Sketching a parametric curve and its orientation

At times you may encounter a situation where you're given separate expressions for the variables x and y in terms of a common variable. That variable is known as a parameter. The parameter that's most commonly used in a situation like this is t, because it's often used to represent time. For example, the expressions for x and y may represent the x and y coordinates of the points on a curve that's being traced out by some physical object as it undergoes motion in the xy-plane over some interval of time.

In this lesson, you're going to learn how to take a pair of parametric equations (one for x in terms of t, and one for y in terms of t), use them to eliminate the parameter t (and get an equation in terms of x and y only), and then sketch the curve in the xy-plane that corresponds to a specified interval of values of t. That curve is known as the parametric curve. You'll also learn how to show (on your graph) the direction in which the curve is traced out as the value of the parameter t increases over the specified interval. We call that direction the orientation of the curve.

Example

Sketch the parametric curve defined by $x = t/2$ and $y = 3t/4$ where $-4 \le t \le 4$.

In this case, we can solve the first equation ($x = t/2$) for t, and then plug the resulting expression for t into the second equation ($y = 3t/4$) to get an equation in x and y only:

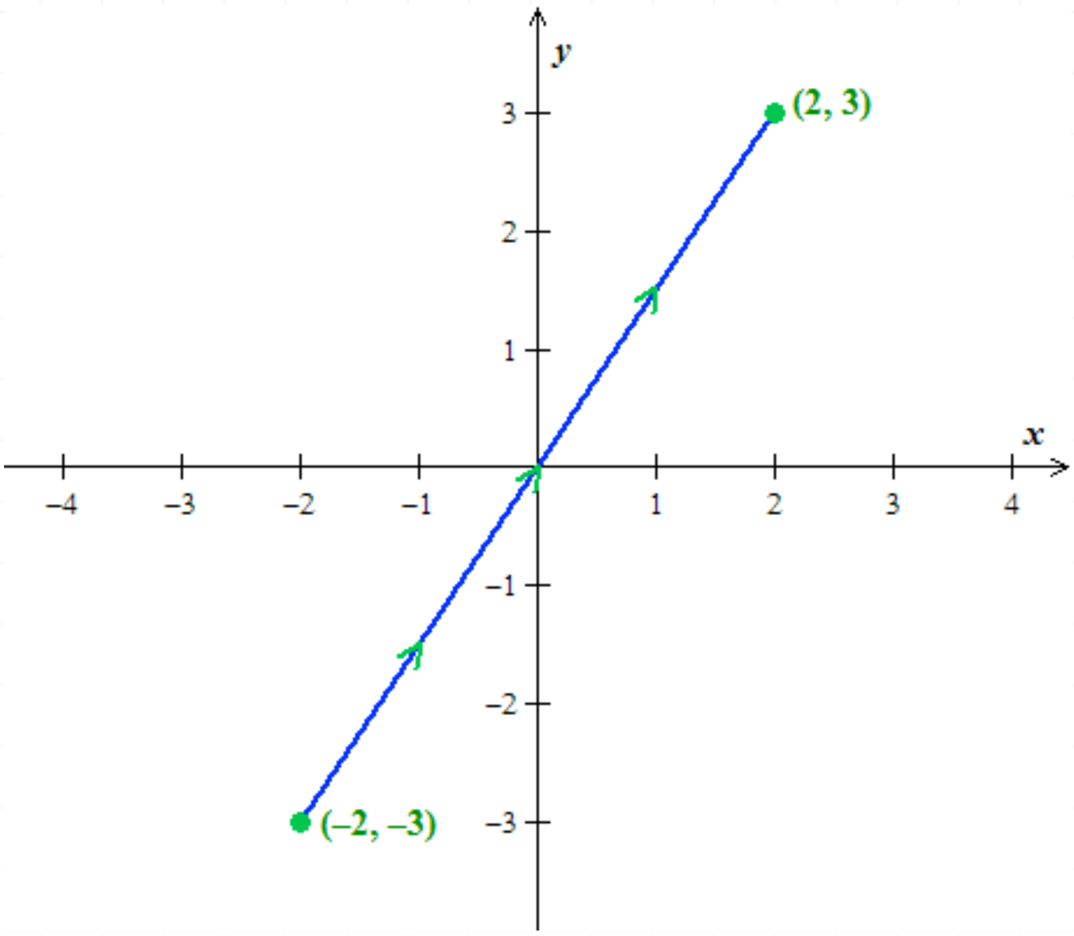
$$x = \frac{t}{2} \Longrightarrow t = 2x, \quad y = \frac{3t}{4} \Longrightarrow y = \frac{3(2x)}{4} = \frac{6x}{4} = \frac{3x}{2}$$

You may recognize $y = 3x/2$ as the equation of the line that passes through the origin and has a slope of $3/2$.

