Graph the polar curve, limacon

Now you're going to learn about a polar curve known as a limaçon. (That word is French. The diacritical mark below the letter c is known as a cedilla, and it indicates that that c is pronounced as an s.)

A limaçon is the graph of a polar equation that has one of the following forms, where a and b are positive numbers and $a \neq b$:

$$r = a + b\cos\theta$$

$$r = a - b\cos\theta$$

$$r = a + b\sin\theta$$

$$r = a - b\sin\theta$$

Recall that a cardioid is the graph of a polar equation that has one of the following forms where a is a positive number:

$$r = a(1 + \cos\theta) = a + a\cos\theta$$

$$r = a(1 - \cos\theta) = a - a\cos\theta$$

$$r = a(1 + \sin\theta) = a + a\sin\theta$$

$$r = a(1 - \sin\theta) = a - a\sin\theta$$

Thus we could think of a cardioid as a special kind of limaçon (one with $a = b$).

Let's take a look at the range of values of r for a limaçon which is the graph of the polar equation $r = a + b\cos\theta$. Well, we know that

$$-1 \leq \cos\theta \leq 1$$

Since b is positive, we can multiply through by b and retain the direction of the inequalities, so

$$-b \leq b\cos\theta \leq b$$

Adding a throughout, we get

$$a - b \leq a + b\cos\theta \leq a + b$$

That is,

$$a - b \leq r \leq a + b$$

Thus r ranges from $a - b$ to $a + b$. You should convince yourself that this is also the range of a limaçon which is the graph of a polar equation of any of the other three types listed earlier.

Since $a + b$ is positive, for every limaçon there is exactly one angle θ in the interval $[0,2\pi)$ at which $r = a + b$.

$$r = a + b\cos\theta: \qquad r = a + b \iff \cos\theta = 1 \iff \theta = 0$$

$$r = a - b\cos\theta: \qquad r = a + b \iff \cos\theta = -1 \iff \theta = \pi$$

$$r = a + b\sin\theta: \qquad r = a + b \iff \sin\theta = 1 \iff \theta = \frac{\pi}{2}$$

$$r = a - b\sin\theta: \qquad r = a + b \iff \sin\theta = -1 \iff \theta = \frac{3\pi}{2}$$

If $a > b$ (hence the lower limit on the value of r is $a - b$, which is positive), there is no angle θ at which $r = 0$ (hence the limaçon doesn't pass through the pole) and no angle θ at which r is negative.

If $a < b$ (hence the lower limit on the value of r is $a - b$, which is negative), there is exactly one angle θ in the interval $[0,2\pi)$ at which $r = a - b$.

$$r = a + b\cos\theta: \qquad r = a - b \iff \cos\theta = -1 \iff \theta = \pi$$

$$r = a - b\cos\theta: \qquad r = a - b \iff \cos\theta = 1 \iff \theta = 0$$

$$r = a + b\sin\theta: \qquad r = a - b \iff \sin\theta = -1 \iff \theta = \frac{3\pi}{2}$$

$$r = a - b\sin\theta: \qquad r = a - b \iff \sin\theta = 1 \iff \theta = \frac{\pi}{2}$$

If $a < b$, there are two angles θ in the interval $[0,2\pi)$ at which $r = 0$ (hence the limaçon passes through the pole twice), and the limaçon has a loop whose endpoints are the points that correspond to those two angles.

To determine the two angles at which $r = 0$, we'll set r to 0 and solve for θ. For example, if $r = a + b\cos\theta$, then

$$r = 0 \iff a + b\cos\theta = 0 \iff \cos\theta = -\frac{a}{b}$$

Since a and b are both positive (and $a < b$), we see that

$$-1 < -\frac{a}{b} < 0$$

hence that $-1 < \cos\theta < 0$ for the two points with $r = 0$.

The cosine function is negative (and greater than -1) in the second and third quadrants. Therefore, there is exactly one angle in the second quadrant (which we'll denote by θ_1) with a cosine of $-a/b$ (and hence $r = 0$), and exactly one angle in the third quadrant (which we'll denote by θ_2) with a cosine of $-a/b$ (and hence $r = 0$).

Recall that the inverse cosine function, $\cos^{-1}(\theta)$, is defined on the interval $[0,\pi]$. Therefore,

$$\theta_1 = \cos^{-1}\left(-\frac{a}{b}\right)$$

By the reference angle theorem,

$$\pi - \theta_1 = \theta_2 - \pi$$

Solving this equation for θ_2, we find that

$$\theta_2 = 2\pi - \theta_1 = 2\pi - \cos^{-1}\left(-\frac{a}{b}\right)$$

For limaçons with $a < b$, let's determine the subinterval(s) of the interval $[0,2\pi)$ on which $r = a + b\cos\theta$ is negative:

$$r = a + b\cos\theta < 0 \iff b\cos\theta < -a \iff \cos\theta < -\frac{a}{b}$$

Given what we've already found in regard to the angles θ_1 and θ_2 at which $r = 0$ (and the fact that $-a/b$ is negative), this shows that (within the interval $[0,2\pi)$) the value of r is negative at the angles θ at which the value of the cosine function is "more negative" than it is at θ_1 and θ_2. That happens precisely on the interval (θ_1, θ_2). Moreover, the loop of the limaçon is “inside” the rest of the limaçon, so the loop corresponds to the angles θ with the smallest values of r - in particular, to those at which $r \leq 0$.

