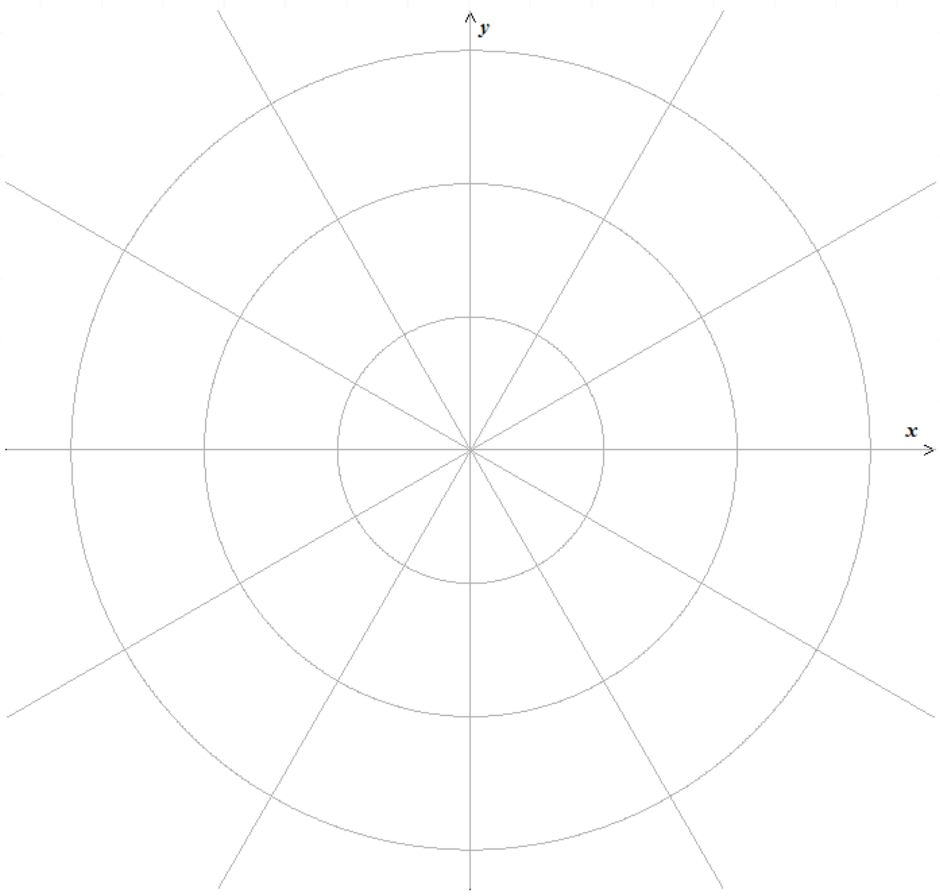
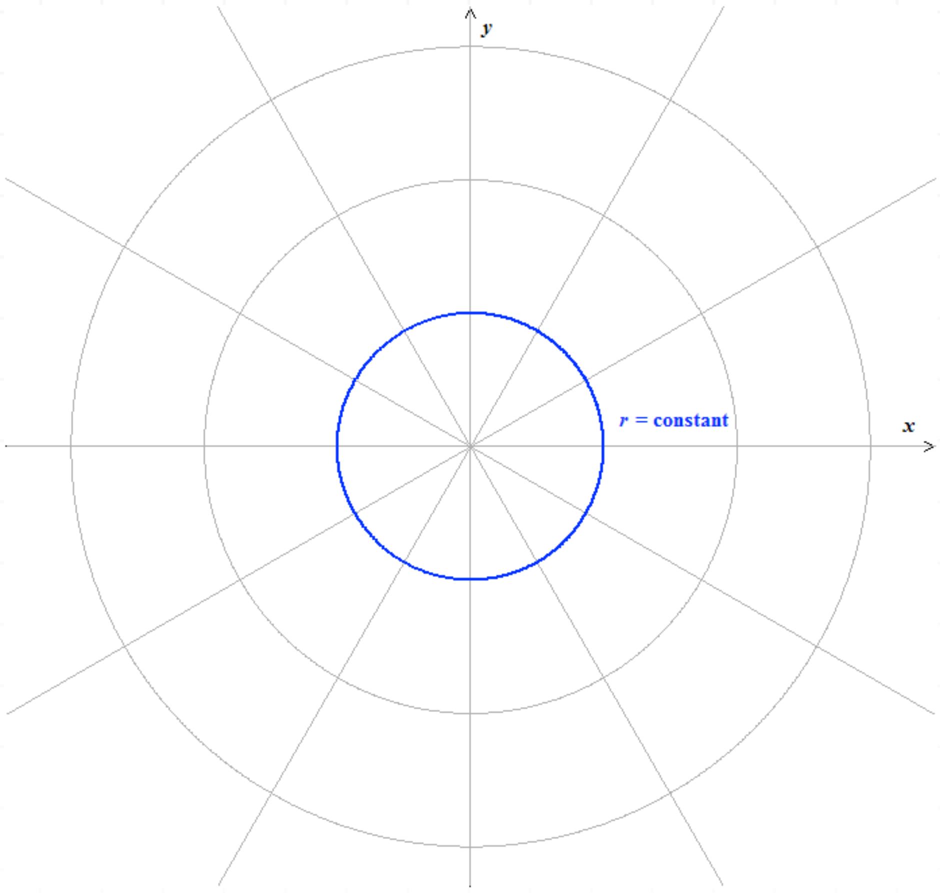
Graph the polar curve, circle

