Transformations

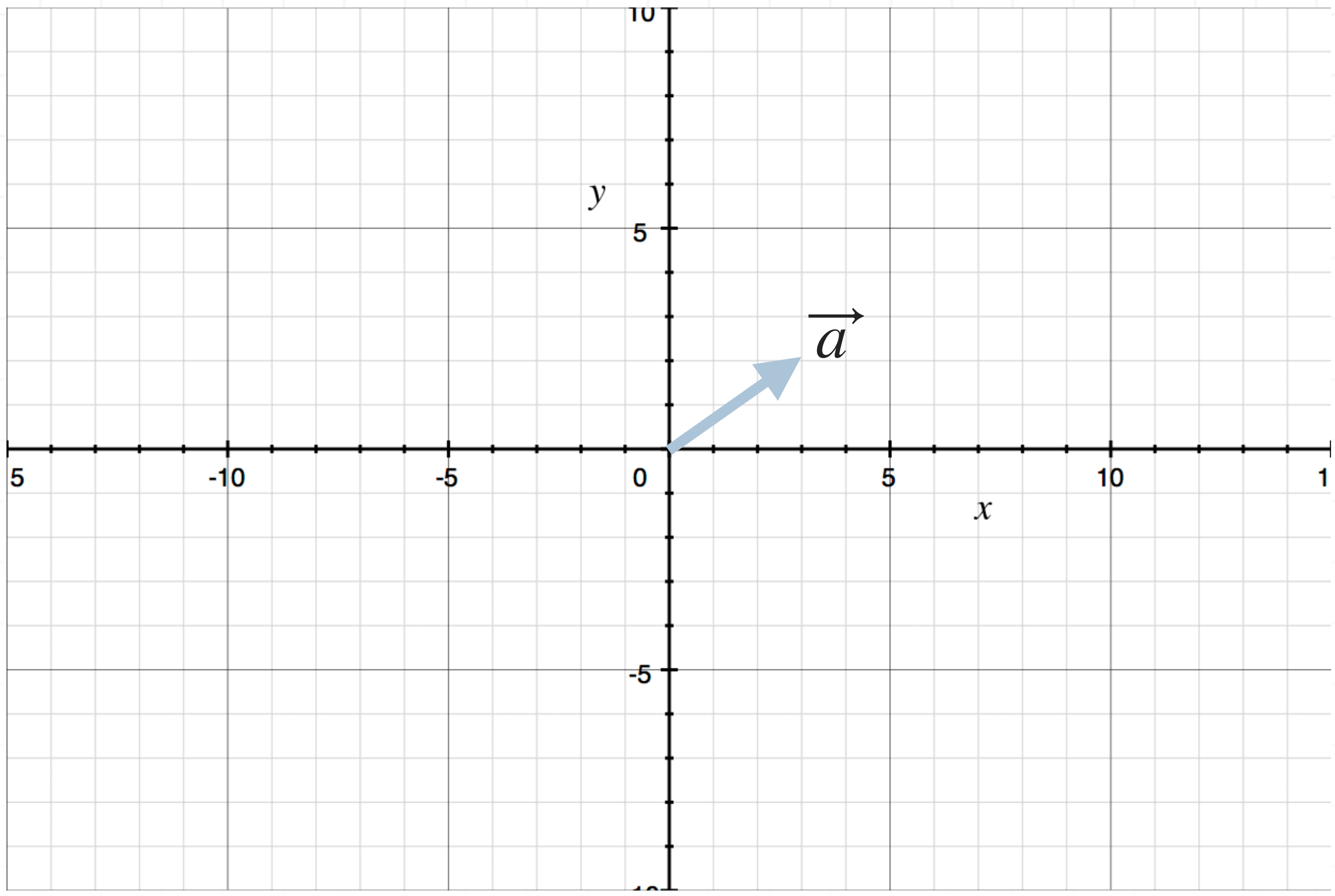
Matrices can be extremely useful when it comes to describing what would otherwise be complex changes to two-dimensional space.

For instance, let's say you want to know what happens if you take every point in the coordinate plane and shift it up vertical by 2 units and stretch it out horizontally by 5 units.

