Law of cosines

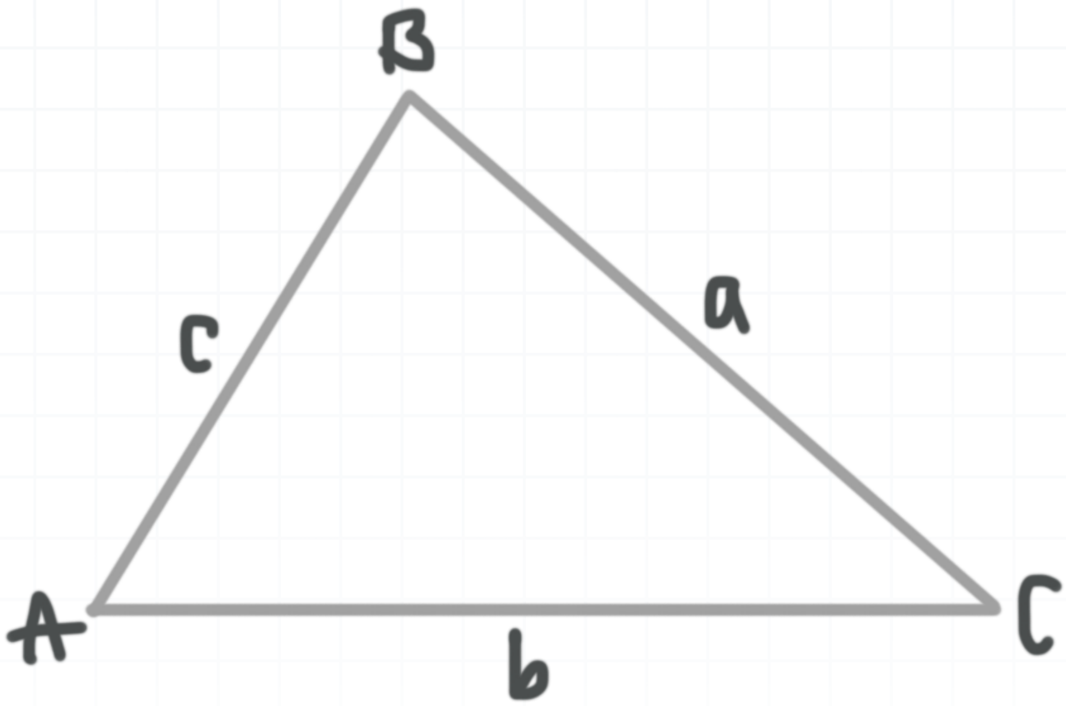
When we first learned about solving oblique triangles, we listed out in the table below the full set of scenarios in which we'd need to use either the law of sines or law of cosines.

Known information	**How to solve**
SAA or ASA One side and two angles	1. Use A+B+C=180° to find the remaining angle 2. Use law of sines to find the remaining sides
SAS Two sides and the included angle	1. Use law of cosines to find the third side 2. Use law of sines to find another angle 3. Use A+B+C=180° to find the remaining angle
SSS Three sides	1. Use law of cosines to find the largest angle 2. Use law of sines to find either remaining angle 3. Use A+B+C=180° to find the remaining angle
SSA Two sides and a non-included angle	**The ambiguous case**. If two triangles exist, use this same set of steps to find both triangles. 1. Use law of sines to find an angle 2. Use A+B+C=180° to find the remaining angle 3. Use law of sines to find the remaining side

We've already talked about the SAA or ASA case and the SSA ambiguous case, which both require the law of sines. In this lesson, we'll talk about the SAS and SSS cases, both of which requires the law of cosines.

The law of cosines

For any triangle with vertices A, B, and C, where side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C,



the **law of cosines** comes in three parts:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

There are several things we want to say about these formulas. First, realize that they all follow the same pattern.

Look at the first formula, which is the one that includes the angle C. For the angle in the formula, we always use its corresponding side on the other side of the formula. So the formula with angle C is set equal to c^2. The other two sides, a and b, are only the right side as $a^2 + b^2 - 2ab$. For the formula that includes the angle B, we have the matching b^2 on the left side, and then the other two sides a and c follow the same $a^2 + c^2 - 2ac$ pattern on the right.

In other words, these three formulas are actually all the same and can be used interchangeably. All that matters is that we get the correct relationships between the angles and sides.

Second, we want to notice what happens when we use a 90° angle. We'll plug into the first formula, and we get

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = a^2 + b^2 - 2ab\cos 90^\circ$$

$$c^2 = a^2 + b^2 - 2ab(0)$$

$$c^2 = a^2 + b^2$$

$$a^2 + b^2 = c^2$$

We've arrived at the Pythagorean theorem, which makes sense. Because $C = 90^\circ$, we have a right triangle, and the Pythagorean theorem gives us the relationship between the side lengths in a right triangle.

