Trigonometry Workbook Solutions

Trig identities

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SUM-DIFFERENCE IDENTITIES FOR SINE AND COSINE

1. Evaluate the expression.

$$\cos\left(\frac{13\pi}{12}\right)$$

Solution:

From just the unit circle, we wouldn't know the values of sine and cosine at $13\pi/12$, but we can rewrite $13\pi/12$ as

$$\frac{13\pi}{12}=\frac{(4+9)\pi}{12}=\frac{4\pi}{12}+\frac{9\pi}{12}=\frac{\pi}{3}+\frac{3\pi}{4}$$

Therefore, by the sum identity for the cosine function,

$$\cos\left(\frac{13\pi}{12}\right)=\cos\left(\frac{\pi}{3}+\frac{3\pi}{4}\right)$$

$$\cos\left(\frac{13\pi}{12}\right)=\left(\cos\frac{\pi}{3}\right)\left(\cos\frac{3\pi}{4}\right)-\left(\sin\frac{\pi}{3}\right)\left(\sin\frac{3\pi}{4}\right)$$

$$\cos\left(\frac{13\pi}{12}\right)=\left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\cos\left(\frac{13\pi}{12}\right)=-\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$$

$$\cos\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{2}+\sqrt{6}}{4}$$

2. Find $\sin 75^\circ$.

Solution:

Rewrite $\sin 75^\circ$ as $\sin(45^\circ + 30^\circ)$, then apply the sum identity for sine.

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$\sin(45^\circ + 30^\circ) = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$\sin(45^\circ + 30^\circ) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\sin(45^\circ + 30^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

3. Simplify the expressions.

$$\cos\left(\frac{\pi}{2} + \theta\right) \text{ and } \cos\left(\frac{\pi}{2} - \theta\right)$$

Solution:

To find cosine of the sum, use the sum identity for cosine.

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=\cos\frac{\pi}{2}\cos\theta-\sin\frac{\pi}{2}\sin\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=0\cdot\cos\theta-1\cdot\sin\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$$

To find cosine of the difference, use a difference identity.

$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=0\cdot\cos\theta+1\cdot\sin\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

■ 4. Find the value of $a-2b$, if a and b are real numbers.

$$\sin(\theta - \alpha) = a \sin\theta \cos\alpha + b \cos\theta \sin\alpha$$

Solution:

For two angles θ and α, the difference identity for sine is

$$\sin(\theta - \alpha) = \sin\theta \cos\alpha - \cos\theta \sin\alpha$$

Therefore $a = 1$ and $b = -1$. Substitute and evaluate the expression.

$$a - 2b$$

$$1 - 2(-1)$$

$$1 + 2$$

$$3$$

■ 5. Find the exact value of the expression.

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\frac{4}{5}\right)$$

Solution:

To find the exact value of the expression, we need to use the difference identity for the cosine function.

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Define α and β.

$$\alpha = \sin^{-1}\frac{\sqrt{3}}{2}$$

$$\beta = \cos^{-1}\frac{4}{5}$$

Using the properties of inverses we can write

$$\sin\alpha = \frac{\sqrt{3}}{2}, \ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$$

$$\cos\beta = \frac{4}{5}, \ 0 \le \beta \le \pi$$

Then using the Pythagorean identities we can find $\cos\alpha$ and $\sin\beta$.

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substitute $\sin\alpha = \sqrt{3}/2$.

$$\cos^2\alpha = 1 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\cos^2\alpha = 1 - \frac{3}{4}$$

$$\cos^2\alpha = \frac{1}{4}$$

$$\cos\alpha = \pm\sqrt{\frac{1}{4}}$$

Since $-\pi/2 \leq \alpha \leq \pi/2$, we can say $\cos\alpha = 1/2$.

$$\sin^2\beta + \cos^2\beta = 1$$

$$\sin^2\beta = 1 - \cos^2\beta$$

Substitute $\cos\beta = 4/5$.

$$\sin^2\beta = 1 - \left(\frac{4}{5}\right)^2$$

$$\sin^2\beta = 1 - \frac{16}{25}$$

$$\sin^2\beta = \frac{9}{25}$$

$$\sin\beta = \pm\sqrt{\frac{9}{25}}$$

Since $0 \leq \beta \leq \pi$, we can say $\sin\beta = 3/5$. Then using the difference formula for cosine, we get

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\frac{4}{5}\right) = \cos(\alpha - \beta)$$

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\frac{4}{5}\right) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\frac{4}{5}\right)=\frac{1}{2}\left(\frac{4}{5}\right)+\frac{\sqrt{3}}{2}\left(\frac{3}{5}\right)$$

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\frac{4}{5}\right)=\frac{4}{10}+\frac{3\sqrt{3}}{10}$$

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\frac{4}{5}\right)=\frac{4+3\sqrt{3}}{10}$$

■ 6. Find the solutions to the equation in the interval $[0,\pi)$.

$$\cos\left(\theta-\frac{\pi}{2}\right)+\sin\left(\theta-\frac{3\pi}{2}\right)=0$$

Solution:

Use the sum-difference identities for cosine,

$$\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha$$

$$\cos\left(\theta-\frac{\pi}{2}\right)=\cos\theta\cos\frac{\pi}{2}+\sin\theta\sin\frac{\pi}{2}$$

$$\cos\left(\theta-\frac{\pi}{2}\right)=\cos\theta(0)+\sin\theta(1)$$

$$\cos\left(\theta-\frac{\pi}{2}\right)=\sin\theta$$

and sine.

$$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$$

$$\sin\left(\theta - \frac{3\pi}{2}\right) = \sin\theta\cos\frac{3\pi}{2} - \cos\theta\sin\frac{3\pi}{2}$$

$$\sin\left(\theta - \frac{3\pi}{2}\right) = \sin\theta(0) - \cos\theta(-1)$$

$$\sin\left(\theta - \frac{3\pi}{2}\right) = \cos\theta$$

Therefore we get

$$\sin\theta + \cos\theta = 0$$

$$\sin\theta = -\cos\theta$$

From the unit circle we know that $\sin\theta = -\cos\theta$ at $\theta = 3\pi/4$ and $\theta = 7\pi/4$. We're looking for solutions over the interval $[0,\pi)$, so the only solution is $\theta = 3\pi/4$.

COFUNCTION IDENTITIES

1. Find an angle θ that satisfies the equation.

$$\tan\left(-\frac{3\pi}{4}\right)=\cot\theta$$

Solution:

The equation we're given tells us that the cotangent of some angle is equivalent to tangent of $-3\pi/4$. Tangent and cotangent are cofunctions, which means we can plug into the cofunction identity for tangent that relates them.

$$\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$$

$$\tan\left(-\frac{3\pi}{4}\right)=\cot\left(\frac{\pi}{2}-\left(-\frac{3\pi}{4}\right)\right)$$

$$\tan\left(-\frac{3\pi}{4}\right)=\cot\left(\frac{\pi}{2}+\frac{3\pi}{4}\right)$$

Find a common denominator.

$$\tan\left(-\frac{3\pi}{4}\right)=\cot\left(\frac{\pi}{2}\left(\frac{2}{2}\right)+\frac{3\pi}{4}\right)$$

$$\tan\left(-\frac{3\pi}{4}\right) = \cot\left(\frac{2\pi}{4} + \frac{3\pi}{4}\right)$$

$$\tan\left(-\frac{3\pi}{4}\right) = \cot\left(\frac{5\pi}{4}\right)$$

So the angle θ that satisfies the equation is $\theta = 5\pi/4$. And this result tells us that tangent of the angle $-3\pi/4$ has the same value as cotangent of the angle $5\pi/4$.

2. Find an acute angle that satisfies the equation.

$$\sin\left(2\alpha - \frac{5\pi}{6}\right) = \cos\left(4\alpha - \frac{\pi}{3}\right)$$

Solution:

Use the cofunction identity for sine.

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

Let $\theta = 2\alpha - (5\pi/6)$. Then

$$4\alpha - \frac{\pi}{3} = \frac{\pi}{2} - \left(2\alpha - \frac{5\pi}{6}\right)$$

$$4\alpha - \frac{\pi}{3} = \frac{\pi}{2} - 2\alpha + \frac{5\pi}{6}$$

$$4\alpha + 2\alpha = \frac{\pi}{2} + \frac{5\pi}{6} + \frac{\pi}{3}$$

Find a common denominator.

$$6\alpha = \frac{\pi}{2}\left(\frac{3}{3}\right) + \frac{5\pi}{6} + \frac{\pi}{3}\left(\frac{2}{2}\right)$$

$$6\alpha = \frac{3\pi}{6} + \frac{5\pi}{6} + \frac{2\pi}{6}$$

$$6\alpha = \frac{10\pi}{6}$$

$$\alpha = \frac{10\pi}{36}$$

$$\alpha = \frac{5\pi}{18}$$

■ 3. What is the value of θ?

$$\tan\left(\frac{\pi}{6} - \theta\right) = \cot\left(\frac{\pi}{6}\right)$$

Solution:

Rewrite the left side of the equation as

$$\tan\left(\frac{\pi}{6} - \theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{3} - \theta\right)$$

