Trigonometry Workbook Solutions

Inverse trig functions

krista king
MATH

INVERSE TRIG RELATIONS

1. In degrees, use the unit circle to find the set of angles whose cosine is $-\sqrt{2}/2$.

Solution:

On the unit circle, we know that the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that cosine of the angle is $-\sqrt{2}/2$, we need to find the angles in the unit circle where the corresponding coordinate point has an x-value equal to $-\sqrt{2}/2$.

Those angles are 135° and 225°. To give the full set of angles, we have to give all of the angles that are coterminal with these two.

$$\theta = 135^\circ + n(360^\circ) \text{ and } \theta = 225^\circ + n(360^\circ)$$

2. In both radians and degrees, use the unit circle to find the set of angles whose sine is -1.

Solution:

On the unit circle, we know that the y-value in the coordinate point is the value that gives us the sine of the angle. Therefore, because we're told that sine of the angle is -1, we need to find the angles in the unit circle where the corresponding coordinate point has a y-value equal to -1.

The only angle that does this is the angle $3\pi/2$. To give the full set of radian angles, we have to give all of the angles that are coterminal with $3\pi/2$.

$$\theta = \frac{3\pi}{2} + 2n\pi$$

We know that $3\pi/2 = 270^\circ$, so the set of angles in degrees will be

$$\theta = 270^\circ + n(360^\circ)$$

3. In both radians and degrees, use the unit circle to find the set of angles whose secant is 2.

Solution:

We remember that

$$\sec\theta = \frac{1}{\cos\theta}$$

So if secant is 2, then cosine will be $1/2$. So to find the set of angles whose secant is 2 we can find the set of angles whose cosine is $1/2$.

On the unit circle, we know that the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that cosine of the angle is $1/2$, we need to find the angles in the unit circle where the corresponding coordinate point has an x-value equal to $1/2$.

Those angles are $\pi/3$ and $5\pi/3$. To give the full set of radian angles, we have to give all of the angles that are coterminal with these two.

$$\theta = \frac{\pi}{3} + 2n\pi \text{ and } \theta = \frac{5\pi}{3} + 2n\pi$$

We know that $\pi/3 = 60^\circ$ and $5\pi/3 = 300^\circ$, so the set of angles in degrees will be

$$\theta = 60^\circ + n(360^\circ) \text{ and } \theta = 300^\circ + n(360^\circ)$$

■ 4. In both radians and degrees, use the unit circle to find the set of angles whose cosecant is 1.

Solution:

We remember that

$$\csc\theta = \frac{1}{\sin\theta}$$

So, if cosecant is 1, then sine will be 1. So to find the set of angles whose cosecant is 1 we can find the set of angles whose sine is 1.

On the unit circle, we know that the y-value in the coordinate point is the value that gives us the sine of the angle. Therefore, because we're told that sine of the angle is 1, we need to find the angles in the unit circle where the corresponding coordinate point has a y-value equal to 1.

The only angle that does this is the angle $\pi/2$. To give the full set of radian angles, we have to give all of the angles that are coterminal $\pi/2$.

$$\theta = \frac{\pi}{2} + 2n\pi$$

We know that $\pi/2 = 90^\circ$, so the set of angles in degrees will be

$$\theta = 90^\circ + n(360^\circ)$$

5. In both radians and degrees, use the unit circle to find the set of angles whose tangent is 1.

Solution:

We remember that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

So if tangent is 1, then sine will be equal to cosine. So to find the set of angles whose tangent is 1 we can find the set of angles whose sine is equal to cosine.

On the unit circle, we know that the y-value in the coordinate point is the value that gives us the sine of the angle and the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that sine of the angle is equal to cosine of the angle, we need to find the angles in the unit circle where the corresponding coordinate point has a y-value equal to the x-value.

Those angles are $\pi/4$ and $5\pi/4$. To give the full set of radian angles, we have to give all of the angles that are coterminal with these two.

$$\theta = \frac{\pi}{4} + 2n\pi \text{ and } \theta = \frac{5\pi}{4} + 2n\pi$$

We can combine this set of angles into just one expression.

$$\theta = \frac{\pi}{4} + n\pi$$

We know that $\pi/4 = 45^\circ$ and $5\pi/4 = 225^\circ$, so the set of angles in degrees will be

$$\theta = 45^\circ + n(360^\circ) \text{ and } \theta = 225^\circ + n(360^\circ)$$

We can combine this set of angles into just one expression.

$$\theta = 45^\circ + n(180^\circ)$$

■ 6. In both radians and degrees, use the unit circle to find the set of angles whose cotangent is 0.

