Graph the polar curve, cardioid

In this lesson, you're going to see how to graph a polar curve that's shaped somewhat like the human heart - and for that reason, such a curve is called a cardioid. The polar equation of every cardioid that we'll deal with has one of the following forms where a is a positive number:

$$r = a(1 + \cos\theta)$$

$$r = a(1 - \cos\theta)$$

$$r = a(1 + \sin\theta)$$

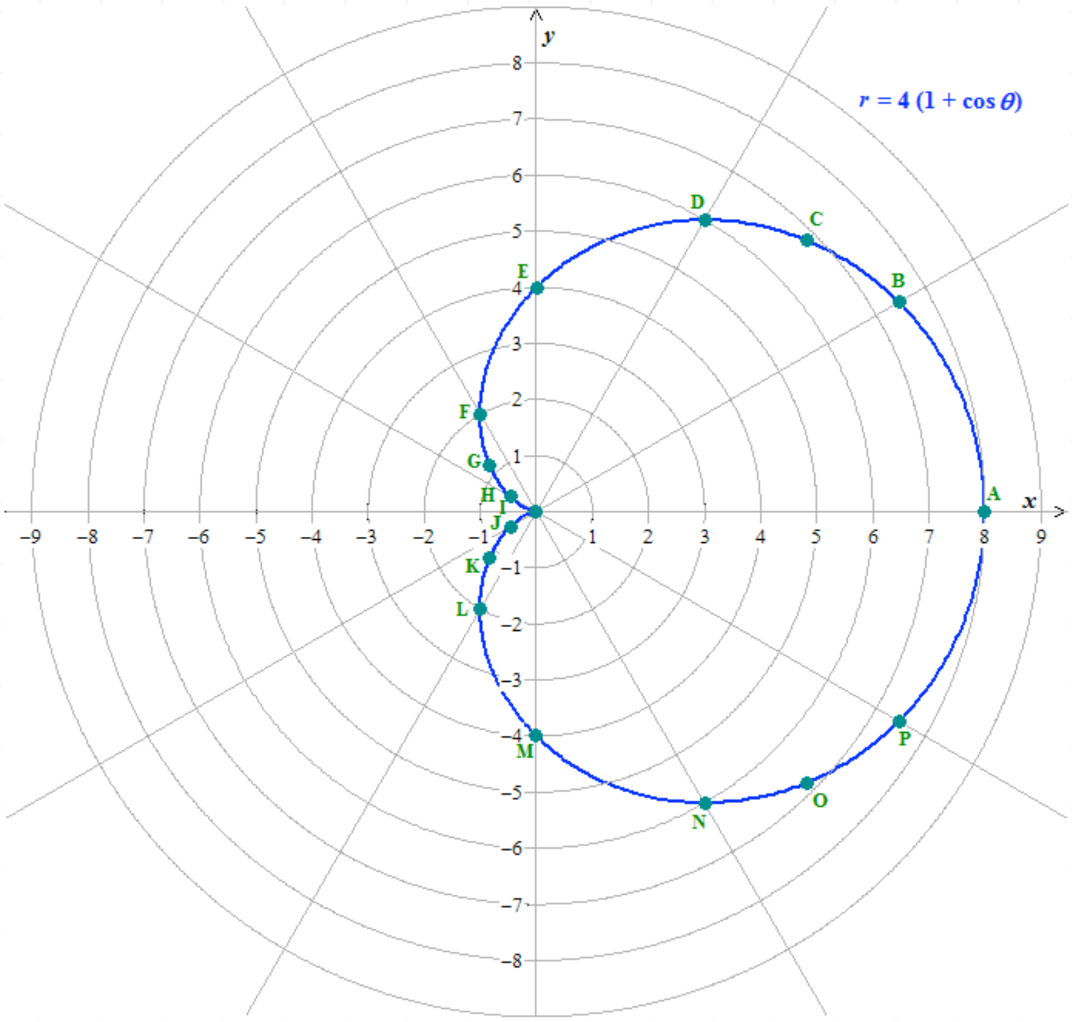
$$r = a(1 - \sin\theta)$$

Example

Graph the polar equation $r = 4(1 + \cos\theta)$.

In the following table, the values of $\cos\theta$ and $r = 4(1 + \cos\theta)$, and the resulting polar coordinates (r, θ), are shown for a number of angles θ in the interval $[0,2\pi)$.

