N-Body Simulations with REBOUND

Lab course protocol

Group 3+10

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Abstract

This is optional, but never longer than half a page.

Contents

1 Introduction

In this laboratory work numerically we solve various N-body problems using integrator software package REBOUND.

2 Theory

2.1 Classical N-Body Problem

2.2 Time Integrators

Computers can calculate equations of motion for N-bodies with time integrators. Time integrators use certain algorithms to solve ODEs with discrete values of time starting from initial time t_n, and computers position and velocity at next time $t_n + \Delta t$. There are various numerical schemes to solve initial value problems in such way, and next subsections will give a general outlook on some of methods, since one of our tasks is to compare different time integrators.

2.2.1 Leapfrog

Leapfrog is second order symplectic integrator. In the first step, it calculates the next value for position at time $\Delta t/2$:

$$\mathbf{r}_{1/2} = \mathbf{r}_0 + \mathbf{v}_0 \frac{\Delta t}{2} \tag{1}$$

From that slope (acceleration) $a_{1/2}$ is calculated, and next regular steps calculate velocity and position at integer and half integer steps:

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \mathbf{a}_{n+1/2}\Delta t \tag{2}$$

$$\mathbf{r}_{n+3/2} = \mathbf{r}_{n+1/2} + \mathbf{v}_{n+1}\Delta t \tag{3}$$

And the last step for position:

$$\mathbf{r}_{n+1} = \mathbf{r}_{n+1/2} + \mathbf{v}_{n+1} \frac{\Delta t}{2} \tag{4}$$

2.2.2 IAS15, WHFast, and Gragg-Bulirsch-Stoer

IAS15 is a non-symplectic 15th order integrator with dynamic step size. This method can compute chaotic encounters in N-body systems down to machine precision, but very computationally demanding.

Wisdom-Holman Fast (WHFast) is second-order integrator with 11th order corrector. Is is fast for integrating long evolutions with conserved energy, so it is best for systems without collisions.

Gragg-Bulirsch-Stoer is a method based on Richardson exrapolation with very high accuracy. It uses adaptive step sizes for higher accuracy, but computationally demanding.

2.3 REBOUND

To simulate the N-body problem in various astrophysical contexts, we use the REBOUND software package developed by Professor Hanno Rein. REBOUND can simulate particles under the influence of their gravities. These particles can represent astrophysical bodies like stars, planets, moons, asteroids, dust particles etc[RL12]. The documentation for REBOUND can be found at: https://rebound.readthedocs.io/en/latest/. It provides convenient tools to study the properties and evolution of an N-body system like the energy, angular momentum and orbital elements.
REBOUND runs natively on windows, mac and linux. This can be run in either C or Python. For our purposes we will stick with the latter. REBOUND for python can be easily installed by `pip install rebound`.

2.3.1 REBOUNDX

3 Experiment

3.1 Two Body Problem

We use the simple two body problem to test various integrators in REBOUND (Leapfrog, IAS15, WHFast, Gragg-Bulirsch-Stoer) and compare the quality of the resulting outputs. We also test the quality of the results as we change the timestep from 1 to 10^{-6}
In this two body problem we simulate a moon orbiting a planet or a planet orbiting a star. Here, one body will be significantly heavier than the other. The energy and the angular momentum of the system should remain constant and are given as:

$$E = -\mu \frac{GM}{2a} \tag{5}$$

$$L = \mu\sqrt{GMa(1-e^2)} \tag{6}$$

Where $\mu = \frac{m_1 m_2}{M}$ is the reduced mass of the system and $M = m_1 + m_2$ is the total mass. From the above equations we can derive that the semi major axis and the eccentricity of

the system should also remain constant as we integrate the system over time.
The two body problem is simulated with REBOUND as per the following procedure:

1. Initialize the simulation with a chosen integrator and timestep
2. Add the two bodies to the system with $m_1 = 1$ and $m_2 = 0.3, a = 1, e = 0.3$
3. The simulation is integrated for one orbit and 250 steps. At each step the positions, energies and orbital parameters of the system are stored.
4. The orbit can be plotted from the stored positions of the two bodies. Various properties of the system can be plotted as a function of time.