Similar considerations enable us to determine the angles θ_1 and θ_2 in the interval $[0,2\pi)$ at which $\theta_1 < \theta_2$ and $r = 0$, and the subinterval(s) of $[0,2\pi)$ on which r is negative, for the other three types of limaçons with $a < b$. These characteristics of limaçons are summarized in the table below. For all four types of limaçons with $a < b$, the loop corresponds to the angles θ at which $r \leq 0$.

Polar equation	θ_1		θ_2		Interval(s) on which $r < 0$
	Measure	Quadrant	Measure	Quadrant	
$r = a + b\cos\theta$	$\cos^{-1}\left(-\frac{a}{b}\right)$	II	$2\pi - \cos^{-1}\left(-\frac{a}{b}\right)$	III	(θ_1, θ_2)
$r = a - b\cos\theta$	$\cos^{-1}\left(\frac{a}{b}\right)$	I	$2\pi - \cos^{-1}\left(\frac{a}{b}\right)$	IV	$(0, \theta_1), (\theta_2, 2\pi)$
$a + b\sin\theta = 0$	$\pi + \sin^{-1}\left(\frac{a}{b}\right)$	III	$2\pi - \sin^{-1}\left(\frac{a}{b}\right)$	IV	(θ_1, θ_2)
$a - b\sin\theta = 0$	$\sin^{-1}\left(\frac{a}{b}\right)$	I	$\pi - \sin^{-1}\left(\frac{a}{b}\right)$	II	(θ_1, θ_2)

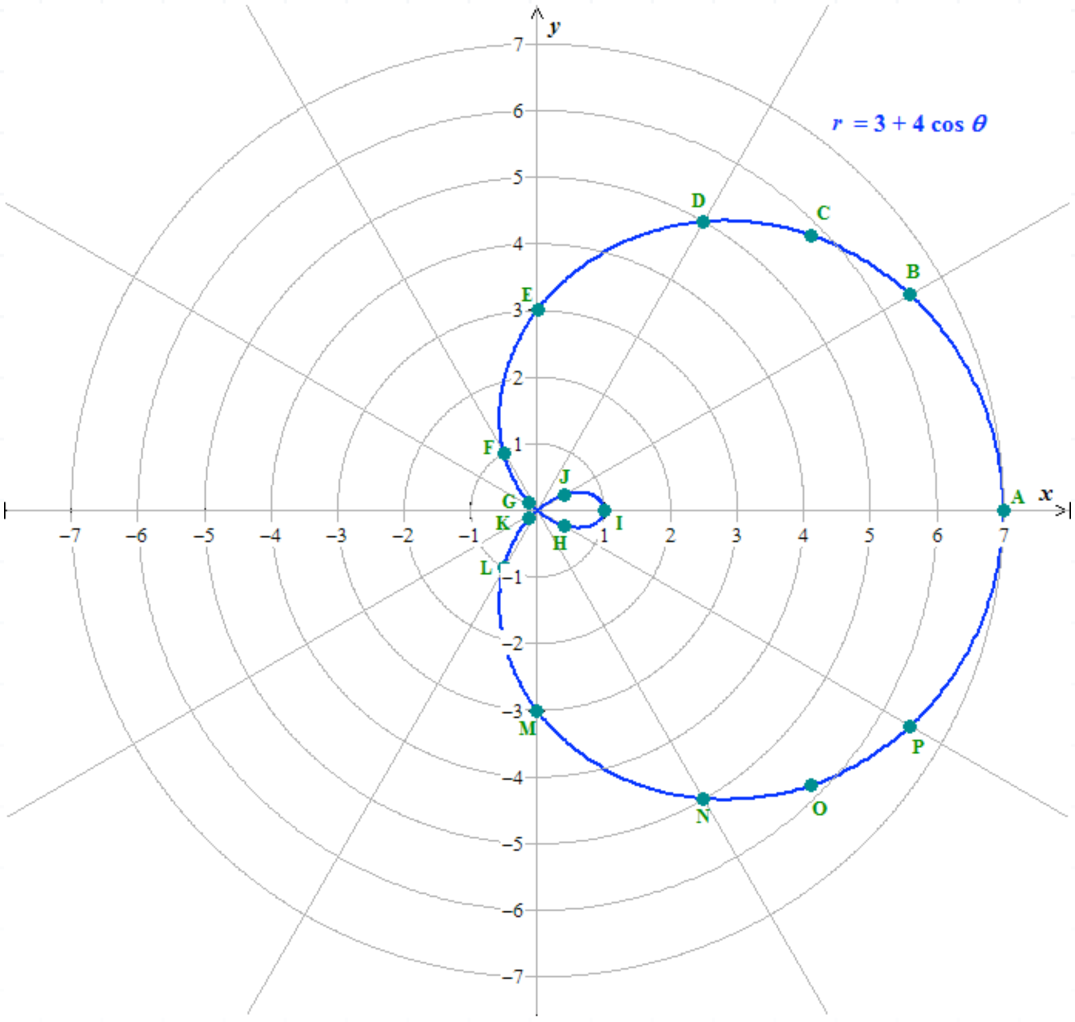
Example

Graph the limaçon $r = 3 + 4\cos\theta$.

In the following table, the values of $\cos\theta$ and $r = 3 + 4\cos\theta$ are shown for a number of angles θ in the interval $[0,2\pi)$. In the table, we also give one pair of polar coordinates, (r, θ), for points where the equation $r = 3 + 4\cos\theta$ yields a positive value of r, and we give two pairs of polar coordinates, (r, θ) and $(-r, \theta + \pi)$, where that equation yields a negative value of r.