Up to now, we don't have a simple way to describe this change mathematically. But that's where matrices come in. They allow us to organize the transformation information into a transformation matrix, which will tell us exactly how every point in the plane should move.

Transformation of a single point

Let's say we have the point in space $(3,2)$. Keep in mind that, for the purpose of this lesson, it's also not very different to think about that point as a vector that starts at the origin and extends out to the point $(3,2)$.



Let's call the vector $\vec{a}$:

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

We can apply a transformation matrix to the vector (or coordinate point), and change it into another vector (or coordinate point). Let's say we use the transformation matrix

$$M = \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix}$$

and apply it to the vector $\vec{a}$. Then the transformation of $\vec{a}$ by M will be the multiplication of $\vec{a}$ by M.

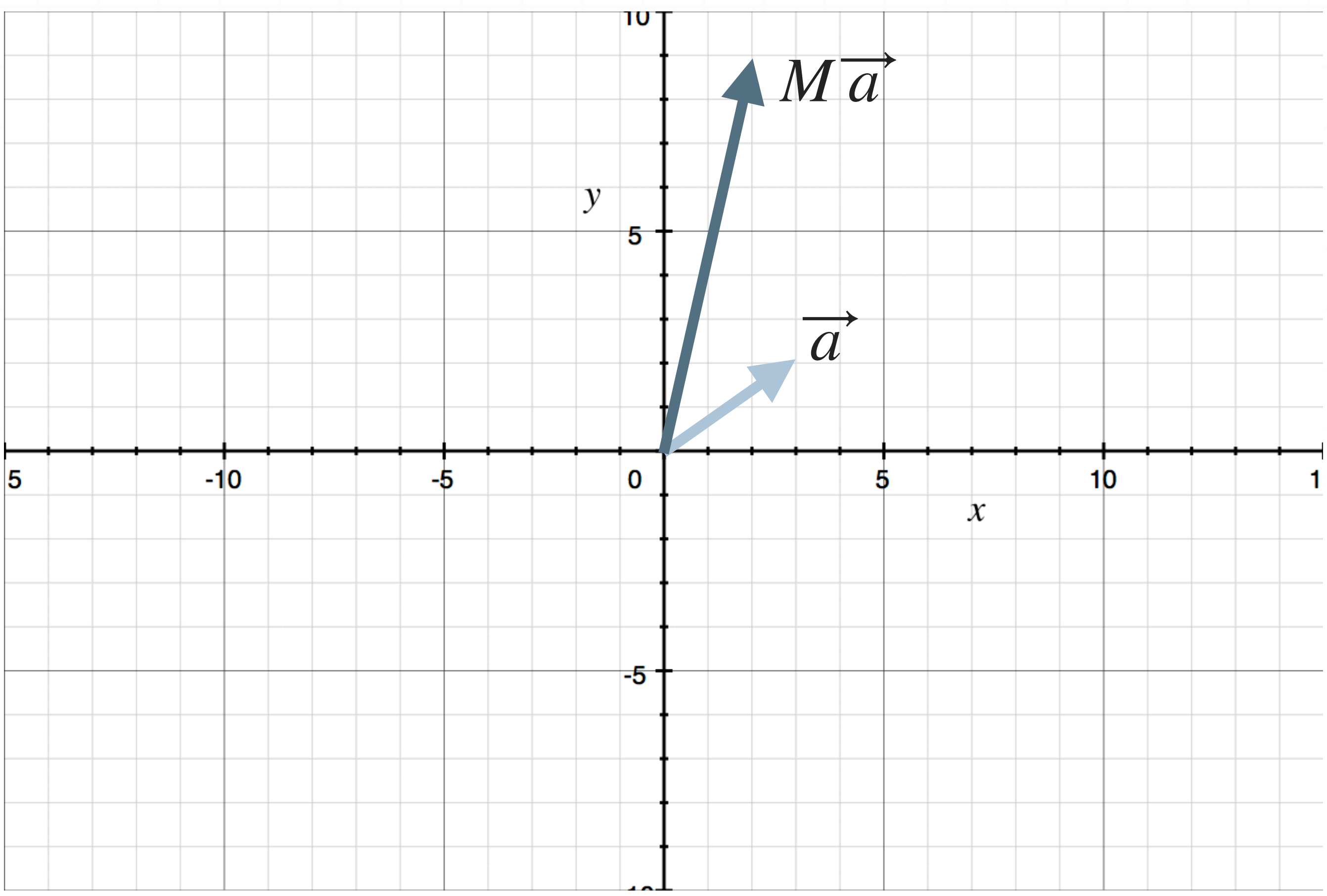
$$M\vec{a} = \begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$M\vec{a} = \begin{bmatrix} -2(3) + 4(2) \\ 3(3) + 0(2) \end{bmatrix}$$