From the cofunction identity for tangent,

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

we get

$$\tan\left(\frac{\pi}{6} - \theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{3} - \theta\right)$$

$$\tan\left(\frac{\pi}{6} - \theta\right) = \tan\left(\frac{\pi}{2} - \left(\frac{\pi}{3} + \theta\right)\right)$$

$$\tan\left(\frac{\pi}{6} - \theta\right) = \cot\left(\frac{\pi}{3} + \theta\right)$$

$$\tan\left(\frac{\pi}{6} - \theta\right) = \cot\left(\frac{\pi}{6}\right)$$

So we get

$$\frac{\pi}{3} + \theta = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{6} - \frac{\pi}{3}$$

$$\theta = -\frac{\pi}{6}$$

■ 4. Find the value of $\cos\theta$.

$$\sin\left(\frac{\pi}{2}-\theta\right)+\frac{1}{4}\csc\left(\frac{\pi}{2}-\theta\right)=1$$

Solution:

From the cofunction identities, we know that

$$\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$$

$$\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$$

So we can rewrite the equation as

$$\sin\left(\frac{\pi}{2}-\theta\right)+\frac{1}{4}\csc\left(\frac{\pi}{2}-\theta\right)=1$$

$$\cos\theta+\frac{1}{4}\sec\theta=1$$

$$\cos\theta+\frac{1}{4\cos\theta}=1$$

$$4\cos^2\theta+1=4\cos\theta$$

$$4\cos^2\theta-4\cos\theta+1=0$$

$$(2\cos\theta-1)^2=0$$

$$2\cos\theta-1=0$$

$$\cos\theta = \frac{1}{2}$$

5. Rewrite the expression as the cosine of an angle in terms of α and β.

$$\sin\left(\frac{\pi}{2} - \alpha - \beta\right)$$

Solution:

Rewrite the expression as

$$\sin\left(\frac{\pi}{2} - (\alpha + \beta)\right)$$

Apply the cofunction identity for cosine,

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

to get

$$\sin\left(\frac{\pi}{2} - (\alpha + \beta)\right) = \cos(\alpha + \beta)$$

Now apply the sum identity for cosine to get

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

6. Find an angle θ that satisfies the equation.

$$\csc\left(\frac{\pi}{5}\right) = \sec\theta$$

Solution:

The equation we're given tells us that the secant of some angle is equivalent to cosecant of $\pi/5$. Secant and cosecant are cofunctions, which means we can plug into the cofunction identity for cosecant that relates them.

$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$

$$\csc\left(\frac{\pi}{5}\right) = \sec\left(\frac{\pi}{2} - \frac{\pi}{5}\right)$$

Find a common denominator.

$$\csc\left(\frac{\pi}{5}\right) = \sec\left(\frac{\pi}{2}\left(\frac{5}{5}\right) - \frac{\pi}{5}\left(\frac{2}{2}\right)\right)$$

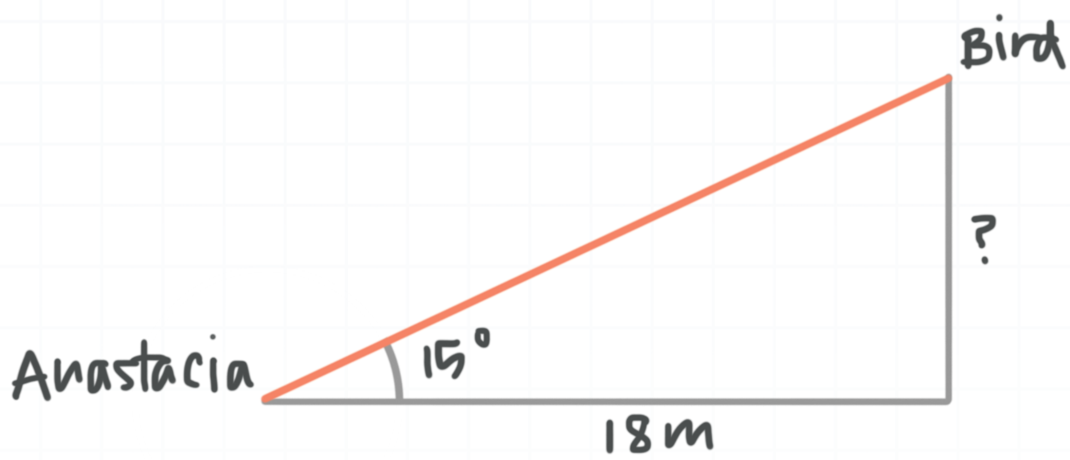
$$\csc\left(\frac{\pi}{5}\right) = \sec\left(\frac{5\pi}{10} - \frac{2\pi}{10}\right)$$

$$\csc\left(\frac{\pi}{5}\right) = \sec\left(\frac{3\pi}{10}\right)$$

So the angle θ that satisfies the equation is $\theta = 3\pi/10$. And this result tells us that cosecant of the angle $\pi/5$ has the same value as secant of the angle $3\pi/10$.

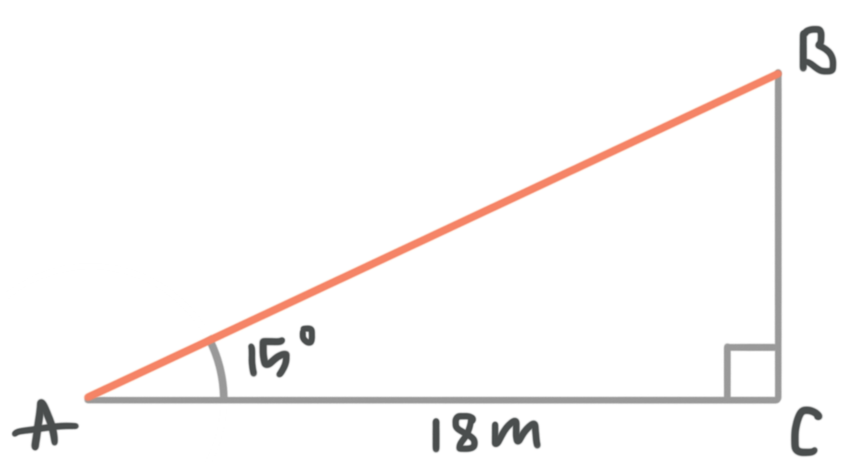
SUM-DIFFERENCE IDENTITIES FOR TANGENT

■ 1. Cara is watching a bird on a tree. She measured the angle of elevation of the bird as 15°, and the distance to the tree as 18 meters. Find the exact altitude of the bird above the ground.



Solution:

Call the right triangle ABC where A is Cara's position, B is the bird's position, and C is the point on the ground just below the bird.



$\overline{BC}$ is the side we're interested in, and by the definition of tangent, we can say

$$\tan A = \frac{\overline{BC}}{\overline{AC}}$$

$$\overline{BC} = \overline{AC} \tan A$$

$$\overline{BC} = 18 \tan 15^\circ$$

To find the exact value of $\tan 15^\circ$, rewrite the angle as the difference $45^\circ - 30^\circ$.

$$\overline{BC} = 18 \tan(45^\circ - 30^\circ)$$

Then apply the difference identity for tangent.

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta \tan\phi}$$

$$\overline{BC} = 18 \cdot \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$\overline{BC} = 18 \cdot \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}}$$

$$\overline{BC} = 18 \cdot \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

Rationalize the denominator using conjugate method.

$$\overline{BC} = 18 \cdot \frac{(3 - \sqrt{3})^2}{(3 + \sqrt{3})(3 - \sqrt{3})}$$

$$\overline{BC} = 18 \cdot \frac{9 - 6\sqrt{3} + 3}{9 - 3}$$

$$\overline{BC} = 18 \cdot \frac{12 - 6\sqrt{3}}{6}$$

$$\overline{BC} = 3(12 - 6\sqrt{3})$$

$$\overline{BC} = 36 - 18\sqrt{3} \text{ m}$$

■ 2. Find the exact value of $\tan 105^\circ$.

Solution:

From just the unit circle, we wouldn't know the value of tangent at 105°, but we can rewrite 105° as

$$105^\circ = 45^\circ + 60^\circ$$

Therefore, by the sum identity for tangent, $\tan 105^\circ = \tan(45^\circ + 60^\circ)$, and

$$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

$$\tan(45^\circ + 60^\circ) = \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ}$$

$$\tan(45^\circ + 60^\circ) = \frac{1 + \sqrt{3}}{1 - (1)(\sqrt{3})}$$

$$\tan(45^\circ + 60^\circ) = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

Rationalize the denominator using conjugate method.

$$\tan(45^\circ + 60^\circ) = \frac{1+\sqrt{3}}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$\tan(45^\circ + 60^\circ) = \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$\tan(45^\circ + 60^\circ) = \frac{1+2\sqrt{3}+3}{1-3}$$

$$\tan(45^\circ + 60^\circ) = -\frac{4+2\sqrt{3}}{2}$$

$$\tan(45^\circ + 60^\circ) = -2-\sqrt{3}$$

■ 3. Find the exact values of $\tan(\theta - \alpha)$ if θ is an angle in the first quadrant whose cosine is $3/5$ and α is an angle in the fourth quadrant whose sine is $-5/13$.