Point	θ	$\cos\theta$	$r=3+4\cos\theta$	Polar coordinates (r,θ)	Polar coordinates $(-r,\theta+\pi)$ r negative
A	0	1	7	$(7,0)$	
B	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$3+2\sqrt{3}$	$\left(3+2\sqrt{3}, \frac{\pi}{6}\right)$	
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3+2\sqrt{2}$	$\left(3+2\sqrt{2}, \frac{\pi}{4}\right)$	
D	$\frac{\pi}{3}$	$\frac{1}{2}$	5	$\left(5, \frac{\pi}{3}\right)$	
E	$\frac{\pi}{2}$	0	3	$\left(3, \frac{\pi}{2}\right)$	
F	$\frac{2\pi}{3}$	$-\frac{1}{2}$	1	$\left(1, \frac{2\pi}{3}\right)$	
G	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3-2\sqrt{2}$	$\left(3-2\sqrt{2}, \frac{3\pi}{4}\right)$	
H	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$3-2\sqrt{3}$	$\left(3-2\sqrt{3}, \frac{5\pi}{6}\right)$	$\left(-3+2\sqrt{3}, \frac{11\pi}{6}\right)$
I	π	-1	-1	$(-1,\pi)$	$(1,2\pi)^*$
J	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$3-2\sqrt{3}$	$\left(3-2\sqrt{3}, \frac{7\pi}{6}\right)$	$\left(-3+2\sqrt{3}, \frac{13\pi}{6}\right)^\dagger$
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3-2\sqrt{2}$	$\left(3-2\sqrt{2}, \frac{5\pi}{4}\right)$	
L	$\frac{4\pi}{3}$	$-\frac{1}{2}$	1	$\left(1, \frac{4\pi}{3}\right)$	
M	$\frac{3\pi}{2}$	0	3	$\left(3, \frac{3\pi}{2}\right)$	
N	$\frac{5\pi}{3}$	$\frac{1}{2}$	5	$\left(5, \frac{5\pi}{3}\right)$	
O	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3+2\sqrt{2}$	$\left(3+2\sqrt{2}, \frac{7\pi}{4}\right)$	
P	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$3+2\sqrt{3}$	$\left(3+2\sqrt{3}, \frac{11\pi}{6}\right)$	

*Still another pair of polar coordinates of point I is $(1,0)$.
†Still another pair of polar coordinates of point J is $\left(-3+2\sqrt{3}, \frac{\pi}{6}\right)$.



As derived earlier (in regard to all limaçons with $a < b$), the value of $r = a + b\cos\theta$ is equal to 0 at exactly two angles in the interval $[0,2\pi)$, namely θ_1 and θ_2, where θ_1 is in the second quadrant, θ_2 is in the third quadrant, and they satisfy the equations

$$\theta_1 = \cos^{-1}\left(-\frac{a}{b}\right)$$

and

$$\theta_2 = 2\pi - \theta_1$$

respectively.

In this case, $a = 3$ and $b = 4$, so with the aid of a calculator (set to radians!), we find that

$$\theta_1 = \cos^{-1}\left(-\frac{3}{4}\right) \approx 2.42 \text{ (radians)} = \left(\frac{2.42}{\pi}\right)\pi \text{ (radians)} \approx 0.77\pi$$

and

$$\theta_2 = 2\pi - \theta_1 \approx 2\pi - 0.77\pi = 1.23\pi$$

Also, we found that the value of r is negative at all θ in the interval (θ_1, θ_2), and positive at all θ in the intervals $[0,\theta_1)$ and $(\theta_2,2\pi)$, and that the points on the loop of the limaçon are those with $r \leq 0$ (equivalently, the points that correspond to angles θ in the interval $[\theta_1, \theta_2]$).

The only points in our table that have a negative value of r are H, I, and J, which are the only points in our table that (on our graph) lie on the loop of the limaçon $r = 3 + 4\cos\theta$. For points H and J (the two points that are on the loop and closest to the pole),

$$r = 3 - 2\sqrt{3} \approx 3 - 2(1.732) = 3 - 3.464 = -0.464 < 0$$

Point H corresponds to $\theta = 5\pi/6$, and point J corresponds to $\theta = 7\pi/6$. Also, point I corresponds to $\theta = \pi$, so

$$r = 3 + 4\cos\pi = 3 + 4(-1) = 3 - 4 = -1 < 0$$

For the points G and K in our table (the two points that on our graph are just outside the loop of the limaçon and closest to the pole),

$$r = 3 - 2\sqrt{2} \approx 3 - 2(1.414) = 3 - 2.828 = 0.172 > 0$$

Points G and K correspond to angles $\theta = 3\pi/4$ and $\theta = 5\pi/4$, respectively.

