Optimal Execution[1]

Algorithmic Trading & Quantitative Strategies

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 If you find any typos, I would much appreciate you email them to me.

Readings

In addition to the lecture notes, the required readings are:

- "Optimal Execution of Portfolio Transactions" by Almgren and Chriss (2001).
- "Optimal Execution with Nonlinear Impact Functions and Trading-Enhanced Risk" by Almgren (2003).

Notation

The objective is to liquidate X units of a security before time T at minimum cost. We will make "minimum cost" more precise later on.

We will use the following notation throughout:

- For some positive integer N let $\tau := T/N$. Then we refer to $t_k := k\tau$, for $k = 0, \ldots, N$, as the *time steps*.
- We refer to $x_0, \ldots, x_N$ as the *holdings*, where x_k is the number of units held of the security at time t_k.
- We assume initial and final holdings are given by $x_0 = X$ and $x_N = 0$.
- $n_1, \ldots, n_N$ is the *trade list*. Here $n_k = x_{k-1} - x_k$ is the number of units that we sell between t_{k-1} and t_k.

With this notation, we have

$$x_k = X - \sum_{j=1}^{k} n_j = \sum_{j=k+1}^{N} n_j\,, \quad \text{for } k = 0, \ldots, N\,. \tag{1}$$

Terminology

By a *trading strategy* we refer to a rule for determining the holdings n_k based on the information available. We distinguish between two types of trading strategies:

- *Static strategies*: These depend on information available before trading start at time t_0, and
- *Dynamic strategies*: These depend on information available up to and including time t_{k-1}.

Model Assumptions I

We assume that the initial security price (at time t_0) is S_0 such that the market value of the portfolio is XS_0.

In addition, we assume there are two kinds of market impact, temporary and permanent.

Model Assumptions II

When we are not trading, we assume security price evolves as the discrete arithmetic random walk

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k \tag{2}$$

for $k = 1, \ldots, N$, where σ is the instantaneous volatility of the security, and ξ_k are independent random variables with zero mean and unit variance, respectively.

Note that by not having a drift term, we have no information about the direction of future price movements.

Permanent Impact in Discrete Time

We incorporate permanent impact as a linear function of trade size into (2), obtaining

$$
\begin{aligned}
S_k &= S_{k-1} + \sigma\tau^{1/2}\xi_k - \gamma n_k &\quad (3)\\
&= S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau\gamma v_k &\quad (4)
\end{aligned}
$$

where $\gamma > 0$ is the the linear permanent impact coefficient and $v_k := n_k/\tau$ is the average rate of trading during $[t_{k-1}, t_k]$.

Temporary Impact in Discrete Time

Next, we incorporate temporary impact in the current period. Let us assume the price per share received for the sale in $[t_{k-1}, t_k]$ is

$$\tilde{S}_k := S_{k-1} - \eta \cdot \frac{n_k}{\tau} = S_{k-1} - \eta \cdot v_k \tag{5}$$

where $\eta > 0$ is the the linear temporary impact coefficient.

Defined in this way, the temporary impact is linear in the rate of trading and it is gone in the next period. Therefore, it does not appear in the next market price S_k

$$S_k = S_{k-1} + \sigma \tau^{1/2} \xi_k - \gamma n_k \,. \tag{6}$$

Price Dynamics, Permanent and Temporary Impact in Continuous Time

In the continuous-time limit, $\tau \to 0$, the rate of trading becomes

$$v(t) := -\dot{x}(t) = \lim_{\tau \to 0} \frac{x(t-\tau) - x(t)}{\tau} \tag{7}$$

and therefore, our price process with permanent impact is given by

$$dS = \sigma dW - \gamma v dt = \sigma dW + \gamma dx \tag{8}$$

where $W(t)$ is a Brownian motion.

Integrating, we obtain

$$S(t) = S_0 + \sigma W(t) - \gamma (X - x(t)) \tag{9}$$

Observe that permanent market impact is cumulative and proportional to the number of shares traded (as it should be).