The code where the following procedure is implemented is given in the appendix 4.
Here is the orbit we obtain with the leapfrog integrator and a timestep of 10^{-3}:

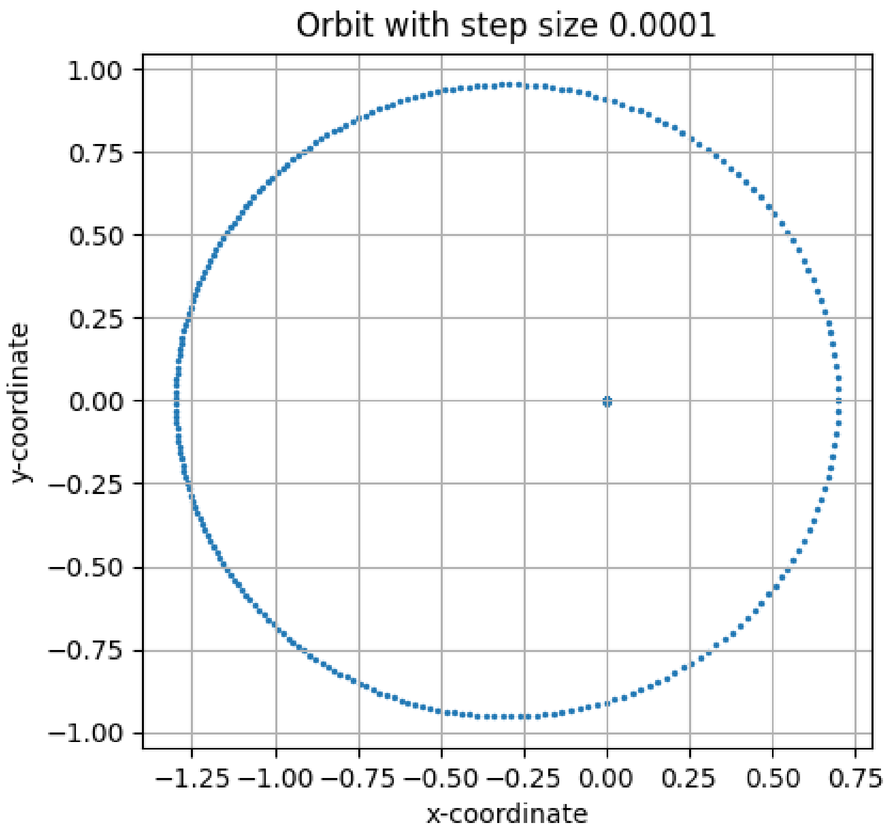
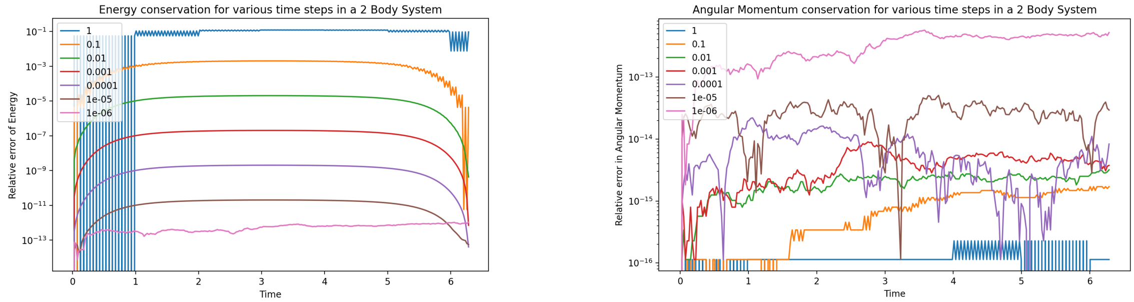


Figure 1: Orbit of a two body system with leapfrog integrator and timestep of 10^{-3}. The number of data points are less dense closer to the pericenter, which means that m_2 is moving faster. This follows Kepler's second law.

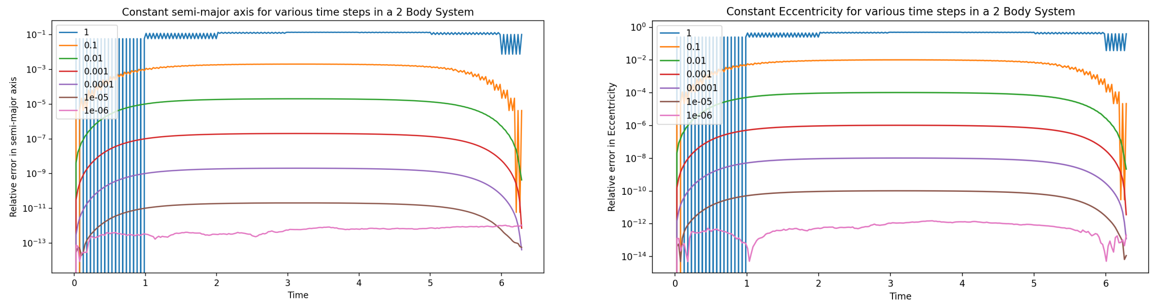
Effect of smaller timesteps is tested with the leapfrog integrator with timesteps of 1, 10^{-1},10^{-2}, 10^{-3},10^{-4}, 10^{-5} and 10^{-6}.



