Trigonometry Workbook Solutions

Angles in circles

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RADIANS AND ARC LENGTH

■ 1. Find the degree measure of the central angle if the length of an arc carved out by this central angle is 9.42 and the radius of the circle is $r = 6$.

Solution:

Use the arc length formula

$$s = r\theta$$

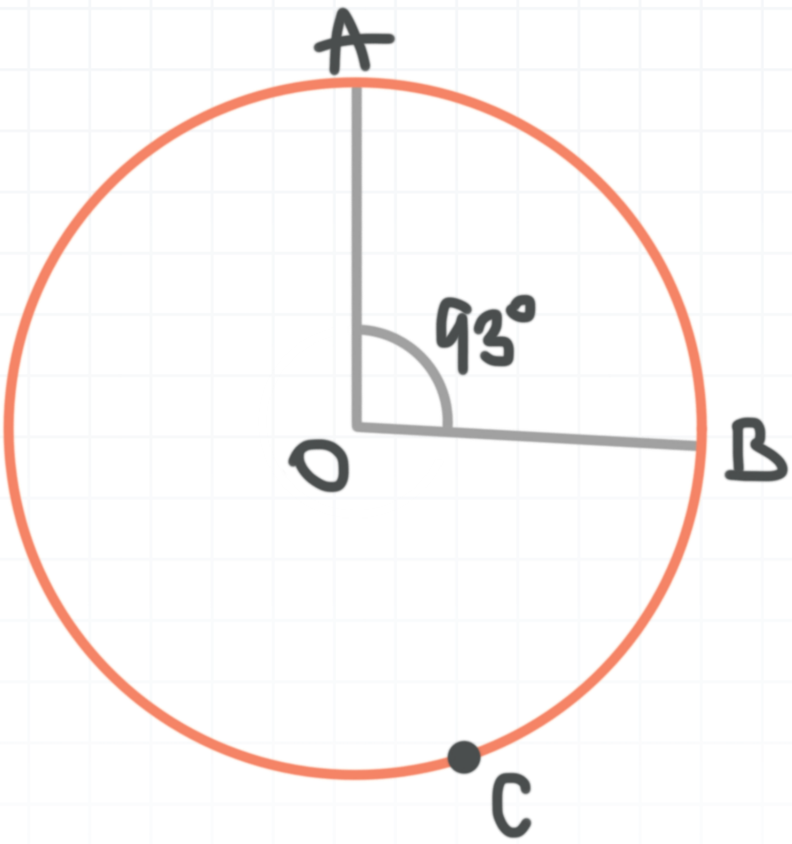
$$9.42 = 6\theta$$

$$\theta = \frac{9.42}{6} = 1.57$$

Now remember that 1 radian is approximately 57.32°, so we get

$$\theta = 1.57(57.32^\circ) = 90^\circ$$

■ 2. In circle O, the diameter is 30 cm, and the measure of arc AB is 93°. Find the length of arc ACB.



Solution:

To find the length of the arc ACB, first we need to find the central angle $m\angle ACB$. We remember that a full circle sweeps out 360°, so

$$m\angle ACB = 360^\circ - m\angle AOB$$

$$m\angle ACB = 360^\circ - 93^\circ$$

$$m\angle ACB = 267^\circ$$

We can only use an angle defined in radians in the arc length formula, so we'll need to convert 267° to radians.

$$267^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) \approx 1.483\pi \text{ radians}$$

Since the diameter is 30 cm, the radius will be $30/2 = 15$ cm. Now we'll plug what we know into the arc length formula.

$$s = r\theta$$

$$s = 15(1.483\pi)$$

$$s \approx 69.85$$

■ 3. A circle has a central angle of $35^\circ 23' 6''$ which subtends an arc of length 3π cm. Find the diameter of the circle to the nearest centimeter.

Solution:

We can only use an angle defined in radians in the arc length formula, so we'll need to convert $35^\circ 23' 6''$ to radians.

