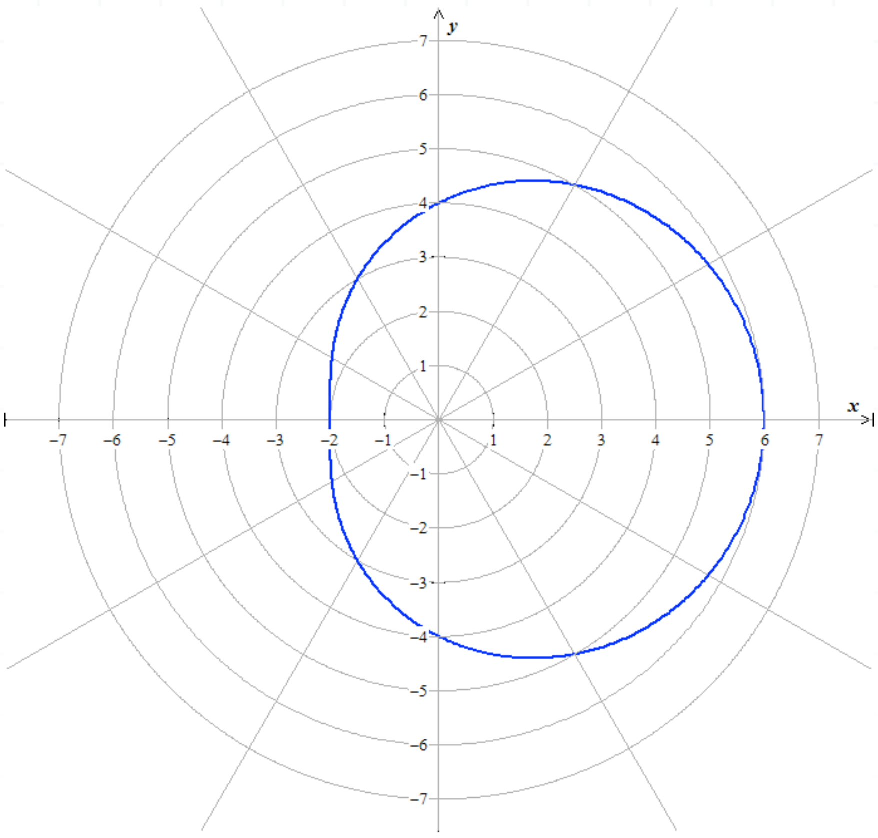
Topic: Graph the polar curve, limacon

Question: The following curve is the graph of one of the polar equations given below. Which polar equation is it?



Answer choices:

A $\quad r = 2 - 4\cos\theta$

B $\quad r = 5 + \sin\theta$

C $\quad r = 1 - 4\sin\theta$

D $\quad r = 4 + 2\cos\theta$

Solution: D

This curve is symmetric with respect to the horizontal axis, so it's the graph of a “cosine” limacon (not a “sine” limacon). Therefore, we can eliminate answer choices B and C.

Also, the curve has no loop, so it's a limacon that satisfies either a polar equation $r = a + b\cos\theta$ or a polar equation $r = a - b\cos\theta$ for some positive numbers a and b with $a > b$. Thus we can eliminate answer choice A, since in that polar equation $a = 2$ and $b = 4$, so $a < b$.

The only answer choice that's left is D. We can verify that this is correct by determining the values of $r = 4 + 2\cos\theta$ for $\theta = 0$ and $\theta = \pi$:

$$\theta = 0 \Longrightarrow r = 4 + 2(\cos 0) = 4 + 2(1) = 6$$

$$\theta = \pi \Longrightarrow r = 4 + 2(\cos \pi) = 4 + 2(-1) = 2$$

Inspection of the given curve shows that the point with polar coordinates $(6,0)$ and the point with polar coordinates $(2,\pi)$ are both on it.

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Question: Which of the following are the angles θ_1, θ_2 in the interval $[0,2\pi)$ such that $\theta_1 < \theta_2$ and (θ_1, θ_2) is the subinterval of $[0,2\pi)$ on which the value of r in the polar equation $r = 1 + 2\sin\theta$ is negative?

