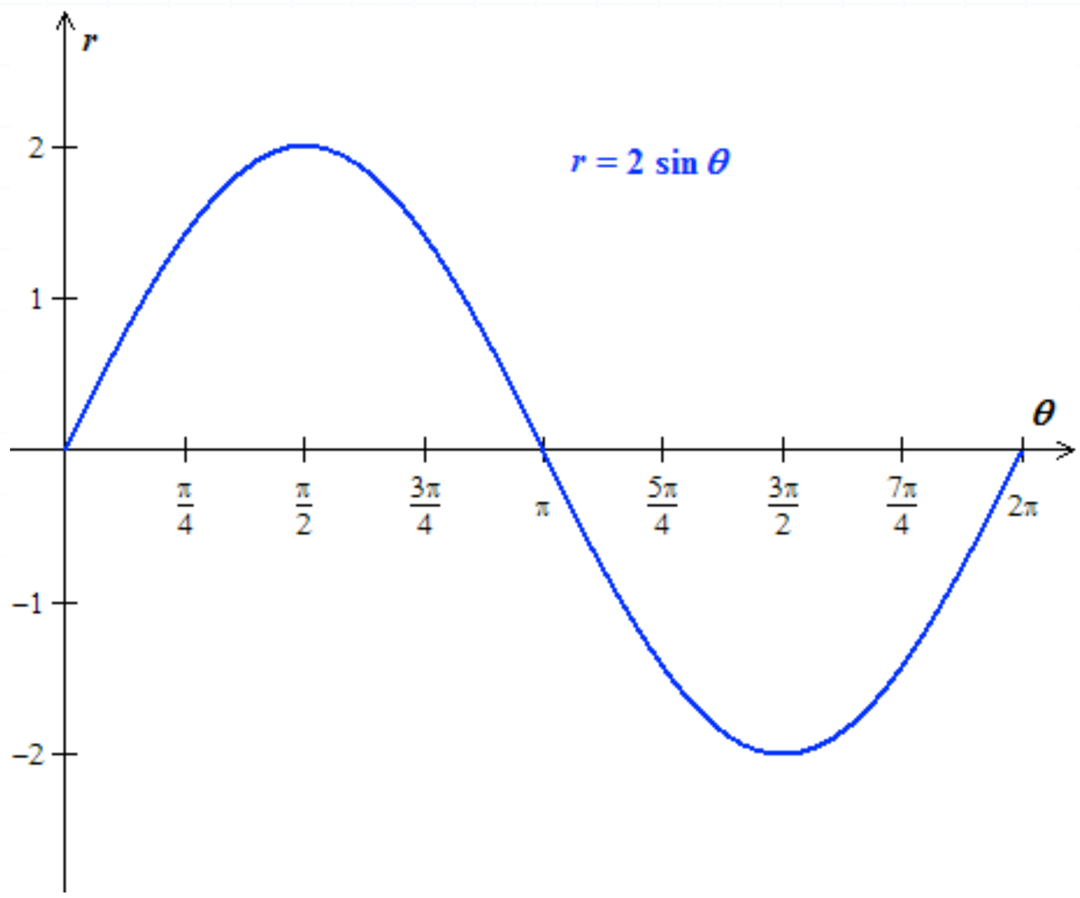
As you may recall, if we have a polar equation of the form $r = a$ where a is a negative constant, then the polar graph is a circle of radius $|a|$ which is centered at the pole. Thus the rectangular graph of this equation is the horizontal line that's located $|a|$ units below the horizontal axis, and the polar graph is the same as that of the equation $r = -a$.

Next, let's consider a polar equation of the form $r = a\sin\theta$ where a is a nonzero constant.

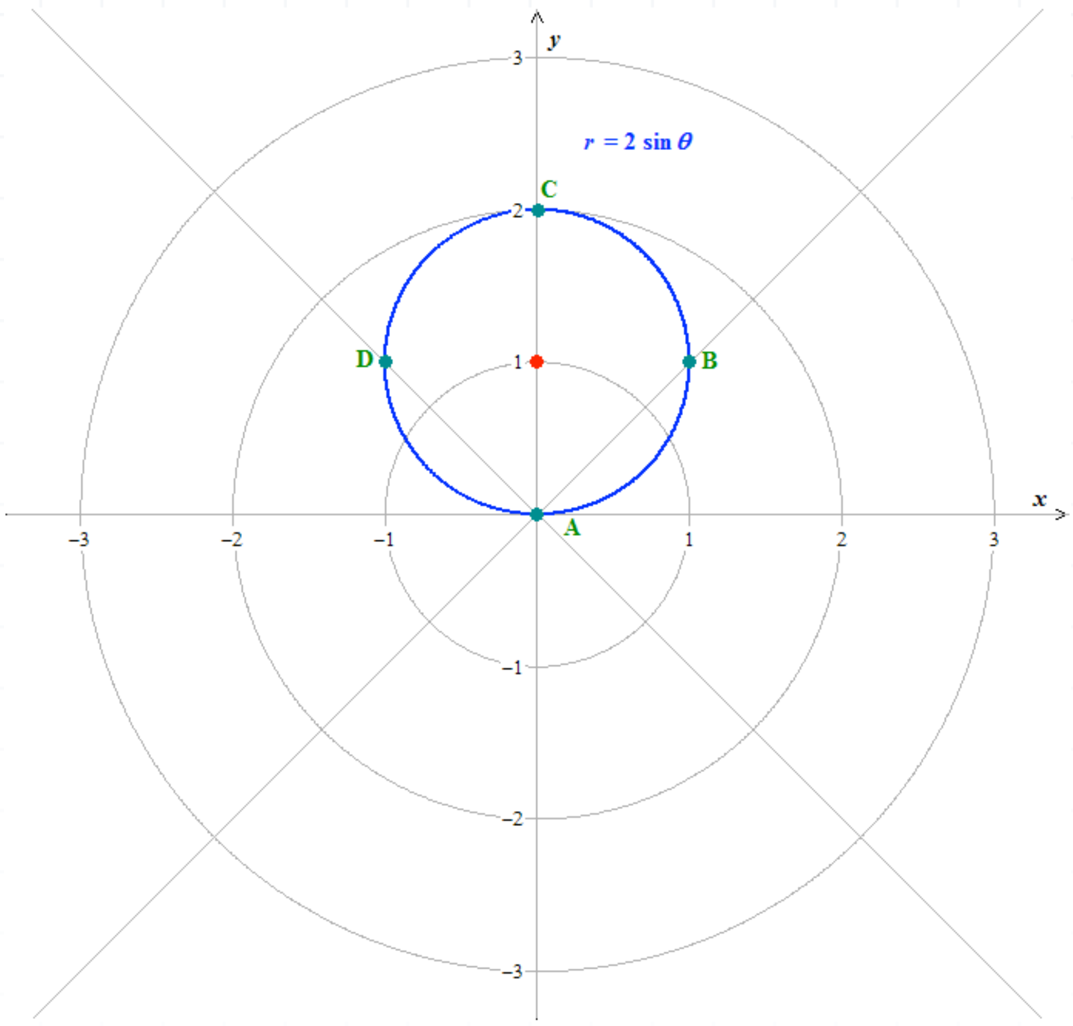
Example

Sketch the rectangular graph of the polar equation $r = 2\sin\theta$ on the interval $[0,2\pi]$.