(a) Conservation of Energy at various timesteps. As we increase the timestep, the energy deviation from the initial value increases.

(b) Conservation of Angular Momentum at various timesteps. Conservation of angular momentum gets worse as we decrease the timestep.

Figure 2: Conservation of Energy and Angular Momentum at various timesteps

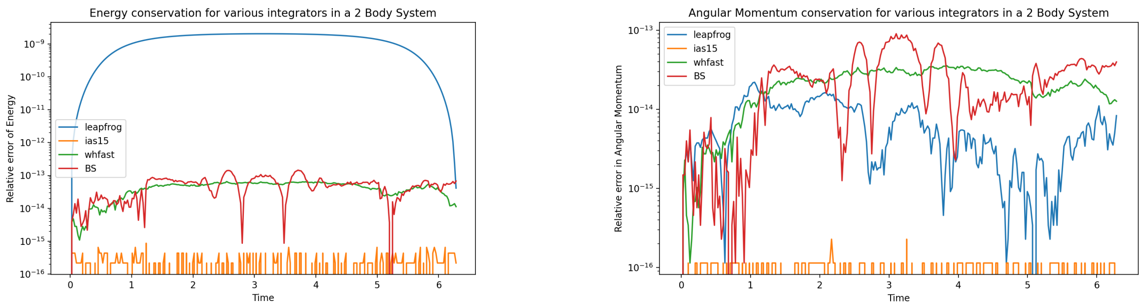


(a) Semi Major Axis at various timesteps

(b) Eccentricity at various timesteps

Figure 3: Constant semi major axis and eccentricity at various timesteps

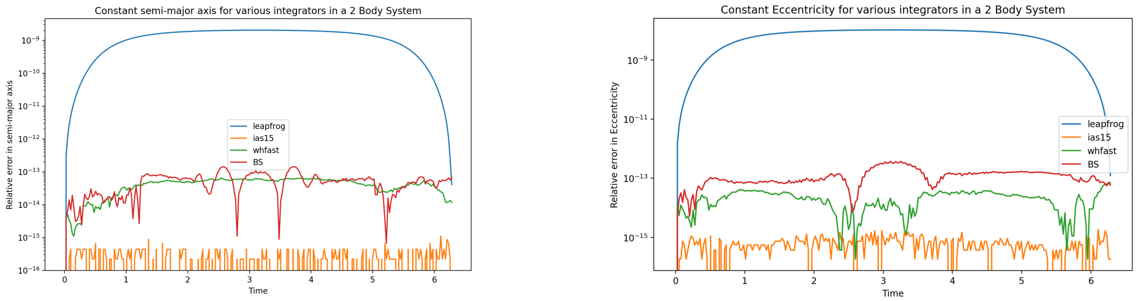
Effect of different integrators is tested with a timestep of 10^{-3} using the leapfrog, IAS15, WHFast and Gragg-Bulirsch-Stoer integrators



(a) Conservation of Energy using different integrators

(b) Conservation of Angular Momentum using different integrators

Figure 4: Conservation of Energy and Angular Momentum using different integrators



(a) Semi Major Axis using different integrators (b) Eccentricity using different integrators

Figure 5: Constant semi major axis and eccentricity using different integrators

3.2 Three Body Problem and Stability of the Planet System

Here we will simulate a coplanar 3 body system with a large central mass (m_1) and two smaller masses (m_2, m_3) in circular orbits. The two small objects will start opposite to each other (at a phase difference of π). We vary the distance between m_2 and m_3 to study the stability of the system.
The eccentricity and the semi major axis of the bodies will vary due to gravitational perturbations from the other bodies. For bodies that are closer, the gravitational perturbations will be larger which will result in stronger changes for the eccentricities and semi major axis. For initially circular orbits, Gladman[Gla93] defined a criterion for the stability of the system:

$$\Delta_c \simeq 2.40(\mu_2 + \mu_3)^{1/3} \tag{7}$$

Where μ is the mass ratio with respect to m_1. At distances greater than Δ_c, the system is considered to be stable. When the system is unstable, there is a possibility of close encounters between m_1 and m_2. A 'close encounter' is when the distance between the two bodies is less than the Hill radius of the larger of the two bodies. Here, the Hill radius is given by:

