Double-angle identities

Very often we'll handle trig functions with an angle of 2θ, instead of just θ, like $\sin 2\theta$. We'll usually want to convert these to get the 2 out of the angle.

But we can't simply say $\sin 2\theta = 2\sin\theta$; that's just not true. 2θ is the argument of the sine function; we're evaluating the sine function at the angle 2θ, so the $\sin$ and 2θ aren't simply multiplied together such that we can pull the 2 out in front of the sine function.

The way we'll get the 2 out of the argument is by using the double-angle identities, which are actually derived from the sum identities we learned about earlier.

Double-angle identities from the sum identities

To build the double-angle identity for sine, we start with the sum identity for sine, $\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha$, and replace α with θ.

$$\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha$$

$$\sin(\theta+\theta)=\sin\theta\cos\theta+\cos\theta\sin\theta$$

$$\sin(2\theta)=\sin\theta\cos\theta+\sin\theta\cos\theta$$

$$\sin(2\theta)=2\sin\theta\cos\theta$$

Similarly, we build the double-angle identity for cosine by starting with the sum identity for cosine, $\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha$, and replacing α with θ.

$$\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha$$

$$\cos(\theta+\theta)=\cos\theta\cos\theta-\sin\theta\sin\theta$$

$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$

Keep in mind that we'll also often see this double-angle cosine identity rewritten using the Pythagorean identity with sine and cosine. From the Pythagorean identity, we can rewrite the double-angle cosine identity by substituting $\cos^2\theta=1-\sin^2\theta$,

$$\cos 2\theta=(1-\sin^2\theta)-\sin^2\theta$$

$$\cos 2\theta=1-\sin^2\theta-\sin^2\theta$$

$$\cos 2\theta=1-2\sin^2\theta$$

or by substituting $\sin^2\theta=1-\cos^2\theta$.

$$\cos 2\theta=\cos^2\theta-(1-\cos^2\theta)$$

$$\cos 2\theta=\cos^2\theta-1+\cos^2\theta$$

$$\cos 2\theta=2\cos^2\theta-1$$

And to build the double-angle identity for tangent, we'll again start with the sum identity for tangent and replace α with θ.

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

To summarize, the **double-angle identities**, including the alternate forms of the double-angle identity for cosine, are

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

All of these give us a way to change the angle 2θ into the angle θ, which we'll be something we'll want to do all the time in Trigonometry and beyond.

Let's do an example with the double-angle cosine identity.

Example

If θ is an angle in the second quadrant with $\cos\theta = -\sqrt{5}/6$, find $\cos 2\theta$ and $\sin 2\theta$.

By the double-angle identity for cosine,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

In order to use the double-angle identity, we first need to find $\sin^2\theta$. By the basic Pythagorean identity,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

Since $\cos\theta = -\sqrt{5}/6$, we get

$$\sin^2\theta = 1 - \left(\frac{\sqrt{5}}{6}\right)^2$$

$$\sin^2\theta = 1 - \frac{5}{36}$$

$$\sin^2\theta = \frac{31}{36}$$

Plus, since we know that $\cos\theta = -\sqrt{5}/6$, we can say

$$\cos^2\theta = \left(-\frac{\sqrt{5}}{6}\right)^2$$

$$\cos^2\theta = \frac{5}{36}$$

Now to find $\cos 2\theta$, we'll substitute into the double-angle identity for cosine.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \frac{5}{36} - \frac{31}{36}$$

$$\cos 2\theta = -\frac{26}{36}$$

$$\cos 2\theta = -\frac{13}{18}$$

To find $\sin 2\theta$ for this same angle θ, we'll use the double-angle identity for sine.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We've already found

$$\sin^2\theta = \frac{31}{36}$$

but we don't yet know the value of $\sin\theta$. Since θ is in the second quadrant, $\sin\theta$ is positive. Therefore,

$$\sin\theta = \sqrt{\frac{31}{36}} = \frac{\sqrt{31}}{\sqrt{36}} = \frac{\sqrt{31}}{6}$$

Now we're ready to apply the double-angle identity for sine.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = 2\left(\frac{\sqrt{31}}{6}\right)\left(-\frac{\sqrt{5}}{6}\right)$$

$$\sin 2\theta = -\frac{2\sqrt{31}\sqrt{5}}{36}$$

$$\sin 2\theta = -\frac{\sqrt{155}}{18}$$

Let's look at an example where we only know the value of $\sin\theta$ and the quadrant of the angle, and we need to find $\sin 2\theta$ and $\cos 2\theta$.

Example

Find $\sin 2\theta$ and $\cos 2\theta$ for an angle θ which lies in the third quadrant and has $\sin\theta = -3/7$.

By the double-angle identity for sine,

$$\sin 2\theta = 2\sin\theta\cos\theta$$

We already know that $\sin\theta = -3/7$, but we need to find $\cos\theta$. We'll plug into the Pythagorean identity with sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(-\frac{3}{7}\right)^2$$

$$\cos^2\theta = 1 - \frac{9}{49}$$

$$\cos^2\theta = \frac{40}{49}$$

$$\cos\theta = \pm\sqrt{\frac{40}{49}}$$

$$\cos\theta = \pm\frac{\sqrt{40}}{7}$$

Since θ is in the third quadrant, we know that $\cos\theta$ is negative, so we can ignore the positive value and say

$$\cos\theta = -\frac{\sqrt{40}}{7}$$

Now, substituting $\cos\theta = -\sqrt{40}/7$ and $\sin\theta = -3/7$ into the double-angle identity for sine, we get

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = 2\left(-\frac{3}{7}\right)\left(-\frac{\sqrt{40}}{7}\right)$$

$$\sin 2\theta = \frac{6\sqrt{40}}{49}$$

$$\sin 2\theta = \frac{12\sqrt{10}}{49}$$

By the double-angle identity for cosine, we get

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \left(-\frac{\sqrt{40}}{7}\right)^2 - \left(-\frac{3}{7}\right)^2$$

$$\cos 2\theta = \frac{40}{49} - \frac{9}{49}$$

$$\cos 2\theta = \frac{31}{49}$$

If we're given just the value of $\tan\theta$ for some angle θ, we can compute the value of $\tan 2\theta$ (even if we don't know the quadrant in which θ lies) by using the double-angle identity for tangent.

Example

Find $\tan 2\theta$ if $\tan\theta = \sqrt{23}$.

By the double-angle identity for tangent,

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{1 - (\sqrt{23})^2}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{1 - 23}$$

$$\tan 2\theta = \frac{2\sqrt{23}}{-22}$$

$$\tan 2\theta = -\frac{\sqrt{23}}{11}$$