Matrix inverses, and invertible and singular matrices

We've talked about matrix addition, subtraction, and multiplication, and now in this section we'll work on matrix division. But we need to address a couple of other things first.

The determinant

The determinant of a matrix is a value we'll use frequently whenever we deal with matrices in any branch of math, so it's important to know how to find it. A matrix is always given in brackets, like this:

$$\begin{bmatrix}-2 & 4\\ 3 & 0\end{bmatrix}$$

but when you want to indicate the determinant instead, you use straight lines.

$$\begin{vmatrix}-2 & 4\\ 3 & 0\end{vmatrix}$$

To calculate the determinant, you multiply the value in the upper left by the value in the lower right, then subtract the product of the lower left and upper right. So the determinant for this matrix would be given by

$$\begin{vmatrix}-2 & 4\\ 3 & 0\end{vmatrix} = (-2)(0) - (3)(4)$$

$$\begin{vmatrix}-2 & 4\\ 3 & 0\end{vmatrix} = 0 - 12$$

$$\begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix} = -12$$

Division as multiplication by the reciprocal

To build toward matrix division, we want to remember that dividing by some value is the same as multiplying by the reciprocal of that value. For instance, dividing by 4 is the same as multiplying by $1/4$. So if k is a real number, then we know that

$$k \cdot \frac{1}{k} = 1$$

If we call $1/k$ the inverse of k and instead write it as k^{-1}, then we could rewrite this equation as

$$kk^{-1} = 1$$

and read this as "k multiplied by the inverse of k is 1." What we want to know now is whether this is also true for matrices. If I divide matrix K by matrix K, or multiply matrix K by its inverse, do I get back to 1? In other words, we're trying to prove that

$$K \cdot \frac{1}{K} = I \text{ or } KK^{-1} = I$$

where I is the identity matrix, which of course is the matrix equivalent of 1.

Matrix inverses

Let's go ahead and give away the surprise up front: matrix division is a valid operation, and multiplying a matrix by its inverse will result in the identity matrix.

So how do we find the inverse of a matrix? Well, now that we know how to find the determinant of a matrix, the formula for the inverse matrix will actually be something we already know how to calculate. Given matrix M as,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

its inverse is given by the formula

$$M^{-1} = \frac{1}{|M|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Notice that the formula for the inverse matrix is a fraction with a numerator of 1 and the determinant as the denominator, multiplied by another matrix. The other matrix is called the **adjugate** of M, and the adjugate is the matrix in which the values a and d have been swapped, and the values b and c have been multiplied by -1.

Example

Find the inverse of matrix K, then find $K \cdot K^{-1}$ and $K^{-1} \cdot K$ to show that you found the correct value for the inverse matrix.

$$K = \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix}$$

To find the inverse of matrix K, we plug into the formula for the inverse of a matrix.

$$K^{-1} = \frac{1}{|K|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$K^{-1} = \frac{1}{\begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix}} \begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = \frac{1}{-2(0) - 4(3)} \begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = -\frac{1}{12} \begin{bmatrix} 0 & -4 \\ -3 & -2 \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} -\frac{0}{12} & \frac{4}{12} \\ \frac{3}{12} & \frac{2}{12} \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix}$$

This is the inverse of K, but we can prove it to ourselves by multiplying K by its inverse. If we've done our math right, we should get the identity matrix when we multiply them.

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} -2(0) + 4\left(\frac{1}{4}\right) & -2\left(\frac{1}{3}\right) + 4\left(\frac{1}{6}\right) \\ 3(0) + 0\left(\frac{1}{4}\right) & 3\left(\frac{1}{3}\right) + 0\left(\frac{1}{6}\right) \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0 + 1 & -\frac{2}{3} + \frac{4}{6} \\ 0 + 0 & 1 + 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K \cdot K^{-1} = I_2$$

When we multiplied K by its inverse, we get the identity matrix. We also want to make the point that we can multiply in the other direction, and we still get the identity matrix.

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0(-2) + \frac{1}{3}(3) & 0(4) + \frac{1}{3}(0) \\ \frac{1}{4}(-2) + \frac{1}{6}(3) & \frac{1}{4}(4) + \frac{1}{6}(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 + 1 & 0 + 0 \\ -\frac{1}{2} + \frac{1}{2} & 1 + 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K^{-1} \cdot K = I_2$$

This example shows how to use the formula to find the inverse matrix, and proves that multiplying by the inverse matrix is commutative. Whether we calculate $K \cdot K^{-1}$ or $K^{-1} \cdot K$, we get back to the identity matrix either way.

Invertible and singular matrices

Not every matrix has an inverse. Given the formula for the inverse matrix,

$$K^{-1} = \frac{1}{|K|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we can probably spot right away that, because a fraction is undefined when its denominator is 0, we have to say that $|K| \neq 0$. In other words, if the determinant is equal to 0, then the inverse matrix is undefined. To be specific, if

$$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the inverse of K is undefined when $ad - bc = 0$, or when $ad = bc$. If we divide both sides by b and d, we get

$$\frac{a}{b} = \frac{c}{d}$$

So if the ratio of a to b (the values in the first row of matrix K) is equal to the ratio of c to d (the values in the second row of matrix K), then you know right away that the matrix K does not have a defined inverse. If the matrix doesn't have an inverse, we call it a **singular matrix**. When the matrix does have an inverse, we say that it's **invertible**.

Example

Say whether the each matrix is invertible or singular.

(a) $M = \begin{bmatrix} 1 & -3 \\ 3 & 5 \end{bmatrix}$

(b) $L = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}$

For each matrix, we'll look at whether or not the ratio of a to b is equal to the ratio of c to d. For matrix M, we get

$$\frac{1}{-3} = -\frac{1}{3} \quad \neq \quad \frac{3}{5}$$

Because these aren't equivalent ratios, matrix M is invertible, which means it has a defined inverse. For matrix L, we get

$$\frac{-6}{2} = -3 \quad = \quad \frac{3}{-1} = -3$$

Because these are equivalent ratios, matrix L is not invertible, which means it does not have a defined inverse, and we can therefore say that it's a singular matrix.