We'll convert the seconds part first. We need to convert $6''$ from seconds to minutes. We know that $1' = 60''$, so we'll multiply $6''$ by $1'/60''$ in order to cancel the seconds and be left with just minutes.

$$6''\left(\frac{1'}{60''}\right)$$

$$\left(\frac{1}{10}\right)' = 0.1'$$

Then the total minutes in $35^\circ 23' 6''$ is

$$23.1'$$

To convert this value for minutes into degrees, we'll multiply by $1^\circ/60'$ in order to cancel the minutes and be left with just degrees.

$$23.1'\left(\frac{1^\circ}{60'}\right)$$

$$0.385^\circ$$

Putting this together with the 35° from the original angle, we get approximately

$$35.385^\circ$$

$$35.385^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right) \approx 0.197\pi \text{ radians}$$

Now we'll plug what we know into the arc length formula.

$$s = r\theta$$

$$3\pi = r(0.197\pi)$$

$$r = \frac{3\pi}{0.197\pi} \approx 15$$

$$d = 2r \approx 2(15) \approx 30$$

■ 4. A circle has a radius of 19 cm. Find the central angle that subtends an arc of length 47.5 cm, rounding the answer to the nearest second.

Solution:

Use the arc length formula

$$s = r\theta$$

$$47.5 = 19\theta$$

$$\theta = \frac{47.5}{19} = 2.5$$

Now remember that 1 radian is approximately 57.32°, so we get

$$\theta = 2.5(57.32^\circ) = 143.3^\circ$$

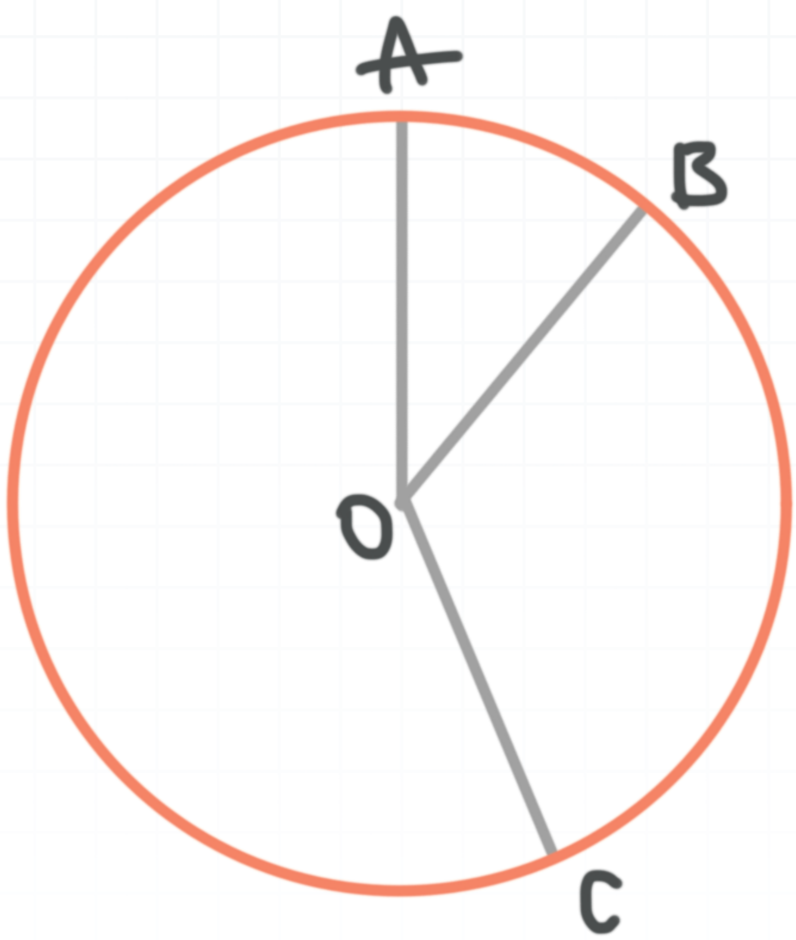
Now we need to convert degrees into DMS. The angle in degrees is 143.3°, so the degrees part in DMS is 143°. All we have to do is convert 0.3° to minutes and seconds. First, we'll convert 0.3° to minutes, and then if we get a decimal value for the minutes, we'll convert the remaining part to seconds.

$$0.3^\circ \left(\frac{60'}{1^\circ}\right)$$

$$18'$$

We've found that 0.3° converts to $18'$. Since 18 is an integer, there's nothing left to convert to seconds, so the angle in DMS is $143^\circ 18'$.

