Designing And Analysing Wave-Guides Made Of Metagradinds

Lior Bar Hillel and Eran Mann

February 16, 2022

Abstract

In this report we will generalize the calculation done in [3] for N particles. After that will use the generalization in order to create an anomalous reflector, and check our design in an EM simulatiom software.After that will analyse a waveguide made of two of these Reflective surfaces, made of metagrating, to check the fusibility of such a waveguide.

The full MATLAB code used to simulate this project is available in Githab.

1 designing a spatial reflective surface using methagrading

1.1 Mathematical modeling of N layers of metagradings

1.1.1 our system

We shall work on a $2D$ system ($\frac{\partial}{\partial x} = 0$), and the electric field involved will be TE ($E_z = E_y = H_x = 0$). Whenever we consider electric field E or current I in this document, we assume a time dependence of $e^{j\omega t}$,where ω is the radial frequency. Electromagnetic wave comes to the $\hat{z}$ direction with angel θ_{in}. The system is composed of N layers of metagradings, where each layer is composed of infinite wires to the $\hat{x}$ direction with distance Λ to the $\hat{y}$ direction between them. The m's layer has a wire in the coordinates $(y, z) = (d_m, h_m)$ with Z_m impedance per unit length. The system is shown in figure 1.

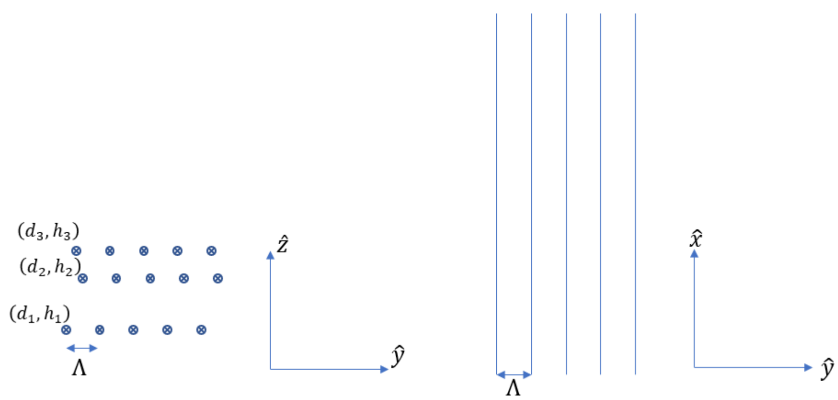

Figure 1: Schematic drawing of the system

1.1.2 Calculation of the electric field in 2D space

Here we follow a similar calculation done in [4]. The electric field generated by a single wire in location (y_0, z_0) and current I is

$$E_x^{wire}(y, z) = -\frac{k\eta}{4} I_0 H_0^{(2)}\left(k\sqrt{(y - y_0)^2 + (z - z_0)^2}\right) \tag{1}$$

Where $H_0^{(2)}$ is the Henkels function of the second, $k = \omega/c$ is the wave number and η is the wave impedance of vacuum. In our case the electromagnetic wave entering the system is

$$E_{in}(y,z) = E_0 e^{-jk(\sin\theta_{in} y + \cos\theta_{in})} \tag{2}$$

And so we have a phase delay of $\varphi = -k\Lambda \sin\theta_{in}$ between the currents. As a result, the total Electric field created by of the m's layer is

$$E_x^m(y,z) = \sum_{n=-\infty}^{\infty} I_0 e^{j\varphi n} H_0^{(2)}\left(k\sqrt{(y-(d_m+\Lambda n))^2 + (z-h_m)^2}\right) \tag{3}$$

Using the Poisson equation

$$\sum_{n=-\infty}^{\infty} f(n\Lambda) = \sum_{s=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(l)\, e^{-j\frac{2\pi s}{\Lambda}}}{\Lambda} dl \tag{4}$$

And the following integral

$$\int_{-\infty}^{\infty} H_0^{(2)}(k\sqrt{(y-(d_m+l))^2 + (z-h_m)^2}) e^{-j\frac{2\pi s}{\Lambda}l} e^{j\frac{\varphi}{\Lambda}l} dl = 2\frac{e^{-j\alpha_s(y-d_m)} e^{-j\beta_s|z-h_m|}}{\beta_s} \tag{5}$$

We get the total electric field created

$$E_x(y,z) = -\frac{k\eta}{2\Lambda} \sum_{m=1}^{N} I_m \sum_{n=-\infty}^{\infty} \frac{e^{-j\alpha_n(y-d_m)} e^{-j\beta_n|z-h_m|}}{\beta_n} \tag{6}$$

Where $\alpha_n = \frac{2\pi s - \varphi}{\Lambda}$ and $\beta_n = \sqrt{k^2 - \alpha_n^2}$ (if β is complex the its imaginary part is defined to be negative).

1.1.3 Finding the electric field on the wires

In order to find the actual currents we need to consider the electric field on the wires to calculate $I_m = \frac{E_m}{Z_m}$. In order to find the electric field on the m's wire we need to find the external field, the field created by the other wire, and the field that the wire creates on itself. Will assume that there is a geometric factor of the wires r_{eff} so that the self induced field is

$$E_{self} = -\frac{k\eta}{4} I_M H_0^{(2)}(kr_{eff}) \tag{7}$$

Since $r_{eff}k \ll 1$ it can be written as

$$E_{self} = -\frac{k\eta}{4} I_M \ln(kr_{eff}) \tag{8}$$

The rest of the fields induced on the wire by it's layer is

$$E_{layer} = -\frac{k\eta}{4} I_M \sum_{n=-\infty, n\neq 0}^{\infty} e^{j\varphi n} H_0^{(2)}(k|\Lambda n|) \tag{9}$$

We shall use the fact that $H_0^{(2)}(x) = J(x) - jY(x)$ and the summations from the book [2] in order to get to the total electric field on the wire

$$\begin{aligned} E_{wire}^{tot} = Z_m I_m = E_0 e^{-jk(\sin\theta_{in} y + \cos\theta_{in})} + j\frac{k\eta}{2\pi} I_M \ln\frac{2\pi r_{eff}}{\Lambda} - \frac{k\eta}{2\Lambda\beta_0} I_M - \frac{k\eta}{2} \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{1}{\Lambda\beta_n} - \frac{1}{2\pi|n|}\right) \\ - \frac{k\eta}{2\Lambda} \sum_{m=1, m\neq M}^{N} I_m \sum_{n=-\infty}^{\infty} \frac{e^{-j\alpha_n(y-d_m)} e^{-j\beta_n|z-h_m|}}{\beta_n} \end{aligned} \tag{10}$$

From this we can find the currents if we know all the other parameters, or we can calculate the impedance array from the currents.

1.2 Designing an anomalous reflector

1.2.1 design specifications

We shall design a system that for a specific electromagnetic wave entering the system with angle of indecent θ_{in} the system will reflect it perfectly to a different wave with angle θ_{out}. We also want a phase difference of ϕ between the original and reflected wave, and no energy loss.

We arbitrary chose $\theta_{in} = 10^o$, $\theta_{out} = 70^o$ and $\phi = 25^o$.

1.2.2 finding Λ

The symmetry of the system demands that a movement of Λ to the $\hat{y}$ direction will only result in a phase change. For that to happen we consider the sum of the original and reflected wave at $z = 0$ and at $y = 0$, $y = \Lambda$

$$1 + A = (e^{-jk\Lambda \sin\theta_{in}} + Ae^{-jk\Lambda \sin\theta_{out}})e^{j\alpha}$$

For some real number α, where A is the ratio between the waves amplitudes.

This can happen only if

$$\Lambda = \frac{2\pi}{k}\frac{1}{|\sin\theta_{in} - \sin\theta_{out}|} \tag{11}$$

1.2.3 Calculating the non fading modes

For a mode of the calculation to not fade, β_n must be a real number. In our case this means that $n = 0, 1, -1$. We shall write the relevant field

$$\begin{aligned} E(y,z) = E_0 e^{-jk(\sin\theta_{in} y + \cos\theta_{in} z)} - \frac{k\eta}{2\Lambda}\sum_{m=1}^{N} I_m \Bigg(&\frac{e^{-j\alpha_0(y-d_m)}e^{-j\beta_0|z-h_m|}}{\beta_0} \\ + \frac{e^{-j\alpha_1(y-d_m)}e^{-j\beta_1|z-h_m|}}{\beta_1} + \frac{e^{-j\alpha_{-1}(y-d_m)}e^{-j\beta_{-1}|z-h_m|}}{\beta_{-1}} \Bigg)& \end{aligned} \tag{12}$$

It is important to notice that $\beta_0 = k\cos\theta_{in}$ and so in $z \to \infty$ it is the same mode as E_{in}, and $\beta_{-1} = k\cos\theta_{out}$ so for $z \to -\infty$ it is the wanted mode.

1.2.4 Finding the reflected wave amplitude

The reflected wave should be

$$E_{out}(y,z) = E_1 e^{j\phi} e^{-jk(\sin\theta_{out} y - \cos\theta_{out} z)} \tag{13}$$

We shall equalize the power getting in the system and the power getting out of it

$$\frac{P_{out}}{P_{in}} = 1 = \frac{E_1^2 \cos\theta_{out}}{E_0^2 \cos\theta_{in}}$$

and so we get

$$E_1 = E_0\sqrt{\frac{\cos\theta_{out}}{\cos\theta_{in}}} \tag{14}$$

1.2.5 Conditions for perfect reflection

We shall require all the non relevant mode to vanish, and the relevant reflected mode to equal in energy to the entering mode to get the following set of equations:

$$
\begin{aligned}
&\sum_{m=1}^{N} I_m e^{j(\alpha_0 d_m + \beta_0 h_m)} = \frac{2}{Z_{in}} E_0 \\
&\sum_{m=1}^{N} I_m e^{j(\alpha_1 d_m + \beta_1 h_m)} = 0 \\
&\sum_{m=1}^{N} I_m e^{j(\alpha_{-1} d_m + \beta_{-1} h_m)} = 0 \\
&\sum_{m=1}^{N} I_m e^{j(\alpha_0 d_m - \beta_0 h_m)} = 0 \\
&\sum_{m=1}^{N} I_m e^{j(\alpha_1 d_m - \beta_1 h_m)} = -\frac{2}{Z_{out}} E_1 e^{j\phi} \\
&\sum_{m=1}^{N} I_m e^{j(\alpha_{-1} d_m + \beta_{-1} h_m)} = 0
\end{aligned}
\tag{15}
$$

Where $Z_{out} = \frac{\eta}{\cos\theta_{out}}$ and $Z_{in} = \frac{\eta}{\cos\theta_{in}}$. We have here 6 Linear equations for the currents, so we want at least 6 layers to design a reflector.