Solution:

We remember that

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

So if cotangent is 0, then cosine will be 0. Therefore, to find the set of angles whose cotangent is 0 we can find the set of angles whose cosine is 0.

On the unit circle, we know that the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that cosine of the angle is 0, we need to find the angles in the unit circle where the corresponding coordinate point has an x-value equal to 0.

Those angles are $\pi/2$ and $3\pi/2$. To give the full set of radian angles, we have to give all of the angles that are coterminal with these two.

$$\theta = \frac{\pi}{2} + 2n\pi \text{ and } \theta = \frac{3\pi}{2} + 2n\pi$$

We can combine this set of angles into just one expression.

$$\theta = \frac{\pi}{2} + n\pi$$

We know that $\pi/2 = 90^\circ$ and $3\pi/2 = 270^\circ$, so the set of angles in degrees will be

$$\theta = 90^\circ + n(360^\circ) \text{ and } \theta = 270^\circ + n(360^\circ)$$

We can combine this set of angles into just one expression.

$$\theta = 90^\circ + n(180^\circ)$$

INVERSE TRIG FUNCTIONS

■ 1. Find the value of the inverse tangent function.

$$\tan^{-1}(0)$$

Solution:

We remember that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

So if tangent is 0, then sine will be 0. If we look at the unit circle, we can see that the sine function is 0 when $\theta = 0$ and when $\theta = \pi$. But because we're dealing with the inverse tangent function, we only want an angle in the interval $[-\pi/2, \pi/2]$.

The angle that works is $\theta = 0$, so we'll say

$$\tan^{-1}(0) = 0$$

■ 2. Find the value of the inverse cotangent function.

$$\cot^{-1}(-1)$$

Solution:

We remember that

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

So if cotangent is -1, then sine will be equal to cosine, but have the opposite sign. If we look at the unit circle, we can see that $\sin\theta = -\cos\theta$ when $\theta = 3\pi/4$ and when $\theta = 7\pi/4$. But because we're dealing with the inverse cotangent function, we only want an angle in the interval $(0,\pi)$.

Remember that the interval $(0,\pi)$ spans the first and second quadrants. The angle $\theta = 3\pi/4$ is in the second quadrant, and the angle $\theta = 7\pi/4$ is in the fourth quadrant. The angle that works is $\theta = 3\pi/4$, so we'll say

$$\cot^{-1}(-1) = \frac{3\pi}{4}$$

■ 3. Find the value of the inverse sine function.

$$\sin^{-1}\left(-\frac{1}{2}\right)$$

Solution:

If we look at the unit circle, we can see that the sine function is $-1/2$ when $\theta = 7\pi/6$ and when $\theta = 11\pi/6$. But because we're dealing with the inverse sine function, we only want an angle in the interval $[-\pi/2,\pi/2]$.

Both $\theta = 7\pi/6$ and $\theta = 11\pi/6$ fall outside the interval $[-\pi/2,\pi/2]$, which means we'll need to find an angle coterminal with either $\theta = 7\pi/6$ or $\theta = 11\pi/6$ that falls within $[-\pi/2,\pi/2]$.

Remember that the interval $[-\pi/2,\pi/2]$ spans the fourth and first quadrants. The angle $\theta = 7\pi/6$ is in the third quadrant, and the angle $\theta = 11\pi/6$ is in the fourth quadrant. Which means the angle we need is one that's coterminal with $\theta = 11\pi/6$, but in the interval $[-\pi/2,\pi/2]$. The angle that works is $\theta = -\pi/6$, so we'll say

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

4. Find the value of the inverse secant function.

$$\sec^{-1}(-2)$$

Solution:

We remember that

$$\sec\theta = \frac{1}{\cos\theta}$$

So, if secant is -2, then cosine will be $-1/2$. If we look at the unit circle, we can see that the cosine function is $-1/2$ when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. But because we're dealing with the inverse secant function, we only want an angle in the interval $[0,\pi/2) \cup (\pi/2,\pi]$.

Remember that the interval $[0,\pi/2) \cup (\pi/2,\pi]$ spans the first and second quadrants. The angle $\theta = 2\pi/3$ is in the second quadrant, and the angle $\theta = 4\pi/3$ is in the third quadrant. The angle that works is $\theta = 2\pi/3$, so we'll say

$$\sec^{-1}(-2) = \frac{2\pi}{3}$$

■ 5. Find the value of the inverse cosine function.