Solution:

Before we can find the values of $\tan(\theta - \alpha)$, we need to find the values of $\sin\theta$ and $\cos\alpha$. To find the sine from the cosine, we'll rewrite the Pythagorean identity with sine and cosine,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

and then substitute $\cos\theta = 3/5$.

$$\sin^2\theta = 1 - \left(\frac{3}{5}\right)^2$$

$$\sin^2\theta = 1 - \frac{9}{25}$$

$$\sin^2\theta = \frac{16}{25}$$

$$\sin\theta = \pm\sqrt{\frac{16}{25}}$$

Since θ is in the first quadrant, we know that $\sin\theta$ is positive. So we can ignore the negative value and say

$$\sin\theta = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Now we need to find the value of $\cos\alpha$. Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substituting $\sin\alpha = -5/13$, we get

$$\cos^2\alpha = 1 - \left(-\frac{5}{13}\right)^2$$

$$\cos^2\alpha = 1 - \frac{25}{169}$$

$$\cos^2\alpha = \frac{144}{169}$$

$$\cos\alpha = \pm\sqrt{\frac{144}{169}}$$

Since α is in the fourth quadrant, we know that $\cos\alpha$ is positive. So we can ignore the negative value and say

$$\cos\alpha = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

Now we can find the values of $\tan\theta$ and $\tan\alpha$. Using the definition of the tangent function, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{5}\left(\frac{5}{3}\right) = \frac{4}{3}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{13}\left(\frac{13}{12}\right) = -\frac{5}{12}$$

By the difference identity for the tangent function,

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

$$\tan(\theta-\alpha)=\frac{\frac{4}{3}-\left(-\frac{5}{12}\right)}{1+\left(\frac{4}{3}\right)\left(-\frac{5}{12}\right)}$$

$$\tan(\theta-\alpha)=\frac{\frac{16+5}{12}}{1-\frac{20}{36}}$$

$$\tan(\theta-\alpha)=\frac{\frac{21}{12}}{\frac{36-20}{36}}$$

$$\tan(\theta-\alpha)=\frac{\frac{21}{12}}{\frac{16}{36}}$$

$$\tan(\theta-\alpha)=\frac{\frac{7}{4}}{\frac{4}{9}}$$

$$\tan(\theta-\alpha)=\frac{7}{4}\left(\frac{9}{4}\right)$$

$$\tan(\theta-\alpha)=\frac{63}{16}$$

■ 4. Simplify the expressions $\tan(\pi+\theta)$ and $\tan(\pi-\theta)$.

Solution:

To find tangent of the sum, use the sum identity for tangent.

$$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

$$\tan(\pi + \theta) = \frac{\tan\pi + \tan\theta}{1 - \tan\pi\tan\theta}$$

$$\tan(\pi + \theta) = \frac{0 + \tan\theta}{1 - (0)\tan\theta}$$

$$\tan(\pi + \theta) = \frac{\tan\theta}{1}$$

$$\tan(\pi + \theta) = \tan\theta$$

To find tangent of the difference, use a difference identity.

$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

$$\tan(\pi - \theta) = \frac{\tan\pi - \tan\theta}{1 + \tan\pi\tan\theta}$$

$$\tan(\pi - \theta) = \frac{0 - \tan\theta}{1 + (0)\tan\theta}$$

$$\tan(\pi - \theta) = \frac{-\tan\theta}{1}$$

$$\tan(\pi - \theta) = -\tan\theta$$

5. Find the exact values of $\tan(\theta+\alpha)$ if θ is an angle in the second quadrant whose cosine is $-4/7$ and α is an angle in the third quadrant whose cosine is $-9/10$.

Solution:

Before we can find the value of $\tan(\theta+\alpha)$, we need to find the values of $\sin\theta$ and $\cos\alpha$. To find the sines from the cosines, we'll rewrite the Pythagorean identity with sine and cosine,

$$\sin^2\theta+\cos^2\theta=1$$

$$\sin^2\theta=1-\cos^2\theta$$

and then substitute $\cos\theta=-4/7$.

$$\sin^2\theta=1-\left(-\frac{4}{7}\right)^2$$

$$\sin^2\theta=1-\frac{16}{49}$$

$$\sin^2\theta=\frac{33}{49}$$

$$\sin\theta=\pm\sqrt{\frac{33}{49}}$$

Since θ is in the second quadrant, we know that $\sin\theta$ is positive. So we can ignore the negative value and say

$$\sin\theta = \sqrt{\frac{33}{49}} = \frac{\sqrt{33}}{7}$$

Now we need to find the value of $\sin\alpha$. Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha = 1 - \cos^2\alpha$$

Substituting $\cos\alpha = -9/10$, we get

$$\sin^2\alpha = 1 - \left(-\frac{9}{10}\right)^2$$

$$\sin^2\alpha = 1 - \frac{81}{100}$$

$$\sin^2\alpha = \frac{19}{100}$$

$$\sin\alpha = \pm\sqrt{\frac{19}{100}}$$

Since α is in the third quadrant, we know that $\sin\alpha$ is negative. So we can ignore the positive value and say

$$\sin\alpha = -\sqrt{\frac{19}{100}} = -\frac{\sqrt{19}}{10}$$

Now we can find the values of $\tan\theta$ and $\tan\alpha$. Using the definition of the tangent function, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{33}}{7}}{-\frac{4}{7}} = \frac{\sqrt{33}}{7}\left(-\frac{7}{4}\right) = -\frac{\sqrt{33}}{4}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-\frac{\sqrt{19}}{10}}{-\frac{9}{10}} = -\frac{\sqrt{19}}{10}\left(-\frac{10}{9}\right) = \frac{\sqrt{19}}{9}$$

By the sum identity for the tangent function,

$$\tan(\theta+\alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

$$\tan(\theta+\alpha) = \frac{-\frac{\sqrt{33}}{4} + \frac{\sqrt{19}}{9}}{1 - \left(-\frac{\sqrt{33}}{4}\right)\left(\frac{\sqrt{19}}{9}\right)}$$

$$\tan(\theta+\alpha) = \frac{-\frac{9\sqrt{33}}{36} + \frac{4\sqrt{19}}{36}}{1 + \frac{\sqrt{33}\sqrt{19}}{36}}$$

$$\tan(\theta+\alpha) = \frac{\frac{4\sqrt{19} - 9\sqrt{33}}{36}}{\frac{36 + \sqrt{33}\sqrt{19}}{36}}$$

$$\tan(\theta+\alpha) = \frac{4\sqrt{19} - 9\sqrt{33}}{36}\left(\frac{36}{36 + \sqrt{33}\sqrt{19}}\right)$$

$$\tan(\theta+\alpha) = \frac{4\sqrt{19} - 9\sqrt{33}}{36 + \sqrt{627}}$$

6. Find the exact value of the expression.

$$\tan\left(\sin^{-1}\frac{1}{2}-\cos^{-1}\frac{1}{2}\right)$$

Solution:

To find the exact value of the expression, we need to use the difference identity for the tangent function.

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

Define α and β.

$$\alpha=\sin^{-1}\frac{1}{2}$$

$$\beta=\cos^{-1}\frac{1}{2}$$

Using the properties of inverses, we can write

$$\sin\alpha=\frac{1}{2},\quad -\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$$

$$\cos\beta=\frac{1}{2},\ 0\le\beta\le\pi$$

Use the Pythagorean identity with sine and cosine to find $\cos\alpha$ and $\sin\beta$.

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substitute $\sin\alpha = 1/2$.

$$\cos^2\alpha = 1 - \left(\frac{1}{2}\right)^2$$

$$\cos^2\alpha = 1 - \frac{1}{4}$$

$$\cos^2\alpha = \frac{3}{4}$$

$$\cos\alpha = \pm\sqrt{\frac{3}{4}}$$

Since $-\pi/2 \le \alpha \le \pi/2$, we can say $\cos\alpha = \sqrt{3}/2$.

$$\sin^2\beta + \cos^2\beta = 1$$

$$\sin^2\beta = 1 - \cos^2\beta$$

Substitute $\cos\beta = 1/2$.

$$\sin^2\beta = 1 - \left(\frac{1}{2}\right)^2$$

$$\sin^2\beta = 1 - \frac{1}{4}$$

$$\sin^2\beta = \frac{3}{4}$$

$$\sin\beta = \pm\sqrt{\frac{3}{4}}$$

Since $0 \le \beta \le \pi$, we can say $\sin\beta = \sqrt{3}/2$. Then using the difference formula for cosine, we get

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \tan(\alpha - \beta)$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right) - \frac{\sqrt{3}}{2}\left(\frac{2}{1}\right)}{1 + \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{1}\right)}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + 1}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{\frac{\sqrt{3}}{3} - \frac{3\sqrt{3}}{3}}{2}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{-\frac{2\sqrt{3}}{3}}{2}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = -\frac{\sqrt{3}}{3}$$

DOUBLE-ANGLE IDENTITIES

1. If θ is an angle in the fourth quadrant whose sine is $-3/5$, what are the values of $\tan 2\theta$?

Solution:

By the double-angle identity for tangent,

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

In order to use the double-angle identity, we first need to find $\cos\theta$ and then $\tan\theta$. By the Pythagorean identity with sine and cosine,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

Since $\sin\theta = -3/5$, we get

$$\cos^2\theta = 1 - \left(-\frac{3}{5}\right)^2$$

$$\cos^2\theta = 1 - \frac{9}{25}$$

$$\cos^2\theta = \frac{16}{25}$$

Since θ is in the fourth quadrant, we know that $\cos\theta$ is positive, so we can ignore the negative value and say

$$\cos\theta = \frac{4}{5}$$

Now, substituting $\cos\theta = 4/5$ and $\sin\theta = -3/5$ into

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

we get

$$\tan\theta = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{5}\left(\frac{5}{4}\right) = -\frac{3}{4}$$

Now to find $\tan 2\theta$, we'll substitute into the double-angle identity for tangent.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 2\theta = \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}$$

$$\tan 2\theta = \frac{-\frac{3}{2}}{1-\frac{9}{16}}$$

$$\tan 2\theta = \frac{-\frac{3}{2}}{\frac{16-9}{16}}$$

$$\tan 2\theta = \frac{-\frac{3}{2}}{\frac{7}{16}}$$

$$\tan 2\theta = -\frac{3}{2}\left(\frac{16}{7}\right)$$

$$\tan 2\theta = -\frac{24}{7}$$

■ 2. If θ is an angle in the third quadrant whose tangent is $3/4$, what are the values of $\cos 2\theta$?