Point	θ	$\cos\theta$	$r=4(1+\cos\theta)$	Polar coordinates (r,θ)
A	0	1	8	$(8,0)$
B	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$4+2\sqrt{3}$	$\left(4+2\sqrt{3},\frac{\pi}{6}\right)$
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$4+2\sqrt{2}$	$\left(4+2\sqrt{2},\frac{\pi}{4}\right)$
D	$\frac{\pi}{3}$	$\frac{1}{2}$	6	$\left(6,\frac{\pi}{3}\right)$
E	$\frac{\pi}{2}$	0	4	$\left(4,\frac{\pi}{2}\right)$
F	$\frac{2\pi}{3}$	$-\frac{1}{2}$	2	$\left(2,\frac{2\pi}{3}\right)$
G	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$4-2\sqrt{2}$	$\left(4-2\sqrt{2},\frac{3\pi}{4}\right)$
H	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$4-2\sqrt{3}$	$\left(4-2\sqrt{3},\frac{5\pi}{6}\right)$
I = pole	π	-1	0	$(0,\pi)$
J	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$4-2\sqrt{3}$	$\left(4-2\sqrt{3},\frac{7\pi}{6}\right)$
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$4-2\sqrt{2}$	$\left(4-2\sqrt{2},\frac{5\pi}{4}\right)$
L	$\frac{4\pi}{3}$	$-\frac{1}{2}$	2	$\left(2,\frac{4\pi}{3}\right)$
M	$\frac{3\pi}{2}$	0	4	$\left(4,\frac{3\pi}{2}\right)$
N	$\frac{5\pi}{3}$	$\frac{1}{2}$	6	$\left(6,\frac{5\pi}{3}\right)$
O	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$4+2\sqrt{2}$	$\left(4+2\sqrt{2},\frac{7\pi}{4}\right)$
P	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$4+2\sqrt{3}$	$\left(4+2\sqrt{3},\frac{11\pi}{6}\right)$



Notice that the value of $r = 4(1 + \cos\theta)$ is positive for every angle θ other than the pole. (At the pole, of course, $r = 0$.) This is a general feature of cardioids of the form $r = a(1 + \cos\theta)$. Here's why:

The value of $\cos\theta$ ranges from -1 to 1:

$$-1 \le \cos\theta \le 1$$

Adding 1 throughout, we obtain

$$0 \le 1 + \cos\theta \le 2$$

Since a is positive, multiplying through by a preserves the direction of the inequalities:

$$0 \le a(1 + \cos\theta) \le 2a$$

That is,

$$0 \le r \le 2a$$

For all cardioids of the form $r = a(1 + \cos\theta)$, the point furthest from the pole corresponds to $\theta = 0$, because the value of $1 + \cos\theta$ is greatest at $\theta = 0$, so that point is located $2a$ units to the right of the pole.

The same is true for cardioids of the form $r = a(1 - \cos\theta)$. Again,

$$-1 \le \cos\theta \le 1$$

In this case, we'll multiply through by -1, which does change the direction of the inequalities:

$$1 \ge -\cos\theta \ge -1$$

Adding 1 throughout:

$$2 \ge 1 - \cos\theta \ge 0$$

Multiplying throughout by a (which is positive, hence we preserve the direction of the inequalities we just obtained), we get

