Topic: Sum-to-product identities

Question: Express $\sin(8\theta) - \sin(20\theta)$ as a product.

Answer choices:

A $\quad -4\cos(7\theta)\sin(3\theta)$

B $\quad -\cos(14\theta)\sin(-6\theta)$

C $\quad -\cos(7\theta)\sin(3\theta)$

D $\quad -2\cos(14\theta)\sin(6\theta)$

Solution: D

Using the sum-to-product identity,

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

we can set $\theta = 8\theta$ and $\alpha = 20\theta$ and rewrite the product as

$$2\cos\left(\frac{8\theta+20\theta}{2}\right)\sin\left(\frac{8\theta-20\theta}{2}\right)$$

$$2\cos\left(\frac{28\theta}{2}\right)\sin\left(\frac{-12\theta}{2}\right)$$

$$2\cos(14\theta)\sin(-6\theta)$$

Using the even-odd identity $\sin(-\theta) = -\sin\theta$ to simplify the negative angle, we get

$$-2\cos(14\theta)\sin(6\theta)$$

Topic: Sum-to-product identities

Question: Find the exact value of the expression.

$$\cos\left(\frac{15\pi}{8}\right) + \cos\left(\frac{7\pi}{8}\right)$$

Answer choices:

A -2

B 1

C 0

D 2

Solution: C

Using the sum-to-product identity,

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

we can set $\theta = 15\pi/8$ and $\alpha = 7\pi/8$ and rewrite the product as

$$2\cos\left(\frac{\frac{15\pi}{8}+\frac{7\pi}{8}}{2}\right)\cos\left(\frac{\frac{15\pi}{8}-\frac{7\pi}{8}}{2}\right)$$

$$2\cos\left(\frac{\frac{22\pi}{8}}{2}\right)\cos\left(\frac{\frac{8\pi}{8}}{2}\right)$$

$$2\cos\left(\frac{\frac{11\pi}{4}}{2}\right)\cos\left(\frac{\pi}{2}\right)$$

$$2\cos\left(\frac{11\pi}{4}\left(\frac{1}{2}\right)\right)\cos\left(\frac{\pi}{2}\right)$$

$$2\cos\left(\frac{11\pi}{8}\right)\cos\left(\frac{\pi}{2}\right)$$

Because $\cos(\pi/2) = 0$, we get

$$2\cos\left(\frac{11\pi}{8}\right)(0) = 0$$

Topic: Sum-to-product identities

Question: Find the exact value of the expression.

$$4\sin 45^\circ + 6\cos 165^\circ + 4\sin 45^\circ - 6\cos 105^\circ$$

Answer choices:

A $5\sqrt{2}$

B $\dfrac{\sqrt{2}}{2}$

C $\sqrt{2}$

D $2+\sqrt{3}$

Solution: C

First we need to rewrite our expression as

$$4\sin 45^\circ + 4\sin 45^\circ + 6\cos 165^\circ - 6\cos 105^\circ$$

$$4(\sin 45^\circ + \sin 45^\circ) + 6(\cos 165^\circ - \cos 105^\circ)$$

Using the sum-to-product identity,

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

we can set $\theta = 45^\circ$ and $\alpha = 45^\circ$ and rewrite the product as

$$\sin 45^\circ + \sin 45^\circ = 2\sin\left(\frac{45^\circ + 45^\circ}{2}\right)\cos\left(\frac{45^\circ - 45^\circ}{2}\right)$$

$$\sin 45^\circ + \sin 45^\circ = 2\sin\left(\frac{90^\circ}{2}\right)\cos\left(\frac{0}{2}\right)$$

$$\sin 45^\circ + \sin 45^\circ = 2\sin 45^\circ \cos 0^\circ$$

$$\sin 45^\circ + \sin 45^\circ = 2\left(\frac{\sqrt{2}}{2}\right)(1)$$

$$\sin 45^\circ + \sin 45^\circ = \sqrt{2}$$

Now using the sum-to-product identity,

$$\cos\theta - \cos\alpha = -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

we can set $\theta = 165^\circ$ and $\alpha = 105^\circ$ and rewrite the product as

$$\cos 165^\circ - \cos 105^\circ = -2\sin\left(\frac{165^\circ + 105^\circ}{2}\right)\sin\left(\frac{165^\circ - 105^\circ}{2}\right)$$

$$\cos 165^\circ - \cos 105^\circ = -2\sin\left(\frac{270^\circ}{2}\right)\sin\left(\frac{60^\circ}{2}\right)$$

$$\cos 165^\circ - \cos 105^\circ = -2\sin 135^\circ \sin 30^\circ$$

$$\cos 165^\circ - \cos 105^\circ = -2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$\cos 165^\circ - \cos 105^\circ = -\frac{\sqrt{2}}{2}$$

Then the value of the original expression is

$$4(\sqrt{2}) + 6\left(-\frac{\sqrt{2}}{2}\right)$$

$$4\sqrt{2} - 3\sqrt{2}$$

$$\sqrt{2}$$