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Solution:

If we look at the unit circle, we can see that the cosine function is $\sqrt{3}/2$ when $\theta = \pi/6$ and when $\theta = 11\pi/6$. But because we're dealing with the inverse cosine function, we only want an angle in the interval $[0,\pi]$.

The angle $\theta = \pi/6$ is the only angle in $[0,\pi]$, so

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

■ 6. Find the value of the inverse cosecant function.

$$\csc^{-1}(-\sqrt{2})$$

Solution:

We remember that

$$\csc\theta = \frac{1}{\sin\theta}$$

So if cosecant is $-\sqrt{2}$, then sine will be $-\sqrt{2}/2$. If we look at the unit circle, we can see that the sine function is $-\sqrt{2}/2$ when $\theta = 5\pi/4$ and when $\theta = 7\pi/4$. But because we're dealing with the inverse cosecant function, we only want an angle in the interval $[-\pi/2,0) \cup (0,\pi/2]$.

Both $\theta = 5\pi/4$ and $\theta = 7\pi/4$ fall outside the interval $[-\pi/2,0) \cup (0,\pi/2]$, which means we'll need to find an angle coterminal with either $\theta = 5\pi/4$ or $\theta = 7\pi/4$ that falls within $[-\pi/2,0) \cup (0,\pi/2]$.

Remember that the interval $[-\pi/2,0) \cup (0,\pi/2]$ spans the fourth and first quadrants. The angle $\theta = 5\pi/4$ is in the third quadrant, and the angle $\theta = 7\pi/4$ is in the fourth quadrant. Which means the angle we need is one that's coterminal with $\theta = 7\pi/4$, but in the interval $[-\pi/2,0) \cup (0,\pi/2]$. The angle that works is $\theta = -\pi/4$, so we'll say

$$\csc^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$$

TRIG FUNCTIONS OF INVERSE TRIG FUNCTIONS

1. Find the value of the expression.

$$\sin\left(\tan^{-1}\left(\frac{1}{3}\right)\right)$$

Solution:

Let θ represent the angle in $(-\pi/2,\pi/2)$ whose tangent is $1/3$. Then we can say

$$\theta = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\tan\theta = \frac{1}{3}$$

Because $\tan\theta$ is positive, θ must be an angle in $(0,\pi/2)$, so θ is a positive angle that lies in quadrant I.

$$\tan\theta = \frac{1 = \text{opposite}}{3 = \text{adjancent}}$$

Given a triangle with opposite leg 1 and adjacent leg 3, the hypotenuse must be

$$a^2 + b^2 = c^2$$

$$1^2 + 3^2 = c^2$$

$$c^2 = 1 + 9$$

$$c^2 = 10$$

$$c = \sqrt{10}$$

Because sine is equivalent to opposite/hypotenuse, we get

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{10}}$$

$$\sin\left(\tan^{-1}\left(\frac{1}{3}\right)\right) = \sin\theta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

■ 2. Find the value of $\tan^{-1}(\sin\pi)$.

Solution:

We first need to find $\sin\pi$.

$$\sin\pi = 0$$

The angle in $(-\pi/2, \pi/2)$ whose tangent is 0 is 0. Therefore,

$$\tan^{-1}(\sin\pi) = \tan^{-1}(0) = 0$$

3. Find the value of the expression.

$$\csc\left(\cot^{-1}\left(\frac{1}{x}\right)\right)$$

Solution:

Set $\theta = \cot^{-1}(1/x)$. Then we can say

$$\theta = \cot^{-1}\left(\frac{1}{x}\right)$$

$$\theta = \cot^{-1}\left(\frac{1 = \text{adjacent}}{x = \text{opposite}}\right)$$

Given a triangle with adjacent leg 1 and opposite leg x, the hypotenuse must be

$$a^2 + b^2 = c^2$$

$$1^2 + x^2 = c^2$$

$$c^2 = 1 + x^2$$

$$c = \sqrt{1 + x^2}$$

Now that we know that the triangle we're describing has adjacent leg 1, opposite leg x, and hypotenuse $\sqrt{1 + x^2}$, we can find the cosecant of the interior angle of that triangle. Because cosecant is equivalent to hypotenuse/opposite, we get

$$\frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{1+x^2}}{x}$$

This expression represents $\csc(\cot^{-1}(1/x))$.