Let's construct a table of values of x and y for several values of t in the interval $[-4,4]$.

t	$x = \frac{t}{2}$	$y = \frac{3t}{4}$
-4	-2	-3
-2	-1	$-\frac{3}{2}$
0	0	0
2	1	$\frac{3}{2}$
4	2	3

Now we'll use the data in the table to sketch the parametric curve. When we do this, we'll use arrows at several points on the curve to show the direction in which it's traced out as t increases from -4 to 4.



Let's look at a slightly more complicated example.

Example

Sketch the parametric curve defined by $x = t^2$ and $y = t - 2$ where $-2 \leq t \leq 1$.

Here, we can solve the second equation ($y = t - 2$) for t, and then plug the resulting expression for t into the first equation ($x = t^2$) to get an equation in x and y only:

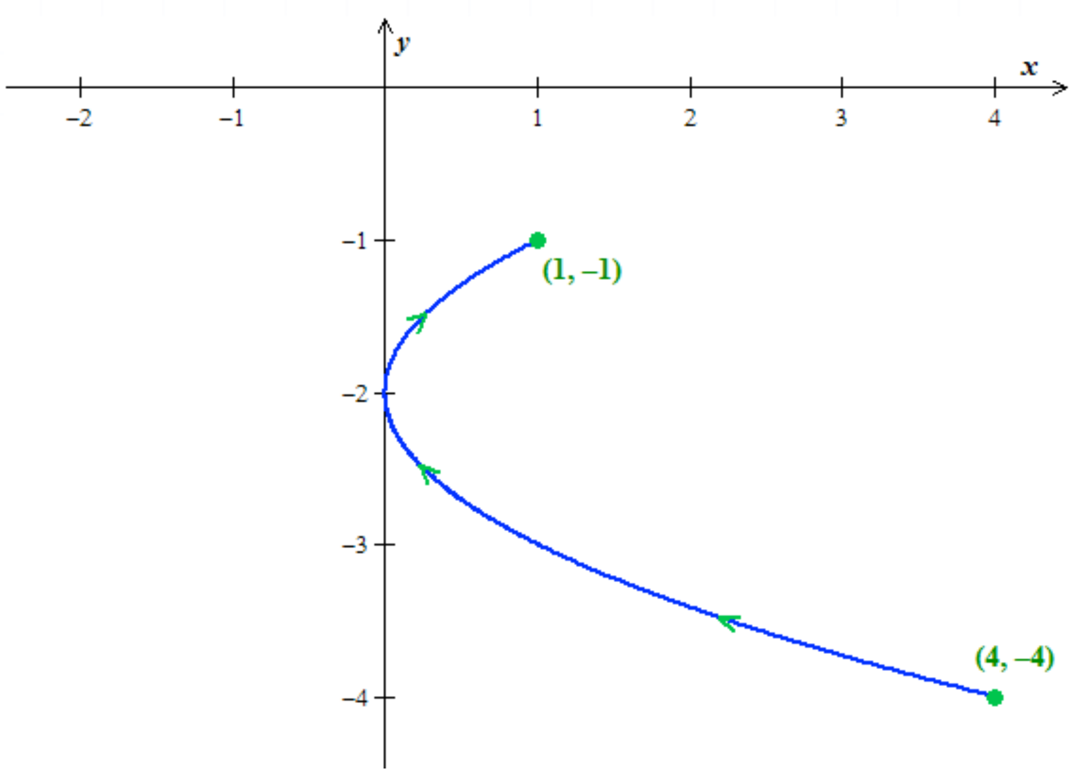
$$y = t - 2 \implies t = y + 2, \quad x = t^2 \implies x = (y + 2)^2$$

You may recall that $x = (y + 2)^2$ is the equation of a parabola that opens to the right and has its vertex at the point $(0, -2)$.

