Solving right triangles

When we talk about "completing a right triangle," or "solving a right triangle" we mean that we're going to try to find all three side lengths, and all three interior angle measures.

The good news is that, once we find a few of these values in the triangle, it gets really easy to find the rest of them.

Formulas for solving right triangles

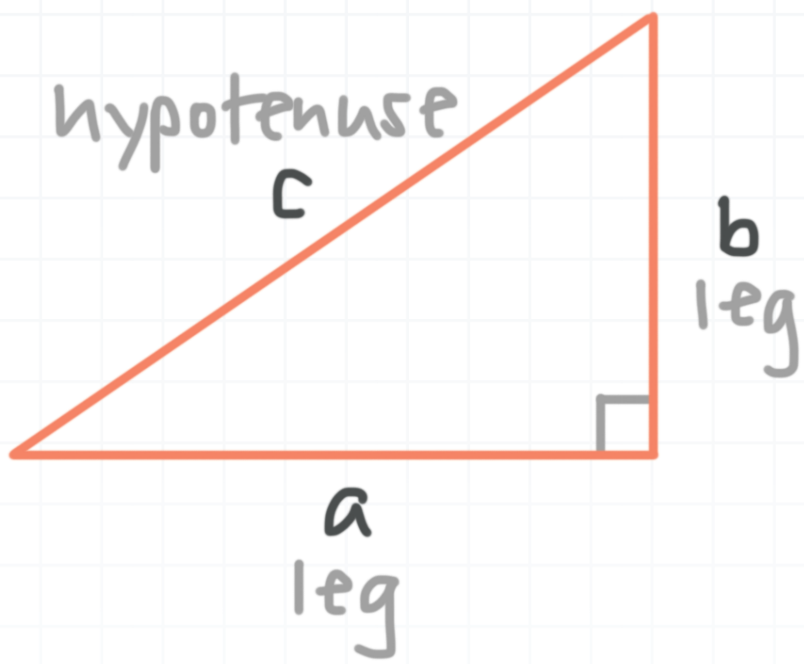
There are two formulas we'll use all the time during this process:

1. $m\angle A + m\angle B + m\angle C = 180^\circ$

2. $a^2 + b^2 = c^2$

The first formula tells us that the three interior angles of any triangle will always sum to 180°. Of course, in a right triangle, one of the angles is 90°, so the other two angles of a right triangle will always sum to 90° as well.

The second formula is the Pythagorean theorem, which tells us that the sum of the squares of the leg lengths, $a^2 + b^2$, is equal to the square of the length of the hypotenuse, c^2. So a and b are the legs and c is the hypotenuse (which is always the longest side, and the side opposite the right angle).



Our general strategy here will be to position the right triangle within the unit circle, but then extend its sides past the unit circle (if the hypotenuse is longer than 1 unit) in order to sketch out the full triangle.