Lastly, these law of cosines formulas can be solved for the cosine functions as

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos B = \frac{a^2 - b^2 + c^2}{2ac}$$

$$\cos A = \frac{-a^2 + b^2 + c^2}{2bc}$$

Sometimes it'll be more convenient to apply the formulas in this form, which is why it's nice to know them. To remember the formulas this way, realize that the negative sign attaches to the side length in the numerator on the right side that corresponds to the angle from the left side.

Let's look at an SSS example where we use the law of cosines to complete a triangle when we know all three side lengths.

Example

Solve the triangle that has side lengths 11, 6, and 9.

We'll plug what we know into the law of cosines in order to find one angle in the triangle. If we let $a = 11$, $b = 6$, and $c = 9$, then for angle A we'll get

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$11^2 = 6^2 + 9^2 - 2(6)(9)\cos A$$

$$121 = 36 + 81 - 108\cos A$$

$$4 = -108\cos A$$

$$\cos A = -\frac{1}{27}$$

$$\cos A \approx -0.0370$$

Solve for A by applying the inverse cosine function to both sides.

$$A \approx \arccos(-0.0370)$$

$$A \approx 92^\circ$$

The easiest way to get the next angle will be to plug into the law of sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 92^\circ}{11} = \frac{\sin B}{6} = \frac{\sin C}{9}$$

Use the first and second parts of this three-part equation to solve for the angle B.

$$\frac{\sin 92^\circ}{11} = \frac{\sin B}{6}$$

$$\sin B = \frac{6 \sin 92^\circ}{11}$$

$$\sin B \approx 0.5451$$

Solve for B by applying the inverse sine function to both sides.

$$B \approx \arcsin(0.5451)$$

$$B \approx 33^\circ$$

Remember that once we have the first two angles, we can simply find the third angle by subtracting the first two from 180°

$$C = 180^\circ - A - B$$

$$C \approx 180^\circ - 92^\circ - 33^\circ$$

$$C \approx 55^\circ$$

To summarize, the given triangle is defined by $a = 11$, $b = 6$, $c = 9$, $A \approx 92^\circ$, $B \approx 33^\circ$, and $C \approx 55^\circ$.

Let's look at another example where only the three side lengths are known. This time though, the triangle won't be defined.

Example

Solve the triangle with side lengths 18, 15, and 2.

Let $a = 18$, $b = 15$, and $c = 2$, plugging into the law of cosines formula that's solved for $\cos A$. We choose the $\cos A$ formula because a is the longest side, so A will be the largest angle.

$$\cos A = \frac{-a^2 + b^2 + c^2}{2bc}$$

$$\cos A = \frac{-18^2 + 15^2 + 2^2}{2(15)(2)}$$

$$\cos A = \frac{-324 + 225 + 4}{30}$$

$$\cos A = -\frac{95}{30}$$

$$\cos A \approx -3.17$$

But remember, the cosine function has a range of $[-1,1]$. So it's impossible to find $\cos A \approx -3.17$, which means that the triangle with the given side lengths actually can't exist.

Finally, let's do an SAS example where we know the length of two sides and the measure of their "included angle," which is the angle between those two sides.

Example

Solve the triangle where two of the sides are 25 and 21 and the measure of their included angle is 70°.

We'll start by using the law of cosines to the find length of the third side. Let $a = 25$ and $b = 21$. The included angle will then be $C = 70^\circ$.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 25^2 + 21^2 - 2(25)(21)\cos 70^\circ$$

$$c^2 = 625 + 441 - 1{,}050\cos 70^\circ$$

$$c^2 = 1{,}066 - 1{,}050\cos 70^\circ$$

$$c^2 \approx 1{,}066 - 1{,}050(0.342)$$

$$c^2 \approx 1{,}066 - 359$$

$$c^2 \approx 707$$

$$c \approx \sqrt{707}$$

$$c \approx 26.6$$

Now that we have this one side length, we'll use the law of sines to find another angle.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{25} = \frac{\sin B}{21} = \frac{\sin 70^\circ}{26.6}$$

Use the first and third parts of this three-part equation to find the measure of angle A.

$$\frac{\sin A}{25} = \frac{\sin 70^\circ}{26.6}$$

$$\sin A = \frac{25 \sin 70^\circ}{26.6}$$

$$\sin A \approx 0.8832$$

$$A \approx \arcsin(0.8832)$$

$$A \approx 62^\circ$$

Now find the measure of the third angle.

$$B = 180^\circ - A - C$$

$$B = 180^\circ - 62^\circ - 70^\circ$$

$$B = 48^\circ$$

To summarize, the given triangle is defined by $a = 25$, $b = 21$, $c \approx 26.6$, $A \approx 62^\circ$, $B = 48^\circ$, and $C = 70^\circ$.