Gauss-Jordan elimination and reduced row-echelon form

We know from algebra that solving a system of linear equations is about finding the value of each variable.

$$-x - 5y + z = 17$$

$$-5x - 5y + 5z = 5$$

$$2x + 5y - 3z = -10$$

Whether by substitution, elimination, or graphing, which are the three methods we've used in the past to solve systems, we need to figure out that the values of x, y, and z that make the above system true are $x = -1$, $y = -4$ and $z = -4$, or $(-1, -4, -4)$.

Row-echelon and reduced row-echelon forms

But now that we know how to use row operations to manipulate matrices, we have a new tool for solving systems of linear equations. In this lesson we want to look at how putting an augmented matrix into row-echelon form or reduced row-echelon form is another way to solve a system.

A matrix is in **row-echelon form** if

1. All rows consisting of only 0s are at the bottom of the matrix.

2. The first non-zero entry in each row sits in a column to the right of the first non-zero entries in all the rows above it. In other words, the non-zero entries sit in a staircase pattern.

For a 3×3 augmented matrix, row-echelon form might look like this:

$$\begin{bmatrix} \mathbf{4} & 1 & 0 & = & 17 \\ 0 & \mathbf{2} & 5 & = & 10 \\ 0 & 0 & -\mathbf{3} & = & 2 \end{bmatrix}$$

The bolded first non-zero entry in each row is called a **pivot**, and each pivot is in a column to the right of the column that houses the first non-zero entry from each row above it. The columns that house the pivots are called **pivot columns**.

If a matrix is in row-echelon form, and if all the pivot entries are equal to 1, and if all of the non-pivot entries in the matrix are equal to 0 (other than the constants in the far-right column), then the matrix is specifically in **reduced row-echelon form**. Reduced row-echelon form for a 3×3 will look like this:

$$\begin{bmatrix} 1 & 0 & 0 & = & C_1 \\ 0 & 1 & 0 & = & C_2 \\ 0 & 0 & 1 & = & C_3 \end{bmatrix}$$

This is what a reduced row-echelon matrix looks like for 2×2, 3×3, and 4×4 matrices:

For 2×2: For 3×3: For 4×4:

$$\begin{bmatrix} 1 & 0 & = & C_1 \\ 0 & 1 & = & C_2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & = & C_1 \\ 0 & 1 & 0 & = & C_2 \\ 0 & 0 & 1 & = & C_3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & = & C_1 \\ 0 & 1 & 0 & 0 & = & C_2 \\ 0 & 0 & 1 & 0 & = & C_3 \\ 0 & 0 & 0 & 1 & = & C_4 \end{bmatrix}$$

Solving systems

If we try to solve the system

$$-x - 5y + z = 17$$

$$-5x - 5y + 5z = 5$$

$$2x + 5y - 3z = -10$$

using the substitution and elimination methods that we learned in algebra, it'll be really tedious. But if we can put this system into an augmented matrix, then we can use all of the row operations that we learned in the previous section in order to transform the original matrix into row-echelon form, or better yet, reduced row-echelon form. Reduced row-echelon form will transform the system into

$$\begin{bmatrix} -1 & -5 & 1 & = & 17 \\ -5 & -5 & 5 & = & 5 \\ 2 & 5 & -3 & = & -10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & = & C_1 \\ 0 & 1 & 0 & = & C_2 \\ 0 & 0 & 1 & = & C_3 \end{bmatrix}$$

Let's talk for a second about why we would want to put the system of linear equations into an augmented matrix and then put the matrix into reduced row-echelon form.

Remember that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & = & C_1 \\ 0 & 1 & 0 & = & C_2 \\ 0 & 0 & 1 & = & C_3 \end{bmatrix}$$

is still representing a system of linear equations. So if we've put the matrix into reduced row-echelon form and then we pull back out the linear equations represented by the reduced row-echelon matrix, we get

$$1x + 0y + 0z = C_1$$

$$0x + 1y + 0z = C_2$$

$$0x + 0y + 1z = C_3$$

or just

$$x = C_1$$

$$y = C_2$$

$$z = C_3$$

In other words, from reduced row-echelon form, we automatically have the values of each variable, and we've solved the system. So what we're saying is that, if we put the matrix into its reduced row-echelon form, then we can pull out the value of each variable directly from the matrix.

Gaussian elimination

So we know that it’s helpful to put a matrix into reduced row-echelon form, and we’ve said that we can use matrix row operations to do this, but is there any systematic, orderly way that we go about these row operations?

Yes! **Gauss-Jordan elimination** is an algorithm (a specific set of steps that can be repeated over and over again) to get the matrix all the way down to reduced row-echelon form. These are the steps:

1. Optional: Pull out any scalars from each row in the matrix.

2. If the first entry in the first row is 0, swap it with another row that has a non-zero entry in its first column. Otherwise, move to step 3.

3. Multiply through the first row by a scalar to make the leading entry equal to 1.

4. Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.

5. Go back step 2 and repeat the process until the matrix is in reduced row-echelon form.

Let’s walk through an example of how to use Gauss-Jordan elimination to change an augmented matrix into reduced row-echelon form and then pull out the values of each variable.

