Sum-to-product identities

In the last lesson, we looked at the product-to-sum identities in order to to break down the product of trig functions into a sum or difference of trig functions.

Now we want to go the other direction, converting the sum or difference of trig functions into the product of trig functions.

To convert in this opposite direction, we'll use the sum-to-product identities, which we can build from the product-to-sum identities we just introduced.

Sum-to-product identities from the product-to-sum identities

As a reminder, the product-to-sum identities are

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)-\sin(\theta-\alpha)\right]$$

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

With these identities in mind, let $x = \theta + \alpha$ and let $y = \theta - \alpha$. Then add these equations,

$$x + y = \theta + \alpha + \theta - \alpha$$

$$x + y = \theta + \theta$$

$$x + y = 2\theta$$

$$\theta = \frac{x + y}{2}$$

and subtract these equations.

$$x - y = \theta + \alpha - (\theta - \alpha)$$

$$x - y = \theta + \alpha - \theta + \alpha$$

$$x - y = \alpha + \alpha$$

$$x - y = 2\alpha$$

$$\alpha = \frac{x - y}{2}$$

Now with these different values for θ and α, we can substitute into the product-to-sum identities. The first identity becomes

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta + \alpha) + \sin(\theta - \alpha)\right]$$

$$\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right) = \frac{1}{2}(\sin x + \sin y)$$

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

the second becomes,

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

$$\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{1}{2}(\sin x - \sin y)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

the third becomes,

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha) + \cos(\theta-\alpha)\right]$$

$$\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}(\cos x + \cos y)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

and the fourth becomes

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha) - \cos(\theta+\alpha)\right]$$

$$\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{1}{2}(\cos y - \cos x)$$

$$\cos y - \cos x = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

or

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

If we summarize what we've just built, we get the four **sum-to-product identities**.

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos\theta - \cos\alpha = -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

Let's do an example where we use the fourth sum-to-product identity to turn the difference of cosine functions into the product of sine functions.

Example

Express $\cos(6\theta) - \cos(15\theta)$ as a product of trig functions.

Because we have the difference of cosine functions, we'll plug into the fourth sum-to-product identity.

$$\cos\theta - \cos\alpha = -2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{6\theta+15\theta}{2}\right)\sin\left(\frac{6\theta-15\theta}{2}\right)$$

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{21\theta}{2}\right)\sin\left(-\frac{9\theta}{2}\right)$$

We could certainly leave the equation this way. But we can further simplify the right side by using the odd identity for the sine function.

$$\cos(6\theta) - \cos(15\theta) = -2\sin\left(\frac{21\theta}{2}\right)\left[-\sin\left(\frac{9\theta}{2}\right)\right]$$

$$\cos(6\theta) - \cos(15\theta) = 2\sin\left(\frac{21\theta}{2}\right)\sin\left(\frac{9\theta}{2}\right)$$

Let's do a different kind of example, where we're asked to use the sum-to-product identities to prove a trig equation. To prove that a trig equation is true, we need to show that both sides are identical, by changing one or both sides until they're identical.

Example

Use a sum-to-product identity to prove the trig equation.

$$\sin\theta = \sin(\pi - \theta)$$

If we subtract $\sin(\pi - \theta)$ from both sides to rewrite the equation as

$$\sin\theta - \sin(\pi - \theta) = 0$$

then we can use the sum-to-product identity for the difference of sine functions,

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta + \alpha}{2}\right)\sin\left(\frac{\theta - \alpha}{2}\right)$$

with $\alpha = \pi - \theta$.

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left(\frac{\theta + (\pi - \theta)}{2}\right)\sin\left(\frac{\theta - (\pi - \theta)}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left(\frac{\theta + \pi - \theta}{2}\right)\sin\left(\frac{\theta - \pi + \theta}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{2\theta - \pi}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 2(0)\sin\left(\frac{2\theta - \pi}{2}\right)$$

$$\sin\theta - \sin(\pi - \theta) = 0$$

Let's do another example.

Example

Use a sum-to-product identity to find the value of $\cos(14\pi/3) + \cos(13\pi/3)$.

We'll use the sum-to-product identity for the sum of cosine functions, with $\theta = 14\pi/3$ and $\alpha = 13\pi/3$.

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{\frac{14\pi}{3}+\frac{13\pi}{3}}{2}\right)\cos\left(\frac{\frac{14\pi}{3}-\frac{13\pi}{3}}{2}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{\frac{27\pi}{3}}{2}\right)\cos\left(\frac{\frac{\pi}{3}}{2}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{27\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2\cos\left(\frac{9\pi}{2}\right)\cos\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 2(0)\cos\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{14\pi}{3}\right) + \cos\left(\frac{13\pi}{3}\right) = 0$$