EEE 424 HW1 FALL 2024-25

Q 1) $x(t) = e^{-2t}u(t)\sum_{n=-\infty}^{\infty}\delta(t-\frac{n}{2})$, Let $x_1(t) = e^{-2t}u(t)$

$x_2(t) = \sum_{n=-\infty}^{\infty}\delta(t-\frac{n}{2})$, $x(t) = x_1(t).x_2(t)$, $X(j\omega) = X_1(j\omega) \ast X_2(j\omega)$

Note that $\sum_{n=-\infty}^{\infty}\delta(t-nT) \xrightarrow{FT} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{2\pi k}{T})$

For $x_2(t)$, $T=\frac{1}{2}$, $X_2(j\omega) = 4\pi\cdot\sum_{k=-\infty}^{\infty}\delta(\omega-4\pi k)$

$u(t)e^{-at} \xrightarrow{FT} \frac{1}{j\omega+a} \Rightarrow e^{-2t}.u(t) \xrightarrow{FT} \frac{1}{j\omega+2} = X_1(j\omega)$

$X(j\omega) = X_1(j\omega) \ast X_2(j\omega) = \frac{1}{j\omega+2} \ast 4\pi\sum_{k=-\infty}^{\infty}\delta(\omega-4\pi k)$

By shifting property, $\boxed{X(j\omega) = 4\pi\sum_{k=-\infty}^{\infty}\frac{1}{j(\omega-4\pi k)+2}}$

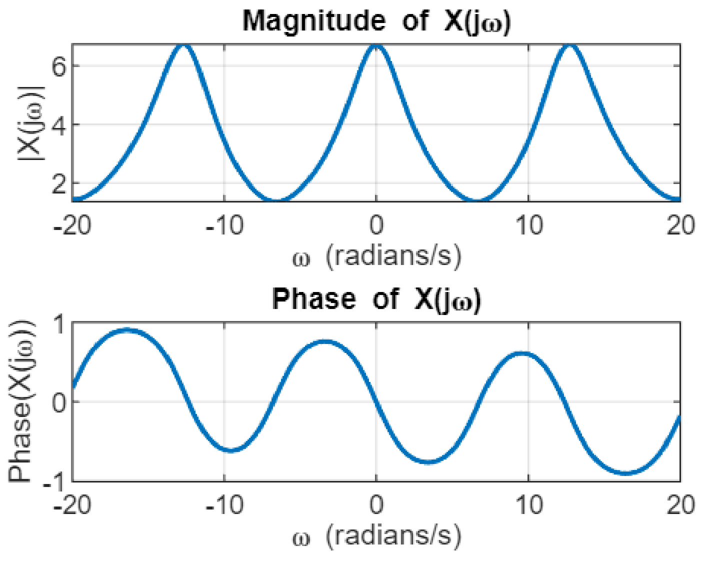


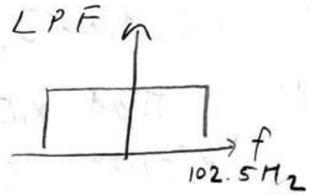
Figure 1 Magnitude and phase plots of CTFT of x(t) for k∈[-1,1]

Check Appendix A for the MATLAB code.

Q2) ⓐ $X(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{\frac{j2\pi k t}{T_p}} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} \cdot e^{j20\pi k t}$

$X(t) = \ldots\ldots \underset{k=0}{1} + \underbrace{\frac{1}{2}\cdot e^{j20\pi t}}_{\substack{k=1 \\ \frac{1}{2}\cos(2\pi\cdot 10 t) \\ 10Hz}} + \underbrace{\frac{1}{4}\cdot e^{j40\pi t}}_{\substack{k=2 \\ \frac{1}{2}\cos(2\pi\cdot 20 t) \\ 20Hz}}$,

