Topic: Converting equations from rectangular to polar

Question: Which of the following equations do we get if we convert the rectangular equation $y = -6x + 5$ to polar coordinates?

Answer choices:

A $r(\sin\theta - 6\cos\theta) = 5$

B $\tan\theta = -\frac{5}{6}$

C $r(\sin\theta + 6\cos\theta) = 5$

D $\tan\theta + 6 = -\frac{5}{\cos\theta}$

Solution: C

Making the replacements $r\cos\theta$ for x, and $r\sin\theta$ for y, in the given rectangular equation, we get

$$r\sin\theta = -6r\cos\theta + 5$$

Adding $6r\cos\theta$ to both sides of this equation, we obtain

$$r\sin\theta + 6r\cos\theta = 5$$

Factoring out an r on the left-hand side gives

$$r(\sin\theta + 6\cos\theta) = 5$$

This shows that answer choice C is correct.

Since the four answer choices all have somewhat different forms, let's check to be sure that none of the other three answer choices is correct.

Answer choice A cannot be correct, because it agrees with answer choice C on everything but the sign of the term that includes $\cos\theta$.

We know that, in general,

$$\frac{y}{x} = \tan\theta$$

Thus the equation given in answer choice B is equivalent to

$$\frac{y}{x} = -\frac{5}{6}$$

This equation can be written as $y = -(5/6)x$, which is the equation of a line that has a slope of $-5/6$. The given rectangular equation, $y = -6x + 5$, is the equation of a line that has a slope of -6. Thus answer choice B cannot be correct.

Finally, suppose answer choice D is correct. Note that one of the points on the line given by the original (rectangular) equation $y = -6x + 5$ has rectangular coordinates $(x, y) = (1, -1)$, since

$$-6(1) + 5 = -6 + 5 = -1$$

One pair of polar coordinates of that point is

$$(r, \theta) = \left(\sqrt{2}, \frac{7\pi}{4}\right)$$

For that point, answer choice D yields

$$\tan\left(\frac{7\pi}{4}\right) + 6 = -\frac{5}{\cos\left(\frac{7\pi}{4}\right)}$$

Since $\tan(7\pi/4) = -1$ and $\cos(7\pi/4) = 1/\sqrt{2}$, that becomes

$$-1 + 6 = -\frac{5}{\left(\frac{1}{\sqrt{2}}\right)}$$

Simplifying, we get

$$5 = -5\sqrt{2}$$

which is absurd. Therefore, answer choice D is incorrect.

Topic: Converting equations from rectangular to polar

Question: Which of the following equations do we get if we convert the rectangular equation $x^2 + (y - 7)^2 = 49$ to polar coordinates?

Answer choices:

A $r = -14 \sin \theta$

B $r^2 = 14 \sin \theta$

C $r = 14 \cos \theta$

D $r = 14 \sin \theta$

Solution: D

Making the replacements $r\cos\theta$ for x, and $r\sin\theta$ for y, we have

$$(r\cos\theta)^2+(r\sin\theta-7)^2=49$$

Expanding the left-hand side (by performing the indicated squaring) yields

$$r^2\cos^2\theta+(r^2\sin^2\theta-14r\sin\theta+49)=49$$

Regrouping some terms, we obtain

$$(r^2\cos^2\theta+r^2\sin^2\theta)-14r\sin\theta+49=49$$

We know that, in general,

$$r^2\cos^2\theta+r^2\sin^2\theta=r^2$$

Substituting this result, we find that

$$r^2-14r\sin\theta+49=49$$

Subtracting 49 from both sides:

$$r^2-14r\sin\theta=0$$

Factoring the left-hand side gives

$$r(r-14\sin\theta)=0$$

This implies that $r=0$ or $r=14\sin\theta$.

The only point for which $r=0$ is the pole, and for any angle θ, $(0,\theta)$ is a pair of polar coordinates for the pole. Thus if we let $\theta=0$, then the pole also satisfies the equation $r = 14 \sin \theta$. Thus the given (rectangular) equation converts to the polar equation $r = 14 \sin \theta$.

