Coterminal angles in a particular interval

We've been looking at finding the coterminal angle within the interval that defines one positive full rotation, namely, $[0^\circ, 360^\circ)$ or $[0, 2\pi)$. But we can actually look for coterminal angles in any interval.

Usually we'll do this by setting up an inequality. To do that, we need to first realize that, given an angle θ, an angle α will be coterminal with θ when $\alpha = \theta + n(360^\circ)$ for any n. In radians, we'd of course express this as $\alpha = \theta + n(2\pi)$.

Let's do an example with an angle in degrees so that we can see how this process works.

Example

Find the angle in the interval $(-900^\circ, -540^\circ]$ that's coterminal with 247°.

The interval $(-900^\circ, -540^\circ]$ is a full 360° rotation. Notice how, because we have a parenthesis around the -900° and a bracket around the -540°, it means that the angle -900° exactly isn't included in the interval, but the angle -540° exactly *is* included.

We'll let $\theta = 247^\circ$, and then we'll say that α is the coterminal angle that lies within $(-900^\circ, -540^\circ]$. Then we can say

$$-900^\circ < \alpha \leq -540^\circ$$

But since α is coterminal with θ, we substitute $\alpha = \theta + n(360^\circ)$ into the inequality.

$$-900^\circ < \theta + n(360^\circ) \leq -540^\circ$$

$$-900^\circ < 247^\circ + n(360^\circ) \leq -540^\circ$$

$$-1{,}147^\circ < n(360^\circ) \leq -787^\circ$$

$$-3.19 < n \leq -2.19$$

Remember, n has to be an integer, which means $n = -3$. And therefore, to find α, we'll substitute $n = -3$ into $\alpha = \theta + n(360^\circ)$.

$$\alpha = 247^\circ + (-3)(360^\circ)$$

$$\alpha = 247^\circ - 1{,}080^\circ$$

$$\alpha = -833^\circ$$

Let's do another example, but this time with a radian angle.

Example

Find the angle in the interval $[-\pi, \pi)$ that's coterminal with $56\pi/3$.

We'll let $\theta = 56\pi/3$ and α be the angle within $[-\pi, \pi)$ that's coterminal with θ. We'll use $\alpha = \theta + n(2\pi)$ and solve for the value of n that makes α lie in that interval.

$$-\pi \leq \alpha < \pi$$

$$-\pi \leq \theta + n(2\pi) < \pi$$

$$-\pi \leq \frac{56\pi}{3} + n(2\pi) < \pi$$

$$-\frac{59\pi}{3} \leq n(2\pi) < -\frac{53\pi}{3}$$

$$-9.83 \leq n < -8.83$$

Because n has to be an integer, we know $n = -9$. To find α, we'll substitute $n = -9$ into $\theta + n(2\pi)$.

$$\alpha = \frac{56\pi}{3} + (-9)(2\pi)$$

$$\alpha = \frac{56\pi}{3} - \frac{54\pi}{3}$$

$$\alpha = \frac{2\pi}{3}$$