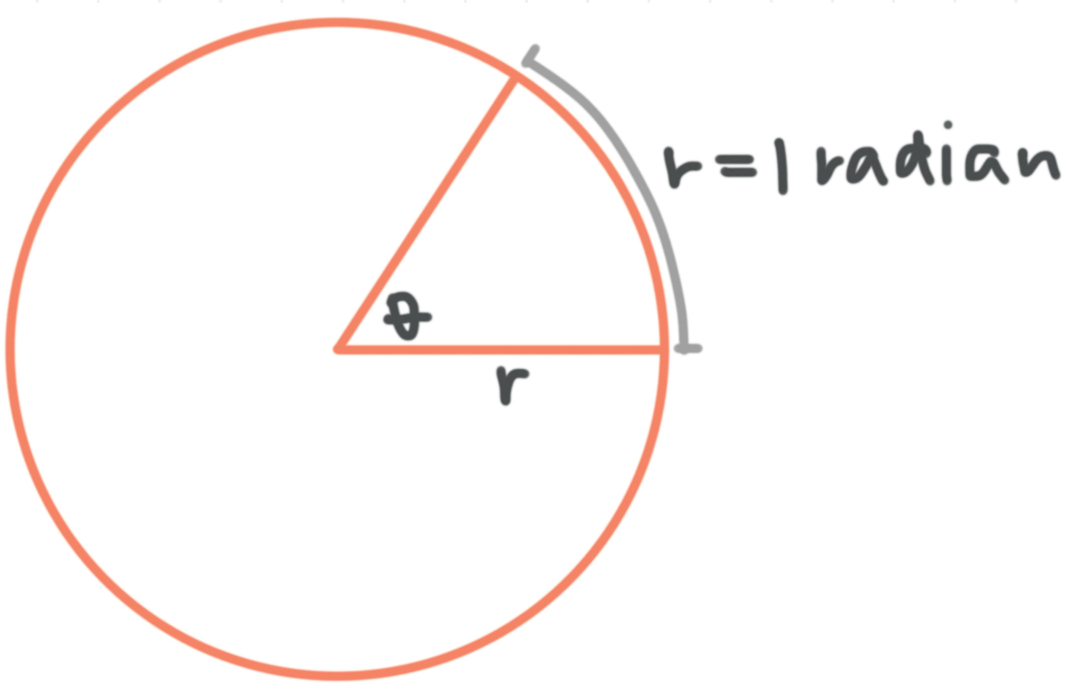
Radians and arc length

We've talked about different units for measuring angles, including degrees, DMS (degrees, minutes, seconds), and radians.

And when we introduced those units, we already defined a degree: it's one 360th of one full rotation. Similarly, in a DMS system, a degree is one 360th of one full rotation, a minute is one 60th of a degree, and a second is one 60th of a minute.