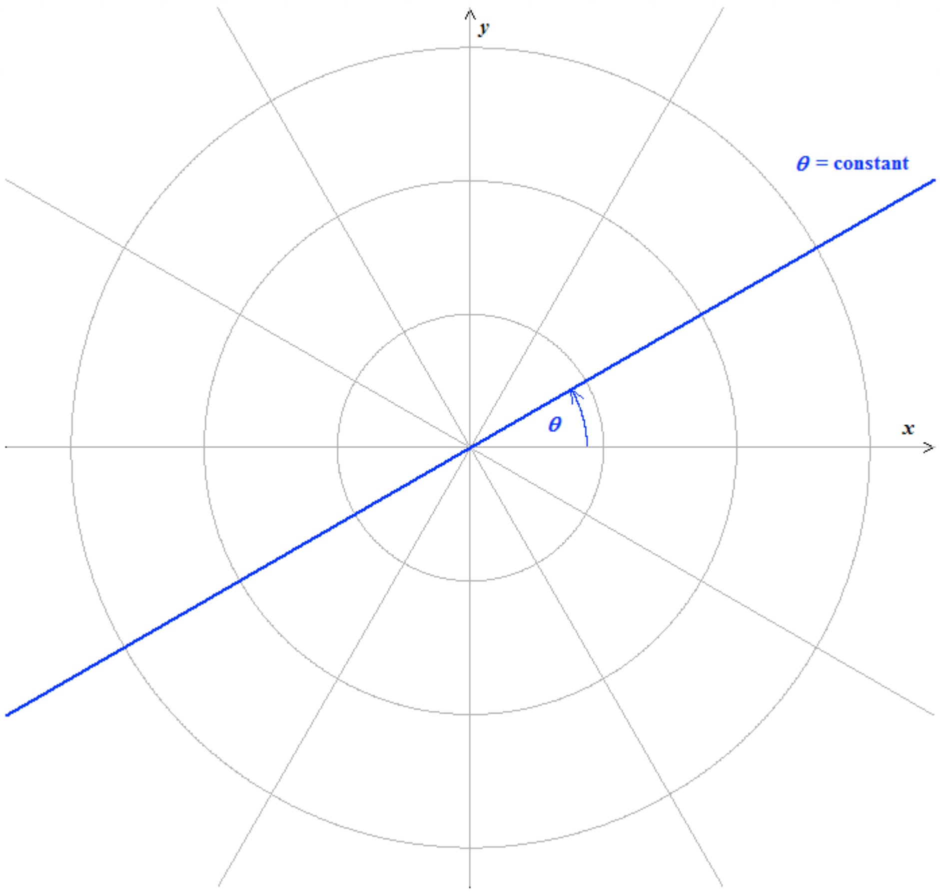
In this lesson, we're going to take a look at how we go about graphing the set of points that satisfy a given polar equation (a polar equation that's expressed entirely in the polar coordinates r, θ). When we graph such an equation, it sometimes helps to use a "polar grid." A polar grid consists of a finite set of circles of equal radius that are centered at the pole and a finite set of lines that pass through the pole.



