Law of sines

Earlier we learned how to solve right triangles by finding all three side lengths and all three interior angle measures.

But we'd also like to be able to solve **oblique triangles**, which are triangles that aren't right (they don't include a right angle). Of course, every triangle has six values we want to find: three side lengths and three interior angles. We'll be able to solve any oblique triangle whenever we know three of these six pieces of information, as long as one of the things we know is one side length.

In other words, the only time we won't be able to solve the triangle is when we only know the three interior angle measures (AAA). That's because knowing three angles gives us the shape of the triangle, but tells us nothing about the size of the triangle. There will be an infinite number of **similar triangles** (triangles with the same shape but different size) that match the three-angle set, but we'll have no way of finding a single set of side lengths.

But for any set of information other than AAA, we'll be able to solve the triangle. We want to address each possible combination of information, but let's go ahead and summarize all of them here:

Known information	How to solve
SAA or ASA One side and two angles	1. Use A+B+C=180° to find the remaining angle 2. Use law of sines to find the remaining sides
SAS Two sides and the included angle	1. Use law of cosines to find the third side 2. Use law of sines to find another angle 3. Use A+B+C=180° to find the remaining angle
SSS Three sides	1. Use law of cosines to find the largest angle 2. Use law of sines to find either remaining angle 3. Use A+B+C=180° to find the remaining angle
SSA Two sides and a non-included angle	**The ambiguous case**. If two triangles exist, use this same set of steps to find both triangles. 1. Use law of sines to find an angle 2. Use A+B+C=180° to find the remaining angle 3. Use law of sines to find the remaining side

If we know one side and two angles (SAA or ASA), or if we know two sides and the included angle (SAS), or if we know all three sides (SSS), then there's exactly one triangle that can be a solution.

But if we know two sides and a non-included angle (SSA), we call this the ambiguous case because it's possible that there are 0, 1, or 2 triangles that satisfy the given conditions. In the ambiguous case, we'll need to first determine how many triangles we have. Then, if we have one or two triangles that satisfy the information, then we'll need to solve for each triangle.

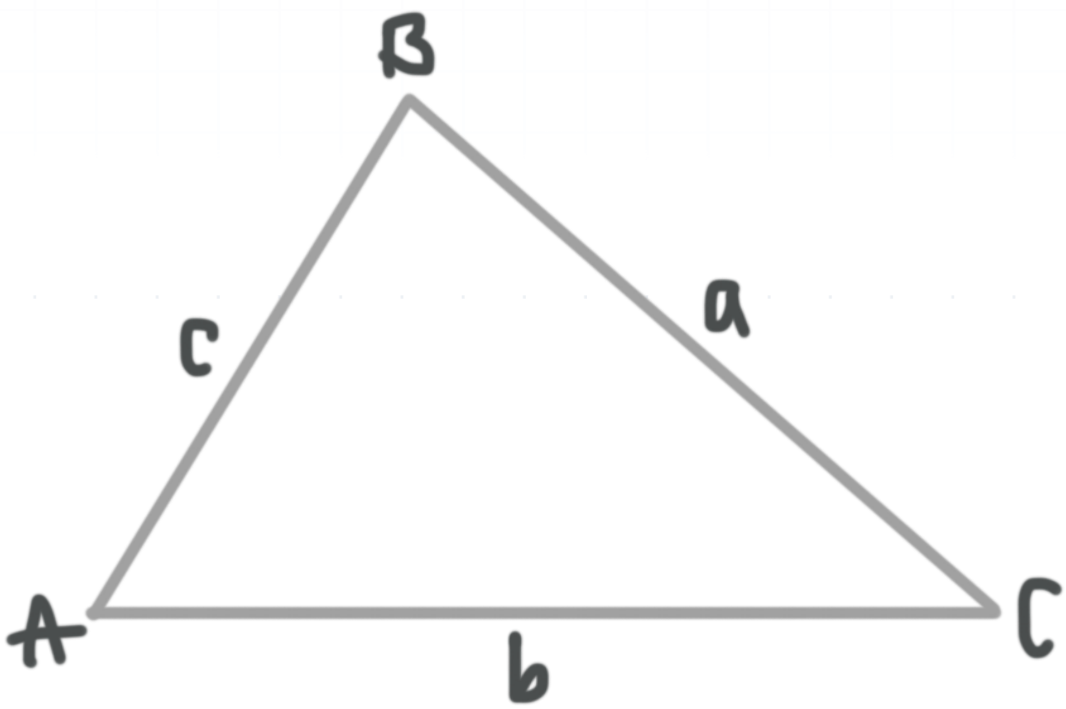
In this lesson, we'll tackle the SAA or ASA case, which will require us to use the law of sines. In the next lesson, we'll look at the SSA ambiguous case since it also uses the law of sines.

