# EEE424 HW2 FALL 2024-25

## Q1)

$\phi_k[n] = e^{\frac{j2\pi kn}{N}}$, $k = 0 \ldots N-1$, $\phi_\ell[n] = e^{\frac{j2\pi \ell n}{N}}$

$\phi_\ell^*[n] = e^{\frac{-j2\pi \ell n}{N}}$, $\sum_{n=0}^{N-1} e^{\frac{j2\pi k \cdot n}{N}} \cdot e^{\frac{-j2\pi \ell n}{N}} = \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi (k-\ell)}{N}}\right)^n$

$= \frac{1 - e^{j2\pi(k-\ell)}}{1 - e^{\frac{j2\pi (k-\ell)}{N}}}$, $\sum_{n=0}^{N-1} \phi_k[n]\,\phi_\ell^*[n] = \begin{cases} N\cdot\delta(k-\ell), & k = \ell \\ 0, & k \neq \ell \end{cases}$

Hence, $\sum_{n=0}^{N-1} \phi_k[n]\,\phi_\ell^*[n] = 0$ for $k \neq \ell$.

## Q2)

$\sin(m\omega t)\cdot\sin(n\omega t) = \frac{1}{2}\cdot\left(\cos((m-n)\omega t) - \cos((m+n)\omega t)\right)$

$\int_{t_0}^{t_0+T} \sin(m\omega t)\cdot\sin(n\omega t)\,dt = \frac{1}{2}\int_{t_0}^{t_0+T} \cos((m-n)\omega t)\,dt - \frac{1}{2}\int_{t_0}^{t_0+T}\cos((m+n)\omega t)\,dt$

if $m \neq n$ the integral becomes $\frac{1}{2}\int_{t_0}^{t_0+T}\cos(k_1\omega t)\,dt - \frac{1}{2}\int_{t_0}^{t_0+T}\cos(k_2\omega t)\,dt$ where $\{k_1, k_2\} \in \mathbb{Z}$. For any $k \in \mathbb{Z}$ $\int_{t_0}^{t_0+T}\cos(k\omega t)\,dt = 0$, hence, $\int_{t_0}^{t_0+T}\sin(m\omega t)\cdot\sin(n\omega t)\,dt = 0$ if $m \neq n$, $m = n \Rightarrow \frac{1}{2}\int_{t_0}^{t_0+T}\cos((m-n)\omega t)\,dt = \frac{1}{2}\int_{t_0} 1\,dt = \frac{T}{2}$

$\int_{t_0}^{t_0+T}\sin(m\omega t)\cdot\sin(n\omega t)\,dt = \frac{T}{2}$ if $m = n$. Similarly,

$\cos(m\omega t)\cdot\cos(n\omega t) = \frac{1}{2}\left(\cos((m+n)\omega t) + \cos((m-n)\omega t)\right)$

$\int_{t_0}^{t_0+T}\cos(m\omega t)\cdot\cos(n\omega t)\,dt = \frac{1}{2}\int_{t_0}^{t_0+T}\cos((m+n)\omega t)\,dt + \frac{1}{2}\int_{t_0}^{t_0+T}\cos((m-n)\omega t)\,dt$, Again let $m+n = k_1$, $m-n = k_2$, $\{k_1, k_2\} \in \mathbb{Z}$, for any $k \in \mathbb{Z}$ $\int_{t_0}^{t_0+T}\cos(k\omega t)\,dt = 0$ hence,

$\int_{t_0}^{t_0+T}\cos(m\omega t)\cdot\cos(n\omega t)\,dt = 0$ if $m \neq n$. $m = n \Rightarrow \frac{1}{2}\int_{t_0}\cos((m-n)\omega t)\,dt = \frac{1}{2}\int_{t_0}^{t_0+T} 1\,dt = \frac{T}{2}$

$\int_{t_0}^{t_0+T}\cos(m\omega t)\cdot\cos(n\omega t)\,dt = \frac{T}{2}$ if $m = n$.

