Converting rectangular coordinates to polar

Now that you know how to convert (what we called) the basic polar coordinates (r, θ) of a point to its rectangular coordinates (x, y), you're going to learn how to go in the opposite direction and convert the rectangular coordinates (x, y) of a point to its basic polar coordinates (r, θ).

Recall that the basic polar coordinate r of a point P is the distance of P from the pole (hence r must be nonnegative), and the basic polar coordinate θ of P is the angle in the interval $[0,2\pi)$ such that θ is in standard position and P is located on the terminal side of θ. Thus we can immediately deduce the following:

P on positive horizontal axis (x positive, $y = 0$) $\Longrightarrow r = x,\ \theta = 0$

P on positive vertical axis ($x = 0$, y positive) $\Longrightarrow r = y,\ \theta = \dfrac{\pi}{2}$

P on negative horizontal axis (x negative, $y = 0$) $\Longrightarrow r = -x,\ \theta = \pi$

P on negative vertical axis ($x = 0$, y negative) $\Longrightarrow r = -y,\ \theta = \dfrac{3\pi}{2}$

For the origin, $r = 0$, and we can take its basic polar coordinate θ to be 0.

Now we'll take a look at how to convert from rectangular coordinates (x, y) to the basic polar coordinates (r, θ) for points in the various quadrants. As you learned in the previous lesson, the equation

$$x^2 + y^2 = r^2$$

must be satisfied for each of those points. Therefore, since the basic polar coordinate r must be nonnegative, we have

$$r = \sqrt{x^2 + y^2}$$

For the angle θ, things are a little more complicated than that, but not overly so. Recall that we said the basic polar coordinate θ must be an angle in the interval $[0,2\pi)$.

In the previous lesson, you learned that

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}$$

Thus

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x}$$

This equation is satisfied by every point (x, y) for which $x \neq 0$, that is, by every point not on the vertical axis. Since we're already dealt with the points on the vertical axis, that aspect of it won't be a problem for us.

Now the question is, how do we use the fact that $\tan\theta = y/x$ (for a point with $x \neq 0$) to get its basic polar coordinate θ? Well, here's where the inverse tangent function, $\tan^{-1}$, comes in handy.

Recall that the range of the inverse tangent function is $(-\pi/2,\pi/2)$, hence that it applies to points in the first and fourth quadrants, but not to points in the second and third quadrants.

For a point in the first quadrant, its basic polar coordinate θ is in the interval $(0,\pi/2)$, and hence in the range of the inverse tangent function. Therefore,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

so the basic polar coordinates of the point in the first quadrant with rectangular coordinates (x, y) are

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example

Find the basic polar coordinates (r, θ) of the point whose rectangular coordinates (x, y) are $(6,11)$.

The basic polar coordinate r is

$$r = \sqrt{x^2 + y^2} = \sqrt{(6)^2 + (11)^2} = \sqrt{36 + 121} = \sqrt{157} \approx 12.5$$

Since both x and y are positive, this point is in the first quadrant, so its basic polar coordinate θ is given by

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{11}{6}\right)$$

With the help of a calculator, we find that

$$\frac{y}{x} = \frac{11}{6} \approx 1.83$$

so

$$\theta \approx \tan^{-1}(1.83) \approx 1.07 \text{ radians}$$

When you use a calculator to compute the value of the inverse tangent function, be sure to set your calculator to radians if you want θ in radians, and to degrees if you want θ in degrees. An alternative way to get θ in degrees is to first compute the value of θ in radians, and then multiply the result by the conversion factor $(180^\circ)/(\pi$ radians$)$ to get θ in degrees. Because of round-off errors, the answers you get with the two methods could be slightly different.

To get the value of the basic polar coordinate θ of the point with rectangular coordinates $(x, y) = (6,11)$ which was found in the example above, you would need to set your calculator to radians. However, if you want to get θ in degrees, you could set your calculator to degrees and compute $\tan^{-1}(y/x)$. If you did that, you would find that

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \approx \tan^{-1}(1.83) \approx 61.3^\circ$$

For a point in the fourth quadrant, θ is in the interval $(3\pi/2,2\pi)$, so θ isn't in the range of the inverse tangent function. However, note that

$$\frac{3\pi}{2} < \theta < 2\pi \Longrightarrow \frac{3\pi}{2} - 2\pi < \theta - 2\pi < 2\pi - 2\pi$$

Now

$$\frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}, \qquad 2\pi - 2\pi = 0$$

so we have the equivalent implication

$$\frac{3\pi}{2} < \theta < 2\pi \Longrightarrow -\frac{\pi}{2} < \theta - 2\pi < 0$$

That is, $\theta - 2\pi$ is in the interval $(-\pi/2,0)$, so $\theta - 2\pi$ is in the range of the inverse tangent function. Since an angle of measure θ is coterminal with an angle of measure $\theta - 2\pi$, we have the following:

$$\sin\theta = \sin(\theta - 2\pi)$$

$$\cos\theta = \cos(\theta - 2\pi)$$

Therefore,

$$\frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin(\theta - 2\pi)}{\cos(\theta - 2\pi)} = \tan(\theta - 2\pi)$$

From this result and the fact that $\theta - 2\pi$ is in the range of the inverse tangent function, we see that

$$\theta - 2\pi = \tan^{-1}\left(\frac{y}{x}\right)$$

Solving this equation for θ, we obtain

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2\pi$$

Thus the basic polar coordinates of the point in the fourth quadrant with rectangular coordinates (x, y) are

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) + 2\pi$$

Example

Find the basic polar coordinates (r, θ) of the point whose rectangular coordinates (x, y) are $(2.4, -8.6)$.