1.3 Numerical method for designing an anomalous reflector

1.3.1 Objective

We want to find ten lengths $h_{m>1}$, $d_{m>1}$ ($h_1 = d_1 = 0$ as a reference point), and six impedances Z_m such that our system will only allow the wanted mode to pass with no energy loss. For reasons we will reveal later, we want the impadances to be of capacitance type.

1.3.2 Methods

All explained in this section is performed in MATLAB. Firstly we choose the distances h_m and Z_m randomly (within appropriate limits). Than we find (using 15) the currents needed for those distances in order for the system to work. Using 10 we find the appropriate impedances. After having the initial Values we found, we ran a MATLAB optimization program that changes the distances and find a local minimum of our optimization function, which is a waited sum of the power lost on the wires, and the power that would be lost due to the physical structure of the wires. We ran this algorithm for many (about a 1000) random initial points until, until finding a satisfying result, only considering results where $\Im\{Z_m\} < 0$ for every m.

1.3.3 Results

After running the MATLAB code we got a few good results. We decided to use the results seen in 3:

Item	Units	layer 1	layer 2	layer 3	layer 4	layer 5	layer 6
h_m	$[mm]$	0	0.38	12.2	16.8	22.0	26.6
d_m	$[mm]$	0	21.0	5.4	27.0	27.0	28.0
Z_m	$-10^5 j\left[\frac{V}{Am}\right]$	0.183	0.927	0.318	1.348	0.306	0.667

Table 1: Optimization Results

These result come with an expected power lost of 3%. The resulting Electric Field (drawn by without the actual structure of the wire, using MATLAB) is shown in the 2:

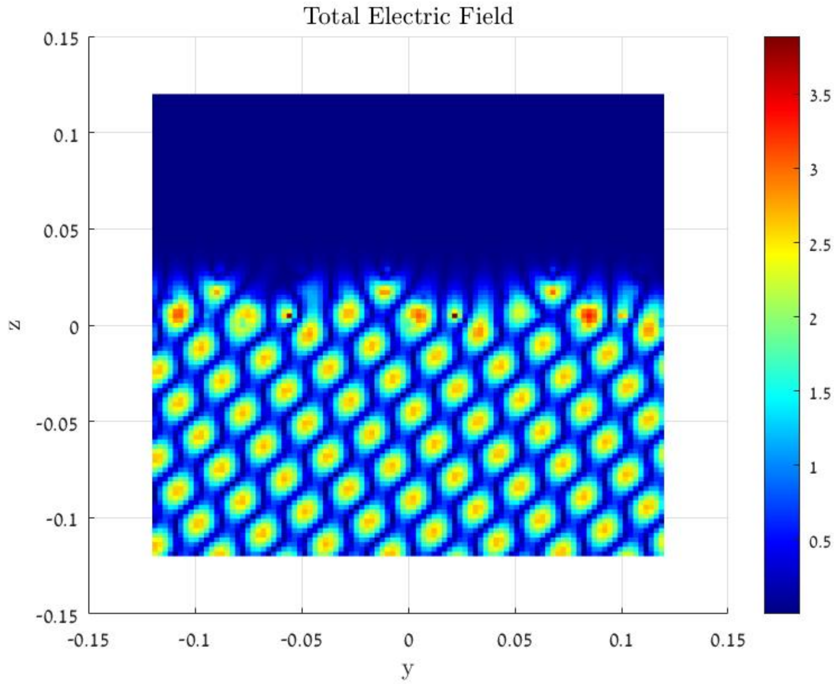

Figure 2: Absolute value of the real part of the expected electric field in the system for $E_0 = 1$

As we can see, there is no refraction, and the reflection seems to be to right direction.

1.4 Electromagnetic simulation of the system

1.4.1 Theory to practice

In order to use our system in the real word, we need to find a good way to create wires with the relevant impedance per unit length. Will use the method shown in [4]. Our wires will be manufactures as shown in 3

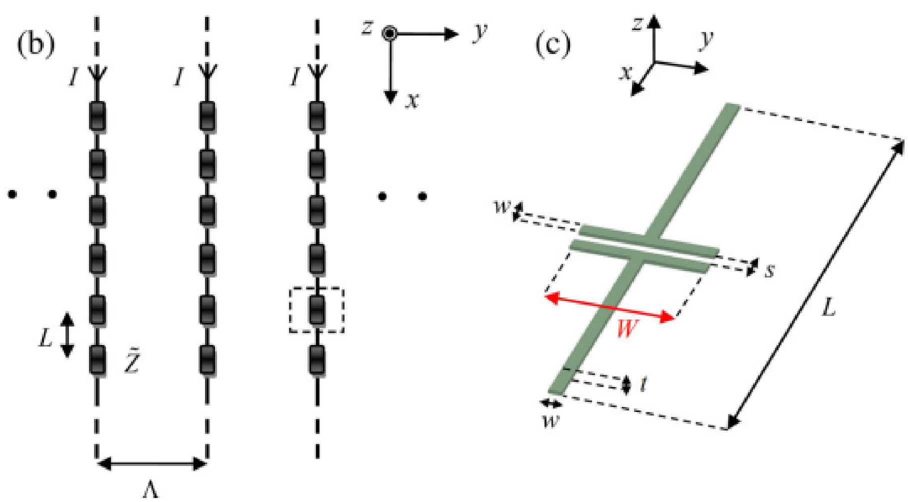

Figure 3: the physical structure of the wires

will use the same manufacturing technique as [4], and so the same physical sizes ($w = s = 3mil$, $t = 18\mu m$, $L = \frac{\lambda}{10}$). The remaining size W is different in each layer, because it controls the impedance. In [4] presented the following relation:

$$W = 2.85 K_{corr} C \left[\frac{mil}{fF}\right] \tag{16}$$

For $f = 10GHz$ the best fit was $K_{corr} = 0.83$. The desired capacitance is

$$C_m = -\frac{1}{2\pi f L \Im\{Z_m\}} \tag{17}$$

to check our results will use a simulation in Ansys electrodynamics.

1.4.2 Simulation results

The resulting magnitude of the electric field seems very similar to our expectations:

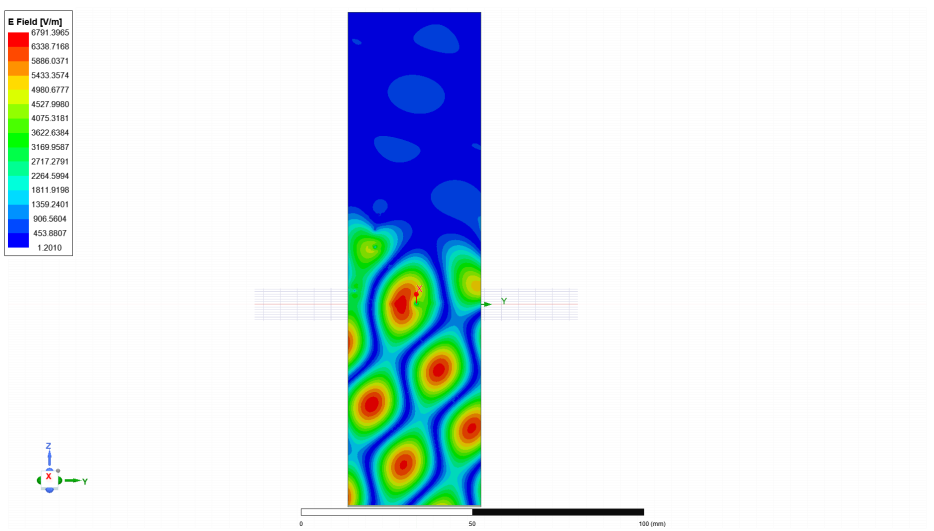

Figure 4: magnitude of simulated electric fields

2 Waveguide Design and investigation

2.1 system introduction

We shall look at a waveguide made of two identical refractive metagradings (similar to the design we have) with some offset D from one another, and a vertical distance $2H$ from one another as shown in 5

