Roots of complex numbers

We know how to raise a complex number to a power, but in this lesson we want to talk about the opposite operation: how to find the roots of a complex number.

To do this, we always want to start with a complex number that's in polar form, $z = r(\cos\theta + i\sin\theta)$. If the complex number is given in rectangular form $z = a + bi$, go ahead and convert it to polar form to get started.

Once the complex number is in polar form, then its nth roots are are given in radians by

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

or in degrees by

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

What do we mean by the nth roots? Well, if we're finding the 5th roots of a complex number, it means we're finding the set of values that, when we raise them to the 5th power, are equal to the complex number.

Keep in mind also that there are always n nth roots. So if we're looking for 5th roots, we'll find 5 of them. If we're looking for 3rd roots, we'll find 3 of them.

And the roots are always defined by the values $k = 0, 1, 2, 3, \dots n - 1$. So the 5th roots will be given by $k = 0, 1, 2, 3, 4$.

Example

Find the 4th roots of the complex number.

$$z = 81(\cos 180^\circ + i \sin 180^\circ)$$

Because we're looking for 4th roots, we know there will be 4 of them, and they'll be given by $k = 0, 1, 2, 3$.

And since this complex number is given in degrees, we'll plug $n = 4$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i \sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[4]{z} = \sqrt[4]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{4}\right) + i \sin\left(\frac{\theta + 360^\circ k}{4}\right)\right]$$

In the complex number we were given, $r = 81$ and $\theta = 180^\circ$, so

$$\sqrt[4]{z} = \sqrt[4]{81}\left[\cos\left(\frac{180^\circ + 360^\circ k}{4}\right) + i \sin\left(\frac{180^\circ + 360^\circ k}{4}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3$.

For $k = 0$:

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{81}\left[\cos\left(\frac{180^\circ + 360^\circ(0)}{4}\right) + i\sin\left(\frac{180^\circ + 360^\circ(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{81}\left(\cos\frac{180^\circ}{4} + i\sin\frac{180^\circ}{4}\right)$$

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{81}\left(\cos 45^\circ + i\sin 45^\circ\right)$$

$$\sqrt[4]{z}_{k=0} = 3\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$\sqrt[4]{z}_{k=0} = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

For $k = 1$:

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{81}\left[\cos\left(\frac{180^\circ + 360^\circ(1)}{4}\right) + i\sin\left(\frac{180^\circ + 360^\circ(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{81}\left(\cos\frac{540^\circ}{4} + i\sin\frac{540^\circ}{4}\right)$$

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{81}\left(\cos 135^\circ + i\sin 135^\circ\right)$$

$$\sqrt[4]{z}_{k=1} = 3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$\sqrt[4]{z}_{k=1} = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{81}\left[\cos\left(\frac{180^\circ + 360^\circ(2)}{4}\right) + i\sin\left(\frac{180^\circ + 360^\circ(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{81}\left(\cos\frac{900^\circ}{4} + i\sin\frac{900^\circ}{4}\right)$$

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{81}\left(\cos 225^\circ + i\sin 225^\circ\right)$$

$$\sqrt[4]{z}_{k=2} = 3\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$\sqrt[4]{z}_{k=2} = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{81}\left[\cos\left(\frac{180^\circ + 360^\circ(3)}{4}\right) + i\sin\left(\frac{180^\circ + 360^\circ(3)}{4}\right)\right]$$