Solution:

By the double-angle identity for cosine,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

or

$$\cos 2\theta = 2\cos^2\theta - 1$$

In order to use the double-angle identity, we first need to find $\cos\theta$. We remember that

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{c}$$

We need to find the hypotenuse using the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$

$$c^2 = 3^2 + 4^2$$

$$c^2 = 9 + 16$$

$$c^2 = 25$$

$$c = 5$$

Therefore

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}$$

To find $\cos 2\theta$, we'll substitute into the double-angle identity for cosine.

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 2\left(\frac{4}{5}\right)^2 - 1$$

$$\cos 2\theta = 2\left(\frac{16}{25}\right) - 1$$

$$\cos 2\theta = \frac{32}{25} - 1$$

$$\cos 2\theta = \frac{7}{25}$$

3. Use a double-angle identity to rewrite the expression.

$$(\sin x + \cos x)^2$$

Solution:

We'll rewrite the expression as

$$(\sin x + \cos x)^2$$

$$(\sin x + \cos x)(\sin x + \cos x)$$

$$\sin^2 x + \sin x \cos x + \cos x \sin x + \cos^2 x$$

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x$$

$$\sin^2 x + \cos^2 x + 2 \sin x \cos x$$

Remember that $\sin^2 x + \cos^2 x = 1$.

$$1 + 2 \sin x \cos x$$

Now use the double-angle identity for sine.

$$1 + \sin 2x$$

4. If θ is an angle in the third quadrant whose sine is $-1/\sqrt{5}$, what is the value of $\sin 2\theta$?

Solution:

In order to use the double-angle identity $\sin 2\theta = 2\sin\theta\cos\theta$, we first need to find $\cos\theta$. By the basic Pythagorean identity with sine and cosine,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

and since $\sin\theta = -1/\sqrt{5}$, we get

$$\cos^2\theta = 1 - \left(-\frac{1}{\sqrt{5}}\right)^2$$

$$\cos^2\theta = 1 - \frac{1}{5}$$

$$\cos^2\theta = \frac{4}{5}$$

Since θ is in the third quadrant, we know that $\cos\theta$ is negative, so we can ignore the positive value and say

$$\cos\theta = -\frac{2}{\sqrt{5}}$$

Substituting $\cos\theta = -2/\sqrt{5}$ and $\sin\theta = -1/\sqrt{5}$ into the double-angle identity for sine, we get

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = 2\left(-\frac{1}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{5}}\right)$$

$$\sin 2\theta = \frac{4}{5}$$

■ 5. If θ is an angle in the third quadrant whose tangent is $7/24$, what is the value of $\tan 2\theta$?

Solution:

To find $\tan 2\theta$, we'll substitute into the double-angle identity for tangent.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 2\theta = \frac{2\left(\frac{7}{24}\right)}{1-\left(\frac{7}{24}\right)^2}$$

$$\tan 2\theta = \frac{\frac{7}{12}}{1-\frac{49}{576}}$$

$$\tan 2\theta = \frac{\frac{7}{12}}{\frac{527}{576}}$$

$$\tan 2\theta = \frac{7}{12}\left(\frac{576}{527}\right)$$

$$\tan 2\theta = \frac{7}{1}\left(\frac{48}{527}\right)$$

$$\tan 2\theta = \frac{336}{527}$$

■ 6. Use a double-angle formula to rewrite the expression.

$$12\sin(4x)\cos(4x)$$

Solution:

Rewrite the expression as

$$12\sin(4x)\cos(4x)$$

$$6(2\sin(4x)\cos(4x))$$

Then the double-angle formula for sine lets us simplify to

$$6\sin(8x)$$

HALF-ANGLE IDENTITIES

■ 1. Use a half-angle identity to find the exact value of the expression.

$$\sin 15^\circ$$

Solution:

Use the the half-angle identity for sine

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

We know that $\theta/2=15^\circ$, so $\theta=2(15^\circ)=30^\circ$. Now substitute to get

$$\sin 15^\circ=\pm\sqrt{\frac{1-\cos 30^\circ}{2}}$$

$$\sin 15^\circ=\pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$\sin 15^\circ=\pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{2}}$$

$$\sin 15^\circ=\pm\sqrt{\frac{2-\sqrt{3}}{4}}$$

$$\sin 15^\circ = \pm\frac{\sqrt{2-\sqrt{3}}}{2}$$

Since 15° is in the first quadrant, sine is positive.

$$\sin 15^\circ = \frac{\sqrt{2-\sqrt{3}}}{2}$$

■ 2. If θ is the angle in Quadrant II with $\sin\theta = 7/25$, what are the values of $\sin(\theta/2)$ and $\cos(\theta/2)$?

Solution:

We'll start by using $\sin\theta = 7/25$ and the Pythagorean identity with sine and cosine to find the corresponding value of $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(\frac{7}{25}\right)^2$$

$$\cos^2\theta = 1 - \frac{49}{625}$$

$$\cos^2\theta = \frac{576}{625}$$

$$\cos\theta = \pm\sqrt{\frac{576}{625}}$$

Since θ is in the second quadrant, the cosine is negative, so

$$\cos\theta = -\sqrt{\frac{576}{625}} = -\frac{\sqrt{576}}{\sqrt{625}} = -\frac{24}{25}$$

Then by the half-angle identity for cosine, we get

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\left(-\frac{24}{25}\right)}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\frac{25-24}{25}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1}{50}}$$

$$\cos\frac{\theta}{2} = \pm\frac{1}{5\sqrt{2}}$$

$$\cos\frac{\theta}{2} = \pm\frac{\sqrt{2}}{10}$$

By the half-angle identity for sine, we get

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\left(-\frac{24}{25}\right)}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{\frac{25+24}{25}}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{49}{50}}$$

$$\sin\frac{\theta}{2}=\pm\frac{7}{5\sqrt{2}}$$

$$\sin\frac{\theta}{2}=\pm\frac{7\sqrt{2}}{10}$$

To find the quadrant of the angle $\theta/2$, we'll divide through the inequality we were given by 2.

$$\frac{\pi}{2}<\theta<\pi$$

$$\frac{\pi}{4}<\frac{\theta}{2}<\frac{\pi}{2}$$

The angle $\pi/4$ is halfway through the first quadrant, and the angle $\pi/2$ is along the positive side of the y-axis, so $\theta/2$ has to be in the first quadrant. Both the cosine function and the sine function are positive for all angles in the first quadrant, so

$$\cos\frac{\theta}{2}=\frac{\sqrt{2}}{10}$$

$$\sin\frac{\theta}{2}=\frac{7\sqrt{2}}{10}$$

■ 3. If θ is the angle in the interval $(0,\pi/2)$ with $\tan\theta=2$, what are the values of $\sin(\theta/2)$ and $\cos(\theta/2)$?

Solution:

Since θ lies in first quadrant, we know that the values of sine, cosine, and tangent for the angle will all be positive.

Because tangent is equivalent to opposite/adjacent, we get

$$\frac{\text{opposite}}{\text{adjacent}}=\frac{2}{1}$$

Given a triangle with adjacent leg 1 and opposite leg 2, the hypotenuse must be

$$a^2+b^2=c^2$$

$$1^2+2^2=c^2$$

$$c^2=1+4$$

$$c^2=5$$

$$c = \sqrt{5}$$

Because cosine is equivalent to adjacent/hypotenuse, we get

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

If we substitute $\cos\theta = \sqrt{5}/5$ into the half-angle identity for cosine, we get

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\frac{\sqrt{5}}{5}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\frac{5+\sqrt{5}}{5}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{5+\sqrt{5}}{10}}$$

If we also substitute $\cos\theta = \sqrt{5}/5$ into the half-angle identity for sine, we get

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\frac{\sqrt{5}}{5}}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{\frac{5-\sqrt{5}}{5}}{2}}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{5-\sqrt{5}}{10}}$$

To figure out the quadrant of $\theta/2$, we'll start with the fact that we were told θ is in the first quadrant, $0<\theta<\pi/2$. We'll divide through the inequality by 2 to change θ into $\theta/2$.

$$0<\theta<\frac{\pi}{2}$$

$$0<\frac{\theta}{2}<\frac{\pi}{4}$$

This inequality tells us that the angle $\theta/2$ falls between 0 (which is along the positive direction of the x-axis) and $\pi/4$ (which is halfway through the first quadrant). Therefore, the bounds $\theta/2=[0,\pi/4]$ define angles only in the first quadrant, so $\theta/2$ must be in the first quadrant.