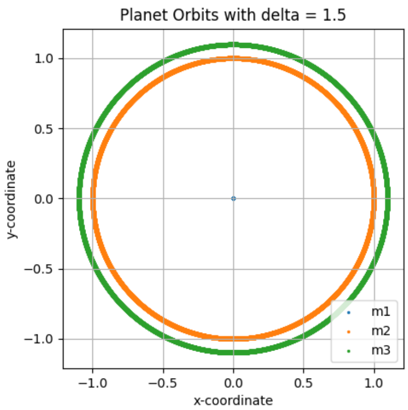
$$R_{in} = a^3\sqrt{\mu/3} \tag{8}$$

If there are no close encounters between the bodies at all times, the system is 'Hill stable' or simply stable.
We simulate the 3 body problem using REBOUND by following a similar procedure to the 2 body problem. Using the IAS15 integrator to achieve high accuracy we simulate the system for 1000 orbits for different separations between the two smaller bodies. REBOUND can automatically detect close encounters between the bodies and handle them accordingly (halting the simulation, merging the bodies, hard sphere collision or a user defined function). For our purposes we will only count the number of close encounters.

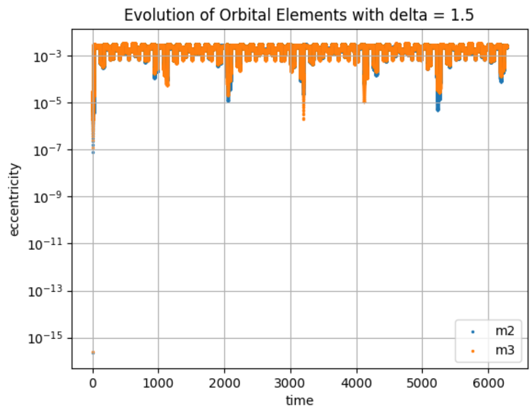
3.2.1 Equal Mass Planets

We simulate the system with $m_1 = 1$, $m_2 = m_3 = 10^-5$,$a_1 = 1$,$a_2 = 1 + \Delta$ where Δ is the separation between the two smaller bodies. The eccentricities of both bodies is initially

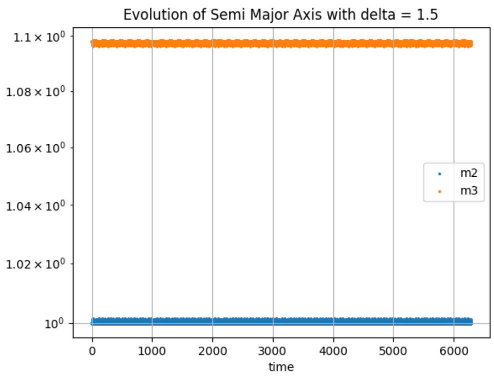
0. We create different simulations for $\Delta = 0.1, 0.5, 1, 1.5$ and 10 times the value of Δ_c for this system. As expected, for $\Delta \geq \Delta_c$ the orbits are stable. For $\Delta = 1\Delta_c$ and $\Delta = 1.5\Delta_c$ the planets are in resonances.



(a) Orbit of a 3 body system with $\Delta = 1.5\Delta_c$. The system is stable and the planets are in a 3:2 resonance



(b) Eccentricity of the planets with $\Delta = 1.5\Delta_c$. It is same for both planets throughout its evolution



(c) Semi Major Axis of the planets with $\Delta = 1.5\Delta_c$. It is basically constant

Figure 6: Properties of a 3 body system with $\Delta = 1.5\Delta_c$

For $\Delta = 10\Delta_c$, we observe the effect of tidal forces and observe the eccentricity of the closer planet increase