$$2a \ge a(1 - \cos\theta) \ge 0$$

Turning this around, we obtain

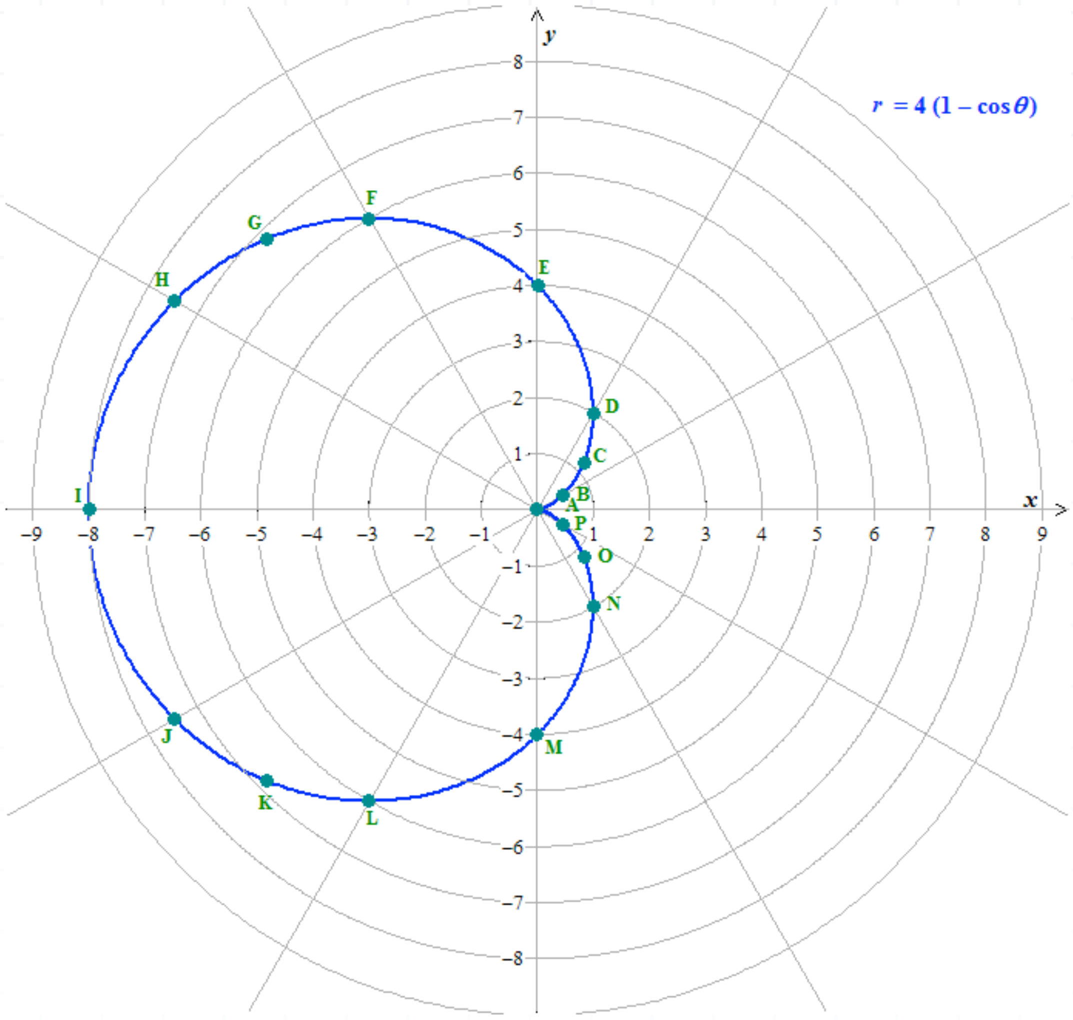
$$0 \le a(1 - \cos\theta) \le 2a$$

For cardioids of the form $r = a(1 - \cos\theta)$, the point furthest from the pole corresponds to $\theta = \pi$, because the value of $1 - \cos\theta$ is greatest at $\theta = \pi$, so that point is located $2a$ units to the left of the pole.

Example

Graph the cardioid $r=4(1-\cos\theta)$.

Point	θ	$\cos\theta$	$r=4(1-\cos\theta)$	Polar coordinates (r,θ)
A = pole	0	1	0	$(0,0)$
B	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$4-2\sqrt{3}$	$\left(4-2\sqrt{3},\frac{\pi}{6}\right)$
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$4-2\sqrt{2}$	$\left(4-2\sqrt{2},\frac{\pi}{4}\right)$
D	$\frac{\pi}{3}$	$\frac{1}{2}$	2	$\left(2,\frac{\pi}{3}\right)$
E	$\frac{\pi}{2}$	0	4	$\left(4,\frac{\pi}{2}\right)$
F	$\frac{2\pi}{3}$	$-\frac{1}{2}$	6	$\left(6,\frac{2\pi}{3}\right)$
G	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$4+2\sqrt{2}$	$\left(4+2\sqrt{2},\frac{3\pi}{4}\right)$
H	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$4+2\sqrt{3}$	$\left(4+2\sqrt{3},\frac{5\pi}{6}\right)$
I	π	-1	8	$(8,\pi)$
J	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$4+2\sqrt{3}$	$\left(4+2\sqrt{3},\frac{7\pi}{6}\right)$
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$4+2\sqrt{2}$	$\left(4+2\sqrt{2},\frac{5\pi}{4}\right)$
L	$\frac{4\pi}{3}$	$-\frac{1}{2}$	6	$\left(6,\frac{4\pi}{3}\right)$
M	$\frac{3\pi}{2}$	0	4	$\left(4,\frac{3\pi}{2}\right)$
N	$\frac{5\pi}{3}$	$\frac{1}{2}$	2	$\left(2,\frac{5\pi}{3}\right)$
O	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$4-2\sqrt{2}$	$\left(4-2\sqrt{2},\frac{7\pi}{4}\right)$
P	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$4-2\sqrt{3}$	$\left(4-2\sqrt{3},\frac{11\pi}{6}\right)$