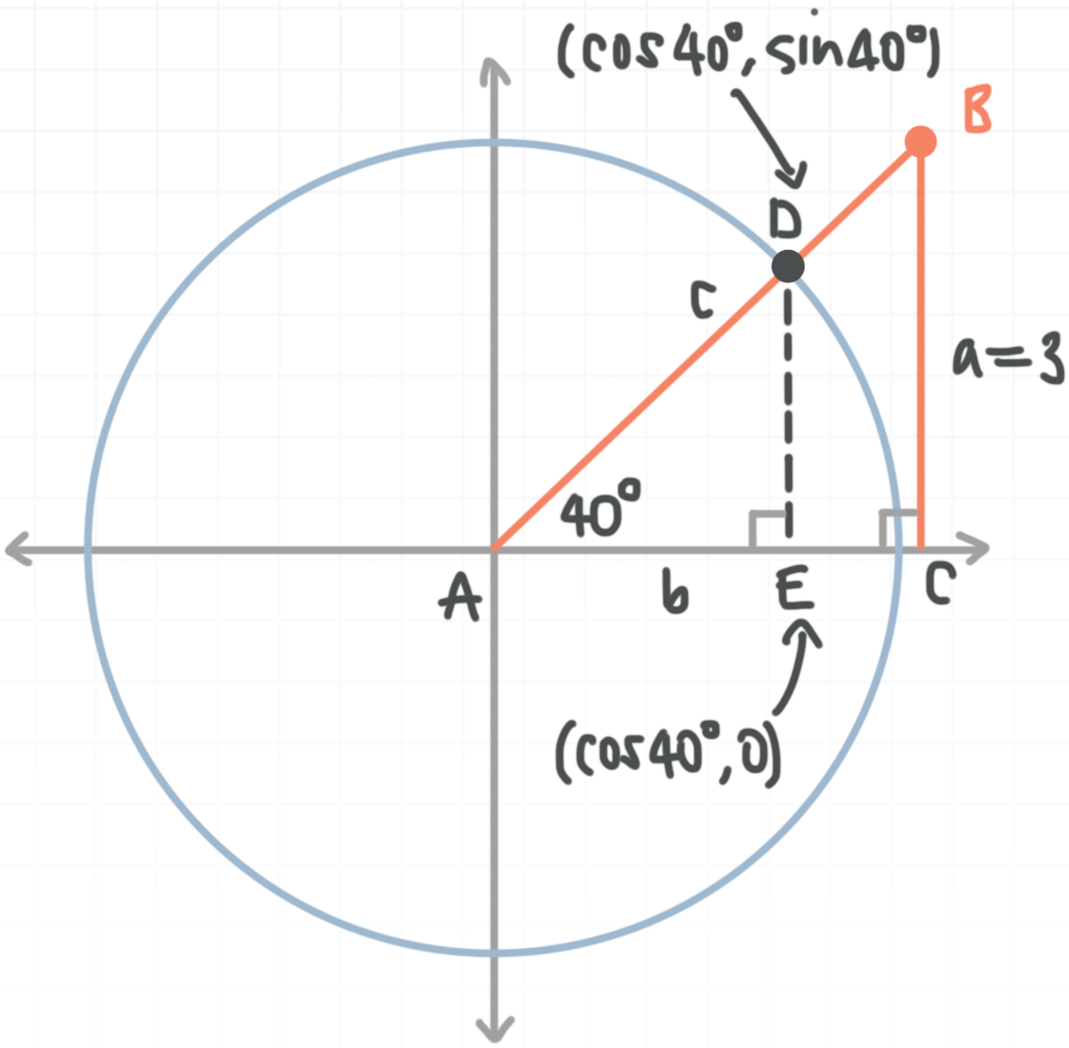
Then we'll treat the full triangle and the triangle within the unit circle as similar triangles.

Let's walk through an example where we're given one angle and the length of one leg.

Example

A right triangle has a leg with length 3. The angle opposite that leg is 40°. Find the measures of all three interior angles and the lengths of all three sides.

It's a good idea to draw a picture of the triangle before we start. Based on what we know, we have



We could just as easily have said $b = 3$ instead of $a = 3$, and therefore, $B = 40^\circ$ instead of $A = 40^\circ$. But it doesn't matter which leg we pick, as long as we make sure that the leg and the 40° angle are opposite each other. It's always helpful to put the right angle out at C so that the hypotenuse can extend out from the origin toward B.

Because we're dealing with a right triangle, we know we have a 90° angle and the 40° angle we were given. So the angle at B must be

$$B = 180^\circ - 90^\circ - 40^\circ$$

$$B = 50^\circ$$

We know the length of side a is $a = 3$, so all we need now is the length of the other two sides. Notice in the diagram that we sketched in $\overline{DE}$ in order to form the small triangle ADE. We do that in order to set up similar triangles. We assume that D is on the unit circle, such that $\overline{AD}$ has length 1. Then triangles ADE and ABC are similar, and we can say

$$\angle DAE = \angle BAC = 40^\circ$$

$$\angle ADE = \angle ABC = 50^\circ$$

$$\angle AED = \angle ACB = 90^\circ$$

Furthermore, corresponding sides of similar triangles are proportional, so

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{\overline{AD}}$$

$$\frac{3}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{1}$$

$$\frac{3}{\overline{DE}} = \frac{b}{\overline{AE}} = c$$

Here's where our sine and cosine functions come in. Since D is on the unit circle, its coordinates (x, y) are $(\cos 40^\circ, \sin 40^\circ)$. Also, the coordinates (x, y) of E are $(\cos 40^\circ, 0)$. Therefore, we can use these coordinates to find the lengths of $\overline{DE}$ and $\overline{AE}$.

