Nonlinear Observability Analysis of Quadrotor Using IMU and Range Measurements

Eranga Fernando

January 28, 2019

1 Model 1 : Without Biases (Quaternion)

1.1 Model

- **States**

$$X = [\underbrace{x,\ y,\ z,}_{\boldsymbol{p}}\ \underbrace{v_x,\ v_y, v_z,}_{\boldsymbol{v_b}}\ \underbrace{q_x,\ q_y,\ q_z,\ q_w,}_{\boldsymbol{q}}\ T]$$

- **Process Model**

$$\dot{\boldsymbol{X}} = \underbrace{\begin{bmatrix} Cv_b \\ -K\boldsymbol{v_b} - T\boldsymbol{e_3} + C^T\boldsymbol{g} \\ 0_{4\times 1} \\ 0 \end{bmatrix}}_{f_0} + \underbrace{\begin{bmatrix} 0_{3\times 4} \\ 0_{3\times 4} \\ \Omega \\ 0 \end{bmatrix}}_{f_1}\omega + \underbrace{\begin{bmatrix} 0_{3\times 1} \\ 0_{3\times 1} \\ 0_{4\times 1} \\ 1 \end{bmatrix}}_{f_2} U_4$$

- **Measurement Model**
  Drag coefficient matrix is expressed as mass and propeller velocity normalized. Thrust is also mass normalized.

$$h = \begin{bmatrix} -K\boldsymbol{v_b} - T\boldsymbol{e_3} \\ \frac{1}{2}||\boldsymbol{r_1}||^2 \\ \frac{1}{2}||\boldsymbol{r_2}||^2 \\ \frac{1}{2}||\boldsymbol{r_3}||^2 \end{bmatrix}$$

  where

$$\boldsymbol{r_i} = \boldsymbol{p} - \boldsymbol{p_i}$$

  $\boldsymbol{p_i}$ is the position of the i^{th} beacon

1.2 Scenarios

1.2.1 General Lie Derivatives

$$\mathcal{L}^0 h = h$$

$$\nabla\mathcal{L}^0 h = \begin{bmatrix} 0_{3\times3} & -K & 0_{3\times4} & -\boldsymbol{e_3} \\ \boldsymbol{R}_i & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \end{bmatrix}_{(3+i)\times 11}$$

$$\boldsymbol{R}_i = [\boldsymbol{r_1},\ ..., \boldsymbol{r_i}]^T : i = 1, 2, 3$$

$$\nabla\mathcal{L}^1_{f_0} h = \begin{bmatrix} 0_{3\times3} & K^2 & -K\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & K\boldsymbol{e_3} \\ \boldsymbol{V}_i C^T & \boldsymbol{R}_i C & \boldsymbol{R}_i \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \end{bmatrix}_{(3+i)\times 11}$$

$$\boldsymbol{V}_i = [\boldsymbol{v_b},\ ..., \boldsymbol{v_b}]^T \in \mathbb{R}^{i\times3}$$

$$\nabla\mathcal{L}^2_{f_0 f_0} h = \begin{bmatrix} 0_{3\times3} & -K^3 & K^2\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & -K^2\boldsymbol{e_3} \\ \dot{\boldsymbol{V}}_{\boldsymbol{b}_i} C^T & 2\boldsymbol{V}_i - \boldsymbol{R}_i C K & \boldsymbol{R}_i \nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R}_i C e_3 \end{bmatrix}_{(3+i)\times 11}$$

$$\dot{\boldsymbol{V}}_{\boldsymbol{b}_i} = [\dot{\boldsymbol{v}}_{\boldsymbol{b}},\ ..., \dot{\boldsymbol{v}}_{\boldsymbol{b}}]^T \in \mathbb{R}^{i\times3}$$

1.2.2 3 Range Measurements upto first order Lie derivatives

- **Observability Condition** : $\mathcal{O}_{3R}$ is full rank

$$\mathcal{O}_{3R} = [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h]$$

- **Observability Matrix**

$$\begin{aligned} \mathcal{O}_{3R} &= [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h] \\ &= \begin{bmatrix} 0_{3\times3} & -K & 0_{3\times4} & -\boldsymbol{e_3} \\ \boldsymbol{R_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & K^2 & -K\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & K\boldsymbol{e_3} \\ \boldsymbol{V_3} C^T & R_3 C & \boldsymbol{R_3} \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \end{bmatrix} \end{aligned}$$