Q3) $\alpha_i = \langle x, \phi_i \rangle$, $x = \sum_{i=0}^{N-1} \alpha_i \phi_i$, $\beta_i = \langle x, \phi_i \rangle$

$\beta_i = \langle \alpha_i \phi_i, \phi_i \rangle$, Let $\underline{\Phi}$ matrix be $[\underline{\Phi}_0 \ \Phi_1 \ \Phi_2 \ \dots \ \underline{\Phi}_{k-1}]$

As matrix vector multiplication $\beta$ can be expressed as

$\underline{\beta} = \underline{\underline{\Phi}}^H \underline{\underline{\Phi}} \, \underline{\alpha}$ , $\langle \underline{\Phi}_i, \underline{\Phi}_j \rangle = \begin{cases} \delta_{ij}, & i=j \\ 0, & i \neq j \end{cases}$ hence, $\underline{\underline{\Phi}}^H \underline{\underline{\Phi}} = \underline{\underline{I}}$

$\underline{\beta} = \underline{\underline{I}} \cdot \underline{\alpha} = \underline{\alpha}$ , hence, $\beta_i = \alpha_i$ for $i = 0, 1, \dots, k-1$

$\| x - \hat{x} \| = \sqrt{\alpha_1^2 \|\phi_1\|^2 + \dots \alpha_{N-1} \|\phi_{N-1}\|^2 - (\beta_1 \|\phi_1\|^2 + \dots + \beta_{k-1} \|\phi_{k-1}\|)}$

$\|\phi_i\|^2 = 1$ due to orthanormality hence $\| x - \hat{x} \| = \sqrt{\sum_{i=0}^{k-1} \alpha_i^2 - \beta_i^2 + c_1}$

where $c_1$ is a constant, $c_1 = \sum_{k}^{N-1} \alpha_i$ , $\| x - \hat{x} \| = \sqrt{\sum_{i=0}^{k-1} (\alpha_i - \beta_i)(\alpha_i + \beta_i) + c_1}$

$\| x - \hat{x} \|$ is minimized if $\left| \sum_{i=0}^{k-1} (\alpha_i - \beta_i)(\alpha_i + \beta_i) \right|$ is minimized

the minimum value can be obtained when $\alpha_i = \beta_i$, $\left| \sum_{i=0}^{k-1} (\alpha_i - \beta_i)(\alpha_i + \beta_i) = 0 \right|$

which results $\| x - \hat{x} \| = \sqrt{c_1}$ .

Q4) a)

(i) We want $\underset{\hat{y}}{\operatorname{argmin}} \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 = \| y - \hat{y} \|^2$ , Let $y - \hat{y} = \underline{e}$

$\|\underline{e}\|$ is minimized if $\langle \underline{e}, f_i \rangle = 0$, Since basis$(S_2) = \{ f_1, f_2, \dots f_M \}$

$\underline{e} \perp S_2$ hence, $y - \hat{y} \perp S_2$

(ii) in part (i) it is shown that $\langle \underline{e}, f_i \rangle = 0$, Since $\{ f_1, f_2, \dots f_M \}$ is basis for $S_2$, $\hat{y} = \sum_{i=1}^{M} \beta_i f_i$, for $B_i$ constants.

$\langle \underline{e}, \hat{y} \rangle = \langle \underline{e}, \beta_i f_i \rangle = \underline{\beta} \underbrace{\langle \underline{e}, f_i \rangle}_{0} = 0$ hence $y - \hat{y} \perp \hat{y}$

(iii) $f_j$ vectors form a basis for $S_2$. $y - \hat{y} \perp S_2$ implies that $\langle y - \hat{y}, f_j \rangle = 0$ due to the fact that $f_j$'s form a basis for $S_2$ hence $y - \hat{y} \perp f_j$ for $j = 1, \dots M$.

b) From part ⓐ it is known that $\langle y-\hat{y}, p_j \rangle = 0$ for $j \in [1, M]$

substitute $\hat{y} = \sum_{i=1}^{M} c_i p_i$ to inner product, $\langle y - \sum_{i=1}^{M} c_i p_i , p_j \rangle = 0$

$\langle y, p_j \rangle - \sum_{i=1}^{M} c_i \langle p_i, p_j \rangle = 0$, $\boxed{\sum_{i=1}^{M} c_i \langle p_i, p_j \rangle = \langle y, p_j \rangle}$

c) $\hat{x} = \underset{x}{\text{argmin}} \left( \|y - Ax\|^2 \right)$, $\underbrace{\|y - Ax\|^2}_{f(x)} = (y - Ax)^T (y - Ax)$

$f(x) = y^T y - y^T A x - \underbrace{(Ax)^T y}_{y^T Ax = x^T A^T y} + x^T A^T A x$, $\nabla_x f(x) = \nabla_x(-2y^T A x) + \nabla_x (x^T A^T A x)$

$\nabla_x f(x) = -2A^T y + 2A^T A x$, equate to zero for $\hat{x}$, $-2A^T y + 2A^T A x = 0$

$A^T A \underset{\hat{x}}{x} = A^T y$, $\boxed{\hat{x} = (A^T A)^{-1} A^T y \text{ where } (A^T A)^{-1} A^T = A_{ps}}$

ⓓ $\hat{x} = (A^T A)^{-1} A^T y$, $A = \begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$ $y = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}_{3\times 1}$