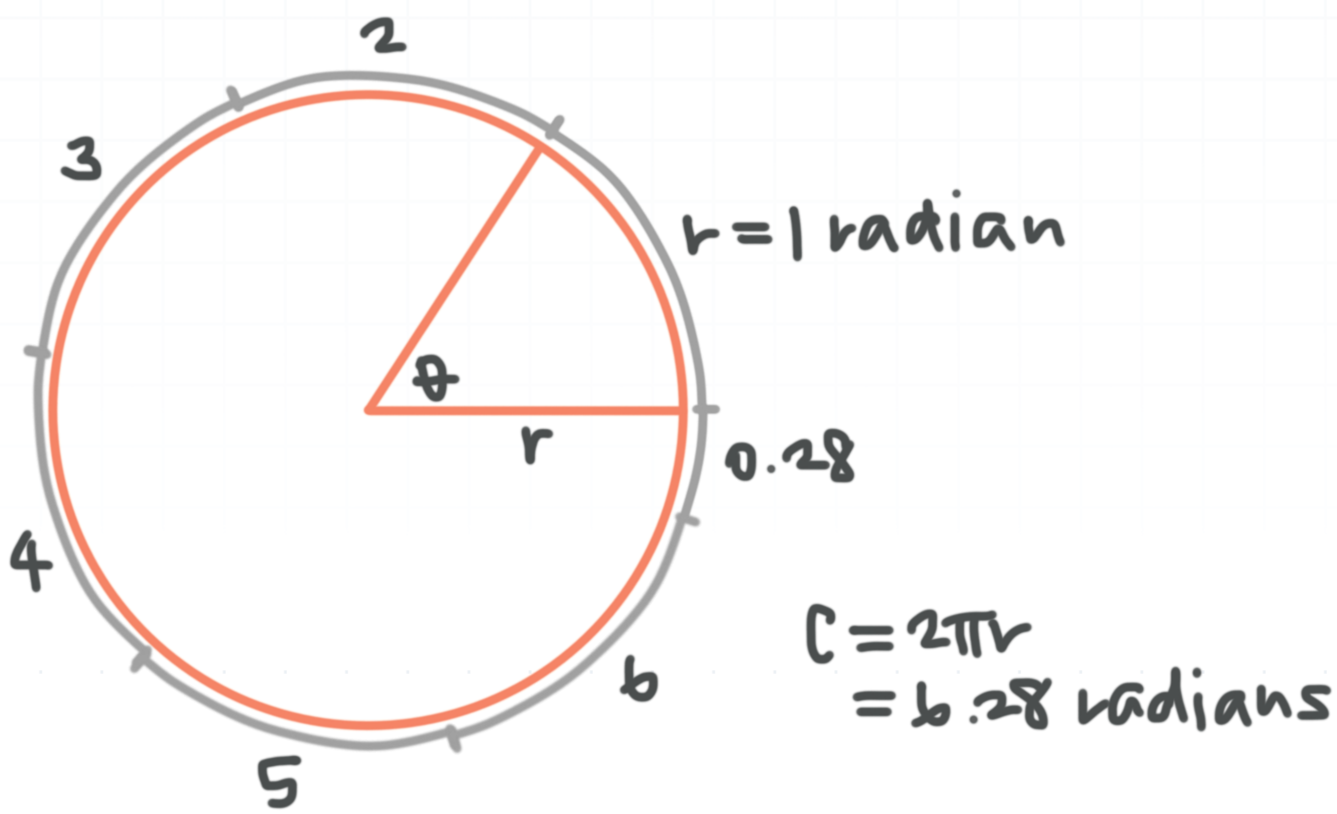
Radian measure

But we never really defined a radian, so let's do that now. Technically, an angle that has its vertex at the center of a circle and that intercepts an arc of the circle equal in length to the circle's radius has an angle measure of **one radian**.



In other words, if you take the radius of a circle and lay the radius out along the edge of the circle, the angle that defines that arc has a measure of 1 radian.

Because we also know that it takes 2π radians to complete one full circle, and $\pi \approx 3.14$, we know there are approximately $2\pi = 2(3.14) = 6.28$ radians in any circle. Put a different way, regardless of the size of the circle, wrapping the radius around the circle about 6.28 times will trace out the full circle.



Of course, this should make sense to us when we think about the equation for the circumference of a circle, $C = 2\pi r$. As we know, the circumference of a circle is the full distance one time around the circle, and the formula here is telling us that the distance one time around the circle is the same as about 6.28 times the radius, which is what we just learned.

And if we know that it takes about 6.28 radians to get around one full circle, and we also know that one full circle is defined by 360°, then we know the degree measure of exactly 1 radian is approximately

$$\frac{360^\circ}{6.28} \approx 57.32^\circ \approx 57^\circ 19' 29''$$

We see that these measures look approximately correct from the image we drew above where we sketched out the 6.28 radians around the circle.

Arc length

This definition of one radian leads us to the idea of arc length. We just said that laying out the length of one radius around the edge of a circle gave us an angle that measures 1 radian.

But what we really did there was create an **arc**, which is just two points connected by a curved line. A **circular arc**, specifically, is when the curve of the arc follows the perimeter of a circle.

Arc length is always given by

$$s = r\theta$$

where s is the length of the arc, r is the radius of the circle, and θ is the central angle that carves out that particular arc.

Keep in mind that whenever we use the formula for arc length, θ has to be in radians; we can't plug in a value for θ in degrees or DMS.

Let's do an example with the arc length formula.

Example

Find the length of an arc carved out by a central angle of 60° in a circle of radius $r = 2$.

We can only use an angle defined in radians in the arc length formula, so we'll need to convert 60° to radians.

$$60^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{\pi}{3} \text{ radians}$$

Now we'll plug what we know into the arc length formula.

$$s = r\theta$$

$$s = 2\left(\frac{\pi}{3}\right)$$

$$s = \frac{2\pi}{3}$$

Let's do another example where we know the circumference of the circle.

Example

If the circumference of a circle is 9π, find the length of an arc that lies on the circle and subtends a central angle of 20°.

If we solve both the arc length formula $s = r\theta$ for r, and the circumference formula $C = 2\pi r$ for r, we get $r = s/\theta$ and $r = C/2\pi$. Then we can set the equations equal to one another.

$$\frac{s}{\theta} = \frac{C}{2\pi}$$

$$\frac{s}{20^\circ} = \frac{9\pi}{2\pi}$$

Solve for arc length.

$$s = \frac{9}{2}(20^\circ)$$

$$s = 90^\circ$$

$$s = \frac{\pi}{2}$$