Coterminal angles

If we sketch two angles in standard position, they're **coterminal** if their terminal sides lie on top of each other. In other words, if both angles finish up at the same place, then they're coterminal.

Coterminal angles will always differ by 360° or 2π radians. So to find a coterminal angle, we just add or subtract 360° or 2π as many times as we want to. For instance, let's say we want to find angles that are coterminal with 45°. Adding 360° one, two, and three times to 45° gives three angles that are all coterminal with 45°, and therefore all coterminal with one another:

$$45^\circ + 360^\circ = 405^\circ$$

$$45^\circ + 2(360^\circ) = 765^\circ$$

$$45^\circ + 3(360^\circ) = 1{,}125^\circ$$

We could also subtract 360° once, twice, and three times to find three more angles that are coterminal with 45°, coterminal with each other, and coterminal with the three positive we just found:

$$45^\circ - 360^\circ = -315^\circ$$

$$45^\circ - 2(360^\circ) = -675^\circ$$

$$45^\circ - 3(360^\circ) = -1{,}035^\circ$$

And this pattern continues indefinitely in both the positive and negative directions.

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We can also do this with radians. Instead of adding or subtracting some multiple of 360°, we'd add or subtract any multiple of 2π. For example, all of these angles are coterminal with $\pi/6$:

$$\frac{\pi}{6}+2\pi=\frac{13\pi}{6}$$

$$\frac{\pi}{6}+2(2\pi)=\frac{25\pi}{6}$$

$$\frac{\pi}{6}+3(2\pi)=\frac{37\pi}{6}$$

...

and

$$\frac{\pi}{6}-2\pi=-\frac{11\pi}{6}$$

$$\frac{\pi}{6}-2(2\pi)=-\frac{23\pi}{6}$$

$$\frac{\pi}{6}-3(2\pi)=-\frac{35\pi}{6}$$

...

Let's do an example where we find a few positive coterminal angles.

Example

Find the three smallest positive angles that are coterminal with 67°.

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We can add or subtract 360° to find coterminal angles for 67°. If we were to subtract any multiple of 360°, we'd get a negative angle, but we were asked for only positive angles. Therefore, we need to add multiples of 360° in order to find the angles we need.

$$67^\circ + 1(360^\circ) = 67^\circ + 360^\circ = 427^\circ$$

$$67^\circ + 2(360^\circ) = 67^\circ + 720^\circ = 787^\circ$$

$$67^\circ + 3(360^\circ) = 67^\circ + 1{,}080^\circ = 1{,}147^\circ$$

These are the three smallest positive coterminal angles for 67°.

Let's do an example with radian angles.

Example

Find the two smallest positive angles that are coterminal with $-3\pi/2$.

We can add or subtract 2π to find coterminal angles for $-3\pi/2$. If we were to subtract any multiple of 2π, we'd get another negative angle, but we were asked for only positive angles. Therefore, we need to add multiples of 2π in order to find the angles we need.

$$-\frac{3\pi}{2} + 1(2\pi) = -\frac{3\pi}{2} + \frac{4\pi}{2} = \frac{\pi}{2}$$

$$-\frac{3\pi}{2} + 2(2\pi) = -\frac{3\pi}{2} + \frac{8\pi}{2} = \frac{5\pi}{2}$$

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These are the two smallest positive coterminal angles for $-3\pi/2$.

Now we'll try an example with negative rotations.

Example

Find four negative angles that are coterminal with $6\pi/5$.

In order to find negative angles, we'll need to subtract multiples of 2π.

$$\frac{6\pi}{5}-1(2\pi)=\frac{6\pi}{5}-\frac{10\pi}{5}=-\frac{4\pi}{5}$$

$$\frac{6\pi}{5}-2(2\pi)=\frac{6\pi}{5}-\frac{20\pi}{5}=-\frac{14\pi}{5}$$

$$\frac{6\pi}{5}-3(2\pi)=\frac{6\pi}{5}-\frac{30\pi}{5}=-\frac{24\pi}{5}$$

$$\frac{6\pi}{5}-4(2\pi)=\frac{6\pi}{5}-\frac{40\pi}{5}=-\frac{34\pi}{5}$$

Let's do two quick examples with angles given in DMS so that we know how to handle those as well.

Example

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Find the angle α that's coterminal with $150^\circ 17' 49''$, if we make two full positive rotations around the origin.

To find coterminal angles for DMS angles, we do the same thing we did with angles given in degrees, and we just carry the minutes and seconds along with us.

Since we were asked to make two full positive rotations from $150^\circ 17' 49''$ to find α, we can say that α is

$$\alpha = 150^\circ 17' 49'' + 2(360^\circ)$$

$$\alpha = 150^\circ 17' 49'' + 720^\circ$$

$$\alpha = (150 + 720)^\circ 17' 49''$$

$$\alpha = 870^\circ 17' 49''$$

In the example we just did, the original angle was positive, and we rotated in the positive direction. So the signs of the angle and the rotation matched; they were both positive.

Things get little more complicated when the signs are different (when the angle is positive and we rotate in the negative direction, or when the angle is negative and we rotate in the positive direction). Let's look at an example like that now.

Example

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Find the angle α that's coterminal with $16^\circ 20' 42''$ if we make three full negative rotations around the origin.

Since we were asked to make three full negative rotations from $16^\circ 20' 42''$ to find α, we can say that α is

$$\alpha = (16^\circ + 20' + 42'') - 3(360^\circ)$$

$$\alpha = 16^\circ + 20' + 42'' - 1{,}080^\circ$$

$$\alpha = (16^\circ - 1{,}080^\circ) + 20' + 42''$$

$$\alpha = -1{,}064^\circ + 20' + 42''$$

The reason we separated the degrees, minutes, and seconds from each in this example, but kept them together in the last example, is because for DMS angles, the three parts all must be positive, or all must be negative. Otherwise, if the signs are mixed, then part of the angle is rotating in the positive direction, while the other is rotating in the negative direction, and we don't want that.

To make sure all the signs match, we can calculate degrees first like we did here, and find that we have a negative value for degrees, but then we need to make both the minutes and seconds negative as well.

To make the minutes part negative, we'll borrow -1° from the $-1{,}064^\circ$, and combine that -1° with the $20'$ by using the fact that $1^\circ = 60'$.

$$\alpha = -1{,}063^\circ + (-1^\circ) + 20' + 42''$$

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$$\alpha = -1{,}063^\circ + (-60') + 20' + 42''$$

$$\alpha = -1{,}063^\circ + (-40') + 42''$$

To make the seconds part negative, we'll borrow $-1'$ from the $-40'$, and combine that $-1'$ with the $42''$ by using the fact that $1' = 60''$.

$$\alpha = -1{,}063^\circ + (-39') + (-1') + 42''$$

$$\alpha = -1{,}063^\circ + (-39') + (-60'') + 42''$$

$$\alpha = -1{,}063^\circ + (-39') + (-18'')$$

Now that all three parts are negative, we can write the coterminal angle as $\alpha = -1{,}063^\circ 39' 18''$. The negative sign in front implies that the entire angle is negative, because the negative sign applies to all three parts (degrees, minutes, and seconds) of the DMS angle.

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