Powers of complex numbers and De Moivre's theorem

In this lesson, we want to build on our understanding of multiplication of complex numbers in polar form, by looking at how to find powers of complex numbers, like z^2, z^3, z^4, etc.

Exponential form for large powers

Given a complex number like

$$z = 3\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$$

we don't know yet how to raise the complex number to a power. For instance, if we want to find z^{12}, it seems like it would be pretty tedious to do it this way:

$$z = \left[3\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\right]^{12}$$

$$z = 3^{12}\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)^{12}$$

We'll actually learn how to do this in the next part of this lesson, but for now, we want to realize that we can convert the complex number from polar form to exponential form. In exponential form, we get

$$z = re^{i\theta}$$

$$z^{12} = \left(re^{i\theta}\right)^{12}$$

$$z^{12} = r^{12}e^{12i\theta}$$

$$z^{12} = 3^{12}e^{12i\cdot\frac{3\pi}{8}}$$

$$z^{12} = 531{,}441e^{\frac{9\pi}{2}i}$$

De Moivre's theorem

We don't always have to convert to exponential form to find the power of a complex number. We can keep them in polar form. Let's talk about how.

We know that when we multiply two complex numbers z_1 and z_2, their product is

$$z_1z_2 = r_1r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

In the previous section, we always multiplied different complex numbers. But what happens when we multiply two equivalent complex numbers? Let's say that

$$z_1 = 3\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$$

and we want to find the product of z_1, multiplied by z_1. We know from the product formula above that the result is

$$z_1z_2 = r_1r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$z_1 z_1 = r_1 r_1 \left[\cos(\theta_1 + \theta_1) + i\sin(\theta_1 + \theta_1)\right]$$

$$(z_1)^2 = (r_1)^2\left[\cos(2\theta_1) + i\sin(2\theta_1)\right]$$

Let’s say we want to take this result and multiply by z_1 again. We’d get

$$(z_1)^2 z_1 = (r_1)^2 r_1 \left[\cos(2\theta_1 + \theta_1) + i\sin(2\theta_1 + \theta_1)\right]$$

$$(z_1)^3 = (r_1)^3\left[\cos(3\theta_1) + i\sin(3\theta_1)\right]$$

If we multiply again by z_1, we get

$$(z_1)^3 z_1 = (r_1)^3 r_1 \left[\cos(3\theta_1 + \theta_1) + i\sin(3\theta_1 + \theta_1)\right]$$

$$(z_1)^4 = (r_1)^4\left[\cos(4\theta_1) + i\sin(4\theta_1)\right]$$

When we put these results together,

$$z^2 = r^2 \left[\cos(2\theta) + i\sin(2\theta)\right]$$

$$z^3 = r^3 \left[\cos(3\theta) + i\sin(3\theta)\right]$$

$$z^4 = r^4 \left[\cos(4\theta) + i\sin(4\theta)\right]$$

we see a pattern emerging. The pattern we’re getting is

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

This is **De Moivre's theorem** (sometimes called De Moivre's formula), and it comes directly from what we already knew about multiplying complex numbers. Looking at the formula, we can see that it tells us that, if we want to raise a complex number to the 5th power, we just raise r to the 5th power, and multiply θ by 5.

Example

Use De Moivre's theorem to find z^3, the third power of z.

$$z = 7\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

To find z^3, we need to use $n = 3$ in De Moivre's theorem. With $n = 3$, and $r = 7$ and $\theta = \pi/6$ from this problem, we plug into De Moivre's theorem to get

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^3 = r^3\left[\cos(3\theta) + i\sin(3\theta)\right]$$

$$z^3 = 7^3\left[\cos\left(3\cdot\frac{\pi}{6}\right) + i\sin\left(3\cdot\frac{\pi}{6}\right)\right]$$

$$z^3 = 343\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

We could leave the complex number in its polar form, or we could simplify into rectangular form.

$$z^3 = 343\left(0 + i(1)\right)$$

$$z^3 = 343i$$

Let's do another example with different n, r, and θ values.

Example

Find z^8.

$$z = \frac{1}{2}\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$$

Plugging $r = 1/2$, $\theta = \pi/7$, and $n = 8$ into De Moivre's theorem, we get

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^8 = r^8\left[\cos(8\theta) + i\sin(8\theta)\right]$$

$$z^8 = \left(\frac{1}{2}\right)^8\left[\cos\left(8\cdot\frac{\pi}{7}\right) + i\sin\left(8\cdot\frac{\pi}{7}\right)\right]$$

$$z^8 = \frac{1}{256}\left(\cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}\right)$$

Power of a rectangular complex number

We know from the previous examples how to find the power of a complex number when the complex number is already in polar form. If we want to find the power of a complex number that's given in rectangular form, we need to first convert the rectangular complex number into a polar complex number.

For instance, given the rectangular complex number $z = 1 - \sqrt{3}i$, we'll first convert to polar form by finding the modulus $|z|$ and the angle θ. The distance from the origin is

$$|z| = r = \sqrt{a^2 + b^2}$$

$$|z| = r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2}$$

$$|z| = r = \sqrt{1 + 3}$$

$$|z| = r = \sqrt{4}$$

$$|z| = r = 2$$

and the angle is

$$\theta = \arctan\frac{b}{a} = \arctan\frac{-\sqrt{3}}{1} = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Then $z = 1 - \sqrt{3}i$ in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

And now that the complex number is in polar form, we know that we can find any power of it using De Moivre's theorem. If we want to find z^4, we'll plug into De Moivre's formula to get

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^4 = r^4\left[\cos(4\theta) + i\sin(4\theta)\right]$$

With $r = 2$ and $\theta = -\pi/3$, this becomes

$$z^4 = 2^4\left[\cos\left(4\left(-\frac{\pi}{3}\right)\right) + i\sin\left(4\left(-\frac{\pi}{3}\right)\right)\right]$$

$$z^4 = 16\left[\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right]$$

We can leave this in polar form, or simplify into rectangular form.

$$z^4 = 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$z^4 = 8\left(-1 + \sqrt{3}i\right)$$

$$z^4 = -8 + 8\sqrt{3}i$$