Express the polar point multiple ways

As mentioned in an earlier lesson, every point has infinitely many pairs of polar coordinates. This is easy to see in the case of the pole, because for every angle θ, $(0,\theta)$ is a pair of polar coordinates of the pole. (The pole is at the vertex of every angle θ in standard position, so the pole lies on the terminal side of every angle θ.)

Since the terminal sides of any two angles that differ in measure by an integer multiple of 2π coincide, you might have guessed that if (what we're calling) the basic polar coordinates of a point P are (r,θ), that is, if r is positive and θ is in the interval $[0,2\pi)$, then for every integer n, $(r,\theta+2n\pi)$ is a pair of polar coordinates of P. Since $2n$ is an even integer, another way of stating this is that for every even integer n, $(r,\theta+n\pi)$ is a pair of polar coordinates of P.

For example, for the point P whose basic polar coordinates are $(3,\pi/4)$, the following are pairs of polar coordinates of P:

$$\ldots, \left(3,-\frac{23\pi}{4}\right), \left(3,-\frac{15\pi}{4}\right), \left(3,-\frac{7\pi}{4}\right), \left(3,\frac{\pi}{4}\right), \left(3,\frac{9\pi}{4}\right), \left(3,\frac{17\pi}{4}\right), \left(3,\frac{25\pi}{4}\right), \ldots$$

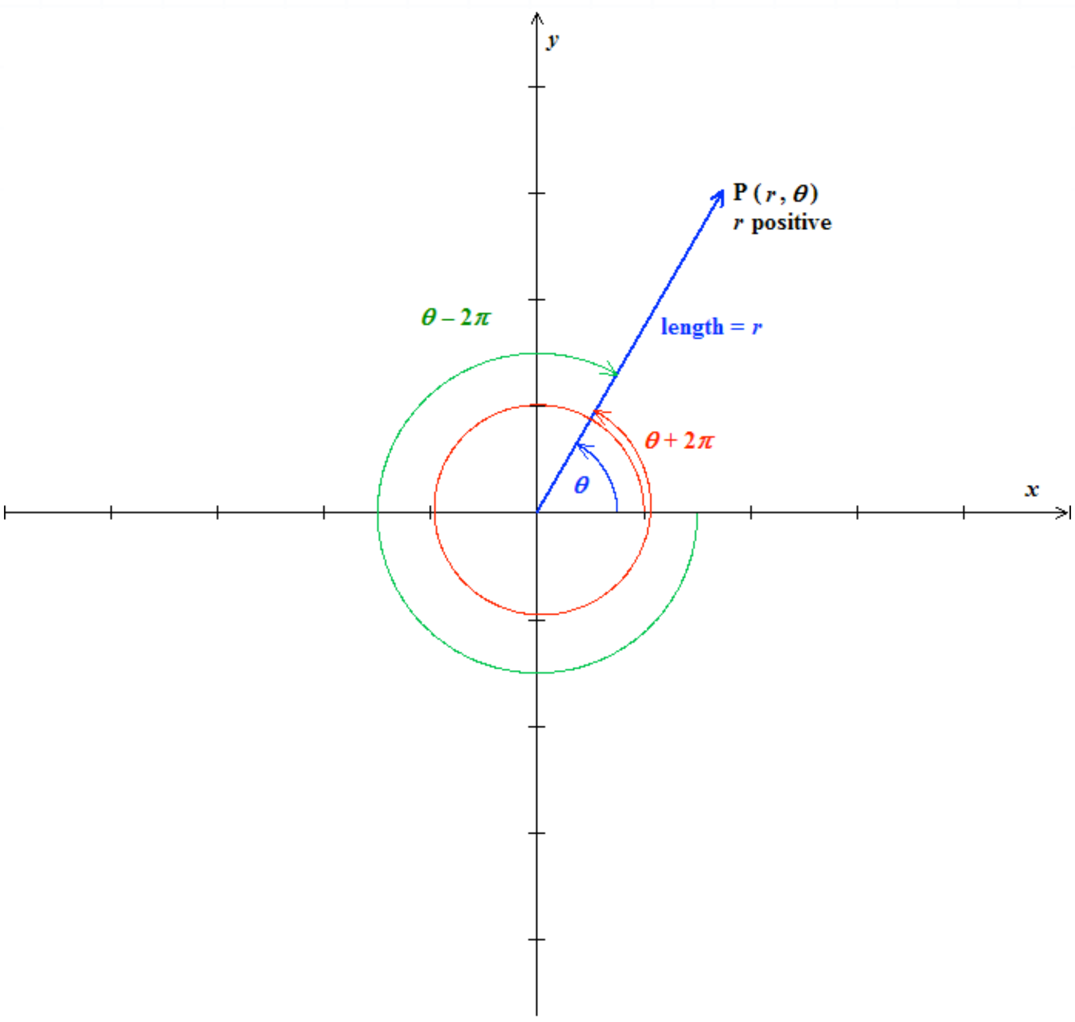
Now there's something that you may not have guessed, and that is that every point other than the pole has infinitely many pairs of polar coordinates with r negative. In particular, if the basic polar coordinates of a point P are (r,θ), that is, if r is positive and θ is in the interval $[0,2\pi)$, then for every integer n, $(-r,\theta+(2n+1)\pi)$ is a pair of polar coordinates of P. Since $2n+1$ is an odd integer, another way of stating this is that for every odd integer n, $(-r,\theta+n\pi)$ is a pair of polar coordinates of P.