$\overline{DE} = \sin 40^\circ - 0 = \sin 40^\circ$ (Since $\overline{DE}$ is perpendicular to the x-axis, we can always find its length as the absolute value of the difference of the y-coordinates of points E and D.)

$\overline{AE} = \cos 40^\circ - 0 = \cos 40^\circ$ (Since $\overline{AE}$ is perpendicular to the y-axis, we can always find its length as the absolute value of the difference of the x-coordinates of points A and E.)

So the proportion becomes

$$\frac{3}{\sin 40^\circ} = \frac{b}{\cos 40^\circ} = c$$

To find b (the length of $\overline{AC}$), we can use

$$\frac{3}{\sin 40^\circ} = \frac{b}{\cos 40^\circ}$$

$$\frac{3}{\sin 40^\circ}(\cos 40^\circ) = b$$

$$b \approx \frac{3}{0.643}(0.766)$$

$$b \approx 3.57$$

To find c (the length of $\overline{AB}$), we can use

$$\frac{3}{\sin 40^\circ} = c$$

$$c \approx \frac{3}{0.643}$$

$$c \approx 4.67$$

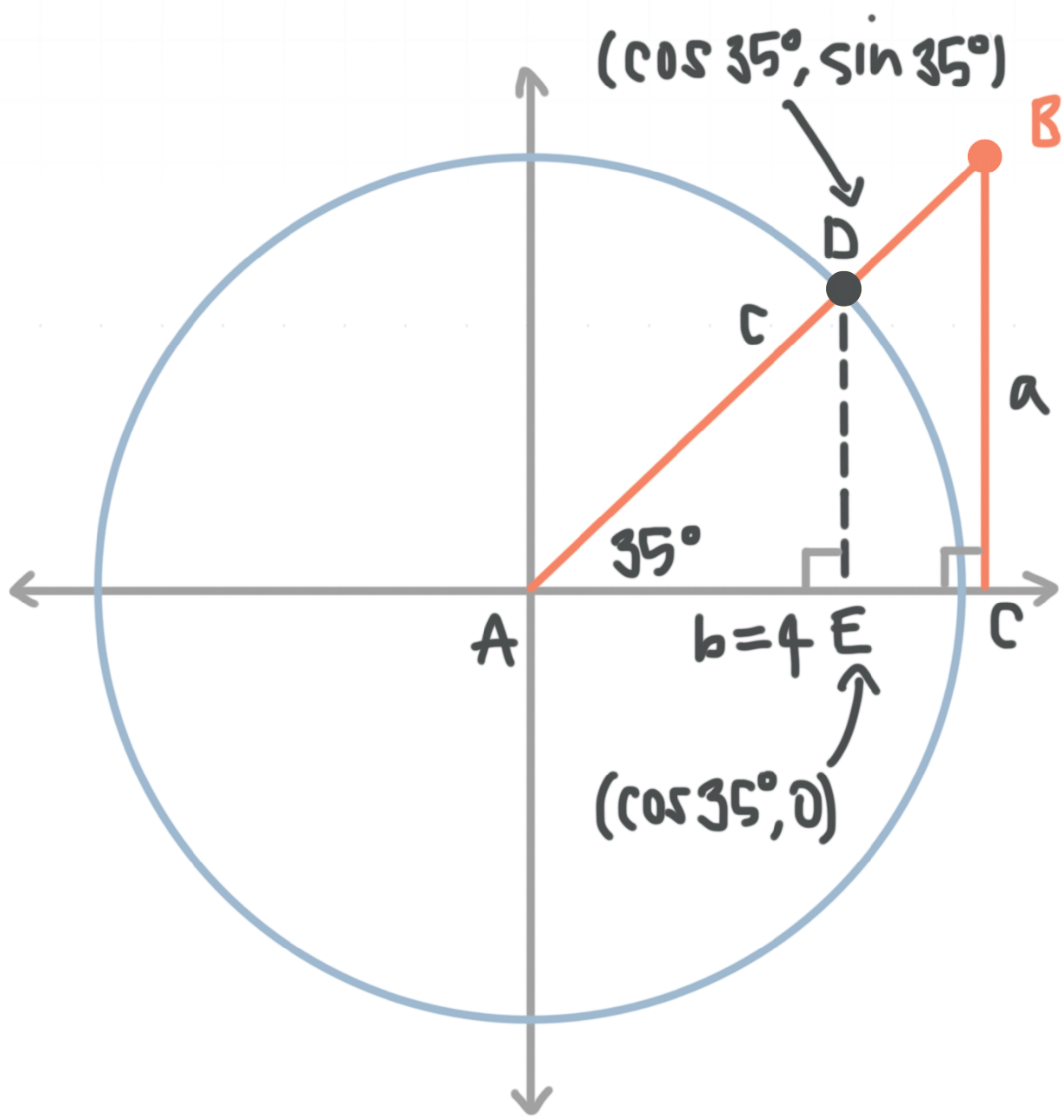
We've solved the right triangle, and we can say that the side lengths are $a = 3$, $b \approx 3.57$, and $c \approx 4.67$, and the angle measures are $A = 40^\circ$, $B = 50^\circ$, and $C = 90^\circ$.