LPF

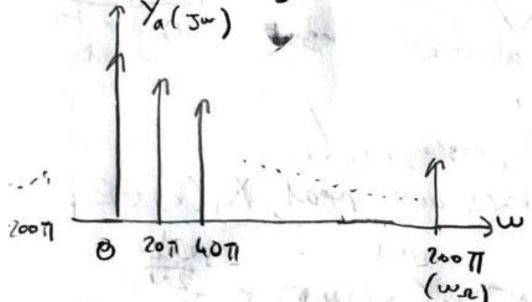


The LPF cuts the frequency content of x(t) up to $\left(\frac{1}{2}\right)^{10} e^{j200\pi t}$ then, $\rightarrow \cos(2\pi\cdot 100 t)$ $100Hz < 102.5Hz$

$y_a(t) = \sum_{k=-10}^{10} \left(\frac{1}{2}\right)^{|k|} \cdot e^{j20\pi k t}$

The frequency content of $y_a(t)$ is a decaying impulse train up to $w_\Omega = 200\pi$

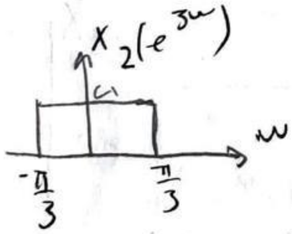
$e^{jw_0 t} \xrightarrow{FT} 2\pi\delta(w-w_0)$



$* \underbrace{400\pi \cdot \sum_{k=-\infty}^{\infty} \delta(w - 400\pi k)}_{\text{sampling}}$ $\rightarrow 2\pi\cdot 200$ 200Hz $\rightarrow$ periodic with with 200Hz

Since $400\pi \geq 2w_\Omega$ no aliasing occurs therefore the magnitude spectrum is $400\pi \cdot |Y_a(jw)|$, where frequency spectrum is $f \in [-100Hz, 100Hz]$ periodic with $\frac{1}{200}$ s (200Hz)

ⓑ $\left(X[n] \sum_{k=-\infty}^{\infty} \delta[n-3k]\right) * \underbrace{\frac{\sin\frac{\pi}{3}n}{\frac{\pi}{3}n}}_{X_2[n]} \xrightarrow{DTFT} \left(\frac{2\pi}{3} X(e^{jw}) * \sum_{k=-\infty}^{\infty} \delta\left(w - \frac{2\pi k}{3}\right)\right) \cdot X_2(e^{jw})$



, $X(e^{jw}) = \frac{2\pi}{3} \cdot \sum_{k=-\infty}^{\infty} X\left(e^{j\left(w - \frac{2\pi}{3}k\right)}\right) \cdot \underbrace{X_2(e^{jw})}_{\substack{c_1,\ |w| < \frac{\pi}{3} \\ 0,\ |w| > \frac{\pi}{3}}}$

For some w^*, $X(e^{jw^*}) = 0$, $X(e^{jw^*}) = \sum_{k=-\infty}^{\infty} X\left(e^{j\left(w^* - \frac{2\pi k}{3}\right)}\right) \cdot \underbrace{X_2(e^{jw})}_{\substack{c_1,\ |w^*| < \frac{\pi}{3} \\ 0,\ |w^*| > \frac{\pi}{3}}}$

if $|w^*| > \frac{\pi}{3}$ the equation becomes $X(e^{jw^*}) = 0$ due to the window function.

$$\boxed{X(e^{jw}) = 0 \quad \text{for } |w| > \frac{\pi}{3}}$$ $\rightarrow$ for guranteed case

Q3)

(a) $a_k = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi k}{N} \cdot n} = \sum_{n=0}^{N-1} \left(a e^{-j\frac{2\pi k}{N}}\right)^n$

$\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$, $r = a \cdot e^{-j\frac{2\pi k}{N}}$, $e^{-2\pi j k} = 1$, $\boxed{a_k = \frac{1-a^N}{1-a e^{-j\frac{2\pi k}{N}}}}$

(b) $b_k = \sum_{n=0}^{N-1} e^{j\frac{2\pi k_0 n}{N}} \cdot e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(k_0-k)}$

$b_k = \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi(k_0-k)}{N}}\right)^n$, use the same property