$$M\vec{a} = \begin{bmatrix} -6 + 8 \\ 9 + 0 \end{bmatrix}$$

$$M\vec{a} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

What this means is that the values in matrix M cause the vector $\vec{a} = (3,2)$ to transform into the vector $M\vec{a} = (2,9)$.



Transforming a figure

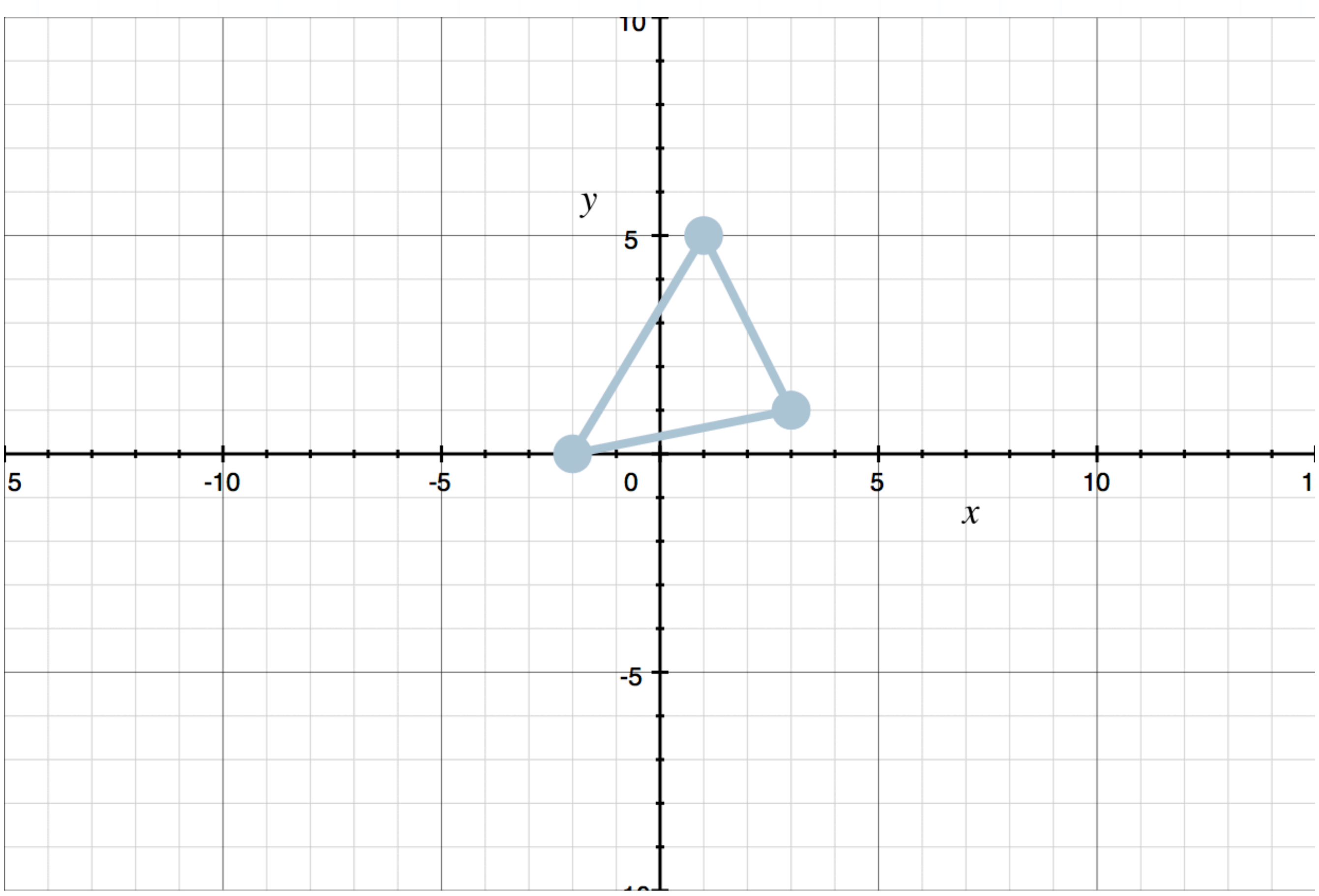
If instead of being given the single point or vector $\vec{a}$, we'd been given a set of points that represent the vertices of a figure, whether that figure is a triangle, quadrilateral, pentagon, etc. (any polygon), we can apply a transformation matrix to the set of points, and transform the figure into a different figure.