Topic: Converting equations from rectangular to polar

Question: Which of the following equations do we get if we convert the equation to polar coordinates?

$$x = \frac{1}{y+3}$$

Answer choices:

A $\quad r\cos^2\theta + 6r^2\sin\theta = -4$

B $\quad r^2\sin(2\theta) + 6r\cos\theta = 2$

C $\quad r^2\cos(2\theta) + 6r\sin\theta = -6$

D $\quad r\sin^2\theta - 6r\sin\theta = 2$

Solution: B

Replacing x with $r\cos\theta$, and y with $r\sin\theta$, we have

$$r\cos\theta = \frac{1}{r\sin\theta + 3}$$

Multiplying both sides of this equation by $r\sin\theta + 3$, we obtain

$$(r\cos\theta)(r\sin\theta + 3) = 1$$

Doing the indicated multiplication on the left-hand side:

$$r^2\cos\theta\sin\theta + 3r\cos\theta = 1$$

By the double-angle formula for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

Thus

$$\cos\theta\sin\theta = \sin\theta\cos\theta = \left(\frac{1}{2}\right)(2\sin\theta\cos\theta) = \left(\frac{1}{2}\right)\sin(2\theta)$$

Substituting this result, we find that

$$r^2\left(\frac{1}{2}\right)\sin(2\theta) + 3r\cos\theta = 1$$

Multiplying both sides of this equation by 2:

$$r^2\sin(2\theta) + 6r\cos\theta = 2$$

This is answer choice B.

Let's check to be sure that none of the other answer choices is correct. Using our original equation,