5. If AOB is a central angle of 53°, the angle $BOC = 122^\circ$, and the radius is 9 cm, then find the length of the arc ABC. Use $\pi = 3.14$ and round the answer to one decimal place.



Solution:

To find the length of the arc ABC, first we need to find the central angle $m\angle ABC$.

$$m\angle ABC = m\angle AOB + m\angle BOC$$

$$m\angle ABC = 53^\circ + 122^\circ$$

$$m\angle ABC = 175^\circ$$

We can only use an angle defined in radians in the arc length formula, so we'll need to convert 175° to radians.

$$175^\circ \left(\frac{\pi \text{ radians}}{180^\circ} \right) = \frac{35\pi}{36} \text{ radians}$$

Now we'll plug what we know into the arc length formula.

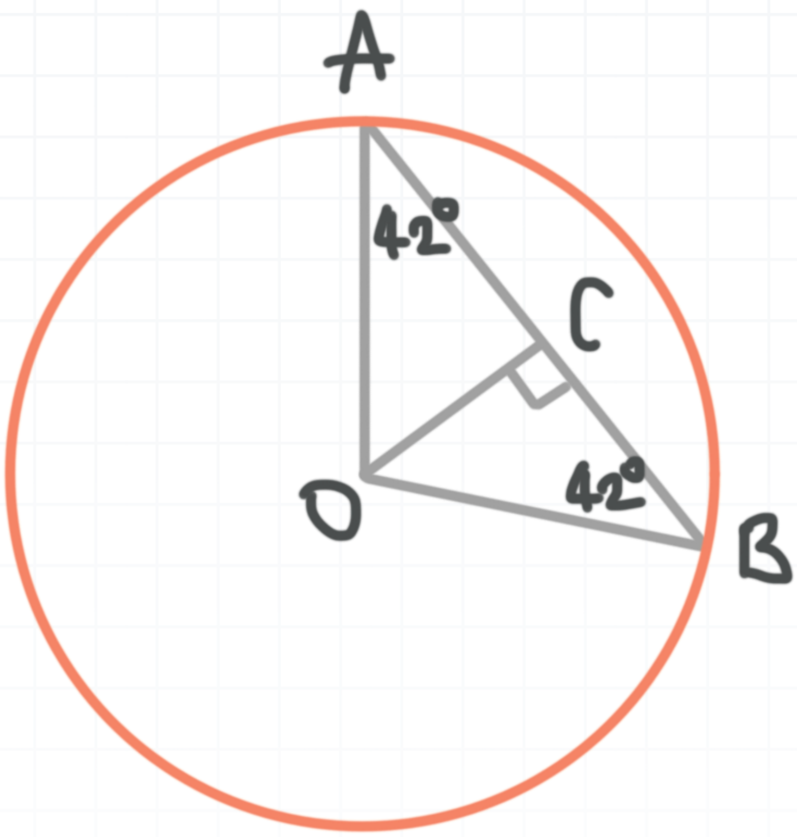
$$s = r\theta$$

$$s = 9 \left(\frac{35\pi}{36} \right)$$

$$s = \frac{35\pi}{4}$$

$$s \approx 27.5$$

■ 6. Find the length of the arc AB given that the height of the triangle is 8 cm. Round the answer to one decimal place.



Solution:

The sum of the interior angles of a triangle is 180°. So the central angle will be

$$m\angle AOB = 180^\circ - 42^\circ - 42^\circ$$

$$m\angle AOB = 96^\circ$$

We can only use an angle defined in radians in the arc length formula, so we'll need to convert 96° to radians.

$$96^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{24\pi}{45} \text{ radians}$$

Now we'll plug what we know into the arc length formula.