Example

A triangle is defined by the set of points $(3,1)$, $(1,5)$, and $(-2,0)$. Apply the transformation matrix M and give the vertices of the triangle after the transformation.

$$M = \begin{bmatrix} 1 & -3 \\ 4 & 0 \end{bmatrix}$$

The graph of the triangle with the given vertices is



First, pull the vertices of the triangle into a matrix. Put the x-values from the coordinate points into the first row, then put corresponding y-values into the second row, such that each column of the matrix represents the coordinate point of one vertex of the figure.

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 5 & 0 \end{bmatrix}$$

Then multiply the transformation matrix by the point-set matrix.

$$M \cdot \begin{bmatrix} 3 & 1 & -2 \\ 1 & 5 & 0 \end{bmatrix}$$

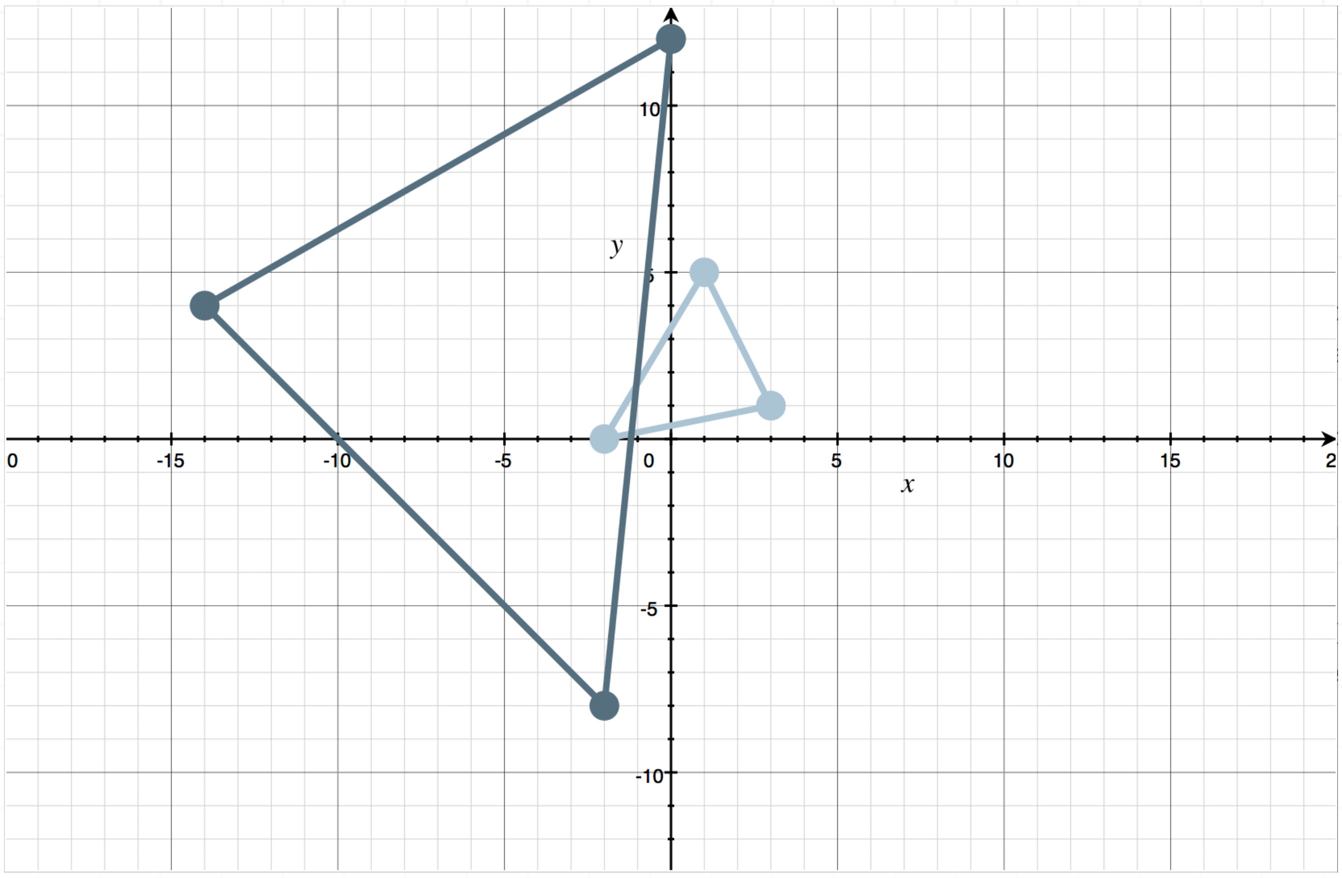
$$\begin{bmatrix} 1 & -3 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & -2 \\ 1 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1(3)-3(1) & 1(1)-3(5) & 1(-2)-3(0) \\ 4(3)+0(1) & 4(1)+0(5) & 4(-2)+0(0) \end{bmatrix}$$

