EEE424 HW5 FALL 2024-25

Q1)

a) $N(t) = 50\,sinc(200t) * w(t) + 75\,sinc(100t) * w(t) = (50 sin(200t) + 75 sinc100t) * w(t)$

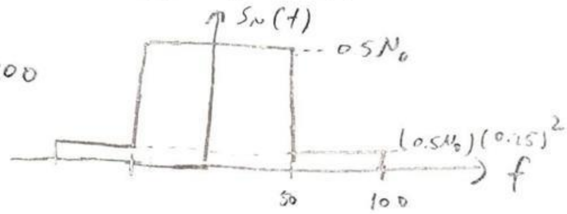
Let $50 sin 200t + 75 sin 100t = h(t)$, $N(t) = h(t) * w(t)$, $R_N(t) = R_{wN}(t) * h(-t)$

$R_N(t) = R_w(t) * h(t) * h(-t)$, $S_w(f)$ is fourier transform of auto-correlation function hence, $S_N(f) = S_w(f) \cdot |H(f)|^2$ Also $sinc(at) \xrightarrow{FT} \frac{1}{a} \cdot rect\left(\frac{f}{a}\right)$

Therefore $H(f) = \frac{50}{200} \cdot rect\left(\frac{f}{200}\right) + \frac{75}{100} \cdot rect\left(\frac{f}{100}\right)$

$H(f) = \begin{cases} 1, & |f| < 50 \\ 0.25, & 50 < |f| < 100 \\ 0, & ow \end{cases}$, $|H(f)|^2 = \begin{cases} 1, & |f| < 50 \\ (0.25)^2, & 50 < |f| < 100 \\ 0, & ow. \end{cases}$

$S_N(f) = \begin{cases} 0.5N_0, & |f| < 50 \\ (0.5N_0)(0.25)^2, & 50 < |f| < 100 \\ 0, & ow \end{cases}$



b) Inverse Fourier Transform of $S_N(f)$ gives autocorrelation function of $N(t)$, For non-zero parts, $S_N(f) = 0.03125N_0\, rect\left(\frac{f}{200}\right) + 0.46875N_0\, rect\left(\frac{f}{100}\right)$

Sums up to $0.5N_0$

$S_N(f) \xrightarrow{IFT} (6.25\,sinc(200t) + 46.875\,sinc(100t))N_0 = R_N(t)$

c) $R_N(0) = \underbrace{E[N(t)^2]}_{\text{average power}} = 46.875N_0 + 6.25N_0 = 53.125N_0$

Q2)

$Var(Y(n)) = E[Y(n)^2] - \underbrace{E[Y(n)]}_{0}^2$, $E[Y(n)] = 0$ because, $\mu_Y = H(e^{j0}) \cdot \underbrace{\mu_X}_{0}$

$E[Y(n)^2] = r_Y(0)$, $r_Y(k) = h(k) * h(-k) * r_X(k)$, first solve for $h(k) * r_X(k)$

$r_X(k) * h(k) = \frac{1}{8}\left(\frac{1}{2}\right)^{|k-3|}$, now convolve the result with $h(-k)$

$r_X(k) * h(k) * h(-k) = \frac{1}{64}\left(\frac{1}{2}\right)^{|-k|} = r_Y(k) \rightarrow$ part b

$Var(Y(n)) = r_Y(0) = \frac{1}{64} \rightarrow$ part a

Q3)

(a) Not valid since we must have $r_x(0) \geq |r_x(1)|$

(b) Power spectrum must be non-negative, Take FT, $3 + 2(e^{-jk\omega} + e^{jk\omega}) \cdot \frac{2}{2}$

$P(e^{j\omega}) = 3 + 4\cos(\omega k)$ has negative parts not valid.

(c) This auto-correlation sequence is valid and usually referred as harmonic process. If β is assumed to be a R.V which is uniformly distributed between $[-\pi, \pi]$, $x(n)$ can be generated by

$$x(n) = e^{j\left(\frac{\pi k}{4} + \beta\right)}$$

(d) If Fourier Transform of this rectangular signal is taken,

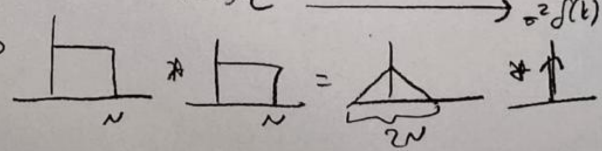
$$r(k) = \sum_{m=-N}^{N} \delta(k-m) \xrightarrow{FT} \sum_{-N}^{N} e^{-j\omega m} = \frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}} \cdot e^{j\omega N} = \frac{e^{-j\omega\frac{(2N+1)}{2}} e^{j\omega N}}{e^{\frac{-j\omega}{2}}} \cdot \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$P(e^{j\omega}) = \frac{\sin\left(\omega\left(N + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$ $\rightarrow$ similar to sinc have zero crossing which makes $P(e^{j\omega})$ negative hence not valid.

(e) when $|k| < N$ $r_x(k) = 1 - \frac{|k|}{N}$ which is a triangular shape. Notice that $r_x(k)$ is symmetric and it is the convolution of two rectangles which means in frequency domain it is a squared and shifted version of $P(e^{j\omega})$ in part (d) which makes the power spectrum non-negative hence this sequence is valid $r_x(k) = h(k) * h(-k) * \underbrace{r_w(k)}_{\text{unit variance white noise}} \longrightarrow \sigma^2\delta(k)$

Sample process generation:

$$h(n) = \begin{cases} 1 & : 0 < n < N \\ 0, & o.w \end{cases} \Rightarrow$$



$= r_x(k)$

Q4) For a process to be WSS, the mean and the auto-correlation functions must be constant.