- **Gaussian Elimination**

$$\mathcal{O}_{3R} = \begin{bmatrix} 0_{3\times3} & -K & 0_{3\times4} & -\boldsymbol{e_3} \\ \boldsymbol{R_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & K^2 & -K\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & K\boldsymbol{e_3} \\ \boldsymbol{V_3} C^T & \boldsymbol{R_3} C & \boldsymbol{R_3} \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \end{bmatrix}$$

$$\begin{aligned} &R_2 \leftarrow \boldsymbol{R_3}^{-1} \times R_2 \\ &R_3 \leftarrow K^{-1} \times R_3 + R_1 \\ &R_4 \leftarrow R_4 - \boldsymbol{V_3} C^T \times R_2 \\ &R_1 \leftarrow -K^{-1} \times R_1 \\ &R_4 \leftarrow \boldsymbol{R_3}^{-1} \times R_4 \end{aligned}$$

$$\mathcal{O}_{3R} = \begin{bmatrix} 0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\ \boldsymbol{I_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & -\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\ 0_{3\times3} & C & \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \end{bmatrix}$$

$$R_4 \leftarrow R_4 - C \times R_1$$
$$R_3 \leftarrow -R_3$$

$$\mathcal{O}_{3R} = \begin{bmatrix} 0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\ \boldsymbol{I_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & -CK^{-1}\boldsymbol{e_3} \end{bmatrix}$$

$$C_{11} \leftarrow C_{11} - K^{-1} C_6$$

$$\mathcal{O}_{3R} = \begin{bmatrix} 0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & 0_{3\times1} \\ \boldsymbol{I_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & -CK^{-1}\boldsymbol{e_3} \end{bmatrix}$$

$$\mathcal{O}_{3R_1} = \mathcal{O}_{3R}(7:12, 7:11)$$

$$= \begin{bmatrix} \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\ \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & -CK^{-1}\boldsymbol{e_3} \end{bmatrix}$$

- **Observability Conditions:**
  - K must be full rank
  - $\boldsymbol{R_3}$ must be full rank. i.e. quadrotor cannot lie on the line between any two anchors. System become rank deficient when following conditions satisfied
    * $\boldsymbol{v_b} = 0$
    * $C\boldsymbol{v_b} = k\boldsymbol{g}$
  - $\boldsymbol{R_3}$ is not full rank. i.e. quadrotor lies on the line between any two anchors ($\boldsymbol{r_i} = k\boldsymbol{r_j}$) or the quadrotor and three anchors are on the same line ($\boldsymbol{r_1} = k\boldsymbol{r_2} = l\boldsymbol{r_3}$). For the following conditions the system is rank deficient
    * If the quadrotor is in the middle of the two beacons i.e. $\boldsymbol{r_i} = -\boldsymbol{r_j}$: This is will not happen. The rank become deficient only if $k = 1$. But under the assumptions that we have made, this cannot happen.
    * Quadrotor moves towards a beacon i.e. $C\boldsymbol{v_b} = k\boldsymbol{r_i}$
    * Quadrotor velocity is zero. i.e.$\boldsymbol{v_b} = 0$
    *
    * $C\boldsymbol{v_b} = k\boldsymbol{g}$

1.2.3 3 Range Measurements upto second order Lie derivatives

- **Observability Condition** : $\mathcal{O}_{3R}$ is full rank

$$\mathcal{O}_{3R} = [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_0} h]$$

- **Observability Matrix**

$$\begin{aligned}
\mathcal{O}_{3R} &= [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_0} h] \\
&= \begin{bmatrix}
0_{3\times3} & -K & 0_{3\times4} & -\boldsymbol{e_3} \\
\boldsymbol{R_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\
0_{3\times3} & K^2 & -K\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & K\boldsymbol{e_3} \\
\boldsymbol{V_3} C^T & R_3 C & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \\
0_{3\times3} & -K^3 & K^2\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & -K^2\boldsymbol{e_3} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b}_3} C^T & 2\boldsymbol{V}_3 - \boldsymbol{R_3} CK & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C\dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R_3} Ce_3
\end{bmatrix}
\end{aligned}$$