Summarizing these results in terms of θ and the sign of r:

	G		Pole		H		I		J		Pole		K
$\theta \Longrightarrow$	$\frac{3\pi}{4}$	$<$	$\theta_1 \approx 0.77\pi$	$<$	$\frac{5\pi}{6}$	$<$	π	$<$	$\frac{7\pi}{6}$	$<$	$\theta_2 \approx 1.23\pi$	$<$	$\frac{5\pi}{4}$
Sign of $r \Longrightarrow$	$+$		0		$-$		$-$		$-$		0		$+$

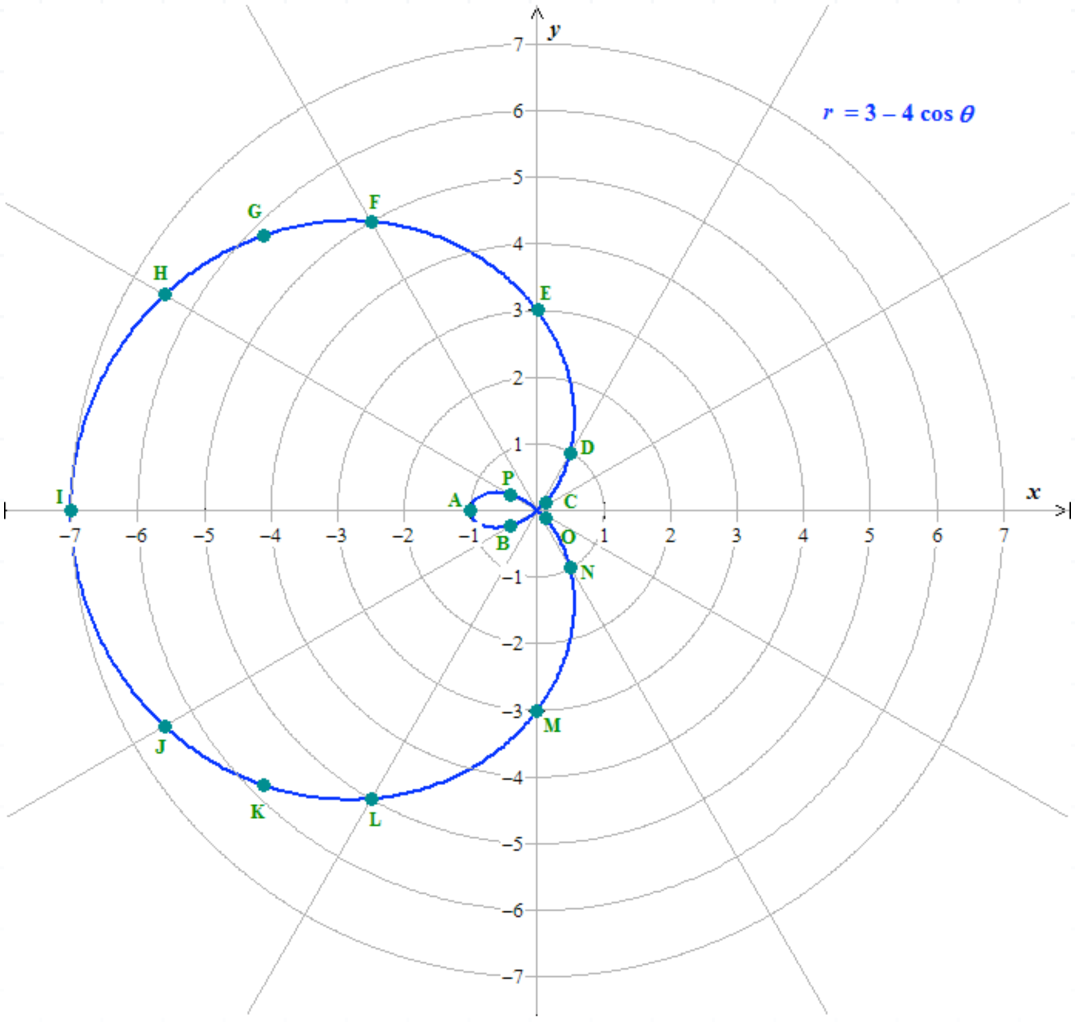
Notice that the graph of the polar equation $r = 3 + 4\cos\theta$ is symmetric with respect to the horizontal axis. Just as for all "cosine" cardioids, this is true for all "cosine" limaçons. Another similarity to "cosine" cardioids is that (as a curve) the limaçon $a - b\cos\theta$ is a reflection in the vertical axis of the limaçon $a + b\cos\theta$. This is true regardless of whether $a > b$ or $a < b$.

Example

Graph the limaçon $r = 3 - 4\cos\theta$.

Point	θ	$\cos\theta$	$r = 3 - 4\cos\theta$	Polar coordinates (r, θ)	Polar coordinates $(-r, \theta + \pi)$ r negative
A	0	1	-1	$(-1, 0)$	$(1, \pi)$
B	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$3 - 2\sqrt{3}$	$\left(3 - 2\sqrt{3}, \frac{\pi}{6}\right)$	$\left(-3 + 2\sqrt{3}, \frac{7\pi}{6}\right)$
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3 - 2\sqrt{2}$	$\left(3 - 2\sqrt{2}, \frac{\pi}{4}\right)$	
D	$\frac{\pi}{3}$	$\frac{1}{2}$	1	$\left(1, \frac{\pi}{3}\right)$	
E	$\frac{\pi}{2}$	0	3	$\left(3, \frac{\pi}{2}\right)$	
F	$\frac{2\pi}{3}$	$-\frac{1}{2}$	5	$\left(5, \frac{2\pi}{3}\right)$	
G	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3 + 2\sqrt{2}$	$\left(3 + 2\sqrt{2}, \frac{3\pi}{4}\right)$	
H	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$3 + 2\sqrt{3}$	$\left(3 + 2\sqrt{3}, \frac{5\pi}{6}\right)$	
I	π	-1	7	$(7, \pi)$	
J	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$3 + 2\sqrt{3}$	$\left(3 + 2\sqrt{3}, \frac{7\pi}{6}\right)$	
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3 + 2\sqrt{2}$	$\left(3 + 2\sqrt{2}, \frac{5\pi}{4}\right)$	
L	$\frac{4\pi}{3}$	$-\frac{1}{2}$	5	$\left(5, \frac{4\pi}{3}\right)$	
M	$\frac{3\pi}{2}$	0	3	$\left(3, \frac{3\pi}{2}\right)$	
N	$\frac{5\pi}{3}$	$\frac{1}{2}$	1	$\left(1, \frac{5\pi}{3}\right)$	
O	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3 - 2\sqrt{2}$	$\left(3 - 2\sqrt{2}, \frac{7\pi}{4}\right)$	
P	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$3 - 2\sqrt{3}$	$\left(3 - 2\sqrt{3}, \frac{11\pi}{6}\right)$	$\left(-3 + 2\sqrt{3}, \frac{17\pi}{6}\right)$*

