Topic: Intersection points of the polar curves

Question: Which of the following is a set of pairs of polar coordinates of all the points of intersection of the graphs of the polar equations $r = 2\cos\theta$ and $r = 3 - 4\cos\theta$?

Answer choices:

A $\left\{\left(\sqrt{3}, \frac{5\pi}{6}\right), (0,0), \left(\sqrt{3}, \frac{7\pi}{6}\right)\right\}$

B $\left\{\left(5, \frac{2\pi}{3}\right), (0,0), \left(5, \frac{4\pi}{3}\right)\right\}$

C $\left\{\left(1, \frac{\pi}{3}\right), (0,0), \left(1, \frac{5\pi}{3}\right)\right\}$

D $\left\{\left(3 - 2\sqrt{3}, \frac{\pi}{6}\right), (0,0), \left(3 - 2\sqrt{3}, \frac{11\pi}{6}\right)\right\}$

Solution: C

The graph of the polar equation $r = 2\cos\theta$ is a circle, and the graph of the polar equation $r = 3 - 4\cos\theta$ is a limacon (one with $a = 3$ and $b = 4$, so $a < b$ and this limacon has a loop).

Equating the expressions for r in the two polar equations gives

$$2\cos\theta = 3 - 4\cos\theta$$

$$6\cos\theta = 3$$

Dividing both sides by 6 gives

$$\cos\theta = \frac{1}{2}$$

The angles θ in the interval $[0,2\pi)$ that have a cosine of $1/2$ are $\pi/3$ and $5\pi/3$. For these two angles, the polar equation of the circle ($r = 2\cos\theta$) gives us the following values of r:

$$\theta = \frac{\pi}{3} \Longrightarrow r = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

$$\theta = \frac{5\pi}{3} \Longrightarrow r = 2\cos\left(\frac{5\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

The polar equation of the limacon $r = 3 - 4\cos\theta$ gives these same values of r for those two angles:

$$\theta = \frac{\pi}{3} \Longrightarrow r = 3 - 4\cos\left(\frac{\pi}{3}\right) = 3 - 4\left(\frac{1}{2}\right) = 3 - 2 = 1$$

$$\theta = \frac{5\pi}{3} \Longrightarrow r = 3 - 4\cos\left(\frac{5\pi}{3}\right) = 3 - 4\left(\frac{1}{2}\right) = 3 - 2 = 1$$

Thus the points with polar coordinates

$$(r, \theta) = \left(1, \frac{\pi}{3}\right) \quad \text{and} \quad (r, \theta) = \left(1, \frac{5\pi}{3}\right)$$

are points of intersection of the circle and the limacon. As always, we want to check to see if the pole (the origin) is a point of intersection.

We know a polar curve will pass through the origin when $r = 0$, so we'll set $r = 0$ in each curve. The circle $r = 2\cos\theta$ goes through the origin at

$$0 = 2\cos\theta$$

$$\cos\theta = 0$$

We know the cosine function is equal to 0 at $\pi/2$ and $3\pi/2$, so the circle $r = 2\cos\theta$ definitely passes through the origin. The limacon $r = 3 - 4\cos\theta$ goes through the origin at