Notice that as a curve, the cardioid $r = 4(1 - \cos\theta)$ is the reflection of the cardioid $r = 4(1 + \cos\theta)$ in the vertical axis. However, this isn't true pointwise (in regard to the specific angles θ and the labels of the points given in the two tables and graphs). For example, the point labeled A in the cardioid $r = 4(1 + \cos\theta)$ corresponds to $\theta = 0$, and the reflection of that point in the vertical axis is the point labeled I (not the point labeled A) in the cardioid $r = 4(1 - \cos\theta)$, which corresponds to $\theta = \pi$.

If a and b are different positive numbers, the only difference between the cardioid $r = a(1 + \cos\theta)$ and the cardioid $r = b(1 + \cos\theta)$ is in their “size.” You could think of the multiplicative constants a and b as different “magnifying factors.” The same is true of the difference between the cardioid $r = a(1 - \cos\theta)$ and the cardioid $r = b(1 - \cos\theta)$.

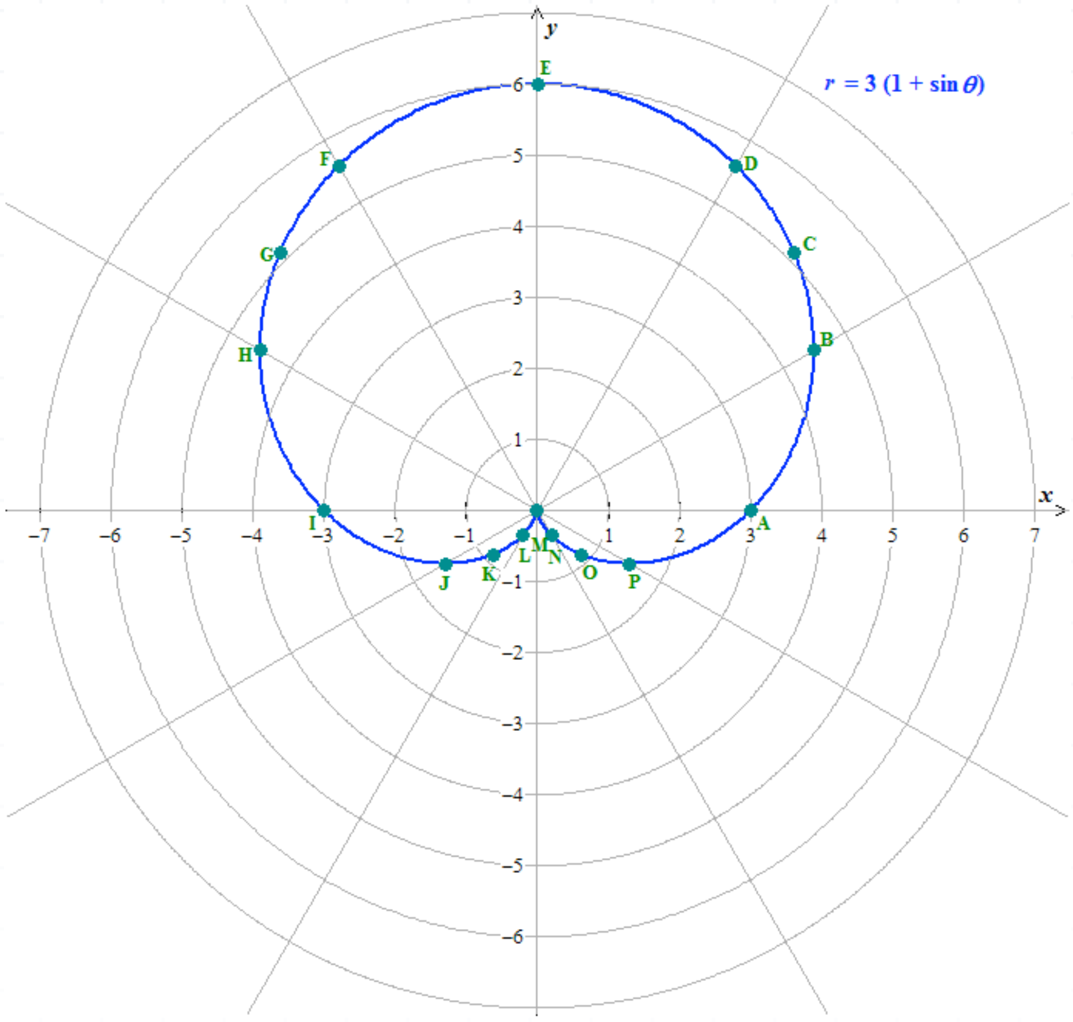
Notice also that cardioids of the form $r = a(1 + \cos\theta)$ or $r = a(1 - \cos\theta)$ are symmetric with respect to the horizontal axis (that is, the part of the cardioid that's above the horizontal axis is a reflection of the part that's below the horizontal axis).

