Topic: Polar equation of an elliptical conic section

Question: Which equation represents the polar equation of the ellipse?

$$x^2 + 4m^2y^2 - 4m^2 = 0$$

Answer choices:

A $\quad r = \dfrac{4m}{\sqrt{\cos^2\theta + 2m^2\sin^2\theta}}$

B $\quad r = \dfrac{4m}{\sqrt{\cos^2\theta + 4m\sin^2\theta}}$

C $\quad r = \dfrac{2m}{\sqrt{\cos^2\theta + 4m^2\sin^2\theta}}$

D $\quad r = \dfrac{2m}{\sqrt{\cos^2\theta + 4m\sin^2\theta}}$

Solution: C

For the equation we've been given

$$x^2+4m^2y^2-4m^2=0$$

we'll use the conversion equations

$$x=r\cos\theta$$

$$y=r\sin\theta$$

to substitute and get

$$(r\cos\theta)^2+4m^2\left(r\sin\theta\right)^2-4m^2=0$$

$$r^2\cos^2\theta+4m^2r^2\sin^2\theta-4m^2=0$$

$$r^2\left(\cos^2\theta+4m^2\sin^2\theta\right)=4m^2$$

$$r^2=\frac{4m^2}{\cos^2\theta+4m^2\sin^2\theta}$$

$$r=\sqrt{\frac{4m^2}{\cos^2\theta+4m^2\sin^2\theta}}$$

$$r=\frac{2m}{\sqrt{\cos^2\theta+4m^2\sin^2\theta}}$$

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Question: The polar equation represents which of the following ellipses?

$$r = \frac{5\cos\theta - 2\sin\theta \pm \sqrt{5(21\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta)}}{5\cos^2\theta + \sin^2\theta}$$

Answer choices:

A $\quad 5x^2 + y^2 + 10x - 2y - 16 = 0$

B $\quad 5x^2 + y^2 - 10x - 2y + 16 = 0$

C $\quad 5x^2 + y^2 - 10x + 4y + 16 = 0$

D $\quad 5x^2 + y^2 - 10x + 4y - 16 = 0$

Solution: D

Choose

$$5x^2 + y^2 - 10x + 4y - 16 = 0$$

from answer choice D. Use the conversion equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

to substitute into the equation.

$$5(r\cos\theta)^2 + (r\sin\theta)^2 - 10(r\cos\theta) + 4(r\sin\theta) - 16 = 0$$

$$5r^2\cos^2\theta + r^2\sin^2\theta - 10r\cos\theta + 4r\sin\theta - 16 = 0$$

$$(5\cos^2\theta + \sin^2\theta)r^2 + 2(-5\cos\theta + 2\sin\theta)r - 16 = 0$$

Now this is a quadratic equation, and we can use the quadratic formula to find its roots. With $a = (5\cos^2\theta + \sin^2\theta)$, $b = 2(-5\cos\theta + 2\sin\theta)$, and $c = -16$, we plug into the quadratic formula and get

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-2(-5\cos\theta + 2\sin\theta) \pm \sqrt{(2(-5\cos\theta + 2\sin\theta))^2 - 4(5\cos^2\theta + \sin^2\theta)(-16)}}{2(5\cos^2\theta + \sin^2\theta)}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{4(-5\cos\theta + 2\sin\theta)(-5\cos\theta + 2\sin\theta) + 64(5\cos^2\theta + \sin^2\theta)}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{4(25\cos^2\theta - 20\sin\theta\cos\theta + 4\sin^2\theta) + 320\cos^2\theta + 64\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{100\cos^2\theta - 80\sin\theta\cos\theta + 16\sin^2\theta + 320\cos^2\theta + 64\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{420\cos^2\theta - 80\sin\theta\cos\theta + 80\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{420\cos^2\theta - 40(2\sin\theta\cos\theta) + 80\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

Using the trig identity $\sin(2x) = 2\sin x\cos x$, we get

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{420\cos^2\theta - 40\sin 2\theta + 80\sin^2\theta}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm \sqrt{20(21\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta)}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{10\cos\theta - 4\sin\theta \pm 2\sqrt{5(21\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta)}}{10\cos^2\theta + 2\sin^2\theta}$$

$$r = \frac{5\cos\theta - 2\sin\theta \pm \sqrt{5(21\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta)}}{5\cos^2\theta + \sin^2\theta}$$

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Question: Name the center of each ellipse.

$$r^2 = \frac{m^4n^4}{m^4\sin^2\theta_1 + n^4\cos^2\theta_1} \text{ and } r^2 = \frac{m^6n^6}{m^6\sin^2\theta_2 + n^6\cos^2\theta_2}$$

Answer choices:

A $(0,0)$ and $(1,0)$

B $(0,0)$ and $(0,0)$

C $(0,0)$ and $(0,1)$

D $(0,0)$ and $(-1,0)$

Solution: B

Rewrite the conversion equations $x = r\cos\theta$ and $y = r\sin\theta$ as

$\cos\theta_1 = \frac{x}{r}$ and $\sin\theta_1 = \frac{y}{r}$

and then make substitutions into the first equation.

$$r^2 = \frac{m^4n^4}{m^4\sin^2\theta_1 + n^4\cos^2\theta_1}$$

$$r^2 = \frac{m^4n^4}{m^4\left(\frac{y}{r}\right)^2 + n^4\left(\frac{x}{r}\right)^2}$$

$$r^2 = \frac{m^4n^4}{\frac{m^4y^2 + n^4x^2}{r^2}}$$

$$r^2\left(\frac{m^4y^2 + n^4x^2}{r^2}\right) = m^4n^4$$

$$m^4y^2 + n^4x^2 = m^4n^4$$

$$\frac{m^4y^2}{m^4n^4} + \frac{m^4x^2}{m^4n^4} = \frac{m^4n^4}{m^4n^4}$$

$$\frac{y^2}{n^4} + \frac{x^2}{n^4} = 1$$

$$\frac{y^2}{(n^2)^2} + \frac{x^2}{(n^2)^2} = 1$$

This is the equation of an ellipse centered at the origin with $a = n^2$ and $b = n^2$.

Now substitute into the second equation.

$$r^2 = \frac{m^6 n^6}{m^6 \sin^2 \theta_2 + n^6 \cos^2 \theta_2}$$

$$r^2 = \frac{m^6 n^6}{m^6 \left(\frac{y}{r}\right)^2 + n^6 \left(\frac{x}{r}\right)^2}$$

$$r^2 = \frac{m^6 n^6}{\frac{m^6 y^2 + n^6 x^2}{r^2}}$$

$$r^2 \left(\frac{m^6 y^2 + n^6 x^2}{r^2}\right) = m^6 n^6$$

$$m^6 y^2 + n^6 x^2 = m^6 n^6$$

$$\frac{m^6 y^2}{m^6 n^6} + \frac{m^6 x^2}{m^6 n^6} = \frac{m^6 n^6}{m^6 n^6}$$

$$\frac{y^2}{n^6} + \frac{x^2}{n^6} = 1$$

$$\frac{y^2}{(n^3)^2} + \frac{x^2}{(n^3)^2} = 1$$

This is the equation of an ellipse centered at the origin with $a = n^3$ and $b = n^3$.