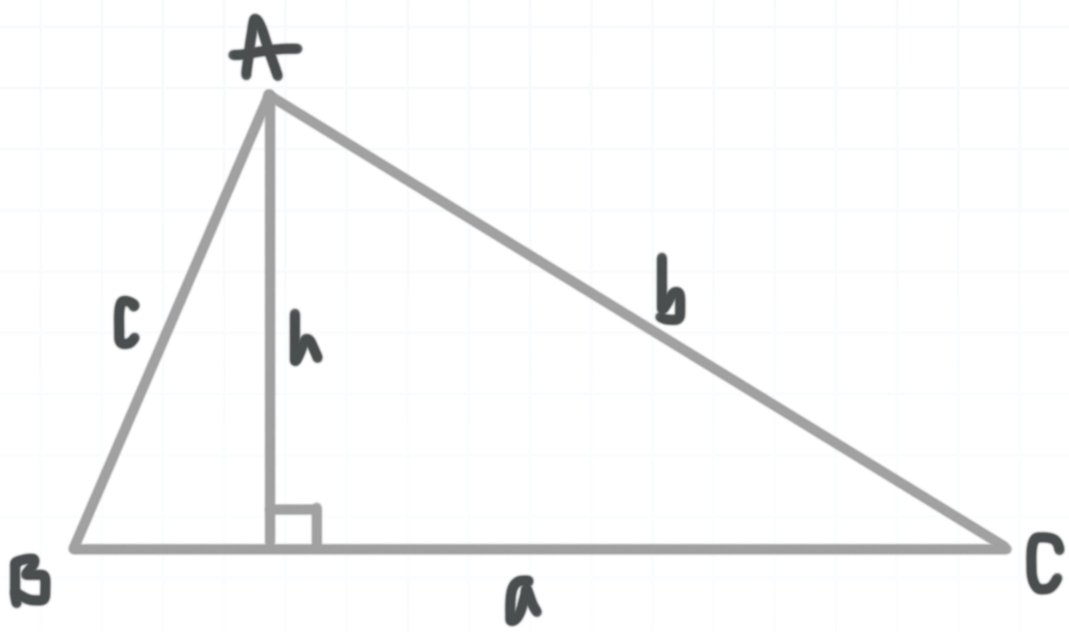
Area from the law of sines

From the law of sines, we can find formulas that give the area of an oblique triangle. Given any oblique triangle with angles A, B, and C, and side lengths a, b, and c,



The area of the triangle will be $A=(1/2)ah$, where a is the base and h is height (when the height is drawn perpendicular to a). By the definition of sine, the height is $h=b\sin C$. Therefore, we can rewrite the area formula as $A=(1/2)ab\sin C$. If we choose either of the other two sides as the triangle's "base," then we can write similar formulas for the area of the triangle in terms of $\sin A$ and $\sin B$:

$$\text{Area}=\frac{1}{2}ab\sin C \qquad \text{Area}=\frac{1}{2}ac\sin B \qquad \text{Area}=\frac{1}{2}bc\sin A$$

Notice that it doesn't matter which sides or angle we use. What matters is that we use two sides, and then the angle we choose should be the one opposite the side we didn't use.

That happens to always be the angle that's included between the two sides we chose. In other words, we always need to use two sides and their included angle.

These area formulas are sometimes called the **law of sines for the area of a triangle**, and we can apply it anytime we know the lengths of two sides of a triangle and the measure of the included angle.

Let's do an example where we apply the law of sines for the area of a triangle.

Example

Find the area of the triangle in which two of the sides have lengths 23 and 5 and the measure of the included angle is 38°.

Let $a = 23$ and $b = 5$, and let angle $C = 38^\circ$ be the included angle. Then we'll use the area formula that includes sides a and b and the angle C.

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}(23)(5)(\sin 38^\circ)$$

$$\text{Area} = \frac{115\sin 38^\circ}{2}$$

$$\text{Area} \approx 35.4$$

Realize here that we could have used any of the area formulas and the result would have come out the same. For instance, we could have used the second formula, setting $a = 23$, $b = 5$, and $B = 38^\circ$.

Notice also what happens to these area formulas when we use an angle of 90° as the included angle. Let's plug $C = 90^\circ$ into the first area formula, naming a and b as the side lengths adjacent to C.

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}ab\sin 90^\circ$$

$$\text{Area} = \frac{1}{2}ab(1)$$

$$\text{Area} = \frac{1}{2}ab$$

Using $C = 90^\circ$ means we're dealing with a right triangle. When that's the case, the area formula simplifies to $(1/2)ab$.

Because a and b are the sides around the right angle, we can think of b as the base of the triangle and a as the height of the triangle. So we could rewrite the area formula as

$$\text{Area} = \frac{1}{2}bh$$

And we recognize that this is the standard formula for the area of a right triangle. So it makes sense that this would be the result when we use $C = 90^\circ$ as the included angle.