As you're about to see now, cardioids of the form $r = a(1 + \sin\theta)$ or $r = a(1 - \sin\theta)$ have the same basic characteristics as cardioids of the form $r = a(1 + \cos\theta)$ or $r = a(1 - \sin\theta)$, except that the “sine cardioids” are symmetric with respect to the vertical axis instead of the horizontal axis.

Example

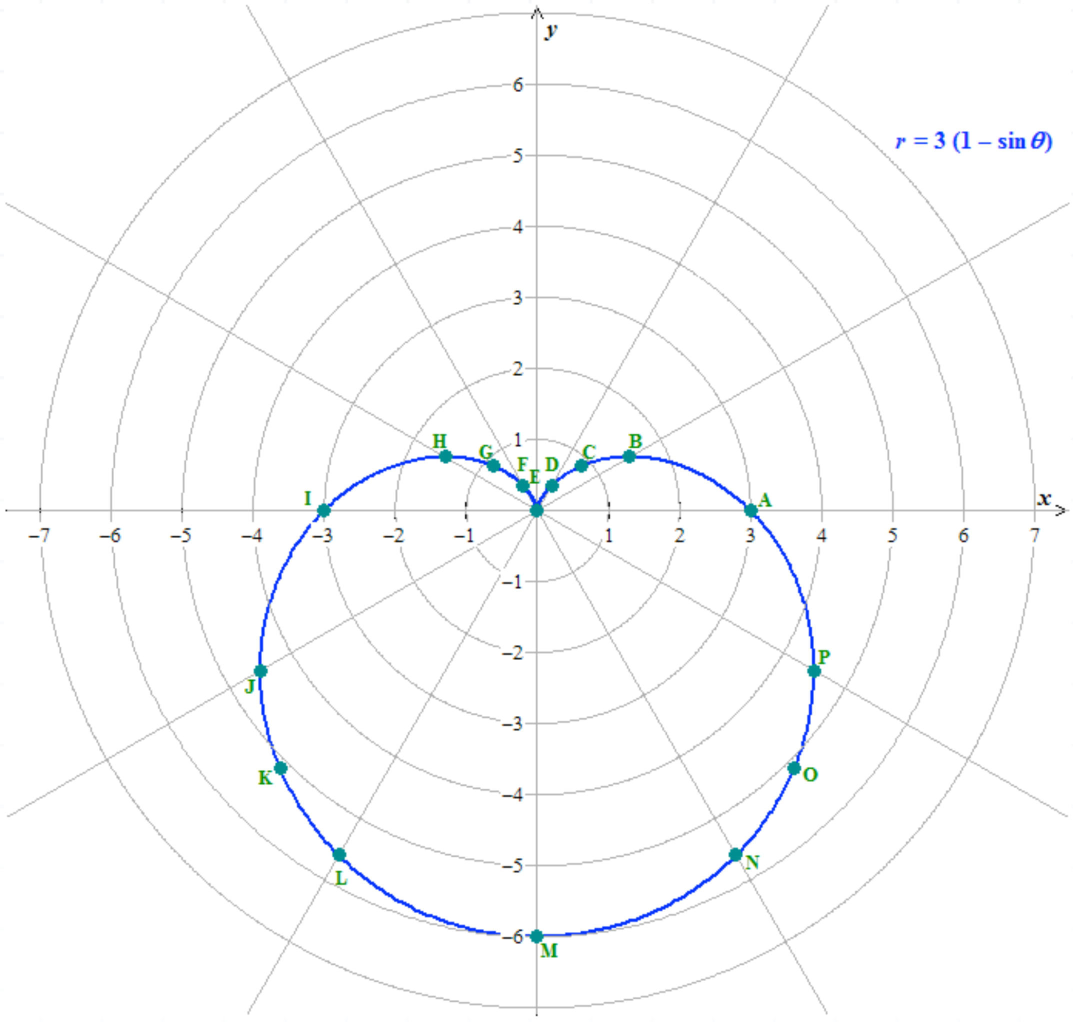
Graph the cardioids $r = 3(1 + \sin\theta)$ and $r = 3(1 - \sin\theta)$.