Answer choices:

A $\theta_1 = \frac{\pi}{6}$ and $\theta_2 = \frac{5\pi}{6}$

B $\theta_1 = \frac{7\pi}{6}$ and $\theta_2 = -\frac{\pi}{6}$

C $\theta_1 = \frac{7\pi}{6}$ and $\theta_2 = \frac{11\pi}{6}$

D $\theta_1 = \frac{5\pi}{6}$ and $\theta_2 = \frac{7\pi}{6}$

Solution: C

Well, $r = 1 + 2\sin\theta$ is the polar equation of the limacon $r = a + b\cos\theta$ with $a = 1$ and $b = 2$. Since $a < b$, this limacon passes through the pole twice and has a loop. Thus the subinterval of $[0,2\pi)$ on which $r = 1 + 2\sin\theta$ is negative is (θ_1, θ_2), where θ_1 and θ_2 are the angles in the interval $[0,2\pi)$ at which $r = 0$ and $\theta_1 < \theta_2$. Now

$$r = 0 \Longrightarrow 1 + 2\sin\theta = 0 \Longrightarrow 2\sin\theta = -1 \Longrightarrow \sin\theta = -\frac{1}{2}$$

Recall that the sine function is negative in the third and fourth quadrants. Thus θ_1 (the smaller of the two angles at which $r = 0$) is in the third quadrant, and θ_2 is in the fourth quadrant.

What we need to do is determine the values of θ_1 and θ_2.

Let's first recall that

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

By the sum identity for sine,

$$\sin(\theta + \pi) = (\sin\theta)(\cos\pi) + (\cos\theta)(\sin\pi)$$

$$\sin(\theta + \pi) = \sin\theta(-1) + \cos\theta(0)$$

$$\sin(\theta + \pi) = -\sin\theta$$

Since the angle of measure $\pi/6$ is in the first quadrant, the angle of measure

$$\frac{\pi}{6} + \pi = \frac{7\pi}{6}$$

is in the third quadrant. Therefore,

$$\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

Note that $7\pi/6$ is in the interval $[0,2\pi)$.

Also, by the odd identity for sine,

$$\sin(-\theta) = -\sin\theta$$

Thus

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

Since the angle of measure $\pi/6$ is in the first quadrant, the angle of measure $-\pi/6$ is in the fourth quadrant, but $-\pi/6$ isn't in the interval $[0,2\pi)$. However, the sine of any angle which differs in measure from $-\pi/6$ by an integer multiple of 2π is also equal to $-1/2$. One such angle is

$$-\frac{\pi}{6} + 2\pi\left(=\frac{11\pi}{6}\right)$$

Note that $11\pi/6$ is in the interval $[0,2\pi)$.

What we have found is that

$$\theta_1 = \frac{7\pi}{6} \text{ and } \theta_2 = \frac{11\pi}{6}$$

Since the sine function is negative in the third and fourth quadrants, it is negative on the interval

$$\left(\frac{7\pi}{6},\frac{11\pi}{6}\right)$$

Moreover, the value of the sine function is less than $-1/2$ on that interval. That is,

$$\theta_1 < \theta < \theta_2 \Longrightarrow \sin\theta < \sin\theta_1 = -\frac{1}{2}$$

From this it follows that

$$r = 1 + 2\sin\theta < 1 + 2\left(-\frac{1}{2}\right) = 1 - 1 = 0$$

so the value of $r = 1 + 2\sin\theta$ is negative on the interval (θ_1, θ_2).

Note that for any angle θ in either the interval $[0,7\pi/6)$ or the interval $(11\pi/6,2\pi)$,

$$\sin\theta > \sin\theta_1 = -\frac{1}{2}$$

so the value of $r = 1 + 2\sin\theta$ in either of these intervals is

$$r = 1 + 2\sin\theta > 1 + 2\left(-\frac{1}{2}\right) = 1 - 1 = 0$$

What we have shown is that the angles θ in the interval $[0,2\pi)$ at which the value of $r = 1 + 2\sin\theta$ is negative are those in the interval

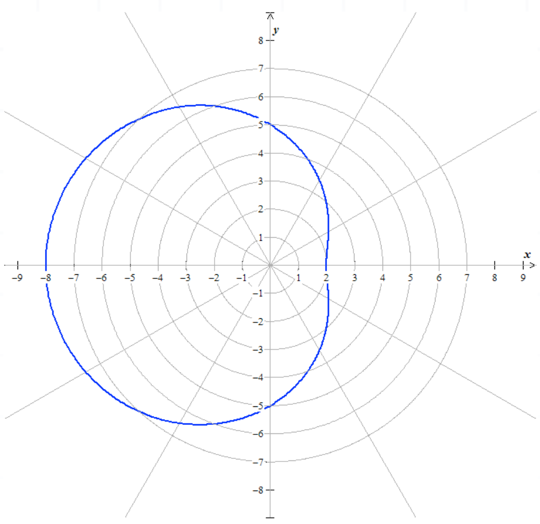
$$(\theta_1, \theta_2) = \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$$