$$\csc\left(\cot^{-1}\frac{1}{x}\right) = \frac{\sqrt{1+x^2}}{x}$$

4. Find the value of the expression.

$$\cos\left(\sec^{-1}\left(-\frac{9}{2}\right)\right)$$

Solution:

Let θ represent the angle in $[0,\pi/2)$ or $(\pi/2,\pi]$ whose secant is $-9/2$. Then we can say

$$\theta = \sec^{-1}\left(-\frac{9}{2}\right)$$

$$\sec\theta = -\frac{9}{2}$$

Because $\sec\theta$ is negative, θ must be an angle in $(\pi/2,\pi]$, so θ is a negative angle that lies in quadrant II.

$$\sec\theta = \frac{9 = \text{hypotenuse}}{-2 = \text{adjancent}}$$

Because cosine is equivalent to adjacent/hypotenuse, we get

$$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{-2}{9} = -\frac{2}{9}$$

$$\cos\left(\sec^{-1}\left(-\frac{9}{2}\right)\right) = \cos\theta = -\frac{2}{9}$$

■ 5. Find the value of the expression.

$$\cot\left[\cos^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)\right]$$

Solution:

We first need to find $\sin(\pi/4)$.

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

The angle in $[0,\pi]$ whose cosine is $\sqrt{2}/2$ is $\pi/4$, so

$$\cos^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Then

$$\cot\left(\frac{\pi}{4}\right)=1$$

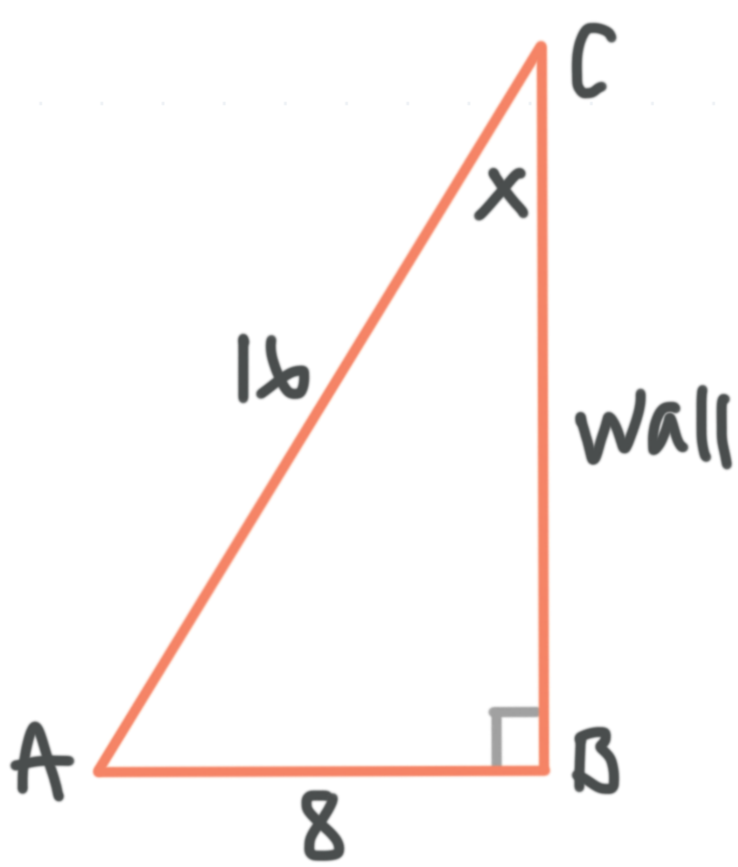
Therefore,

$$\cot\left[\cos^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)\right]=1$$

■ 6. A 16-foot ladder leans against a brick wall. The base of the ladder is 8 feet from the wall. Find the angle the ladder makes with the wall.

Solution:

Let's sketch the scenario.



We can set up an equation using the sine function.

$$\sin x = \frac{\overline{AB}}{\overline{AC}}$$

$$\sin x = \frac{8}{16}$$

$$\sin x = \frac{1}{2}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

Using the inverse property $\sin^{-1}(\sin x) = x$, we get

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

The angle in $[-\pi/2,\pi/2]$ whose sine is $1/2$ is $\pi/6$, so $x = \pi/6$.

W W W . K R I S T A K I N G M A T H . C O M