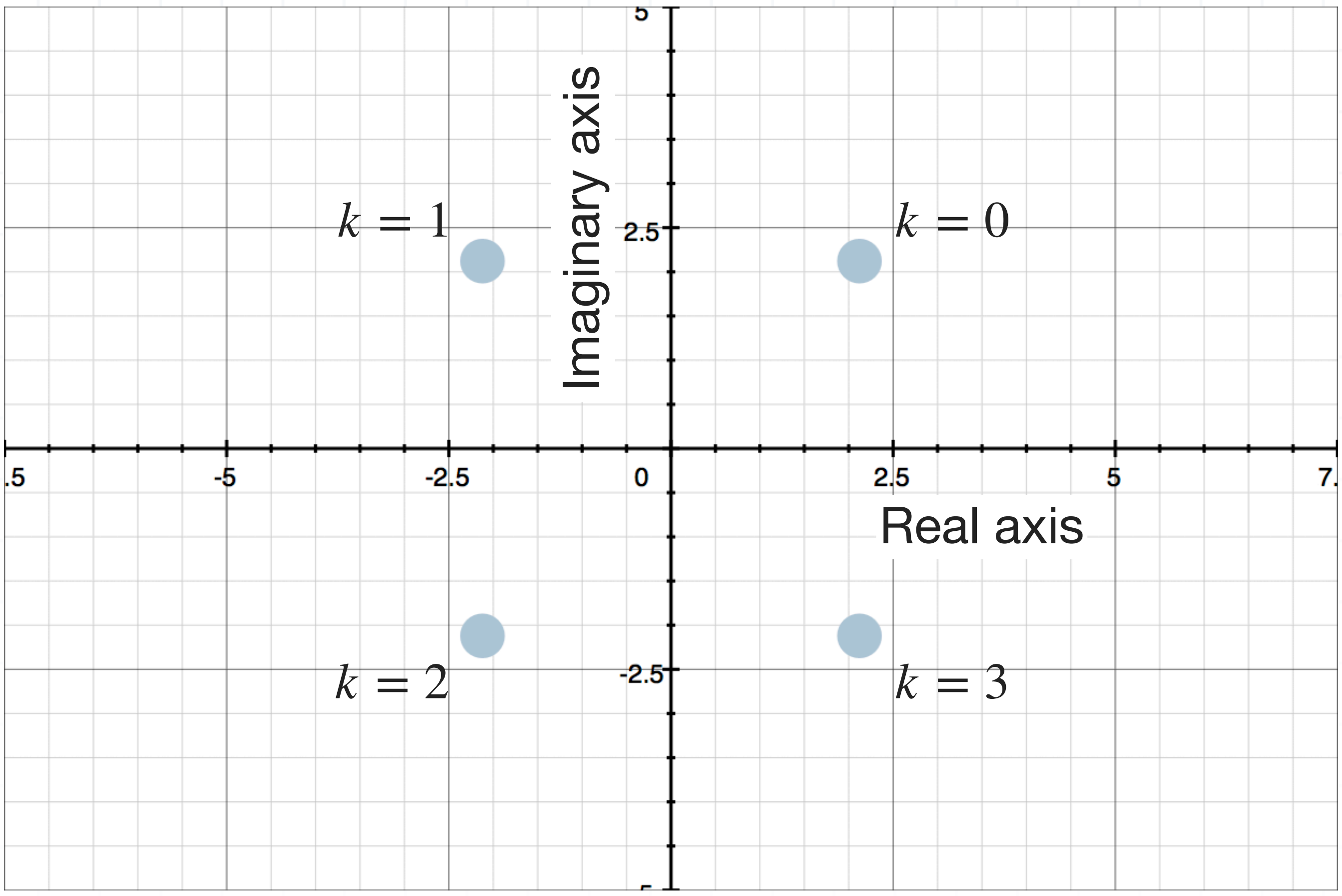
$$\sqrt[4]{z}_{k=3} = \sqrt[4]{81}\left(\cos\frac{1{,}260^\circ}{4} + i\sin\frac{1{,}260^\circ}{4}\right)$$

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{81}\left(\cos 315^\circ + i\sin 315^\circ\right)$$

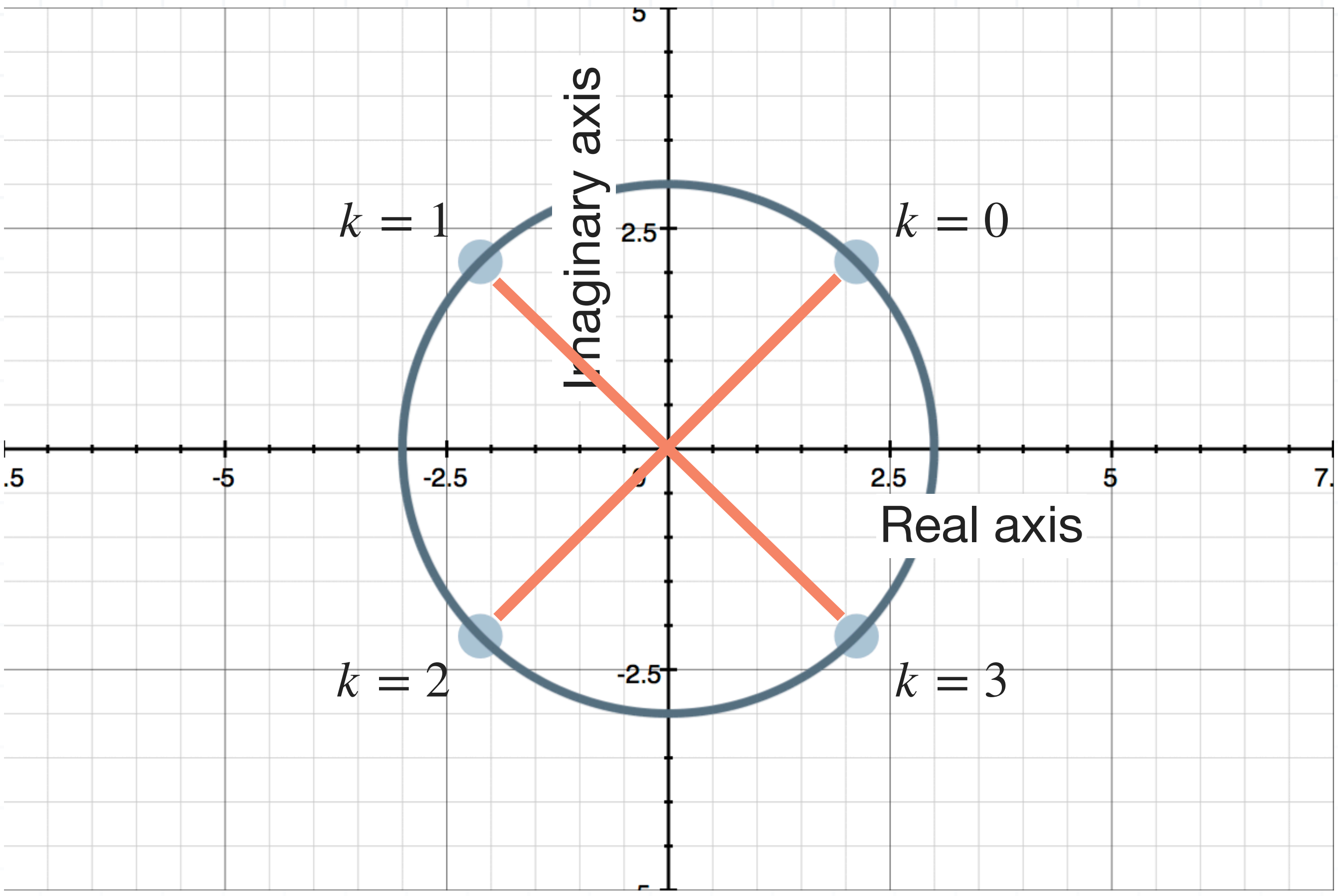
$$\sqrt[4]{z}_{k=3} = 3\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$\sqrt[4]{z}_{k=3} = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$

We found four 4th roots. If we graph them in the complex plane, we can see what they look like.



Notice how these fourth roots of the complex number (in which $r = 3$), divide the circle with radius 3 into four equal parts.



Imaginary axis
Real axis
k = 0
k = 1
k = 2
k = 3
5
2.5
-2.5
-5
.5
-5
-2.5
0
2.5
5
7.