As you may recall, the function $2\sin\theta$ is periodic and has period 2π. Therefore, we get just one period of this function when we sketch its rectangular graph on the interval $[0,2\pi]$.



If we take this rectangular graph and convert it to a polar graph, we get the circle of radius 1 (note: one-half of the coefficient $a = 2$) whose center lies on the vertical axis and is located 1 unit above the pole. You may recall learning this in the lesson on polar graphs of circles.



One thing you may not realize is that if were to you sketch the polar graph of the equation $r = 2\sin\theta$ on the interval $[0,\pi]$, you would get the very same curve as if you had sketched the polar graph of that equation on the interval $[0,2\pi]$. To see this, note that for every angle θ in the interval $(\pi,2\pi)$, the value of $r = 2\sin\theta$ is negative. For each of those angles, one pair of polar coordinates of the corresponding point of the polar graph of $r = 2\sin\theta$ is $(2\sin\theta, \theta)$, but another pair is $(-2\sin\theta, \theta - \pi)$. Now the difference identity for sine tells us that

$$\sin(\theta - \pi) = (\sin\theta)(\cos\pi) - (\cos\theta)(\sin\pi)$$

$$\sin(\theta - \pi) = (\sin\theta)(-1) - (\cos\theta)(0)$$

$$\sin(\theta - \pi) = -\sin\theta$$

Therefore,

$$-2\sin\theta = 2(-\sin\theta) = 2\sin(\theta - \pi)$$

Thus if θ is in angle in the interval $(\pi, 2\pi)$, the first of the two indicated pairs of polar coordinates (given above) of the point which corresponds to θ on the polar graph of the equation $r = 2\sin\theta$ is $(2\sin\theta, \theta)$, and the other pair is

$$(-2\sin\theta, \theta - \pi) = (2\sin(\theta - \pi), \theta - \pi)$$

Note that $\theta - \pi$ is in the interval $(0, \pi)$, so the point with polar coordinates $(2\sin(\theta - \pi), \theta - \pi)$ is also a point of the polar graph of $r = 2\sin\theta$. Thus every point of the polar graph of $r = 2\sin\theta$ that corresponds to some angle in the interval $(\pi, 2\pi)$ is also a point of the polar graph of $r = 2\sin\theta$ that corresponds to some angle in the interval $(0, \pi)$.

The following table lists the angle(s) θ in the interval $[0, 2\pi]$ to which the points labeled A, B, C, and D on our polar graph of $r = 2\sin\theta$ correspond.

Point on polar graph	Angle(s) θ
A	$0, \pi, 2\pi$
B	$\frac{\pi}{4}, \frac{5\pi}{4}$
C	$\frac{\pi}{2}, \frac{3\pi}{2}$
D	$\frac{3\pi}{4}, \frac{7\pi}{4}$

Analogous behavior can be shown to occur in the case of the rectangular and polar graphs of polar equations of the form $r = a\cos\theta$ where a is a nonzero constant. You should convince yourself that it suffices to sketch the polar graph on the interval $[0,\pi]$ for these polar equations as well.

Now suppose we consider a slightly more complicated type of polar equation:

$$r = a\cos(b\theta)$$

where a is a nonzero constant and b is an integer with $|b| \geq 2$. Here, we're looking at a "cosine function," but similar behavior occurs in the case of functions of the form $r = a\sin(b\theta)$.

Example

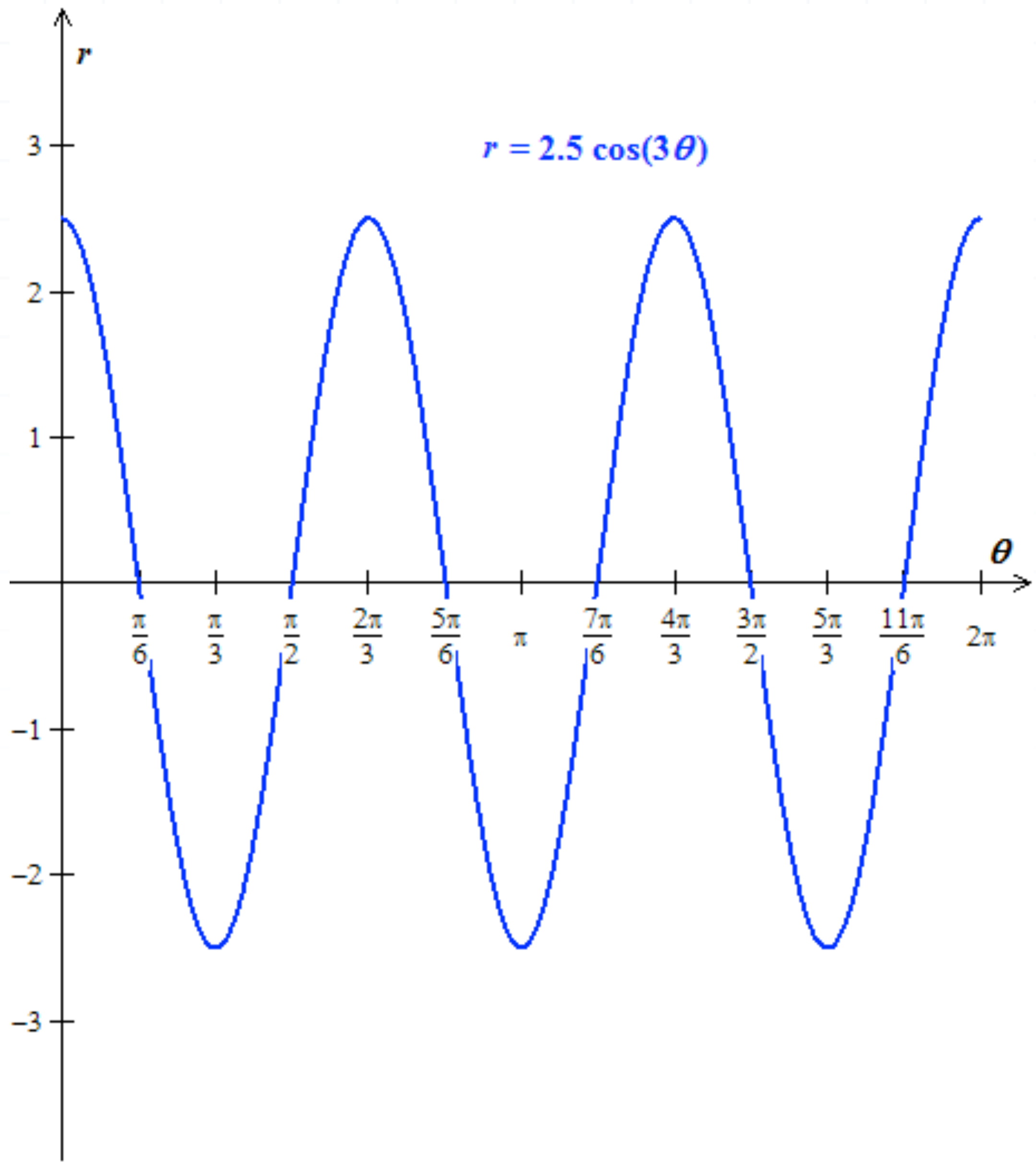
Sketch the rectangular graph of the polar equation $r = 2.5\cos(3\theta)$ on the interval $[0,2\pi)$.