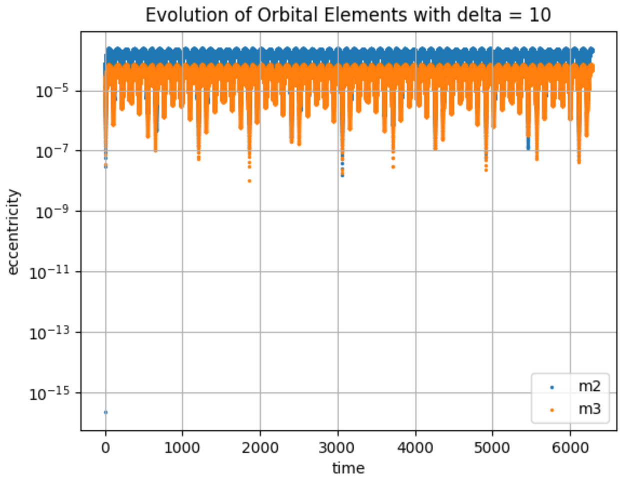
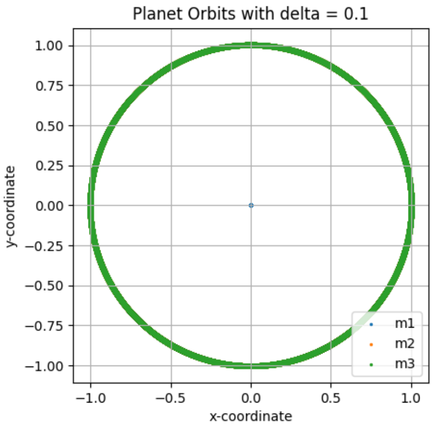
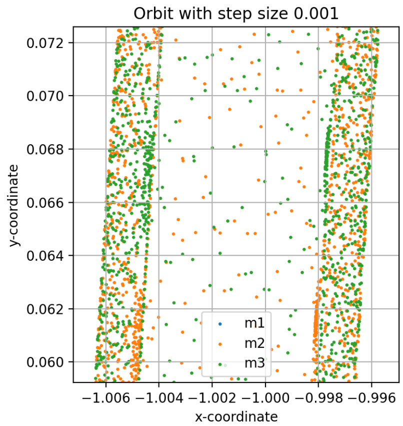


Figure 7: Eccentricity of the planets with $\Delta = 10\Delta_c$. An increase in eccentricity for the planet closer to the star(m_2) is observed

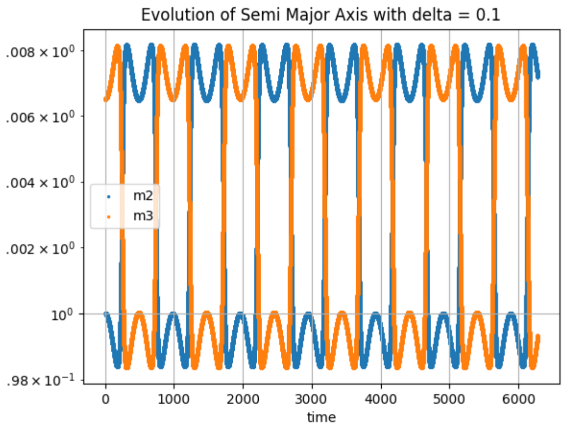
For $\Delta = 0.1\Delta_c$ we observe that the planets periodically exchange their orbits. Even though this is less than Δ_c, there are no close encounters detected by REBOUND.



(a) Orbit of a 3 body system with $\Delta = 0.1\Delta_c$. In this plot both of the planet orbits are on top of each other



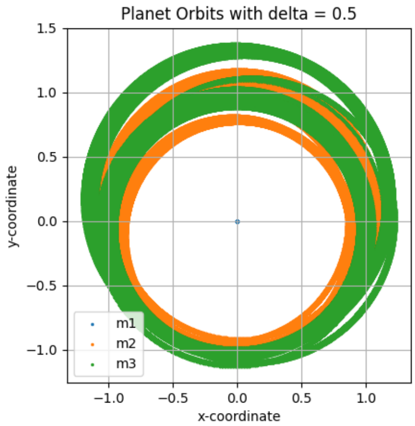
(b) A closer view of the orbit. The exchange of orbits is clearly observed here



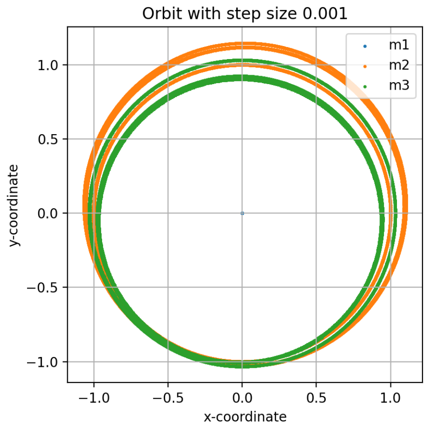
(c) Change in semi major axis of the planets with $\Delta = 0.1\Delta_c$. The orbit exchange is clearly observed here