$A^T A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$, $\det(A^T A) = 14$

$(A^T A)^{-1} = \frac{1}{14} \cdot \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$, $(A^T A)^{-1} A^T = \frac{1}{14} \cdot \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}_{2\times 3}$

$= \frac{1}{14} \cdot \begin{bmatrix} 1 & -4 & 5 \\ -4 & 2 & 8 \end{bmatrix}$, $(A^T A)^{-1} A^T y = \frac{1}{14} \cdot \begin{bmatrix} 1 & -4 & 5 \\ -4 & 2 & 8 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 13 \\ -10 \end{bmatrix}$

$\boxed{\hat{x} = \frac{1}{14} \cdot \begin{bmatrix} 13 \\ -10 \end{bmatrix}}$

ⓔ $\hat{y} = A\hat{x} = \frac{1}{14} \begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ -10 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 23 \\ -36 \\ 3 \end{bmatrix}$

$y - \hat{y} = \begin{bmatrix} 4 - \frac{23}{14} \\ -1 + \frac{36}{14} \\ 1 - \frac{3}{14} \end{bmatrix} = \begin{bmatrix} 33/14 \\ 22/14 \\ 11/14 \end{bmatrix}$,

$\langle y - \hat{y}, \hat{y} \rangle = \begin{bmatrix} \frac{23}{14} & -\frac{36}{14} & \frac{3}{14} \end{bmatrix} \begin{bmatrix} \frac{33}{14} \\ \frac{22}{14} \\ \frac{11}{14} \end{bmatrix} = 0$ hence $y - \hat{y} \perp \hat{y}$.

Q5)

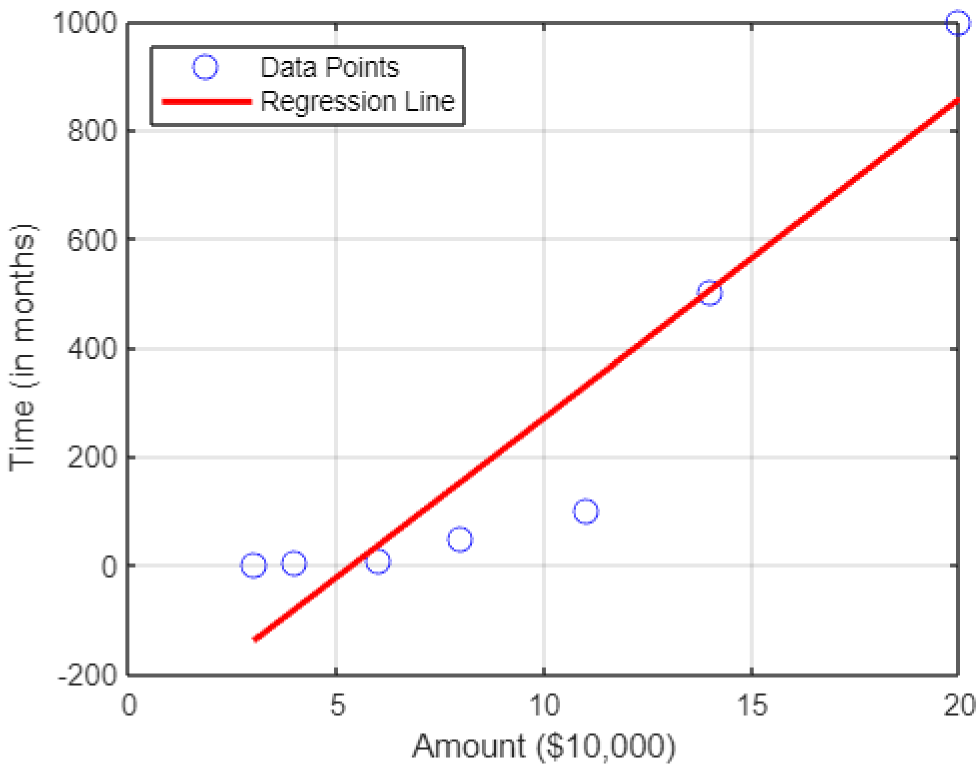


Figure 1 The amount of time for approval vs amount of fund with linear regression model and original data