A given circle of a polar grid consists of all the points that are at a constant distance r from the pole.



A given line of a polar grid is oriented at a constant angle θ relative to the horizontal axis.

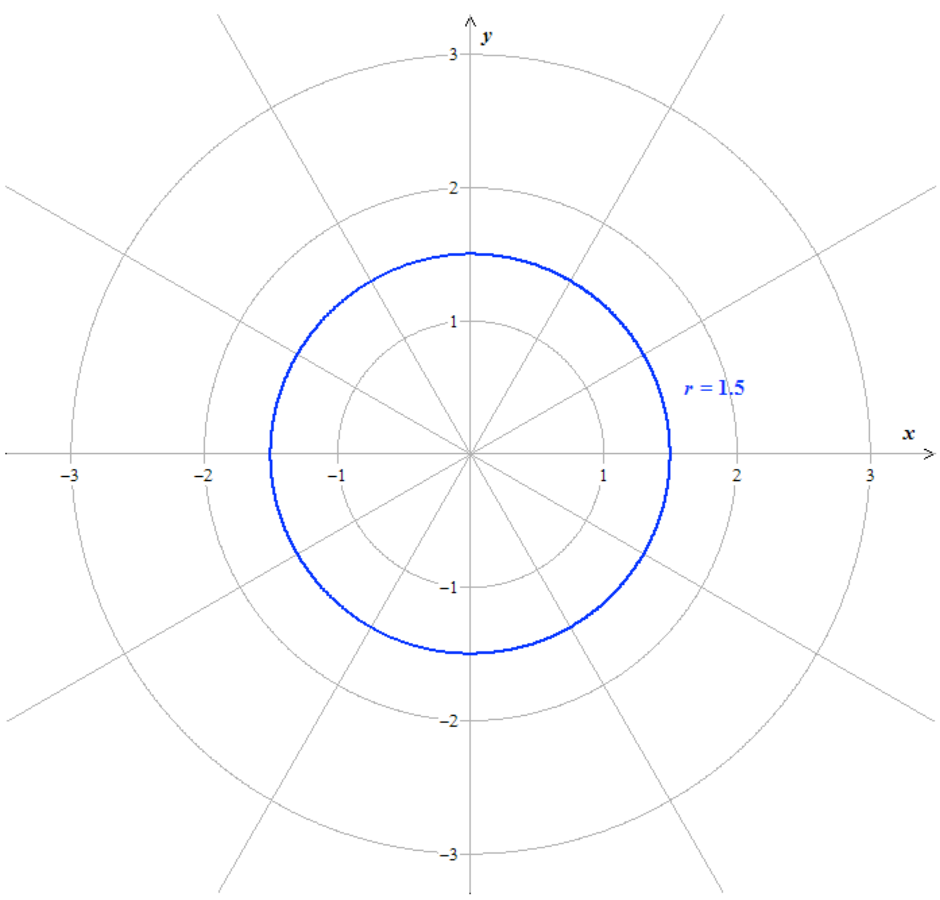


In this lesson, we'll focus on graphing circles. The simplest type of circle is one that's centered at the pole. The polar equation of such a circle has the form $r = a$ for some real number a. If $a = 0$, the "circle" reduces to a single point (namely, the origin). However, what we're really interested in are circles with $a \neq 0$. If a is positive (i.e., if we have the polar equation $r = a$ for some positive constant a), the curve is the set of all points on the circle that's centered at the pole and has radius a.

Example

Graph the curve that satisfies the polar equation $r = 1.5$.

Even if a is negative, we can graph the polar curve that satisfies the equation $r = a$. The solution of this equation is the set of points that have a pair of polar coordinates (r, θ) with $r = a$ and some angle θ.

