The set of all possible angles

Previously we defined the set of angles coterminal with θ as

$\alpha = \theta + n(360^\circ)$ or $\alpha = \theta + n(2\pi)$

where n is any integer. We've also said that, from what we understand about the symmetry of x- and y-values across axes, and from the even-odd identities, sine and cosine and the other trig functions can have the same value at different angles.

Given all that, in this lesson we want to take a brief moment to realize that the angles θ that satisfy a trig equation will always include both

1. the complete set of coterminal angles, and
2. the complete set of coterminal angles at any other angle that satisfies the equation.

The easiest way to see how to find the complete solution to a trig equation is to work through an example, so let's do one with a cosine equation.

Example

Solve the equation $\cos\theta = 0$ for all possible values of θ.

We know that the cosine function represents the x-value in a coordinate point, so we might recognize that $\cos\theta = 0$ is telling us $x = 0$. From there, we know along the unit circle that $x = 0$ at $\theta = 90^\circ$ and $\theta = 270^\circ$, so we may be tempted to give the solution as that set of two angles.

But we can't forget to account for all the angles that are coterminal with these two! We can think of the coterminal angles in two sets, one for $\theta = 90^\circ$ and one for $\theta = 270^\circ$. Don't forget to include both positive and negative angles.

$$90^\circ, 90^\circ \pm 360^\circ, 90^\circ \pm 720^\circ, \ \dots$$

$$270^\circ, 270^\circ \pm 360^\circ, 270 \pm 720^\circ, \ \dots$$

If we use interval notation, we could express these two sets together as

$$\theta = \{90^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\} \cup \{270^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

$$\theta = \{90^\circ + (360^\circ)k: \ k \in \mathbb{Z}\} \cup \{270^\circ + (360^\circ)k: \ k \in \mathbb{Z}\}$$

where $\mathbb{Z}$ is the set of all integers. But since we know that $270^\circ = 90^\circ + 180^\circ$, we can rewrite this as

$$\theta = \{90^\circ + (180^\circ)k: \ k \in \mathbb{Z}\}$$

This notation is telling us that the set of all possible solutions for θ is given by 90°, plus or minus any k number of full 360° rotations, together with (∪) 270°, plus or minus any k number of full 360° rotations. Defining k as any whole number, together with the ± sign, let's us express the infinite set of coterminal angles in a really compact way.

Because 90° is $\pi/2$ and 270° is $3\pi/2$, we could also express the complete solution set in radians as

$$\left\{\frac{\pi}{2} \pm 2\pi k: \ k = 0,1,2,\dots\right\} \cup \left\{\frac{3\pi}{2} \pm 2\pi k: \quad k = 0,1,2,\dots\right\}$$

$$\left\{\frac{\pi}{2} + 2\pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{3\pi}{2} + 2\pi k: \quad k \in \mathbb{Z}\right\}$$

But since we know that $\frac{3\pi}{2} = \frac{\pi}{2} + \pi$, we can rewrite this as

$$\theta = \left\{\frac{\pi}{2} + \pi k: \ k \in \mathbb{Z}\right\}$$

Let's look at another example with cosine, but this time we'll have a value that doesn't fall on one of the major axes.

Example

Solve the equation $\cos\theta = \sqrt{2}/2$ for all possible values of θ.

From the unit circle, we know that

$$\cos 45^\circ = \frac{\sqrt{2}}{2}$$

So 45° is one solution, but we need to include all other angles that are coterminal with 45°.

$$\{45^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

Remember in the last example that $\cos\theta = 0$ at both 90° and 270°. Similarly in this example, we have to realize that $\cos\theta = \sqrt{2}/2$ is also true at $\theta = 315^\circ$, since cosine represents the x-value from the coordinate point, and x is positive in both the first (45°) and fourth (315°) quadrants.

The complete set of angles that are coterminal with $\theta = 315^\circ$ is

$$\{315^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

Combining our results, we say that the complete set of possible solutions to $\cos\theta = \sqrt{2}/2$ is

$$\theta = \{45^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\} \cup \{315^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

$$\theta = \{45^\circ + (360^\circ)k: \ k \in \mathbb{Z}\} \cup \{315^\circ + (360^\circ)k: \ k \in \mathbb{Z}\}$$

or in radians,

$$\left\{\frac{\pi}{4} \pm 2\pi k: \ k = 0,1,2,\dots\right\} \cup \left\{\frac{7\pi}{4} \pm 2\pi k: \ k = 0,1,2,\dots\right\}$$

$$\left\{\frac{\pi}{4} + 2\pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{7\pi}{4} + 2\pi k: \ k \in \mathbb{Z}\right\}$$

Let's do an example where the value of cosine is negative.