$$0 = 3 - 4\cos\theta$$

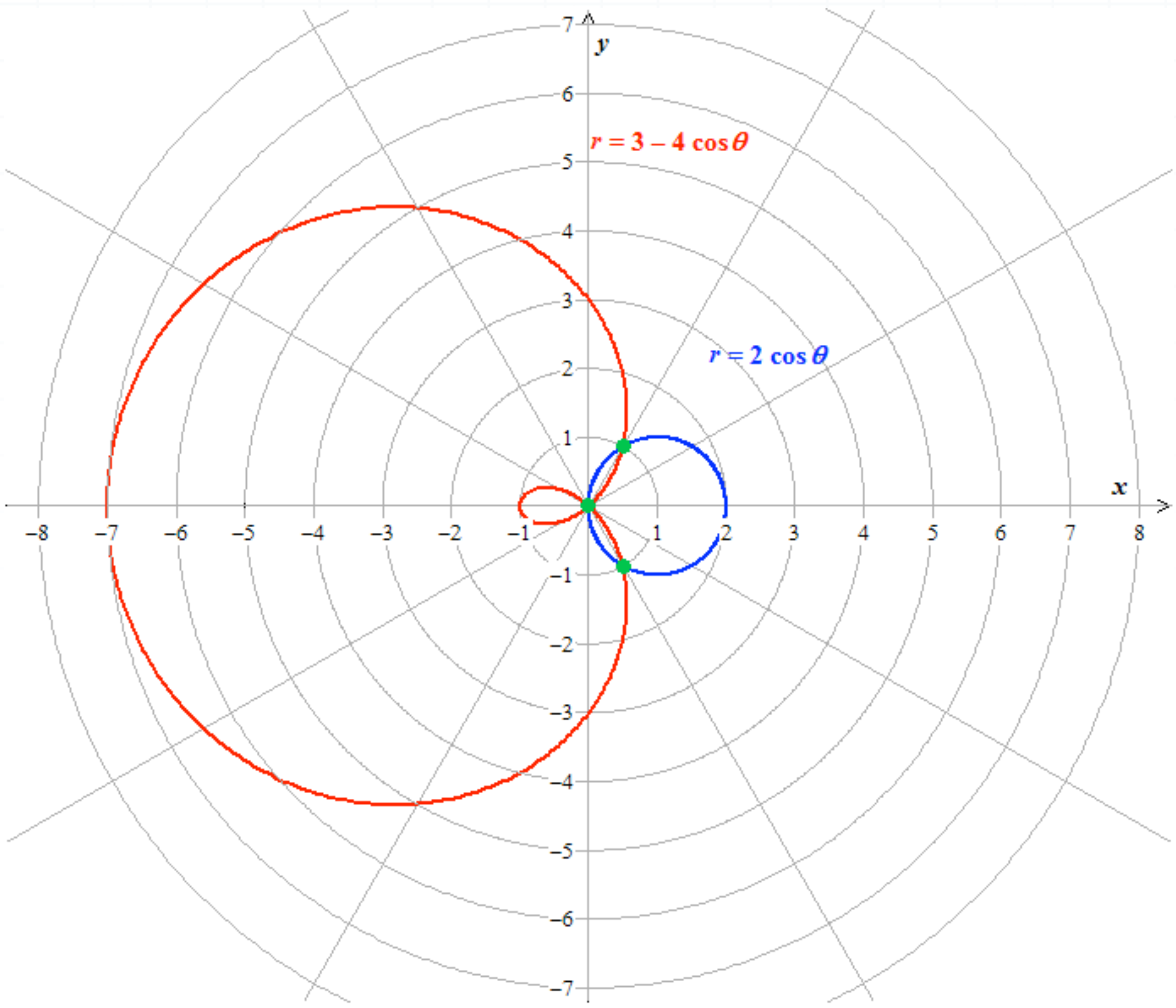
$$4\cos\theta = 3$$

$$\cos\theta = \frac{3}{4}$$

Because the values of the cosine function oscillate back and forth between 0 and 1, we know the value of the cosine function will be equal to $3/4$ at some point. We said that the limacon $r = 3 - 4\cos\theta$ will pass through the origin when $\cos\theta = 3/4$, and since we know that $\cos\theta$ will equal $3/4$ at some point, we know that the limacon also passes through the origin.

Since both curves pass through the origin, we can say that $(0,0)$ is also an intersection point.

We have found three points of intersection. To be sure we haven't missed any points of intersection, we'll graph the two polar equations.



From the graph, we see that there are indeed only three points of intersection.

Topic: Intersection points of the polar curves

Question: At how many points does the graph of the polar equation $r = 5\sin(3\theta)$ intersect the graph of the polar equation $r = 4$?

Answer choices:

A One

B Three

C Six

D Eight

Solution: C

The graph of the polar equation $r = 5\sin(3\theta)$ is a three-petal rose, and the graph of the polar equation $r = 4$ is the circle of radius 4 that has its center at the pole.