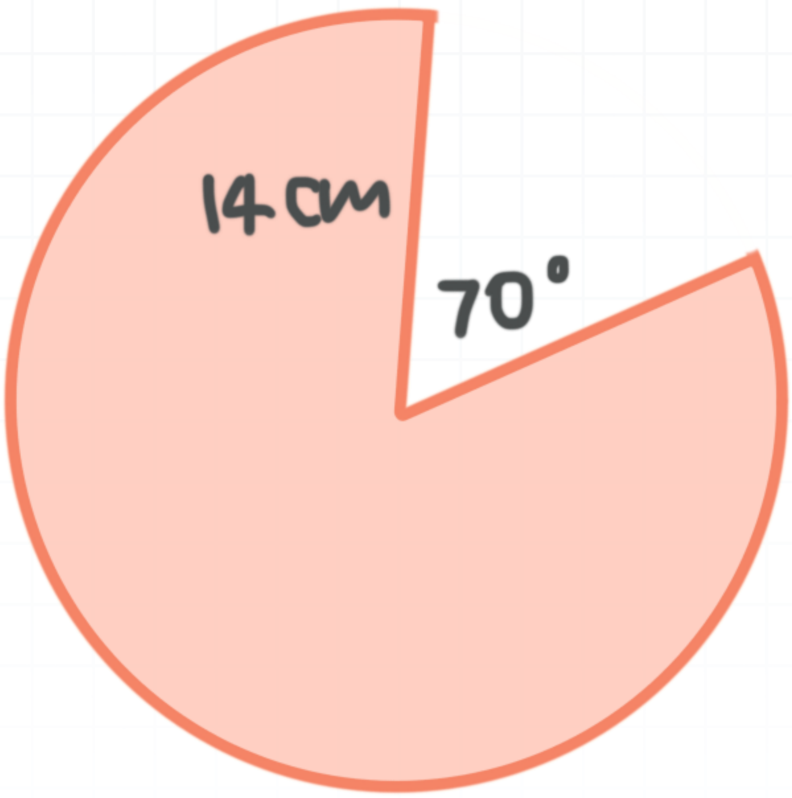
$$s = r\theta$$

$$s = 12\left(\frac{24\pi}{45}\right)$$

$$s = \frac{288\pi}{45}$$

AREA OF A CIRCULAR SECTOR

1. Find the area of the shaded region.



Solution:

The angle of the circular sector is

$$\theta = 360^\circ - 70^\circ$$

$$\theta = 290^\circ$$

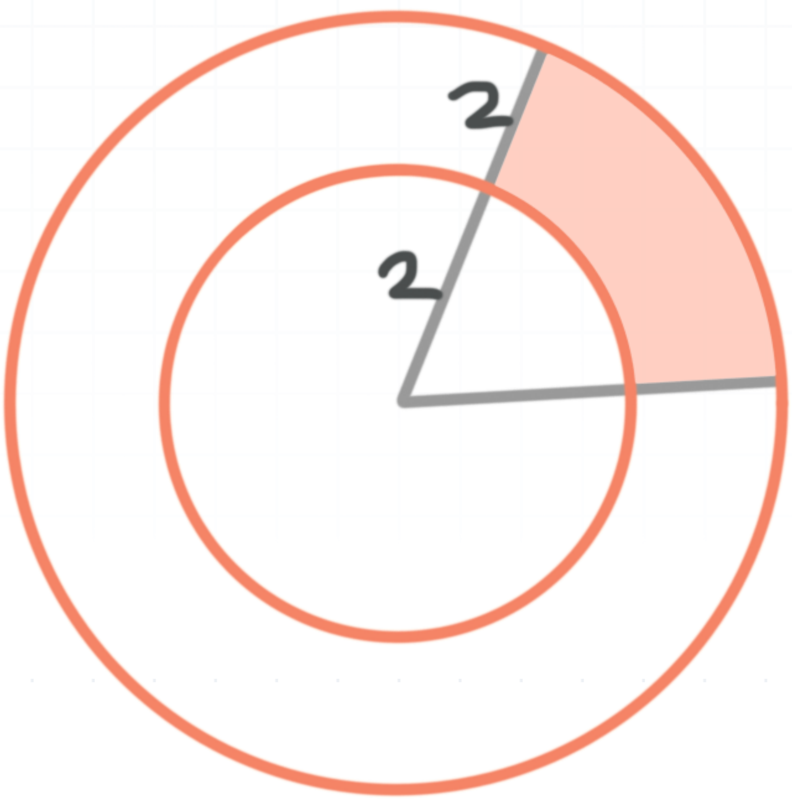
Plugging this angle and the radius into the formula for the area of a circular sector (in degrees) gives

$$A = \left(\frac{\pi}{360}\right) r^2\theta$$

$$A=\left(\frac{\pi}{360}\right)(14)^2(290)$$

$$A=\frac{1{,}421\pi}{9}$$

■ 2. Find the area of the shaded region between the concentric circles, if the angle that subtends the arc is 80°.