Let's construct a table of values of x and y for a few values of t in the interval $[-2,1]$.

t	$x=t^2$	$y=t-2$
-2	4	-4
-1	1	-3
0	0	-2
1	1	-1

Now we'll use those data to sketch the parametric curve, and we'll use arrows at several points on the curve to show the direction in which it's traced out as t increases from -2 to 1.



Parametric equations often involve trig functions.

Example

Sketch the parametric curve defined by $x = 4\cos t$ and $y = 4\sin t$ where $0 \le t \le \pi$.

Solving these equations for $\cos t$ and $\sin t$, respectively, we get

$$\cos t = \frac{x}{4}, \quad \sin t = \frac{y}{4}$$

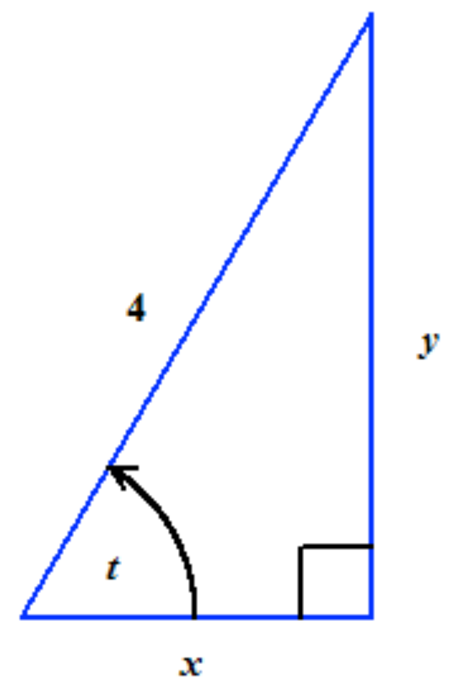
Let's construct a right triangle to determine the equation of the parametric curve. We'll let t be one of the acute angles of the triangle. Recall that

$$\cos t = \frac{\text{adj}}{\text{hyp}}$$

where "adj" is the leg adjacent to angle t and "hyp" is the hypotenuse. In this case, $\cos t = x/4$, so we'll let x be the leg adjacent to angle t, which means that the hypotenuse is 4. Similarly,

$$\sin t = \frac{\text{opp}}{\text{hyp}}$$

where "opp" is the leg opposite angle t. In this case, $\sin t = y/4$. Since 4 is the hypotenuse, we find that y is the leg opposite angle t.



By the Pythagorean theorem,

$$x^2 + y^2 = 4^2$$

This is the equation of the circle of radius 4 which is centered at the origin.

Note that the value of $\sin t$ is nonnegative on the entire interval $[0,\pi]$, so the value of $y = 4\sin t$ is nonnegative on that entire interval.

Now we'll tabulate values of x and y for several values of t in the interval $[0,\pi]$.

