Complete solution set of the equation

Much earlier on in Trigonometry, when we introduced coterminal angles, we showed how, if one angle satisfied a trig equation, then every angle coterminal with the original angle would also satisfy the equation.

For example, given the equation $\cos\theta = 1$, we know from the unit circle that the cosine function is 1 when $\theta = 0$, so $\theta = 0$ is a solution. But we also know that every angle coterminal with $\theta = 0$ is a solution, so we gave the complete solution set of $\cos\theta = 1$ as

$$\theta = 0 + 2n\pi$$

$$\theta = 2n\pi$$

where n is any integer. Which means that the equation $\cos\theta = 1$ is satisfied by an infinitely large set of angles:

$$\ldots -8\pi,\ -6\pi,\ -4\pi,\ -2\pi,\ 0,\ 2\pi,\ 4\pi,\ 6\pi,\ 8\pi\ldots$$

In this lesson, we want to build on this same idea, but we'll be dealing with trig equations that are more complex than equations like $\cos\theta = 1$. The equations in this lesson may require us to apply trig identities first to simplify the equation, and then find the full solution set of coterminal angles.

Let's work through some examples, starting with a sine equation.

Example

Find all the values of θ that satisfy $\sin(2\theta) = \sin\theta$.

We'll start by using the double-angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to rewrite the left side of the equation.

$$\sin(2\theta) = \sin\theta$$

$$2\sin\theta\cos\theta = \sin\theta$$

Subtract $\sin\theta$ from both sides, then factor out a $\sin\theta$.

$$2\sin\theta\cos\theta - \sin\theta = 0$$

$$\sin\theta(2\cos\theta - 1) = 0$$

The only way the left side of the equation is 0 is if $\sin\theta = 0$, $2\cos\theta - 1 = 0$, or both. So we need to solve these equations individually to find the values of θ that satisfy the equation. We get

$$\sin\theta = 0$$

and

$$2\cos\theta - 1 = 0$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

The equation $\sin\theta = 0$ is true when $\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$, which is just all multiples of π. So the solution set of $\sin\theta = 0$ is $\theta = n\pi$, where n is the set of all integers.

The equation $\cos\theta = 1/2$ is true at $\theta = \pi/3$ and $\theta = 5\pi/3$. But the set of all angles coterminal with these two angles is

$$\theta = \frac{\pi}{3} + 2n\pi \text{ and } \theta = \frac{5\pi}{3} + 2n\pi$$

Putting all these sets together, we can say that the complete solution set of $\sin(2\theta) = \sin\theta$ includes all of these, where n is any integer:

$$\theta = n\pi$$

$$\theta = \frac{\pi}{3} + 2n\pi$$

$$\theta = \frac{5\pi}{3} + 2n\pi$$

In some cases, we may need to find the roots of a polynomial to get the solutions of a trig equation. Let's look at an example like this in which we're asked to limit the solutions to the interval $[0,2\pi)$.

Example

Find the set of angles in the interval $[0,2\pi)$ which satisfies the trig equation.

$$\tan^2\theta + 2\tan\theta + 1 = 0$$

Let's substitute $u = \tan\theta$ to rewrite the equation.

$$u^2 + 2u + 1 = 0$$

In this form, we can see that the equation will factor as $(u+1)^2$. Which means the original equation will factor as

$$(\tan\theta + 1)^2 = 0$$

$$\tan\theta + 1 = 0$$

$$\tan\theta = -1$$

Tangent of an angle will be -1 when the sine and cosine values are equal, but with opposite signs. This happens in the second and fourth quadrants at $\theta = 3\pi/4$ and $\theta = 7\pi/4$.

$$\tan\left(\frac{3\pi}{4}\right) = -1$$

$$\tan\left(\frac{7\pi}{4}\right) = -1$$

And of course, every angle that's coterminal with these two also has a tangent of -1, which means they are solutions as well. So θ is a solution of $\tan^2\theta + 2\tan\theta + 1 = 0$ when n is any integer for all of these:

$$\theta = \frac{3\pi}{4} + 2n\pi$$

$$\theta = \frac{7\pi}{4} + 2n\pi$$

Or since the period of tangent is π, we see that $3\pi/4$ and $7\pi/4$ differ by π, which means that these two expressions can be combined to just

$$\theta = \frac{3\pi}{4} + n\pi$$

While this is the complete solution set, we were only asked for the solutions in $[0,2\pi)$. Of the full solution set we found, the only angles that lie in the interval are the original angles, $\theta = 3\pi/4$ and $\theta = 7\pi/4$.

Let's do one more example in which we need to first apply a trig identity.

Example

Find all the solutions of the trig equation, then list only the solutions that lie in the interval $[0,2\pi)$.

$$\sin^2\theta + \cos(2\theta) = 1$$

If we use the double-angle identity for cosine, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we can rewrite the equation as

$$\sin^2\theta + \cos^2\theta - \sin^2\theta = 1$$

$$\cos^2\theta = 1$$

Take the square root of both sides.

$$\cos\theta = \pm\sqrt{1}$$

$$\cos\theta = \pm 1$$

From the unit circle, we know that $\cos\theta = 1$ at

$$\theta = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$

and that $\cos\theta = -1$ at

$$\theta = \pi,\ 3\pi,\ 5\pi,\ 7\pi,\ \dots$$

Putting these sets of solutions together, we get

$$\theta = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ 5\pi,\ 6\pi,\ \dots$$

The only angles in this set that fall in the interval $[0,2\pi)$ are the angles $\theta = 0$ and $\theta = \pi$. The interval notation $[0,2\pi)$ tells us that $\theta = 0$ is included in the interval (because there's a square bracket around the 0), but that $\theta = 2\pi$ is excluded from the interval (because there's a parenthesis around the 2π). That's why we say $\theta = 0$ falls in the interval, while $\theta = 2\pi$ does not.