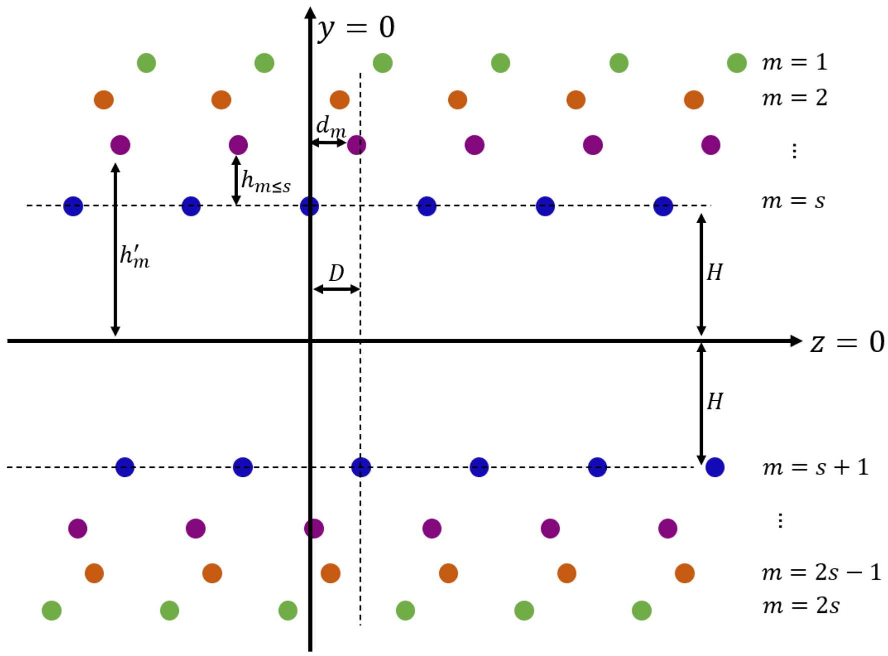

Figure 5: A schematic drawing of the waveguide

The relations between $h_1,\dots,h_s,d_1,\dots,d_s,Z_1,\dots,Z_s$, the parameters of the reflective surface as designed and $h'_1,\dots,h'_{2s},d'_1,\dots,d'_{2s},Z'_1,\dots,Z'_{2s}$, the parameters of the waveguide, is (for $1\le i\le s$):

$$\begin{aligned} h'_i &= H + h_{s+1-i}, \qquad & d'_i &= d_{s+1-i}, \qquad & Z'_i &= Z_{s+1-i} \\ h'_{s+i} &= -H - h_i, \qquad & d'_{s+i} &= D - d_i, \qquad & Z'_{s+i} &= Z_i \end{aligned} \tag{18}$$

2.2 Current source in K_y space

Since it is convenient to solve the reaction of the metagradings layers to plane wave (in a similar formalism to the designing stage), we want to look at the field created by our source as a sum of plane wave. To do that we want to convert our source to the Fourier space. Our source is $\vec{J}(y,z) = I_0\delta(z-z_0)\,\delta(y-y_0)\,\hat{x}$. Will make it's Fourier transform:

$$\vec{J}(k_y,z) = \int_{-\infty}^{\infty} \vec{J}(y,z)\,e^{jk_y y}dy = I_0\delta(z-z_0)\,e^{jk_y y_0}\hat{x} \tag{19}$$

So we can define the current's "spacial phasor"

$$\vec{J}(k_y,z)_{ph} = I_0\delta(z-z_0)e^{-jk_y(y-y_0)}\hat{x} \tag{20}$$

This phasor is a plane of current, and we can calculate it's electric field (in the $\hat{x}$ direction) to be:

$$E_{k_y} = -\frac{k\eta I_0}{2}\frac{e^{-j(k_z|z-z_0|+k_y(y-y_0))}}{k_z} \tag{21}$$

Where $k_z = \sqrt{k^2-k_y^2}$

2.3 waveguide reaction to current source

The method used here is very similar to, and was based on [1] We Know from equation 10 the connection between the currents on the wire to the external electric field. This is a solvable set of equations for the currents which allows us to find the currents $I_m^{norm}(k_y)$, for an external electric field.

$$E_{ext}(y,z) = e^{-j(k_y y + k_z z)} \tag{22}$$

Notice that the electric field is unitless (Amplitude of one), and so the resulting current is normalized (current per electric field).

The matrix equation that will use to numerically solve the current for each plane wave, derived from equation 10 is:

$$A\left(k_y\right) I_{norm}\left(k_y\right) = V\left(k_y\right) \tag{23}$$

The matrix A has a diagonal:

$$A_{MM} = Z_M - j\frac{k\eta}{2}\left(\frac{1}{\pi}\ln\left(\frac{2\pi r_{eff}}{\Lambda}\right) + \frac{1}{\Lambda\beta_0} + \sum_{n=-\infty}^{\infty}\left[\frac{1}{\Lambda\beta_n} - \frac{j}{2\pi\left|n\right|}\right]\right) \tag{24}$$

From the current, and using equation 6, we can find the electric field resulting in space. From [1] We know that a current source of $I_0\delta\left(y-y_0\right)\delta\left(z-z_0\right)$, (a wire in $(y,z)=(y_0,z_0)$) creates the following electric field in

and the rest of it:

$$A_{Mm} = \frac{k\eta}{2\Lambda}\sum_{n=-\infty}^{\infty}\frac{1}{\beta_n}e^{-j(\alpha_n(d_M-d_m)+\beta_m|h_M-h_m|)} \tag{25}$$

And the vector V:

$$V_M = e^{-j(k_y d_M + k_z h_M)} \tag{26}$$

After numerically inverting the matrix and solving for the current, we can sum it with the relevant weights (21) to find the total field generated by the waveguide in response of the current source:

$$E_{gen}(y,z) = \frac{k^2\eta^2 I_0}{4\Lambda}\int_{-\infty}^{\infty}\frac{e^{-j(k_z|z-z_0|+k_y(y-y_0))}}{k_z}\sum_{m=1}^{N} I_m^{norm}\left(k_y\right)\sum_{n=-\infty}^{\infty}\frac{e^{-j(\alpha_n(y-d_m)+\beta_n|z-h_m|)}}{\beta_n}dk_y \tag{27}$$

This can be numerically calculated.

2.4 interesting cases and expectations

Will look at the special angels for which the refractive surface was designed specifically to reflect, and will assume perfect reflection of light as a geometric optic beam of light. The journey this light beam goes through is as shown in figure 6.

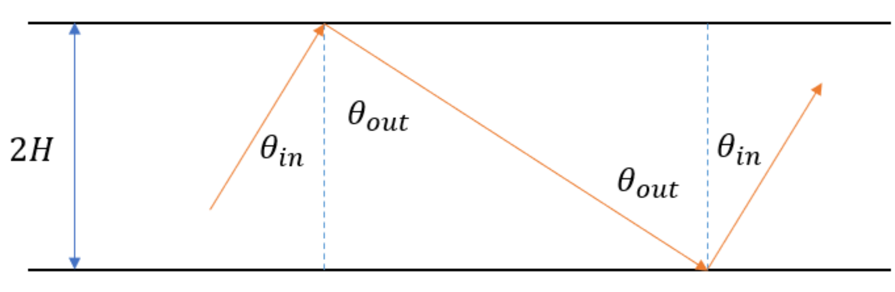

Figure 6: schematic drawing of a light beam in a waveguide

As we can see, the path done by the beam is

$$d = 2H\left(\frac{1}{\cos\theta_{in}} + \frac{1}{\cos\theta_{out}}\right) \tag{28}$$

In order to have a constructive interference of this mode, we need this distance to contain a whole number of wave lengths, and for the interference to be destructive it needs to contain a half number of wave lengths (The phase difference does not play a roll because it reverses if we between the reflection from θ_{in} to θ_{out} to the reflection from θ_{out} to θ_{in}). This allows us to choose two different interesting sizes for H, one where we expect the relevant mode to be dominate, and one to be neglected:

We will choose the distance $D = 0$, since its impact is not clear, and does not impact our prediction.

$$
\begin{aligned}
H_{\text{const}} &= \frac{\lambda \cos\theta_{in} \cdot \cos\theta_{out}}{2 \cdot (\cos\theta_{in} + \cos_{out})} \\
H_{dist} &= \frac{\lambda \cos\theta_{in} \cdot \cos\theta_{out}}{4 \cdot (\cos\theta_{in} + \cos_{out})}
\end{aligned}
\tag{29}
$$

In the case of the constructive interference we expect this mode (an interference pattern of waves up in θ_{in} and down in θ_{out}, similar to shown in figures 2 and 4) will be conducted well in the waveguide, and in a certain distance from the current source, the pattern will be apparent (all the other modes will decay). For the case of the destructive interference, we expect this mode to decay very fast, so the pattern will see far away (if the electric field wont decay completely), wont include this interference pattern.

2.5 Simulation results and discussion

2.5.1 waveguide simulation

will simulate first the constructive interference waveguide, using equation 27. Picture shows the Real part of the electric field generated by the waveguise in response to the particle not including the field generated by the particle itself. The y,z axis are normalized by the inner wavegude length H.

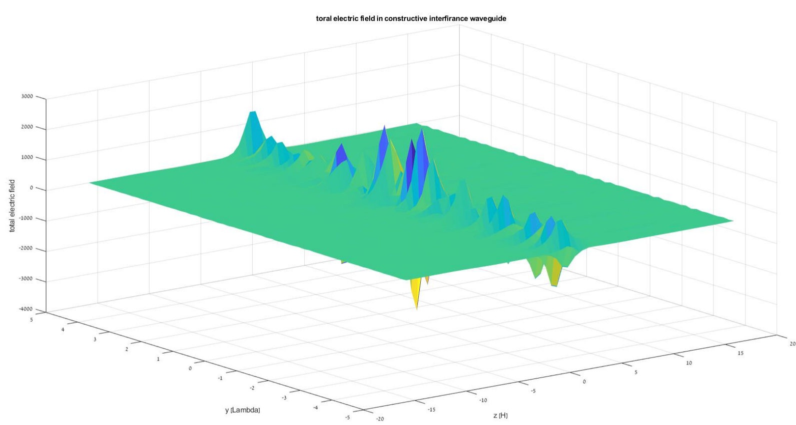

Figure 7: The total electric field in a constructive Waveguide in response to current source in the middle

Now lets look at the firs "destructive interference" waveguide:

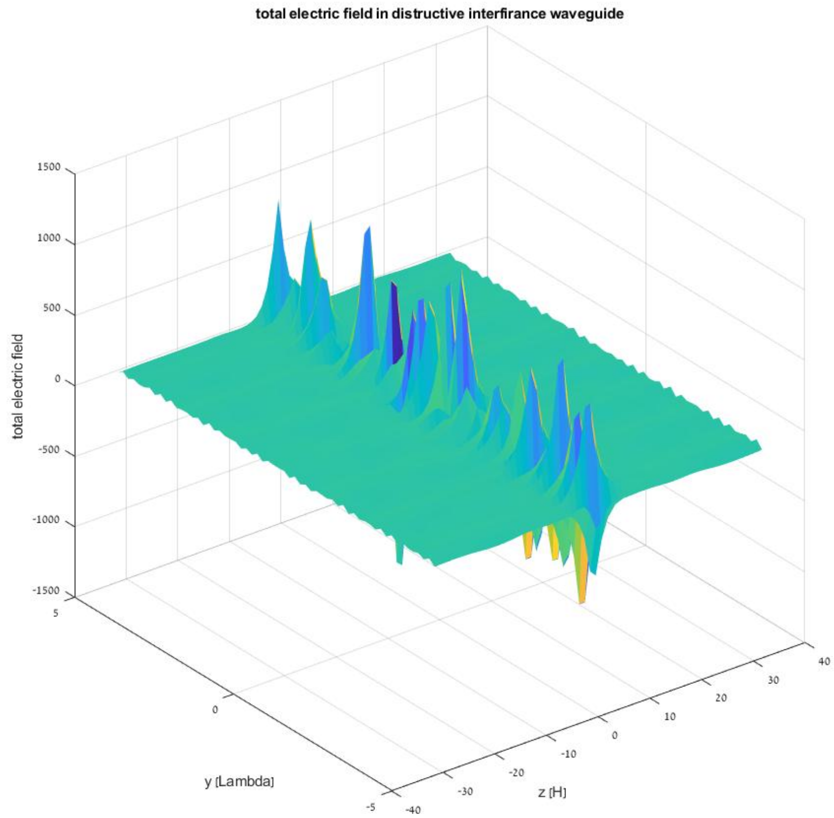

Figure 8: The total electric field in a destructive Waveguide in response to current source in the middle

We can see that here as well there are propagating modes (thou they are about 3 times weaker).This means that either there are several different angles that are reflected well from our original layer, or there is something else interesting going on here.

In order to farther investigate will search for self sustaining modes, by looking at waves that can exist with zero input - modes that the A matrix isn't ineverable

It is clear that there is a propagating wave mode (or modes) in the waveguide. In order to further examine it's behavior we shall look at the magnitude of the field at several cross sections of the waveguide:

2.5.2 examination of modes

We have equation 23 to calculate the currents in space exited by a plane wave of ω and k_y. This is a matrix equation which have 1 solution for A matrix with non zero determinant. For $det(A) = 0$ there is a family of self sustaining solutions, which can exist without any excitation, and can infinitely propagate with excitation. We would like to find those modes if existing, or modes that are close to that, which decay slowly.

For that reason we shall plot $\det(A)$ as a function of k_y, and other interesting waveguide parameters, such as H and D. In order to be able to notice between different mode with wildly different determinant, will plot $\log_{10}(\det(A))$:

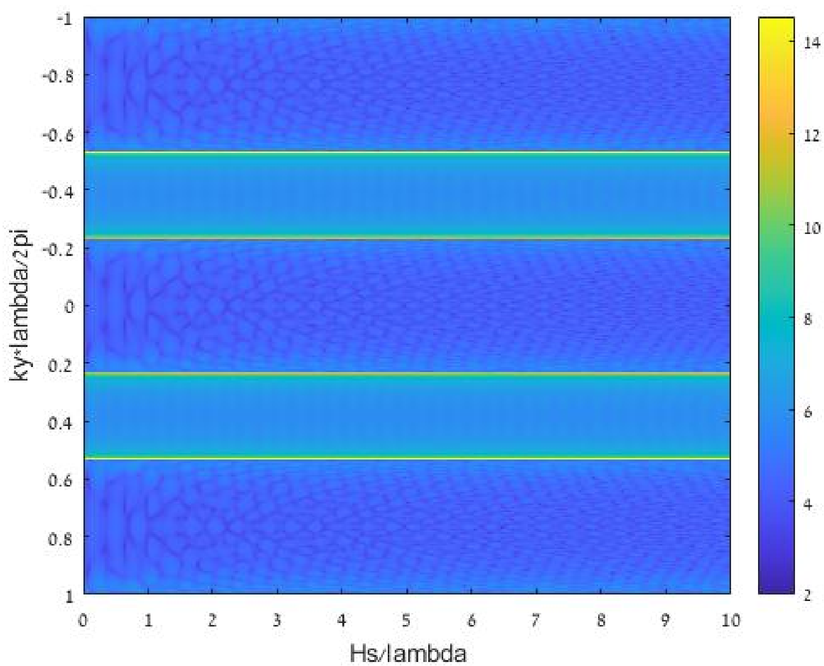

Figure 9: log of determinant of the A matrix as a function of H and k_y for $D = 0$

we can see that there is a symmetry about the $k_y = 0$. we can see 2 "forbidden" areas for k_y, and 3 areas in which there can be guided modes, depending on the distance. We also that for small H there are fewer guided modes than for large H.

Our expected constructive interference length is $H_{const} = 0.12n\lambda$

will look at a more closeup picture for small H, normalized by H_{const}

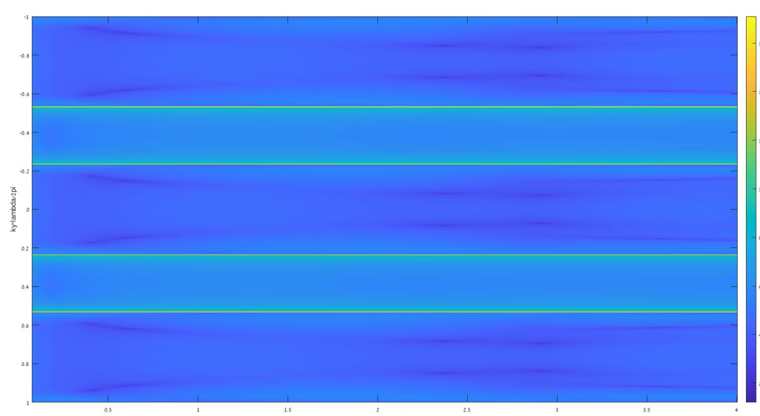

Figure 10: log of determinant of the A matrix as a function of H and k_y for $D = 0$, for small Hs normalized by the "constructive length"

This unfortunately does not gives ass much incite. We can notice an unexpected symetry between the center and edges of k_y In order to explore more Will also look at the determinant as a function of D for few intresting cases: $H = \lambda$

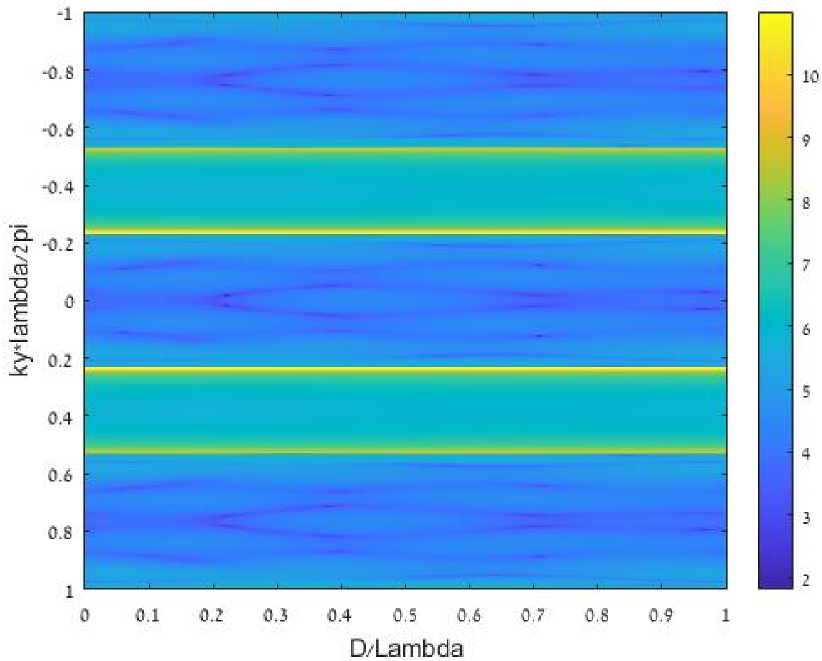

Figure 11: log of determinant of the A matrix as a function of D and k_y for $H = \lambda$

$H = H_{const}$

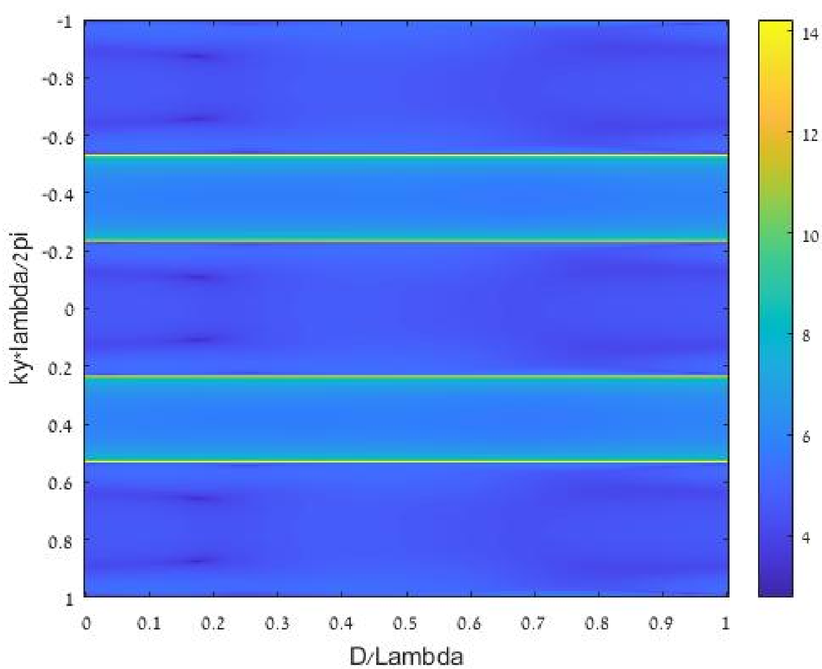

Figure 12: log of determinant of the A matrix as a function of D and k_y for H_{const}

