Complex number operations

Just like we know how to do with real numbers, we want to be able to add, subtract, multiply, and divide complex numbers.

Adding and subtracting complex numbers

To add two complex numbers, we add their real parts, and then separately add their imaginary parts. There's no logical way to combine the real and imaginary parts themselves, so the result is still a complex number.

Example

Find the sum of the complex numbers $-2+17i$ and $6-8i$.

Add the real parts together, and then separately add the imaginary parts.

$$(-2+17i)+(6-8i)$$

$$(-2+6)+(17i-8i)$$

$$(-2+6)+(17-8)i$$

$$4+9i$$

Just like complex number addition, to subtract two complex numbers, subtract their real parts and their imaginary parts separately.

Example

Find the difference of the complex numbers $13 + 4i$ and $5 + 9i$.

Taking the difference of their real parts and imaginary parts separately, we get

$$(13 + 4i) - (5 + 9i)$$

$$(13 - 5) + (4i - 9i)$$

$$(13 - 5) + (4 - 9)i$$

$$8 - 5i$$

Let's do another example with both addition and subtraction.

Example

Find the sum and difference of $21 + 16i$ and $-9 + 4i$.

The sum of the complex numbers is

$$(21 + 16i) + (-9 + 4i)$$

$$(21 - 9) + (16i + 4i)$$

$$(21 - 9) + (16 + 4)i$$

$$12 + 20i$$

Their difference is

$$(21 + 16i) - (-9 + 4i)$$

We have to be a little careful here with the subtraction in the middle, and distribute it across the second complex number.

$$21 + 16i + 9 - 4i$$

$$(21 + 9) + (16i - 4i)$$

$$30 + 12i$$

Multiplying and dividing complex numbers

In algebra, you learned to multiplying two binomials by "FOILing them out": multiplying the first terms, outer terms, inner terms, and last terms. Complex numbers in the form $z = a + bi$ are binomials, too, so you can use the same FOIL process to multiply them.

Here's a quick shortcut guide to what happens when you multiply the real and imaginary parts of complex numbers:

Real $\times$ real = real	$3 \times 4 = 12$
Real $\times$ imaginary = imaginary	$3 \times 4i = 12i$
Imaginary $\times$ imaginary = imaginary	$3i \times 4i = 12i^2$

For instance, the product of 25 and $8i$ is

$$(25)(8i)$$

$$(25 \cdot 8)i$$

$$200i$$

And the product of i and $-i$ is

$$(i)(-i)$$

$$-(i^2)$$

$$-(-1)$$

$$1$$

We can also multiply complex numbers by 0, and just like real numbers, the result will be 0. The product of $-36 + 11i$ and 0 is

$$(-36 + 11i)(0)$$

$$(-36)(0) + (11i)(0)$$

$$0 + (11)(0)i$$

$$0 + 0i$$

$$0$$

Let's do an example of multiplying two full complex numbers as binomials $z = a + bi$ to see how to use FOIL to find the product.

Example

Find the product of $-6 + 2i$ and $4 - 5i$.

Using FOIL, we'll multiply the first, outer, inner, and last terms.

$$(-6 + 2i)(4 - 5i)$$

$$(-6)(4) + (-6)(-5i) + (2i)(4) + (2i)(-5i)$$

$$-24 + 30i + 8i - 10i^2$$

$$-24 + 38i - 10i^2$$

Because $i^2 = -1$, we get

$$-24 + 38i - 10(-1)$$

$$-24 + 38i + 10$$

$$(-24 + 10) + 38i$$

$$-14 + 38i$$

When we divide a complex number by a real number, it'll look like this:

$$\frac{a+bi}{c} \quad \to \quad \frac{a}{c}+\frac{bi}{c} \quad \to \quad \frac{a}{c}+\frac{b}{c}i$$

As you can hopefully tell, the way the fraction splits up makes a/c the real part, and b/c the imaginary part.

When we divide a complex number by an imaginary number, a similar thing happens:

$$\frac{a+bi}{ci} \quad \to \quad \frac{a}{ci}+\frac{bi}{ci} \quad \to \quad \frac{a}{c}i^{-1}+\frac{b}{c} \quad \to$$

$$\frac{a}{c}(-i)+\frac{b}{c} \quad \to \quad -\frac{a}{c}i+\frac{b}{c} \quad \to \quad \frac{b}{c}-\frac{a}{c}i$$

The way fraction splits up makes b/c the real part, and $-a/c$ the imaginary part.

But in both of these cases, it's fairly simple to split up the fraction. But what about when we divide the full complex binomial $z=a+bi$ by another complex binomial $z=c+di$?

$$\frac{a+bi}{c+di}$$

This is where conjugates come in. The **conjugate** of a binomial is the same two terms in the binomial, but with the opposite sign between them. The **complex conjugate** is the same thing, but when the binomial is a complex number. To give you some examples, the complex conjugate of $3+4i$ is $3-4i$, the complex conjugate of $11-6i$ is $11+6i$, and the complex conjugate of $a+bi$ is $a-bi$.

Example

Find the complex conjugate of each of the complex numbers.

$$13+5i$$

$$7-4i$$

$$-6+i$$

For each of these, we keep the real part (13, 7, or -6) and change the sign of the imaginary part (from 5 to -5, from -4 to 4, or from 1 to -1):

The complex conjugate of $13+5i$ is $13-5i$.

The complex conjugate of $7-4i$ is $7+4i$.

The complex conjugate of $-6+i$ is $-6-i$.

To simplify the quotient of two complex numbers, we need to multiply both the numerator and denominator of the fraction by the complex conjugate of the denominator.

$$\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}$$

$$\frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

Then we FOIL the numerator, and separately FOIL the denominator.

$$\frac{ac - adi + bci - bdi^2}{c^2 - cdi + cdi - d^2i^2}$$

When you multiply by the complex conjugate like this, the two terms in the middle of the denominator will always cancel with one another.

$$\frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2}$$

$$\frac{ac - adi + bci - bd(-1)}{c^2 - d^2(-1)}$$

$$\frac{ac - adi + bci + bd}{c^2 + d^2}$$

$$\frac{(ac + bd) + (-ad + bc)i}{c^2 + d^2}$$

$$\frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

$$\frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

If you look at the format of this result, it's just a complex number in standard form. The first fraction is the real number part, and the second fraction is the imaginary number part. So when you divide two complex number binomials, the result will be a complex number binomial.

Example

Find the quotient of $7 + 3i$ and $9 - 4i$.

First, we'll write the quotient as

$$\frac{7+3i}{9-4i}$$

and then multiply top and bottom by the complex conjugate of $9-4i$, which is $9+4i$.

$$\left(\frac{7+3i}{9-4i}\right)\left(\frac{9+4i}{9+4i}\right)$$

$$\frac{(7+3i)(9+4i)}{(9-4i)(9+4i)}$$

FOIL across the numerator and denominator, then simplify.

$$\frac{63+28i+27i+12i^2}{81+36i-36i-16i^2}$$

$$\frac{63+55i+12i^2}{81-16i^2}$$

$$\frac{63+55i+12(-1)}{81-16(-1)}$$

$$\frac{63+55i-12}{81+16}$$

$$\frac{51+55i}{97}$$

Split the fraction to write this in standard form of a complex number.

$$\frac{51}{97} + \frac{55}{97}i$$