Propositional and Predicate Logic

Schema Operatores:

- ❑ **Propositional Logic**
  - Conjunction and Disjunction
  - Implication and equivalence
  - Negations, tautology and contradictions

- ❑ **Predicate Logic**
  - Predicate calculus
  - Quantifiers: Universal and Existential
  - Satisfaction and Validity

Propositional Logic

- Introduction
- Conjunction and Disjunction
- Implication and equivalence
- Negations, tautology and contradictions

Introduction

Logic is the study of ***reasoning*** and ***the validity of arguments***, and it is concerned with the truth of statements (propositions) and the nature of truth.

Formal logic is concerned with the form of arguments and the principles of valid inference

Propositional logic is the study of propositions, where a proposition is a declarative statement that can result in either true or false. The statement must be a constant thus the value cannot change. Propositions may be combined with other propositions(with a logical connective) to form compound propositions.

Propositional logic may **be used** to encode simple arguments that are expressed in natural language, and to determine their validity

Example 2.1 The following statements are propositions:

- A tomato is a fruit.
- An orange is a fruit.
- Oranges are not the only fruit

Example 2.2

Statement	Result
Babcock University is located in Ogun state	true
Babcock University is a State University	false

The following table describes five propositional connectives, arranged in descending order of operator precedence:

Precedence	Symbol	Meaning	Example
1	( )		
2	⌐ ~	NOT	Fail means NOT Pass
3	^	AND Conjunction	Hard work AND good attitude
4	v	OR Disjunction	Code in Java OR Code in C++
5	→	Conditional Implies	If you pass then you get reward
6	↔	Equals Bi-directional Bi-implication	Pass if and only if marks above 40
	⊗	Different Exclusive	Success is different from Failure

TRUTH TABLE

Truth tables is used to tell whether a propositional statement is true or false not only for one (1) instance but for all possible instances of the variable.

Proposition: A = Ebube is a boy.
Possible value: true or false therefore in a truth table.

(A) Ebube is a buy
T
F

Proposition: A = Ebube is NOT a boy or NOT (Ebube is a boy)
Possible value: true or false therefore in a truth table.

With an NOT operator true becomes false and false become true.

A	Result (⌐A)
T	F
F	T

Proposition: Ali is a boy and Mary is a girl.

Possible value: true or false for A and **possible value**: true or false for M

With an **AND** operator both statement must be true then the join statement will be true

A	M	**Result** (A ^ M)
T	T	T
T	F	F
F	T	F
F	F	F

A	M	**Result** (A v M)
T	T	T
T	F	T
F	T	T
F	F	F

Conjuction(∧) − −*AND*

Disjunction (∨) − *OR*

A more complex proposition

$(P \rightarrow Q) \vee (Q \rightarrow P)$

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	**Result** $(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Using the notion of precedence, we can see that the proposition

$$\neg p \wedge q \vee r \Leftrightarrow q \Rightarrow p \wedge r$$

is equivalent to the parenthesised version

$$(((\neg p) \wedge q) \vee r) \Leftrightarrow (q \Rightarrow (p \wedge r))$$

Example 2.2

- ¬(jaffar cakes are biscuits.)
- your cat is rich ∧ your dog is good looking
- *FG is winning the BokoHaram war* ∨ the minister is lying
- Tochi is thirty-something =>Tochi is under forty
- Tochi is thirty-something<=> Tochi is under forty

Notes: The truth of a compound proposition is uniquely determined by the truth of its constituent parts.

Conjunction and Disjunction

The conjunction $p \wedge q$ is true exactly when p is true and q is true. Using a truth table we have

p	q	$p \wedge q$
t	t	t
t	f	f
f	t	f
f	f	f

Example: Consider proposition A given by "An orange is a fruit" and the proposition B given by "2 + 2 = 5", then A is true and B is false. Therefore

Implies:

A	M	**Result** ($A \rightarrow M$)
T	T	T
T	F	F
F	T	T
F	F	T

Equals:

A	M	**Result** ($A \leftrightarrow M$)
T	T	T
T	F	F
F	T	F
F	F	T

Different:

A	M	**Result** ($A \otimes M$)
T	T	F
T	F	T
F	T	T
F	F	F

Properties of Propositional Logic

Common properties of the propositional calculus include the commutative, associative and distributive properties.
These help in the evaluation of complex expressions and allow logical expressions to be simplified.