$H = H_{dist}$

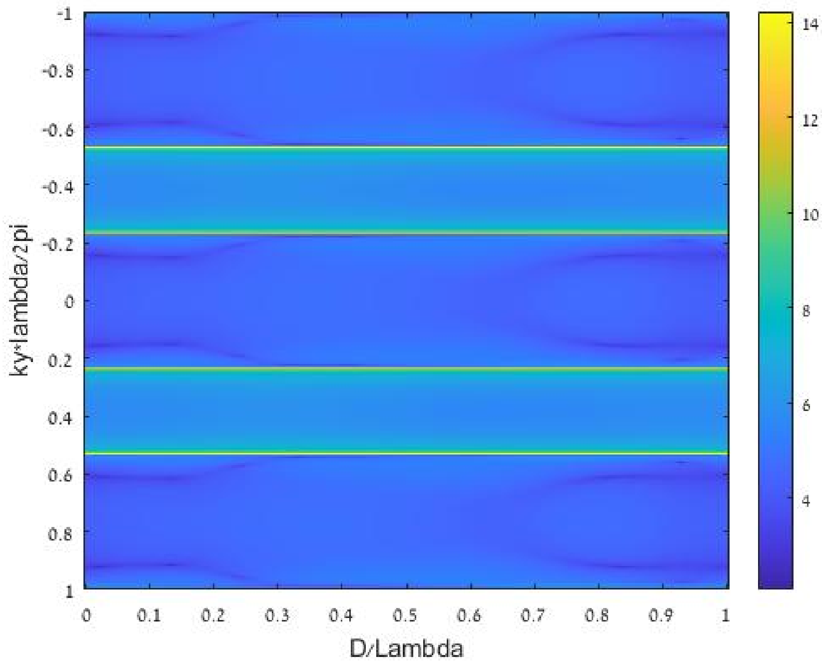

Figure 13: log of determinant of the A matrix as a function of D and k_y for H_{dist}

$H = \Lambda$

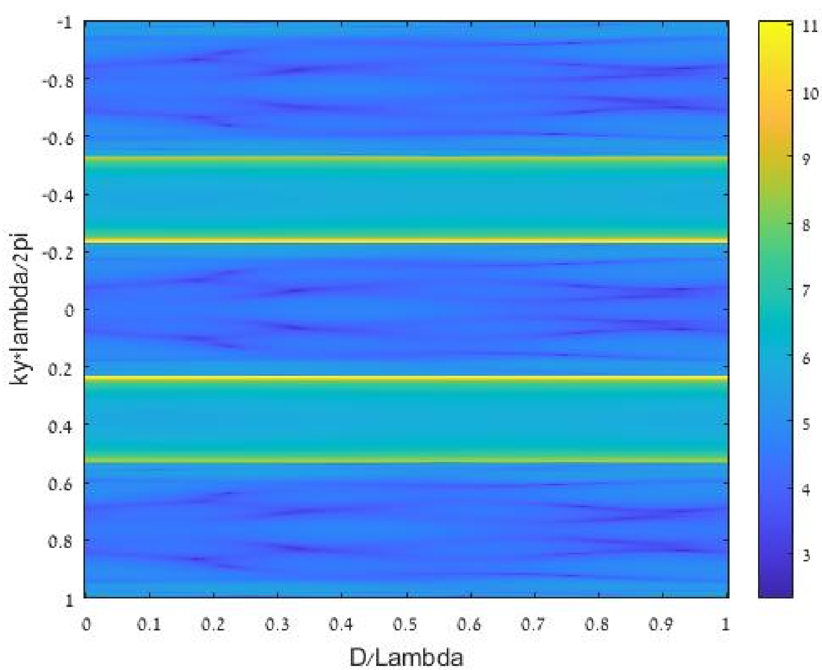

Figure 14: log of determinant of the A matrix as a function of D and k_y for $H = \Lambda$"

We can see that there is a dependency of the modes in D, which is an extra degree of freedom we don't have with regular wavrguides. We also see that for good conductivity' and for complexity we need larger H.

2.5.3 Laplace space

will want to see the "laplace space" propagating modes,The modes that may decay and be eigenstates of the system. for that will want to consider complex k_y values. Will return to eqaution 5, and calculate it for complex k:

$$\int_{-\infty}^{\infty} H_0^{(2)}\left(k\sqrt{(y-(d_m+l))^2+(z-h_m)^2}\right) e^{-j\frac{2\pi s}{\lambda}l} e^{-j\Re(k)\sin\theta_{in}l} e^{-\Im(k)\sin\theta_{in}l} dl = 2\frac{e^{-j\alpha_s(y-d_m)}e^{-j\beta_s|z-h_m|}}{\beta_s} \tag{30}$$

2.5.4 Farther examination

It can bee seen from the k_y apace analysis that there are interesting patterns to be examined, such as the "forbidden zones" that exist - the places where $det\,(A)$ are very large, and no waves are guided, and the interesting symmetry that appears. We would like to know the reasons for these patterns.

- Some explanations may be:`\item` command.
- It is due to the specific design of the reflective surface
- It is due to the large thickness of the reflective surface
- It is due to the specific angels for which the surface was designed to reflect
- It is a generic property of waveguides made of metagradings

In order to check it, will designe some more reflective surfaces according to the method intreduced in part 1. First to check the first two reasons will create a new, mach thinner reflective surface, with the same angels:

Item	Units	layer 1	layer 2	layer 3	layer 4	layer 5	layer 6
h_m	$[mm]$	0	0.5	1.0	1.6	2.2	2.5
d_m	$[mm]$	0	28.4	17	23.6	11.3	6.5
Z_m	$-10^4 j\left[\frac{V}{Am}\right]$	8.39	8.48	7.41	7.39	7.21	7.77

Table 2: Thin reflective surface

The waves guided are in this waveguide are:

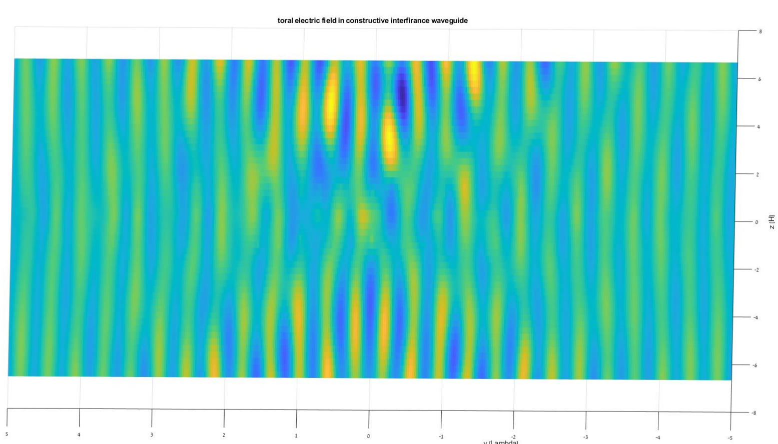

Figure 15: log of determinant of the A matrix as a function of D and k_y for H_{const}

It seems that it guides much less energy, and lets most of the power escape. Will look at the fields farther from source:

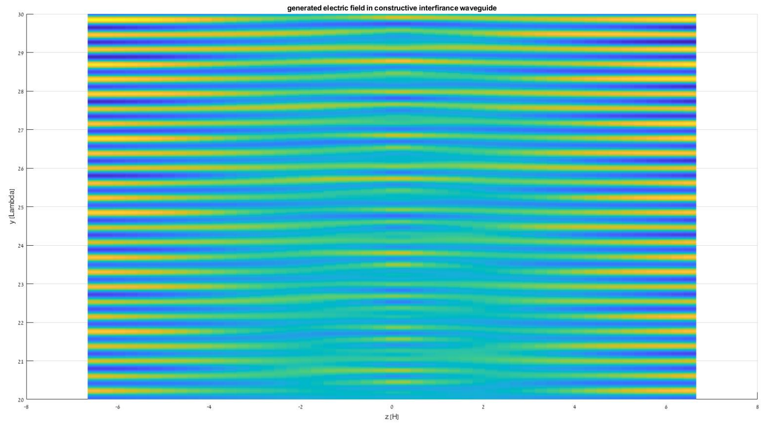

Figure 16: log of determinant of the A matrix as a function of D and k_y for H_{const}

Will look at the k_y space picture by H:

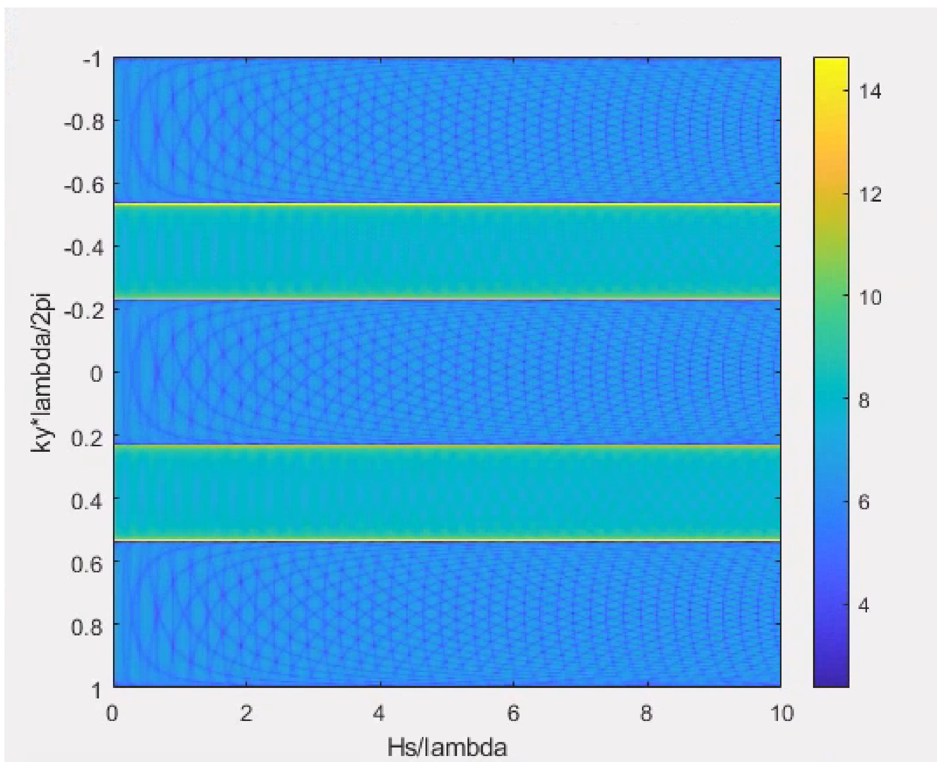

Figure 17: log of determinant of the A matrix as a function of H and k_y for H_{const}, and small thickness waveguide

and as a function of D:

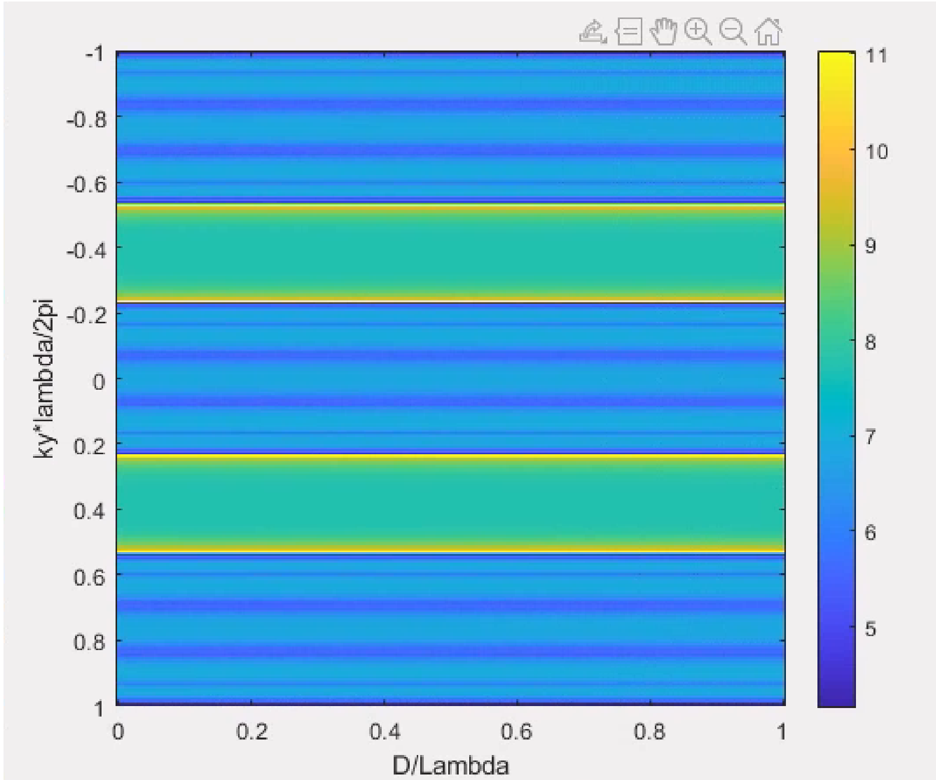

Figure 18: log of determinant of the A matrix as a function of H and k_y for H_{const}, and small thickness waveguide

From here it appears that for small thickness reflective surface there is no D dependency of the modes.

2.5.5 Thickness role in conduction

It is clear that this waveguide does not conduct waves very well. The comparison between thin and thick waveguides suggest that maybe the waves are guided inside the reflective layers - maybe the thickness of the waveguide is important.

to Check that will design mane reflective surfaces with different thicknesses. For each will simulate the k space $det\,(A)$ vs k_y and H, for $D = 0$. We will take the $det\,(A)$ value of the best conducted mode for each surface, and plot it against the thickness. The expected result if the waves are conducted mainly whitin the conductive surface is a declining trend of $det\,(A)$ by the thickness. The code plotting the graph takes very long to run and so we wont have amazing resolution or averaging over many reflective surfaces.

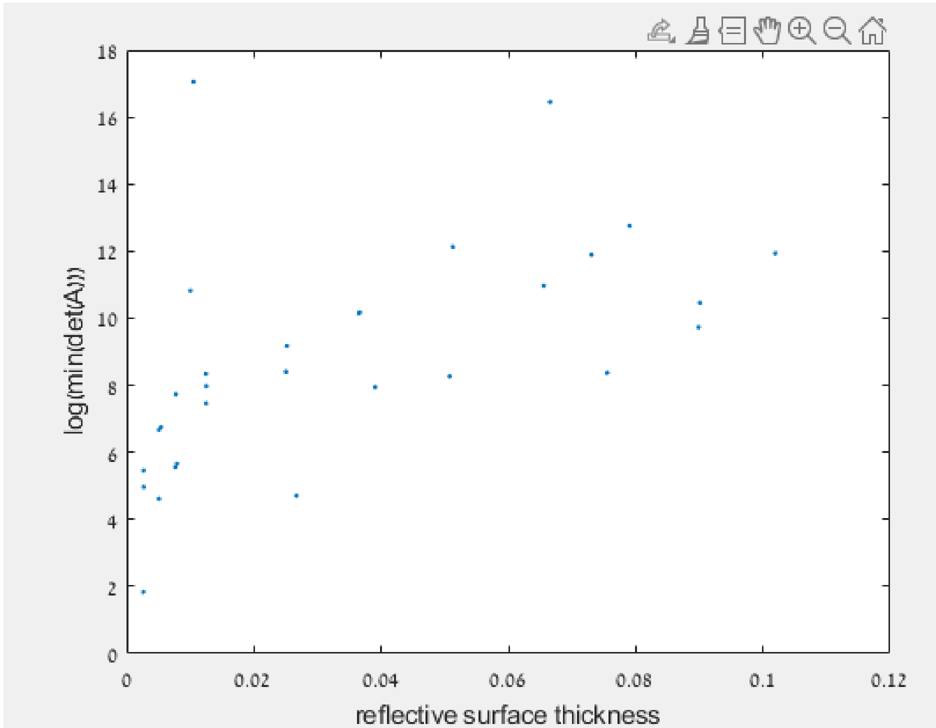

Figure 19: minimum of log of determinant of the A matrix as a function the waveguide thickness

From here we can see that (unexpectedly) the conductivity mainly improves (the determinant is smaller) for low thickness waveguides. we also find a large variation of conductivity (as expected, since there are many more parameters than waveguide thickness). We assume that we had an unlucky bad conductive waveguide for our first try with small thickness and a very lucky good conductivity waveguide for our first try with large thickness waveguide.

It is important to notice here that all these waveguides were designed for the same reflection criteria. This shows that we have a more freedom in desining the reflective surface.

2.6 Power flow

The time averaged Pointying Vector is:

$$\langle S \rangle = \frac{1}{2\eta}\Re\left(\vec{E} \times \vec{H}\right) \tag{31}$$

Will calculate the relevent magnetic fields in the system. The Magnetic field is calculated from the following Maxwell elation:

$$\vec{H} = \frac{j}{\mu\omega}\left(\frac{\partial E}{\partial z}\hat{y} - \frac{\partial E}{\partial y}\hat{z}\right) \tag{32}$$

The derivatives:

$$\begin{aligned} \frac{\partial E}{\partial z} &= -j\frac{k^2\eta^2 I_0}{4\Lambda}\int_{-\infty}^{\infty}\frac{e^{-j(k_z|z-z_0|+k_y(y-y_0))}}{k_z}\sum_{m=1}^{N} I_m^{norm}\left(k_y\right) \\ &\sum_{n=-\infty}^{\infty}\frac{e^{-j(\alpha_n(y-d_m)+\beta_n|z-h_m|)}}{\beta_n}\left[k_z sign\left(z-z_0\right)+\beta_n sign\left(z-h_m\right)\right]dk_y \end{aligned} \tag{33}$$

so the magnetic Field is:

$$\vec{H} = \frac{k\eta I_0}{4\Lambda} \int_{-\infty}^{\infty} \frac{e^{-j(k_z|z-z_0|+k_y(y-y_0))}}{k_z} \sum_{m=1}^{N} I_m^{norm}(k_y) \sum_{n=-\infty}^{\infty} \frac{e^{-j(\alpha_n(y-d_m)+\beta_n|z-h_m|)}}{\beta_n} \tag{34}$$
$$[-(k_y+\alpha_n)\hat{z} + (k_z sign(z-z_0) + \beta_n sign(z-h_m))\hat{y}]\, dk_y$$

For the wires magnetic field:

$$H_\phi = \frac{jI_0}{4}\frac{\partial}{\partial\rho}\left(H_0^{(2)}(k\rho)\right) = \frac{jkI_0}{4}\left(H_1^{(2)}(k\rho)\right) \tag{35}$$

3 Simpler case

It seems that we couldn't analyze all of our waveguide k space and place space data in an informative way. We assume that the complexity of the waveguide system we designed is too great for now, since there is a lack of intuition of what happens, and so a lack of intuition of what we search for. To solve that will try to get insights on a simpler case. One complicating factor in the waveguide is the energy dissipation. Since the waveguide is completely open, most of the energy flows outside of it into free space. To eliminate this factor Will try to use a reflective surface, made of methagradings layers, and a PEC.