When we sketch the rectangular graph of this polar equation, we want to be sure that all the values of θ with $|\cos(3\theta)| = 1$ and all those with $\cos(3\theta) = 0$ are displayed as tick marks on the horizontal axis. Now $|\cos(3\theta)| = 1$ if and only if 3θ is an integer multiple of π, and $\cos(3\theta) = 0$ if and only if 3θ is an odd integer multiple of $\pi/2$.

The spacing between tick marks on the horizontal scale of our graph has to be positive, so we'll consider the least positive angle θ such that $|\cos(3\theta)| = 1$ (namely, the angle θ that satisfies the equation $3\theta = \pi$) and the least positive angle θ such that $\cos(3\theta) = 0$ (namely, the angle θ that satisfies the equation $3\theta = \pi/2$), and we'll use the smaller of those two angles θ for our spacing:

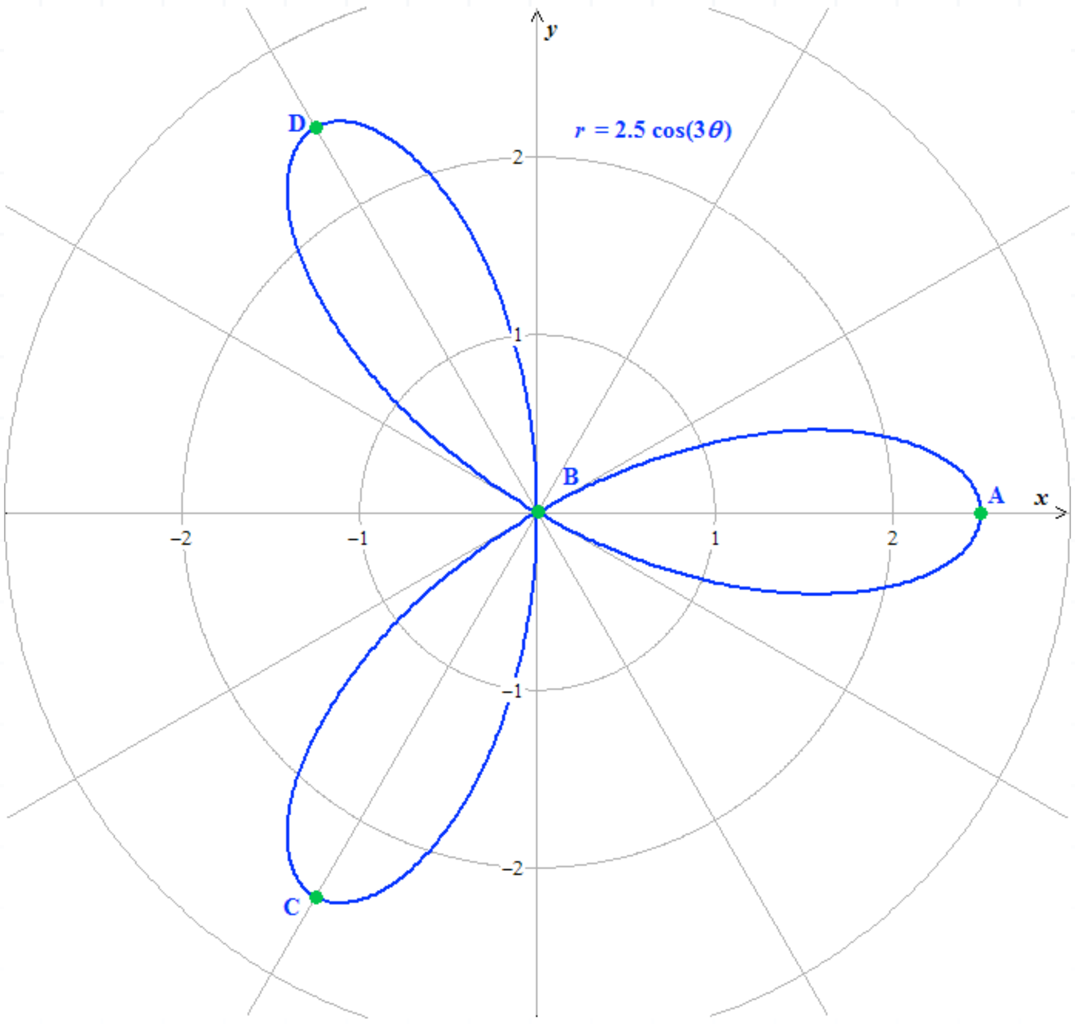
$$3\theta = \pi \Longrightarrow \theta = \frac{\pi}{3}, \qquad 3\theta = \frac{\pi}{2} \Longrightarrow \theta = \frac{\pi}{6}$$

The smaller of $\pi/3$ and $\pi/6$ is $\pi/6$. Thus the values of θ that we'll mark off along the horizontal axis of our rectangular graph of $r = 2.5\cos(3\theta)$ are the multiples of $\pi/6$.