$b_k = \frac{1-e^{j2\pi(k_0-k)}}{1-e^{j\frac{2\pi(k_0-k)}{N}}}$ if $k=k_0$ $b_k = N$

if $k \neq k_0$ $b_k = 0$ $(\cos(2\pi k^* = 0))$

Hence $\boxed{b_k = N \cdot \delta(k-k_0)}$

(c) $x[n] = \cos\left(\frac{2\pi}{N} k_0 n\right) = \frac{1}{2} \underbrace{e^{j\frac{2\pi k_0 n}{N}}}_{y[n]} + \frac{1}{2} \underbrace{e^{-j\frac{2\pi k_0 n}{N}}}_{h[n]}$

$c_k = \underbrace{\frac{1}{2}\sum_{n=0}^{N-1} y[n] \cdot e^{-j\frac{2\pi k n}{N}}}_{S_{1k}} + \underbrace{\frac{1}{2}\sum_{n=0}^{N-1} h[n] \cdot e^{-j\frac{2\pi k n}{N}}}_{S_{2k}}$

from part b, $S_1 k = \frac{N}{2} \cdot \delta(k-k_0)$, similarly $S_2 k = \frac{1}{2} \frac{1-e^{j2\pi(k_0+k)}}{1-e^{j\frac{2\pi(k_0-k)}{N}}}$

if $k_0 = -k \Rightarrow S_2 k = \frac{N}{2}$, if $k \neq -k \Rightarrow S_2 k = 0$

hence, $S_2 k = \frac{N}{2} \delta(k+k_0)$, $\boxed{c_k = \frac{N}{2}\delta(k-k_0) + \frac{N}{2}\delta(k+k_0)}$

(d) $x[n] = \begin{cases} 1, & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad 0 \le n \le \underbrace{N-1}_{\text{even}}$

$X(k) = \sum_{n=0}^{N-1} x[n] \cdot e^{-\frac{j2\pi k n}{N}}$, let $n = 2m$

$a_k = \sum_{m=0}^{\frac{N-1}{2}} e^{-\frac{j4\pi k m}{N}} = \sum_{m=0}^{\frac{N-1}{2}} \left(e^{-\frac{j4\pi k}{N}}\right)^m = \frac{1 - e^{-\frac{j2\pi k(N+1)}{N}}}{1 - e^{-\frac{j4\pi k}{N}}}$

$a_k = \frac{1 - \overbrace{e^{-j2\pi k}}^{1} \cdot e^{-\frac{j2\pi k}{N}}}{1 - e^{-\frac{j4\pi k}{N}}} = \frac{1 - e^{-\frac{j2\pi k}{N}}}{1 - e^{-\frac{j4\pi k}{N}}}$

$$\boxed{a_k = \begin{cases} 1/2 & k = 0 \\ \frac{1 - e^{-\frac{j2\pi k}{N}}}{1 - e^{-\frac{j4\pi k}{N}}} & k \neq 0 \end{cases}}$$

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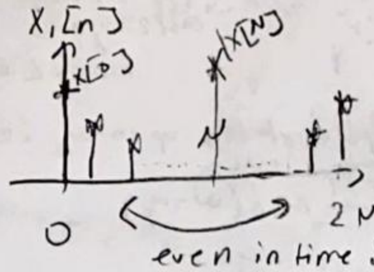
Q4) (a) For $n=0$ and $n=N$ $x_1[0]=x[0]$, $x_1[N]=(-1)^k \cancel{x}[N]$

for $n \in [1, N-1]$, $x_1[n] = c \cdot x[n]$, for $n \in [N+1, 2N-1]$
$x_1[n] = c \cdot x[2N-m]$

Notice that $x_1[n]$ is symetric around $n=N$

ie. $x_1[N-1] = c \cdot x[N-1]$, $x_1[N+1] = c \cdot x[N-1]$ which is even symetry.