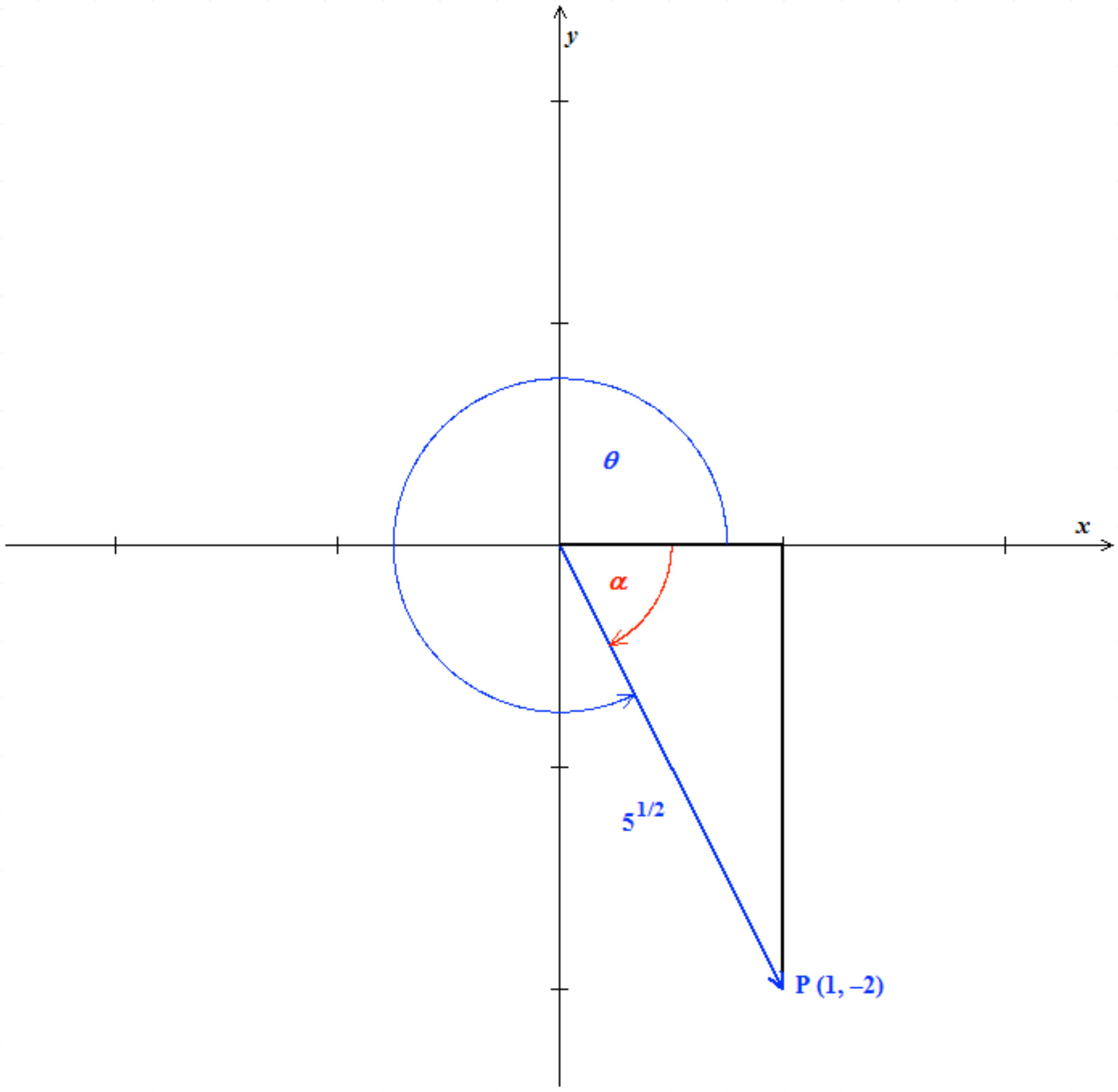
$$x = \frac{1}{y+3}$$

let's set x to 1 and solve for the corresponding value of y:

$$1 = \frac{1}{y+3}$$

Multiplying both sides by $y+3$, we get $1(y+3) = 1$, which gives $y+3=1$, so $y = 1-3 = -2$. This point, with rectangular coordinates $(x, y) = (1, -2)$, is in the fourth quadrant.

We'll construct a right triangle, with one leg along the positive horizontal axis (and of length $x = 1$), the other leg parallel to the negative vertical axis (and of length $|y| = |-2| = 2$), and the acute interior angle α as shown in the figure.



Then the hypotenuse is of length

$$\sqrt{(1)^2+(2)^2}=\sqrt{1+4}=\sqrt{5}$$

Since our point is in the fourth quadrant, we can use an angle θ in the interval $[3\pi/2, 2\pi)$, so

$$\theta = 2\pi - \alpha$$

and r (the distance of our point $(x, y) = (1, -2)$ from the pole) is equal to the length of the hypotenuse (namely, $\sqrt{5}$).

By our tried-and-true formulas for right triangles:

$$\cos\alpha = \frac{\text{length of leg adjacent to } \alpha}{\text{length of hypotenuse}} = \frac{1}{\sqrt{5}}$$

$$\sin\alpha = \frac{\text{length of leg opposite } \alpha}{\text{length of hypotenuse}} = \frac{2}{\sqrt{5}}$$

By the difference identity for cosine,

$$\cos(2\pi - \alpha) = \cos(2\pi)\cos\alpha + \sin(2\pi)\sin\alpha$$

$$\cos(2\pi - \alpha) = (1)\cos\alpha + (0)\sin\alpha$$

$$\cos(2\pi - \alpha) = \cos\alpha$$

$$\cos(2\pi - \alpha) = \frac{1}{\sqrt{5}}$$

And by the difference identity for sine,

$$\sin(2\pi - \alpha) = \sin(2\pi)\cos\alpha - \cos(2\pi)\sin\alpha$$

$$\sin(2\pi - \alpha) = (0)\cos\alpha - (1)\sin\alpha$$

$$\sin(2\pi - \alpha) = -\sin\alpha$$

$$\sin(2\pi - \alpha) = -\frac{2}{\sqrt{5}}$$

Therefore,

$$\cos\theta = \cos(2\pi - \alpha) = \cos\alpha = \frac{1}{\sqrt{5}}$$

and

$$\sin\theta = \sin(2\pi - \alpha) = -\sin\alpha = -\frac{2}{\sqrt{5}}$$

Let's plug these values of r, $\cos\theta$, and $\sin\theta$ for our chosen point into the equations given in the other three answer choices, and we'll see that none of those equations is satisfied.

