Topic: Converting equations from polar to rectangular

Question: What is the solution set of the polar equation in terms of rectangular coordinates?

$$5r\sin\theta + 10r\cos\theta = 15$$

Answer choices:

A The solution set is the set of all points on the line with slope -5 and y-intercept $(0,10)$.

B The solution set is the set of all points on the line with slope -2 and y-intercept $(0,3)$.

C The solution set is the set of all points on the line with slope -10 and y-intercept $(0,5)$.

D The solution set is the set of all points on the line with slope -4 and y-intercept $(0,-3)$.

Solution: B

Using the conversion equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

we find that the given polar equation, $5r\sin\theta + 10r\cos\theta = 15$, becomes

$$5y + 10x = 15$$

Dividing both sides by 5 gives

$$y + 2x = 3$$

Solving for y, we find that $y = -2x + 3$. This is the equation of the line with slope -2 and y-intercept $(0,3)$.

We'll now show that every point on the line $y = -2x + 3$ is a solution of the polar equation

$$5r\sin\theta + 10r\cos\theta = 15$$

Using the those same conversion equations and starting from the equation $y = -2x + 3$, we get

$$r\sin\theta = -2r\cos\theta + 3$$

$$r\sin\theta + 2r\cos\theta = 3$$

Multiplying both sides by 5 yields the given polar equation

$$5r\sin\theta + 10r\cos\theta = 15$$

Topic: Converting equations from polar to rectangular

Question: What is the solution set of the polar equation in terms of rectangular coordinates?

$$r = -6\sin\theta$$

Answer choices:

A The solution set is the set of all points on the circle $(x-3)^2 + y^2 = 9$.

B The solution set is the set of all points on the circle $(x+3)^2 + y^2 = 9$.

C The solution set is the set of all points on the circle $x^2 + (y-3)^2 = 9$.

D The solution set is the set of all points on the circle $x^2 + (y+3)^2 = 9$.

Solution: D

We have the conversion equation

$$y = r\sin\theta$$

$$\sin\theta = \frac{y}{r}$$

and we can substitute this result into the given polar equation, $r = -6\sin\theta$.

$$r = -6\sin\theta$$

$$r = -6\left(\frac{y}{r}\right)$$

Multiplying both sides by r yields

$$r^2 = -6y$$

Using the general equation $r^2 = x^2 + y^2$, we get

$$x^2 + y^2 = -6y$$

$$x^2 + y^2 + 6y = 0$$

Now we'll complete the square on the y part of the expression $x^2 + y^2 + 6y$ (that is, on $y^2 + 6y$). To do that, we need to add 9, because

$$y^2 + 6y + 9 = (y+3)^2$$

If we add 9, we also have to subtract 9, so

$$y^2 + 6y = (y^2 + 6y) + 9 - 9 = (y^2 + 6y + 9) - 9 = (y+3)^2 - 9$$

Substituting this expression for $y^2 + 6y$ in the equation $x^2 + y^2 + 6y = 0$, we get

$$x^2 + (y + 3)^2 - 9 = 0$$

Adding 9 to both sides:

$$x^2 + (y + 3)^2 = 9$$

$$(x - 0)^2 + [y - (-3)]^2 = 3^2$$

This is the equation of the circle whose center has rectangular coordinates $(0, -3)$ and whose radius is 3.

It would be easy to check, by working backwards, that every point on that circle is indeed a solution of the given polar equation, $r = -6\sin\theta$.

Topic: Converting equations from polar to rectangular

Question: What is the solution set of the polar equation in terms of rectangular coordinates?

$$r^2 \sin(2\theta) = -8$$

Answer choices:

A The solution set is the set of all points on the hyperbola $y = -4/x$.

B The solution set is the set of all points on the hyperbola $y = 4/x$.

C The solution set is the set of all points on the parabola $y = x^2/4$.

D The solution set is the set of all points on the circle $x^2 + (y - 1/2)^2 = 4$.

Solution: A

By the double-angle formula for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Substituting this result into the given polar equation, $r^2\sin(2\theta) = -8$, we obtain

$$r^2(2\sin\theta\cos\theta) = -8$$

Using the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}$$

Substituting these results gives

$$r^2\left[2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)\right] = -8$$

If we cancel the r^2 in the numerator against the two factors of r in the denominator, we're left with the equation $2(yx) = -8$. Solving this equation for y yields

$$y = -\frac{4}{x}$$

This is the equation of a hyperbola. Because of the minus sign on the right-hand side of this equation, x and y must be of opposite sign, so this hyperbola resides in the second and fourth quadrants.

If we were to start from the equation $y = -4/x$ and work backwards, we would easily find that every point on this hyperbola is a solution of the given polar equation, $r^2 \sin(2\theta) = -8$.