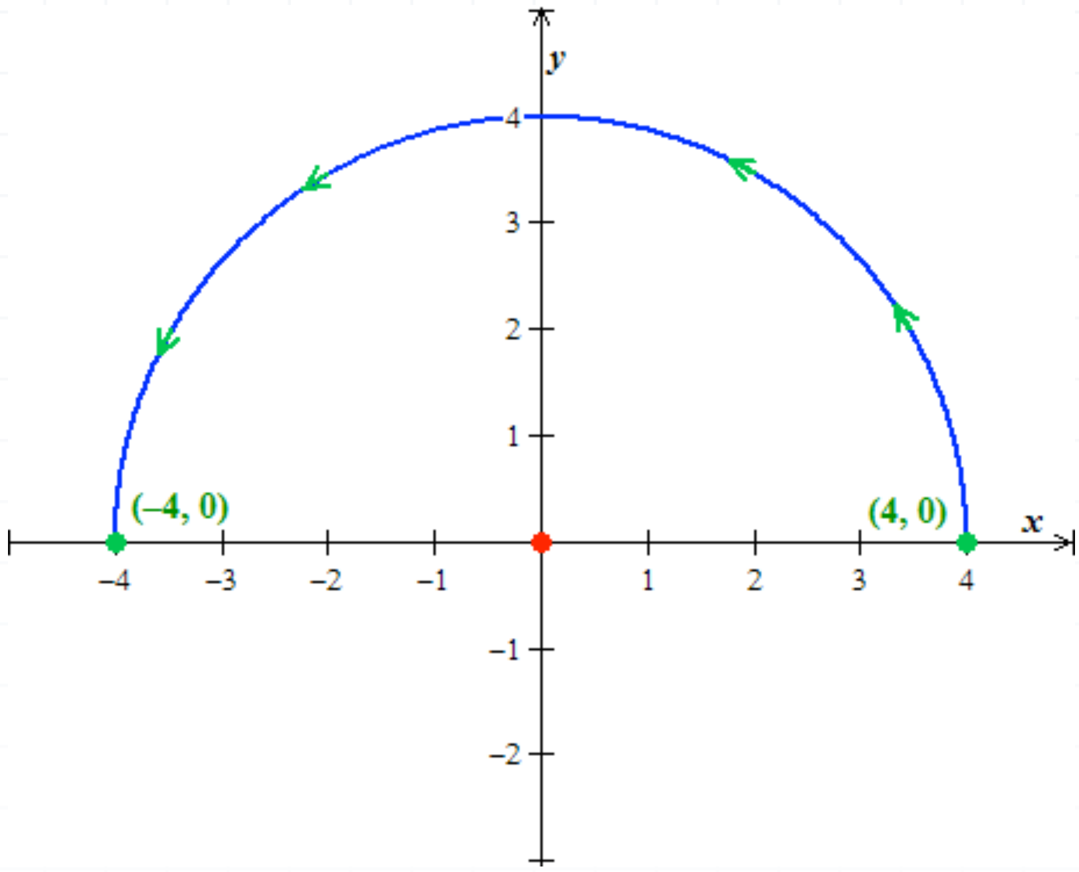
t	$\cos t$	$\sin t$	$x = 4\cos t$	$y = 4\sin t$
0	1	0	4	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$2\sqrt{2}$	$2\sqrt{2}$
$\frac{\pi}{2}$	0	1	0	4

$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-2\sqrt{2}$	$2\sqrt{2}$
π	-1	0	-4	0

As t ranges over the interval $[0,\pi]$, x starts at $4\cos(0) = 4$ (at $t = 0$) and decreases to $4(0) = 0$ (at $t = \pi/2$) and then continues to decrease to $4(-1) = -4$ (at $t = \pi$). y starts at $4(0) = 0$ (at $t = 0$) and increases to $4(1) = 4$ (at $t = \pi/2$), and then decreases to $4(0) = 0$ (at $t = \pi$). Thus the parametric curve which is traced out as t ranges from 0 to π starts at the point $(4,0)$ and ends at the point $(-4,0)$.

Using those data, we can sketch the parametric curve.

Since y is nonnegative throughout, this parametric curve is the upper half of the circle $x^2 + y^2 = 4^2$. Notice that this curve is traced out in the counterclockwise direction as t goes from 0 to π. When we sketch the curve, we'll use arrows at several points on the curve to indicate its orientation.



In this example, an alternative way to get the equation $x^2 + y^2 = 4^2$ would be to apply the basic Pythagorean identity ($\cos^2 t + \sin^2 t = 1$) after solving the equations $x = 4\cos t$ and $y = 4\sin t$ for $\cos t$ and $\sin t$, respectively:

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

$$\frac{x^2}{4^2} + \frac{y^2}{4^2} = 1$$

Multiplying both sides of this equation by 4^2 yields

$$x^2 + y^2 = 4^2$$

Now suppose we wanted to use a parameter to obtain this same curve but with the opposite orientation (starting at $(-4,0)$ and ending at $(4,0)$, instead of starting at $(4,0)$ and ending at $(-4,0)$).

As you will see, we can do this by replacing t in the equations $x = 4\cos t$ and $y = 4\sin t$ with $\pi - t$.

First, let's see what happens to the range of values of $\pi - t$ when t is in the interval $[0,\pi]$.