Point	θ	$\sin\theta$	$r=3(1+\sin\theta)$	Polar coordinates (r,θ)
A	0	0	3	$(3,0)$
B	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{9}{2}$	$\left(\frac{9}{2},\frac{\pi}{6}\right)$
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3+\frac{3\sqrt{2}}{2}$	$\left(3+\frac{3\sqrt{2}}{2},\frac{\pi}{4}\right)$
D	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$3+\frac{3\sqrt{3}}{2}$	$\left(3+\frac{3\sqrt{3}}{2},\frac{\pi}{3}\right)$
E	$\frac{\pi}{2}$	1	6	$\left(6,\frac{\pi}{2}\right)$
F	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$3+\frac{3\sqrt{3}}{2}$	$\left(3+\frac{3\sqrt{3}}{2},\frac{2\pi}{3}\right)$
G	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3+\frac{3\sqrt{2}}{2}$	$\left(3+\frac{3\sqrt{2}}{2},\frac{3\pi}{4}\right)$
H	$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{9}{2}$	$\left(\frac{9}{2},\frac{5\pi}{6}\right)$
I	π	0	3	$(3,\pi)$
J	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\frac{3}{2}$	$\left(\frac{3}{2},\frac{7\pi}{6}\right)$
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3-\frac{3\sqrt{2}}{2}$	$\left(3-\frac{3\sqrt{2}}{2},\frac{5\pi}{4}\right)$
L	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$3-\frac{3\sqrt{3}}{2}$	$\left(3-\frac{3\sqrt{3}}{2},\frac{4\pi}{3}\right)$
M = pole	$\frac{3\pi}{2}$	-1	0	$\left(0,\frac{3\pi}{2}\right)$
N	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$3-\frac{3\sqrt{3}}{2}$	$\left(3-\frac{3\sqrt{3}}{2},\frac{5\pi}{3}\right)$
O	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3-\frac{3\sqrt{2}}{2}$	$\left(3-\frac{3\sqrt{2}}{2},\frac{7\pi}{4}\right)$
P	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{3}{2}$	$\left(\frac{3}{2},\frac{11\pi}{6}\right)$



$r = 3(1 + \sin\theta)$
y
x
A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P

Point	θ	$\sin\theta$	$r=3(1-\sin\theta)$	Polar coordinates (r,θ)
A	0	0	3	$(3,0)$
B	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\left(\frac{3}{2},\frac{\pi}{6}\right)$
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3-\frac{3\sqrt{2}}{2}$	$\left(3-\frac{3\sqrt{2}}{2},\frac{\pi}{4}\right)$
D	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$3-\frac{3\sqrt{3}}{2}$	$\left(3-\frac{3\sqrt{3}}{2},\frac{\pi}{3}\right)$
E = pole	$\frac{\pi}{2}$	1	0	$\left(0,\frac{\pi}{2}\right)$
F	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$3-\frac{3\sqrt{3}}{2}$	$\left(3-\frac{3\sqrt{3}}{2},\frac{2\pi}{3}\right)$
G	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$3-\frac{3\sqrt{2}}{2}$	$\left(3-\frac{3\sqrt{2}}{2},\frac{3\pi}{4}\right)$
H	$\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	$\left(\frac{3}{2},\frac{5\pi}{6}\right)$
I	π	0	3	$(3,\pi)$
J	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\frac{9}{2}$	$\left(\frac{9}{2},\frac{7\pi}{6}\right)$
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3+\frac{3\sqrt{2}}{2}$	$\left(3+\frac{3\sqrt{2}}{2},\frac{5\pi}{4}\right)$
L	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$3+\frac{3\sqrt{3}}{2}$	$\left(3+\frac{3\sqrt{3}}{2},\frac{4\pi}{3}\right)$
M	$\frac{3\pi}{2}$	-1	6	$\left(6,\frac{3\pi}{2}\right)$
N	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$3+\frac{3\sqrt{3}}{2}$	$\left(3+\frac{3\sqrt{3}}{2},\frac{5\pi}{3}\right)$
O	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3+\frac{3\sqrt{2}}{2}$	$\left(3+\frac{3\sqrt{2}}{2},\frac{7\pi}{4}\right)$
P	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{9}{2}$	$\left(\frac{9}{2},\frac{11\pi}{6}\right)$