MATLAB code :

```
x= [3 ;4; 6; 8; 11; 14; 20];
y = [1; 5; 10; 50; 100; 500; 1000];
A = [ones(length(x), 1), x]; % design matrix
R = transpose(A) * A;
R1 = inv(R);
q = transpose(A) * y;
c = (R1)* q;
B0 =c(1); % B0
B1= c(2); % B1
y_predict = B0 + B1*x; % test prediction
figure;
plot(x, y, 'o', 'MarkerSize', 8, 'MarkerEdgeColor', 'b', 'DisplayName', 'Data
Points');
hold on;
x_line = linspace(min(x), max(x), 100);
y_line = B0 + B1 * x_line;
plot(x_line, y_line, '-r', 'LineWidth', 2, 'DisplayName', 'Regression Line');
xlabel('Amount ($10,000)');
ylabel('Time (in months)');
legend('Location', 'NorthWest');
grid on;
hold off;
```

Q6)

| | 1 |
|---|---|
| 1 | 0.0020 |
| 2 | 0.0039 |
| 3 | 1.2732 |
| 4 | 0.0039 |
| 5 | -6.2958e-06 |
| 6 | 0.0039 |
| 7 | 0.4244 |
| 8 | 0.0039 |
| 9 | -1.2592e-05 |
| 10 | 0.0039 |
| 11 | 0.2546 |
| 12 | 0.0039 |
| 13 | -1.8888e-05 |
| 14 | 0.0039 |
| 15 | 0.1819 |
| 16 | 0.0039 |
| 17 | -2.5184e-05 |
| 18 | 0.0039 |
| 19 | 0.1415 |
| 20 | 0.0039 |
| 21 | -3.1481e-05 |

