Converting equations from polar to rectangular

There may be times when you'll be given an equation in polar coordinates that you need to solve, and you'll want to convert the equation from polar coordinates to rectangular coordinates, perhaps so that you can get a more intuitive sense of what the solution is. When you do that, you'll make use of one or more of the following relationships between polar coordinates (r, θ) and rectangular coordinates (x, y):

$$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad \tan\theta = \frac{y}{x}$$

Example

Convert the polar equation $r = 7$ to rectangular coordinates and describe its solution geometrically.

Since r is positive and $r^2 = x^2 + y^2$, we know that $r = \sqrt{x^2 + y^2}$, so we get the equation

$$\sqrt{x^2 + y^2} = 7$$

Squaring both sides of this equation:

$$\left(\sqrt{x^2 + y^2}\right)^2 = 7^2$$

Since $\left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$, we see that

$$x^2+y^2=7^2$$

You may recognize this as the equation of the circle that's centered at the origin and has a radius of 7. (The general equation of the circle that's centered at the point with rectangular coordinates (h,k) and has radius r is $(x-h)^2+(y-k)^2=r^2$. Here, $h=0$, $k=0$, and $r=7$.)

This solution should make sense, since the polar coordinate r is constant and positive (in particular, equal to 7), which means that a point P is in the solution set of the polar equation $r=7$ if and only if 7 is the distance of P from the pole. Thus the solution set of this equation is precisely the set of all points on the circle that's centered at the pole/origin and has a radius of 7.

The given equation in that example contained the variable r but not the variable θ. Let's look at an equation that contains the variable θ but not the variable r.

Example

Convert the polar equation $\theta=\pi/3$ to rectangular coordinates and describe its solution geometrically.

Taking tangents of the expressions on both sides of the equation $\theta=\pi/3$, we have

$$\tan\theta = \tan\left(\frac{\pi}{3}\right)$$

Since $\tan\theta = y/x$, we obtain

$$\frac{y}{x} = \tan\left(\frac{\pi}{3}\right)$$

Now

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Substituting this result, we get

$$\frac{y}{x} = \sqrt{3}$$

Solving for y, we find that $y = \sqrt{3}x$. Notice that this is the equation of the line that passes through the origin and has slope $\sqrt{3}$.

In this case, we were told only the value of the second polar coordinate (θ) of the points in the solution set. We weren't told the value (or the sign) of the first polar coordinate (r) of those points.

If r is positive for some point P that satisfies the polar equation $\theta = \pi/3$, then P is located on the terminal side of the angle of measure $\pi/3$ (in standard position), hence P is in the first quadrant. Thus it should be clear that P lies on the line $y = \sqrt{3}x$, and that every point in the first quadrant that satisfies the equation $y = \sqrt{3}x$ is a solution of the polar equation $\theta = \pi/3$.

If r is negative for some point P that satisfies the polar equation $\theta = \pi/3$, then one pair of polar coordinates of P is

$$\left(r, \frac{\pi}{3}\right)$$

Now recall that since r is negative, another pair of polar coordinates of P is

$$\left(-r, \frac{\pi}{3} + \pi\right)$$

Since $-r$ is positive, P is located on the terminal side of the angle of measure $(\pi/3) + \pi$ (in standard position), hence P is in the third quadrant. Now

$$\tan\left(\frac{\pi}{3} + \pi\right) = \tan\left(\frac{\pi}{3}\right)$$

To see this, use the sum identities for sine and cosine to get $\sin((\pi/3) + \pi)$ and $\cos((\pi/3) + \pi)$, and then compute their ratio to get $\tan((\pi/3) + \pi)$. Thus it should be clear that P lies on the line $y = \sqrt{3}x$, and that every point in the third quadrant that satisfies the equation $y = \sqrt{3}x$ is a solution of the polar equation $\theta = \pi/3$.

Finally, it's clear that the origin lies on the line $y = \sqrt{3}x$. Also, since the pole has a pair of polar coordinates $(0,\theta)$ for any angle θ, the pole (i.e., the origin) is a solution of the polar equation $\theta = \pi/3$.

What we have shown is that a point P is a solution of the polar equation $\theta = \pi/3$ if and only if P is a solution of the equation $y = \sqrt{3}x$.

Now let's consider a more complicated-looking equation, one that contains both r and θ as variables.

Example

Convert the polar equation $r = 8\cos\theta$ to rectangular coordinates and describe its solution geometrically.

By the general equation $x = r\cos\theta$, we get

$$\cos\theta = \frac{x}{r}$$

Substituting this result into the given polar equation, $r = 8\cos\theta$, we get

$$r = 8\left(\frac{x}{r}\right)$$

Multiplying both sides by r yields

$$r^2 = 8x$$

Using the general equation $r^2 = x^2 + y^2$, we obtain

$$x^2 + y^2 = 8x$$

Subtracting $8x$ from both sides, we have

$$x^2 - 8x + y^2 = 0$$

The solution set of this equation ($x^2 - 8x + y^2 = 0$) may not be transparent, so we'll proceed to work on it further, to see if we can get it into an equivalent form that does shed light on the solution.

We'll complete the square on the "x part" of the expression $x^2 - 8x + y^2$ (that is, on $x^2 - 8x$). To do that, we need to add 16, because

$$x^2 - 8x + 16 = (x - 4)^2$$

Now if we add 16, we also need to subtract 16 (to avoid changing the value of anything), that is,

$$x^2 - 8x = (x^2 - 8x) + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16$$

Substituting this expression for $x^2 - 8x$ in the equation $x^2 - 8x + y^2 = 0$, we get

$$(x - 4)^2 - 16 + y^2 = 0$$

Adding 16 to both sides, we obtain

$$(x - 4)^2 + y^2 = 16$$

Equivalently,

$$(x - 4)^2 + (y - 0)^2 = 4^2$$

You should recognize this as the equation of the circle which is centered at the point with rectangular coordinates $(4,0)$ and has a radius of 4. Thus if a point P satisfies the polar equation $r = 8\cos\theta$, then P is a point on the circle whose center has rectangular coordinates $(4,0)$ and whose radius is 4. We'll now prove the converse: If P is a point on that circle, then P satisfies the polar equation $r = 8\cos\theta$.

Well, if P is a point on the circle whose center has rectangular coordinates $(4,0)$ and whose radius is 4, then it satisfies the equation

$$(x-4)^2 + (y-0)^2 = 4^2$$

Carrying out the indicated squaring in this equation, we get

$$x^2 - 8x + 16 + y^2 = 16$$

Subtracting 16 from both sides yields

$$x^2 - 8x + y^2 = 0$$

Rearranging terms:

$$x^2 + y^2 - 8x = 0$$

Using the general equations $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we obtain

$$r^2 - 8r\cos\theta = 0$$

Factoring the left-hand side gives

$$r(r - 8\cos\theta) = 0$$

This implies that $r = 0$ or $r = 8\cos\theta$.

If $r = 8\cos\theta$, we're done, so we can assume that $r = 0$. Now the only point with $r = 0$ is the pole. It is easy to check that the pole, which has rectangular coordinates $(x, y) = (0,0)$, satisfies the equation

$$x^2 + y^2 - 8x = 0$$

Since $(0,\theta)$ is a pair of polar coordinates of the pole for any angle θ, we can set θ to $\pi/2$, so for $r = 0$ we get

$$r = 0 = 8(0) = 8(\cos(\pi/2)) = 8\cos\theta$$

Thus the pole also satisfies the equation $r = 8\cos\theta$.