$$\begin{bmatrix} 3-3 & 1-15 & -2-0 \\ 12+0 & 4+0 & -8+0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -14 & -2 \\ 12 & 4 & -8 \end{bmatrix}$$

This transformed matrix gives us the vertices of the transformed triangle, which are $(0,12)$, $(-14,4)$, and $(-2,-8)$, and the graphs of the original and transformed triangles together are



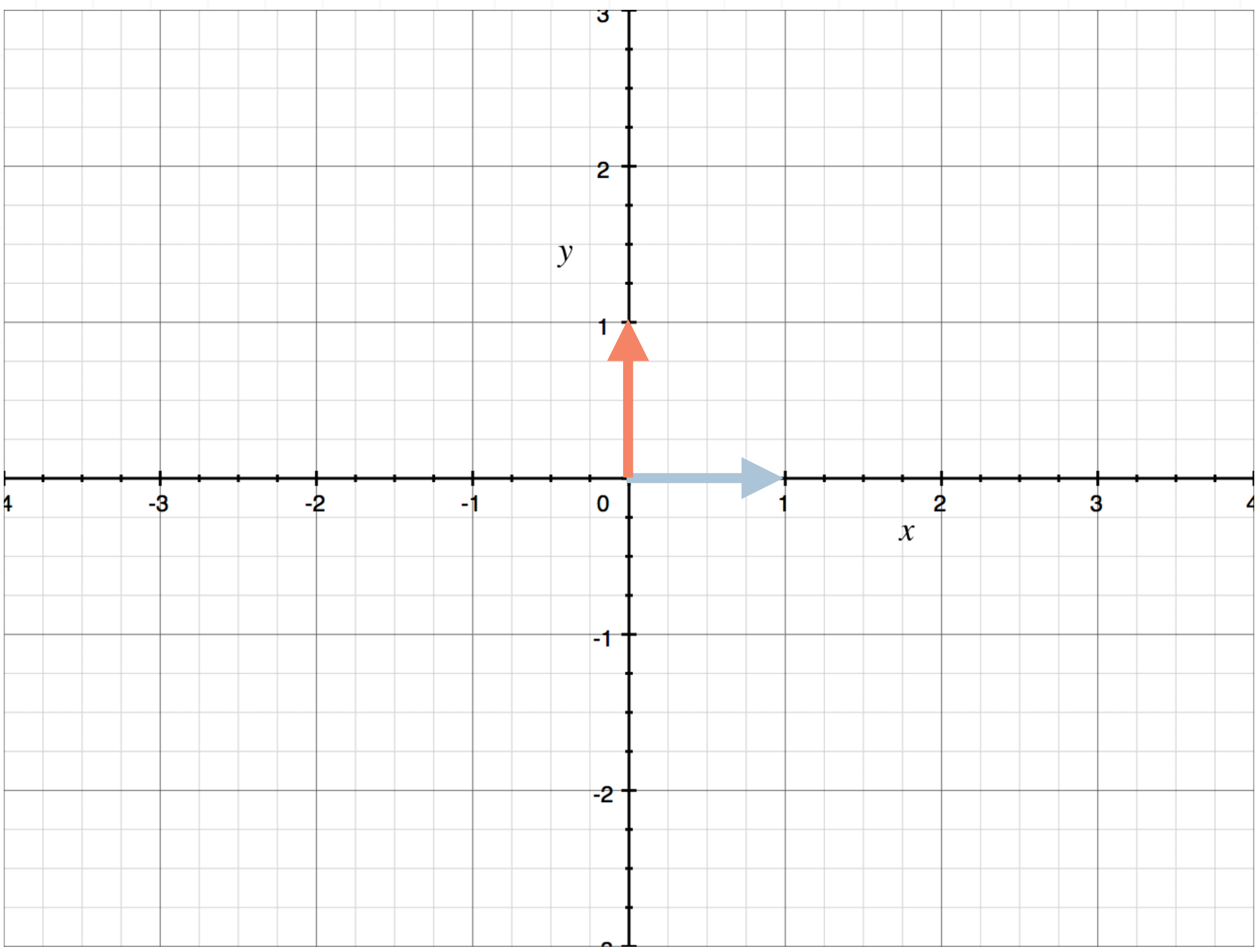