Notice that the rectangular graph of $r = 2.5\cos(3\theta)$ on the interval $[0,2\pi]$ contains exactly three complete periods of the cosine function. This is consistent with what you learned some time ago, namely, that the period of a function of the form $a\cos(b\theta)$ where $a, b \neq 0$ is $2\pi/|b|$. In this case, $b = 3$, so the period is $2\pi/3$.

Now we'll take the rectangular graph of the equation $r = 2.5\cos(3\theta)$ and convert it to a polar graph.



As you can see here (and as you may recall from an earlier lesson), the graph of the polar equation $r = 2.5\cos(3\theta)$ is a 3-petal rose.

For every angle θ in the interval $[0,2\pi]$ such that $r = 2.5\cos(3\theta)$ is negative, there is some angle α (either $\alpha = \theta + \pi$ or $\alpha = \theta - \pi$) such that α is in the interval $[0,2\pi]$ and $r = 2.5\cos(3\alpha)$ is positive and equal to $-2.5\cos(3\theta)$. You could derive this result with the help of the sum and difference identities for cosine, which yield the following:

$$\cos(3(\theta + \pi)) = -\cos(3\theta) \quad \text{and} \quad \cos(3(\theta - \pi)) = -\cos(3\theta)$$

Thus (just as in the previous example) if you were to sketch the polar graph of the equation $r = 2.5\cos(3\theta)$ on the interval $[0,\pi]$, you would get the very same curve as if you had sketched the polar graph of that equation on the interval $[0,2\pi]$.

The following table lists the angle(s) θ in the interval $[0,2\pi]$ to which the points labeled A, B, C, and D on our polar graph of $r = 2.5\cos(3\theta)$ correspond.

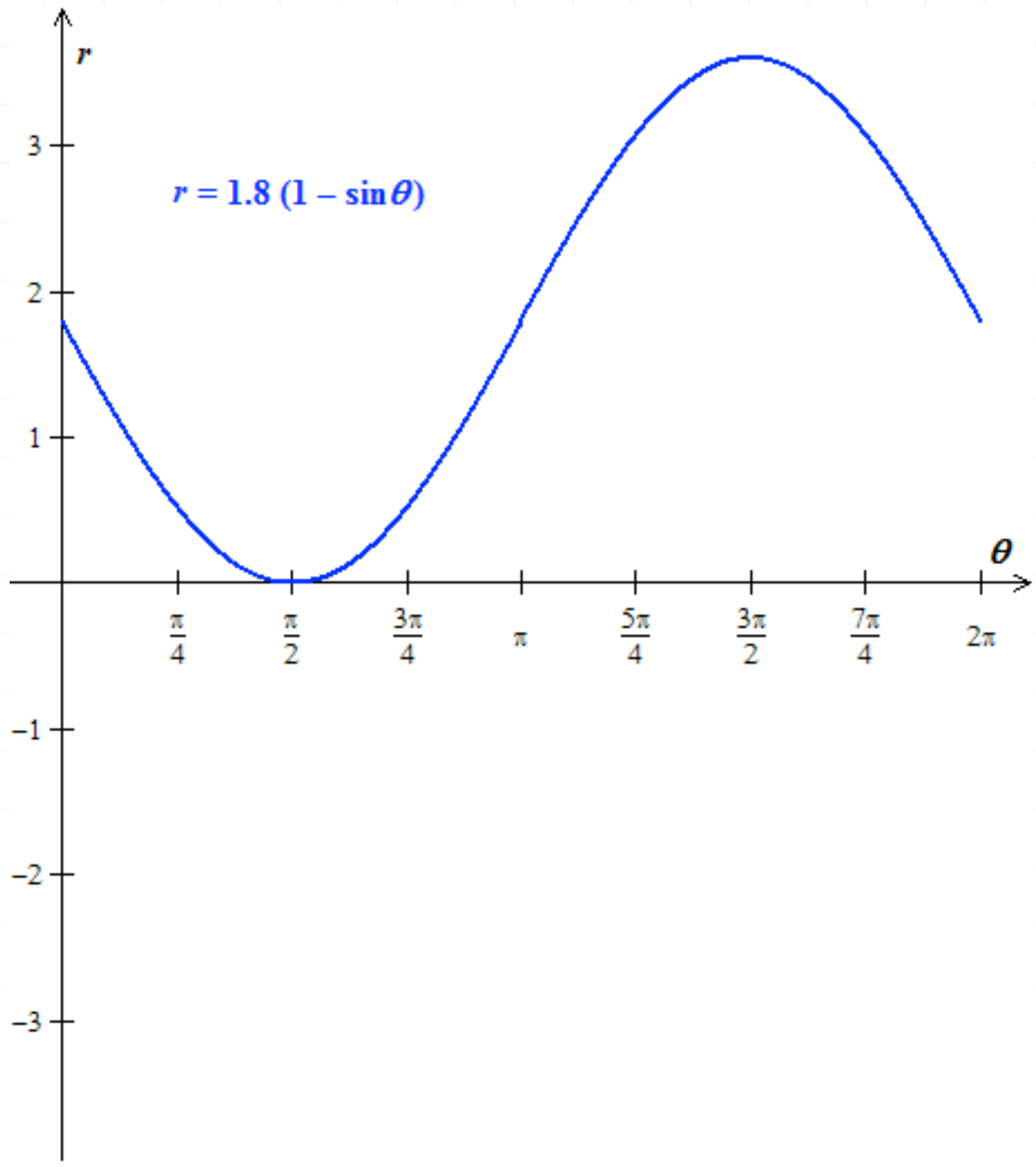
Point on polar graph	Angle(s) θ
A	$0,\ \pi,\ 2\pi$
B	$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$
C	$\frac{\pi}{3}, \frac{4\pi}{3}$
D	$\frac{2\pi}{3}, \frac{5\pi}{3}$

For the sake of variety, we'll next consider a “sine function” - in particular, a function of the form $r = a(1-\sin(\theta))$ where $a > 0$.