3.1 reflective surface design

Will look at a PEC at $z=0$ plane, with an infinite wire to the $\hat{x}$ direction at $(y,z)=(y_0,-a)$ and phasor currant I_0. In the $z<0$ half space we know from image theory this is the same problem as a current source I_0 in $(y,z)=(y_0,-a)$ and a current source $-I_0$ in $(y,z)=(y_0,a)$. Now lets look at N methagradings layer with h_i, d_i, I_i, where $1 \leq i << N$, and y_i is the placement of the "first" wire of each layer in the $\hat{y}$ direction (as shown in figure). @@@@@@@@@@@@@@ adding figure @@@@@@@@@@@@@ Will use the image theory to get the following methagradings layers instead.

We can use equation 6 to write the fields in the relevant part of space:

$$E_x(y,z) = -\frac{k\eta}{2\Lambda}\sum_{m=1}^{N} I_m \sum_{n=-\infty}^{\infty} \frac{e^{-j\alpha_n(y-d_m)}e^{-j\beta_n|z-h_m|} - e^{-j\alpha_n(y-d_m)}e^{-j\beta_n|z+h_m|}}{\beta_n} \tag{36}$$

and from equation 10 we get:

$$\begin{aligned} E_{wire}^{tot} &= Z_M I_M = E_{outsidefield} + E_{otherlayers} + E_{reflectionoflayer} + E_{restoflayer} + E_{self} = \\ & E_0\left[e^{-jk(\sin\theta_{in}d_m+\cos\theta_{in}h_m)} - e^{-jk(\sin\theta_{in}d_m-\cos\theta_{in}h_m)}\right] \\ & -\frac{k\eta}{2\Lambda}\sum_{\substack{m=1\\m\neq M}}^{N} I_m \sum_{n=-\infty}^{\infty}\frac{e^{-j\alpha_n(y-d_m)}e^{-j\beta_n|z-h_m|} - e^{-j\alpha_n(y-d_m)}e^{-j\beta_n|z+h_m|}}{\beta_n} \\ & +\frac{k\eta I_M}{2\Lambda}\left[\frac{j\Lambda}{\pi}\ln\frac{2\pi\cdot r_{eff}}{\Lambda} - \frac{1}{\beta_0} - \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(\frac{1}{\beta_n} - \frac{\Lambda}{2\pi|n|}\right) + \sum_{n=-\infty}^{\infty}\frac{e^{-2j\beta_n H_M}}{\beta_n}\right] \end{aligned} \tag{37}$$

It is important to notice that the outside field is the sum of the incoming field and the field the field reflected from the PEC. Will calculate α_i, β_i again and find that in our case (in normalized units):

Item	expression	$i=0$	$i=1$	$i=-1$	$i=2$	$i=-2$
α_i	$\frac{2\pi i-\varphi}{\Lambda}$	0.1736	0.9397	-0.5924	1.7057	-1.3584
β_i	$\sqrt{k^2-\alpha_n^2}$	0.9848	0.3420	0.8056	$-1.382j$	$-0.919j$

Table 3: first modes

We can see that the far field modes that do not decay are $i = 0, 1, -1$. In this simpler case We don't really care about the phase shift, and so we can enforce that the two unwanted modes $i = 0, -1$, and through the optimization algorithm enforce energy conservation. So will enforce what happens for $z \to -\infty$ to these modes:

$$\frac{jk\eta}{\Lambda} \sum_{i=-1,0,1} \sum_{m=1}^{M} I_m \frac{e^{-j\alpha_i(y-d_m)} e^{j\beta_i z}}{\beta_i} \sin(\beta_i h_m) - E_0 e^{j(\beta_0 z - \alpha_0 y)} = E_1 e^{j(\beta_1 z - \alpha_1 y)} \tag{38}$$

We used $\alpha_0 = k \cdot \sin(\theta_{in})$ and $\alpha_1 = k \cdot \sin(\theta_{out})$. We Will enforce the conditions on α_{-1}, α_0, and the condition on α_1 will be enforced using energy conservation. Note the E_1 ideally contains the entire energy entering the system. The conditions We get are:

$$\frac{jk\eta}{\Lambda} \sum_{m=1}^{M} I_m \frac{e^{j\alpha_i d_m}}{\beta_i} \sin(\beta_i h_m) = E_0 \delta_{i0} \tag{39}$$

We can see that there are two constrains and so two layers of methagrading should be enough. @@@@@@@@@@@@ chose one of the following options@@@@@@@@@@@@@@@@@

@@@@@@@@@@@@ option 1 @@@@@@@@@@@@@@
For the optimization Will chose random h_m, d_m, calculate the currents that satisfy one of the conditions 39, find the appropriate impedances to create those currents, and optimize for minimal power loss, and for the physical impedances.

@@@@@@@@@@@@ option 2 @@@@@@@@@@@@@@
For the optimization Will chose random h_m, d_m, Z_m,calculate the resulting currents using equation 37, and try to optimize by minimizing deviations from the conditions in equation 39.

@@@@@@@@@@@@@@@@ add the numerical results @@@@@@@@@@@@@@@

3.2 Waeguide studing

@@@@@@@@@@@@ describing the waveguide appearance @@@@@@@@@@@@@

3.2.1 analytical formulation

Will start by finding an analytical expiration for a single methagrading layer inside a waveguide made of two different PEC's. Will look at a waveguide made of a PEC in $z = 0$ plane, and a PEC in $z = -H$ Plane. between them there is a wire with current I_0 in $(y, z) = (y_0, -a)$. According to image theory the problem inside the waveguide is the same as the waveguide in figure 20

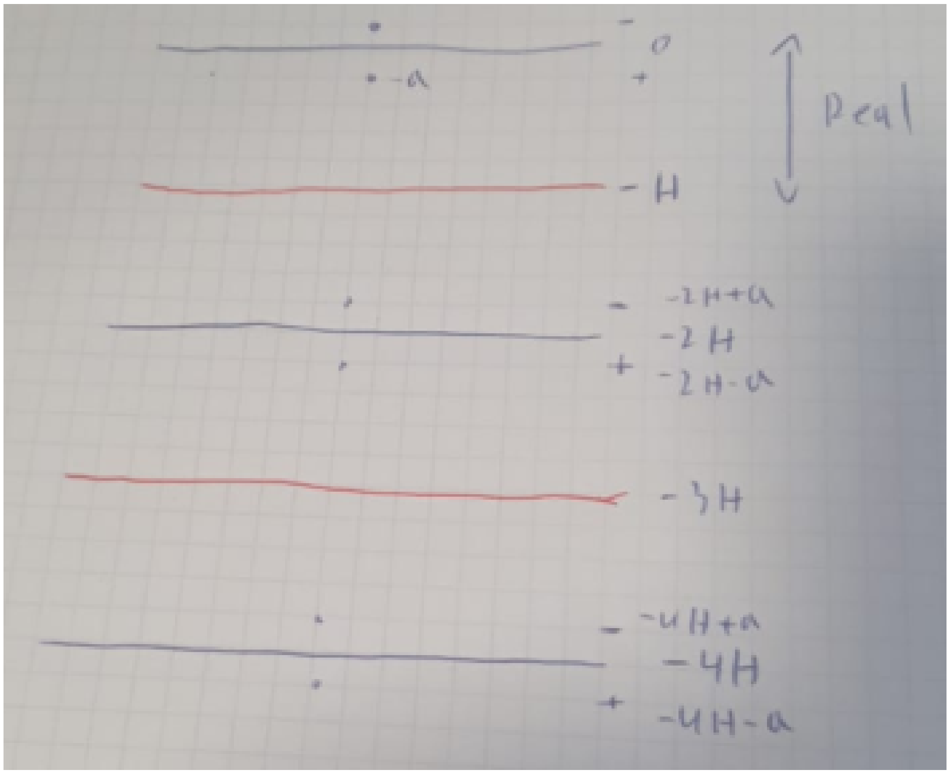

Figure 20: PEC wavegude with image theory

The equivalent problem is a current source I_0 in $z = 2nH - a$ and a current source $-I_0$ in $z = 2nH + a$, for whole number n. In our case we look at a layer and not a wire. So we can write the fields in the waveguide:

$$E_x(y,z) = \frac{kI_0\eta}{2\Lambda} \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{e^{-j\alpha_n(y-y_0)}e^{-j\beta_n|z-(2lH+a)|} - e^{-j\alpha_n(y-y_0)}e^{-j\beta_n|z-(2lH-a)|}}{\beta_n} \tag{40}$$

In our case $-H < z < 0$ and so we can write:

$$\begin{aligned} f_n^{(a,H)}(z) &= \sum_{l=-\infty}^{\infty} e^{-j\beta_n|z-(2lH+a)|} = \\ &= e^{-j\beta_n|z-a|} + \sum_{l=1}^{\infty} e^{j\beta_n(z-(2lH+a))} + \sum_{l=1}^{\infty} e^{-j\beta_n(z-(2(-l)H+a))} = \\ &= e^{-j\beta_n|z-a|} + e^{j\beta_n(z-a)} \cdot \sum_{l=1}^{\infty} e^{-2jlH\beta_n} + e^{-j\beta_n(z-a)} \cdot \sum_{l=1}^{\infty} e^{-2jlH\beta_n} = \\ &= +e^{j\beta_n(z-a)} \cdot \frac{e^{-2jH\beta_n}}{1-e^{-2jH\beta_n}} + e^{-j\beta_n(z-a)} \cdot \frac{e^{-2jH\beta_n}}{1-e^{-2jH\beta_n}} = \\ &= e^{-j\beta_n|z-a|} - je^{-jH\beta_n}\frac{\cos(\beta_n(z-a))}{\sin(\beta_n H)} \end{aligned} \tag{41}$$

And so the total field created by the layer is:

$$\begin{aligned} E_x(y,z) &= \frac{kI_0\eta}{2\Lambda} \sum_{n=-\infty}^{\infty} \frac{e^{-j\alpha_n(y-y_0)}}{\beta_n} \left(f_n^{(a,H)}(z) - f_n^{(-a,H)}(z)\right) \\ &= \frac{kI_0\eta}{2\Lambda} \sum_{n=-\infty}^{\infty} \frac{e^{-j\alpha_n(y-y_0)}}{\beta_n} \left(e^{-j\beta_n|z-a|} - e^{-j\beta_n|z+a|} - 2j\frac{e^{-j\beta_n H}}{\sin(\beta_n H)}\sin(\beta_n z)\sin(\beta_n a)\right) \end{aligned} \tag{42}$$

We would like to also calculate the field on the wire itself. Inside $f_n^{(-a)}(-a)$ the element $e^{-j\beta_n|z+a|}$ comes from the effect of the layer on itself. Will have to replace it with the real effect of a layer on itself:

$$
\begin{aligned}
E_M &= Z_M I_M = E_{out} + E_{otherlayers} + E_{reflectionsoflayer} + E_{self} = \\
&= E_{out} \\
&+ \frac{k\eta}{2\Lambda} \sum_{\substack{m=1 \\ m\neq M}}^{N} I_m \sum_{n=-\infty}^{\infty} \frac{e^{-j\alpha_n(d_M-d_m)}}{\beta_n} \left(e^{-j\beta_n|h_M+h_m|} - e^{-j\beta_n|h_M-h_m|} + 2j\frac{e^{-j\beta_n H}}{\sin(\beta_n H)} \sin(\beta_n h_M)\sin(\beta_n h_m) \right) \\
&+ \frac{k\eta}{2\Lambda} I_M \sum_{n=-\infty}^{\infty} \frac{1}{\beta_n} \left(e^{-2j\beta_n|h_M|} + 2j\frac{e^{-j\beta_n H}}{\sin(\beta_n H)} \sin^2(\beta_n h_M) \right) + \\
&+ \frac{k\eta}{2\Lambda} I_M \left[\frac{j\Lambda}{\pi} \ln\frac{2\pi\cdot r_{eff}}{\Lambda} - \frac{1}{\beta_0} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{\beta_n} - \frac{\Lambda}{2\pi|n|} \right) \right]
\end{aligned}
\tag{43}
$$

@@@@@@@@@@@@@@@@ calculation of the fields induced by a wire @@@@@@@@@@@@@@@@@@@@@@@@@@@ For simplicity will calculate the induced field in the waveguide by an infinite wire in the **center** of the waveguide. Image theory tells us that this problem is equivalent to a single perpendicular layer, with equivalent $\Lambda = H$ and $\phi = \pi$. The relevant field induced is according to equation 6:

$$E(y,z) = -\frac{k\eta}{2H} I_0 \sum_{n=-\infty}^{\infty} \frac{e^{-j\alpha_n(z+H/2)} e^{-j\beta_n|y|}}{\beta_n} \tag{44}$$

For $\alpha_n = \pi\frac{2n-1}{H}$ and $\beta_n = \sqrt{k^2-\alpha_n^2}$

3.2.2 numerical results

4 Unitless equations

In this section will create unitless equations for easier code and understanding. This is mainly for ourselves s it wont have many worded explanations. For lengths we will normalize everything by the wavelength:

$$\Lambda^* = \frac{\Lambda}{\lambda} = \frac{1}{|\sin(\theta_{in}) - \sin(\theta_{out})|} \tag{45}$$

The wire Impedance will normalize by the appropriate units:

$$Z^* = Z\cdot\frac{\lambda}{\eta} \tag{46}$$

We will also normalize all of the currents and current densities by I_0, and all wave-numbers (k, α, β) by $\frac{1}{\lambda}$:

$$k^* = k\cdot\lambda \tag{47}$$

After all this we can write our normalized equations:

$$E^*_{k_y} = -\frac{k^*}{2}\frac{e^{-j\left(k_z^*|z^*-z_0^*| + k_y^*(y^*-y_0^*)\right)}}{k_z^*} \tag{48}$$

for $E^* = E\cdot\frac{\lambda}{\eta I_0}$
The Matrix equations to find I_{norm} will be:

$$A^*\left(k_y^*\right) I^*_{norm}\left(k_y^*\right) = V^*\left(k_y^*\right) \tag{49}$$

$$A^*_{MM} = Z^*_M - j\pi\left(\frac{1}{\pi}\ln\left(\frac{2\pi r^*_{eff}}{\Lambda^*}\right) + \frac{1}{\Lambda^*\beta^*_0} + \sum_{n=-\infty}^{\infty}\left[\frac{1}{\Lambda^*\beta^*_n} - \frac{j}{2\pi\left|n\right|}\right]\right) \tag{50}$$

$$A^*_{Mm} = \frac{\pi}{\Lambda^*}\sum_{n=-\infty}^{\infty}\frac{1}{\beta^*_n}e^{-j(\alpha^*_n(d^*_M-d^*_m)+\beta^*_m|h^*_M-h^*_m|)} \tag{51}$$

$$V^*_M = e^{-j\left(k^*_y d^*_M + k^*_z h^*_M\right)} \tag{52}$$

for $V^* = V$, $A^* = A\cdot\frac{\lambda}{\eta}$, and $I^*_{norm} = I_{norm}\cdot\frac{\eta}{\lambda}$.
Now we can write the final integral:

$$E^*_{gen}(y,z) = \frac{\pi^2}{\Lambda^*}\int_{-\infty}^{\infty}\frac{e^{-j\left(k^*_z|z^*-z^*_0|+k^*_y(y^*-y^*_0)\right)}}{k^*_z}\sum_{m=1}^{N}I^*_{norm}\left(k^*_y\right)\sum_{n=-\infty}^{\infty}\frac{e^{-j(\alpha^*_n(y^*-d^*_m)+\beta^*_n|z^*-h^*_m|)}}{\beta^*_n}dk^*_y \tag{53}$$

And of course the regular generated field

$$E^*_{wire}\left(y^*,z^*\right) = -\frac{1}{2\pi}H^{(2)}_0\left(2\pi\sqrt{\left(y^*-y^*_0\right)^2+\left(z^*-z^*_0\right)^2}\right) \tag{54}$$

The generated magnetic field is ($H^* = H\cdot\frac{\lambda}{I_0}$):

$$\begin{aligned}\vec{H}^* = & \frac{k^*}{4\Lambda^*}\int_{-\infty}^{\infty}\frac{e^{-j(k_z|z-z_0|+k_y(y-y_0))}}{k^*_z}\sum_{m=1}^{N}I^{norm*}_m\left(k_y\right)\sum_{n=-\infty}^{\infty}\frac{e^{-j(\alpha_n(y-d_m)+\beta_n|z-h_m|)}}{\beta^*_n} \\ & \left[\left(k^*_z sign\left(z-z_0\right)+\beta^*_n sign\left(z-h_m\right)\right)\hat{y} - \left(k^*_y+\alpha^*_n\right)\hat{z}\right]dk^*_y\end{aligned} \tag{55}$$

For the wires magnetic field:

$$H^*_\phi = \frac{jk^*}{4}\left(H^{(2)}_1(k^*\rho^*)\right) \tag{56}$$

5 Future research

5.1 Laplace space studding examination of modes

We would like to change the k_y space mode studying we have done part 2.5.2 to a Laplace space analysis - do the analysis for complex K_y. for that to happen we need to be able to write the solution for the integral in equation 5, for complex φ. The other option is to the original sum in 3 in the matlab code, but this sum converges very slowly, and so either the code will take very long time to run or the results wont be accurate. The reasoning behind the complex k_y search is to find the decaying modes that are eigenvalues of the system - to find how much each $\Re(k_y)$ value decays.

5.2 different reflective surfaces

We found here that the conducted modes don't necessarily depend on the angles the reflective surface was designed to.So It may be interesting to sea what happens with different orientations of methagrading layers: two different reflective surfaces as waveguides, reflective surfaces of other angles and phase, methagradings that where not designed to reflect waves,and so on.

5.3 optimization improvement in reflective surface design

We saw during the work, that there are many reflective surfaces that satisfy our specific design of reflection law. This implies that maybe we have some more power and control in the design than we thought.There is a possibility to explore, that we can add some more functionality to the surface, without adding more layers of methagradings. This functionality can maybe be control of the reflection of another angle, control of geometrical parameters to ease manufacturing, or simply reducing the number of methagradings.

5.4 Run time improvement

Our numerical methods took a significant amount of time to run on our computer. Designing a good reflective surface can take about 20 minutes, plotting the electric field of a waveguide about 10, and so on. This constricts the amount of different waveguide variations we are able to check. The time it takes to run can break down to a few elements. First element is the complexity of the analytical expression we try to calculate. The second element is the neatness of the actual code. The third element is the hardware used. We assume each of those elements can be improved.

References

[1] Mohammad Memarian George V. Eleftheriades. "Evanescent-to-Propagating Wave Conversion in Sub-Wavelength Metal-Strip Gratings". In: *IEEE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES* 60.12 (2012).

[2] I. S. Gradshtein and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Academic Press, New York, 2015.

[3] Ariel Epstien Oshri Rabinovich. "Perfect Anomalous Refraction with Metagratings". In: (2018).

[4] Ariel Epstien Oshri Rabinovich. "Unveiling the Properties of Metagratings via a Detailed Analytical Model for Synthesis and Analysis". In: *PHYSICAL REVIEW APPLIED* 8.054037 (2017).