Recall that we have

$$0 \leq t \leq \pi$$

Suppose we multiply through by -1 (a negative number, which means that the direction of the inequalities must be reversed!):

$$0 \geq -t \geq -\pi$$

Adding π throughout, we obtain

$$\pi \geq \pi - t \geq 0$$

Turning this around, we have

$$0 \leq \pi - t \leq \pi$$

Thus the range of values of $\pi - t$ is the same as the range of values of t.

Now let's let $x = 4\cos(\pi - t)$ and $y = 4\sin(\pi - t)$. By the difference identities for cosine and sine, we have the following:

$$\cos(\pi - t) = (\cos\pi)(\cos t) + (\sin\pi)(\sin t) = (-1)(\cos t) + (0)(\sin t) = -\cos t$$

$$\sin(\pi - t) = (\sin\pi)(\cos t) - (\cos\pi)(\sin t) = (0)(\cos t) - (-1)(\sin t) = \sin t$$

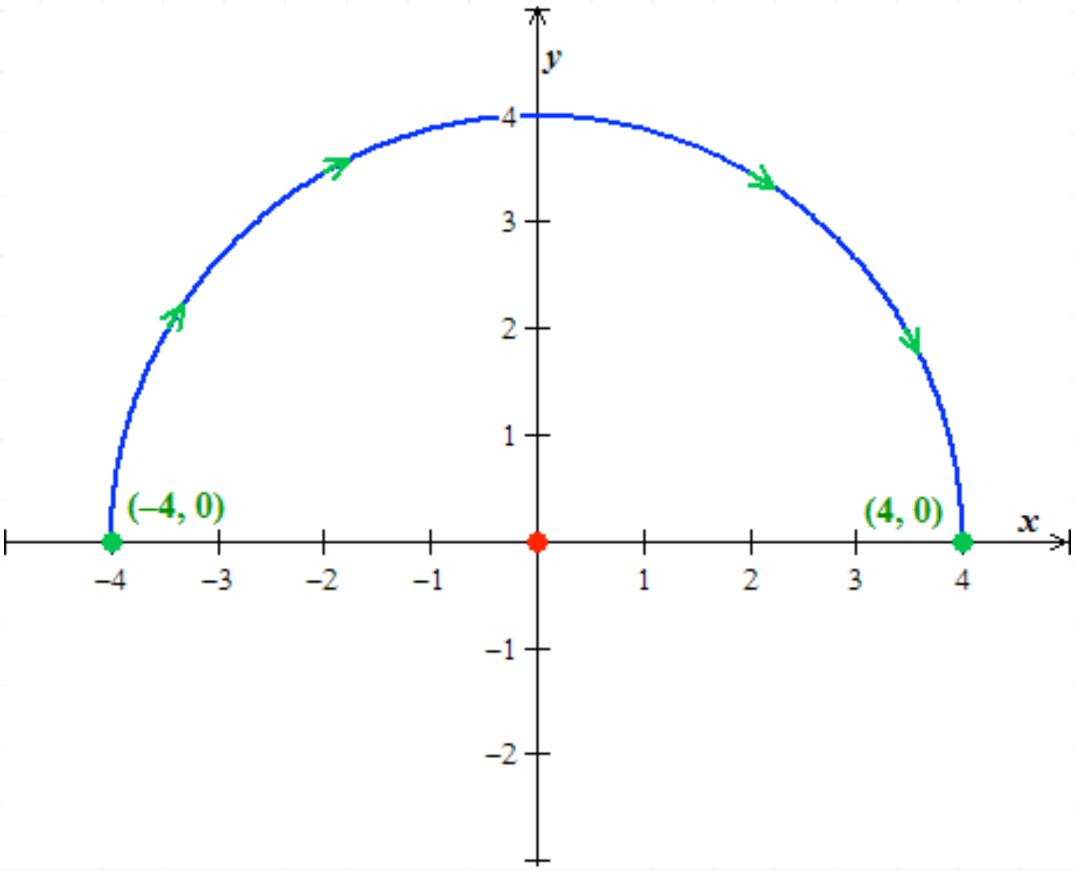
Using these results, we get

$$x = 4\cos(\pi - t) = -4\cos t, \quad y = 4\sin(\pi - t) = 4\sin t$$

Next, we'll tabulate values of $x = -4\cos t$ and $y = 4\sin t$ for various values of t in the interval $[0,\pi]$.

t	$\cos t$	$\sin t$	$x = -4\cos t$	$y = 4\sin t$
0	1	0	-4	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-2\sqrt{2}$	$2\sqrt{2}$
$\frac{\pi}{2}$	0	1	0	4
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$2\sqrt{2}$	$2\sqrt{2}$
π	-1	0	4	0

Using those data, we can sketch the parametric curve.