For example, if P is the point whose basic polar coordinates are $(3,\pi/4)$, the following are the pairs of polar coordinates of P with a negative value of r:

$$\dots,\left(-3,-\frac{19\pi}{4}\right),\left(-3,-\frac{11\pi}{4}\right),\left(-3,-\frac{3\pi}{4}\right),\left(-3,\frac{5\pi}{4}\right),\left(-3,\frac{13\pi}{4}\right),\left(-3,\frac{21\pi}{4}\right),\dots$$

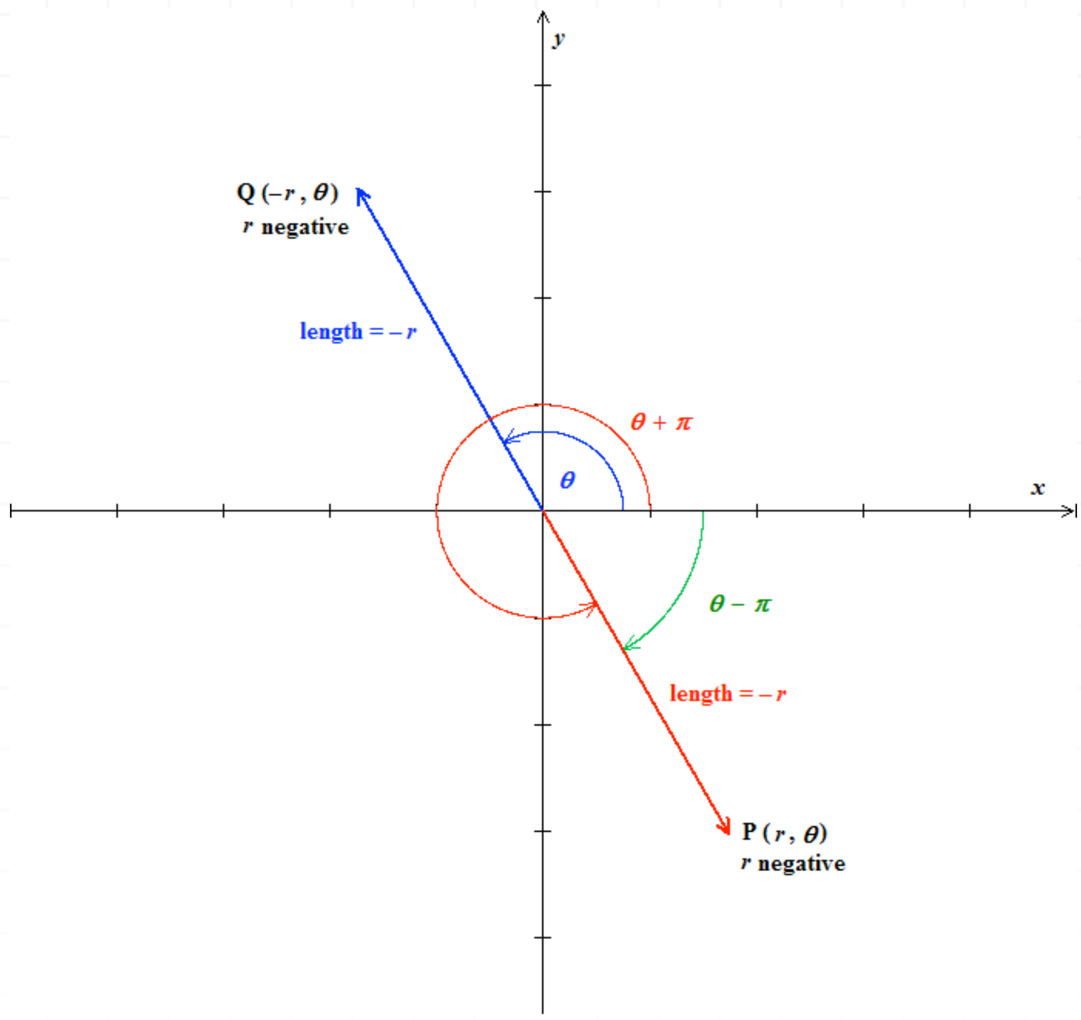
Here's how to graph a point P other than the pole by using a pair of polar coordinates (r,θ) of P such that r is positive. If you draw a line segment of length r, starting at the pole and going along the terminal side of the angle θ, then P is the point at the end of that segment. Note that P lies on the terminal side of every angle $\theta+2n\pi$ where n is an integer (including on the terminal side of the angles $\theta+2\pi$ and $\theta-2\pi$).