*Still another pair of polar coordinates of point P is $\left(-3 + 2\sqrt{3}, \frac{5\pi}{6}\right)$.



Notice that (as a curve) the limaçon $r = 3 - 4\cos\theta$ is indeed a reflection of the limaçon $r = 3 + 4\cos\theta$ in the vertical axis, and that both curves are symmetric with respect to the horizontal axis. Also, in both curves, the maximum value of r is

$$a + b = 3 + 4 = 7$$

and the minimum value of r is

$$a - b = 3 - 4 = -1$$

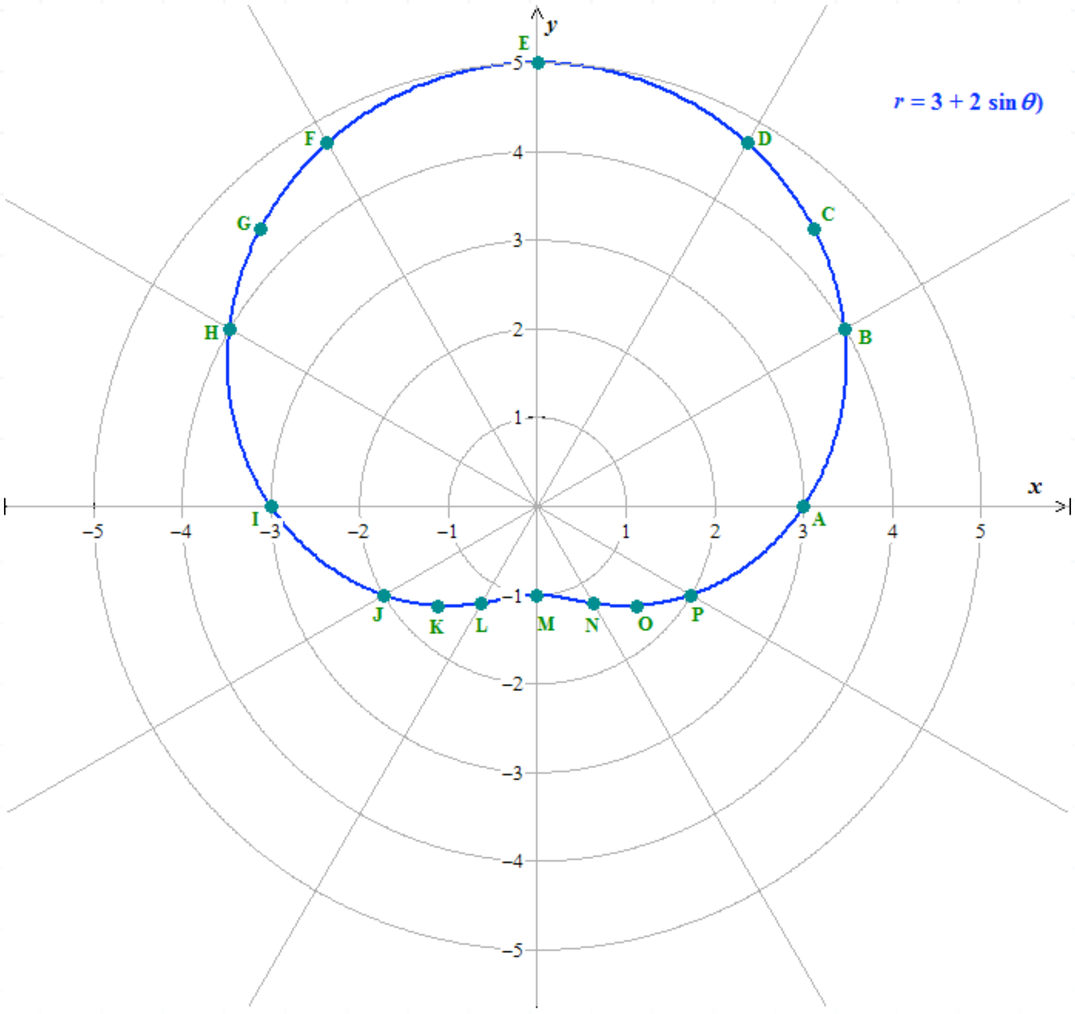
In the limaçon $r = 3 + 4\cos\theta$, the maximum and minimum values of r occur at $\theta = 0$ and $\theta = \pi$, respectively, whereas just the opposite is true of the locations of the maximum and minimum values of r in the limaçon $r = 3 - 4\cos\theta$.

Now we'll consider limaçons with $a > b$. In particular, we'll look at a pair of "sine" limaçons, to get a flavor for what such curves look like and how they differ from limaçons with $a < b$ and from "cosine" limaçons.