The points of the rose that are furthest from the pole are the tips of the petals, each of which is at a distance of 5 units from the pole. Also, every petal of the rose intersects the pole, hence every petal contains a point which is at a distance of 0 units from the pole.

Each "edge" of every petal of the rose $5\sin(3\theta)$ is a continuous curve, so for every real number x such that $0 < x < 5$, each edge of every petal of this rose contains a point which is at a distance of x units from the pole. This assures us that each edge of every petal of this rose contains a point which is at a distance of 4 units from the pole, hence a point which is on the circle $r = 4$.

Since the rose $r = 5\sin(3\theta)$ has three petals, and each petal has two edges, the number of points of intersection of the rose and the circle is $3 \times 2 = 6$.

To locate the points of intersection, we can equate the expressions for r in the polar equations of the two curves:

$$5\sin(3\theta) = 4$$

Dividing both sides of the equation $5\sin(3\theta) = 4$ by 5 gives

$$\sin(3\theta) = \frac{4}{5}$$

Since

$$0 < \frac{4}{5} < 1$$

there are points that satisfy this equation, and (because the value of $\sin(3\theta)$ is positive) all the angles 3θ that have a sine of $4/5$ must be in either the first quadrant or the fourth quadrant.

One such point has an angle coordinate θ such that 3θ is in the interval $(0,\pi/2)$, namely,

$$3\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

where $\sin^{-1}$ denotes the inverse sine function. Thus one point of intersection of the rose and the circle is

$$(r,\theta) = \left(4, \frac{\sin^{-1}\left(\frac{4}{5}\right)}{3}\right)$$

By the reference angle theorem, $\sin(\pi - 3\theta) = \sin(3\theta)$, so there is an angle θ such that 3θ is in the interval $(\pi/2,\pi)$ and $\sin(3\theta) = 4/5$:

$$3\theta = \pi - \sin^{-1}\left(\frac{4}{5}\right)$$

Thus another point of intersection is

$$(r,\theta) = \left(4, \frac{\pi - \sin^{-1}\left(\frac{4}{5}\right)}{3}\right)$$

The other four points of intersection correspond to angles θ with the following values of 3θ:

$$3\theta = \sin^{-1}\left(\frac{4}{5}\right) + 2\pi$$

$$3\theta = \left(\pi - \sin^{-1}\left(\frac{4}{5}\right)\right) + 2\pi$$

$$3\theta = \sin^{-1}\left(\frac{4}{5}\right) + 4\pi$$

$$3\theta = \left(\pi - \sin^{-1}\left(\frac{4}{5}\right)\right) + 4\pi$$

Thus these four points are

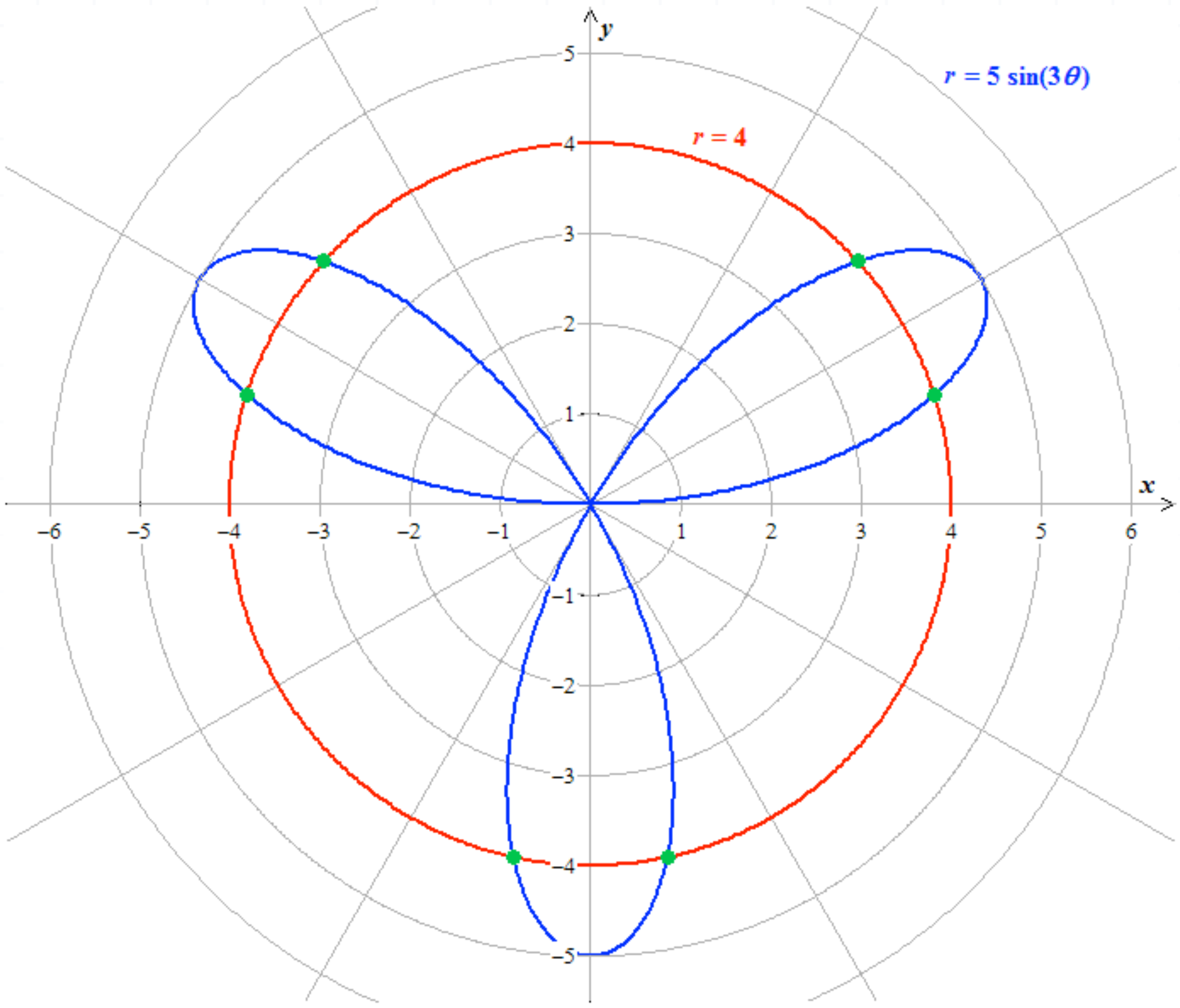
$$(r, \theta) = \left(4, \frac{\sin^{-1}\left(\frac{4}{5}\right) + 2\pi}{3}\right)$$

$$(r, \theta) = \left(4, \frac{3\pi - \sin^{-1}\left(\frac{4}{5}\right)}{3}\right)$$

$$(r, \theta) = \left(4, \frac{\sin^{-1}\left(\frac{4}{5}\right) + 4\pi}{3}\right)$$

$$(r,\theta)=\left(4,\frac{5\pi-\sin^{-1}\left(\frac{4}{5}\right)}{3}\right)$$

Now let's look at the graphs of these polar equations to confirm the points of intersection we've found and make sure that there are no others.



Topic: Intersection points of the polar curves

Question: At how many points do the graphs of the polar equations and intersect each other?

$$r = 2\sin(2\theta)$$

$$r^2 = 9\sin(2\theta)$$

Answer choices:

A Zero

B One

C Two

D Three

Solution: B

The graph of the polar equation $r = 2\sin(2\theta)$ is a four-petal rose, and the graph of the polar equation $r^2 = 9\sin(2\theta)$ is a lemniscate. As we know, every lemniscate passes through the pole, as does every rose, so the pole is a point of intersection of these two curves. Thus what we need to show is that they have no other points of intersection.