Figure 8: Orbit of a 3 body system with $\Delta = 0.1\Delta_c$

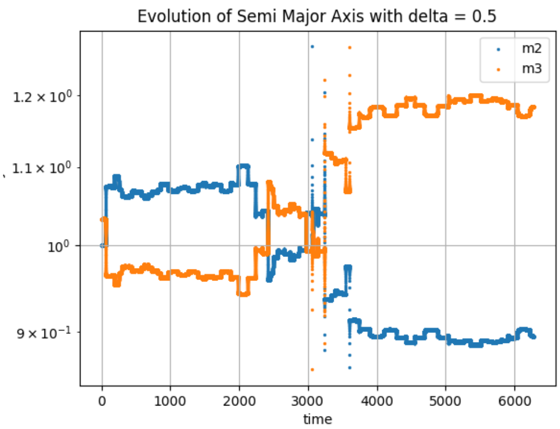
For $\Delta = 0.5\Delta_c$ the system is unstable and REBOUND detects several collisions here (in hour case$\sim$ 11,000 collisions but this can vary depending on the type of integrator or time step).



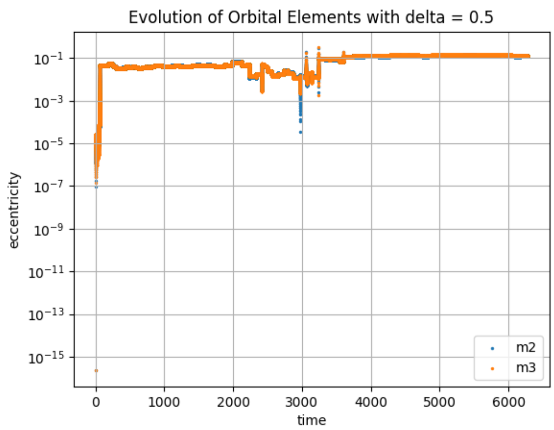
(a) Orbit of a 3 body system with $\Delta = 0.5\Delta_c$



(b) The orbit plotted for a shorter time



(c) Change in semi major axis of the planets with $\Delta = 0.5\Delta_c$

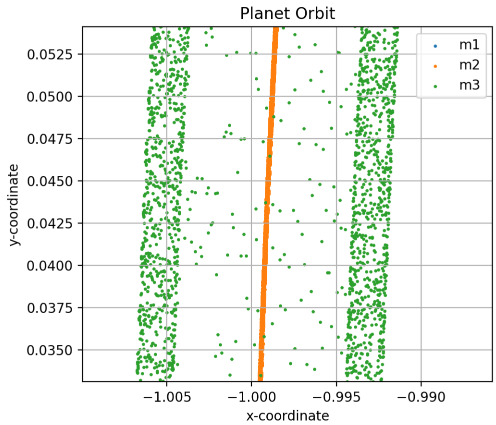


(d) Change in eccentricity of the planets with $\Delta = 0.5\Delta_c$

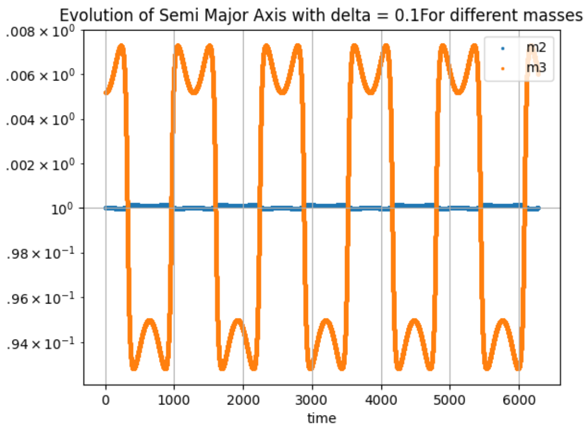
Figure 9: Properties of a 3 body system with $\Delta = 0.5\Delta_c$

3.2.2 Unequal Mass Planets

The above exercise was repeated by changing one of the masses $m_3 = 10^{-7}$.
For $\Delta = 0.1\Delta_c$ the orbit exchange is still observed, but now the smaller planet m_3 has a larger variation in semi major axis as compared to m_2.



(a) A closer view of the orbit. The exchange of orbits is clearly observed here



(b) Change in semi major axis of the planets with $\Delta = 0.1\Delta_c$ for planets of different masses. The orbit exchange is clearly observed here

Figure 10: Orbit of a 3 body system with $\Delta = 0.1\Delta_c$

Now, REBOUND detects collisions for both $\Delta = 0.5\Delta_c$ and $\Delta = 1\Delta_c$.
Systems with higher Δ_c are still stable

3.3 Jupiter and Kirkwood Gaps

In this part of the lab work we must observe Kirkwood gaps - gaps in the asteroid belt - by simulationg and observing the evolution of Sun-Jupiter-Mars system with 10000 test particles. Using REBOUND we can simulate these objects and get semi-major axis and eccentricity of all bodies up to 1 million years of evolution. Additionally, we check the change of distibution at every 100000 years.