Possible interperation of $x_1[n]$ just for demonstration purposes



$$X_1[n] = x[0] + (-1)^k x[N] + \underset{even}{x[n]}$$

$$X_1[n] = x[0] + x[N] + \sum x_k \cdot \cos(w_k n) + j \underbrace{\sum x_i \sin(w_k}_{0 \text{ due to being even}}$$

all $\in R$

$$x_1[n] = \left(x[0] e^{\frac{-j2\pi 0k}{N}} + (-1)^k \cancel{x}[N]\right) + x_1 \cdot \cos(w_1 n) + x_2 \cos(w_2 n) \ldots$$

Since every coeffitient of basis vectors are real, $X_1(k) \in R$.

(b) $X_1(k) = x[0] + c \sum_{n=1}^{N-1} x[n] \cdot e^{-j\frac{2\pi kn}{2N}} + (-1)^k N + c \sum_{n=N+1}^{2N-1} x[2N-n] \, e^{-j\frac{2\pi kn}{2N}}$

$$\sum_{n=N+1}^{2N-1} x[2N-n] \cdot e^{-j\frac{2\pi kn}{2N}} \longrightarrow \sum^{N-1} x[n] \cdot \underbrace{e^{-j\frac{2\pi k(2N-n)}{2N}}}_{\underbrace{e^{-j2\pi k}}_{1} \cdot e^{+j\frac{2\pi kn}{2N}}}$$

$$X_1(k) = x[0] + (-1)^k x[N] + 2c \sum_{n=1}^{N-1} \frac{1}{2} x[n] \cdot \left(e^{-j\frac{2\pi kn}{2N}} + e^{j\frac{2\pi kn}{2N}}\right), \quad \frac{e^{-j\frac{2\pi kn}{2N}} + e^{j\frac{2\pi kn}{2N}}}{2} = \cos\left(\frac{2\pi kn}{2N}\right)$$

divide and mutiply by 2

$$X_1(k) = x[0] + (-1)^k x[N] + 2c \sum_{n=1}^{N-1} x[n] \cdot \cos\left(\frac{2\pi kn}{2N}\right)$$

$$X_1(k) = \sum_{n=0}^{N} x[n] \, \phi_k[n] \Rightarrow \boxed{\phi_k[n] = \begin{cases} 1, & n=0 \\ 2c\cos\left(\frac{\pi kn}{N}\right) & n=[1, N-1] \\ (-1)^k, & n=N \end{cases}}$$

(c) $X_1(-k) = \sum_{n=0}^{2N-1} x_1[n] \cdot e^{-j\frac{2\pi}{2N}(-k)n}$

$X(-k) = \sum_{n=0}^{2N-1} x_1[n] \cdot e^{j\frac{2\pi}{2N}kn}$ According to Hermitian Symetry, $X_1(k) = X_1^*(-k \bmod 2N)$

In part (a) it is proven that $X_1(k) \in R$

Hence, $X_1(k) = X_1(-k \bmod 2N)$

Q5)