Notice that as a curve, the cardioid $r = 3(1 - \sin\theta)$ is a reflection of the cardioid $r = 3(1 + \sin\theta)$ in the horizontal axis, and that both of these cardioids are symmetric with respect to the vertical axis (that is, the part of the cardioid that's to the left of the vertical axis is a reflection of the part that's to the right of the vertical axis).

For cardioids of the form $r = a(1 + \sin\theta)$, the point furthest from the pole corresponds to $\theta = \pi/2$, because the value of $1 + \sin\theta$ is greatest at $\theta = \pi/2$, so that point is located $2a$ units above the pole. For cardioids of the form $r = a(1 - \sin\theta)$, the point furthest from the pole corresponds to $\theta = 3\pi/2$, because the value of $1 - \sin\theta$ is greatest at $\theta = 3\pi/2$, so that point is located $2a$ units below the pole.

You may be wondering why we stated that the number a in the polar equation of the cardioids that we're considering in this lesson is positive.

The short answer is that we get no new curves if a is negative. In particular, the following are true:

- If a is negative, the curve which is the graph of the cardioid $a(1+\cos\theta)$ is identical to the curve which is the graph of the cardioid $-a(1-\cos\theta)$, and the curve which is the graph of the cardioid $a(1-\cos\theta)$ is identical to the curve which is the graph of the cardioid $-a(1+\cos\theta)$.

- If a is negative, the curve which is the graph of the cardioid $a(1+\sin\theta)$ is identical to the curve which is the graph of the cardioid $-a(1-\sin\theta)$, and the curve which is the graph of the cardioid $a(1-\sin\theta)$ is identical to the curve which is the graph of the cardioid $r=-a(1+\sin\theta)$.

Example

Graph the cardioid $r=-4(1+\cos\theta)$.

We'll tabulate the values of $\cos\theta$ and $r=-4(1+\cos\theta)$, as well as the polar coordinates

$$(r,\theta)=(-a(1+\cos\theta),\theta)$$

for a number of angles θ in the interval $[0,2\pi)$.

Now for every angle $\theta\in[0,2\pi)$, let's define an angle α as follows:

$$\alpha=\begin{cases}\theta+\pi, & \theta\in[0,\pi)\\ \theta-\pi, & \theta\in[\pi,2\pi)\end{cases}$$

This definition guarantees that for every angle θ the angle α is in the interval $[0,2\pi)$, and that every angle $\alpha \in [0,2\pi)$ corresponds to exactly one angle $\theta \in [0,2\pi)$.

Now note that since one pair of polar coordinates of a point of the cardioid $r = -4(1+\cos\theta)$ is

$$(r,\theta) = (-4(1+\cos\theta),\theta)$$

and since $-4(1+\cos\theta)$ is nonpositive (i.e., either negative or 0) for every θ in the interval $[0,2\pi)$, another pair of polar coordinates of that point is

$$(-r,\theta+\pi) = (4(1+\cos\theta),\theta+\pi)$$

Notice, in addition, that the angles $\theta+\pi$ and $\theta-\pi$ differ by 2π, so

$$(-r,\theta-\pi) = (4(1+\cos\theta),\theta-\pi)$$

is still another pair of polar coordinates of that point.

