The Bootstrap

Algorithmic Trading and Quantitative Strategies[1]

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 If you find any typos, I would much appreciate you email them to me.

Readings I

In addition to the lecture notes, the following is required reading:

- Wikipedia: Resampling (statistics): Helps you compare and understand the difference between the bootstrap, jackknife and cross-validation.
- Sections 7.11 and 8.2 in Friedman, Hastie, and Tibshirani (2017).
- Sections 4 and 5 in MacKinnon (2002) (you can skip "Simultaneous equations models")

Readings II

Optional reading:

- Wikipedia: Bootstrapping (statistics) provides a simple overview of different bootstrapping technqiues.
- Shao and Tu (2012) provide a systematic treatment of the theory and applications of the bootstrap, jackknife and other resampling methods. This is perhaps the most complete reference on these techniques currently in print. Chapter 7 and 8 deal with linear and nonlinear models, respectively.
- Other nice references on these topics include Efron (1982), Wu (1986), Cameron and Trivedi (2005, Chapter 11).

Background and Overview

The Bootstrap I

First introduced by Efron (1979) and Efron (1982), the *bootstrap* is a *resampling technique* and similar in spirit to Monte Carlo methods you may have used.

Bootstrapping is the practice of estimating properties of an estimator (such as its variance) by measuring those properties by sampling from an approximating distribution. A common choice for the approximating distribution is the empirical distribution of the observed data.

Bootstrapping allows estimation of the sample distribution of almost any statistic using computer simulation. It is also frequently used in constructing hypothesis tests and determining confidence intervals.

The Bootstrap II

The bootstrap is a convenient alternative to inference based on parametric assumptions when those assumptions are in doubt, or when parametric inference is impossible or requires complicated formulas for calculating standard errors.

In addition, exact *finite-sample* results are unavailable for most statistical estimators[2] and their test statistics. Most of the analysis of methods of statistical inference rely on *large-sample* (asymptotic) theory that usually results in normal and chi-square distributions.

The Bootstrap III

There is a wide range of bootstrap methods. Broadly, they can be divided into two broad approaches:

1. Simple bootstrap methods that can perform statistical inference when conventional methods such as standard error calculations are difficult or not available.
2. More sophisticated bootstrap methods that can provide asymptotic refinements that may lead to a better approximations in finite samples.

We will focus on the first point in this lecture.

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[2] The finite-sample theory for the classical linear regression model under normality is an exception.

Bootstrap for Linear Regression

Bootstrap for Linear Regression

In this lecture, we provide an introduction to the (1) *residual* and (2) *paired bootstrap* for estimating standard errors of regressions coefficients in OLS. These algorithms extend (in the obvious way) to nonlinear problems.

As before, we consider the linear regression problem

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + \varepsilon_i\,, \quad i = 1, \ldots, n, \tag{1}$$

where $y_i \in \mathbb{R}$ is the i-th response, $\mathbf{x}_i \in \mathbb{R}^p$ are the explanatory variables, $\boldsymbol{\beta} \in \mathbb{R}^p$ are unknown parameters, and $\varepsilon_i \in \mathbb{R}$ is the i-th residual.

The Residual Bootstrap I

Steps:

1. Compute $\widehat{\boldsymbol{\beta}} := \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$ and $\widehat{\boldsymbol{\varepsilon}} := (\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_n)' = \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}$.
2. With replacement, randomly draw n samples from the set $\{\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_n\}$, by assuming a probability of $\frac{1}{n}$ for each $\widehat{\varepsilon}_i$.[3] Denote the *bootstrapped residuals* by $\boldsymbol{\varepsilon}^* := (\varepsilon_1^*, \ldots, \varepsilon_n^*)'$.
3. Compute the bootstrapped dependent variable $\mathbf{y}^* := \mathbf{X}\widehat{\boldsymbol{\beta}} + \boldsymbol{\varepsilon}^*$.
4. Compute $\widehat{\boldsymbol{\beta}}^* := \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}^*$.
5. Repeat steps 2 through 4 for m times. Each time in step 2, we create a new sample from random draws. When finished, we have $\widehat{\boldsymbol{\beta}}^{*(1)}, \ldots, \widehat{\boldsymbol{\beta}}^{*(m)} \in \mathbb{R}^p$.
6. Calculate the bootstrapped standard error for each element of $\widehat{\boldsymbol{\beta}}$ via the standard formula, i.e. $\mathrm{SE}(\widehat{\beta}_j) = \sqrt{\frac{1}{m}\sum_{k=1}^{m}\left(\widehat{\beta}_j^{*(k)} - \frac{1}{m}\sum_{l=1}^{m}\widehat{\beta}_j^{*(k)}\right)^2}$.

The Residual Bootstrap II

Remarks:

- In the case of classical linear regression (homoskedasticity), it can be shown that $\text{SE}(\widehat{\beta}_j)$ converges in probability to $\sqrt{\frac{\widehat{\varepsilon}'\widehat{\varepsilon}}{n}(\mathbf{X}'\mathbf{X})^{-1}_{jj}}$. In other words, the residual bootstrap procedure produces consistent standard errors in this case.
- However, note that the residual boostrap is downward biased for the linear model. While this rarely matters for large sample sizes, there is a simple approach by Efron (1982) to correct for the bias if needed.
- In the case of heteroskedasticity, the residual bootstrap procedure does not yield consistent standard errors.

[3] In other words, it is possible to pick the same $\widehat{\varepsilon}_i$ more than once.

The Paired Bootstrap I

Steps:

1. Compute $\widehat{\boldsymbol{\beta}} := \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{y}$.
2. With replacement, randomly draw n pairs from the set $\{(y_i, \mathbf{x}_i')\}_{i=1}^n$, by assuming a probability of $\frac{1}{n}$ for each pair.[4] Denote the bootstrapped data by $\mathbf{X}^* := (\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*)'$ and $\mathbf{y}^* := (y_1^*, \ldots, y_n^*)'$.
3. Compute $\widehat{\boldsymbol{\beta}}^* := \left((\mathbf{X}^*)'\mathbf{X}^*\right)^{-1} (\mathbf{X}^*)'\mathbf{y}^*$.
4. Repeat steps 2 through 3 for m times. Each time in step 2, we create a new data sample from random draws. When finished, we have $\widehat{\boldsymbol{\beta}}^{*(1)}, \ldots, \widehat{\boldsymbol{\beta}}^{*(m)} \in \mathbb{R}^p$.
5. Calculate the bootstrapped standard error for each element of $\widehat{\boldsymbol{\beta}}$ via the standard formula, i.e.
$\mathrm{SE}^*(\widehat{\beta}_j) = \sqrt{\frac{1}{m}\sum_{k=1}^m \left(\widehat{\beta}_j^{*(k)} - \frac{1}{m}\sum_{l=1}^m \widehat{\beta}_j^{*(k)}\right)^2}$.

The Paired Bootstrap II

Remark:

- It can be shown that paired bootstrap produces heteroscedasticity robust standard errors for OLS.

[4] Hence, it is possible to pick the same pair more than once.

Bootstrap for Nonlinear Regression

Bootstrap for Nonlinear Regression

The two bootstrap procedures we discussed in the context of OLS (i.e. the residual and paired bootstrap) can be applied to the nonlinear regression problem

$$y = f(\mathbf{x}, \boldsymbol{\beta}) + \varepsilon\,,\ \boldsymbol{\beta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^p\,. \tag{2}$$

Their properties for NLS are similar to that of OLS. In particular, the paired bootstrap is robust to heteroskedasticity, while the residual bootstrap is not.

References I

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- Friedman, Jerome, Trevor Hastie, and Robert Tibshirani (2017). *The Elements of Statistical Learning*. Second Edition. Springer Series in Statistics Springer, Berlin.
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- Shao, Jun and Dongsheng Tu (2012). *The Jackknife And Bootstrap*. Springer Science & Business Media.

References II

Wu, Chien-Fu Jeff (1986). "Jackknife, Bootstrap And Other Resampling Methods In Regression Analysis". In: *The Annals of Statistics* 14.4, pp. 1261–1295.