Then, later on, we'll look at the law of cosines and the SAS and SSS cases that require that law.

For now, let's introduce the law of sines.

The law of sines

For any triangle with vertices A, B, and C, where side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C,



the **law of sines** says

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

It's equivalent to write this equation as

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Which of these two equations we use depends on the value we're trying to solve for. It's usually easiest to set up the equation such that the value we're trying to find is in the numerator. Which means that we'll prefer the first equation if we're trying to solve for a side length, and we'll prefer the second equation if we're trying to solve for an angle.

But which equation we choose really doesn't matter, because we'll always be able to use algebra to rewrite the equation and solve for the value we need, regardless of whether we start with the unknown value in the numerator or denominator.

The idea of these law of sines equations is that we'll have the same ratio between the angles and sides of the triangle.

And even though the law of sines is a three-part equation, we can always pull apart the equation and include only two pieces of it. So each of these equations is also valid:

$$\frac{a}{\sin A}=\frac{b}{\sin B} \qquad \frac{\sin A}{a}=\frac{\sin B}{b}$$

$$\frac{a}{\sin A}=\frac{c}{\sin C} \qquad \frac{\sin A}{a}=\frac{\sin C}{c}$$

$$\frac{b}{\sin B}=\frac{c}{\sin C} \qquad \frac{\sin B}{b}=\frac{\sin C}{c}$$

SAA or ASA

Now let's tackle the first case in our table, which is when we know the measures of any two interior angles and the length of one of the sides.

In the SAA case, we know two adjacent angles and a side opposite one of those angles. In the ASA case, we know two adjacent angles and the side between them.

Either way, to solve these kinds of triangles, we'll start by using $A + B + C = 180^\circ$ to find the measure of the remaining unknown angle. Once we know all three angles, we'll use the law of sines to find the length of the two remaining unknown sides.

Let's do an example with an ASA triangle.

Example

Solve the triangle that has angles 38° and 64°, where the length of the side opposite the third angle is 55.

We'll let angle $A = 38^\circ$ and angle $B = 64^\circ$, then we'll find the measure of the third angle.

$$A + B + C = 180^\circ$$

$$38^\circ + 64^\circ + C = 180^\circ$$

$$C = 180^\circ - 38^\circ - 64^\circ$$

$$C = 78^\circ$$

The known side is opposite this third angle C, so we'll say $c = 55$. Plugging this side length and all three angle measures into the law of sines gives

$$\frac{a}{\sin 38^\circ} = \frac{b}{\sin 64^\circ} = \frac{55}{\sin 78^\circ}$$

We'll use just the first and third parts of the three-part equation in order to solve for a.

$$\frac{a}{\sin 38^\circ} = \frac{55}{\sin 78^\circ}$$

$$a = \frac{55 \sin 38^\circ}{\sin 78^\circ}$$

$$a \approx 34.6$$

To solve for b, we'll use just the second and third parts of the three-part equation.

$$\frac{b}{\sin 64^\circ} = \frac{55}{\sin 78^\circ}$$

$$b = \frac{55 \sin 64^\circ}{\sin 78^\circ}$$

$$b \approx 50.6$$