Permanent Impact & Trading Strategies

Recall

$$S(t) = S_0 + \sigma W(t) - \gamma\,(X - x(t))\ . \tag{10}$$

As a warm-up exercise, we calculate total dollar cost incurred from permanent impact, $\mathcal{C}_{\text{perm}}$, over the whole the time period $[0, T]$ as follows

$$\begin{aligned}
\mathcal{C}_{\text{perm}} &= \int_0^T \gamma(X - x(t))v(t)dt && (11)\\
&= -\int_0^T \gamma(X - x(t))\dot{x}(t)dt && (12)\\
&= -\gamma X x(t)|_0^T + \frac{1}{2}\gamma x^2(t)\Big|_0^T && (13)\\
&= \frac{1}{2}\gamma X^2\,, && (14)
\end{aligned}$$

as $x(0) = X$ and $x(T) = 0$. In other words, permanent impact is independent of the specific choice of $v(t)$ and therefore does not effect the optimal strategy.

Revenues of the Trading Strategy I

Recall

$$S(t) = S_0 + \sigma W(t) - \gamma\,(X - x(t))\ . \tag{15}$$

Including temporary impact into (15), we obtain

$$S(t) = S_0 + \sigma W(t) - \gamma\,(X - x(t)) + \eta\dot{x}(t)\,, \tag{16}$$

or equivalently

$$dS = \sigma dW + \gamma\dot{x}(t)dt + \eta\ddot{x}(t)dt\,. \tag{17}$$

Revenues of the Trading Strategy II

We calculate the total revenues $\mathcal{R}$ from the trading strategy as follows

$$
\begin{aligned}
\mathcal{R} &= \int_0^T v(t)S(t)dt \qquad (18)\\
&= \int_0^T -\dot{x}(t)S(t)dt \\
&= -x(t)S(t)|_0^T + \int_0^T x(t)\frac{dS(t)}{dt}dt \\
&= X(S_0 + \eta\dot{x}(0)) + \int_0^T x(t)dS(t) \\
&= X(S_0 + \eta\dot{x}(0)) + \sigma\int_0^T x(t)dW + \int_0^T x(t)\left(\gamma\dot{x}(t) + \eta\ddot{x}(t)\right)dt \\
&= X(S_0 + \eta\dot{x}(0)) + \sigma\int_0^T x(t)dW - \frac{\gamma}{2}X^2 + \eta x(t)\dot{x}(t)|_0^T \\
&\qquad - \eta\int_0^T \left(\dot{x}(t)\right)^2 dt
\end{aligned}
$$

Implementation Shortfall I

Therefore, the implementation shortfall, $\mathcal{C}$, becomes

$$\begin{aligned}
\mathcal{C} &= XS_0 - R && (20)\\
&= -\sigma \int_0^T x(t)dW + \frac{\gamma}{2}X^2 + \eta \int_0^T \left(\dot{x}(t)\right)^2 dt && (21)\\
&= -\sigma \int_0^T x(t)dW + \eta \int_0^T \left(\dot{x}(t)\right)^2 dt + B && (22)
\end{aligned}$$

where $B := \frac{\gamma}{2}X^2$ is a constant.

Implementation Shortfall II

Reall

$$\mathcal{C} = -\sigma \int_0^T x(t) dW + \eta \int_0^T (\dot{x}(t))^2 \, dt + B \,. \tag{23}$$

$\mathcal{C}$ is a random variable with expectation and variance given by

$$\mathbb{E}[\mathcal{C}] = \eta \int_0^T (\dot{x}(t))^2 \, dt + B \,, \tag{24}$$

$$\begin{aligned} \mathrm{var}[\mathcal{C}] &= \mathbb{E}\left[(\mathcal{C} - \mathbb{E}[\mathcal{C}])^2\right] \\ &= \mathbb{E}\left[\left(-\sigma \int_0^T x(t) dW\right)^2\right] = \mathbb{E}\left[\sigma^2 \int_0^T (x(t) dW)^2\right] \\ &= \sigma^2 \int_0^T (x(t))^2 \, dt \,, \end{aligned} \tag{25}$$

where the last equality follows from Itô's lemma.

The Optimal Execution Problem

Almgren and Chriss (2001) propose solving the following problem

$$\min_{x(t)} \mathbb{E}[\mathcal{C}] + \lambda \mathrm{var}[\mathcal{C}] \tag{26}$$

$$\text{s.t. } x(0) = X \tag{27}$$

$$x(T) = 0 \tag{28}$$

where $\mathcal{C}$ is the implementation shortfall, and $\lambda \geq 0$ is the risk aversion to execution risk.

