Negative angles and angles more than one rotation

In the last lesson, we looked at how to find the values of all six trig functions at some positive angle along the unit circle.

In this lesson, we want to look at the values of trig functions for negative angles in the unit circle, and at the values of trig functions for angles outside of one full rotation, either positive or negative.

The key here is to realize that the value of a trig function is the same for any set of coterminal angles. For instance, $\pi/4$ and $9\pi/4$ are coterminal angles. Which means the value of sine is the same at both angles, the value of cosine is the same at both angles, the value of tangent is the same at both angles, etc.

$$\sin\frac{\pi}{4}=\sin\frac{9\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\csc\frac{\pi}{4}=\csc\frac{9\pi}{4}=\sqrt{2}$$

$$\cos\frac{\pi}{4}=\cos\frac{9\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\sec\frac{\pi}{4}=\sec\frac{9\pi}{4}=\sqrt{2}$$

$$\tan\frac{\pi}{4}=\tan\frac{9\pi}{4}=1$$

$$\cot\frac{\pi}{4}=\cot\frac{9\pi}{4}=1$$

Because the value of the trig functions are equivalent for coterminal angles, if we're given a negative angle, or any angle (positive or negative) outside of one full rotation, we just need to find the positive angle inside $[0^\circ,360^\circ)$ in degrees or $[0,2\pi)$ in radians that's coterminal with it, and then find the values of the trig functions at that coterminal angle.

Trig functions at negative angles

Let's think about the angle -30°. If we start along the positive horizontal axis in the unit circle, and rotate 30° in the *clockwise* direction (negative direction), we'll arrive at 330°. At that angle,

$$\sin(-30^\circ) = \sin(330^\circ) = -\frac{1}{2}$$

$$\cos(-30^\circ) = \cos(330^\circ) = \frac{\sqrt{3}}{2}$$

Given the sine and cosine of $\theta = -30^\circ$, we can use reciprocal identities to find cosecant and secant of the angle,

$$\sin(-30^\circ) = \sin(330^\circ) = -\frac{1}{2} \qquad \csc(-30^\circ) = \csc(330^\circ) = -2$$

$$\cos(-30^\circ) = \cos(330^\circ) = \frac{\sqrt{3}}{2} \qquad \sec(-30^\circ) = \sec(330^\circ) = \frac{2\sqrt{3}}{3}$$

and then use the quotient identities to find tangent and cotangent.

$$\sin(-30^\circ) = \sin(330^\circ) = -\frac{1}{2} \qquad \csc(-30^\circ) = \csc(330^\circ) = -2$$

$$\cos(-30^\circ) = \cos(330^\circ) = \frac{\sqrt{3}}{2} \qquad \sec(-30^\circ) = \sec(330^\circ) = \frac{2\sqrt{3}}{3}$$

$$\tan(-30^\circ) = \tan(330^\circ) = -\frac{\sqrt{3}}{3} \qquad \cot(-30^\circ) = \cot(330^\circ) = -\sqrt{3}$$

What this shows is that the values of the six trig functions are equivalent at -30° and 330°. And of course, the reason this is true is because -30° and 330° are coterminal, and they're coterminal because they differ by 360°.

What we want to be able to do now is what we just did with -30°: we want to be able to take any angle *outside* of $[0^\circ,360^\circ)$ or $[0,2\pi)$ and find its coterminal angle that falls *inside* $[0^\circ,360^\circ)$ or $[0,2\pi)$. That way, we'll be able to use our unit circle to find values of the six trig functions at that coterminal angle.

And as we've seen by converting -30° to 330°, we can find an angle within $[0^\circ,360^\circ)$ or $[0,2\pi)$ that's coterminal with a negative angle simply by adding multiples of 360° or 2π to the negative angle until we have an angle within $[0^\circ,360^\circ)$ or $[0,2\pi)$.

