Topic: Eccentricity and directrix of the conic section

Question: The total of the eccentricities of the following functions is equal to 6. Which type of curve is represented by r_4?

$$r_1 = \frac{16}{4 - 5\cos\theta_1}$$

$$r_2 = \frac{12}{2 - 7\cos\theta_2}$$

$$r_3 = \frac{24}{8 - 3\cos\theta_3}$$

$$r_4 = \frac{15}{6 - m\cos\theta_4}$$

Answer choices:

A A circle

B An ellipse

C A hyperbola

D A parabola

Solution: B

Find the eccentricity of each function.

For the eccentricity of r_1:

$$r_1 = \frac{16}{4 - 5\cos\theta_1}$$

$$r_1 = \frac{\frac{16}{4}}{\frac{4}{4} - \frac{5}{4}\cos\theta_1}$$

$$r_1 = \frac{4}{1 - \frac{5}{4}\cos\theta_1}$$

$$e_1 = \frac{5}{4}$$

For the eccentricity of r_2:

$$r_2 = \frac{12}{2 - 7\cos\theta_2}$$

$$r_2 = \frac{\frac{12}{2}}{\frac{2}{2} - \frac{7}{2}\cos\theta_2}$$

$$r_2 = \frac{6}{1 - \frac{7}{2}\cos\theta_2}$$

$$e_2 = \frac{7}{2}$$

For the eccentricity of r_3:

$$r_3 = \frac{24}{8 - 3\cos\theta_3}$$

$$r_3 = \frac{\frac{24}{8}}{\frac{8}{8} - \frac{3}{8}\cos\theta_3}$$

$$r_3 = \frac{3}{1 - \frac{3}{8}\cos\theta_3}$$

$$e_3 = \frac{3}{8}$$

For the eccentricity of r_4:

$$r_4 = \frac{15}{6 - m\cos\theta_4}$$

$$r_4 = \frac{\frac{15}{6}}{\frac{6}{6} - \frac{m}{6}\cos\theta_4}$$

$$r_4 = \frac{\frac{5}{2}}{1 - \frac{m}{6}\cos\theta_4}$$

$$e_4 = \frac{m}{6}$$

The sum of these eccentricities is therefore

$$\frac{5}{4} + \frac{7}{2} + \frac{3}{8} + \frac{m}{6} = 6$$

Which means that m is

$$\frac{30}{24} + \frac{84}{24} + \frac{9}{24} + \frac{4m}{24} = 6$$

$$30 + 84 + 9 + 4m = 144$$

$$4m = 21$$

$$m = \frac{21}{4}$$

So e_4 is

$$e_4 = \frac{\frac{21}{4}}{6}$$

$$e_4 = \frac{21}{24}$$

$$e_4 = \frac{7}{8}$$

Since $0 < e_4 < 1$, then r_4 represents an ellipse.

Topic: Eccentricity and directrix of the conic section

Question: Which polar curves have the same directrix?

$$r_1 = \frac{6}{1 - 7\cos\theta_1}$$

$$r_2 = \frac{12}{9 - 5\cos\theta_2}$$

$$r_3 = \frac{12}{5 - 14\cos\theta_3}$$

$$r_4 = \frac{5}{7 - \cos\theta_4}$$

Answer choices:

A The directrices of r_1 and r_4 are parallel.

B The directrices of r_1 and r_3 are parallel.

C The directrices of r_2 and r_3 are parallel.

D The directrices of r_2 and r_4 are parallel.

Solution: B

The directrix of

$$r_1 = \frac{6}{1 - 7\cos\theta_1}$$

is

$$d = \frac{6}{7}$$

The directrix of

$$r_3 = \frac{12}{5 - 14\cos\theta_3}$$

$$r_3 = \frac{\frac{12}{5}}{1 - \frac{14}{5}\cos\theta_3}$$

is

$$d = \frac{\frac{12}{5}}{\frac{14}{5}} = \frac{6}{7}$$

Therefore, r_1 and r_3 have the same directrix.

Topic: Eccentricity and directrix of the conic section

Question: The following polar functions are given, where e_1 and d_1 are the eccentricity and directrix of the function r_1, and e_2 and d_2 are the eccentricity and directrix of the function r_2. If $\theta_1 = \theta_2$, $e_1 = 3e_2$ and $3d_1 = d_2$, then which statement is true about the positions of the graphs of the given functions.

$$r_1 = \frac{a}{b - c\cos\theta_1}$$

$$r_2 = \frac{c}{b - a\cos\theta_2}$$

Answer choices:

A The graphs of r_1 and r_2 are the same.

B The graphs of r_1 and r_2 don't overlap.

C The graph of r_1 is 3 units above the graph of r_2.

D The graph of r_1 is 3 units below the graph of r_2.

Solution: A

We'll rewrite r_1.

$$r_1 = \frac{a}{b - c\cos\theta_1}$$

$$r_1 = \frac{\frac{a}{b}}{\frac{b}{b} - \frac{c}{b}\cos\theta_1}$$

$$r_1 = \frac{\frac{a}{b}}{1 - \frac{c}{b}\cos\theta_1}$$

We'll rewrite r_2.

$$r_2 = \frac{c}{b - a\cos\theta_2}$$

$$r_2 = \frac{\frac{c}{b}}{\frac{b}{b} - \frac{a}{b}\cos\theta_2}$$

$$r_2 = \frac{\frac{c}{b}}{1 - \frac{a}{b}\cos\theta_2}$$

Now we can say

$$e_1d_1 = \frac{a}{b}$$

and

$$e_2d_2 = \frac{c}{b}$$

Divide these equations side-by-side, and substitute $e_1 = 3e_2$ and $3d_1 = d_2$.

$$\frac{e_1 d_1}{e_2 d_2} = \frac{a}{c}$$

$$\frac{\left(3e_2\right) d_1}{e_2 \left(3d_1\right)} = \frac{a}{c}$$

$$\frac{1}{1} = \frac{a}{c}$$

$$a = c$$

Replacing $a = c$ in the given functions gives

$$r_1 = \frac{a}{b - a\cos\theta_1}$$

$$r_2 = \frac{a}{b - a\cos\theta_2}$$

Since $\theta_1 = \theta_2$, the functions are the same.