Example

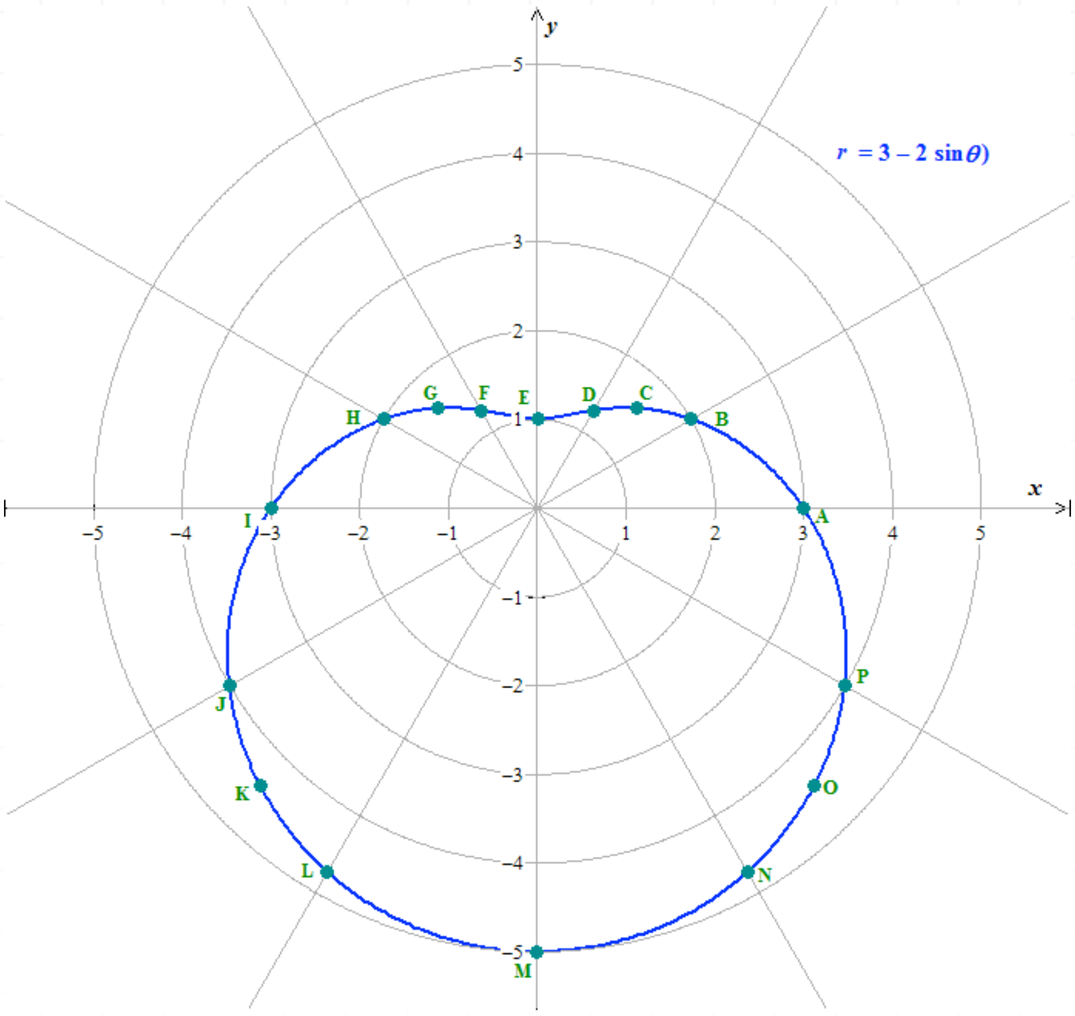
Graph the limaçons $r = 3 + 2\sin\theta$ and $r = 3 - 2\sin\theta$.

Point	θ	$\sin\theta$	$r=3+2\sin\theta$	Polar coordinates (r,θ)
A	0	0	3	$(3,0)$
B	$\frac{\pi}{6}$	$\frac{1}{2}$	4	$\left(4,\frac{\pi}{6}\right)$
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3+\sqrt{2}$	$\left(3+\sqrt{2},\frac{\pi}{4}\right)$
D	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$3+\sqrt{3}$	$\left(3+\sqrt{3},\frac{\pi}{3}\right)$
E	$\frac{\pi}{2}$	1	5	$\left(5,\frac{\pi}{2}\right)$
F	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$3+\sqrt{3}$	$\left(3+\sqrt{3},\frac{2\pi}{3}\right)$
G	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3+\sqrt{2}$	$\left(3+\sqrt{2},\frac{3\pi}{4}\right)$
H	$\frac{5\pi}{6}$	$\frac{1}{2}$	4	$\left(4,\frac{5\pi}{6}\right)$
I	π	0	3	$(3,\pi)$
J	$\frac{7\pi}{6}$	$-\frac{1}{2}$	2	$\left(2,\frac{7\pi}{6}\right)$
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3-\sqrt{2}$	$\left(3-\sqrt{2},\frac{5\pi}{4}\right)$
L	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$3-\sqrt{3}$	$\left(3-\sqrt{3},\frac{4\pi}{3}\right)$
M	$\frac{3\pi}{2}$	-1	1	$\left(1,\frac{3\pi}{2}\right)$
N	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$3-\sqrt{3}$	$\left(3-\sqrt{3},\frac{5\pi}{3}\right)$
O	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3-\sqrt{2}$	$\left(3-\sqrt{2},\frac{7\pi}{4}\right)$
P	$\frac{11\pi}{6}$	$-\frac{1}{2}$	2	$\left(2,\frac{11\pi}{6}\right)$