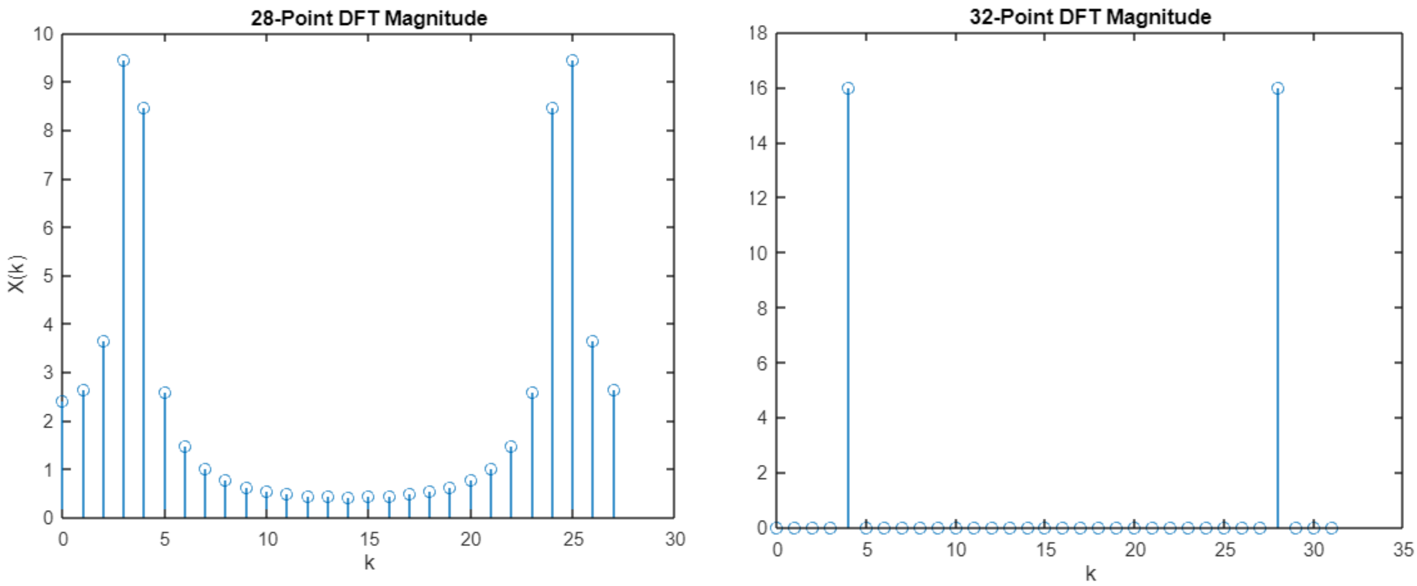


Figure 2 28 point and 32 point DFT of sin(πt/4)

In the MATLAB code the signal is sampled 28 and 32 times respectively in order to compute the 28 and 32 point DFT. In addition, in an N point DFT X(k) 0 X(k+N) hence each of the plots are plotted up to k=28 and k=32 due to periodicity.

The signal sin(π/4) can be expressed as difference of two complex exponentials analytically with corresponding coefficients o the terms when k=4 and k=28. Meaning that, it is expected to have only two frequency elements in a DFT. The main difference between 28 point 32 point DFT is that N=32 is such a integer that the frequency π/4 can be experessed by using only two term (i.e. πk/16). Whereas, in the N=28 (i.e. πk/14) case, multiple k values necessary to construct the desired frequency. Therefore multiple k values observed in 28 point DFT plot.

Check Appendix B for the MATLAB code.

Appendix A

MATLAB code for Q1:

```
num_terms = 3;
omega = linspace(-20, 20, 1000);
X = zeros(size(omega));
for k = -num_terms:num_terms
    X = X + 4 * pi ./ (2 + 1j * (omega - 4 * k * pi));
end
figure;
subplot(2,1,1);
plot(omega, abs(X), 'LineWidth', 1.5);
xlabel('\omega (radians/s)');
ylabel('|X(j\omega)|');
title('Magnitude of X(j\omega)');
grid on;


subplot(2,1,2);
plot(omega, angle(X), 'LineWidth', 1.5);
xlabel('\omega (radians/s)');
ylabel('Phase(X(j\omega))');
title('Phase of X(j\omega)');
grid on;
```

Appendix B

MATLAB code for Q5:

```
f = 1/8;
T = 1;
Fs1 = 32;
Fs2 = 28;
t_32 = (0:Fs1-1)*T;
 t_28 = (0:27)*T;
x_32 = sin(2*pi*f*t_32);
x_28 = sin(2*pi*f*t_28);
N_32 = 32;
N_28 = 28;
X_32 = zeros(1, N_32);
X_28 = zeros(1, N_28);
for k = 0:N_32-1
    for n = 0:N_32-1
        X_32(k+1) = X_32(k+1) + x_32(n+1) * exp(-1j*2*pi*k*n/N_32);
    end
end
for k = 0:N_28-1
    for n = 0:N_28-1
        X_28(k+1) = X_28(k+1) + x_28(n+1) * exp(-1j*2*pi*k*n/N_28);
    end
end
```

```
figure;
stem(0:N_32-1, abs(X_32));
title('32-Point DFT Magnitude');
xlabel('k');
ylabel('X(k)');
figure;
stem(0:N_28-1, abs(X_28));
title('28-Point DFT Magnitude');
xlabel('k');
ylabel('X(k)');
```