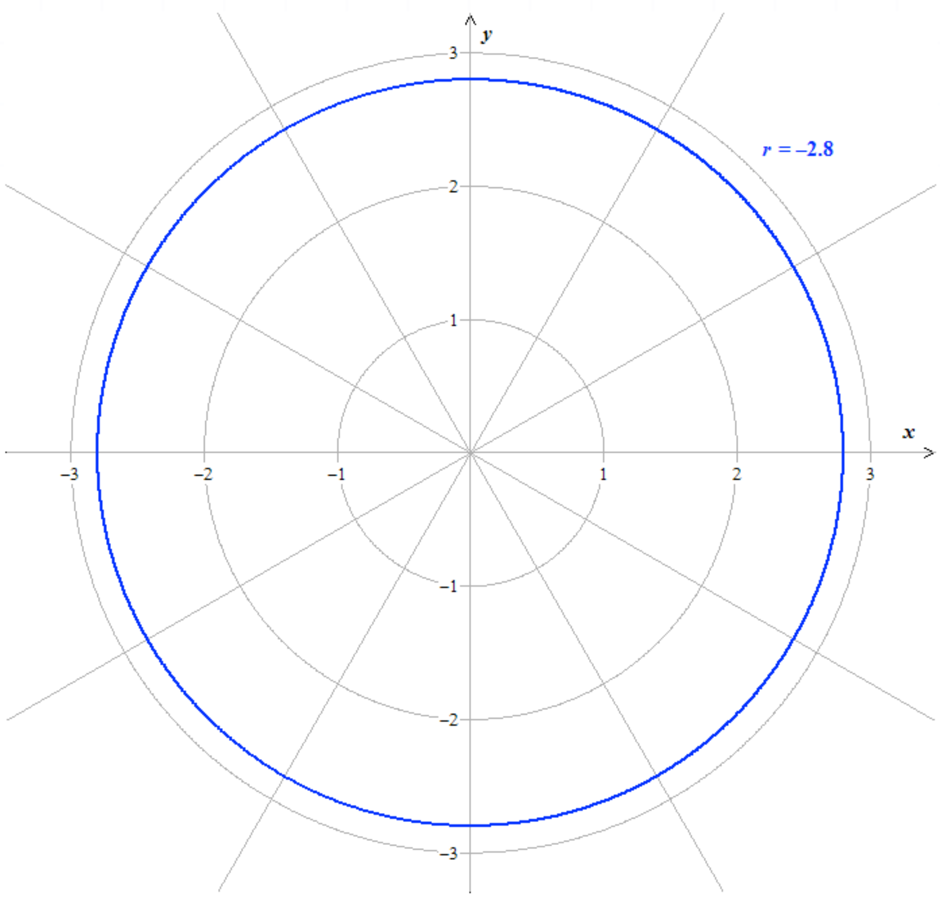


Now recall that such a point also has polar coordinates $(-a, \theta + \pi)$. There is a one-to-one correspondence between the set of angles $\theta + \pi$ and the set of angles θ. Thus if a is negative, the curve that satisfies the polar equation $r = a$ is identical to the curve that satisfies the polar equation $r = -a$ (which is, in turn, just the circle of radius $|a|$ that's centered at the pole).

Example

Graph the curve that satisfies the polar equation $r = -2.8$.

This curve is just the circle of radius 2.8, hence it's identical to the curve that satisfies the polar equation $r = 2.8$.



Circles that are centered somewhere other than at the pole don't have as nice a form as those that are centered at the pole. To determine the polar equations of certain classes of circles that are not centered at the pole but whose center is on either the horizontal axis or the vertical axis, we're going to go back to the general equation of a circle in rectangular coordinates.

In rectangular coordinates, the general form of the equation of a circle is

$$(x-h)^2+(y-k)^2=c^2$$

where (h, k) are the rectangular coordinates of the center of the circle and c is its radius. (Note that c is positive.)

The equation of a circle whose center is on the x axis (but not at the origin) has the following form in rectangular coordinates:

$$(x-h)^2+y^2=c^2$$

In polar coordinates in general, we have

$$x=r\cos\theta$$

$$y=r\sin\theta$$

Therefore, a point (x, y) is on such a circle if and only if

$$(r\cos\theta-h)^2+(r\sin\theta)^2=c^2$$

Expanding the left-hand side, we get

$$(r^2\cos^2\theta-2rh\cos\theta+h^2)+(r^2\sin^2\theta)=c^2$$

Regrouping:

$$(r^2\cos^2\theta + r^2\sin^2\theta) - 2rh\cos\theta + h^2 = c^2$$

Now

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

where the last equality follows from the Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Thus the equation of our circle becomes