$r = 3 + 2\sin\theta$)
y
x
A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P

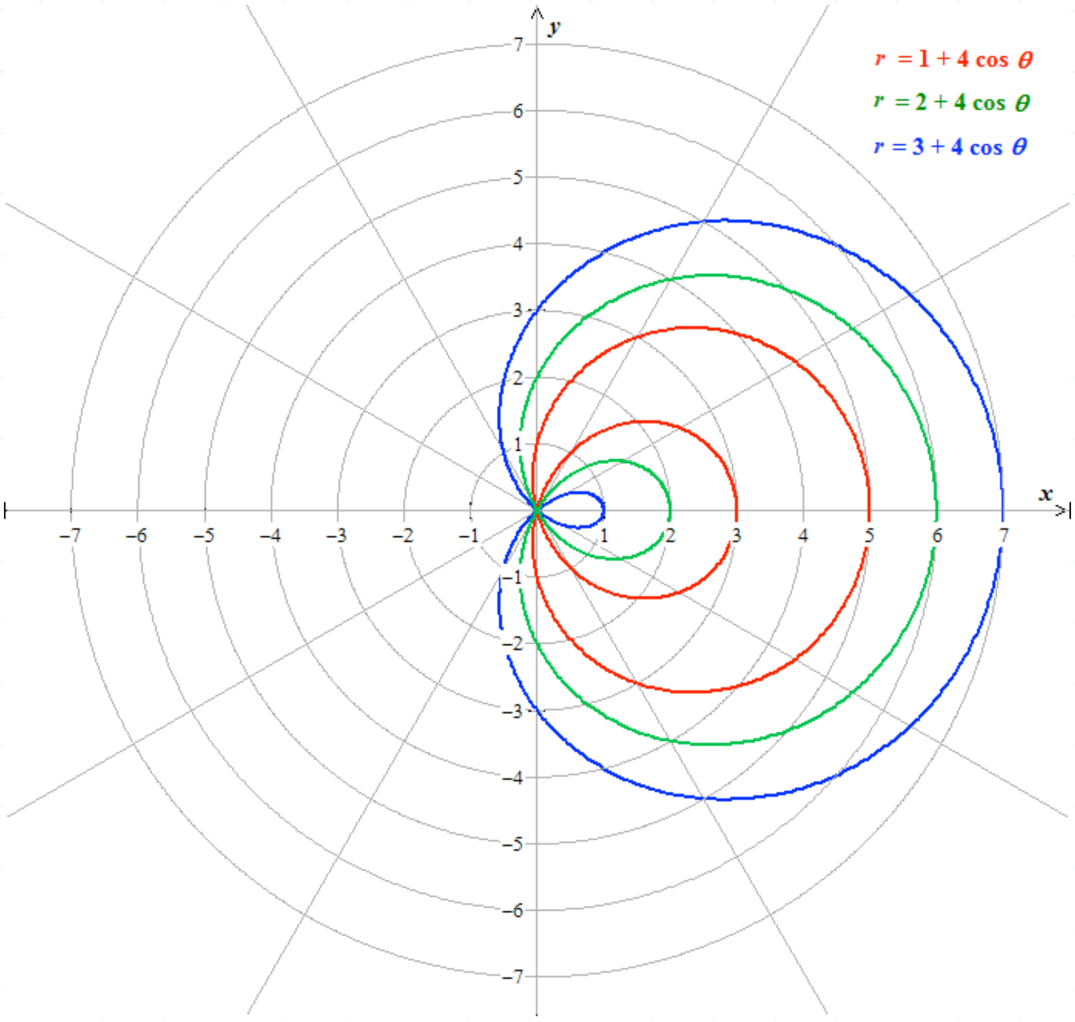
Point	θ	$\sin\theta$	$r = 3 - 2\sin\theta$	Polar coordinates (r, θ)
A	0	0	3	$(3, 0)$
B	$\frac{\pi}{6}$	$\frac{1}{2}$	2	$\left(2, \frac{\pi}{6}\right)$
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3 - \sqrt{2}$	$\left(3 - \sqrt{2}, \frac{\pi}{4}\right)$
D	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$3 - \sqrt{3}$	$\left(3 - \sqrt{3}, \frac{\pi}{3}\right)$
E	$\frac{\pi}{2}$	1	1	$\left(1, \frac{\pi}{2}\right)$
F	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$3 - \sqrt{3}$	$\left(3 - \sqrt{3}, \frac{2\pi}{3}\right)$
G	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3 - \sqrt{2}$	$\left(3 - \sqrt{2}, \frac{3\pi}{4}\right)$
H	$\frac{5\pi}{6}$	$\frac{1}{2}$	2	$\left(2, \frac{5\pi}{6}\right)$
I	π	0	3	$(3, \pi)$
J	$\frac{7\pi}{6}$	$-\frac{1}{2}$	4	$\left(4, \frac{7\pi}{6}\right)$
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3 + \sqrt{2}$	$\left(3 + \sqrt{2}, \frac{5\pi}{4}\right)$
L	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$3 + \sqrt{3}$	$\left(3 + \sqrt{3}, \frac{4\pi}{3}\right)$
M	$\frac{3\pi}{2}$	-1	5	$\left(5, \frac{3\pi}{2}\right)$
N	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$3 + \sqrt{3}$	$\left(3 + \sqrt{3}, \frac{5\pi}{3}\right)$
O	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3 + \sqrt{2}$	$\left(3 + \sqrt{2}, \frac{7\pi}{4}\right)$
P	$\frac{11\pi}{6}$	$-\frac{1}{2}$	4	$\left(4, \frac{11\pi}{6}\right)$



As you may have expected (given your knowledge of “sine” cardioids), all “sine” limaçons are symmetric with respect to the vertical axis, and the limaçon $r = a - b\sin\theta$ is the reflection in the horizontal axis of the limaçon $r = a + b\sin\theta$.