Therefore, for every angle θ in the interval $[0,2\pi)$, the polar coordinates of a point of the cardioid $r = -4(1+\cos\theta)$ can be expressed as

$$(r,\theta) = (-4(1+\cos\theta),\theta)$$

and as

$$(-r,\alpha) = \begin{cases} (4(1+\cos\theta),\theta+\pi), & \theta \in [0,\pi) \\ (4(1+\cos\theta),\theta-\pi), & \theta \in [\pi,2\pi) \end{cases}$$

As stated earlier, the curve which is the graph of the cardioid $r = -4(1+\cos\theta)$ is identical to the curve which is the graph of the cardioid $r = 4(1-\cos\theta)$. In fact, for every $\theta \in [0,2\pi)$ there is some point in the table that we presented (earlier in this lesson) for the cardioid $r = 4(1 - \cos\theta)$ which has polar coordinates $(-r, \alpha)$. Thus in our table for the cardioid $r = -4(1 + \cos\theta)$, we're including not only the polar coordinates (r, θ) of every point but also the polar coordinates $(-r, \alpha)$ for it and the label of that point from our table for the cardioid $r = 4(1 - \cos\theta)$.

Point	θ	$\cos\theta$	$r = -4(1+\cos\theta)$	Polar coordinates (r, θ)	Polar coordinates $(-r, \alpha)^*$	Label from table for $r = 4(1-\cos\theta)$
A	0	1	-8	$(-8, 0)$	$(8, \pi)$	I
B	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-4-2\sqrt{3}$	$\left(-4-2\sqrt{3}, \frac{\pi}{6}\right)$	$\left(4+2\sqrt{3}, \frac{7\pi}{6}\right)$	J
C	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-4-2\sqrt{2}$	$\left(-4-2\sqrt{2}, \frac{\pi}{4}\right)$	$\left(4+2\sqrt{2}, \frac{5\pi}{4}\right)$	K
D	$\frac{\pi}{3}$	$\frac{1}{2}$	-6	$\left(-6, \frac{\pi}{3}\right)$	$\left(6, \frac{4\pi}{3}\right)$	L
E	$\frac{\pi}{2}$	0	-4	$\left(-4, \frac{\pi}{2}\right)$	$\left(4, \frac{3\pi}{2}\right)$	M
F	$\frac{2\pi}{3}$	$-\frac{1}{2}$	-2	$\left(-2, \frac{2\pi}{3}\right)$	$\left(2, \frac{5\pi}{3}\right)$	N
G	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-4+2\sqrt{2}$	$\left(-4+2\sqrt{2}, \frac{3\pi}{4}\right)$	$\left(4-2\sqrt{2}, \frac{7\pi}{4}\right)$	O
H	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-4+2\sqrt{3}$	$\left(-4+2\sqrt{3}, \frac{5\pi}{6}\right)$	$\left(4-2\sqrt{3}, \frac{11\pi}{6}\right)$	P
I = pole	π	-1	0	$(0, \pi)$	$(0, 0)$	A = pole
J	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-4+2\sqrt{3}$	$\left(-4+2\sqrt{3}, \frac{7\pi}{6}\right)$	$\left(4-2\sqrt{3}, \frac{\pi}{6}\right)$	B
K	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-4+2\sqrt{2}$	$\left(-4+2\sqrt{2}, \frac{5\pi}{4}\right)$	$\left(4-2\sqrt{2}, \frac{\pi}{4}\right)$	C
L	$\frac{4\pi}{3}$	$-\frac{1}{2}$	-2	$\left(-2, \frac{4\pi}{3}\right)$	$\left(2, \frac{\pi}{3}\right)$	D
M	$\frac{3\pi}{2}$	0	-4	$\left(-4, \frac{3\pi}{2}\right)$	$\left(4, \frac{\pi}{2}\right)$	E
N	$\frac{5\pi}{3}$	$\frac{1}{2}$	-6	$\left(-6, \frac{5\pi}{3}\right)$	$\left(6, \frac{2\pi}{3}\right)$	F
O	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-4-2\sqrt{2}$	$\left(-4-2\sqrt{2}, \frac{7\pi}{4}\right)$	$\left(4+2\sqrt{2}, \frac{3\pi}{4}\right)$	G
P	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-4-2\sqrt{3}$	$\left(-4-2\sqrt{3}, \frac{11\pi}{6}\right)$	$\left(4+2\sqrt{3}, \frac{5\pi}{6}\right)$	H

*For θ in the interval $[0, \pi)$, $\alpha = \theta + \pi$. For θ in the interval $[\pi, 2\pi)$, $\alpha = \theta - \pi$.

Now take a look at the accompanying graph of the cardioid $r = -4(1 + \cos\theta)$, and you'll see that it's identical (as a curve) to the graph of the cardioid $r = 4(1 - \cos\theta)$ that we presented earlier in this lesson, but that the labels of the individual points plotted in the two graphs are different.