For instance, given -300°, we can add 360° to find a coterminal angle.

$$-300^\circ + 360^\circ$$

$$60^\circ$$

Or given -420°, we'll add 360° twice to find a coterminal angle. Adding 360° only once doesn't put us within $[0^\circ,360^\circ)$, which is why we add 360° twice.

$$-420^\circ + 2(360^\circ)$$

$$-420^\circ + 720^\circ$$

$$300^\circ$$

Let's do another example with an angle that's more than one full rotation.

Example

Find the angle in the interval $[0^\circ, 360^\circ)$ that's coterminal with -539°.

Let $\theta = -539^\circ$, and let α be the angle that lies in the interval $[0^\circ, 360^\circ)$ and is coterminal with θ. To find α, let's add 360° to $\theta = -539^\circ$ until we get to an angle that lies in the interval $[0^\circ, 360^\circ)$.

$$-539^\circ + 360^\circ = -179^\circ$$

$$-179^\circ + 360^\circ = 181^\circ$$

This 181° angle lies within $[0^\circ, 360^\circ)$ and is coterminal with $\theta = -539^\circ$.

Trig functions at angles of more than one full rotation

In the process of talking about negative angles, we've already looked at some angles that are more than one full rotation in the negative direction.

But none of the angles we've tackled so far have been very many rotations away from the interval $[0^\circ, 360^\circ)$ or $[0, 2\pi)$. This won't always be the case. If we're further away from one full rotation, it helps to start by dividing the given angle by either 360° or 2π.

Let's work through an example.

Example

Find the angle in the interval $[0,2\pi)$ that's coterminal with $\theta = -61\pi/4$.

To find the number of full rotations included in $\theta = -61\pi/4$, we'll divide the angle by 2π.

$$\frac{-\frac{61\pi}{4}}{2\pi}$$

$$-\frac{61\pi}{4} \cdot \frac{1}{2\pi}$$

$$-\frac{61\pi}{8\pi}$$

$$-7.625$$

So $\theta = -61\pi/4$ is 7 full rotations in the negative direction, and then an additional 0.625 of one more rotation in the negative direction. So to find a coterminal angle, we'll get rid of the 7 full rotations by adding $7(2\pi)$ to the angle.

$$-\frac{61\pi}{4} + 7(2\pi)$$

$$-\frac{61\pi}{4} + 14\pi$$

$$-\frac{61\pi}{4} + \frac{56\pi}{4}$$

$$-\frac{5\pi}{4}$$

Now we have an angle that's less than one full rotation, but we'd still like to find a positive coterminal angle that's less than one full rotation. So we'll add 2π one more time.

$$-\frac{5\pi}{4}+2\pi$$

$$-\frac{5\pi}{4}+\frac{8\pi}{4}$$

$$\frac{3\pi}{4}$$

Therefore, we can say that $3\pi/4$ is coterminal with $\theta = -61\pi/4$.

Evaluating trig functions at these angles

As we've said, the value of a trig function is equivalent for coterminal angles. So, referencing the last example, we found that $\theta = -61\pi/4$ is coterminal with the angle $3\pi/4$.

Therefore, if we were asked to find the values of all six trig functions at $\theta = -61\pi/4$, we'd simply find the coterminal angle $3\pi/4$, and then evaluate the trig functions at $3\pi/4$, by pulling values from the unit circle. The values we get will be the same as the values of the trig functions at $\theta = -61\pi/4$.

$$\sin\frac{3\pi}{4}=\sin\left(-\frac{61\pi}{4}\right)=\frac{\sqrt{2}}{2} \qquad \csc\frac{3\pi}{4}=\csc\left(-\frac{61\pi}{4}\right)=\sqrt{2}$$

$$\cos\frac{3\pi}{4}=\cos\left(-\frac{61\pi}{4}\right)=-\frac{\sqrt{2}}{2} \qquad \sec\frac{3\pi}{4}=\sec\left(-\frac{61\pi}{4}\right)=-\sqrt{2}$$

$$\tan\frac{3\pi}{4}=\tan\left(-\frac{61\pi}{4}\right)=-1 \qquad \cot\frac{3\pi}{4}=\cot\left(-\frac{61\pi}{4}\right)=-1$$