Example

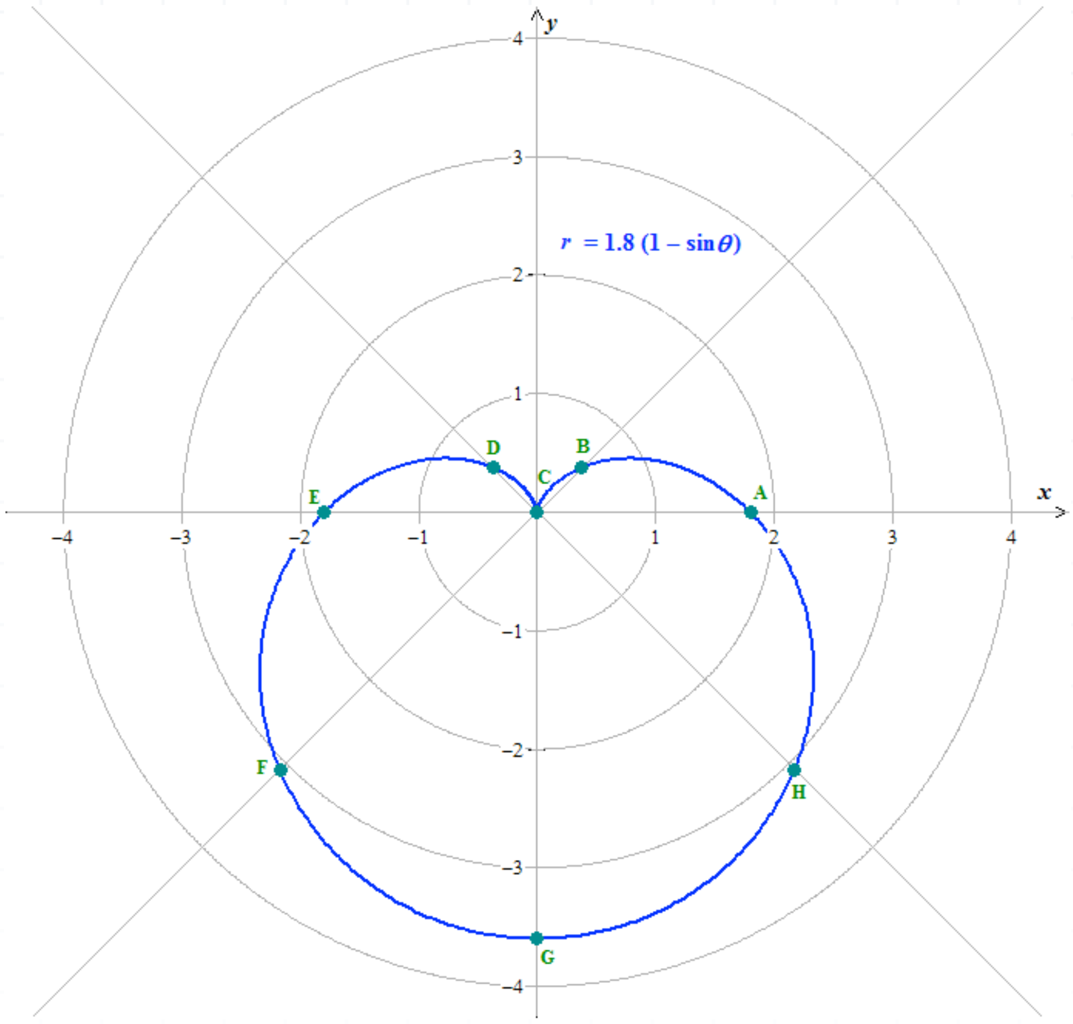
Sketch the rectangular graph of the polar equation $r = 1.8(1-\sin\theta)$ on the interval $[0,2\pi]$.

Since the sine function is periodic and has period 2π, the function $r = 1.8(1-\sin\theta)$ is also periodic and has period 2π. Thus we get just one period of this function when we sketch its rectangular graph on the interval $[0,2\pi]$.



Notice that for every angle θ, the value of r is nonnegative. Thus there is no pair of angles in the interval $[0,\pi]$ that differ by π and have values of r that are equal in absolute value but opposite in sign. This means that when we convert this rectangular graph to a polar graph, we can't take shortcuts and graph it on some interval of length less than 2π and get all the points (i.e., the points for all the angles in the interval $[0,2\pi]$).

Let's go ahead and sketch the polar graph of $r = 1.8(1 - \sin\theta)$.



As you can see here (and as you may recall from an earlier lesson), the polar graph of the equation $r = 1.8(1 - \sin\theta)$ is a cardioid that's symmetric with respect to the vertical axis. The following table shows the angle(s) θ in the interval $[0,2\pi]$ to which the points labeled A through H on our polar graph of $r = 1.8(1 - \sin\theta)$ correspond.

Point on polar graph	Angle(s) θ
A	$0,\ 2\pi$
B	$\frac{\pi}{4}$
C	$\frac{\pi}{2}$
D	$\frac{3\pi}{4}$

E	π
F	$\frac{5\pi}{4}$
G	$\frac{3\pi}{2}$
H	$\frac{7\pi}{4}$

Now let's go to a somewhat more complicated polar equation:

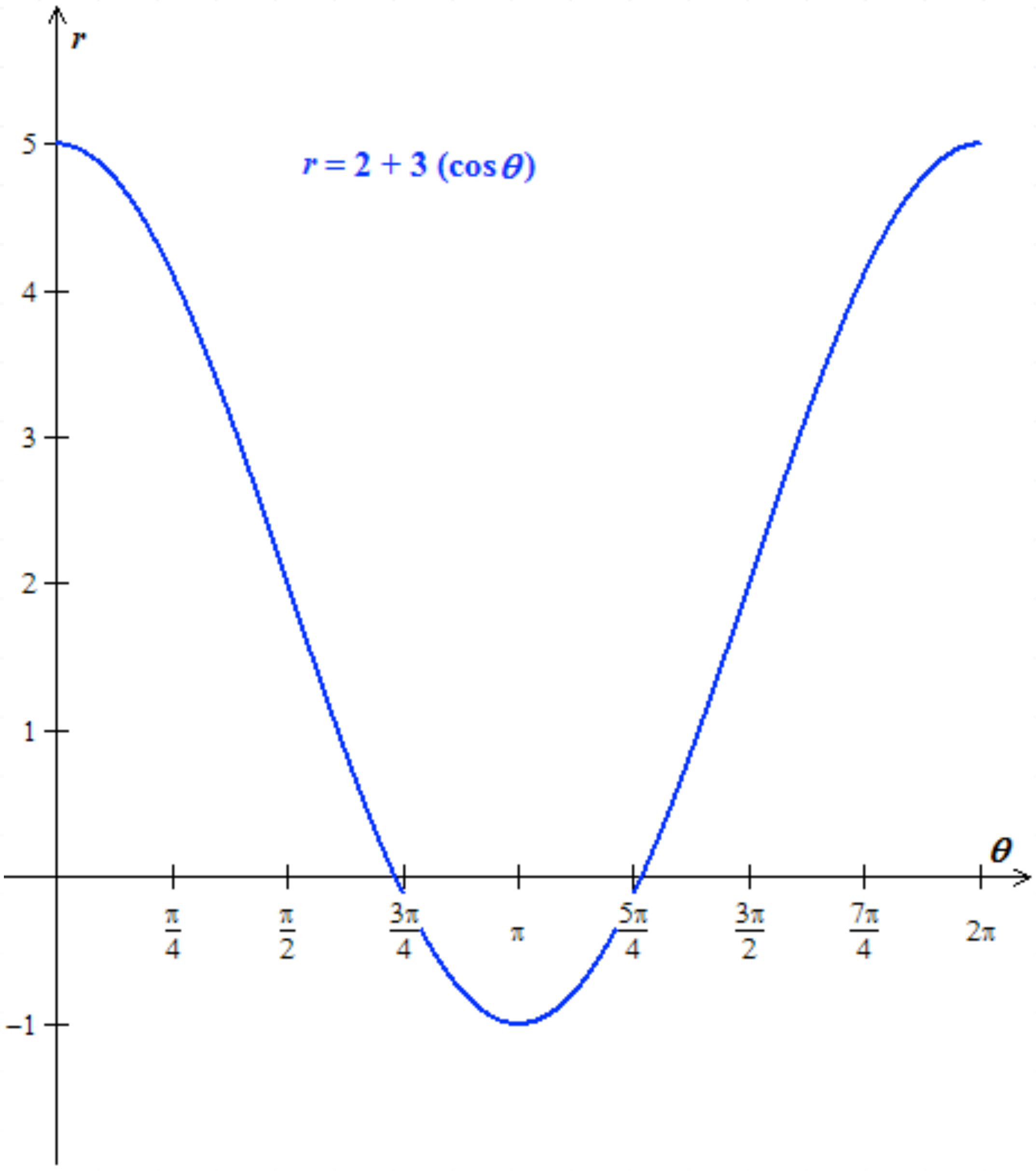
$$r = a + b\cos\theta$$

where $0 < a < b$. Here, we're looking at a "cosine function," but similar behavior occurs in the case of "sine functions" of this same type.

Example

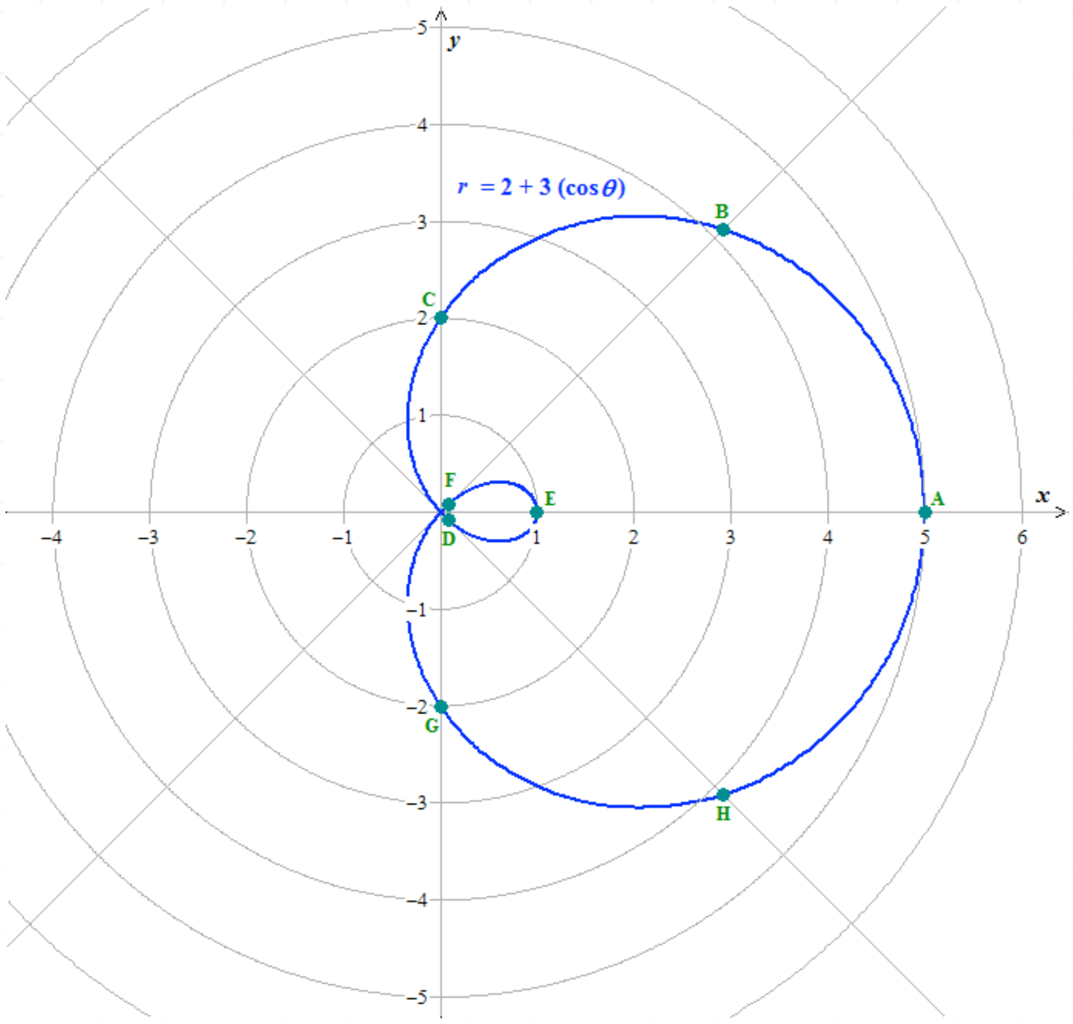
Sketch the rectangular graph of the polar equation $r = 2 + 3\cos\theta$.

Since the cosine function is periodic and has period 2π, the function $r = 2 + 3\cos\theta$ is also periodic and has period 2π. Thus we get just one period of this function when we sketch its rectangular graph on the interval $[0,2\pi]$.