Constant Mean: $E[X(t)] = E[A\cos(\omega t)] + E[B\sin(\omega t)]$, for constant mean $E[A]$ and $E[B]$ must be finite

Constant auto-correlation: $E[X_1(t)\, X_2(t_2)] = E\left[\left(A\cos(\omega t_1)+B\sin(\omega t_1)\right)\left(A\cos\omega t_2 + B\sin\omega_2 t\right)\right]$

After simplification it is clear that there are A^2 B^2 and AB terms hence, for constant auto-correlation $E[A^2]$, $E[B^2]$, $E[AB]$ must be finite.

Q5)

(a) Mean of $Z(t) \rightarrow E[Z(t)] = \mu_x \cdot E_f[\underbrace{E_\theta[\sin(2\pi f+\theta)]}_{\text{integral over a period is 0}}] = 0$

Auto-correlation $\rightarrow R_Z(\tau) = E[Z(t) \cdot Z(t+\tau)]$

$R_Z(\tau) = E\left[X(t) \cdot X(t+\tau) \cdot \sin(2\pi f t+\theta)\sin(2\pi f t + 2\pi f \tau + \theta)\right] = R_X(\tau) \cdot E[\sin(\varphi)\sin(\varphi+2\pi f\tau)]$

$= E_f\left[E_\theta\left[\sin\varphi \sin \varphi+2\pi f\tau\right]\right]$, $\sin(\varphi)\sin(\varphi+2\pi f\tau) = \dfrac{\cos(-2\pi f\tau) - \cos(2\varphi+2\pi f\tau)}{2}$ $\varphi = 2\pi f t + \theta$

Use $\sin A \sin B = \dfrac{\cos(A-B) - \cos(A+B)}{2}$

$\cos(-2\pi f\tau)$ does not depend on θ, integral over $\cos(2\varphi + 2\pi f\tau)$ terms results as 0

The equation becomes $\frac{1}{2} E_f[\cos(2\pi f \tau)] = \frac{1}{2} \cdot \int_0^{100} \frac{1}{100} \cdot \cos(2\pi f\tau)\, d\tau = \dfrac{\sin(200\pi\tau)}{400\pi\tau}$

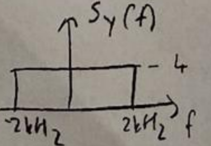
$R_Z(\tau) = R_X(\tau) \cdot \dfrac{\sin(200\pi\tau)}{400\pi\tau}$, shift invariant auto correlation and mean hence $R_Z(\tau)$ is WSS.

(b) $R_{ZY}(\tau) = E[Z(t)\, Y(t+\tau)]$, after multiplication averaging over θ in calculating E_θ will vanish any cross term hence $R_{ZY}(\tau) = 0, \forall \tau$ makes them uncorrelated processes.

(c) $E[Y(t)] = 2E[X(t)] + \frac{d}{dt} E[X(t)] = 2\mu_x + \underbrace{\frac{d}{dt}\mu_x}_{0} = 2\mu_x$

(d) In frequency domain, $Y(f) = 2X(f) + j2\pi f X(t) = (2 + j2\pi f) X(f)$

$S_Y(f) = |2 + j2\pi f|^2 S_X(f) = 4 + 4\pi^2 f^2 \cdot \frac{1}{1+\pi^2 f^2} = 4$ for $|f| < 2kHz$



, $S_Y(f) \xrightarrow{IFT} R_Y(\tau) = 4 \cdot \dfrac{\sin(\omega_c \tau)}{\pi\tau} = 4\,\dfrac{\sin(4000\pi\tau)}{\pi\tau}$

FT of sinc is rectangle

Q6) (a) $Y(t) = (X(t) + N(t)) * h(t)$, $Z(t) = (X(t) + N(t)) * h(t) - X(t)$

$Z(t) = X(t) * (h(t) - \delta(t)) + N(t) * h(t)$, $X(t)$ and $N(t)$ are zero mean WSS and mutually uncorrelated, let $h(t) - \delta(t) = h_1(t)$

$R_Z = R_X * h_1(\tau) * h_1(-\tau) + R_N * h(\tau) * h(-\tau)$, take the FT

$S_Z(f) = S_X(f) \, |H(f) - 1|^2 + S_N(f) \, |H(f)|^2$

(b) $E[Z(t)^2] = R_Z(0) = \int_{-\infty}^{\infty} S_Z(f) \cdot e^{j2\pi f \tau}\big|_{\tau=0} \, df = \int_{-\infty}^{\infty} S_Z(f) \cdot df$

$= (a-1)^2 \int S_X(f) \cdot df + a^2 \int S_N(f) \cdot df = (a-1)^2 \cdot R_X(0) + a^2 R_N(0) = L(a)$

$\underset{a}{\operatorname{argmin}} \, L(a) = 2(a^* - 1) R_X(0) + 2a^* R_N(0) = 0$, $a^* = \dfrac{R_X(0)}{R_X(0) + R_N(0)}$