Solution:

If we find the area of the sector for the larger circle, given that its interior angle measure is 80° and its radius is 4, we get

$$A=\left(\frac{\pi}{360}\right)r^2\theta$$

$$A = \left(\frac{\pi}{360}\right)(4)^2(80)$$

$$A = \frac{32\pi}{9}$$

The area of the sector for the smaller circle, given that its interior angle measure is 80° and its radius is 2, is

$$A = \left(\frac{\pi}{360}\right) r^2\theta$$

$$A = \left(\frac{\pi}{360}\right)(2)^2(80)$$

$$A = \frac{8\pi}{9}$$

So the area of the shaded region is

$$A = \frac{32\pi}{9} - \frac{8\pi}{9}$$

$$A = \frac{24\pi}{9}$$

$$A = \frac{8\pi}{3}$$

■ 3. A circle has radius 13. Find the area A of a sector of the circle that has a central angle of $2\pi/5$.

Solution:

The area of the circular sector with radius $r = 13$ and central angle $2\pi/5$ is

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(13)^2\left(\frac{2\pi}{5}\right)$$

$$A = \frac{169\pi}{5}$$

■ 4. A pizza with 16 inch diameter is sliced into 8 equal slices. Find the area of one of the pizza slices.

Solution:

Let's think of θ in degrees. The central angle associated with each slice is $360^\circ/8 = 45^\circ$. Then the area of each slice is

$$A = \left(\frac{\pi}{360}\right)r^2\theta$$

$$A = \left(\frac{\pi}{360}\right)(8)^2(45)$$

$$A = 8\pi$$

5. Find the area of a sector of a circle that has diameter $\overline{GH}$ with $G(-1,-1)$ and $H(5,7)$ if the arc which bounds that sector subtends a central angle of $4\pi/9$. Use the distance formula for d to find the length of the diameter.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Solution:

Start by finding the length of the diameter.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$d=\sqrt{(7-(-1))^2+(5-(-1))^2}$$

$$d=\sqrt{8^2+6^2}$$

$$d=\sqrt{100}$$

$$d=10$$

The length of the diameter is 10 units, so the radius is 5 units.

$$A=\frac{1}{2}r^2\theta$$

$$A=\frac{1}{2}(5)^2\left(\frac{4\pi}{9}\right)$$

$$A = \frac{50\pi}{9}$$

6. The area of a sector of a circle is formed with a central angle $3\pi/4$ and has area 54π. Find the diameter of the circle.

Solution:

Substitute the area and central angle into the circular sector area formula,

$$A = \frac{1}{2}r^2\theta$$

$$54\pi = \frac{1}{2}r^2\left(\frac{3\pi}{4}\right)$$

then solve for the radius.

$$54\pi = r^2\left(\frac{3\pi}{8}\right)$$

$$54\pi\left(\frac{8}{3\pi}\right) = r^2$$

$$18(8) = r^2$$

$$r^2 = 144$$

$$r = 12$$

Then the diameter of the circle is double the radius, so $d = 2r = 2(12) = 24$.

TRIG FUNCTIONS OF REAL NUMBERS

■ 1. Find $\sec 1.56$ using a calculator to evaluate only cosine. Round the result to three decimal places.

Solution:

We'll use a calculator to evaluate $\cos 1.56$, making sure the calculator is set to radian mode.

$$\cos 1.56 \approx 0.011$$

Then the value of $\sec 1.56$ is

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\sec 1.56 \approx \frac{1}{0.011}$$

$$\sec 1.56 \approx 90.909$$

■ 2. Find $\cot 0.567$ using a calculator to evaluate only sine and cosine. Round the result to four decimal places.

$$\cot 0.567$$

Solution:

We'll use a calculator to evaluate $\sin 0.567$ and $\cos 0.567$, making sure the calculator is set to radian mode.

$$\sin 0.567 \approx 0.5371$$

$$\cos 0.567 \approx 0.8435$$

Then the value of $\cot 0.567$ is

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot 0.567 \approx \frac{0.8435}{0.5371}$$

$$\cot 0.567 \approx 1.5705$$

■ 3. Find the value of all six circular functions at $a = 1.273$ using a calculator to evaluate only sine and cosine.