$$r^2 - 2rh\cos\theta + h^2 = c^2$$

Subtracting h^2 from both sides, we get

$$r^2 - 2rh\cos\theta = c^2 - h^2$$

We're going to limit our discussion of this to the “simplest” circles that satisfy this equation, namely, those with $h = \pm c$, so the equation

$$r^2 - 2rh\cos\theta = c^2 - h^2$$

becomes

$$r^2 - 2r(\pm c)\cos\theta = 0$$

Equivalently,

$$r^2 \mp 2rc\cos\theta = 0$$

Factoring the left-hand side gives

$$r(r \mp 2c\cos\theta) = 0$$

Thus each point of the circle must satisfy at least one of the equations

$$r = 0, \qquad r = \pm 2c \cos\theta$$

The fact that $h = \pm c$ (where c is positive) implies that the rectangular coordinates of the center of the circle are

$$(x, y) = (c, 0)$$

if h is positive (that is, if $h = c$), and

$$(x, y) = (-c, 0)$$

if h is negative (i.e., if $h = -c$).

If $h = c$, each point of the circle satisfies the polar equation

$$r = 2c \cos\theta$$

and one pair of polar coordinates of the center of the circle is

$$(r, \theta) = (h, 0)$$

If $h = -c$, each point of the circle satisfies the polar equation

$$r = -2c \cos\theta$$

and one pair of polar coordinates of the center of the circle is

$$(r, \theta) = (-h, \pi)$$

The pole is one of the points of the circle, since it's at a distance of $|h| = c$ units from the center of the circle. To see that the pole satisfies the equation

$$r = \pm 2c\cos\theta$$

regardless of whether $h = c$ or $h = -c$, note that for $\theta = \pi/2$ we get

$$r = \pm 2c\cos\left(\frac{\pi}{2}\right) = \pm 2c(0) = 0$$

The general result we have derived in this process is that if a is a nonzero constant, we can set a equal to $2h$ (hence $h = a/2$), so

$$r = a\cos\theta$$

is the polar equation of the circle that's centered at the point

$$(x, y) = (h,0) = \left(\frac{a}{2},0\right)$$

and has a radius of

$$c = \frac{|a|}{2}$$

One pair of polar coordinates of the center of this circle is

$$(r, \theta) = (h,0) = \left(\frac{a}{2},0\right)$$

if h is positive, and

$$(r, \theta) = (-h, \pi) = \left(-\frac{a}{2}, \pi\right)$$

if h is negative.

Example

Graph the curve that satisfies the polar equation $r = 6\cos\theta$.

Since the coefficient of $\cos\theta$ is $a = 6$, we can set a equal to $2h$ (hence $h = a/2 = 3$), so

$$r = 6\cos\theta$$

is the polar equation of the circle that's centered at the point

$$(x, y) = (h,0) = \left(\frac{a}{2},0\right) = (3,0)$$

and has a radius of

$$c = \frac{|a|}{2} = 3$$

Since a is positive, one pair of polar coordinates of the center of this circle is