Figure 2 The Fourier coefficients for 10 harmonics

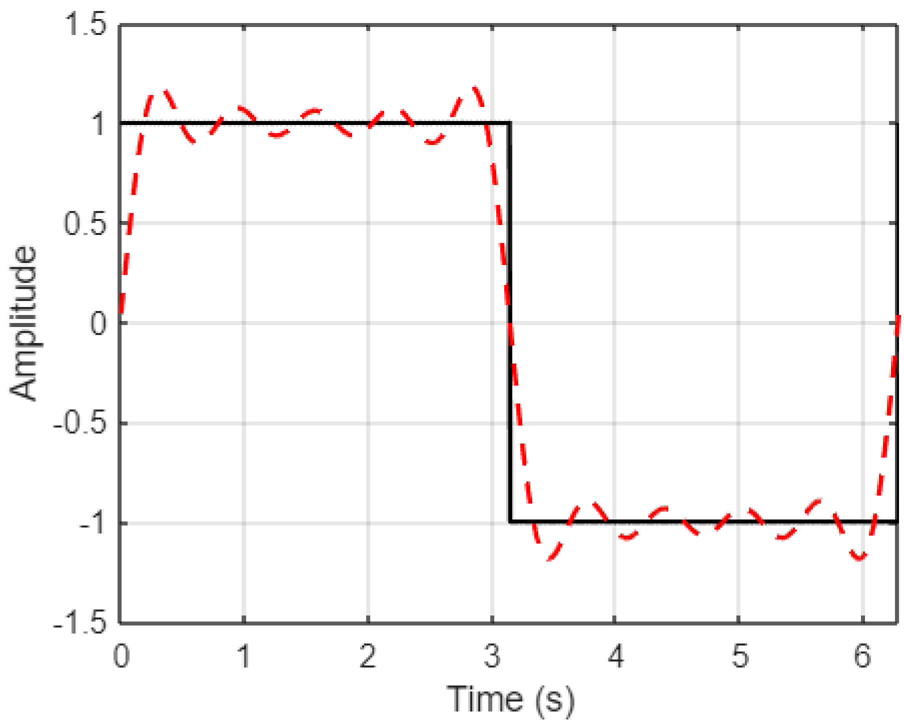


Figure 3 Approximation of the square wave with 10 harmonics

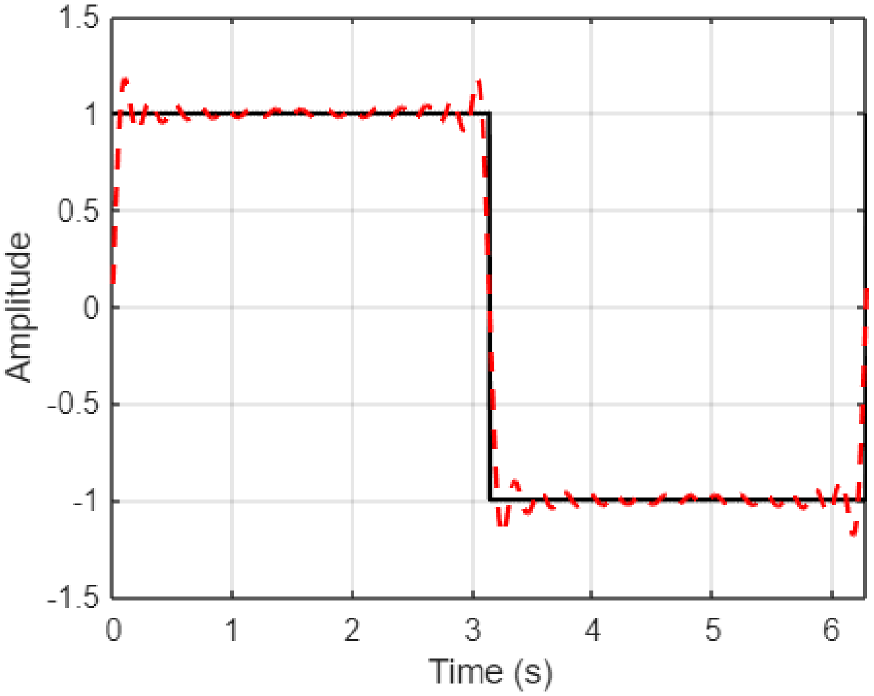


Figure 4 Approximation of the square wave with 30 harmonics

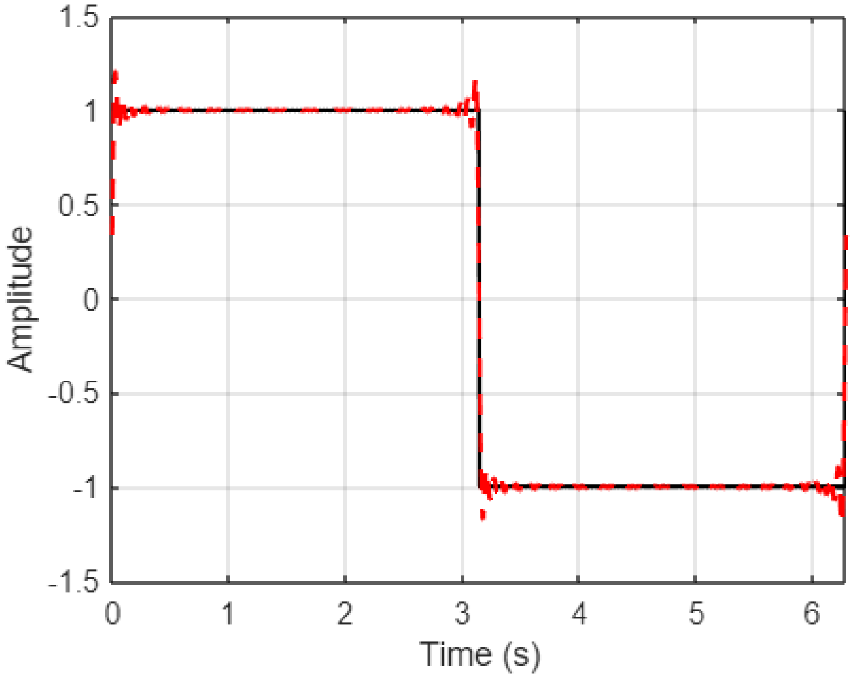


Figure 5 Approximation of the square wave with 100 harmonics

Three different figures are provided above in which 10, 30, and, 100 harmonics are used respectively in order to approximate the square wave periodic with 2π and with a ranging amplitude from -1 to 1. It is clearly observable from the figures that, as number of harmonics increases the the approximation fits better to the square wave. To explain this, think the design matrix A as the feature matrix where each of the harmonics represent a feature which are independent from each other. As the number of features increase the model is more likely to fit to the features of the desired square wave shape in a logical sense. Hence, as we increase the number of harmonics the approximation becomes more accurate.

MATLAB code :

```
T = 2 * pi;
omega = 2 * pi / T;
t = linspace(0, T, 1000);
num_coeff = 100;
square_wave = zeros(size(t));
for i = 1:length(t)
    if mod(t(i), T) < T/2
        square_wave(i) = 1;
    else
        square_wave(i) = -1;
    end
end
A = ones(length(t), 2 * num_coeff + 1);
for k = 1:num_coeff
    A(:, 2*k) = cos(k * omega * t);
    A(:, 2*k + 1) = sin(k * omega* t);
end
R = transpose(A) * A;
R1 = inv(R);
q = transpose(A) * transpose(square_wave);
coeff = (R1) * q;
approx = A* coeff;
plot(t, square_wave, 'k', 'LineWidth', 1.5, 'DisplayName', 'Original Square
Wave');
hold on;
plot(t, approx, 'r--', 'LineWidth', 1.5, 'DisplayName', sprintf('Approximation
with %d Harmonics', num_coeff));
xlabel('Time (s)');
ylabel('Amplitude');
grid on;
hold off;
```