Here again, there is no pair of angles in the interval $[0,\pi]$ that differ by π and have values of $r = 2 + 3\cos\theta$ that are equal in absolute value but opposite in sign. Thus here again, we can't sketch the polar graph of the equation $r = 2 + 3\cos\theta$ on some interval of length less than 2π and get the entire set of points (i.e., all the points for angles θ in the interval $[0,2\pi]$).

Let's sketch the polar graph of $r = 2 + 3\cos\theta$.



As you can see here (and as you may recall from an earlier lesson), the polar graph of the equation $r = 2 + 3\cos\theta$ is a limaçon that's symmetric with respect to the horizontal axis and has a loop. The following table shows the angle(s) θ in the interval $[0,2\pi]$ to which the points labeled A through H on our polar graph of $r = 2 + 3\cos\theta$ correspond.

Point on polar graph	Angle(s) θ
A	$0,\ 2\pi$
B	$\frac{\pi}{4}$
C	$\frac{\pi}{2}$
D	$\frac{3\pi}{4}$

E	π
F	$\frac{5\pi}{4}$
G	$\frac{3\pi}{2}$
H	$\frac{7\pi}{4}$

If you were to sketch the polar graph of any limaçon that has a loop (i.e., a limaçon which is the polar graph of any of the types given below and where the constants a, b satisfy $0 < a < b$), you would find that you have to graph the polar equation on an interval of length 2π to get the complete curve:

$$r = a + b\sin\theta$$

$$r = a - b\sin\theta$$

$$r = a + b\cos\theta$$

$$r = a - b\cos\theta$$

The same would be true if you were to graph a limaçon that has a depression instead of a loop (i.e., where $0 < b < a$ and everything else is the same as above).

Finally, let's look at a polar equation of the form $r^2 = a^2\sin(2\theta)$ where a is positive.

Example

Sketch the rectangular graph of the polar equation $r^2 = 9\sin(2\theta)$.

Here we have r^2 (not r) as a function of θ. Since r^2 is always nonnegative, $\sin(2\theta)$ must also be nonnegative. Thus 2θ must be in the interval $[0,\pi]$ or in some interval that differs from that by an integer multiple of 2π.

By the sum identity for sine, we have

$$\sin(2\theta + 2\pi) = \sin(2\theta)\cos(2\pi) + \cos(2\theta)\sin(2\pi)$$

$$\sin(2\theta + 2\pi) = \sin(2\theta)(1) + \cos(2\theta)(0)$$

$$\sin(2\theta + 2\pi) = \sin(2\theta)$$

Therefore, we can restrict our attention to angles θ such that 2θ is in the interval $[0,\pi]$; equivalently, we can restrict our attention to angles θ in the interval $[0,\pi/2]$. The only angles θ in that interval at which $\sin(2\theta) = 0$ are 0 and $\pi/2$. Thus for $\theta = 0$ and $\theta = \pi/2$ there is just one value of r such that $r^2 = 9\sin(2\theta)$, namely, $r = 0$. This means that for every angle θ in the interval $(0,\pi/2)$, there are two values of r such that $r^2 = 9\sin(2\theta)$:

$$r = 3\sqrt{\sin(2\theta)} \quad \text{and} \quad r = -3\sqrt{\sin(2\theta)}$$

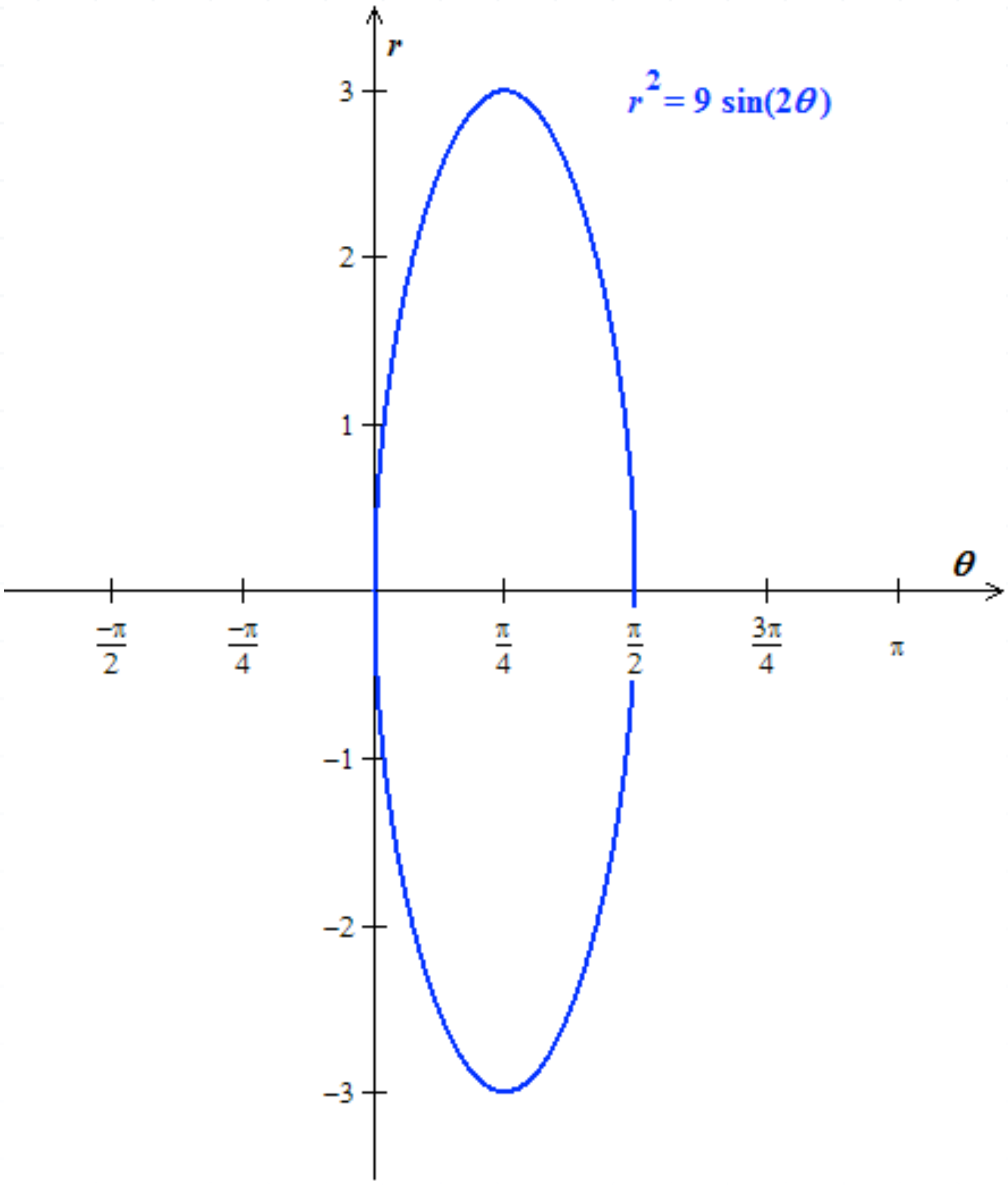
Using that information, let's go ahead and sketch the rectangular graph of the polar equation $r^2 = 9\sin(2\theta)$. Similarly to what we did when we sketched the rectangular graph of the polar equation $r = 2.5\cos(3\theta)$, we want to be sure that all the values of θ in the interval $[0,\pi/2]$ with $|\sin(2\theta)| = 1$ and all those with $\sin(2\theta) = 0$ are displayed as tick marks on the horizontal axis.