Solution:

We'll use a calculator to evaluate $\sin 1.273$ and $\cos 1.273$, making sure the calculator is set to radian mode.

$$\sin 1.273 \approx 0.9560$$

$$\cos 1.273 \approx 0.2934$$

Now we can find $\tan 1.273$ using the quotient identity for tangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan 1.273 \approx \frac{0.9560}{0.2934}$$

$$\tan 1.273 \approx 3.2584$$

Then the reciprocal identities give the values of cosecant, secant, and cotangent.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \csc 1.273 \approx \frac{1}{0.9560} \approx 1.0460$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \sec 1.273 \approx \frac{1}{0.2934} \approx 3.4083$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \cot 1.273 \approx \frac{1}{3.2584} \approx 0.3069$$

■ 4. Find the value of all six circular functions at $t = -0.2489$.

Solution:

We'll use a calculator to evaluate the circular functions at -0.2489, making sure the calculator is set to radian mode.

$$\sin(-0.2489) \approx -0.2463 \qquad \csc(-0.2489) \approx -4.0601$$

$$\cos(-0.2489) \approx 0.9692 \qquad \sec(-0.2489) \approx 1.0318$$

$$\tan(-0.2489) \approx -0.2542 \qquad \cot(-0.2489) \approx -3.9339$$

■ 5. Find $\tan 3.49$ using a calculator to evaluate only sine and cosine. Round the result to two decimal places.

Solution:

We'll use a calculator to evaluate $\sin 3.49$ and $\cos 3.49$, making sure the calculator is set to radian mode.

$$\sin 3.49 \approx -0.3414$$

$$\cos 3.49 \approx -0.9399$$

Then the value of $\tan 3.49$ is

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan 3.49 \approx \frac{-0.3414}{-0.9399}$$

$$\tan 3.49 \approx 0.36$$

■ 6. Find the value of all six circular functions at $s = -4.5$, using a calculator to evaluate only sine and cosine.

Solution:

We'll use a calculator to evaluate the circular functions at -4.5, making sure the calculator is set to radian mode.

$$\sin(-4.5) \approx 0.9775$$

$$\cos(-4.5) \approx -0.2108$$

Now we can find $\tan(-4.5)$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan(-4.5) \approx \frac{0.9775}{-0.2108}$$

$$\tan(-4.5) \approx -4.6371$$

Then use the reciprocal identities to find cosecant, secant, and cotangent.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \csc(-4.5) \approx \frac{1}{0.9775} \approx 1.0230$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \sec(-4.5) \approx \frac{1}{-0.2108} \approx -4.7438$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \cot(-4.5) \approx \frac{1}{-4.6371} \approx -0.2157$$

LINEAR AND ANGULAR VELOCITY

■ 1. What is the angular velocity, in radians per second, of a wheel that rotates at a constant rate and sweeps out an angle of $33\pi/4$ radians in 0.6 seconds?

Solution:

The angular velocity is

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{\frac{33\pi}{4}}{0.6}$$

$\omega = 13.75\pi$ radians per second

■ 2. The wind turbine has a circular blade with diameter 154 meters that rotates at 18 rotations per minute. Find the angular velocity of the blade in degrees per second.

Solution:

Find angular velocity in revolutions per minute.

$$\omega = 18\ \frac{\text{rev}}{\text{min}}$$

Convert revolutions per minute to radians per second.

$$\omega = 18\ \frac{\text{rev}}{\text{min}} \times 2\pi\ \frac{\text{rad}}{\text{rev}} \times \frac{1\ \text{min}}{60\ \text{sec}}$$

$$\omega = \frac{36\pi}{60}\ \frac{\text{rad}}{\text{sec}}$$

$$\omega = \frac{3\pi}{5}\ \frac{\text{rad}}{\text{sec}}$$

Convert from radians per second to degrees per second.

$$\frac{3\pi}{5}\ \frac{\text{rad}}{\text{sec}} \times \frac{180^\circ}{\pi\ \text{rad}}$$

108° per second

■ 3. What is the angular velocity, in radians per second, of a wheel that rotates at a constant rate and sweeps out an angle of $21\pi/5$ radians in 0.85 seconds?