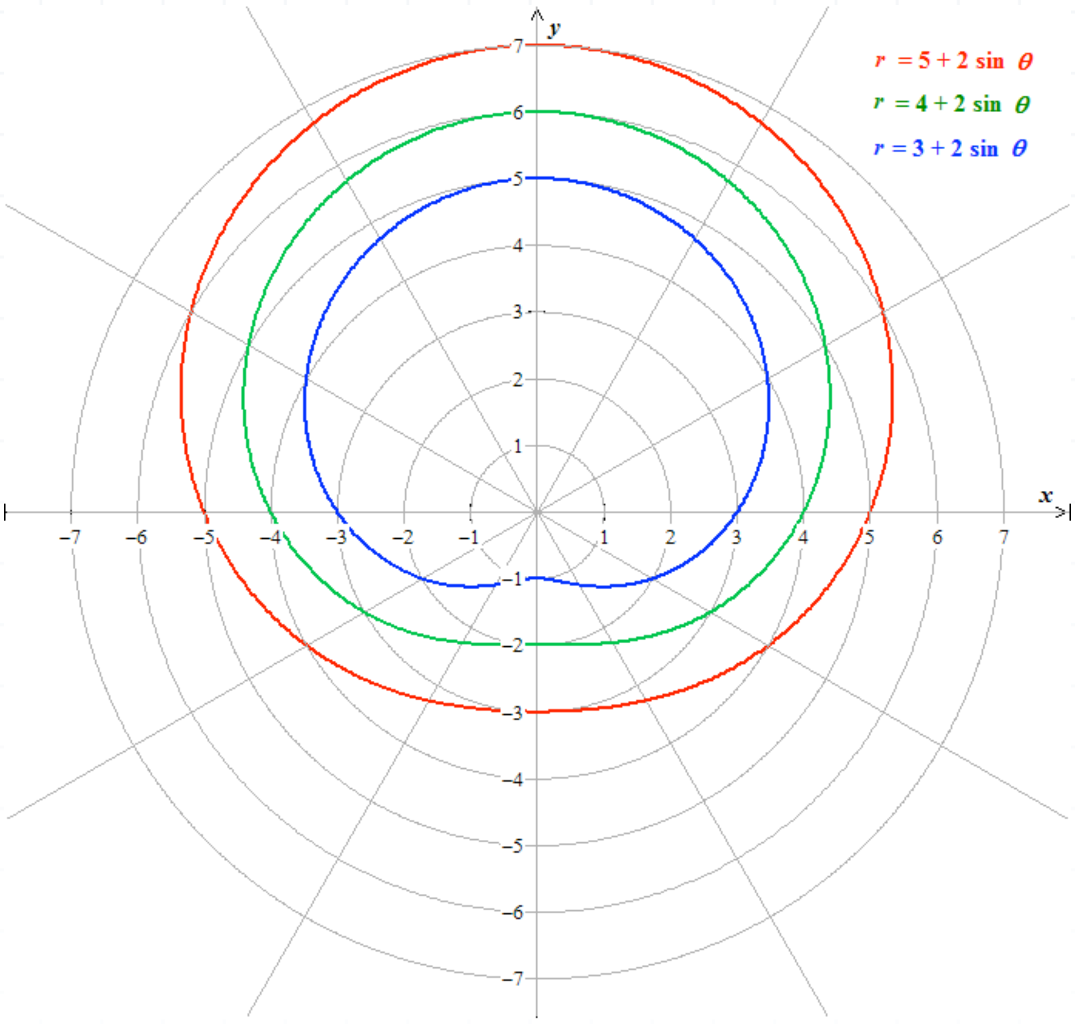
Perhaps the most noticeable feature of limaçons with $a > b$ is that they have no loop. Instead, they have what could perhaps best be described as a depression.

You might be wondering what happens if we have limaçons with the same value of b and different values of the ratio a/b. In particular, let's graph several “cosine” limaçons with $b = 4$ and different values of $a < b$ (namely, $a = 1$, $a = 2$, and $a = 3$).



Notice that as a increases from 1 to 2, and then to 3, the size of the loop decreases. In fact, it would decrease to a single point if a got all the way up to b. Therefore, if we have a sequence of limaçons with the same value of b and larger and larger values of $a < b$, the limaçons look more and more like a "cosine" cardioid.

Now let's graph several "sine" limaçons with $b = 2$ and different values of $a > b$ (namely, $a = 5$, $a = 4$, and $a = 3$).



As a decreases from 5 to 4, and then to 3, these limaçons have depressions that are more and more pronounced, and the limaçons look more and more like a "sine" cardioid as a gets closer and closer to b.

We can conclude from these considerations that limaçons for which the ratio a/b gets closer and closer to 1 look more and more like cardioids, which is what you might expect because the distinguishing characteristic of a cardioid is that $a = b$.