The left-hand side of the equation in answer choice A is

$$r\cos^2\theta + 6r^2\sin\theta$$

Substituting the indicated values of r, $\cos\theta$, and $\sin\theta$, we get

$$r\cos^2\theta + 6r^2\sin\theta = \sqrt{5}\left(\frac{1}{\sqrt{5}}\right)^2 + 6(\sqrt{5})^2\left(-\frac{2}{\sqrt{5}}\right)$$

$$r\cos^2\theta + 6r^2\sin\theta = \sqrt{5}\left(\frac{1}{5}\right) + 6(5)\left(-\frac{2}{\sqrt{5}}\right)$$

$$r\cos^2\theta + 6r^2\sin\theta = \sqrt{5}\left(\frac{1}{5}\right) - 12\sqrt{5}$$

$$r\cos^2\theta + 6r^2\sin\theta = \sqrt{5}\left(\frac{1}{5} - 12\right)$$

This is not equal to the number on the right-hand side of the equation in answer choice A (namely, -4).

The left-hand side of the equation in answer choice C is

$$r^2\cos(2\theta) + 6r\sin\theta$$

By the double-angle formula for cosine,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{1}{\sqrt{5}}\right)^2 - \left(-\frac{2}{\sqrt{5}}\right)^2 = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$$

Substituting the indicated values of r, $\cos\theta$, and $\sin\theta$, the left-hand side of that equation gives

$$r^2\cos(2\theta) + 6r\sin\theta = \left(\sqrt{5}\right)^2\left(-\frac{3}{5}\right) + 6\sqrt{5}\left(-\frac{2}{\sqrt{5}}\right)$$

$$r^2\cos(2\theta) + 6r\sin\theta = 5\left(-\frac{3}{5}\right) - 12$$

$$r^2\cos(2\theta) + 6r\sin\theta = -3 - 12$$

$$r^2\cos(2\theta) + 6r\sin\theta = -15$$

This is not equal to the number on the right-hand side of the equation in answer choice C (namely, -6).

The left-hand side of the equation in answer choice D is

$$r\sin^2\theta - 6r\sin\theta$$

Substituting the indicated values of r, $\cos\theta$, and $\sin\theta$, the left-hand side of that equation gives

$$r\sin^2\theta - 6r\sin\theta = \sqrt{5}\left(-\frac{2}{\sqrt{5}}\right)^2 - 6\sqrt{5}\left(-\frac{2}{\sqrt{5}}\right)$$

$$r\sin^2\theta - 6r\sin\theta = \sqrt{5}\left(\frac{4}{5}\right) + 12$$

This is not equal to the number on the right-hand side of the equation in answer choice D (namely, 2).

We have shown that answer choices A, C, and D are incorrect. For the sake of completeness, let's show that the equation in answer choice B is satisfied by the values of r, $\cos\theta$, and $\sin\theta$ that correspond to our point $(x, y) = (1, -2)$.

The left-hand side of the equation in answer choice B is

$$r^2\sin(2\theta) + 6r\cos\theta$$

By the double-angle formula for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = -\frac{4}{\left(\sqrt{5}\right)^2} = -\frac{4}{5}$$

Therefore,

$$r^2\sin(2\theta) + 6r\cos\theta = \left(\sqrt{5}\right)^2\left(-\frac{4}{5}\right) + 6\sqrt{5}\left(\frac{1}{\sqrt{5}}\right)$$

$$r^2\sin(2\theta) + 6r\cos\theta = 5\left(-\frac{4}{5}\right) + 6$$

$$r^2\sin(2\theta) + 6r\cos\theta = -4 + 6$$

$$r^2\sin(2\theta) + 6r\cos\theta = 2$$

This is indeed the number on the right-hand side of the equation in answer choice B.