Matrix multiplication

We talked before about scalar multiplication, which is when we multiply a matrix by a real-number value. But **matrix multiplication** is what we do when we multiply two matrices together.

Dimensions matter

Based on what we've already learned about matrix addition and subtraction, you'd think that multiplying matrices is just a matter of multiplying corresponding entries, since matrix addition is just a matter of adding corresponding entries, and matrix subtraction is just a matter of subtracting corresponding entries.

But in fact, we follow an entirely different process to multiply matrices, and we'll walk through exactly what that is in this section.

First, when you multiply two matrices A and B together, the order matters. So $A \cdot B$ doesn't have the same result as $B \cdot A$. Which means that matrices do not follow the commutative property of multiplication.

The reason the order matters is because of the way we multiply the matrices, which really depends on the dimensions. Here's the thing to remember about dimensions:

> you need the same number of columns in the first matrix as you have rows in the second matrix.

So for example, you can multiply a 3×2 matrix by any of these:

$$2 \times 1$$

$$2 \times 2$$

$$2 \times 3$$

$$2 \times 4$$

$$\dots$$

That's because, when we multiply one matrix by another, we multiply the rows in the first matrix by the columns in the second matrix. Let's say we want to multiply a 2×2 matrix called A by a 2×2 matrix called B.

$$A = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

If we call the first and second rows in A rows R_1 and R_2, and call the first and second columns in B columns C_1 and C_2,

$$A = \begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix}$$

$$B = \begin{array}{c} \begin{array}{cc} C_1 & C_2 \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix} \end{array}$$

then the product of A and B is

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$$

Let's look at each entry in the product:

$(A \cdot B)_{1,1}$ is the product of the **first row** and **first column**

$(A \cdot B)_{2,1}$ is the product of the **second row** and **first column**

$(A \cdot B)_{1,2}$ is the product of the **first row** and **second column**

$(A \cdot B)_{2,2}$ is the product of the **second row** and **second column**

The easy way to tell whether or not you can multiply matrices is to line up their dimensions. For instance, given matrix A is a 2×3 and matrix B is a 3×4, then line up the product AB this way:

$$AB: \ 2 \times \boxed{3 \quad 3} \times 4$$

If the middle numbers match like they do here (they're both 3), then you can multiply the matrices to get a valid result, because you have the same number of columns in the first matrix as rows in the second matrix, which is what we said we needed. If you wanted to multiply B by A, you'd line up the product this way:

$$BA: \ 3 \times \boxed{4 \quad 2} \times 3$$

Because those middle numbers don't match (one is 4, the other is 2), you can't multiply the matrices. You don't have the same number of columns in the first matrix as rows in the second matrix, so the product isn't even defined.

Dimensions of the product

Now that you know how to determine whether or not the product of two matrices will be defined, let's talk about the dimensions of the product.

We said before that, because we have the same number of columns in the first matrix as rows in the second matrix, AB will be defined in this case:

$$AB: \ 2\times\boxed{3 \quad 3}\times 4$$

Once you know that the product AB is defined, you can also quickly know the dimensions of the resulting product. To get those dimensions, just take the number of rows from the first matrix by the number of columns from the second matrix.

$$AB: \boxed{2}\times 3 \quad 3\times\boxed{4}$$

So the dimensions of the product AB will be 2×4. In other words, a 2×3 matrix multiplied by a 3×4 matrix will always result in a 2×4 matrix.

Example

If matrix A is 2×2 and matrix B is 4×2, say whether AB or BA is defined, and give the dimensions of the product if it is defined.

Line up the dimensions for the products AB and BA.

$$AB: \ 2\times\boxed{2 \quad 4}\times 2$$

$$BA: \ 4\times\boxed{2 \quad 2}\times 2$$

For AB, the middle numbers don't match, so that product isn't defined. For BA, the middle numbers match, so that product is defined.

The dimensions of BA are given by the outside numbers,

$$BA: \ \boxed{4}\times 2 \quad 2\times\boxed{2}$$

so the dimensions of BA will be 4×2.

Using the dot product to multiply matrices

The **dot product** is the tool we'll use to multiply an entire row by an entire column. When you're calculating a dot product, you want to think about ordered pairs. For instance, we said that when we take the product of A and B, the first entry we'll need to find is the product of the first row in A and the first column in B. The first row in A is the ordered pair $(2,6)$, and the first column in B is the ordered pair $(-4,1)$.

To take the dot product of these ordered pairs, we take the product of the first values, and then add that result to the product of the second values. In other words, the dot product of $(2,6)$ and $(-4,1)$ is

$2(-4)+6(1)$

$-8+6$

-2

Therefore, to find the product of matrices A and B, we get

$$A \cdot B = \begin{bmatrix} 2 & 6 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 & -2 \\ 1 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2(-4)+6(1) & 2(-2)+6(0) \\ 3(-4)+(-1)(1) & 3(-2)+(-1)(0) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} -8+6 & -4+0 \\ -12+(-1) & -6+0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} -2 & -4 \\ -13 & -6 \end{bmatrix}$$

Properties of matrix multiplication

When it comes to multiplication, matrices do not follow the same rules as real numbers.

Matrix multiplication **is not commutative**. The fact that it's not commutative means that you can't multiply matrices in a different order and still get the same answer.

$$AB \neq BA$$

Matrix multiplication **is associative**. The fact that it's associative means that you can group the multiplication in different ways, and still get the same answer, as long as you don't change the order.

$$(AB)C = A(BC)$$

Matrix multiplication **is distributive**. The fact that it's distributive means that you can distribute multiplication across another value.

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$A(B - C) = AB - AC$$

$$(B - C)A = BA - CA$$

When it comes to the zero matrix, it doesn't matter whether you multiply a matrix by the zero matrix, or multiply the zero matrix by a matrix; you'll get the O matrix either way. But the dimensions of the zero matrix may change, depending on whether it's the first or second matrix in the multiplication.

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