- **Gaussian Elimination**

$$\begin{aligned}
\mathcal{O}_{3R} &= [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_0} h] \\
&= \begin{bmatrix}
0_{3\times3} & -K & 0_{3\times4} & -\boldsymbol{e_3} \\
\boldsymbol{R_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\
0_{3\times3} & K^2 & -K\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & K\boldsymbol{e_3} \\
\boldsymbol{V_3} C^T & \boldsymbol{R_3} C & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \\
0_{3\times3} & -K^3 & K^2\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & -K^2\boldsymbol{e_3} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b}_3} C^T & 2\boldsymbol{V}_3 - \boldsymbol{R_3} CK & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C\dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R_3} Ce_3
\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&R_2 \leftarrow \boldsymbol{R_3}^{-1} \times R_2 \\
&R_4 \leftarrow R_4 - \boldsymbol{V_3} C^T \times R_2 \\
&R_4 \leftarrow \boldsymbol{R_3}^{-1} \times R_4 \\
&R_5 \leftarrow R_5 + K \times R_3 \\
&R_3 \leftarrow R_3 + K \times R_1 \\
&R_1 \leftarrow -K^{-1} \times R_1
\end{aligned}$$

$$\mathcal{O}_{3R} = \begin{bmatrix}
0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\
\boldsymbol{I_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\
0_{3\times3} & 0_{3\times3} & -\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\
0_{3\times3} & C & \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b}_3} C^T & 2\boldsymbol{V}_3 - \boldsymbol{R_3} CK & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C\dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R_3} Ce_3
\end{bmatrix}$$

$$\begin{aligned}
&R_4 \leftarrow R_4 - C \times R_1 \\
&R_3 \leftarrow -R_3 \\
&R_6 \leftarrow R_6 - \dot{\boldsymbol{V}}_{\boldsymbol{b}_3} C^T \times R_2 \\
&R_6 \leftarrow R_6 - (2\boldsymbol{V}_3 - \boldsymbol{R_3} CK) \times R_1
\end{aligned}$$

$$
\mathcal{O}_{3R} = \begin{bmatrix} 0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\ \boldsymbol{I_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & -CK^{-1}\boldsymbol{e_3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -2\boldsymbol{V}_3 K^{-1}\boldsymbol{e_3} \end{bmatrix}
$$

$$
R55 \leftarrow \boldsymbol{R_3}^{-1} \times R_6
$$

$$
\mathcal{O}_{3R} = \begin{bmatrix} 0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\ \boldsymbol{I_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & -CK^{-1}\boldsymbol{e_3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -2\boldsymbol{R_3}^{-1}\boldsymbol{V}_3 K^{-1}\boldsymbol{e_3} \end{bmatrix}
$$

- **Observability Conditions:**
  - K must be full rank

- **If the quadrotor lies on a line connecting two range beacons**

$$
\begin{aligned}
\mathcal{O}_{3R} &= [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_0} h] \\
&= \begin{bmatrix}
0_{3\times3} & -K & 0_{3\times4} & -\boldsymbol{e_3} \\
\boldsymbol{R_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\
0_{3\times3} & K^2 & -K\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & K\boldsymbol{e_3} \\
\boldsymbol{V_3} C^T & \boldsymbol{R_3} C & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \\
0_{3\times3} & -K^3 & K^2\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & -K^2\boldsymbol{e_3} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b}_3} C^T & 2\boldsymbol{V}_3 - \boldsymbol{R_3} CK & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R_3} C e_3
\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&R_3 \leftarrow R_3 + K \times R_1 \\
&R_3 \leftarrow -R_3 \\
&R_5 \leftarrow R_5 - K^2 \times R_1 \\
&R_3 \leftarrow -K^{-1} \times R_1
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}_{3R} &= [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_0} h] \\
&= \begin{bmatrix}
0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\
\boldsymbol{R_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\
0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\
\boldsymbol{V_3} C^T & \boldsymbol{R_3} C & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{3\times1} \\
0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b}_3} C^T & 2\boldsymbol{V}_3 - \boldsymbol{R_3} CK & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R_3} C e_3
\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&R_4 \leftarrow R_5 - R_3 \\
&R_4 \leftarrow R_4 - \boldsymbol{R_3} C \times R_1 \\
&R_6 \leftarrow R_6 - (2\boldsymbol{V}_3 - \boldsymbol{R_3} CK) \times R_1
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}_{3R} &= [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_0} h] \\
&= \begin{bmatrix}
0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\
\boldsymbol{R_3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\
0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\
\boldsymbol{V_3} C^T & 0_{3\times3} & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & -CK^{-1}\boldsymbol{e_3} \\
0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b}_3} C^T & 0_{3\times3} & \boldsymbol{R_3}\nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -2\boldsymbol{V}_3 K^{-1}\boldsymbol{e_3}
\end{bmatrix}
\end{aligned}
$$

