Heron's formula

We've already looked at how to use the law of sines for finding the area of a triangle, but that's not the only formula we can use to find area.

Heron's formula for area of a triangle

When we know the lengths of all three sides of a triangle, we can find its area using Heron's formula. And this area formula works for all oblique triangles, not just right triangles. **Heron's formula** is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the lengths of the sides of the triangle and

$$s = \frac{1}{2}(a+b+c)$$

which is half the perimeter of the triangle.

Let's look at an example where we use the three side lengths in Heron's formula to calculate area.

Example

Apply Heron's formula to find the area of the triangle that has side lengths 16, 19, and 7.

Let $a=16$, $b=19$, and $c=7$, then calculate s.

$$s=\frac{1}{2}(a+b+c)$$

$$s=\frac{1}{2}(16+19+7)$$

$$s=\frac{1}{2}(35+7)$$

$$s=\frac{1}{2}(42)$$

$$s=21$$

Now we'll plug into Heron's formula.

$$\text{Area}=\sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Area}=\sqrt{21(21-16)(21-19)(21-7)}$$

$$\text{Area}=\sqrt{21(5)(2)(14)}$$

$$\text{Area}=\sqrt{21(140)}$$

$$\text{Area}=\sqrt{2{,}940}$$

$$\text{Area}\approx 54.2$$

What we've shown is that, if we know the lengths of two sides of a triangle and the measure of the included angle, we can use the law of sines for the area of a triangle to compute the area. And if we know the lengths of all three sides of a triangle, we can use Heron's formula to compute area.

But how do we get the area of a triangle if we don't have either of these particular information sets?

In that's the case, we might need to apply the law of sines or cosines first to find whatever information we're missing, and then go on to find area once we have everything we need.

Example

Find the area of the triangle that has side lengths 10 and 5 and where the angle opposite the side with length 10 is 40°.

Let $a=10$ and $b=5$. Then the angle opposite the side of length 10 is $A=40^\circ$, the angle opposite the side of length 5 is B, and the included angle is C.

We want to be able to eventually apply the law of sines for the area of a triangle, so we'll first find the measure of B, and then use that to get the measure of C.

By the law of sines, we get

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\frac{10}{\sin 40^\circ}=\frac{5}{\sin B}=\frac{c}{\sin C}$$

Use the first and second parts of this three-part equation to solve for B.

$$\frac{10}{\sin 40^\circ} = \frac{5}{\sin B}$$

$$\frac{10}{\sin 40^\circ}(\sin B) = 5$$

$$10 \sin B = 5 \sin 40^\circ$$

$$\sin B = \frac{5 \sin 40^\circ}{10}$$

$$\sin B \approx 0.322$$

$$B \approx \arcsin(0.322)$$

$$B \approx 19^\circ$$

Then the measure of angle C is

$$C \approx 180^\circ - 40^\circ - 19^\circ$$

$$C \approx 121^\circ$$

Now we're ready to compute the area of the triangle. Plugging what we know into the law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}ab \sin C$$

$$\text{Area} \approx \frac{1}{2}(10)(5)\sin 121^\circ$$

$$\text{Area} \approx 25(0.857)$$

Area ≈ 21.4