The basic polar coordinate r is

$$r = \sqrt{x^2 + y^2} = \sqrt{(2.4)^2 + (-8.6)^2} = \sqrt{5.76 + 73.96} = \sqrt{79.72} \approx 8.9$$

Since x is positive and y is negative, this point is in the fourth quadrant, so its basic polar coordinate θ is given by

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2\pi = \tan^{-1}\left(\frac{-8.6}{2.4}\right) + 2\pi$$

With the help of a calculator, we find that

$$\frac{y}{x} = \frac{-8.6}{2.4} \approx -3.58$$

hence that

$$\tan^{-1}\left(\frac{y}{x}\right) \approx -1.30 \text{ radians}$$

Thus $\theta \approx -1.30 + 2\pi$ radians, that is, $\theta \approx 4.98$ radians.

For that computation, you would need to set your calculator to radians, since $\tan^{-1}(y/x)$ and 2π were computed in radians. However, if you want to get θ in degrees, you could set your calculator to degrees and then (since 2π radians is equivalent to 360°) calculate the value of $(\tan^{-1}(y/x)) + 360^\circ$. If you did that, you would get $\tan^{-1}(y/x) \approx -74.4^\circ$, hence that

$$\theta \approx -74.4^\circ + 360^\circ \approx 286^\circ$$

Now you're probably wondering how in the world you could get the basic polar coordinate θ for a point in the second or third quadrant. Well, the basic polar coordinate θ for a point in the second quadrant is in the interval $(\pi/2, \pi)$, hence θ isn't in the range of the inverse tangent function, but note that

$$\frac{\pi}{2} < \theta < \pi \Longrightarrow \frac{\pi}{2} - \pi < \theta - \pi < \pi - \pi$$

Now

$$\frac{\pi}{2} - \pi = -\frac{\pi}{2}, \qquad \pi - \pi = 0$$

so we have the equivalent implication

$$\frac{\pi}{2} < \theta < \pi \Longrightarrow -\frac{\pi}{2} < \theta - \pi < 0$$

That is, $\theta - \pi$ is in the interval $(-\pi/2, 0)$, so $\theta - \pi$ is in the range of the inverse tangent function.

By the difference identity for sine,

$$\sin(\theta - \pi) = (\sin\theta)(\cos\pi) - (\cos\theta)(\sin\pi) = (\sin\theta)(-1) - (\cos\theta)(0) = -\sin\theta$$

And by the difference identity for cosine,

$$\cos(\theta - \pi) = (\cos\theta)(\cos\pi) + (\sin\theta)(\sin\pi) = (\cos\theta)(-1) + (\sin\theta)(0) = -\cos\theta$$

Therefore,

$$\frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sin(\theta - \pi)}{-\cos(\theta - \pi)} = \frac{\sin(\theta - \pi)}{\cos(\theta - \pi)} = \tan(\theta - \pi)$$

From this result and the fact that $\theta - \pi$ is in the range of the inverse tangent function, we see that

$$\theta - \pi = \tan^{-1}\left(\frac{y}{x}\right)$$

Solving this equation for θ, we obtain

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

Thus the basic polar coordinates of the point in the second quadrant with rectangular coordinates (x, y) are

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

Similar techniques can be used to show that the basic polar coordinate θ for a point in the third quadrant is

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

Thus the basic polar coordinates of the point in the third quadrant with rectangular coordinates (x, y) are

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

Example

Find the basic polar coordinates (r, θ) of the point whose rectangular coordinates (x, y) are $(-5.7, 9.2)$.

The basic polar coordinate r is

$$r = \sqrt{x^2 + y^2} = \sqrt{(-5.7)^2 + (9.2)^2} = \sqrt{32.49 + 84.64} = \sqrt{117.13} \approx 10.8$$

Since x is negative and y is positive, this point is in the second quadrant, so its basic polar coordinate θ is given by

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi = \tan^{-1}\left(\frac{9.2}{-5.7}\right) + \pi$$

With the help of a calculator, we find that

$$\frac{y}{x} = \frac{9.2}{-5.7} \approx -1.61$$

hence that

$$\tan^{-1}\left(\frac{y}{x}\right) \approx -1.01 \text{ radians}$$

Therefore, $\theta \approx (-1.01 + \pi)$ radians, that is, $\theta \approx 2.13$ radians.

For that computation, you would need to set your calculator to radians, since $\tan^{-1}(y/x)$ and π were computed in radians. However, if you want to get θ in degrees, you could set your calculator to degrees and then (since π radians is equivalent to 180°) calculate the value of $(\tan^{-1}(y/x)) + 180^\circ$. If you did that, you would find that $\tan^{-1}(y/x) \approx -58.2^\circ$, hence that

$$\theta \approx -58.2^\circ + 180^\circ \approx 122^\circ$$