Solution:

The angular velocity is

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{\frac{21\pi}{5}}{0.85}$$

$\omega \approx 4.94\pi$ radians per second

4. Suppose a frisbee rotates at a constant rate of 105 revolutions per minute. What is its angular velocity ω in radians per second?

Solution:

To convert from revolutions per minute to radians per second, we'll set up a conversion equation.

$$\omega = \left(105 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)$$

$\omega = 3.5\pi$ radians per second

5. Find angular velocity, in radians per minute, of an object that rotates at a constant rate and sweeps out an angle of 985° in 8.4 seconds.

Solution:

We'll first convert the angle 985° to radians.

$$\theta = 985^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right)$$

$$\theta = \frac{197\pi}{36} \text{ radians}$$

Now we'll find angular velocity.

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{\frac{197\pi}{36} \text{ radians}}{8.4 \text{ seconds}}$$

$$\omega = \frac{197\pi}{36(8.4)} \text{ radians per second}$$

$$\omega \approx 0.651\pi \text{ radians per second}$$

Convert this value to radians per minute.

$$\omega \approx \frac{0.651\pi \text{ radians}}{1 \text{ second}} \left(\frac{60 \text{ seconds}}{1 \text{ minute}}\right)$$

$$\omega \approx 39.06\pi \text{ radians per minute}$$

■ 6. A cylinder with a 3.4 ft radius is rotating at 150 rpm. Give the angular velocity in rad/sec and in degrees per second.

Solution:

We already know angular velocity, we just need to convert the units.

$$\omega = 150\ \frac{\text{revolutions}}{\text{minute}}$$

$$\omega = \left(150\ \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)$$

$$\omega = \frac{150(2\pi)}{60}\ \text{radians per second}$$

$$\omega \approx 5\pi\ \text{radians per second}$$

Convert from radians to degrees.

$$\omega \approx 5\pi\ \frac{\text{rad}}{\text{sec}} \times \frac{180^\circ}{\pi\ \text{rad}}$$

$$\omega \approx 900^\circ\ \text{per second}$$

RELATING LINEAR AND ANGULAR VELOCITY

1. A saw has a circular blade with diameter 10 inches that rotates at $5{,}000$ revolutions per minute. Find the approximate linear velocity of the saw teeth (in ft/sec) as they contact the wood being cut.

Solution:

Find angular velocity.

$$\omega = 5{,}000 \frac{\text{rev}}{\text{min}}$$

$$\omega = 5{,}000 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{1 \text{ min}}{60 \text{ sec}}$$

$$\omega = 5{,}000 \times 2\pi \times \frac{1}{60} \times \frac{\text{rad}}{\text{sec}}$$

$$\omega = 166.67\pi \text{ rad/sec}$$

Because the diameter of the blade is 10 inches, its radius is 5 inches, so linear velocity is

$$v = \omega r$$

$$v = \left(166.67\pi \ \frac{\text{rad}}{\text{sec}}\right)\left(5 \text{ in } \frac{1 \text{ ft}}{12 \text{ in}}\right)$$

$$v = \left(166.67\pi \frac{\text{rad}}{\text{sec}}\right)\left(\frac{5\text{ ft}}{12}\right)$$

$$v \approx 218.2\text{ ft/sec}$$

■ 2. A car's tire has a radius of 12.5 inches and turns with an angular velocity of 84.5 radians per second. Find the approximate linear velocity of the car in miles per hour. (Use the fact that there are 12 inches in 1 foot, and approximately $5{,}280$ feet in 1 mile.)

Solution:

The radius is given as $r = 12.5$ in, and angular velocity is given as $\omega = 84.5$ rad/sec. We want the velocity of the car in miles per hour, so we'll start by finding inches traveled in an hour.

