Proving the trig equation

Now that we've introduced so many trig identities, we want to talk about one way we'll use them, which is to prove trig equations. As a reminder, here's a full list of the identities we've looked at so far:

Reciprocal identities	Cofunction identities
Quotient identities	Double-angle identities
Pythagorean identities	Half-angle identities
Even-odd identities	Product-to-sum identities
Sum-difference identities	Sum-to-product identities

When we use trig identities to prove a trig equation, we often start with one side of the equation and manipulate it using algebra and one or more trig identities in order to get the expression on the other side of the equation.

Mostly, these kinds of problems just take lots of practice, so that's what we'll focus on in this lesson. But there are some general tips that we want to keep in mind when we're trying to figure out the best way to approach the problem.

1. Try to express every trig function in the equation in terms of sine and cosine. For cosecant and secant, we'll do this with the reciprocal identities, and for tangent and cotangent we'll do this with the quotient identities.

2. Make sure all the angles are the same. For example, when we have $\sin(2\alpha)$ on one side of the equation, and $\sin\alpha$ on the other side, it's difficult to prove the equation. The same applies for addition and subtraction: don't try working with $\sin(\alpha+\beta)$ and $\sin\alpha$.

3. Try rewriting the more complicated side of the equation in order to match the simpler side.

4. If we need to add more powers or remove them, we can use the Pythagorean identities like $\cos^2 x + \sin^2 x = 1$. We can always multiply by 1 without changing the meaning, so we can always multiply by $\cos^2 x + \sin^2 x$.

5. Look for fractions that can be combined or pulled apart, and consider whether or not the equation might be factorable.

6. Look for a trig function that links the trig functions in the equation. For instance, if we have sine and cotangent in an equation, we know that tangent is a linking function, because tangent=sine/cosine, and tangent=1/cotangent.

7. If we ever have a value which is the sum or difference of a constant and a trig function, like $1+\cos\theta$, consider multiplying by the conjugate. The conjugate is the same two terms, but with the opposite sign between them. So the conjugate of $1+\cos\theta$ is $1-\cos\theta$.

These tips aren't the only things we can try, but they *are* common operations we'll use. Above anything, don't be afraid to try different things! Some people feel paralyzed because they don't know where to start. But there's no harm in starting somewhere, even if we don't know where it'll lead us.

So when we tackle these kinds of problems, we just need to start *somewhere*, and try *something*. It we feel at any point like we're running into a dead-end, we can just back up and try something else. We might have to work through a couple different approaches before we find the one that works.

With all this in mind, let's get some practice by working through a few examples.

Example

Prove the trig equation.

$$\frac{\cot\theta}{\csc\theta} = \cos\theta$$

We can't simplify the right side at all, so we'll try to rewrite the left side. Our goal would be to rewrite the left side to eventually show that it's equivalent to $\cos\theta$.

Let's start with our first tip in the list, which is to put everything in terms of sine and cosine. Since $\cot\theta = \cos\theta/\sin\theta$, and $\csc\theta = 1/\sin\theta$, we'll rewrite the equation as

$$\frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \cos\theta$$

$$\frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{1}\right)=\cos\theta$$

The $\sin\theta$ will cancel from the numerator and denominator, leaving just

$$\cos\theta=\cos\theta$$

Let's try one where we use quotient and Pythagorean identities.

Example

Prove the trig equation.

$$\frac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=1-2\cos^2\theta$$

We'll start with the left side, and try to show that it's equal to the right side. We can use the quotient identities

$$\tan\theta=\frac{\sin\theta}{\cos\theta}\text{ and }\cot\theta=\frac{\cos\theta}{\sin\theta}$$

to put everything on the left in terms of sine and cosine.

$$\frac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=1-2\cos^2\theta$$

$$\frac{\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}}=1-2\cos^2\theta$$

Find common denominators in both the numerator and denominator, then simplify them.

$$\frac{\frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\sin\theta}\right)-\frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)}{\frac{\sin\theta}{\cos\theta}\left(\frac{\sin\theta}{\sin\theta}\right)+\frac{\cos\theta}{\sin\theta}\left(\frac{\cos\theta}{\cos\theta}\right)}=1-2\cos^2\theta$$

$$\frac{\frac{\sin^2\theta}{\sin\theta\cos\theta}-\frac{\cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}}=1-2\cos^2\theta$$

$$\frac{\frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}}=1-2\cos^2\theta$$

$$\frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}\left(\frac{\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta}\right)=1-2\cos^2\theta$$

The $\sin\theta\cos\theta$ will cancel from the numerator and denominator, leaving just

$$\frac{\sin^2\theta-\cos^2\theta}{\sin^2\theta+\cos^2\theta}=1-2\cos^2\theta$$

With the Pythagorean identity $\sin^2\theta+\cos^2\theta=1$, we get

$$\frac{\sin^2\theta-\cos^2\theta}{1}=1-2\cos^2\theta$$

$$\sin^2\theta-\cos^2\theta=1-2\cos^2\theta$$

Using one of the other forms of the basic Pythagorean identity, $\sin^2\theta=1-\cos^2\theta$, we'll make a substitution for $\sin^2\theta$.

$$(1-\cos^2\theta)-\cos^2\theta=1-2\cos^2\theta$$

$$1-\cos^2\theta-\cos^2\theta=1-2\cos^2\theta$$

$$1-2\cos^2\theta=1-2\cos^2\theta$$

Sometimes it's easier or more convenient to work on the expressions on both sides of the equation separately (one after the other), and prove that they're both equal to some other third expression.

