Topic: Half-angle identities

Question: Given the inequality, which pair of inequalities is true?

$$-\frac{27\pi}{11} < \theta < -\frac{23\pi}{11}$$

Answer choices:

A $\quad \cos\frac{\theta}{2} > 0 \qquad \sin\frac{\theta}{2} > 0$

B $\quad \cos\frac{\theta}{2} > 0 \qquad \sin\frac{\theta}{2} < 0$

C $\quad \cos\frac{\theta}{2} < 0 \qquad \sin\frac{\theta}{2} > 0$

D $\quad \cos\frac{\theta}{2} < 0 \qquad \sin\frac{\theta}{2} < 0$

Solution: C

Dividing through the inequality by 2.

$$-\frac{27\pi}{11} < \theta < -\frac{23\pi}{11}$$

$$-\frac{27\pi}{22} < \frac{\theta}{2} < -\frac{23\pi}{22}$$

$$-1.23\pi < \frac{\theta}{2} < -1.05\pi$$

Find coterminal angles for the bounds on this interval, in order to make the angles positive.

$$-1.23\pi + 2\pi = 0.77\pi$$

$$-1.05\pi + 2\pi = 0.95\pi$$

The value 0.77π falls in the second quadrant, and so does 0.95π. Which means the angle $\theta/2$ falls in the second quadrant, where sine must be positive and cosine must be negative.

Topic: Half-angle identities

Question: If θ is the angle in the interval $(3\pi/2, 2\pi)$ with $\sin\theta = -2/3$, what are the values of $\cos(\theta/2)$ and $\sin(\theta/2)$?

Answer choices:

A $\cos\frac{\theta}{2} = -\sqrt{\frac{3-\sqrt{7}}{6}}$ $\sin\frac{\theta}{2} = \sqrt{\frac{3+\sqrt{7}}{6}}$

B $\cos\frac{\theta}{2} = \sqrt{\frac{3+\sqrt{7}}{6}}$ $\sin\frac{\theta}{2} = -\sqrt{\frac{3-\sqrt{7}}{6}}$

C $\cos\frac{\theta}{2} = -\sqrt{\frac{3+\sqrt{5}}{6}}$ $\sin\frac{\theta}{2} = \sqrt{\frac{3-\sqrt{5}}{6}}$

D $\cos\frac{\theta}{2} = \sqrt{\frac{3-\sqrt{5}}{6}}$ $\sin\frac{\theta}{2} = \sqrt{\frac{3+\sqrt{5}}{6}}$

Solution: C

Substitute $\sin\theta = -2/3$ into the rewritten form of the Pythagorean identity with sine and cosine.

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(-\frac{2}{3}\right)^2$$

$$\cos^2\theta = 1 - \frac{4}{9}$$

$$\cos^2\theta = \frac{5}{9}$$

$$\cos\theta = \pm\sqrt{\frac{5}{9}}$$

We know θ is in the interval $(3\pi/2,2\pi)$, which means θ is in the fourth quadrant, and therefore that $\cos\theta$ is positive.

$$\cos\theta = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}$$

By the half-angle identities for cosine and sine,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{1+\frac{\sqrt{5}}{3}}{2}} = \pm\sqrt{\frac{\frac{3}{3}+\frac{\sqrt{5}}{3}}{2}} = \pm\sqrt{\frac{3+\sqrt{5}}{6}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}=\pm\sqrt{\frac{1-\frac{\sqrt{5}}{3}}{2}}=\pm\sqrt{\frac{\frac{3}{3}-\frac{\sqrt{5}}{3}}{2}}=\pm\sqrt{\frac{3-\sqrt{5}}{6}}$$

Because θ is in the interval $(3\pi/2,2\pi)$, we know that $\theta/2$ must be in the interval

$$\left(\frac{\frac{3\pi}{2}}{2},\frac{2\pi}{2}\right)=\left(\frac{3\pi}{4},\pi\right)$$

The entire interval $(3\pi/4,\pi)$ is in the second quadrant, where sine is positive and cosine is negative, so

$$\cos\frac{\theta}{2}=-\sqrt{\frac{3+\sqrt{5}}{6}}$$

$$\sin\frac{\theta}{2}=\sqrt{\frac{3-\sqrt{5}}{6}}$$

Topic: Half-angle identities

Question: If θ is the angle in the interval $(17\pi/2,9\pi)$ with $\cos\theta = -3/7$, what are the values of $\sin(\theta/2)$ and $\cos(\theta/2)$?

Answer choices:

A $\sin\frac{\theta}{2} = -\sqrt{\frac{1}{7}}$ $\cos\frac{\theta}{2} = \sqrt{\frac{6}{7}}$

B $\sin\frac{\theta}{2} = \sqrt{\frac{5}{7}}$ $\cos\frac{\theta}{2} = \sqrt{\frac{2}{7}}$

C $\sin\frac{\theta}{2} = \sqrt{\frac{3}{7}}$ $\cos\frac{\theta}{2} = \sqrt{\frac{4}{7}}$

D $\sin\frac{\theta}{2} = -\sqrt{\frac{5}{7}}$ $\cos\frac{\theta}{2} = -\sqrt{\frac{2}{7}}$

Solution: B

Substitute $\cos\theta = -3/7$ into the half-angle identities for cosine and sine,

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

we get

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\left(-\frac{3}{7}\right)}{2}} = \pm\sqrt{\frac{\frac{7(1)+1(-3)}{7}}{2}} = \pm\sqrt{\frac{7-3}{14}} = \pm\sqrt{\frac{4}{14}} = \pm\sqrt{\frac{2}{7}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\left(-\frac{3}{7}\right)}{2}} = \pm\sqrt{\frac{\frac{7(1)+1(3)}{7}}{2}} = \pm\sqrt{\frac{7+3}{14}} = \pm\sqrt{\frac{10}{14}} = \pm\sqrt{\frac{5}{7}}$$

Because θ is in the interval $(17\pi/2, 9\pi)$, the half angle is in the interval

$$\frac{17\pi}{4} < \frac{\theta}{2} < \frac{9\pi}{2}$$

Find coterminal angles for the bounds on this interval.

$$\frac{17\pi}{4} = \frac{16\pi}{4} + \frac{\pi}{4} = 4\pi + \frac{\pi}{4}$$

$$\frac{9\pi}{2} = \frac{8\pi}{2} + \frac{\pi}{2} = 4\pi + \frac{\pi}{2}$$

Therefore, we can say

$$\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$$

So $\theta/2$ is in the first quadrant, which means both $\sin(\theta/2)$ and $\cos(\theta/2)$ are positive.

$$\sin\frac{\theta}{2} = \sqrt{\frac{5}{7}}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{2}{7}}$$