For any angle in the first quadrant, sine and cosine are positive.

$$\cos\frac{\theta}{2}=\sqrt{\frac{5+\sqrt{5}}{10}}$$

$$\sin\frac{\theta}{2}=\sqrt{\frac{5-\sqrt{5}}{10}}$$

4. If θ is the angle in the interval $(3\pi/2,2\pi)$ with $\sin\theta = -15/17$, what are the values of $\tan(\theta/2)$ and $\cot(\theta/2)$?

Solution:

We'll start by using $\sin\theta = -15/17$ and the Pythagorean identity with sine and cosine to find the corresponding value of $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(-\frac{15}{17}\right)^2$$

$$\cos^2\theta = 1 - \frac{225}{289}$$

$$\cos^2\theta = \frac{64}{289}$$

$$\cos\theta = \pm\sqrt{\frac{64}{289}}$$

Since θ is in the fourth quadrant, the cosine of every angle in the fourth quadrant is positive, so

$$\cos\theta = \sqrt{\frac{64}{289}} = \frac{\sqrt{64}}{\sqrt{289}} = \frac{8}{17}$$

Then by the half-angle identity for tangent, we get

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2}=\frac{-\frac{15}{17}}{1+\frac{8}{17}}$$

$$\tan\frac{\theta}{2}=\frac{-\frac{15}{17}}{\frac{17+8}{17}}$$

$$\tan\frac{\theta}{2}=\frac{-\frac{15}{17}}{\frac{25}{17}}$$

$$\tan\frac{\theta}{2}=-\frac{15}{17}\left(\frac{17}{25}\right)$$

$$\tan\frac{\theta}{2}=-\frac{3}{5}$$

We remember that

$$\cot\theta=\frac{1}{\tan\theta}$$

$$\cot\frac{\theta}{2}=\frac{1}{\tan\frac{\theta}{2}}$$

$$\cot\frac{\theta}{2}=\frac{1}{-\frac{3}{5}}$$

$$\cot\frac{\theta}{2}=-\frac{5}{3}$$

5. Use a half-angle identity to find the exact value of the expression.

$$\sec\left(\frac{7\pi}{8}\right)$$

Solution:

First we need to find the cosine of $7\pi/8$. We know that $\theta/2 = 7\pi/8$, so

$$\theta = 2\left(\frac{7\pi}{8}\right) = \frac{7\pi}{4}$$

Using the half-angle identity for cosine, we get

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{1+\cos\left(\frac{7\pi}{4}\right)}{2}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{2+\sqrt{2}}{4}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\frac{\sqrt{2+\sqrt{2}}}{2}$$

Since $7\pi/8$ is in the second quadrant, cosine is negative.

$$\cos\left(\frac{7\pi}{8}\right) = -\frac{\sqrt{2+\sqrt{2}}}{2}$$

Then by the reciprocal identity for secant,

$$\sec\left(\frac{7\pi}{8}\right) = \frac{1}{\cos\left(\frac{7\pi}{8}\right)}$$

$$\sec\left(\frac{7\pi}{8}\right) = \frac{1}{-\frac{\sqrt{2+\sqrt{2}}}{2}}$$

$$\sec\left(\frac{7\pi}{8}\right) = -\frac{2}{\sqrt{2+\sqrt{2}}}$$

$$\sec\left(\frac{7\pi}{8}\right) = -\frac{2}{\sqrt{2+\sqrt{2}}}\cdot\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$

$$\sec\left(\frac{7\pi}{8}\right)=-\frac{2\sqrt{2-\sqrt{2}}}{\sqrt{2}}$$

$$\sec\left(\frac{7\pi}{8}\right)=-\frac{2\sqrt{2-\sqrt{2}}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$

$$\sec\left(\frac{7\pi}{8}\right)=-\frac{2\sqrt{4-2\sqrt{2}}}{2}$$

$$\sec\left(\frac{7\pi}{8}\right)=-\sqrt{4-2\sqrt{2}}$$

6. Prove the identity.

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

Solution:

We'll rewrite the right side of the half-angle identity for tangent.

$$\tan\frac{\theta}{2}=\pm\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$\tan\frac{\theta}{2}=\pm\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\cdot\frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos^2\theta}}{\sqrt{(1+\cos\theta)^2}}$$

Use $\sin^2\theta+\cos^2\theta=1$, or $\sin^2\theta=1-\cos^2\theta$ to simplify the numerator.

$$\tan\frac{\theta}{2}=\frac{\sqrt{\sin^2\theta}}{1+\cos\theta}$$

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

PRODUCT-TO-SUM IDENTITIES

1. Rewrite $\cos(x-y)\cos(x+y)$ as a sum.

Solution:

Using the product-to-sum identity,

$$\cos\theta\cos\alpha=\frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

set $\theta=x-y$ and $\alpha=x+y$ to get

$$\frac{1}{2}\left[\cos(x-y+x+y)+\cos(x-y-(x+y))\right]$$

$$\frac{1}{2}\left[\cos(2x)+\cos(x-y-x-y)\right]$$

$$\frac{1}{2}\left[\cos(2x)+\cos(-2y)\right]$$

Using the even-odd identity $\cos(-\theta)=\cos\theta$, we get

$$\frac{1}{2}\left[\cos(2x)+\cos(2y)\right]$$

2. Rewrite $\cos(x-15^\circ)\sin(x+15^\circ)$ as a sum.

Solution:

Using the product-to-sum identity,

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

set $\theta = x - 15^\circ$ and $\alpha = x + 15^\circ$ to get

$$\frac{1}{2}\left[\sin(x - 15^\circ + x + 15^\circ) - \sin(x - 15^\circ - (x + 15^\circ))\right]$$

$$\frac{1}{2}\left[\sin(2x) - \sin(x - 15^\circ - x - 15^\circ)\right]$$

$$\frac{1}{2}\left[\sin(2x) - \sin(-30^\circ)\right]$$

Using the even-odd identity $\sin(-\theta) = -\sin\theta$, we get

$$\frac{1}{2}\left[\sin(2x) + \sin(30^\circ)\right]$$

■ 3. Find a sum equivalent to $\cos^3 x$.

Solution:

Rewrite $\cos^3 x$ as

$$\cos^3 x = \cos x \cdot \cos x \cdot \cos x$$

Consider just the first two factors, $\cos x \cdot \cos x$, and apply the product-to-sum identity

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

With $\theta = x$ and $\alpha = x$, we get

$$\frac{1}{2}\left[\cos(x+x)+\cos(x-x)\right]$$

$$\frac{1}{2}\left[\cos(2x)+\cos(0)\right]$$

$$\frac{1}{2}\left[\cos(2x)+1\right]$$

So $\cos^3 x$ is

$$\cos^3 x = \frac{1}{2}\left[\cos(2x)+1\right]\cos x$$

$$\cos^3 x = \left[\frac{1}{2}\cos(2x)+\frac{1}{2}\right]\cos x$$

$$\cos^3 x = \frac{1}{2}\cos(2x)\cos x + \frac{1}{2}\cos x$$

To rewrite $\cos(2x)\cos x$, we'll again use

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

with $\theta = 2x$ and $\alpha = x$.

$$\cos(2x)\cos x = \frac{1}{2}\left[\cos(2x + x) + \cos(2x - x)\right]$$

$$\cos(2x)\cos x = \frac{1}{2}\left[\cos(3x) + \cos x\right]$$

Substitute this into the expression for $\cos^3 x$.

$$\cos^3 x = \frac{1}{2}\left[\frac{1}{2}\left[\cos(3x) + \cos x\right]\right] + \frac{1}{2}\cos x$$

$$\cos^3 x = \frac{1}{4}\left[\cos(3x) + \cos x\right] + \frac{1}{2}\cos x$$

$$\cos^3 x = \frac{1}{4}\cos(3x) + \frac{1}{4}\cos x + \frac{1}{2}\cos x$$

$$\cos^3 x = \frac{1}{4}\cos(3x) + \frac{1}{4}\cos x + \frac{2}{4}\cos x$$

$$\cos^3 x = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos x$$

■ 4. Find the exact value of each expression.

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right)$$

$$\sin^2\left(\frac{3\pi}{8}\right)$$

$$\cos^2\left(\frac{3\pi}{8}\right)$$

Solution:

To compute $(\sin(3\pi/8))(\cos(3\pi/8))$, we can use the product-to-sum identity for the product of sine and cosine.