Example

Prove the trig equation.

$$\csc\theta+\cot\theta=\frac{\sin\theta}{1-\cos\theta}$$

We'll start by working on the left side. Using the reciprocal and quotient identities

$\csc\theta=\dfrac{1}{\sin\theta}$ and $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$

we'll rewrite the left side as

$$\csc\theta+\cot\theta$$

$$\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}$$

$$\frac{1+\cos\theta}{\sin\theta}$$

With the left side in this form, the full equation is currently

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta}{1-\cos\theta}$$

Now we'll work on the right side. We'll multiply both the numerator and denominator by the conjugate of the denominator, $1+\cos\theta$. (Alternatively, we could continue working on the left side by multiplying both the numerator and denominator by the conjugate of the numerator, $1-\cos\theta$.)

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta}{1-\cos\theta}\left(\frac{1+\cos\theta}{1+\cos\theta}\right)$$

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta(1+\cos\theta)}{1+\cos\theta-\cos\theta-\cos^2\theta}$$

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}$$

Use the rewritten form $1-\cos^2\theta=\sin^2\theta$ of the Pythagorean identity with sine and cosine to substitute for the denominator on the right side.

$$\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$$

On the right side, we can cancel one factor of $\sin\theta$ from both the numerator and denominator.

$$\frac{1+\cos\theta}{\sin\theta}=\frac{1+\cos\theta}{\sin\theta}$$

Let's do an example using the double- and half-angle identities.

Example

Prove the trig equation.

$$\cot\left(\frac{\theta}{2}\right)=\frac{\sin\theta}{1-\cos\theta}$$

We'll start with the expression on the left side. Using the quotient identity, we'll rewrite the cotangent function in terms of sine and cosine.

$$\frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

Multiplying both the numerator and denominator by $2\sin(\theta/2)$.

$$\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

By the double-angle identity for sine, we know that the numerator of the left side is equivalent to

$$2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)=\sin\left(2\cdot\frac{\theta}{2}\right)$$

So we'll replace the numerator on the left side.

$$\frac{\sin\left(2 \cdot \frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{\sin\theta}{2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{\sin\theta}{2\sin^2\left(\frac{\theta}{2}\right)} = \frac{\sin\theta}{1-\cos\theta}$$

With the half-angle identity for sine,

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

the left side of the equation can be rewritten.

$$\frac{\sin\theta}{2\left(\frac{1-\cos\theta}{2}\right)} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{\sin\theta}{1-\cos\theta} = \frac{\sin\theta}{1-\cos\theta}$$

In the next example, we'll apply sum, double-angle, and Pythagorean identities.

Example

Prove the trig equation.

$$\cos(3\theta) = \cos\theta(1 - 4\sin^2\theta)$$

If we think of 3θ as $2\theta + \theta$, then we can apply the sum identity for cosine to the left side.

$$\cos(3\theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos(2\theta + \theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos(2\theta)(\cos\theta) - \sin(2\theta)(\sin\theta) = \cos\theta(1 - 4\sin^2\theta)$$

The double-angle identity for cosine,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

can be applied to the left side to get

$$(\cos^2\theta - \sin^2\theta)(\cos\theta) - \sin(2\theta)(\sin\theta) = \cos\theta(1 - 4\sin^2\theta)$$

Then the double-angle identity for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

can be applied to the left side to get

$$(\cos^2\theta - \sin^2\theta)(\cos\theta) - (2\sin\theta\cos\theta)(\sin\theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta = \cos\theta(1 - 4\sin^2\theta)$$

We can see that $\cos\theta$ is a common factor, so we'll factor it out.

$$\cos\theta(\cos^2\theta - \sin^2\theta - 2\sin^2\theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos\theta(\cos^2\theta - 3\sin^2\theta) = \cos\theta(1 - 4\sin^2\theta)$$

If we express the Pythagorean identity with sine and cosine as $\cos^2\theta = 1 - \sin^2\theta$, the left side becomes

$$\cos\theta(1 - \sin^2\theta - 3\sin^2\theta) = \cos\theta(1 - 4\sin^2\theta)$$

$$\cos\theta(1 - 4\sin^2\theta) = \cos\theta(1 - 4\sin^2\theta)$$

Let's do one more example, this one with quotient, difference, and cofunction identities.

Example

Prove the trig equation.

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

We'll start by working on the left side. By the quotient identity for tangent,

$$\frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \cot\theta$$

The difference identity for sine tells us that

$$\sin\left(\frac{\pi}{2}-\theta\right)=\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)$$

so we'll replace the numerator of the left side of our equation.

$$\frac{\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)}{\cos\left(\frac{\pi}{2}-\theta\right)}=\cot\theta$$

And the difference identity for cosine tells us that

$$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\left(\frac{\pi}{2}\right)(\cos\theta)+\sin\left(\frac{\pi}{2}\right)(\sin\theta)$$

so we'll use this value to replace the denominator of the left side.

$$\frac{\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)}{\cos\left(\frac{\pi}{2}\right)(\cos\theta)+\sin\left(\frac{\pi}{2}\right)(\sin\theta)}=\cot\theta$$

$$\frac{(1)(\cos\theta)-(0)(\sin\theta)}{(0)(\cos\theta)+(1)(\sin\theta)}=\cot\theta$$

$$\frac{\cos\theta}{\sin\theta}=\cot\theta$$

By the quotient identity for cotangent, the left side becomes

$$\cot\theta=\cot\theta$$