Example

Use Gauss-Jordan elimination to solve for the value of each variable.

$$\begin{bmatrix} -1 & -5 & 1 & = & 17 \\ -5 & -5 & 5 & = & 5 \\ 2 & 5 & -3 & = & -10 \end{bmatrix}$$

Remember first that this augmented matrix represents the linear system

$$-x - 5y + z = 17$$

$$-5x - 5y + 5z = 5$$

$$2x + 5y - 3z = -10$$

where the entries in the first column are the coefficients of x, the entries in the second column are the coefficients of y, and the entries in the third column are the coefficients of z. The entries in the fourth column are the constants.

Step 1:

Starting with the optional first step from Gauss-Jordan elimination, we could divide through the second row by 5, and that would reduce those values. After $(1/5)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} -1 & -5 & 1 & = & 17 \\ -1 & -1 & 1 & = & 1 \\ 2 & 5 & -3 & = & -10 \end{bmatrix}$$

Step 2 (with the first row):

The first entry in the first row is non-zero, so there's no need to swap it with another row.

Step 3 (with the first row):

Multiply row 1 by -1 to get a leading 1 in the first row. After $-R_1 \to R_1$, the matrix is

$$\begin{bmatrix} -(-1) & -(-5) & -(1) & = & -(17) \\ -1 & -1 & 1 & = & 1 \\ 2 & 5 & -3 & = & -10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & -1 & = & -17 \\ -1 & -1 & 1 & = & 1 \\ 2 & 5 & -3 & = & -10 \end{bmatrix}$$

Step 4 (with the first row):

Replace row 2 with the sum of rows 1 and 2. After $R_1 + R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & = & -17 \\ 1-1 & 5-1 & -1+1 & = & -17+1 \\ 2 & 5 & -3 & = & -10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & -1 & = & -17 \\ 0 & 4 & 0 & = & -16 \\ 2 & 5 & -3 & = & -10 \end{bmatrix}$$

Replace row 3 with row 3 minus (2 times row 1). After $R_3 - 2R_1 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & = & -17 \\ 0 & 4 & 0 & = & -16 \\ 2-2(1) & 5-2(5) & -3-2(-1) & = & -10-2(-17) \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & -1 & = & -17 \\ 0 & 4 & 0 & = & -16 \\ 0 & -5 & -1 & = & 24 \end{bmatrix}$$

We now have 1, 0, 0 in the first column, which is exactly what we want. It's time to go back to step 2, but this time with the second row.

Step 2 (with the second row):

The second entry in the second row is non-zero, so there's no need to swap it with another row.

Step 3 (with the second row):

Multiply row 2 by $1/4$ to get a leading 1 in the second row. After $(1/4)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & = & -17 \\ \frac{1}{4}(0) & \frac{1}{4}(4) & \frac{1}{4}(0) & = & \frac{1}{4}(-16) \\ 0 & -5 & -1 & = & 24 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & -1 & = & -17 \\ 0 & 1 & 0 & = & -4 \\ 0 & -5 & -1 & = & 24 \end{bmatrix}$$

Step 4 (with the second row):

Replace row 1 with the sum of (-5 times row 2) and row 1. After $-5R_2 + R_1 \to R_1$, the matrix is

$$\begin{bmatrix} -5(0)+1 & -5(1)+5 & -5(0)-1 & = & -5(-4)-17 \\ 0 & 1 & 0 & = & -4 \\ 0 & -5 & -1 & = & 24 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & = & 3 \\ 0 & 1 & 0 & = & -4 \\ 0 & -5 & -1 & = & 24 \end{bmatrix}$$

Replace row 3 with the sum of (5 times row 2) and row 3. After $5R_2 + R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 5 & -1 & = & -17 \\ 0 & 1 & 0 & = & -4 \\ 5(0)+0 & 5(1)-5 & 5(0)-1 & = & 5(-4)+24 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & = & 3 \\ 0 & 1 & 0 & = & -4 \\ 0 & 0 & -1 & = & 4 \end{bmatrix}$$

We now have 0, 1, 0 in the second column, which is exactly what we want. It's time to go back to step 2, but this time with the third row.

Step 2 (with the third row):

The third entry in the third row is non-zero, so there's no need to swap it with another row.

Step 3 (with the third row):

Multiply row 3 by -1 to get a leading 1 in the third row. After $-R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -1 & = & 3 \\ 0 & 1 & 0 & = & -4 \\ -(0) & -(0) & -(-1) & = & -(4) \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & = & 3 \\ 0 & 1 & 0 & = & -4 \\ 0 & 0 & 1 & = & -4 \end{bmatrix}$$

Step 4 (with the third row):

Replace row 1 with the sums of rows 1 and 3. After $R_1 + R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1+0 & 0+0 & -1+1 & = & 3-4 \\ 0 & 1 & 0 & = & -4 \\ 0 & 0 & 1 & = & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & = & -1 \\ 0 & 1 & 0 & = & -4 \\ 0 & 0 & 1 & = & -4 \end{bmatrix}$$

We now have 0, 0, 1 in the third column, which is exactly what we want. The matrix is now in reduced row-echelon form since the leading 1 is the first non-zero value in each row, and the leading 1 in each row is to the right of the leading 1 from all the rows above it, and all other values are 0.

From this resulting matrix, we get the solution set

$$1x + 0y + 0z = -1$$

$$0x + 1y + 0z = -4$$

$$0x + 0y + 1z = -4$$

or simplified, we get

$$x = -1$$

$$y = -4$$

$$z = -4$$

That's all it took to find that the solution to the system is $x = -1$, $y = -4$, and $z = -4$.