Topic: Polar equation of a hyperbolic conic section

Question: The polar functions of two hyperbolas are given. What is the ratio of x_1y_1 to x_2y_2?

$$r_1^2 = \frac{1}{4\sin 2\theta_1}$$

$$r_2^2 = \frac{1}{6\sin 2\theta_2}$$

Answer choices:

A $\quad \dfrac{x_1y_1}{x_2y_2} = \dfrac{2}{3}$

B $\quad \dfrac{x_1y_1}{x_2y_2} = \dfrac{3}{2}$

C $\quad \dfrac{x_1y_1}{x_2y_2} = \dfrac{3}{4}$

D $\quad \dfrac{x_1y_1}{x_2y_2} = \dfrac{4}{3}$

Solution: B

Rewrite both equations.

$$r_1^2 = \frac{1}{4\sin 2\theta_1}$$

$$r_1^2\left(4\sin 2\theta_1\right) = 1$$

$$r_1^2\left(8\sin\theta_1\cos\theta_1\right) = 1$$

$$8\left(r_1\cos\theta_1\right)\left(r_1\sin\theta_1\right) = 1$$

$$8x_1y_1 = 1$$

and

$$r_2^2 = \frac{1}{6\sin 2\theta_2}$$

$$r_2^2\left(6\sin 2\theta_2\right) = 1$$

$$r_2^2\left(12\sin\theta_2\cos\theta_2\right) = 1$$

$$12\left(r_2\cos\theta_2\right)\left(r_2\sin\theta_2\right) = 1$$

$$12x_2y_2 = 1$$

Now pair these two equations together in a ratio.

$$\frac{8x_1y_1}{12x_2y_2} = \frac{1}{1}$$

$$\frac{x_1y_1}{x_2y_2} = \frac{12}{8} = \frac{3}{2}$$

Topic: Polar equation of a hyperbolic conic section

Question: Which conic section is defined by the polar function?

$$r - 4r\cos\theta - 5 = 0$$

Answer choices:

A Circle

B Ellipse

C Parabola

D Hyperbola

Solution: D

Solve the equation for r.

$$r - 4r\cos\theta - 5 = 0$$

$$(1 - 4\cos\theta)\, r - 5 = 0$$

$$(1 - 4\cos\theta)\, r = 5$$

$$r = \frac{5}{1 - 4\cos\theta}$$

Therefore the conic section is a hyperbola.

Topic: Polar equation of a hyperbolic conic section

Question: What are the eccentricity and directrix of the hyperbola?

$$3r - 5r\cos\theta - 9 = 0$$

Answer choices:

A $e = \frac{5}{3}$ and $d = \frac{27}{5}$

B $e = \frac{3}{5}$ and $d = \frac{27}{5}$

C $e = \frac{5}{3}$ and $d = \frac{9}{5}$

D $e = \frac{3}{5}$ and $d = \frac{9}{5}$

Solution: C

Transform the equation to standard polar form.

$$3r - 5r\cos\theta - 9 = 0$$

$$(3 - 5\cos\theta)r - 9 = 0$$

$$(3 - 5\cos\theta)r = 9$$

$$r = \frac{9}{3 - 5\cos\theta}$$

$$r = \frac{3}{1 - \frac{5}{3}\cos\theta}$$

In this form, we can see that the eccentricity and directrix are

$$e = \frac{5}{3}$$

$$d = 3 \div \frac{5}{3} = \frac{9}{5}$$