In this case, we can't directly equate the expressions for r, because the polar equation of the rose gives us an expression for r but the polar equation of the lemniscate gives us an expression for r^2. However, we can square both sides of the polar equation for the rose to get an expression for r^2:

$$r = 2\sin(2\theta) \Longrightarrow r^2 = 4\sin^2(2\theta)$$

Now we can equate the two expressions for r^2:

$$4\sin^2(2\theta) = 9\sin(2\theta)$$

$$4\sin^2(2\theta) - 9\sin(2\theta) = 0$$

Factoring out $\sin(2\theta)$ on the left-hand side, we obtain

$$\sin(2\theta)\left[4\sin(2\theta) - 9\right] = 0$$

Thus every solution of this equation would have to satisfy either $\sin(2\theta) = 0$ or $4\sin(2\theta) - 9 = 0$.

First, we'll consider the equation $\sin(2\theta) = 0$. Evaluating the expression for r^2 for the lemniscate, we obtain

$$\sin(2\theta) = 0 \implies r^2 = 9\sin(2\theta) = 9(0) = 0 \implies r = 0$$

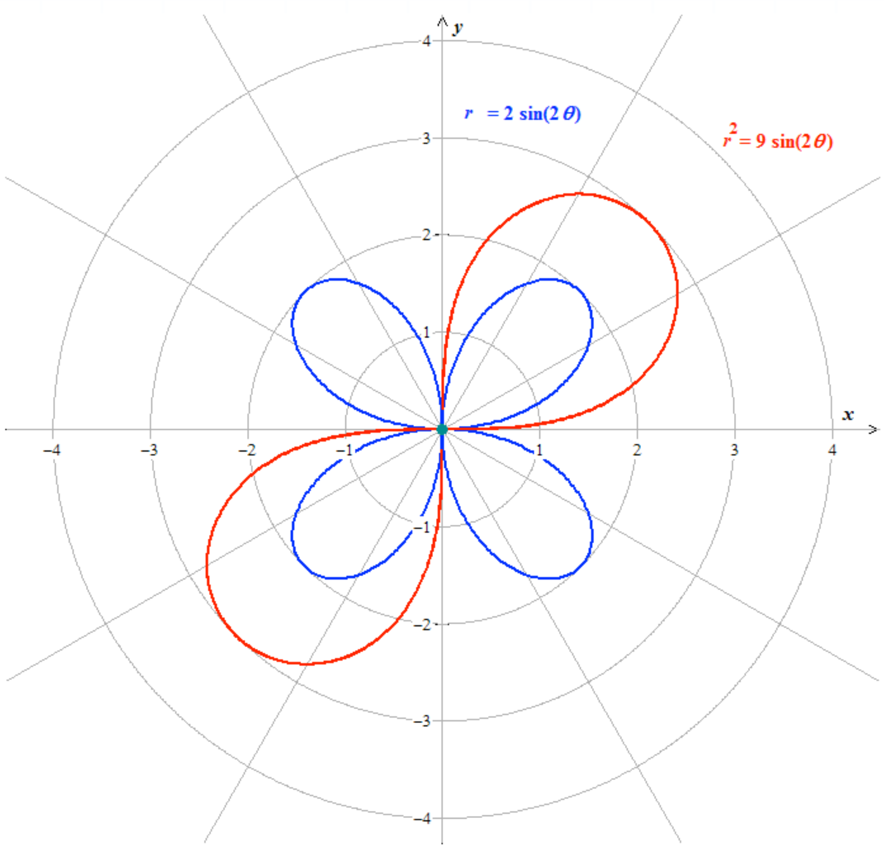
And evaluating the expression for r for the rose, we find that

$$\sin(2\theta) = 0 \implies r = 2\sin(2\theta) = 2(0) = 0$$

In both cases, we find that $r = 0$, so this is just the pole, which we have already stated to be a point of intersection of the two curves.

Next, we'll consider the equation $4\sin(2\theta) - 9 = 0$; equivalently, $\sin(2\theta) = 9/4$. Note that this equation has no solutions, because $9/4 > 1$, hence $\sin(2\theta)$ cannot be equal to $9/4$. (In fact, there is no angle whose sine is equal to $9/4$.)

What we have found is that the graphs of the polar equations $r = 2\sin(2\theta)$ and $r^2 = 9\sin(2\theta)$ have just one point of intersection, namely the pole. To check this, we'll take a look at the graphs of those equations.



Sure enough, the pole is indeed the only point of intersection.