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right) = \frac{1}{2}\left[\sin\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)+\sin\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)\right]$$

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right) = \frac{1}{2}\left(\sin\frac{3\pi}{4}+\sin 0\right)$$

Pulling the values of sine on the right side from the unit circle, we get

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2}+0\right)$$

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right) = \frac{\sqrt{2}}{4}$$

To find $\sin^2(3\pi/8)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

$$\left(\sin\frac{3\pi}{8}\right)\left(\sin\frac{3\pi}{8}\right) = \frac{1}{2}\left[\cos\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)-\cos\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)\right]$$

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}\left(\cos 0-\cos\frac{3\pi}{4}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}\left(1-\left(-\frac{\sqrt{2}}{2}\right)\right)$$

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}+\frac{\sqrt{2}}{4}$$

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{2}{4}+\frac{\sqrt{2}}{4}$$

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{2+\sqrt{2}}{4}$$

To get the value of $\cos^2(3\pi/8)$, we'll use the product-to-sum identity for the product of two cosine functions.

$$\cos\theta\cos\alpha=\frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

$$\left(\cos\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right)=\frac{1}{2}\left[\cos\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)+\cos\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)\right]$$

$$\cos^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}\left(\cos\frac{3\pi}{4}+\cos 0\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\cos^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}\left(-\frac{\sqrt{2}}{2}+1\right)$$

$$\cos^2\left(\frac{3\pi}{8}\right)=-\frac{\sqrt{2}}{4}+\frac{1}{2}$$

$$\cos^2\left(\frac{3\pi}{8}\right)=-\frac{\sqrt{2}}{4}+\frac{2}{4}$$

$$\cos^2\left(\frac{3\pi}{8}\right)=\frac{2-\sqrt{2}}{4}$$

5. Simplify the expression.

$$\sin(x-y)\cos y+\cos(x-y)\sin y$$

Solution:

Consider just the first two factors, $\sin(x-y)\cos y$, and apply the product-to-sum identity

$$\sin\theta\cos\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

With $\theta=x-y$ and $\alpha=y$, we get

$$\sin(x-y)\cos y=\frac{1}{2}\left[\sin(x-y+y)+\sin(x-y-y)\right]$$

$$\sin(x-y)\cos y = \frac{1}{2}\left[\sin x + \sin(x-2y)\right]$$

$$\sin(x-y)\cos y = \frac{1}{2}\sin x + \frac{1}{2}\sin(x-2y)$$

To rewrite $\cos(x-y)\sin y$, we'll use

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

With $\theta = x - y$ and $\alpha = y$, we get

$$\cos(x-y)\sin y = \frac{1}{2}\left[\sin(x-y+y) - \sin(x-y-y)\right]$$

$$\cos(x-y)\sin y = \frac{1}{2}\left[\sin x - \sin(x-2y)\right]$$

$$\cos(x-y)\sin y = \frac{1}{2}\sin x - \frac{1}{2}\sin(x-2y)$$

Then the original expression can be written as

$$\frac{1}{2}\sin x + \frac{1}{2}\sin(x-2y) + \frac{1}{2}\sin x - \frac{1}{2}\sin(x-2y)$$

$$\sin x$$

■ 6. Find the value of the expression.

$$\sin^2\left(\frac{\pi}{12}\right) + \sin^2\left(\frac{3\pi}{12}\right) + \sin^2\left(\frac{5\pi}{12}\right)$$

Solution:

To find $\sin^2(\pi/12)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

With $\theta = \pi/12$ and $\alpha = \pi/12$, we get

$$\left(\sin\frac{\pi}{12}\right)\left(\sin\frac{\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{12}-\frac{\pi}{12}\right)-\cos\left(\frac{\pi}{12}+\frac{\pi}{12}\right)\right]$$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(\cos 0-\cos\frac{\pi}{6}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)$$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}-\frac{\sqrt{3}}{4}$$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{2}{4}-\frac{\sqrt{3}}{4}$$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{2-\sqrt{3}}{4}$$

To find $\sin^2(3\pi/12)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

With $\theta = 3\pi/12$ and $\alpha = 3\pi/12$, we get

$$\left(\sin\frac{3\pi}{12}\right)\left(\sin\frac{3\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{3\pi}{12}-\frac{3\pi}{12}\right)-\cos\left(\frac{3\pi}{12}+\frac{3\pi}{12}\right)\right]$$

$$\sin^2\left(\frac{3\pi}{12}\right) = \frac{1}{2}\left(\cos 0 - \cos\frac{\pi}{2}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\left(\frac{3\pi}{12}\right) = \frac{1}{2}(1-0)$$

$$\sin^2\left(\frac{3\pi}{12}\right) = \frac{1}{2}$$

To find $\sin^2(5\pi/12)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

With $\theta = 5\pi/12$ and $\alpha = 5\pi/12$, we get

$$\left(\sin\frac{5\pi}{12}\right)\left(\sin\frac{5\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{5\pi}{12}-\frac{5\pi}{12}\right)-\cos\left(\frac{5\pi}{12}+\frac{5\pi}{12}\right)\right]$$

$$\sin^2\left(\frac{5\pi}{12}\right) = \frac{1}{2}\left(\cos 0 - \cos\frac{5\pi}{6}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\left(\frac{5\pi}{12}\right) = \frac{1}{2}\left(1 - \left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$\sin^2\left(\frac{5\pi}{12}\right) = \frac{1}{2} + \frac{\sqrt{3}}{4}$$

$$\sin^2\left(\frac{5\pi}{12}\right) = \frac{2}{4} + \frac{\sqrt{3}}{4}$$

$$\sin^2\left(\frac{5\pi}{12}\right) = \frac{2+\sqrt{3}}{4}$$

Then the original expression can be rewritten as

$$\frac{2-\sqrt{3}}{4} + \frac{1}{2} + \frac{2+\sqrt{3}}{4}$$

$$\frac{2-\sqrt{3}}{4} + \frac{2}{4} + \frac{2+\sqrt{3}}{4}$$

$$\frac{6}{4}$$

$$\frac{3}{2}$$

SUM-TO-PRODUCT IDENTITIES

■ 1. Rewrite the function as a product.

$$f(x) = \sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right)$$

Solution:

Apply the sum-to-product identity

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta + \alpha}{2}\right)\sin\left(\frac{\theta - \alpha}{2}\right)$$

to get

$$f(x) = 2\cos\left(\frac{x + \frac{\pi}{6} + x - \frac{\pi}{6}}{2}\right)\sin\left(\frac{x + \frac{\pi}{6} - \left(x - \frac{\pi}{6}\right)}{2}\right)$$

$$f(x) = 2\cos\left(\frac{2x}{2}\right)\sin\left(\frac{x + \frac{\pi}{6} - x + \frac{\pi}{6}}{2}\right)$$

$$f(x) = 2\cos x\sin\left(\frac{\frac{2\pi}{6}}{2}\right)$$

$$f(x) = 2\cos x \sin\left(\frac{2\pi}{12}\right)$$

$$f(x) = 2\cos x \sin\left(\frac{\pi}{6}\right)$$

From the unit circle, the value of $\sin(\pi/6)$ is $1/2$, so

$$f(x) = 2\cos x\left(\frac{1}{2}\right)$$

$$f(x) = \cos x$$

■ 2. Find a product equal to $\sin(x+y)+\sin(x-y)$.

Solution:

Apply the sum-to-product identity

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to get

$$2\sin\left(\frac{x+y+x-y}{2}\right)\cos\left(\frac{x+y-(x-y)}{2}\right)$$

$$2\sin\left(\frac{2x}{2}\right)\cos\left(\frac{x+y-x+y}{2}\right)$$

$$2\sin x\cos\left(\frac{2y}{2}\right)$$

$$2\sin x\cos y$$

3. Find the exact value of the expression.

$$\frac{\cos 93^\circ + \cos 27^\circ}{\cos 33^\circ}$$

Solution:

To the numerator, apply the sum-to-product identity

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to get

$$2\cos\left(\frac{93^\circ + 27^\circ}{2}\right)\cos\left(\frac{93^\circ - 27^\circ}{2}\right)$$

$$2\cos\left(\frac{120^\circ}{2}\right)\cos\left(\frac{66^\circ}{2}\right)$$

$$2\cos 60^\circ \cos 33^\circ$$

Substitute this value into the original fraction.

$$\frac{2\cos 60^\circ \cos 33^\circ}{\cos 33^\circ}$$

$$2\cos 60^\circ$$

$$2\left(\frac{1}{2}\right)$$

$$1$$

4. Simplify the expression.

$$\frac{\sin(7\theta)+\sin(3\theta)}{\cos(7\theta)+\cos(3\theta)}$$

Solution:

Apply the sum-to-product identity

$$\sin\theta+\sin\alpha=2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to the numerator to get

$$2\sin\left(\frac{7\theta+3\theta}{2}\right)\cos\left(\frac{7\theta-3\theta}{2}\right)$$

$$2\sin\left(\frac{10\theta}{2}\right)\cos\left(\frac{4\theta}{2}\right)$$

$$2\sin(5\theta)\cos(2\theta)$$

Apply the sum-to-product identity

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to the denominator to get

$$2\cos\left(\frac{7\theta+3\theta}{2}\right)\cos\left(\frac{7\theta-3\theta}{2}\right)$$

$$2\cos\left(\frac{10\theta}{2}\right)\cos\left(\frac{4\theta}{2}\right)$$

$$2\cos(5\theta)\cos(2\theta)$$

Then the original expression becomes

$$\frac{2\sin(5\theta)\cos(2\theta)}{2\cos(5\theta)\cos(2\theta)}$$

$$\frac{\sin(5\theta)}{\cos(5\theta)}$$

$$\tan(5\theta)$$

■ 5. Find a product equal to $\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta)$.

