Reference angles

We've talked about coterminal angles a couple of times now. First, we looked at the set of coterminal angles in general for an angle θ. And we said that the set of all possible coterminal angles of θ could be defined by

$\alpha = \theta + n(360^\circ)$ or $\alpha = \theta + n(2\pi)$

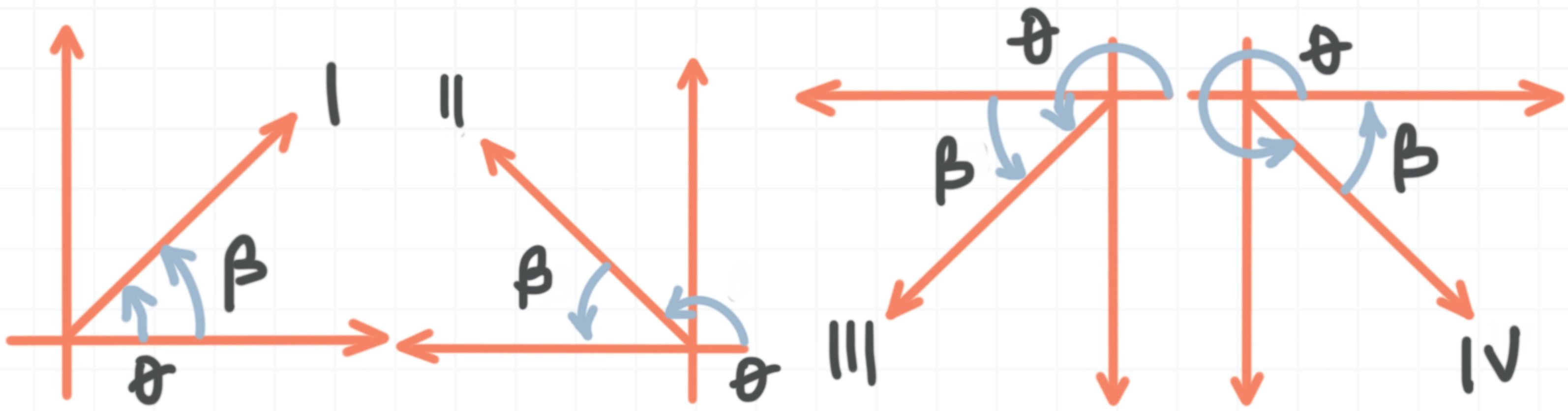
for any integer n. Second, we looked at how to find a coterminal angle in a particular interval. It's true that we usually want the coterminal angle that's both positive, and within one full rotation $[0^\circ, 360^\circ)$ or $[0, 2\pi)$, but other times we'll want to find a coterminal angle within some other specified angle interval.

The reference angle

And now that we're familiar with coterminal angles, and how to find them for both positive and negative angles, and for angles that are within one full rotation and outside of one full rotation, we want to turn our attention toward a closely related concept: the reference angle.

A **reference angle** for an angle θ in standard position is the positive acute angle formed by the x-axis and the terminal side of θ. Because reference angles are always positive and always acute, that means reference angles will always measure between 0° and 90°, or between 0 and $\pi/2$ radians.

If β is the reference angle for θ, we can sketch examples of θ and β in each quadrant.



As we can see from the figure, the measure of the reference angle will depend on the quadrant of the angle. For example, for an angle in quadrant II, $\beta = 180^\circ - \theta$ in degrees or $\beta = \pi - \theta$ in radians. But for an angle in quadrant III, $\beta = \theta - 180^\circ$ in degrees or $\beta = \theta - \pi$ in radians.

Let's summarize these reference angle formulas in a table.

θ's quadrant	**β in radians**	**β in degrees**
I	$\beta = \theta$	$\beta = \theta$
II	$\beta = \pi - \theta$	$\beta = 180^\circ - \theta$
III	$\beta = \theta - \pi$	$\beta = \theta - 180^\circ$
IV	$\beta = 2\pi - \theta$	$\beta = 360^\circ - \theta$

Notice that all of the θ angles in the figure are positive angles (they rotate in the positive, counterclockwise direction). In order to use the equations in the table above to find the reference angle, we need θ to be positive. If we have an angle θ that's negative, then we need to first find the positive coterminal angle, and then use that positive angle to find the reference angle.

Let's do an example where we find a reference angle in radians.

Example

What is the reference angle for $\theta = 2\pi/3$?

The angle $\theta = 2\pi/3$ is in the second quadrant, which means the reference angle β is

$$\beta = \pi - \theta$$

$$\beta = \pi - \frac{2\pi}{3}$$

$$\beta = \frac{3\pi}{3} - \frac{2\pi}{3}$$

$$\beta = \frac{\pi}{3}$$

Let's do an example with an angle in degrees.

Example

What is the reference angle for $\theta = -750^\circ$?

The angle $\theta = -750^\circ$ is two full rotations of 360° in the negative direction, and then an extra 30° in the negative direction, which means the angle is coterminal with $\theta = -30^\circ$. We want to convert this to a positive angle, which we can do by adding the negative angle to 360°.

$$\alpha = 360^\circ + (-30^\circ)$$

$$\alpha = 360^\circ - 30^\circ$$

$$\alpha = 330^\circ$$

So the angle $\theta = -750^\circ$ is coterminal with $\theta = -30^\circ$, which is coterminal with $\alpha = 330^\circ$. Now that we have a positive coterminal angle, we can find the reference angle.

Since $\alpha = 330^\circ$ is in the fourth quadrant, the reference angle β is

$$\beta = 360^\circ - \alpha$$

$$\beta = 360^\circ - 330^\circ$$

$$\beta = 30^\circ$$