Now $|\sin(2\theta)| = 1$ if and only if 2θ is an odd integer multiple of $\pi/2$, and $\sin(2\theta) = 0$ if and only if 2θ is an odd integer multiple of $\pi/2$.

The spacing between tick marks on the horizontal scale of our graph has to be positive, so we'll consider the least positive angle θ in the interval $[0,\pi/2]$ such that $|\sin(2\theta)| = 1$ (namely, the angle θ that satisfies the equation $2\theta = \pi/2$) and the least positive angle θ such that $\sin(2\theta) = 0$ (namely, the angle θ that satisfies the equation $2\theta = \pi$), and we'll use the smaller of those two angles θ for our spacing:

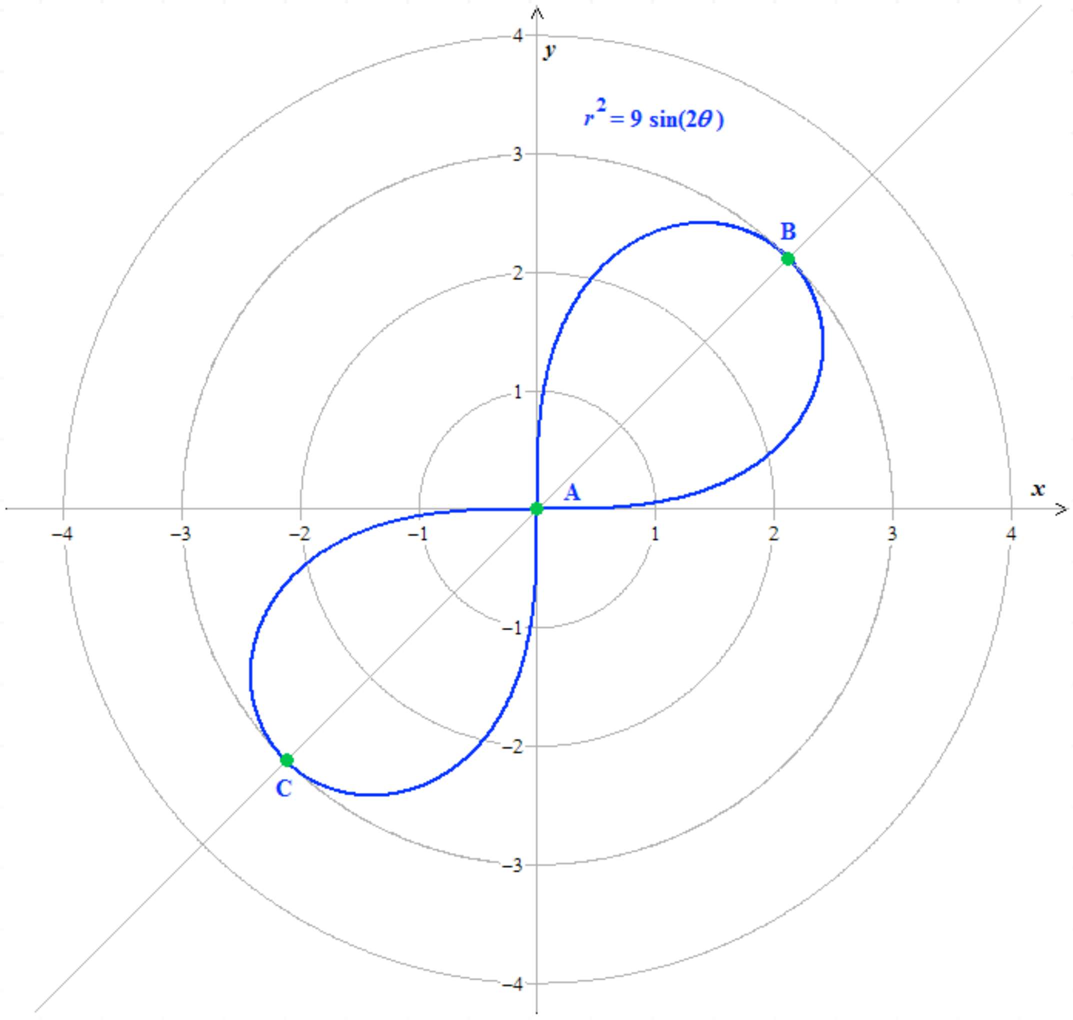
$$2\theta = \frac{\pi}{2} \Longrightarrow \theta = \frac{\pi}{4}, \qquad 2\theta = \pi \Longrightarrow \theta = \frac{\pi}{2}$$

The smaller of $\pi/4$ and $\pi/2$ is $\pi/4$. With this as a guide, the values of θ that we'll be marking off along the horizontal axis of our rectangular graph of $r^2 = 9\sin(2\theta)$ are the multiples of $\pi/4$. We'll sketch the rectangular graph of the polar equation on the interval $[0,\pi/2]$ only.



There is no pair of angles in the interval $[0,\pi/2]$ that differ by π (let alone any pair of this type for which the values of r are equal in absolute value and opposite in sign), so we can't sketch the polar graph of the equation $r^2 = 9\sin(2\theta)$ on some interval of length less than $\pi/2$ and get all the points for angles θ in the interval $[0,\pi/2]$.

Let's sketch the polar graph of $r^2 = 9\sin(2\theta)$.



You may recognize the polar graph of the equation $r^2 = 9\sin(2\theta)$ as a lemniscate that's symmetric with respect to the pole (but not with respect to either of the axes). In this polar graph, the point labeled A corresponds to the angles $\theta = 0$ and $\theta = \pi/2$, and the points labeled B and C both correspond to the angle $\theta = \pi/4$.