Solution:

Apply the sum-to-product identity

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to rewrite $\cos(3\theta) + \cos(5\theta)$. We can set $\theta = 3\theta$ and $\alpha = 5\theta$ and rewrite the product as

$$\cos(3\theta) + \cos(5\theta) = 2\cos\left(\frac{3\theta+5\theta}{2}\right)\cos\left(\frac{3\theta-5\theta}{2}\right)$$

$$\cos(3\theta) + \cos(5\theta) = 2\cos\left(\frac{8\theta}{2}\right)\cos\left(\frac{-2\theta}{2}\right)$$

$$\cos(3\theta) + \cos(5\theta) = 2\cos(4\theta)\cos(-\theta)$$

Using the even-odd identity $\cos(-\theta) = \cos\theta$, we get

$$\cos(3\theta) + \cos(5\theta) = 2\cos(4\theta)\cos(\theta)$$

Substitute into the expression, then simplify.

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 2\cos(4\theta)\cos(\theta) - 2\cos(\theta)\cos(8\theta)$$

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 2\cos(\theta)(\cos(4\theta) - \cos(8\theta))$$

Apply the sum-to-product identity

$$\cos\theta - \cos\alpha = -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

to rewrite $\cos(4\theta) - \cos(8\theta)$. We can set $\theta = 4\theta$ and $\alpha = 8\theta$ and rewrite the product as

$$\cos(4\theta) - \cos(8\theta) = -2\sin\left(\frac{4\theta+8\theta}{2}\right)\sin\left(\frac{4\theta-8\theta}{2}\right)$$

$$\cos(4\theta) - \cos(8\theta) = -2\sin\left(\frac{12\theta}{2}\right)\sin\left(\frac{-4\theta}{2}\right)$$

$$\cos(4\theta) - \cos(8\theta) = -2\sin(6\theta)\sin(-2\theta)$$

Using the odd-even identity $\sin(-\theta) = -\sin\theta$, we get

$$\cos(4\theta) - \cos(8\theta) = 2\sin(6\theta)\sin(2\theta)$$

And finally

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 2\cos(4\theta)\cos(\theta) - 2\cos(\theta)\cos(8\theta)$$

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 2\cos(\theta)(\cos(4\theta) - \cos(8\theta))$$

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 2\cos\theta(2\sin(6\theta)\sin(2\theta))$$

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 4\cos\theta\sin(6\theta)\sin(2\theta)$$

■ 6. Find the exact value of the expression.

$$16\sin 390^\circ + 22\sin 240^\circ + 16\sin 150^\circ - 22\sin 120^\circ$$

Solution:

First we need to rewrite the expression as

$$16\sin 390^\circ + 16\sin 150^\circ + 22\sin 240^\circ - 22\sin 120^\circ$$

$$16(\sin 390^\circ + \sin 150^\circ) + 22(\sin 240^\circ - \sin 120^\circ)$$

Using the sum-to-product identity,

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

we can set $\theta = 390^\circ$ and $\alpha = 150^\circ$ and rewrite the product as

$$\sin 390^\circ + \sin 150^\circ = 2\sin\left(\frac{390^\circ+150^\circ}{2}\right)\cos\left(\frac{390^\circ-150^\circ}{2}\right)$$

$$\sin 390^\circ + \sin 150^\circ = 2\sin\left(\frac{540^\circ}{2}\right)\cos\left(\frac{240}{2}\right)$$

$$\sin 390^\circ + \sin 150^\circ = 2\sin 270^\circ \cos 120^\circ$$

$$\sin 390^\circ + \sin 150^\circ = 2(-1)\left(-\frac{1}{2}\right)$$

$$\sin 390^\circ + \sin 150^\circ = 1$$

Now using the sum-to-product identity,

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

we can set $\theta = 240^\circ$ and $\alpha = 120^\circ$ and rewrite the product as

$$\sin 240^\circ - \sin 120^\circ = 2\cos\left(\frac{240^\circ+120^\circ}{2}\right)\sin\left(\frac{240^\circ-120^\circ}{2}\right)$$

$$\sin 240^\circ - \sin 120^\circ = 2\cos\left(\frac{360^\circ}{2}\right)\sin\left(\frac{120^\circ}{2}\right)$$

$$\sin 240^\circ - \sin 120^\circ = 2\cos 180^\circ \sin 60^\circ$$

$$\sin 240^\circ - \sin 120^\circ = 2(-1)\left(\frac{\sqrt{3}}{2}\right)$$

$$\sin 240^\circ - \sin 120^\circ = -\sqrt{3}$$

Then the value of the original expression is

$$16(1) + 22(-\sqrt{3})$$

$$16 - 22\sqrt{3}$$

PROVING THE TRIG EQUATION

1. Prove the trig equation.

$$\tan\left(\frac{x}{2}\right)=\frac{1-\cos x}{\sin x}$$

Solution:

Start by working on the numerator $1-\cos x$, using a half-angle identity.

$$\sin^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{2}$$

$$2\sin^2\left(\frac{x}{2}\right)=1-\cos x$$

Set this aside for a moment, and work on the denominator $\sin x$, using the double-angle identity $\sin 2\theta=2\sin\theta\cos\theta$. Make a substitution of $\theta=x/2$.

$$\sin x=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

Substitute the two values we've just found into the original expression.

$$\tan\left(\frac{x}{2}\right)=\frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$\tan\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

$$\tan\left(\frac{x}{2}\right) = \tan\left(\frac{x}{2}\right)$$

2. Prove the trigonometric equation.

$$\frac{\sin(5x) - \sin x}{\cos(5x) + \cos x} = \tan(2x)$$

Solution:

Apply a sum-to-product identity to the numerator on the left side.

$$\sin(5x) - \sin x = 2\cos\left(\frac{5x + x}{2}\right)\sin\left(\frac{5x - x}{2}\right)$$

$$\sin(5x) - \sin x = 2\cos\left(\frac{6x}{2}\right)\sin\left(\frac{4x}{2}\right)$$

$$\sin(5x) - \sin x = 2\cos(3x)\sin(2x)$$

Set this aside for a moment, and apply a sum-to-product identity to the denominator.

$$\cos(5x) + \cos x = 2\cos\left(\frac{5x + x}{2}\right)\cos\left(\frac{5x - x}{2}\right)$$

$$\cos(5x) + \cos x = 2\cos\left(\frac{6x}{2}\right)\cos\left(\frac{4x}{2}\right)$$

$$\cos(5x) + \cos x = 2\cos(3x)\cos(2x)$$

Substitute the two values we've just found into the original expression.

$$\frac{2\cos(3x)\sin(2x)}{2\cos(3x)\cos(2x)} = \tan(2x)$$

$$\frac{\sin(2x)}{\cos(2x)} = \tan(2x)$$

$$\tan(2x) = \tan(2x)$$

3. Prove the trigonometric equation.

$$\sin(x - \pi)\sin(x + \pi) = \sin^2 x$$

Solution:

Apply a product-to-sum identity to the left side of the equation.

$$\frac{1}{2}\left[\cos(x - \pi - (x + \pi)) - \cos(x - \pi + x + \pi)\right] = \sin^2 x$$

$$\frac{1}{2}\left[\cos(-2\pi) - \cos(2x)\right] = \sin^2 x$$

$$\frac{1}{2}\left[1 - \cos(2x)\right] = \sin^2 x$$

$$\frac{1-\cos(2x)}{2}=\sin^2 x$$

Set this aside for a moment, and then starting with the half-angle identity

$$\sin^2\left(\frac{\theta}{2}\right)=\frac{1-\cos\theta}{2}$$

substitute $x=\theta/2$ to get

$$\sin^2 x=\frac{1-\cos(2x)}{2}$$

Substitute this value into the left side of the equation we left off with.

$$\frac{1-\cos(2x)}{2}=\sin^2 x$$

$$\sin^2 x=\sin^2 x$$

■ 4. Prove the trigonometric equation.

$$\sin(-x)\cos(-x)\tan(-x)\csc(-x)=-\sin x$$

Solution:

Rewrite the tangent and cosecant functions in terms of sine and cosine, then cancel factors.

$$\sin(-x)\cos(-x)\left(\frac{\sin(-x)}{\cos(-x)}\right)\left(\frac{1}{\sin(-x)}\right)=-\sin x$$

$$\sin(-x)\sin(-x)\left(\frac{1}{\sin(-x)}\right) = -\sin x$$

$$\sin(-x) = -\sin x$$

Apply the even-odd identity $\sin(-x) = -\sin x$.

$$-\sin x = -\sin x$$

■ 5. Prove the trigonometric equation.

$$(\sin t + \cos t)^2 - 1 = \sin(2t)$$

Solution:

Expand the expression.

$$\sin^2 t + 2\sin t\cos t + \cos^2 t - 1 = \sin(2t)$$

$$(\sin^2 t + \cos^2 t) + 2\sin t\cos t - 1 = \sin(2t)$$

Apply the Pythagorean identity $\sin^2 t + \cos^2 t = 1$ to get

$$1 + 2\sin t\cos t - 1 = \sin(2t)$$

$$2\sin t\cos t = \sin(2t)$$

Apply the double-angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$\sin(2t) = \sin(2t)$$

6. Prove the trigonometric equation.

$$\frac{\cos(270^\circ + x)}{\sin(180^\circ - x)} = 1$$

Solution:

Apply the sum-difference identity $\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$ to the numerator, setting $\theta = 270^\circ$ and $\alpha = x$.

$$\cos(270^\circ + x) = \cos 270^\circ \cos x - \sin 270^\circ \sin x$$

$$\cos(270^\circ + x) = (0)\cos x - (-1)\sin x$$

$$\cos(270^\circ + x) = \sin x$$

Apply the sum-difference identity $\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$ to the denominator, setting $\theta = 180^\circ$ and $\alpha = x$.