- **Observability Conditions**

1.2.4 2 Range Measurements

- **Observability Condition** : $\mathcal{O}_{2R}$ is full rank

$$\mathcal{O}_{2R} = [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_0} h]$$

- **Observability Matrix**

$$\begin{aligned}
\mathcal{O}_{2R} &= [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_0} h] \\
&= \begin{bmatrix}
0_{3\times3} & -K & 0_{3\times4} & -\boldsymbol{e_3} \\
\boldsymbol{R_2} & 0_{2\times3} & 0_{2\times4} & 0_{2\times1} \\
0_{3\times3} & K^2 & -K\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & K\boldsymbol{e_3} \\
\boldsymbol{V_2} C^T & \boldsymbol{R_2} C & \boldsymbol{R_2} \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{2\times1} \\
0_{3\times3} & -K^3 & K^2 \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & -K^2 \boldsymbol{e_3} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b}_2} C^T & 2\boldsymbol{V}_2 - \boldsymbol{R}_2 CK & \boldsymbol{R}_2 \nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R}_2 C \boldsymbol{e_3}
\end{bmatrix}
\end{aligned}$$

- **Gaussian Elimination**

$$\mathcal{O}_{2R} = \begin{bmatrix}
0_{3\times3} & -K & 0_{3\times4} & -\boldsymbol{e_3} \\
\boldsymbol{R_2} & 0_{2\times3} & 0_{2\times4} & 0_{2\times1} \\
0_{3\times3} & K^2 & -K\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & K\boldsymbol{e_3} \\
\boldsymbol{V_2} C^T & \boldsymbol{R_2} C & \boldsymbol{R_2} \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{2\times1} \\
0_{3\times3} & -K^3 & K^2 \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & -K^2 \boldsymbol{e_3} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b}_2} C^T & 2\boldsymbol{V}_2 - \boldsymbol{R}_2 CK & \boldsymbol{R}_2 \nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R}_2 C \boldsymbol{e_3}
\end{bmatrix}$$

$$\begin{aligned}
R_3 &\leftarrow K^{-1} R_3 \\
R_3 &\leftarrow R_1 + R_3 \\
R_5 &\leftarrow K^{-2} R_5 \\
R_5 &\leftarrow R_5 - R_1 \\
R_1 &\leftarrow -K^{-1} \times R_1
\end{aligned}$$

$$\mathcal{O}_{2R} = \begin{bmatrix}
0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\
\boldsymbol{R_2} & 0_{2\times3} & 0_{2\times4} & 0_{2\times1} \\
0_{3\times3} & 0_{3\times3} & -\nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\
\boldsymbol{V_2} C^T & \boldsymbol{R_2} C & \boldsymbol{R_2} \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & 0_{2\times1} \\
0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\
\dot{\boldsymbol{V}}_{\boldsymbol{b_2}} C^T & 2\boldsymbol{V}_2 - \boldsymbol{R_2} CK & \boldsymbol{R}_2 \nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -\boldsymbol{R}_2 C \boldsymbol{e_3}
\end{bmatrix}$$

$$\begin{aligned}
R_4 &\leftarrow R_4 - \boldsymbol{R_2} C R_1 \\
R_6 &\leftarrow R_6 - (2\boldsymbol{V}_2 - \boldsymbol{R}_2 CK) R_1 \\
R_5 &\leftarrow R_5 + R_3 \\
R_3 &\leftarrow -R_3
\end{aligned}$$