- Commutative Property: i) $A \wedge B = B \wedge A$ ii) $A \vee B = B \vee A$
- Associative Property: i).$(A \wedge B) \wedge C = A \wedge (B \wedge C$ ii). $A \vee (B \vee C) = (A \vee B) \vee C$
- Distributive Property: i). $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ ii). $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$
- Idempotent Property: i).$(A \vee A) = A$ ii). $A \wedge A = A$

Implication and Equivalence

Implication

The *implication* p=>q is *true* in every case *except* when *p* is *true* and *q* is *false*

p	q	$p \Rightarrow q$
t	t	t
t	f	f
f	t	t
f	f	t

The implication *p* => *q* is true *if and only* if we can prove *q* by assuming *p*. Thus, in order to prove that *p*=>*q*, we may assume that *p* is true and then prove that *q* is true also.

Example: Safety **requirements can often be expressed as implications. A *safety requirement*** expresses that some unsafe situation must not occur. In a radiation therapy clinic it is necessary that the radiation beam must not turn on when the treatment room door is open, in order to protect staff and visitors from scattered radiation

Equivalence

The equivalence $p \Leftrightarrow q$ means that p and q are of the same strength; thus it might also be called bi-implication:

$$p \Leftrightarrow q \text{ means that both } p \Rightarrow q \text{ and } q \Rightarrow p.$$

The *equivalence* $p \Leftrightarrow q$ is *true* when p and q have the *same* truth value, whether it is *true or false.*

p	q	$p \Leftrightarrow q$
t	t	t
t	f	f
f	t	f
f	f	t

The equation *C1* = C2 is *true* when expressions *C1*and *C2* have the same value. Most programming languages use equality(usually =, but == in C) to express logical equivalence

Question: Write the table of situations that satisfies this equivalence:

$$(D_1 + D_2) = 7 \Leftrightarrow (D_1 < D_2)$$

“if and only if”

Negations, tautology and contradictions

Negations: The negation $\neg p$ is true if and only if p is false. The truth table is simple:

p	$\neg p$
t	f
f	t

Tautologies and Contradictions

Propositions which evaluate to t in every combinations of their propositional variables are know as tautologies; they are always true. If, on the other hands they evaluate to f in every combinations they are known to be contradictions. The negation of a contradiction is a tautology

Tautologies:

$$p \vee \neg p$$
$$p \Rightarrow p$$
$$p \Rightarrow (q \Rightarrow p)$$

Contradictions:

$$p \wedge \neg p$$
$$p \Leftrightarrow \neg p$$
$$\neg(p \Rightarrow (q \Rightarrow p))$$

“Logic will get you from A to B. Imagination will take you every-where”

A. Einstein

Predicate Logic

Consider the following example:

- Every SE student must study discrete mathematics.
- Nwachukwu is a SE student

It looks “logical” to deduce that therefore, Nwachukwu must study discrete mathematics. However, this cannot be expressed by propositional logic. Predicate logic is a richer system than propositional logic, and it allows complex facts about the world to be represented.

Predicate calculus consists of ***predicates, variables, constants and quantifiers***.

Definition:

A ***predicate*** is a statement that contains variables (predicate variables), and they may be true or false depending on the values of these variables

Example: P(x) = “x^2 is greater than x” is a predicate. It contains one predicate variable x. If we choose x= 1,P(1) is “1 is greater than 1”, which is a proposition (always false)

Discussion: Predicates

- Is the statement "x^2 is greater than x" a proposition?

Define P(x) = 'x^2 is greater than x'

Is P(1) a proposition? P(1)="1^2 is greater than 1" (F)

Since a predicate takes value true or false once instantiated (that is, once its variables are taking values), we may alternatively say that a predicate instantiated becomes a proposition.

Predicates Domain:

The **domain of a predicate** variable is the collection of all ***possible values*** that the variable may take

Consider the predicate P(x) = "x^2 is greater than x". Then the ***domain*** of x could be for example the set **Z** of all integers. It could alternatively be the set **R** of real numbers. Whether instantiations of a predicate are true or false may depend on the domain considered.

When several predicate variables are involved, they may or not have different domains.