In the figure, we illustrate this for a point P in the first quadrant.



Here's a way to graph a point P other than the pole by using a pair of polar coordinates (r, θ) of P with r negative: First, graph the point Q whose polar coordinates are $(-r, \theta)$, including drawing a line segment of length $-r$, starting from the pole and going along the terminal side of the angle θ. (Since r is negative, $-r$ is positive, so it makes sense to say "of length $-r$.") The point at the end of that segment is Q. Then draw another line segment of length $-r$, this time starting at the pole and going along the terminal side of the angle $\theta + \pi$. The point at the end of that segment is point P. (Note that P is also on the terminal side of the angle $\theta - \pi$.) Thus P is at a distance of $-r$ units from the pole, and at the end of the line segment which is in the direction opposite that of the segment you drew from the pole to point Q, so $(-r, \theta + \pi)$ is a pair of polar coordinates of P in which the first coordinate ($-r$) is positive. Note that P lies on the terminal side of every angle $\theta + (2n + 1)\pi$ where n is an integer.

In the figure, we illustrate this for a point P in the fourth quadrant, so point Q is in the second quadrant and (because r is negative) the terminal side of angle θ passes through Q (not through P).



You may be wondering how to get the rectangular coordinates (x, y) of a point P from a pair of its polar coordinates in which the first coordinate is negative.

First, recall that if (r, θ) are the basic polar coordinates of a point P other than the pole, then r is positive and the following hold:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Those equations hold for the rectangular coordinates (x, y) for any other pair of polar coordinates of P in which the first coordinate is positive, because the angle in that other pair must differ from the angle θ by an integer multiple of 2π. Thus the values of the cosine function for the two angles are identical, as are the values of the sine function for the two angles.

As it turns out (and strange though it may seem), the equations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

also yield the rectangular coordinates (x, y) of a point P that has polar coordinates (r, θ) with r negative.

To see this, recall that if a point P has polar coordinates (r, θ) with r negative, then $(-r, \theta + \pi)$ is a pair of polar coordinates of P in which the first coordinate $(-r)$ is positive. Thus by the formulas given above, the rectangular coordinates (x, y) of P are

$$x = (-r)\cos(\theta + \pi)$$

$$y = (-r)\sin(\theta + \pi)$$

By the sum identities for sine and cosine:

$$\cos(\theta + \pi) = (\cos\theta)(\cos\pi) - (\sin\theta)(\sin\pi) = (\cos\theta)(-1) - (\sin\theta)(0) = -\cos\theta$$

$$\sin(\theta + \pi) = (\sin\theta)(\cos\pi) + (\cos\theta)(\sin\pi) = (\sin\theta)(-1) + (\cos\theta)(0) = -\sin\theta$$

Substituting these results, we get

$$x = (-r)\cos(\theta + \pi) = (-r)(-\cos\theta) = r\cos\theta$$

and

$$y = (-r)\sin(\theta + \pi) = (-r)(-\sin\theta) = r\sin\theta$$

Example

Find the rectangular coordinates (x, y) of a point that has polar coordinates $(-6{,}2\pi/5)$.

