Complex number equations

Let's talk about another application of powers of complex numbers. We can use what we've learned in this section to solve equations like $z^4 = 16$. In an equation like this one, z represents a complex number, which means we're looking for the complex numbers that would satisfy the equation. We'll do this using a system of equations.

First, we want to focus on the left side of $z^4 = 16$. Using De Moivre's theorem, we know we can rewrite z^4 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^4 = r^4 \left[\cos(4\theta) + i\sin(4\theta)\right]$$

Then looking just at the right side of $z^4 = 16$. We know we can rewrite 16 as the complex number $16 + 0i$ in rectangular form. If we find the modulus and angle of this complex number, we get

$$r = \sqrt{16^2 + 0^2}$$

$$r = \sqrt{16^2}$$

$$r = 16$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{16} = \arctan 0 = 0$$

Keep in mind though, that this arctan equation is true at an angle of 0, but it's also true at coterminal angles to 0, including 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 16\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 16\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 16\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^4 = 16$, we can start making substitutions.

$$z^4 = 16$$

$$r^4\left[\cos(4\theta) + i\sin(4\theta)\right] = 16$$

$$r^4\left[\cos(4\theta) + i\sin(4\theta)\right] = 16\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^4 = 16$$

$$4\theta = 360^\circ k$$

From these equations, we get

$r^4 = 16$, so $r = 2$

$4\theta = 360^\circ k$, so $\theta = 90^\circ k$

To $\theta = 90^\circ k$, if we plug in $k = 0, 1, 2, 3, \ldots$, we get

For $k = 0$, $\theta = 90^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 90^\circ(1) = 90^\circ$

For $k = 2$, $\theta = 90^\circ(2) = 180^\circ$

For $k = 3$, $\theta = 90^\circ(3) = 270^\circ$

...

We could keep going for $k = 4, 5, 6, 7, \ldots$, but $k = 4$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ$.

Plugging these four angles and $r = 2$ into the formula for polar form of a complex number, we'll get the solutions to $z^4 = 16$.

$$z_1 = 2\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 2\left[1 + i(0)\right] = 2$$

$$z_2 = 2\left[\cos(90^\circ) + i\sin(90^\circ)\right] = 2\left[0 + i(1)\right] = 2i$$

$$z_3 = 2\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 2\left[-1 + i(0)\right] = -2$$

$$z_4 = 2\left[\cos(270^\circ) + i\sin(270^\circ)\right] = 2\left[0 + i(-1)\right] = -2i$$

We can double-check that these are all roots of 16, by substituting these complex number solutions into $z^4 = 16$ for z.

For $z_1 = 2$, we get

$$2^4 = 16$$

$$16 = 16$$

For $z_2 = 2i$, we get

$$(2i)^4 = 16$$

$$16i^4 = 16$$

$$16i^2i^2 = 16$$

$$16(-1)(-1) = 16$$

$$16 = 16$$

For $z_3 = -2$, we get

$$(-2)^4 = 16$$

$$16 = 16$$

For $z_4 = -2i$, we get

$$(-2i)^4 = 16$$

$$16i^4 = 16$$

$$16i^2i^2 = 16$$

$$16(-1)(-1) = 16$$

$$16 = 16$$

And if we graph these complex numbers, we see the four solutions to $z^4 = 16$ in the complex plane.