Kirkwood gaps are caused by the resonant interaction between the asteroids and Jupiter [Gre21]. The resonant interaction leads to the instability of the orbits of the asteroids. Moreover, the instabilities cause the increase of eccentricity of the asteroids, and the asteroids with higher eccentricity are more likely to collide with inner planets. The main Kirkwood gaps are at 2.50, 2.82, 2.95, 3.27 AU corresponding to the resonant periods of 3:1, 5:2, 7:3, 2:1 respectively.

First, Sun, Jupiter and Mars are added with their corresponding masses, semi-major axes and eccentricities. Next, we add 10000 test particles in a range of 2 to 4 AU and random true anomaly and eccentricities. It is important to make only three major objects active with `sim.N_active` = `3` line so all test particles do not affect the system. Additionally, we simplify the solution by using leap-frog integrator due to the complicatied computation. The code records the values of semi-major axis a and eccentricity a of each body to .txt file for further visualization.

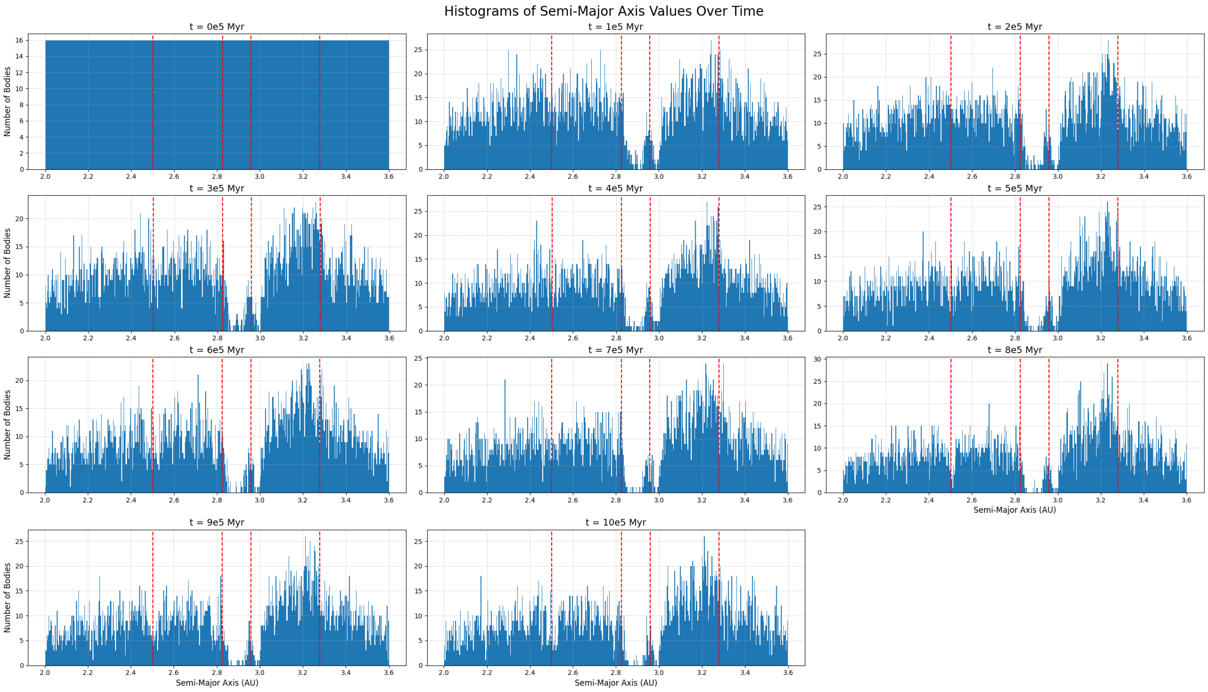


Figure 11: Histograms of the number of asteroids at various distances every 10^5 years. Red vertical lines correspond to actual Kirkwood gaps at 2.50, 2.82, 2.95, 3.27 AU.

Fig. 11 shows the histograms of the number of asteroids at various distances every 10^5 years. End result does not show the exact Kirkwood gaps due to the short timespan of the simulation and simplifications in the model and integrator. However, there are smaller number of asteroids near the expected respnance locations, especially near 2.82 AU and 2.95 AU.

