The normal distribution

Definition of normal

- If risk factors follow GBM, then log-returns should be independent normal
- Is this the case?
- A variable x is normal if it has density:

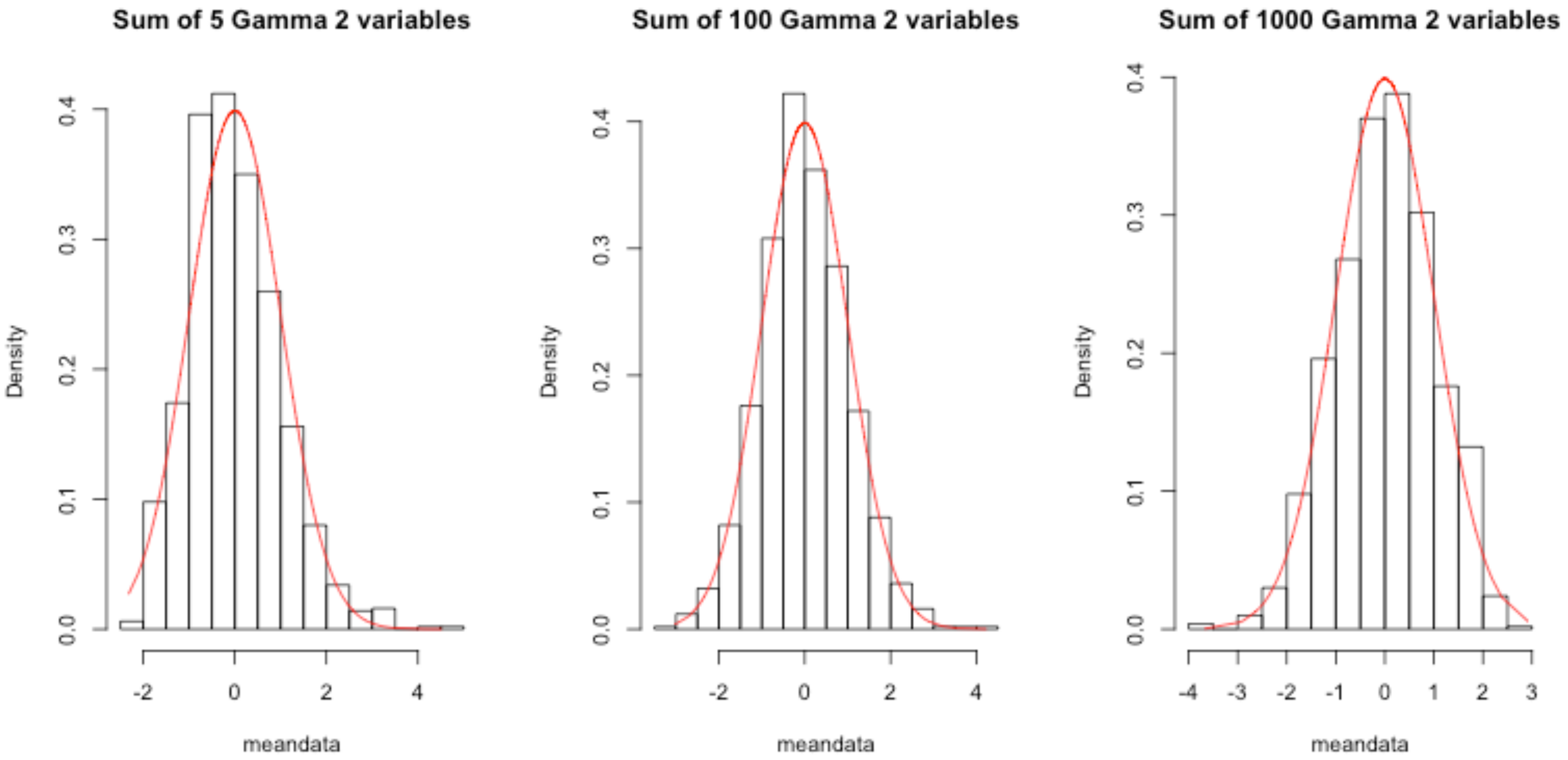
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{\sigma^2}}$$

Depends on two parameters: μ and σ

Properties of the normal

- μ is the mean and σ^2 is the variance
- Usual notation: $X \sim N(\mu, \sigma^2)$
- Parameters easily estimated from data
- Sum of 2+ independent normal variables is also normal

Central limit theorem (CLT)



How to estimate a normal distribution

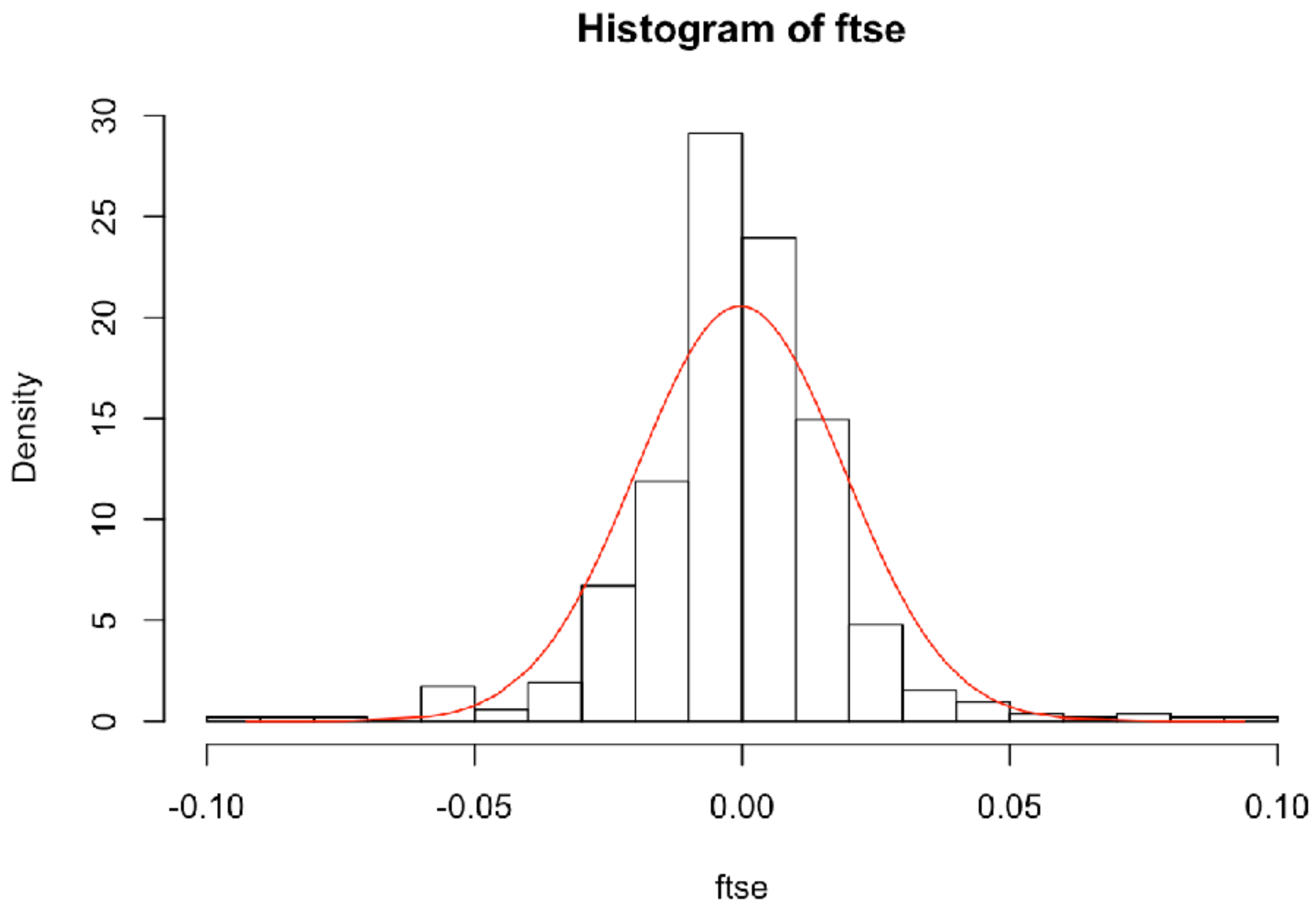
- Data: $X_1, \ldots, X_n$
- **Method of moments:** $$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} X_t$$ $$\hat{\sigma}_u^2 = \frac{1}{n-1}\sum_{t=1}^{n}(X_t - \hat{\mu})^2$$
- Application to FTSE log-returns from 2008-09

FTSE example

```
> head(ftse)
[1] -0.09264548 -0.08178433 -0.07428657 -0.05870079 -0.05637430
-0.05496918

> tail(ftse)
[1] 0.05266208 0.06006960 0.07742977 0.07936751 0.08469137
0.09384244

> mu <- mean(ftse)
> sigma <- sd(ftse)
> c(mu, sigma)
[1] -0.0003378627  0.0194090385
```

Displaying the fitted normal

```
> hist(ftse, nclass = 20, probability = TRUE)
> lines(ftse, dnorm(ftse, mean = mu, sd = sigma), col = "red")
```



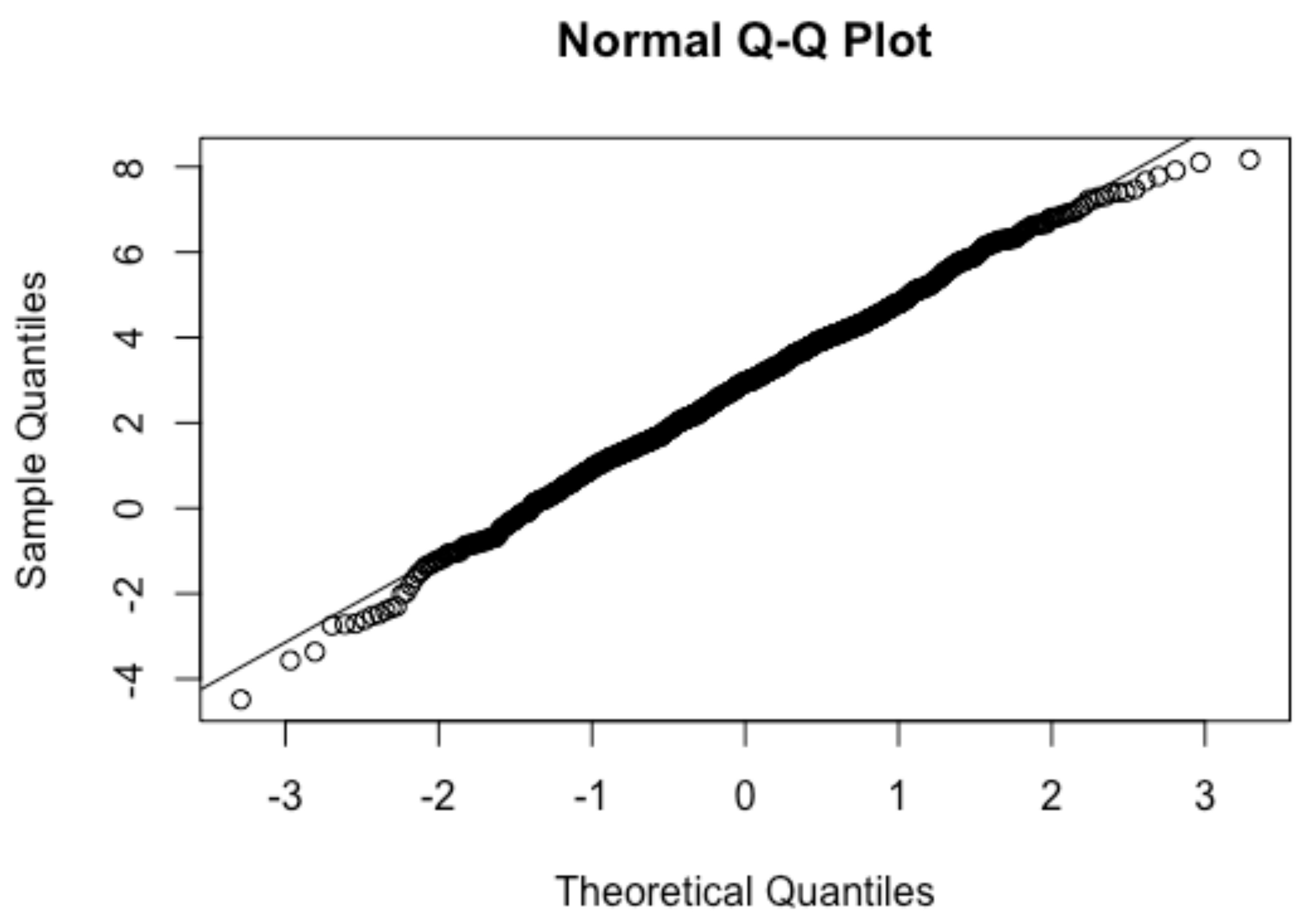
Let’s practice!

Testing for normality

How to test for normality

- Use the **quantile-quantile plot** (Q-Q plot)
- Sample quantiles of data versus theoretical quantiles of a normal distribution

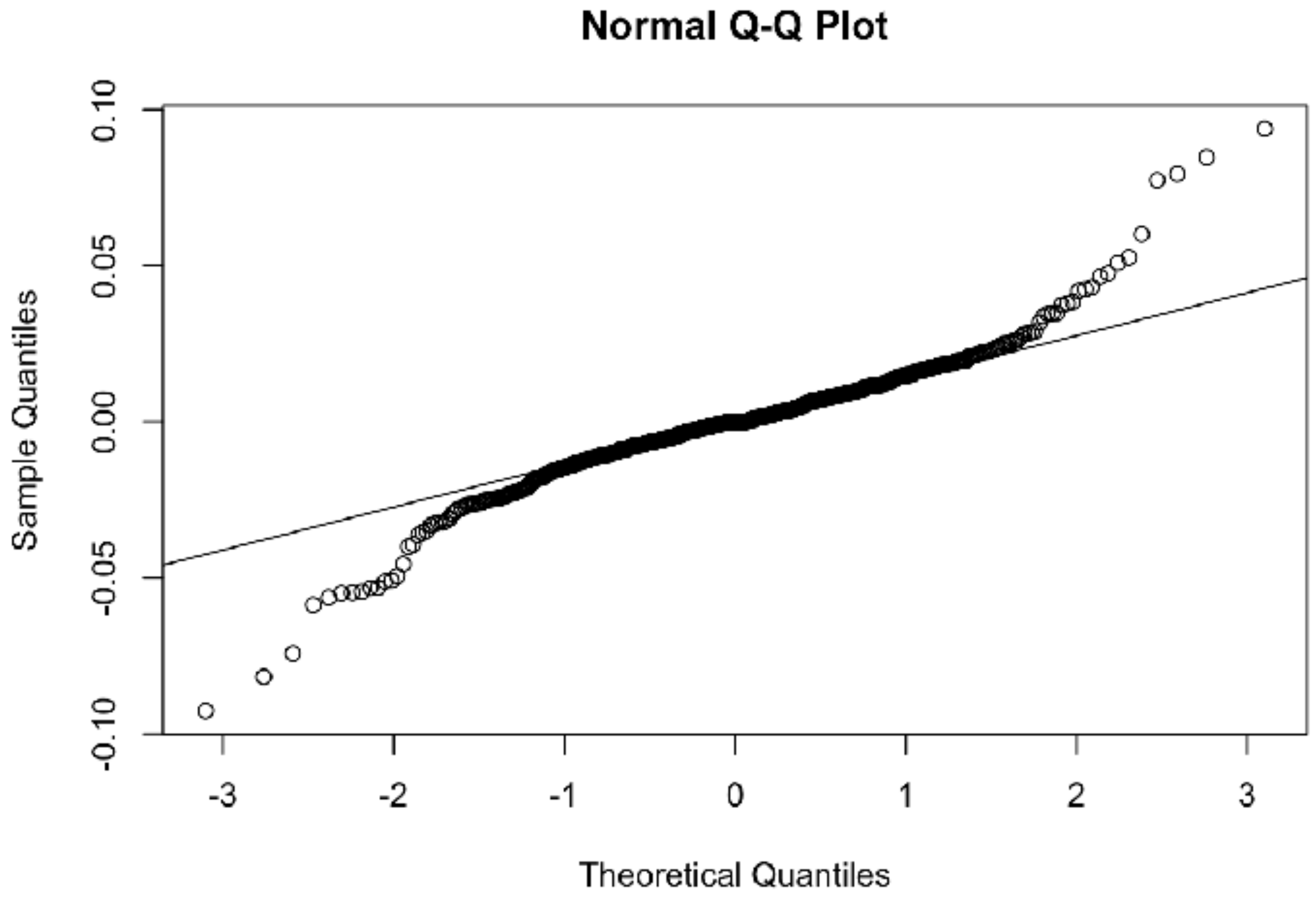
```
> data <- rnorm(1000,
                mean = 3,
                sd = 2)
> qqnorm(data)
> qqline(data)
```



Interpreting the Q-Q plot

- Data with heavier tails than normal: inverted S shape
- Data with lighter tails than normal: S shape
- Data from a very skewed distribution: curved shape

```
> qqnorm(ftse)
> qqline(ftse)
```

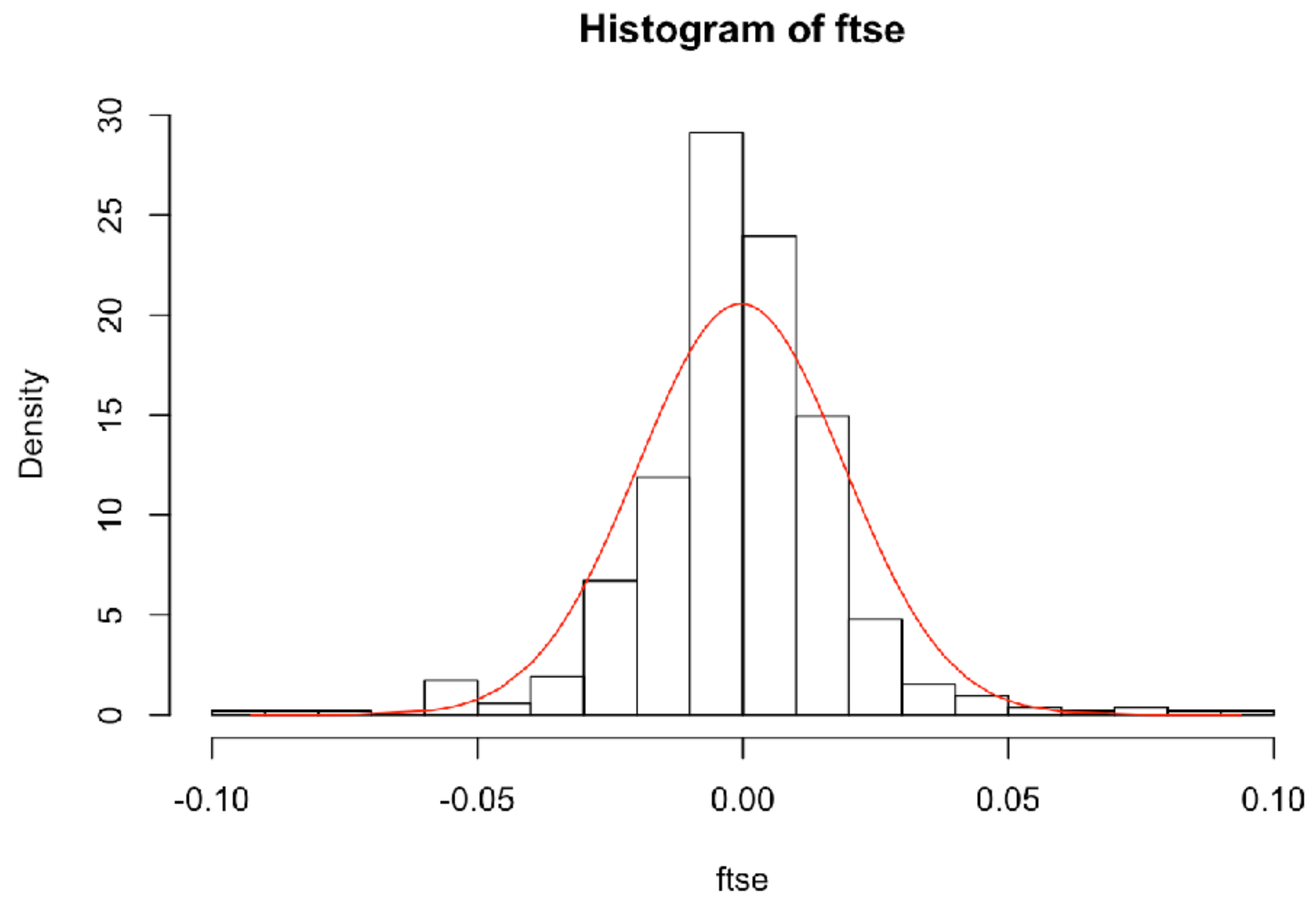


Let's practice!

Skewness, kurtosis and the Jarque-Bera test

Skewness and kurtosis

- **Skewness** (b) is a measure of asymmetry
- **Kurtosis** (k) is a measure of heavy-tailedness
- Skewness and kurtosis of normal are 0 and 3, respectively

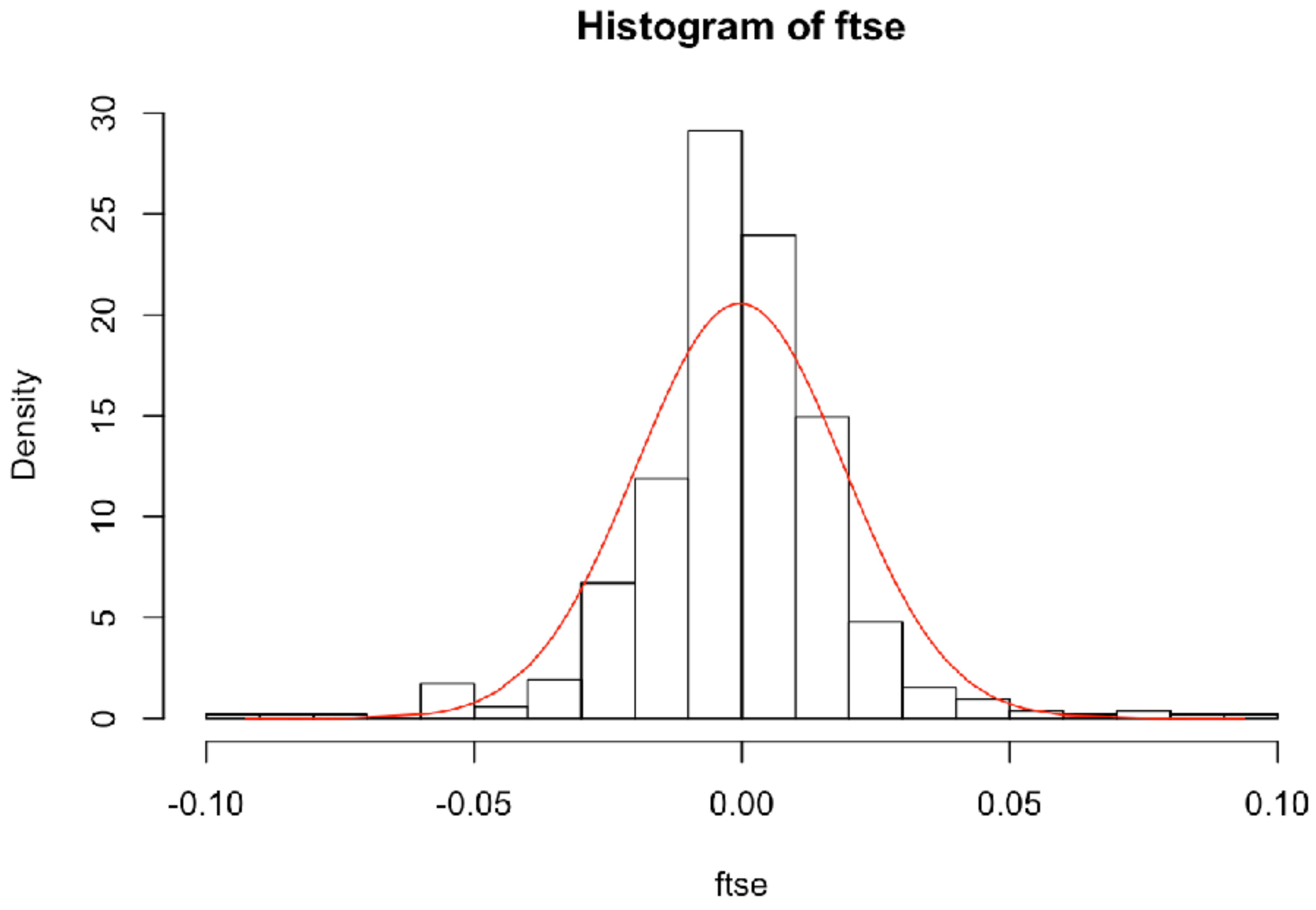


$$= \frac{1}{n} \frac{\sum_{t=1}^{n} (X_t - \hat{\mu})^3}{\hat{\sigma}^3}$$

$$= \frac{1}{n} \frac{\sum_{t=1}^{n} (X_t - \hat{\mu})^4}{\hat{\sigma}^4}$$

Skewness and kurtosis (II)

```
> library(moments)

> skewness(ftse)
[1] -0.01187921

> kurtosis(ftse)
[1] 7.437121
```



The Jarque-Bera test

- Compares skewness and kurtosis of data with theoretical normal values (0 and 3)
- Detects skewness, heavy tails, or both

$$T = \frac{1}{6}n\left(b^2 + \frac{1}{4}(k-3)^2\right)$$

```
> jarque.test(ftse)

  Jarque-Bera Normality Test

data:   ftse
JB = 428.23, p-value < 2.2e-16
alternative hypothesis: greater
```

Longer-interval and overlapping returns

- Daily returns are usually very non-normal
- What about longer-intervals returns?
- Weekly, monthly, quarterly returns obtained by summation
- Recall CLT - suggests they may be more normal
- Reduce quantity of data so tests are weaker
- Can also analyze **overlapping** or **moving sums** of returns

Let’s practice!

The Student t distribution

The Student t distribution

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sigma\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{(x-\mu)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

- This distribution has three parameters: μ, σ, ν
- Small values of ν give heavier tails
- As ν gets larger the distribution tends to normal

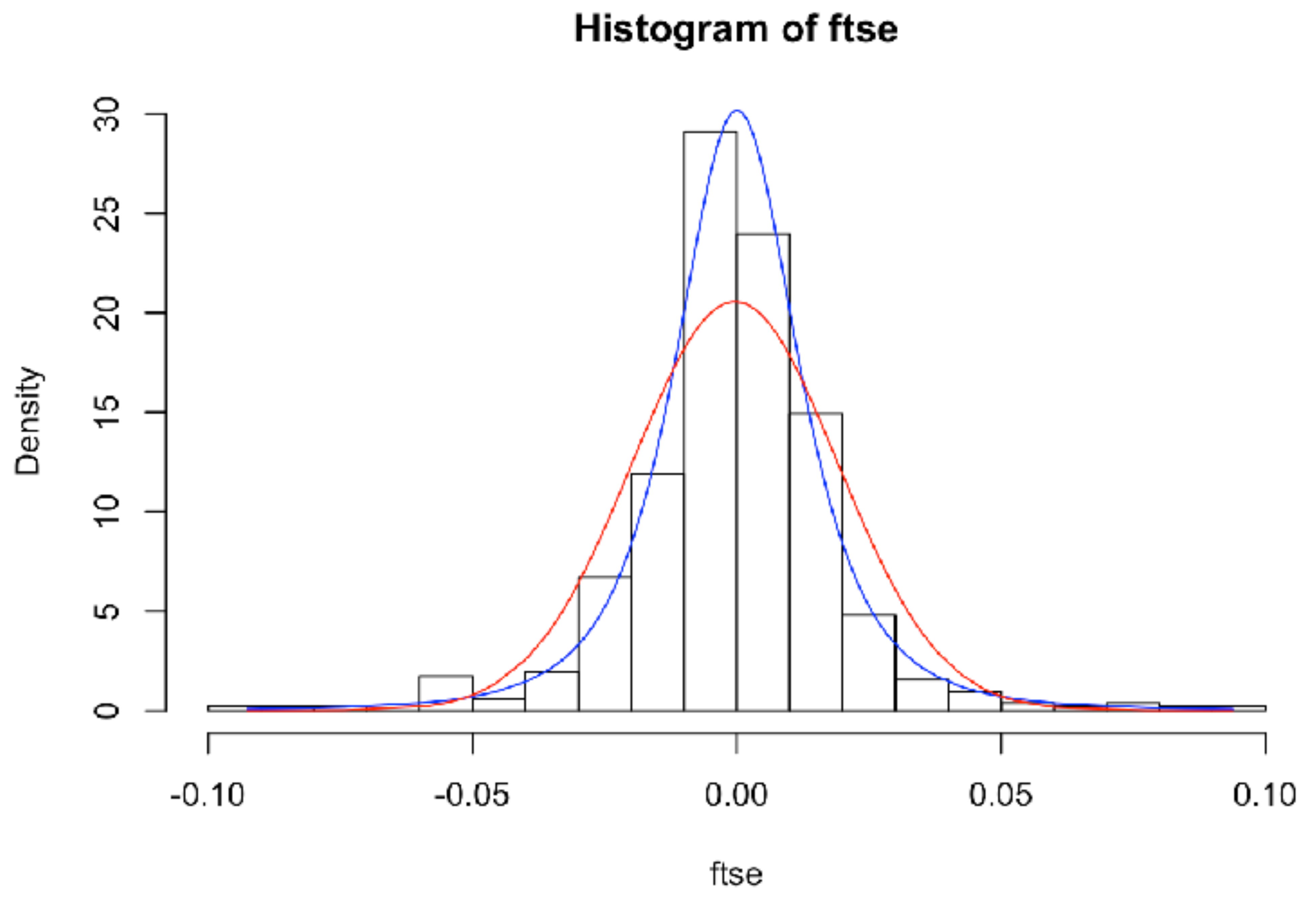
Fitting the Student t distribution

- Method of **maximum likelihood** (ML)
- `fit.st()` in QRM package
- Small ν value (2.95) for FTSE log-returns from 2008-09

```
> library(QRM)

> tfit <- fit.st(ftse)
> tpars <- tfit$par.ests
> tpars
          nu           mu        sigma
2.949514e+00 4.429863e-05 1.216422e-02

> nu <- tpars[1]
> mu <- tpars[2]
> sigma <- tpars[3]
```

Displaying the fitted Student t distribution

```
> hist(ftse, nclass = 20, probability = TRUE)
> lines(ftse, dnorm(ftse, mean = mean(ftse), sd = sd(ftse)), col = "red")

> yvals <- dt((ftse - mu)/sigma, df = nu)/sigma
> lines(ftse, yvals, col = "blue")
```



Let’s practice!