Understanding the transformation matrix

Think about transformations as a series of shifts, stretches, compressions, rotations, etc., that move a point from one spot to another. For instance, we know from the last example that the transformation matrix

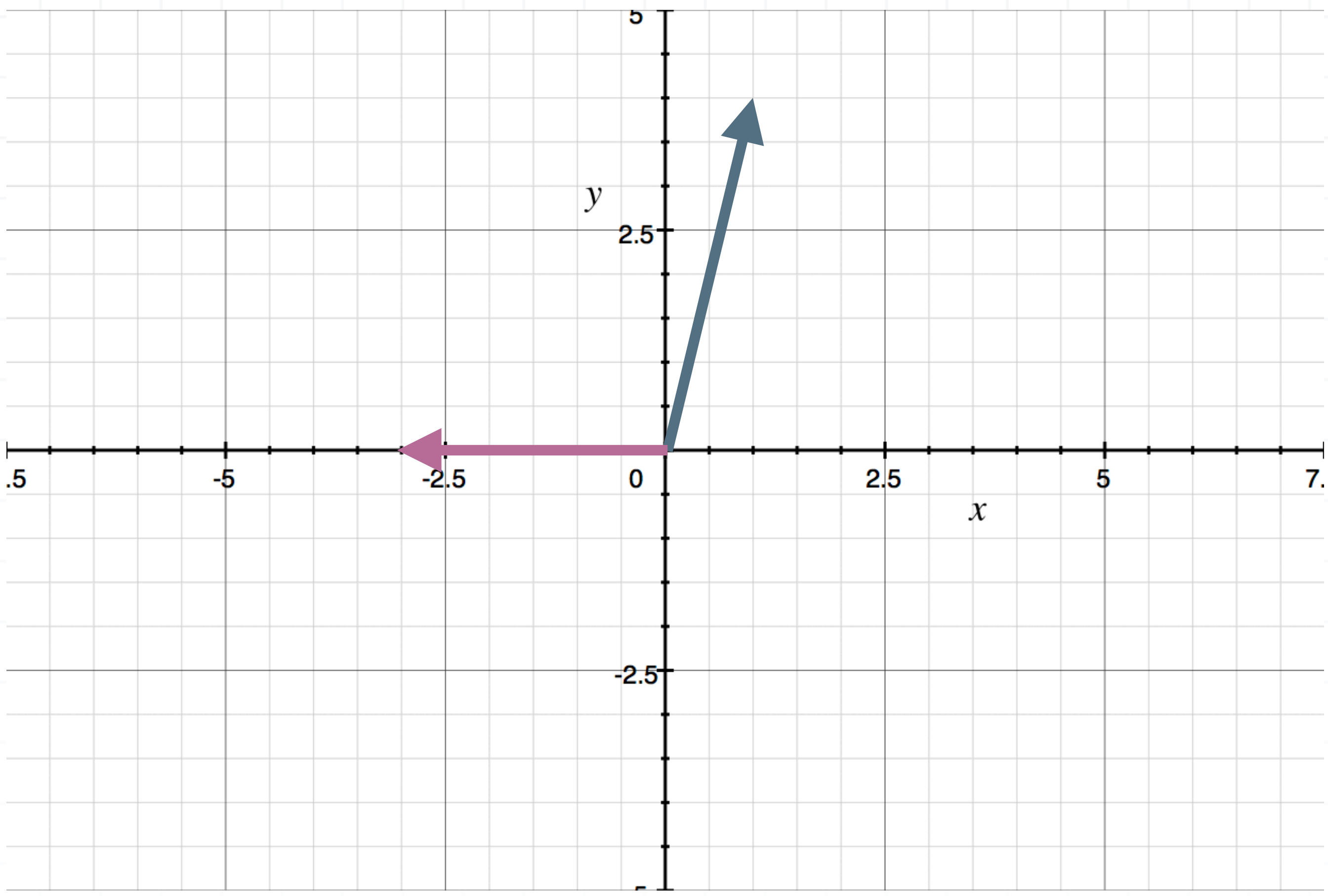
$$M = \begin{bmatrix} 1 & -3 \\ 4 & 0 \end{bmatrix}$$