$$\sin(180^\circ - x) = \sin 180^\circ \cos x - \cos 180^\circ \sin x$$

$$\sin(180^\circ - x) = (0)\cos x - (-1)\sin x$$

$$\sin(180^\circ - x) = \sin x$$

Substituting the values we've found into the given expression, we get

$$\frac{\cos(270^\circ + x)}{\sin(180^\circ - x)} = 1$$

$$\frac{\sin x}{\sin x} = 1$$

$$1 = 1$$

COMPLETE SOLUTION SET OF THE EQUATION

1. Find the complete solution set of the equation $\cos^2 x - 3\cos x + 2 = 0$.

Solution:

The equation is a quadratic equation in terms of $\cos x$, so we can factor it as

$$(\cos x - 1)(\cos x - 2) = 0$$

$\cos x - 1 = 0$ or $\cos x - 2 = 0$

$\cos x = 1$ or $\cos x = 2$

The equation $\cos x = 2$ has no solutions since the cosine function is only defined on the range $[-1,1]$ and therefore can't be equal to 2. The first equation is true at $x = 0$, and every angle coterminal with 0.

$$x = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \dots$$

$x = 2\pi n$ where n is any integer

2. Find all the solutions of the trig equation, then list only the solutions that lie in the interval $[0,2\pi)$.

$$3\csc^2\theta - 2\cot^2\theta - 4 = 0$$

Solution:

If we use the Pythagorean identity for cotangent, $\cot^2\theta = \csc^2\theta - 1$, we can rewrite the equation as

$$3\csc^2\theta - 2(\csc^2\theta - 1) - 4 = 0$$

$$3\csc^2\theta - 2\csc^2\theta + 2 - 4 = 0$$

$$\csc^2\theta - 2 = 0$$

$$\csc^2\theta = 2$$

$$\csc\theta = \pm\sqrt{2}$$

$$\frac{1}{\sin\theta} = \pm\sqrt{2}$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}$$

$$\sin\theta = \pm\frac{\sqrt{2}}{2}$$

This is the value of sine at $\theta = \pi/4$, $\theta = 3\pi/4$, $\theta = 5\pi/4$, $\theta = 7\pi/4$, and every angle coterminal with those.

$$\theta = \left\{\frac{\pi}{4} + \pi k \ \mid \ k \in \mathbb{Z}\right\} \cup \left\{\frac{3\pi}{4} + \pi k \ \mid \ k \in \mathbb{Z}\right\}$$

In the interval $[0,2\pi)$, only $\theta = \pi/4$, $\theta = 3\pi/4$, $\theta = 5\pi/4$, and $\theta = 7\pi/4$ will satisfy the equation.

3. Find the complete solution set of the equation.

$$4\cos^3\theta - 2\cos^2\theta - 2\cos\theta + 1 = 0$$

Solution:

We can rewrite the equation as

$$2\cos^2\theta(2\cos\theta - 1) - (2\cos\theta - 1) = 0$$

$$(2\cos\theta - 1)(2\cos^2\theta - 1) = 0$$

The only way the left side of the equation is 0 is if $2\cos\theta - 1 = 0$, $2\cos^2\theta - 1 = 0$, or both. So we need to solve these equations individually to find the values of θ that satisfy the equation. We get

$$2\cos\theta - 1 = 0$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

and

$$2\cos^2\theta - 1 = 0$$

$$2\cos^2\theta = 1$$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{\sqrt{2}}{2}$$

The equation $\cos\theta = 1/2$ is true at $\theta = \pi/3$ and $\theta = 5\pi/3$. But the set of all angles coterminal with these two angles is

$$\theta = \frac{\pi}{3} + 2n\pi \text{ and } \theta = \frac{5\pi}{3} + 2n\pi$$

The equation $\cos\theta = \sqrt{2}/2$ is true at $\theta = \pi/4$ and $\theta = 7\pi/4$. But the set of all angles coterminal with these two angles is

$$\theta = \frac{\pi}{4} + 2n\pi \text{ and } \theta = \frac{7\pi}{4} + 2n\pi$$

Putting all these sets together, we can say that the complete solution set of $4\cos^3\theta - 2\cos^2\theta - 2\cos\theta + 1 = 0$ includes all of these, where n is any integer:

$$\theta = \frac{\pi}{4} + 2n\pi$$

$$\theta = \frac{7\pi}{4} + 2n\pi$$

$$\theta = \frac{\pi}{3} + 2n\pi$$

$$\theta = \frac{5\pi}{3} + 2n\pi$$

4. Find all the solutions of the trig equation, then list only the solutions that lie in the interval $[0,2\pi)$.

$$\cos\theta + 1 = \sin\theta$$

Solution:

First we need to square both sides.

$$(\cos\theta + 1)^2 = \sin^2\theta$$

If we use the Pythagorean identity for sine, $\sin^2\theta = 1 - \cos^2\theta$, we can rewrite the equation as

$$(\cos\theta + 1)^2 = 1 - \cos^2\theta$$

$$\cos^2\theta + 2\cos\theta + 1 = 1 - \cos^2\theta$$

$$2\cos^2\theta + 2\cos\theta = 0$$

$$2\cos\theta(\cos\theta + 1) = 0$$

The only way the left side of the equation is 0 is if $2\cos\theta = 0$, $\cos\theta + 1 = 0$, or both. So we need to solve these equations individually to find the values of θ that satisfy the equation. We get

$$2\cos\theta = 0$$

$$\cos\theta = 0$$

and

$$\cos\theta + 1 = 0$$

$$\cos\theta = -1$$

The equation $\cos\theta = 0$ is true at $\theta = \pi/2$ and $\theta = 3\pi/2$, and the equation $\cos\theta = -1$ is true at $\theta = \pi$. The equation will also be true at all angles coterminal with these.

$$\theta = \left\{\frac{\pi}{2} + \pi k \ \mid \ k \in \mathbb{Z}\right\} \cup \left\{\pi + 2\pi k \ \mid \ k \in \mathbb{Z}\right\}$$

Of this set, the only solutions in the interval $[0,2\pi)$ are $\theta = \pi/2$, $\theta = \pi$, and $\theta = 3\pi/2$.

■ 5. Find all the solutions of the trig equation, then list only the solutions that lie in the interval $[0,2\pi)$.

$$2(\sin^2\theta - \cos^2\theta) = \sqrt{3}$$

Solution:

We can rewrite the equation as

$$\sin^2\theta - \cos^2\theta = \frac{\sqrt{3}}{2}$$

$$\cos^2\theta - \sin^2\theta = -\frac{\sqrt{3}}{2}$$

Use the double-angle identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to rewrite the left side of the equation.

$$\cos(2\theta) = -\frac{\sqrt{3}}{2}$$

From the unit circle, we know that cosine is $-\sqrt{3}/2$ at $5\pi/6$ and $7\pi/6$. Therefore, we need to solve two equations:

$$2\theta = \frac{5\pi}{6} + 2n\pi$$

$$\theta_1 = \frac{5\pi}{12} + n\pi$$

and

$$2\theta = \frac{7\pi}{6} + 2n\pi$$

$$\theta_2 = \frac{7\pi}{12} + n\pi$$

The full solution set is

$$\theta = \left\{\frac{5\pi}{12} + n\pi \ \middle|\ n \in \mathbb{Z}\right\} \cup \left\{\frac{7\pi}{12} + n\pi \ \middle|\ n \in \mathbb{Z}\right\}$$

and the solution set in the interval $[0,2\pi)$ is just $\theta = 5\pi/12$ and $\theta = 7\pi/12$.

6. Find the complete solution set of the equation.

$$4\sin\left(\theta-\frac{\pi}{3}\right)\cos\left(\theta-\frac{\pi}{3}\right)=\sqrt{3}$$

Solution:

Divide through both sides of the equation by 2.

$$2\sin\left(\theta-\frac{\pi}{3}\right)\cos\left(\theta-\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

Use the double-angle identity $\sin(2\theta)=2\sin\theta\cos\theta$ to rewrite the left side of the equation.

$$\sin\left[2\left(\theta-\frac{\pi}{3}\right)\right]=\frac{\sqrt{3}}{2}$$

From the unit circle, we know that sine is $\sqrt{3}/2$ at $\pi/3$ and $2\pi/3$. Therefore, we need to solve two equations:

$$2\left(\theta-\frac{\pi}{3}\right)=\frac{\pi}{3}+2n\pi$$

$$\theta-\frac{\pi}{3}=\frac{\pi}{6}+n\pi$$

$$\theta=\frac{\pi}{6}+\frac{\pi}{3}+n\pi$$

$$\theta=\frac{\pi}{2}+n\pi$$

and

$$2\left(\theta - \frac{\pi}{3}\right) = \frac{2\pi}{3} + 2n\pi$$

$$\theta - \frac{\pi}{3} = \frac{\pi}{3} + n\pi$$

$$\theta = \frac{2\pi}{3} + n\pi$$

Putting all these sets together, we can say that the complete solution set of the equation includes both angle sets.

$$\theta = \left\{\frac{\pi}{2} + n\pi \ \middle|\ n \in \mathbb{Z}\right\} \cup \left\{\frac{2\pi}{3} + n\pi \ \middle|\ n \in \mathbb{Z}\right\}$$

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