In this case, x starts at $-4(1) = -4$ (at $t = 0$) and increases to $-4(0) = 0$ (at $\pi/2$), and then goes on up to $-4(-1) = 4$ (at $t = \pi$). y starts at $4(0) = 0$ (at $t = 0$) and increases to $4(1) = 4$ (at $t = \pi/2$), and then goes back down to $4(0) = 0$ (at $t = \pi$). Notice that y behaves the same way as before, but x goes from -4 to 4 (instead of going from 4 to -4). Thus the parametric curve starts at the point with coordinates $(x, y) = (-4,0)$ (at $t = 0$) and ends at the point with coordinates $(x, y) = (4,0)$.

What we have found is that if we use the parametric equations

$$x = 4\cos(\pi - t)\ (= -4\cos t)$$

and

$$x = 4\sin(\pi - t) = 4\sin t$$

where $0 \leq t \leq \pi$, the parametric curve is again the upper half of the circle that satisfies the equation $x^2 + y^2 = 4^2$, but now it's traced out in the clockwise direction as t increases from 0 to π.

Example

Sketch the parametric curve defined by $x = 1 + \cos t$ and $y = 2 + \sin t$ where $0 \leq t \leq 3\pi/2$.

One way to approach this is to get equations for $\cos t$ and $\sin t$ from the equations $x = 1 + \cos t$ and $y = 2 + \sin t$, respectively, and then use the basic Pythagorean identity to get an equation in x and y only.

$$x = 1 + \cos t \implies \cos t = x - 1$$

$$y = 2 + \sin t \implies \sin t = y - 2$$

By the basic Pythagorean identity,

$$\cos^2 t + \sin^2 t = 1$$

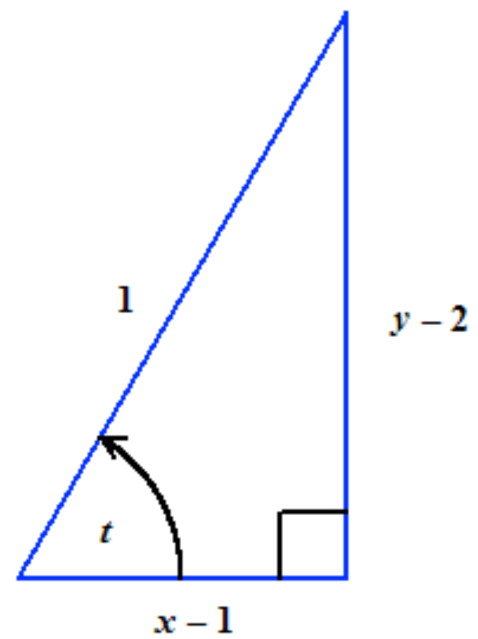
Substituting $x - 1$ and $y - 2$ for $\cos t$ and $\sin t$, respectively, we obtain

$$(x - 1)^2 + (y - 2)^2 = 1$$

This is the equation of the circle of radius 1 with center at the point $(1,2)$.