transformed the light blue triangle into the dark blue triangle. In this section, we want to look at the entries in the transformation matrix M to see how they work together to change the light blue triangle into the dark blue triangle.

In a 2×2 transformation matrix, the first column (in this case $(1,4)$) tells us where the unit vector $(1,0)$ will land after the transformation. The second column (in this case $(-3,0)$) tells us where the unit vector $(0,1)$ will land after the transformation. In other words, given the unit vectors $(1,0)$ in light blue and $(0,1)$ in red,



the transformation matrix M changes the light blue vector $(1,0)$ into the dark blue vector $(1,4)$, and changes the red vector $(0,1)$ into the purple vector $(-3,0)$.

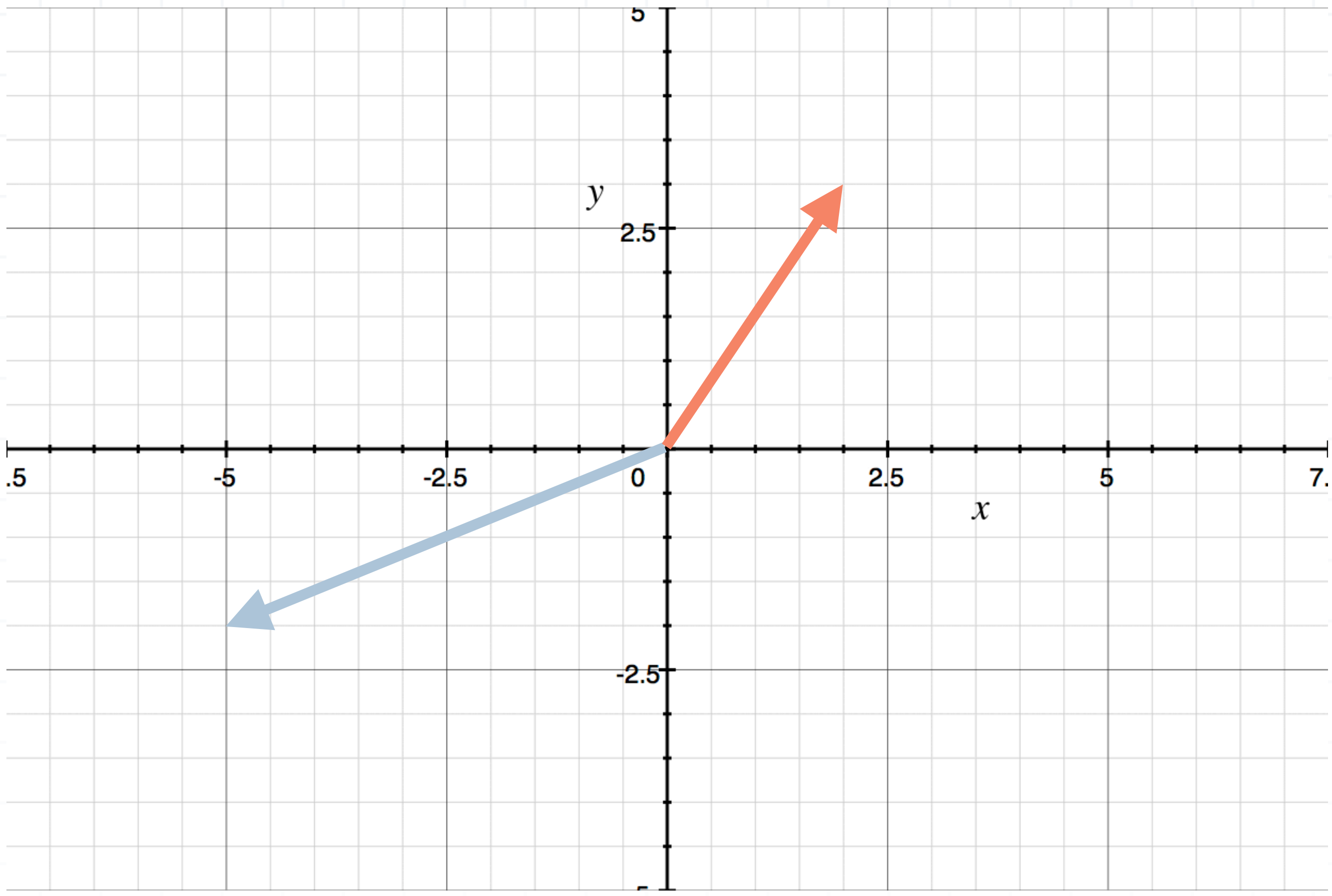


We already know how to take the transformation matrix and apply it to a point or vector, like we did when we transformed the light blue triangle into the dark blue triangle.

But we can also work backwards from a transformed figure to the original figure to figure out what the transformation matrix must have been.

Example

The graph shows the light blue unit vector $(1,0)$ and the red unit vector $(0,1)$ after a transformation has been applied. Find the transformation matrix that did the transformation.



The light blue vector now points to $(-5,-2)$. Since the first column of the transformation matrix represents where the unit vector $(1,0)$ lands after a transformation, we can fill in the first column of the transformation matrix.