Further, we can plot the semi-major axis and eccentricity of the test particles to see the evolution of the system:

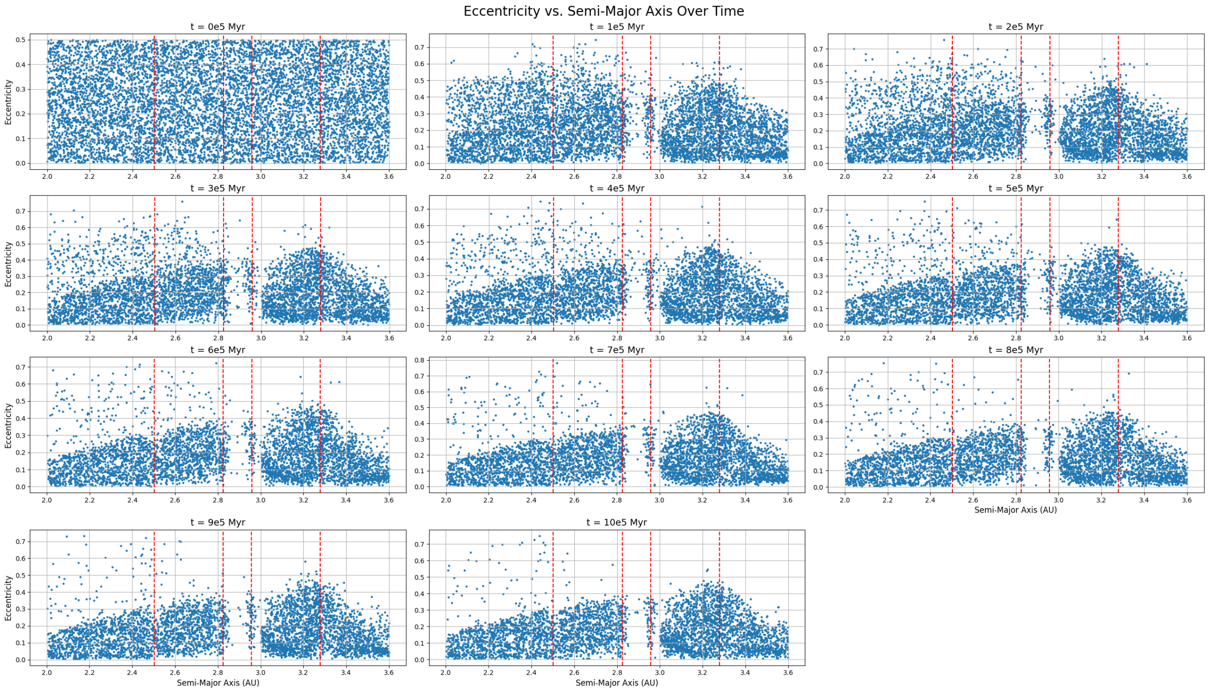


Figure 12: Eccentricity vs Semi-Major Axis of asteroids over time. The gaps and smaller eccentricity at smaller semi-major axis can be seen.

Fig. 12 shows the eccentricity vs semi-major axis of the asteroids over time. One can see gaps near 2.82 and 2.95 AU as well as decrease of the eccentricity at smaller orbits. At smaller semi-major axis, objects with higher eccentricity more likely to interact with Mars, but at some even bigger eccentricities can keep being in the system at least for 10^6 million years.

3.4 Resonant Capture of a Planet

4 Conclusions

An important section in which you should critically review the experiment and its results. Mention also parts that did not work out as expected, but keep a neutral to positive view. This can span from a few sentences to half a page.

References

[Gla93] Brett Gladman. “Dynamics of Systems of Two Close Planets”. In: *Icarus* 106 (Nov. 1993), pp. 247–263. DOI: 10.1006/icar.1993.1169.

[Gre21] Sarah Greenstreet. “Asteroids in the inner solar system”. In: *Physics Today* 74.7 (2021), pp. 42–47. DOI: 10.1063/PT.3.4794.

[RL12] H. Rein and S. -F. Liu. "REBOUND: an open-source multi-purpose N-body code for collisional dynamics". In: *aap* 537, A128 (Jan. 2012), A128. DOI: `10.1051/0004-6361/201118085`. arXiv: `1110.4876 [astro-ph.EP]`.

Appendix

Code

Please attach here your original handwritten notes and other documents created during the experiment.