$$\mathcal{O}_{2R} = \begin{bmatrix} 0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\ \boldsymbol{R_2} & 0_{2\times3} & 0_{2\times4} & 0_{2\times1} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\ \boldsymbol{V_2} C^T & 0_{2\times3} & \boldsymbol{R_2} \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & -\boldsymbol{R_2} C K^{-1} \boldsymbol{e3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ \dot{\boldsymbol{V}}_{\boldsymbol{b}_2} C^T & 0_{2\times3} & \boldsymbol{R}_2 \nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -2\boldsymbol{V}_2 K^{-1} \boldsymbol{e_3} \end{bmatrix}$$

$$R_2 \rightleftarrows R_3$$
$$R_4 \rightleftarrows R_5$$

$$\mathcal{O}_{2R} = \begin{bmatrix} 0_{3\times3} & \boldsymbol{I_3} & 0_{3\times4} & K^{-1}\boldsymbol{e_3} \\ 0_{3\times3} & 0_{3\times3} & \nabla_{\boldsymbol{q}} C^T \boldsymbol{g} & 0_{3\times1} \\ \boldsymbol{R_2} & 0_{2\times3} & 0_{2\times4} & 0_{2\times1} \\ \boldsymbol{V_2} C^T & 0_{2\times3} & \boldsymbol{R_2} \nabla_{\boldsymbol{q}} C \boldsymbol{v_b} & -\boldsymbol{R_2} C K^{-1} \boldsymbol{e3} \\ \dot{\boldsymbol{V}}_{\boldsymbol{b}_2} C^T & 0_{2\times3} & \boldsymbol{R}_2 \nabla_{\boldsymbol{q}} C \dot{\boldsymbol{v}}_{\boldsymbol{b}} & -2\boldsymbol{V}_2 K^{-1} \boldsymbol{e_3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \end{bmatrix}$$

- **Observability conditions**
  - If the quadrotor does not lie on the same line connecting the two beacons. For following scenarios the system become rank deficient.
    - Quadrotor velocity and acceleration in the world frame is towards a beacon. i.e $C\boldsymbol{v_b} = k_1\boldsymbol{r_i}$ and $C\dot{\boldsymbol{v}}_{\boldsymbol{b}} = k_1\boldsymbol{r_j}$.
    - Quadrotor velocity is zero and acceleration is towards a beacons. i.e. $\boldsymbol{v_b} = 0$ and $C\dot{\boldsymbol{v}}_{\boldsymbol{b}} = k_1\boldsymbol{r_i}$
    - Quadrotor acceleration is zero and velocity in the world frame is towards a beacon $\dot{\boldsymbol{v}}_{\boldsymbol{b}} = 0$ and $C\boldsymbol{v_b} = k_1\boldsymbol{r_i}$
    - Quadrotor is stationary and acceleration is zero. $\dot{\boldsymbol{v}}_{\boldsymbol{b}} = 0$ and $C\boldsymbol{v_b} = 0$
  - If the quadrotor lies on the same line as the beacons ($\boldsymbol{r_1} = k\boldsymbol{r_2}$). For following scenarios the system become rank deficient.
    - If quadrotor is in the middle of the two beacons. i.e.$k = -1$
    - Quadrotor is stationary, $\boldsymbol{v_b} = 0$
    - Quadrotor moves toward a beacon, $C\boldsymbol{v_b} = k_1\boldsymbol{r_i}$
    - Quadrotor has no acceleration, $\dot{\boldsymbol{v}}_{\boldsymbol{b}} = 0$
    - Quadrotor accelerates towards a beacon, $C\dot{\boldsymbol{v}}_{\boldsymbol{b}} = k_1\boldsymbol{r_i}$

1.2.5 1 Range Measurement

- **Observability Condition** : $\mathcal{O}_{1R}$ is full rank

$$\mathcal{O}_{1R} = [\nabla\mathcal{L}^0 h; \nabla\mathcal{L}^1_{f_0} h; \nabla\mathcal{L}^2_{f_0 f_1} h]$$