Example

Solve the equation $\cos\theta = -\sqrt{3}/2$ for all possible values of θ.

The equation is telling us that the value of cosine is negative, which means the value of the x-coordinate is negative. Therefore, the solution will be limited to values of θ in the second and third quadrants.

From the unit circle, we know that

$$\cos 150^\circ = -\frac{\sqrt{3}}{2}$$

So 150° is one solution, but we need to include all other angles that are coterminal with 150°.

$$\{150^\circ \pm (360^\circ)k:\ k = 0,1,2,\dots\}$$

We can find the other set of angles by symmetry. If 150° gives us one set of angles, and 150° is 30° "above" the x-axis, then the other set of angles is at 30° "below" the x-axis, so the other angle is $(180 + 30)^\circ = 210^\circ$. Which means the set

$$\{210^\circ \pm (360^\circ)k:\ k = 0,1,2,\dots\}$$

also satisfies the equation. Combining these results, the complete solution set is

$$\theta = \{150^\circ \pm (360^\circ)k:\ k = 0,1,2,\dots\} \cup \{210^\circ \pm (360^\circ)k:\ k = 0,1,2,\dots\}$$

$$\theta = \{150^\circ + (360^\circ)k:\ k \in \mathbb{Z}\} \cup \{210^\circ + (360^\circ)k:\ k \in \mathbb{Z}\}$$

or in radians,

$$\left\{\frac{5\pi}{6} \pm 2\pi k:\ k = 0,1,2,\dots\right\} \cup \left\{\frac{7\pi}{6} \pm 2\pi k:\ k = 0,1,2,\dots\right\}$$

$$\left\{\frac{5\pi}{6}+2\pi k:\ k\in\mathbb{Z}\right\}\cup\left\{\frac{7\pi}{6}+2\pi k:\ k\in\mathbb{Z}\right\}$$

Now let's do some examples with the sine function, starting with a positive value for sine.

Example

Solve the equation $\sin\theta=1/2$ for all possible values of θ.

The value of sine is positive, which means the y-coordinate is positive, and therefore that angles which satisfy the equation are found in the first and second quadrants.

From the unit circle, we know that

$$\sin 30^\circ=\frac{1}{2}$$

So 30° is one solution, but we need to include all other angles that are coterminal with 30°.

$$\{30^\circ\pm(360^\circ)k:\ k=0,1,2,\ldots\}$$

But angles in the first and second quadrant have equivalent sine values, which means 150° is also a solution, as well as all angles that are coterminal with 150°:

$$\{150^\circ\pm(360^\circ)k:\ k=0,1,2,\ldots\}$$

Combining these results, the complete solution set is

$$\theta = \{30^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\} \cup \{150^\circ \pm (360^\circ)k: \ k = 0,1,2,\ldots\}$$

$$\theta = \{30^\circ + (360^\circ)k: \ k \in \mathbb{Z}\} \cup \{150^\circ + (360^\circ)k: \ k \in \mathbb{Z}\}$$

or in radians,

$$\left\{\frac{\pi}{6} \pm 2\pi k: \ k = 0,1,2,\ldots\right\} \cup \left\{\frac{5\pi}{6} \pm 2\pi k: \ k = 0,1,2,\ldots\right\}$$

$$\left\{\frac{\pi}{6} + 2\pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{6} + 2\pi k: \ k \in \mathbb{Z}\right\}$$

Finally, we'll look at the solution set when sine has a negative value.

Example

Solve the equation $\sin\theta = -\sqrt{3}/2$ for all possible values of θ.

The value of sine is negative, which means the y-coordinate is negative, and therefore that angles which satisfy the equation are found in the third and fourth quadrants.

From the unit circle, we know that

$$\sin 240^\circ = -\frac{\sqrt{3}}{2}$$

So 240° is one solution, but we need to include all other angles that are coterminal with 240°.

$$\{240^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

But angles in the third and fourth quadrant have equivalent sine values, which means 300° is also a solution, as well as all angles that are coterminal with 300°:

$$\{300^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

Combining these results, the complete solution set is

$$\theta = \{240^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\} \cup \{300^\circ \pm (360^\circ)k: \ k = 0,1,2,\dots\}$$

$$\theta = \{240^\circ + (360^\circ)k: \ k \in \mathbb{Z}\} \cup \{300^\circ + (360^\circ)k: \ k \in \mathbb{Z}\}$$

or in radians,

$$\left\{\frac{4\pi}{3} \pm 2\pi k: \ k = 0,1,2,\dots\right\} \cup \left\{\frac{5\pi}{3} \pm 2\pi k: \ k = 0,1,2,\dots\right\}$$

$$\left\{\frac{4\pi}{3} + 2\pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{3} + 2\pi k: \ k \in \mathbb{Z}\right\}$$