$$(r, \theta) = (h,0) = \left(\frac{a}{2},0\right) = (3,0)$$

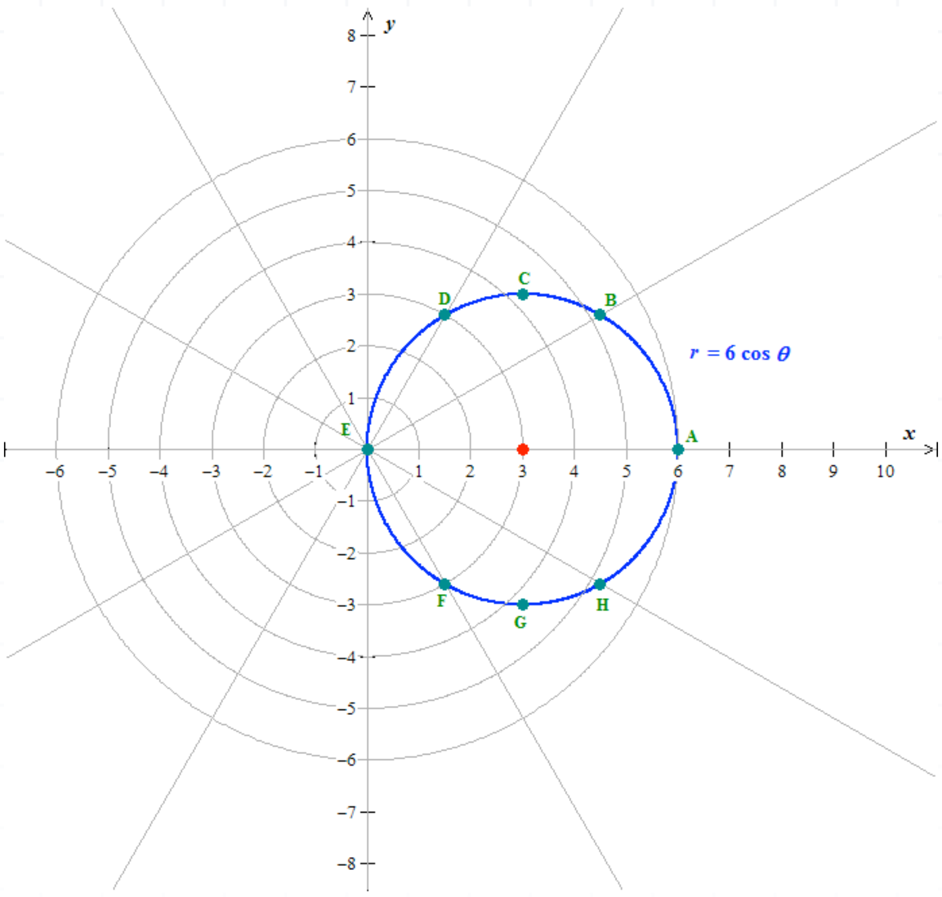
We can think of $r = 6\cos\theta$ as a function, with θ as the independent variable and r as the dependent variable. To graph the circle, we could first find the values of r for a few angles θ, then plot those points (r, θ), and use them to draw the whole circle.

In the following table, the values of $\cos\theta$ and $r = 6\cos\theta$ for a few angles θ in the interval $[0,\pi)$ are shown. In the table, we also give one pair of polar coordinates, (r,θ), for points where the equation $r = 6\cos\theta$ gives us a positive value of r, and we give two pairs of polar coordinates, (r,θ) and $(-r,\theta+\pi)$, for points where the equation $r = 6\cos\theta$ gives us a negative value of r.

The uppercase letters in the column headed "Point" designate the points shown with a green dot on the accompanying graph. You should convince yourself that if you were to tabulate data for the angles $\theta = \pi$, $\theta = 7\pi/6$, $\theta = 5\pi/4$, $\theta = 4\pi/3$, $\theta = 3\pi/2$, $\theta = 5\pi/3$, $\theta = 7\pi/4$, and $\theta = 11\pi/6$, you would again get the points A through H, respectively (perhaps represented by different pairs of polar coordinates than in the existing table).

Point	θ	$\cos\theta$	$r = 6\cos\theta$	(r,θ)	$(-r,\theta+\pi)$
A	0	1	6	$(6,0)$	
B	$\pi/6$	$\sqrt{3}/2$	$3\sqrt{3}$	$\left(3\sqrt{3},\pi/6\right)$	
C	$\pi/4$	$\sqrt{2}/2$	$3\sqrt{2}$	$\left(3\sqrt{2},\pi/4\right)$	
D	$\pi/3$	$1/2$	3	$(3,\pi/3)$	
E	$\pi/2$	0	0	$(0,\pi/2)$	
F	$2\pi/3$	$-1/2$	-3	$(-3,2\pi/3)$	$(3,5\pi/3)$
G	$3\pi/4$	$-\sqrt{2}/2$	$-3\sqrt{2}$	$\left(-3\sqrt{2},3\pi/4\right)$	$\left(3\sqrt{2},7\pi/4\right)$

H	$5\pi/6$	$-\sqrt{3}/2$	$-3\sqrt{3}$	$\left(-3\sqrt{3},5\pi/6\right)$	$\left(3\sqrt{3},11\pi/6\right)$



Example

Graph the curve that satisfies the polar equation $r = -7\cos\theta$.

