Complex numbers in polar form

We've learned how to graph complex numbers in the complex plane, also called the **Argand plane**. But we'd also like to be able to graph complex numbers in polar coordinates. We can't do it without first learning how to find the absolute value of a complex number.

Absolute value

Remember that **absolute value** (also called the **magnitude**) really just means "distance from the origin." The distance from the origin of every complex number can be found using the Pythagorean theorem with the real part and the imaginary part of the complex number.

For instance, given $z = -2 - i$, the distance of the real part from the origin is 2 units, and the distance of the imaginary part from the origin is 1 unit. Therefore, the distance of $z = -2 - i$ from the origin is

$$|z| = \sqrt{2^2 + 1^2}$$

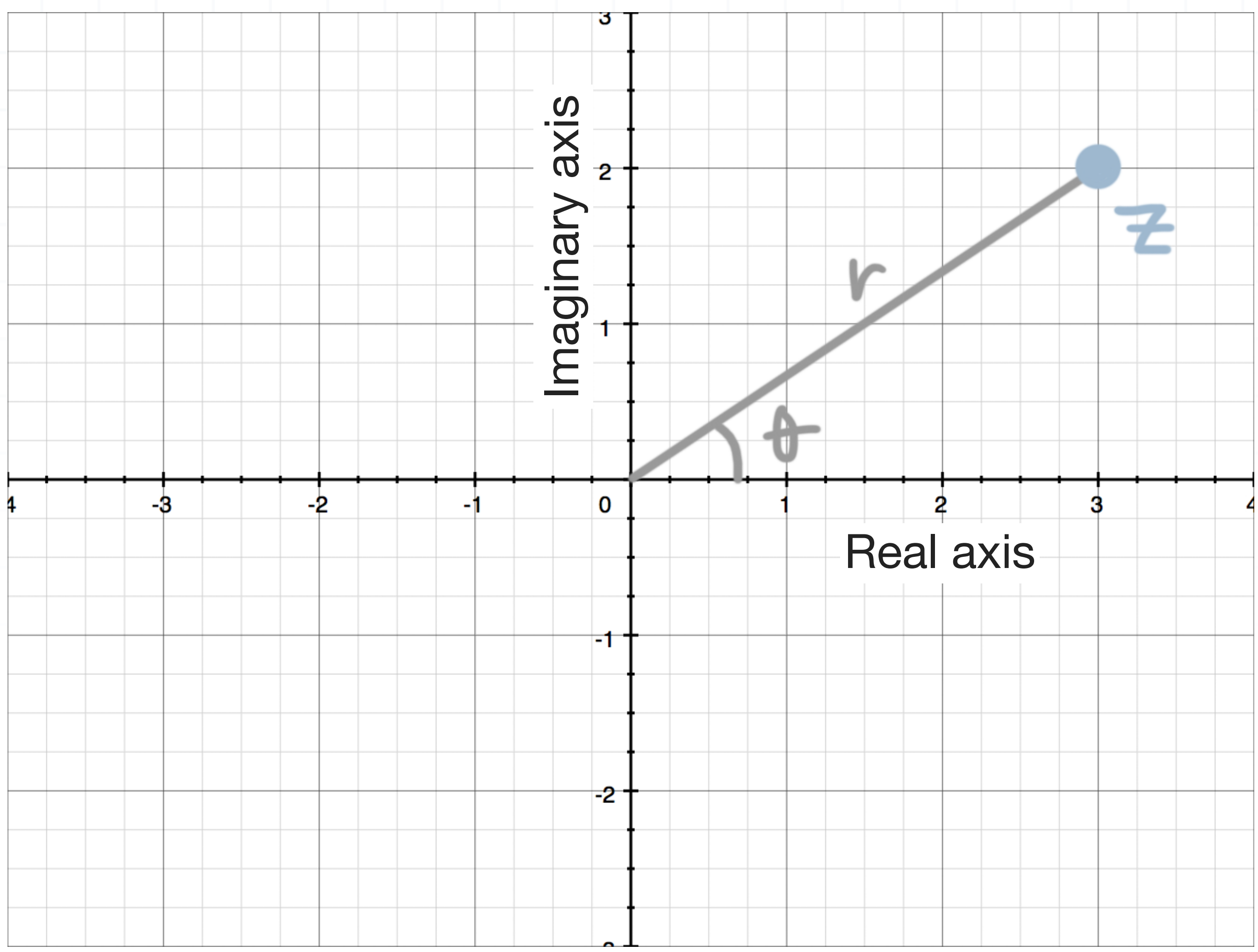
$$|z| = \sqrt{4 + 1}$$

$$|z| = \sqrt{5}$$

Polar form of complex numbers

We know that rectangular coordinates (x, y) always locate a position in coordinate space as the horizontal distance from the origin x, and the vertical distance from the origin y.