$$\begin{bmatrix} -5 & \\ -2 & \end{bmatrix}$$

The red vector now points to $(2,3)$. Since the second column of a transformation matrix represents where the unit vector $(0,1)$ lands after a transformation, we can fill in the second column of the transformation matrix.

$$\begin{bmatrix} -5 & 2 \\ -2 & 3 \end{bmatrix}$$

This is the matrix that describes the transformation happening in coordinate space when $(1,0)$ moves to $(-5,-2)$ and when $(0,1)$ moves to $(2,3)$.

There's a really important conclusion to this last example problem. The transformation matrix we found doesn't just transform the individual points $(1,0)$ and $(0,1)$, it models the transformation of every point in the coordinate plane!

Therefore, armed with this transformation matrix, you can now figure out the transformed location of any other point. For instance, let's say you want to know what happens to $(5,3)$. Simply multiply by the transformation matrix:

$$\begin{bmatrix} -5 & 2 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -5(5)+2(3) \\ -2(5)+3(3) \end{bmatrix}$$

$$\begin{bmatrix} -25+6 \\ -10+9 \end{bmatrix}$$

$$\begin{bmatrix} -19 \\ -1 \end{bmatrix}$$

Under this transformation, $(5,3)$ will go to $(-19,-1)$. And you could use the transformation matrix to transform any other point in the coordinate space that you were interested in.