Since the coefficient of $\cos\theta$ is $a = -7$, we can set a equal to $2h$ (hence $h = a/2 = -7/2$), so

$$r = -7\cos\theta$$

is the polar equation of the circle that's centered at the point

$$(x, y) = (h,0) = \left(\frac{a}{2},0\right) = \left(-\frac{7}{2},0\right)$$

and has a radius of

$$c = \frac{|a|}{2} = \frac{7}{2}$$

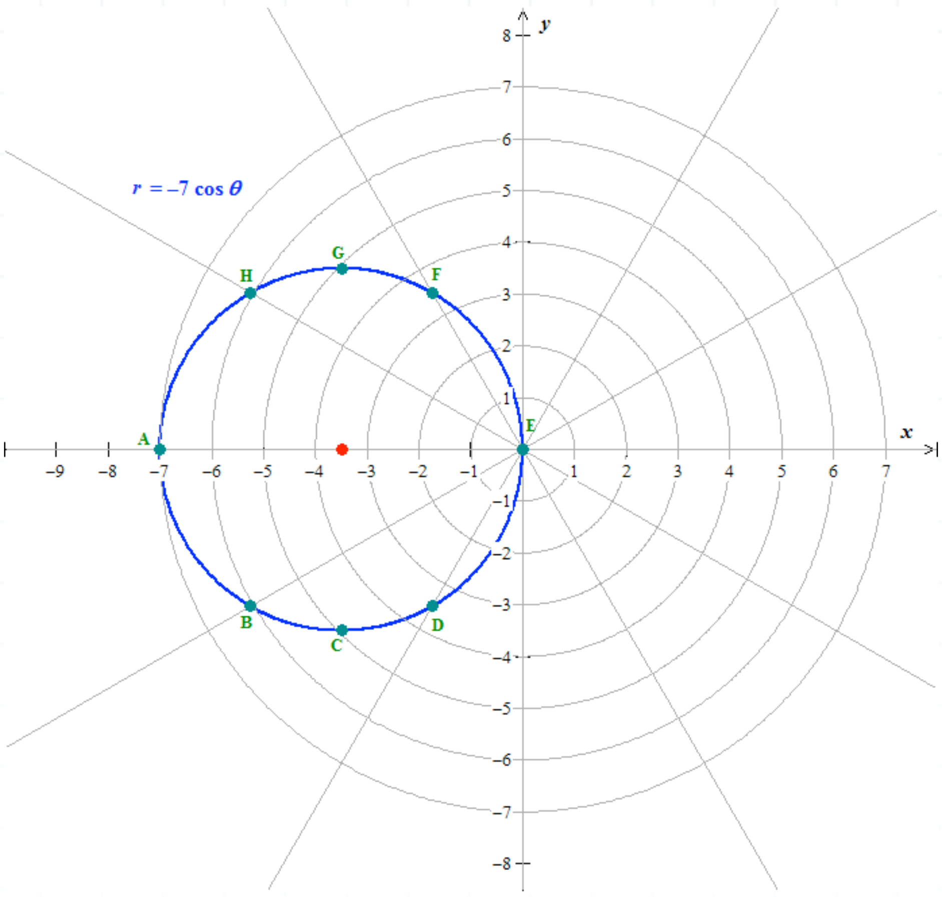
Since a is negative, one pair of polar coordinates of the center of this circle is

$$(r, \theta) = (-h, \pi) = \left(-\frac{a}{2}, \pi\right) = \left(\frac{7}{2}, \pi\right)$$

Thus we get the following table and graph, of the same type we presented for the previous example, with one pair of polar coordinates, (r, θ), for points where the equation $r = -7\cos\theta$ gives us a positive value of r, and two pairs of polar coordinates, (r, θ) and $(-r, \theta + \pi)$, for points where the equation $r = -7\cos\theta$ gives us a negative value of r.