Topic: Graph the polar curve, limacon

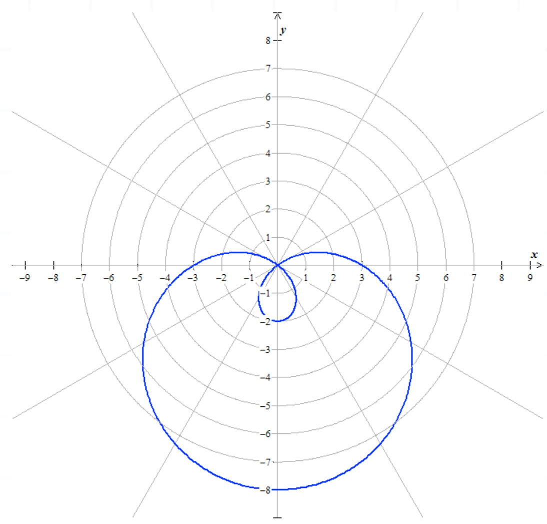
Question: Which of the following curves is the graph of the limacon?

$$r = 3 - 5\cos\theta$$

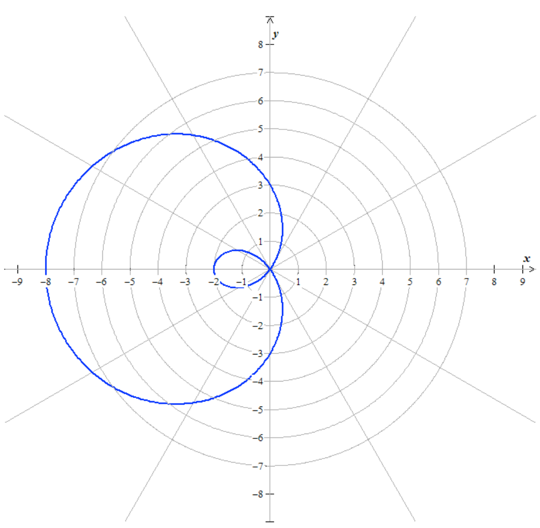
Answer choices:



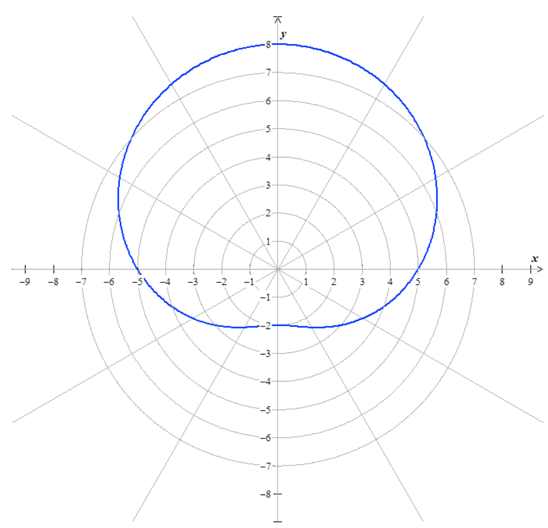
A



C



B



D

Solution: B

The equation $r = 3 - 5\cos\theta$ is a "cosine cardioid", so its graph is symmetric with respect to the horizontal axis. Therefore, we can eliminate answer choices C and D, because the curves given in those answer choices are symmetric with respect to the vertical axis.

Moreover, the equation $r = 3 - 5\cos\theta$ is in the form $a - b\cos\theta$ where $a = 3$ and $b = 5$. Thus $a < b$, so this limacon has a loop. This enables us to eliminate answer choice A, because the curve given in that answer choice has a depression, not a loop.

To check that the curve given in answer choice B is indeed the graph of the equation $r = 3 - 5\cos\theta$, we'll evaluate $r = 3 - 5\cos\theta$ at $\theta = 0$ and $\theta = \pi$:

$$\theta = 0 \implies r = 3 - 5(\cos 0) = 3 - 5(1) = -2$$

$$\theta = \pi \implies r = 3 - 5(\cos\pi) = 3 - 5(-1) = 8$$

This would mean that the given curve includes the point with polar coordinates $(-2,0)$ and the point with polar coordinates $(8,\pi)$. Inspection of the curve tells us that the point with polar coordinates $(8,\pi)$ is definitely on it. Note that the point with polar coordinates $(-2,0)$ also has polar coordinates $(2,\pi)$. Clearly, this point is also on the given curve. (In fact, it's the leftmost point of the loop.)