Substitute $\theta = \omega t$ into $s = r\theta$ to get

$$s = r\omega t$$

$$s = 12.5(84.5)(3{,}600)$$

$$s = 3{,}802{,}500\text{ inches}$$

The wheel travelled $3{,}802{,}500$ inches in one hour, or

$$\frac{3{,}802{,}500}{12} = 316{,}875\text{ ft}$$

or

$$\frac{316{,}875}{5{,}280} \approx 60 \text{ miles}$$

So the wheel was traveling at a speed of approximately 60 miles per hour.

3. A bicycle tire with a diameter of 26 inches turns with an angular velocity of 2 radians per seconds. Find the distance traveled in 5 minutes by a point on the tire.

Solution:

Because the diameter of the tire is 26 inches, its radius is 13 inches, so linear velocity is

$$v = \omega r$$

$$v = \left(2\ \frac{\text{rad}}{\text{sec}}\right)(13 \text{ in})$$

$$v = 26 \text{ in/sec}$$

Find distance.

$$s = vt$$

$$s = \left(26\ \frac{\text{in}}{\text{sec}}\right)(5 \text{ min})\left(60\ \frac{\text{sec}}{\text{min}}\right)$$

$$s = 7{,}800 \text{ in}$$

■ 4. A tire with a radius of 0.75 feet is rotating at 36 miles per hour. Find the angular velocity of a point on its rim, expressed in revolutions per minute.

Solution:

Since the radius is given in feet, we need to convert miles per hour to feet per hour.

$$v = \left(36 \ \frac{\text{mi}}{\text{hr}}\right)\left(5{,}280 \ \frac{\text{ft}}{\text{mi}}\right)$$

$$v = 190{,}080 \ \frac{\text{ft}}{\text{hr}}$$

Linear velocity is $v = \omega r$, so

$$\omega = \frac{v}{r}$$

$$\omega = \frac{190{,}080 \ \frac{\text{ft}}{\text{hr}}}{0.75 \text{ ft}}$$

$$\omega = 253{,}440 \ \frac{\text{rad}}{\text{hr}}$$

Now we need to convert radians per hour to revolutions per minute.

$$\omega = \left(253{,}440 \frac{\text{rad}}{\text{hr}}\right)\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{\text{hr}}{60 \text{ min}}\right)$$

$$\omega = \frac{253{,}440}{2\pi(60)} \text{ revolutions per minute}$$

If we say $\pi \approx 3.14$, we get

$$\omega \approx 672.6 \text{ revolutions per minute}$$

■ 5. The carousel at the county fair makes 3.5 revolutions per minute. The linear speed of a person riding inside the carousel is 2.9 ft/sec. How far is this person from the carousel's center?

Solution:

First we need to convert revolutions per minute to radians per second.

$$\omega = \left(3.5 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)$$

$$\omega = \frac{3.5(2\pi)}{60} \text{ radians per second}$$

$$\omega \approx 0.117\pi \text{ radians per second}$$

Linear velocity is $v = \omega r$, so

$$r = \frac{v}{\omega}$$

$$r = \frac{2.9}{0.117\pi}$$

$$r \approx 7.89 \text{ ft}$$

■ 6. A disk is spinning at 27 rpm. If a fly is sitting 9 cm from the center of the disk, what is the angular velocity of the fly in radians/sec? What is the speed of the fly in cm/sec? After 2 min, how far has the fly traveled?

Solution:

First we need to convert the angular velocity from revolutions per minute to radians per second.

$$\omega = 27 \frac{\text{rev}}{\text{min}}$$

$$\omega = \left(27 \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)$$

$$\omega = \frac{27(2\pi)}{60} \text{ radians per second}$$

$$\omega \approx 0.9\pi \text{ radians per second}$$

Now we need to find the linear velocity of the fly.

$$v = r\omega$$

$$v \approx (9\ \text{cm})\left(\frac{0.9\pi}{\text{sec}}\right)$$

$$v \approx 9(0.9\pi)\ \text{centimeters per second}$$

$$v \approx 25\ \text{centimeters per second}$$

To find how far the fly travels in 2 min, we use

$$s = vt$$

$$s = \left(25\ \frac{\text{cm}}{\text{sec}}\right)(2\ \text{min})\left(60\ \frac{\text{sec}}{\text{min}}\right)$$

$$s = 3{,}000\ \text{cm}$$

W W W . K R I S T A K I N G M A T H . C O M