Alternatively, we can construct a right triangle to determine the equation of the parametric curve. We'll let t be one of the acute angles of the triangle. In this case, $\cos t = x - 1 = (x - 1)/1$, so we'll let $x - 1$ be the leg adjacent to angle t, which means that the hypotenuse is 1. Also, $\sin t = y - 2 = (y - 2)/1$. Since 1 is the hypotenuse, we find that $y - 2$ is the leg opposite angle t. Then by the Pythagorean theorem, we have

$$(x-1)^2 + (y-2)^2 = 1^2$$

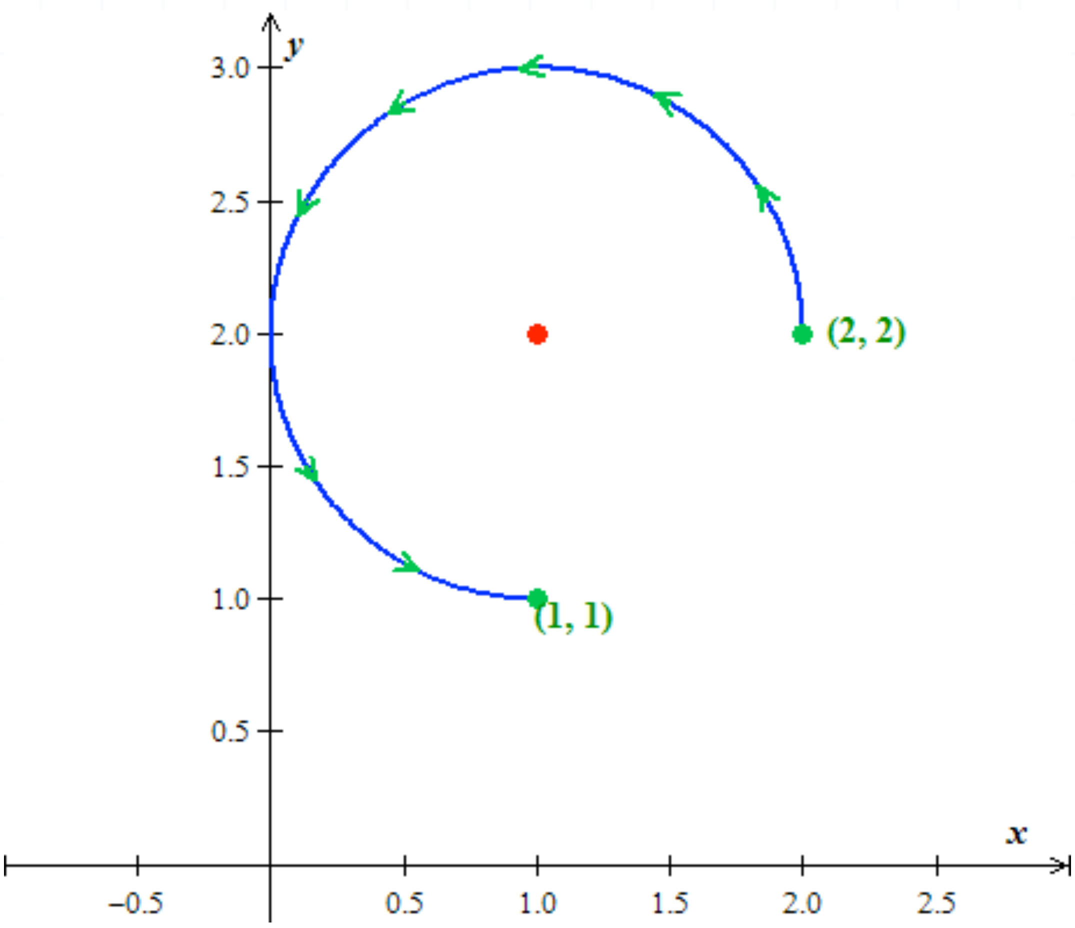


Let's tabulate values of x and y for a number of values of t in the interval $[0,3\pi/2]$.

t	$\cos t$	$\sin t$	$x = 1 + \cos t$	$y = 2 + \sin t$
0	1	0	2	2
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$1 + \frac{\sqrt{2}}{2}$	$2 + \frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1	1	3
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$1 - \frac{\sqrt{2}}{2}$	$2 + \frac{\sqrt{2}}{2}$

π	-1	0	0	2
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$1-\frac{\sqrt{2}}{2}$	$2-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0	-1	1	1

Using those data, we can sketch the parametric curve.



As you can see, the initial point of the parametric curve (the point for $t=0$) has coordinates $(x,y)=(2,2)$, the terminal point (the point for $t=3\pi/2$) has coordinates $(x,y)=(1,1)$, and the curve is traced out in the counterclockwise direction as t increases from 0 to $3\pi/2$.