Example:

Consider the predicate P(x,y) = “x > y”, in two predicate variables. We have **Z** (the set of integers) as domain for both of them

- Take x= 4,y= 3, then P(4,3) = “4 > 3”, which is a proposition taking the value true.
- Take x= 1,y= 2, then P(1,2) = “1 > 2”, which is a proposition taking the value false
- Note that in general P(x,y) $\neq$ P(y,x)!

Question:

Let Q(x, y, z) = “$x^2 + y^2 = z^2$" be a predicate in three variables. What is the truth value of Q(3, 4, 5)? What is the truth value of Q(2, 2, 3)? How many values of (x, y, z) make the predicate true?

A predicate becomes a proposition when fixed values are assigned to it. For instance, the predicate `$x > 10$' is still not a proposition until a value is assigned to x in order to determine whether the proposition is true or false. However, another way to make a predicate into a proposition is to quantify it.

That is, the ***predicate is true (or false)*** for all ***possible values*** in the universe of discourse or for ***some value(s)*** in the universe of discourse

Such quantification can be done with two quantifiers: the ***universal quantifier*** and the ***existential quantifier.***

Quantifiers: Universal and Existential

Universal Quantifiers

The ***universal quantifier*** of a predicate P(x) is the proposition "P(x) is true for ***all values*** of x in the universe of discourse" .

That is "$\forall\, x \in$ D P(x) is true" iff "*P*(*x*) is true for every *x* in *D*".

E.g: The square of any real number is nonnegative. That is $\forall\, x \in \mathbb{R}, x^2 \geq 0$

If the universe of discourse is finite, say $\{n_1, n_2, \ldots \ldots \ldots . n_k\}$, then the universal quantifier is simply the conjunction of all elements:

$$\forall\, xP(x) \Leftrightarrow P(n1) \land P(n2) \land \cdots \land P(nk)$$

Truth Value of Quantified Statements

Statement	When true	When false
$\forall x \in D, P(x)$	$P(x)$ is true for every x.	There is one x for which $P(x)$ is false.
$\exists x \in D, P(x)$	There is one x for which $P(x)$ is true.	$P(x)$ is false for every x.

Assume that D consists of $x_1, x_2, \ldots, x_n$

- $\forall x \in D, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
- $\exists x \in D, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Example:

- Let $P(x)$ be the predicate "*x must take a discrete mathematics course*" and let $Q(x)$ be the predicate "*x is a computer science student* "

- The universe of discourse for both $P(x)$ and $Q(x)$ is all BU students.

Expressing the statement "Every computer science student must take a discrete mathematics course" we have

$$\forall x\, Q(x) \Longrightarrow P(x)$$

Expressing the statement "Everybody must take a discrete mathematics course or be a computer science student" we have

$$\forall x (Q(x) \vee P(x))$$

Question: Express the statement "for every x and for every y, $x+y > 10$"

Existential Quantifier

Definition

The *existential quantification* of a predicate $P(x)$ is the proposition "There exists an x in the universe of discourse such that $P(x)$ is true." We use the notation

$$\exists x P(x)$$

which can be read "there exists an x"

Again, if the universe of discourse is finite, $\{n_1, n_2, \ldots, n_k\}$,

then the existential quantifier is simply the disjunction of all elements:

$$\exists x P(x) \Longleftrightarrow P(n_1) \vee P(n_2) \vee \cdots \vee P(n_k)$$

Satisfaction and Validity

A predicate with free variables or `spaces' is neither true nor false; it cannot be assigned a truth value until values are chosen for these variables or the spaces are filled.
Some predicates will become true whatever values are chosen: these are said to be ***valid*** predicates.

Example: If n denotes a natural number, then the predicate $n \geq 0$

is valid: it will be true whichever value is chosen from the list 0, 1, 2,3; …….

A predicate that is true for some, but not necessarily all, choices of values is said to be ***satisfiable*.**

Example: If n denotes a natural number,
then the predicate $n \geq 5$ is satisfiable. There are natural numbers greater than or equal to 5.

A predicate that is false for all choices is said to be ***unsatisfiable*****.** Valid, satisfiable, and unsatisfiable predicates are the analogues of tautologies, contingencies, and contradictions in the language of propositions.

EXERCISES: