Topic: Complete solution set of the equation

Question: Find the complete solution set of the equation.

$$\cos\left(\theta+\frac{3\pi}{2}\right)=\frac{\sqrt{3}}{2}$$

Answer choices:

A $\theta=\left\{\frac{2\pi}{3}+2n\pi\right\}\cup\left\{\frac{4\pi}{3}+2n\pi\right\}$ where n is any integer

B $\theta=\left\{\frac{\pi}{6}+2n\pi\right\}\cup\left\{\frac{5\pi}{6}+2n\pi\right\}$ where n is any integer

C $\theta=\left\{\frac{\pi}{4}+2n\pi\right\}\cup\left\{\frac{3\pi}{4}+2n\pi\right\}$ where n is any integer

D $\theta=\left\{\frac{\pi}{3}+2n\pi\right\}\cup\left\{\frac{2\pi}{3}+2n\pi\right\}$ where n is any integer

Solution: D

By the sum identity for cosine,

$$\cos\left(\theta+\frac{3\pi}{2}\right)=\cos\theta\cos\left(\frac{3\pi}{2}\right)-\sin\theta\sin\left(\frac{3\pi}{2}\right)$$

$$\cos\left(\theta+\frac{3\pi}{2}\right)=(\cos\theta)(0)-(\sin\theta)(-1)$$

$$\cos\left(\theta+\frac{3\pi}{2}\right)=\sin\theta$$

Replacing the left side of the equation with $\sqrt{3}/2$, we realize that the equation we need to solve is

$$\sin\theta=\frac{\sqrt{3}}{2}$$

From the unit circle, we know this equation is true at $\pi/3$ and $2\pi/3$, so

$$\theta=\frac{\pi}{3}+2n\pi$$

$$\theta=\frac{2\pi}{3}+2n\pi$$

Therefore, the complete solution set is

$$\theta=\left\{\frac{\pi}{3}+2n\pi\right\}\cup\left\{\frac{2\pi}{3}+2n\pi\right\}$$ where n is any integer

Topic: Complete solution set of the equation

Question: Find every angle in the interval $[0,2\pi)$ that satisfies $\csc^2\theta + \csc\theta = 2$.

Answer choices:

A $\quad \theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{\pi}{2}$

B $\quad \theta = \dfrac{2\pi}{3}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

C $\quad \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

D $\quad \theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{\pi}{2}$

Solution: A

We'll start be rewriting $\csc^2\theta + \csc\theta = 2$ as

$$\csc^2\theta + \csc\theta - 2 = 0$$

The left side of this equation is a quadratic, which means it can be factored as

$$(\csc\theta + 2)(\csc\theta - 1) = 0$$

Now we can set each factor equal to 0 individually, and find the angles in the principal interval that satisfy each equation. We get

$$\csc\theta + 2 = 0$$

$$\csc\theta = -2$$

$$\frac{1}{\sin\theta} = -2$$

$$1 = -2\sin\theta$$

$$\sin\theta = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

and

$$\csc\theta - 1 = 0$$

$$\csc\theta = 1$$

$$\frac{1}{\sin\theta} = 1$$

$$\sin\theta = 1$$

$$\theta = \frac{\pi}{2}$$

Therefore, the full solution set of angles in $[0,2\pi)$ is $\pi/2$, $7\pi/6$, $11\pi/6$.

Topic: Complete solution set of the equation

Question: Find every angle in the interval $[0,2\pi)$ that satisfies $\tan(4\theta+\pi)=-1$.

Answer choices:

A $\quad \theta=\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$

B $\quad \theta=\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

C $\quad \theta=\frac{3\pi}{16}, \frac{7\pi}{16}, \frac{11\pi}{16}, \frac{15\pi}{16}, \frac{19\pi}{16}, \frac{23\pi}{16}, \frac{27\pi}{16}, \frac{31\pi}{16}$

D $\quad \theta=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

Solution: C

Using the sum identity for tangent, we can rewrite the left side of $\tan(4\theta + \pi) = -1$ as

$$\frac{\tan(4\theta) + \tan(\pi)}{1 - \tan(4\theta)\tan(\pi)}$$

$$\frac{\tan(4\theta) + 0}{1 - \tan(4\theta)(0)}$$

$$\frac{\tan(4\theta)}{1}$$

$$\tan(4\theta)$$

Therefore, the equation we need to solve is $\tan(4\theta) = -1$. The tangent of an angle will be -1 when the sine and cosine of the angle are equal, but with opposite signs, which will happen in the second quadrant at $3\pi/4$ and in the fourth quadrant at $7\pi/4$. So the angles that satisfy the equation will be these two, and any angles coterminal with these.

Therefore, we need to solve two equations:

$$4\theta = \frac{3\pi}{4} + 2n\pi$$

$$\theta_1 = \frac{3\pi}{16} + \frac{n\pi}{2}$$

and

$$4\theta = \frac{7\pi}{4} + 2n\pi$$

$$\theta_2 = \frac{7\pi}{16} + \frac{n\pi}{2}$$

Now we need to test different values of n to find angles that satisfy the equation in the interval $[0,2\pi)$. At $n = 0$, the angle equations give

$$\theta_1 = \frac{3\pi}{16} + \frac{0\pi}{2} = \frac{3\pi}{16}$$

$$\theta_2 = \frac{7\pi}{16} + \frac{0\pi}{2} = \frac{7\pi}{16}$$

At $n = 1$, the angle equations give

$$\theta_1 = \frac{3\pi}{16} + \frac{1\pi}{2} = \frac{3\pi}{16} + \frac{8\pi}{16} = \frac{11\pi}{16}$$

$$\theta_2 = \frac{7\pi}{16} + \frac{1\pi}{2} = \frac{7\pi}{16} + \frac{8\pi}{16} = \frac{15\pi}{16}$$

At $n = 2$, the angle equations give

$$\theta_1 = \frac{3\pi}{16} + \frac{2\pi}{2} = \frac{3\pi}{16} + \frac{16\pi}{16} = \frac{19\pi}{16}$$

$$\theta_2 = \frac{7\pi}{16} + \frac{2\pi}{2} = \frac{7\pi}{16} + \frac{16\pi}{16} = \frac{23\pi}{16}$$

At $n = 3$, the angle equations give

$$\theta_1 = \frac{3\pi}{16} + \frac{3\pi}{2} = \frac{3\pi}{16} + \frac{24\pi}{16} = \frac{27\pi}{16}$$

$$\theta_2 = \frac{7\pi}{16} + \frac{3\pi}{2} = \frac{7\pi}{16} + \frac{24\pi}{16} = \frac{31\pi}{16}$$

At $n = 4$, the angle equations give

$$\theta_1 = \frac{3\pi}{16} + \frac{4\pi}{2} = \frac{3\pi}{16} + \frac{32\pi}{16} = \frac{35\pi}{16}$$

$$\theta_2 = \frac{7\pi}{16} + \frac{4\pi}{2} = \frac{7\pi}{16} + \frac{32\pi}{16} = \frac{39\pi}{16}$$

These $35\pi/16$ and $39\pi/16$ angles are the first angles we found outside the interval $[0,2\pi)$, so we'll exclude these from the solution set. But all the angles we found previously are within the interval $[0,2\pi)$, so the full solution set is

$$\theta = \frac{3\pi}{16}, \frac{7\pi}{16}, \frac{11\pi}{16}, \frac{15\pi}{16}, \frac{19\pi}{16}, \frac{23\pi}{16}, \frac{27\pi}{16}, \frac{31\pi}{16}$$