Here, $r = -6$ and $\theta = 2\pi/5$. By the formulas given above,

$$x = r\cos\theta = -6\cos\left(\frac{2\pi}{5}\right)$$

$$y = r\sin\theta = -6\sin\left(\frac{2\pi}{5}\right)$$

Use of a calculator tells us that $\cos(2\pi/5) \approx 0.309$ and $\sin(2\pi/5) \approx 0.951$. Therefore,

$$x \approx -6(0.309) \approx -1.85$$

$$y \approx -6(0.951) \approx -5.71$$

Notice that both x and y are negative, which means that P is in the third quadrant. This is precisely what we would expect, because P also has polar coordinates

$$(-r, \theta + \pi) = \left(6, \frac{2\pi}{5} + \pi\right) = \left(6, \frac{1(2\pi) + 5(\pi)}{5}\right) = \left(6, \frac{2\pi + 5\pi}{5}\right) = \left(6, \frac{7\pi}{5}\right)$$

and (the terminal side of) an angle of $7\pi/5$ radians is in the third quadrant.

It should now be clear that the angles θ (the second coordinates) in any two pairs of polar coordinates for the same point must differ by some integer multiple of π. Furthermore, the values of r (the first coordinates) must be of the same sign if the angles θ differ by $n\pi$ for some even integer n, and of opposite sign if the angles θ differ by $n\pi$ for some odd integer n.

Example

If one pair of polar coordinates of a point is $(14,31\pi/7)$, find the pair of polar coordinates (r,θ) of that point if θ is in the interval $[\pi,2\pi)$.

If (r,θ) is the pair of coordinates for this point such that θ is in the interval $[\pi,2\pi)$, then θ must differ from $31\pi/7$ by $n\pi$ for some integer n. Note that

$$4\pi = \frac{28\pi}{7} < \frac{31\pi}{7} < \frac{35\pi}{7} = 5\pi$$

Now

$$\pi = 4\pi - 3\pi, \qquad 2\pi = 5\pi - 3\pi$$

Therefore,

$$\theta = \frac{31\pi}{7} - 3\pi = \frac{1(31\pi) - 7(3\pi)}{7} = \frac{31\pi - 21\pi}{7} = \frac{10\pi}{7}$$

Since 3 is an odd integer and the first coordinate in the given pair of polar coordinates of this point is 14 (hence positive), r must be negative (hence $r = -14$). Thus the indicated pair of polar coordinates of this point is $(-14,10\pi/7)$.

Example

If one pair of polar coordinates of a point is $(-20, -18\pi/11)$, find the pair of polar coordinates (r,θ) of that point if θ is in the interval $[6\pi,7\pi)$.

If (r,θ) is the pair of coordinates for this point such that θ is in the interval $[6\pi,7\pi)$, then θ must differ from $-18\pi/11$ by $n\pi$ for some integer n. Note that

$$-2\pi = -\frac{22\pi}{11} < -\frac{18\pi}{11} < -\frac{11\pi}{11} = -\pi$$

Now

$$6\pi = -2\pi + 8\pi, \qquad 7\pi = -\pi + 8\pi$$

Therefore,

$$\theta = -\frac{18\pi}{11} + 8\pi = \frac{1(-18\pi) + 11(8\pi)}{11} = \frac{-18\pi + 88\pi}{11} = \frac{70\pi}{11}$$

Since 8 is an even integer and the first coordinate in the given pair of polar coordinates of this point is -20 (hence negative), r must also be negative (hence $r = -20$). Thus the indicated pair of polar coordinates of this point is $(-20,70\pi/11)$.

If you're given the rectangular coordinates (x,y) of a point and you want to get the pair of its polar coordinates with θ in a specified interval, you can first convert its rectangular coordinates to its basic polar coordinates (as you learned to do in an earlier lesson) and then apply techniques such as those used in the two examples above to get its polar coordinates (r, θ) with θ in a specified interval.