In the last example we had the length of the leg opposite the known angle. Let's do another example where we have the length of the leg adjacent to the known angle.

Example

A right triangle has one leg with length 4 and an interior angle opposite the other leg that measures 35°. Complete the triangle.

We can say $b = 4$ and $A = 35^\circ$, or $a = 4$ and $B = 35^\circ$. Either will work, but we'll use the same orientation as the last example. Let's also sketch in the unit circle and put D on the unit circle.



With $C = 90^\circ$ and $A = 35^\circ$, B must be

$$B = 180^\circ - 90^\circ - 35^\circ$$

$$B = 55^\circ$$

Because triangles ABC and ADE are similar, we know

$$\angle DAE = \angle BAC = 35^\circ$$

$$\angle ADE = \angle ABC = 55^\circ$$

$$\angle AED = \angle ACB = 90^\circ$$

Furthermore, corresponding sides of similar triangles are proportional, so

$$\frac{a}{\overline{DE}} = \frac{b}{\overline{AE}} = \frac{c}{\overline{AD}}$$

$$\frac{a}{\overline{DE}} = \frac{4}{\overline{AE}} = \frac{c}{1}$$

Point D is again on the unit circle, and its coordinates (x, y) are $(\cos 35^\circ, \sin 35^\circ)$. Also, the coordinates (x, y) of E are $(\cos 35^\circ, 0)$. Therefore, we can use these coordinates to find the lengths of $\overline{DE}$ and $\overline{AE}$.

$$\overline{DE} = \sin 35^\circ - 0 = \sin 35^\circ$$

$$\overline{AE} = \cos 35^\circ - 0 = \cos 35^\circ$$

So the proportion becomes

$$\frac{a}{\sin 35^\circ} = \frac{4}{\cos 35^\circ} = c$$

To find a (the length of $\overline{BC}$), we can use

$$\frac{a}{\sin 35^\circ} = \frac{4}{\cos 35^\circ}$$

$$a = \frac{4}{\cos 35^\circ}(\sin 35^\circ)$$

$$a \approx \frac{4}{0.819}(0.574)$$

$$a \approx 2.80$$

To find c (the length of $\overline{AB}$), we can use

$$\frac{4}{\cos 35^\circ} = c$$

$$c \approx \frac{4}{0.819}$$

$$c \approx 4.88$$

We've solved the right triangle, and we can say that the side lengths are $a \approx 2.80$, $b = 4$, and $c \approx 4.88$, and the angle measures are $A = 35^\circ$, $B = 55^\circ$, and $C = 90^\circ$.