Point	θ	$\cos\theta$	$r = -7\cos\theta$	(r, θ)	$(-r, \theta + \pi)$
A	0	1	-7	$(-7,0)$	$(7,\pi)$
B	$\pi/6$	$\sqrt{3}/2$	$-7\sqrt{3}/2$	$\left(-7\sqrt{3}/2,\pi/6\right)$	$\left(7\sqrt{3}/2,7\pi/6\right)$
C	$\pi/4$	$\sqrt{2}/2$	$-7\sqrt{2}/2$	$\left(-7\sqrt{2}/2,\pi/4\right)$	$\left(7\sqrt{2}/2,5\pi/4\right)$
D	$\pi/3$	$1/2$	$-7/2$	$(-7/2,\pi/3)$	$(7/2,4\pi/3)$
E	$\pi/2$	0	0	$(0,\pi/2)$	
F	$2\pi/3$	$-1/2$	$7/2$	$(7/2,2\pi/3)$	
G	$3\pi/4$	$-\sqrt{2}/2$	$7\sqrt{2}/2$	$\left(7\sqrt{2}/2,3\pi/4\right)$	

H	$5\pi/6$	$-\sqrt{3}/2$	$7\sqrt{3}/2$	$(7\sqrt{3}/2,5\pi/6)$



The equation of a circle whose center is on the y-axis (but not at the pole) has the following form in rectangular coordinates:

$$x^2 + (y-k)^2 = c^2$$

The rectangular coordinates of the center of the circle are $(0,k)$, and c is its radius.

To get the equation of such a circle in polar coordinates, we could use reasoning which is completely analogous to that which we used in deriving the polar equation of a circle whose center is on the x axis. In doing so, we would find that if a is a nonzero constant, then we can set a equal to $2k$ (hence $k = a/2$), so

$$r = a\sin\theta$$

is the polar equation of the circle that's centered at the point

$$(x, y) = (0,k) = \left(0,\frac{a}{2}\right)$$

and has a radius of

$$c = \frac{|a|}{2}$$

One pair of polar coordinates of the center of this circle is

$$(r, \theta) = \left(k,\frac{\pi}{2}\right) = \left(\frac{a}{2},\frac{\pi}{2}\right)$$

if k is positive, and

$$(r, \theta) = \left(-k,\frac{3\pi}{2}\right) = \left(-\frac{a}{2},\frac{3\pi}{2}\right)$$

if k is negative.

Example

Graph the curve that satisfies the polar equation $r = -10\sin\theta$.

Since the coefficient of $\sin\theta$ is $a = -10$, we can set a equal to $2k$ (hence $k = a/2 = -5$), so

$$r = -10\sin\theta$$

is the polar equation of the circle that's centered at the point

$$(x, y) = (0,k) = \left(0,\frac{a}{2}\right) = (0, -5)$$

and has a radius of

$$c = \frac{|a|}{2} = 5$$

Since a is negative, one pair of polar coordinates of the center of this circle is

$$(r, \theta) = \left(-k,\frac{3\pi}{2}\right) = \left(-\frac{a}{2},\frac{3\pi}{2}\right) = \left(5,\frac{3\pi}{2}\right)$$

Thus we get the following table, of the same type we presented for the two previous examples (with two pairs of polar coordinates for points where the equation $r = -10\sin\theta$ gives us a negative value of r), and the accompanying graph.

Point	θ	$\cos\theta$	$r = -10\sin\theta$	(r, θ)	$(-r, \theta + \pi)$
A	0	0	0	$(0,0)$	
B	$\pi/6$	$1/2$	-5	$(-5,\pi/6)$	$(5,7\pi/6)$
C	$\pi/4$	$\sqrt{2}/2$	$-5\sqrt{2}$	$\left(-5\sqrt{2}, \pi/4\right)$	$\left(5\sqrt{2},5\pi/4\right)$
D	$\pi/3$	$\sqrt{3}/2$	$-5\sqrt{3}$	$\left(-5\sqrt{3}, \pi/3\right)$	$\left(5\sqrt{3},4\pi/3\right)$
E	$\pi/2$	1	-10	$(-10,\pi/2)$	$(10,3\pi/2)$
F	$2\pi/3$	$\sqrt{3}/2$	$-5\sqrt{3}$	$\left(-5\sqrt{3},2\pi/3\right)$	$\left(5\sqrt{3},5\pi/3\right)$

G	$3\pi/4$	$\sqrt{2}/2$	$-5\sqrt{2}$	$\left(-5\sqrt{2},3\pi/4\right)$	$\left(5\sqrt{2},7\pi/4\right)$
H	$5\pi/6$	$1/2$	-5	$\left(-5,5\pi/6\right)$	$\left(5,11\pi/6\right)$