But polar coordinates locate points differently. They define points by distance from the origin r, and angle between the point and the positive side of the horizontal axis θ.



To convert a complex number in **rectangular form** $z = a + bi$ into polar form, we can use what we know geometrically about the complex number.

Remember that the real part of the complex number a is the horizontal distance from the origin to z, and the imaginary part of the complex number b is the vertical distance from the origin to z. So a, b, and r form a right triangle between the origin and z, where r is the hypotenuse of the triangle.

We already said that the magnitude, or the absolute value, of the complex number $|z|$ is the distance of z from the origin. So $r=|z|$ and we can say

$$r=|z|=\sqrt{a^2+b^2}$$

To find a formula for θ, we need to remember the SOH-CAH-TOA rule from trigonometry:

Sine = Opposite / Hypotenuse

Cosine = Adjacent / Hypotenuse

Tangent = Opposite / Adjacent

We know that the side opposite the angle θ is b, and that the side adjacent to the angle θ is a, so given the opposite and adjacent side lengths as the imaginary and real parts of the complex number, respectively, we can use the tangent rule to write a formula for θ.

$$\tan\theta=\frac{b}{a}$$

$$\arctan(\tan\theta)=\arctan\frac{b}{a}$$

$$\theta=\arctan\frac{b}{a}$$

That's all assuming that we have values for a and b. But sometimes we'll have values for r and θ instead. In that case, we go back to SOH-CAH-TOA, and build these two equations:

$\cos\theta=\frac{a}{r}$, so $a=r\cos\theta$

$$\sin\theta = \frac{b}{r}, \text{ so } b = r\sin\theta$$

Then, if we plug those values for a and b into the complex number equation, we get the **polar form** of a complex number.

$$z = a + bi$$

$$z = r\cos\theta + (r\sin\theta)i$$

$$z = r(\cos\theta + i\sin\theta)$$

This is a topic for a different time, but the value inside the parentheses $(\cos\theta + i\sin\theta)$ is equal to $e^{i\theta}$, where e is Euler's number $e \approx 2.718$ and i is the imaginary number. Which means the complex number can also be written in **exponential form** as

$$z = re^{i\theta}$$

Let's do an example where we convert a complex number into polar form.

Example

Write the complex number in polar form.

$$z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

In the complex number, the real part is $a = \sqrt{2}/2$ and the imaginary part is $b = -\sqrt{2}/2$, so the value of r will be

$$r=\sqrt{a^2+b^2}$$

$$r=\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{\sqrt{2}}{2}\right)^2}$$

$$r=\sqrt{\frac{1}{2}+\frac{1}{2}}$$

$$r=\sqrt{1}$$

$$r=1$$

The value of θ is

$$\tan\theta=\frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=-1$$

$$\arctan(\tan\theta)=\arctan(-1)$$

$$\theta=\arctan(-1)$$

$$\theta=-\frac{\pi}{4}$$

Then the complex number written in polar form is

$$z=r(\cos\theta+i\sin\theta)$$

$$z=1\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$$

Let’s do another where we change the complex number into polar form.

Example

Write the complex number in polar form.

$$z = 6i$$

We can write the complex number $6i$ as $0 + 6i$, so its real part is $a = 0$ and its imaginary part is $b = 6$. The distance of $6i$ from the origin in the complex plane is

$$r = \sqrt{a^2 + b^2} = \sqrt{0^2 + 6^2} = \sqrt{0 + 36} = \sqrt{36} = 6$$

Since the imaginary part of $6i$ is 6, that means the complex number is located on the positive imaginary axis, so $\theta = \pi/2$. In polar form, the complex number is

$$r(\cos\theta + i\sin\theta)$$

$$6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

The takeaway here is that $a + bi$ (the **rectangular form**) and $r\cos\theta + r\sin\theta i$ (the **polar form**) are two different ways to express the same thing. That’s why we call both of them z.

$$z = a + bi$$

$$z = r\cos\theta + r\sin\theta i$$

Comparing these equations to each other gives us these two equations:

$$a = r\cos\theta$$

$$b = r\sin\theta$$

To convert back and forth between rectangular and polar forms, we use the conversion equations

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\frac{b}{a}$$