Substituting the functional forms for $\mathbb{E}[\mathcal{C}]$ and $\mathrm{var}[\mathcal{C}]$, the mean-variance problem (26)–(28) becomes

$$\min_{x(t)} \int_0^T (\eta \dot{x}^2 + \lambda \sigma^2 x^2) dt \tag{29}$$

$$\text{s.t. } x(0) = X \tag{30}$$

$$x(T) = 0\,. \tag{31}$$

The Optimal Execution Problem: Variational Form

This is a variational problem of the form

$$\min_{x(t)} \int_0^T F(x, \dot{x}, t)dt \tag{32}$$

$$\text{s.t. } x(0) = X\,, \quad x(T) = 0\,, \tag{33}$$

where

$$F(x, \dot{x}, t) = \eta\dot{x}^2 + \lambda\sigma^2 x^2\,. \tag{34}$$

It is well-known from variational calculus that the necessary condition for this problem to have an extremum for a given function x is that the *Euler-Lagrange equation* is satisfied[2]

$$F_x - \frac{d}{dt}F_{\dot{x}} = 0 \tag{35}$$

with boundary conditions $x(0) = X$ and $x(T) = 0$.

[2] Standard texts on calculus of variations include Gelfand and Fomin (1963) and Dacorogna (2014). A brief introduction to the topic is given in Figueroa-O'Farrill (no year).

The Optimal Execution Problem: Solution I

Notice that in our situation F does not explicitly depend on t. That is, $F(x, \dot{x}, t) \equiv F(x, \dot{x})$ so $\frac{\partial F}{\partial t} = 0$. Therefore,

$$F_x - \frac{d}{dt} F_{\dot{x}} = F_x - \dot{x} F_{x\dot{x}} - \ddot{x} F_{\dot{x}\dot{x}} \,. \tag{36}$$

Multiplying (36) by $\dot{x}$, we obtain

$$\begin{aligned} 0 &= \dot{x} F_x - \dot{x}\dot{x} F_{x\dot{x}} - \dot{x}\ddot{x} F_{\dot{x}\dot{x}} && (37) \\ &= \frac{d}{dt}(F - \dot{x} F_{\dot{x}}) \,. && (38) \end{aligned}$$

Hence

$$F - \dot{x} F_{\dot{x}} = K \tag{39}$$

where K is a constant.

The Optimal Execution Problem: Solution II

Recall

$$F(x,\dot{x}) = \eta\dot{x}^2 + \lambda\sigma^2x^2\,. \tag{40}$$

Inserting (40) and $F_{\dot{x}} = 2\eta\dot{x}$ into (39), we obtain

$$\begin{aligned} F - \dot{x}F_{\dot{x}} &= \eta\dot{x}^2 + \lambda\sigma^2x^2 - 2\eta\dot{x}^2 & (41)\\ &= \lambda\sigma^2x^2 - \eta\dot{x}^2 & (42)\\ &= K\,. & (43)\end{aligned}$$

We recognize (42)-(43) is the first order inhomogeneous ODE

$$\lambda\sigma^2x^2 - \eta\dot{x}^2 = K\,. \tag{44}$$

The Optimal Execution Problem: Solution III

It is straightforward to solve the homogenous ODE

$$\lambda\sigma^2 x^2 - \eta\dot{x}^2 = 0 \tag{45}$$

by "taking square roots" to obtain

$$\dot{x} = \kappa x \,, \qquad \kappa := \sqrt{\frac{\lambda\sigma^2}{\eta}} \,. \tag{46}$$

Therefore

$$x = Ae^{\kappa t} \tag{47}$$

where A is a constant.

The Optimal Execution Problem: Solution IV

The solution to the inhomogenous ODE (45) is therefore of the form

$$x = Ae^{\kappa t} + Be^{-\kappa t} \tag{48}$$

for some constants A and B.

From the boundary conditions

$$\begin{aligned} x(0) &= A + B \equiv X & (49) \\ x(T) &= Ae^{\kappa T} + Be^{-\kappa T} \equiv 0\,, & (50) \end{aligned}$$

we obtain

$$\begin{aligned} A &= X\frac{-e^{-\kappa T}}{e^{\kappa T} - e^{-\kappa T}} & (51) \\ B &= X\frac{e^{\kappa T}}{e^{\kappa T} - e^{-\kappa T}}\,. & (52) \end{aligned}$$

The Optimal Execution Problem: Solution V

Altogether, the solution to our variational problem is

$$
\begin{aligned}
x(t) &= \frac{X}{e^{\kappa T} - e^{-\kappa T}} \left(-e^{-\kappa(T-t)} + e^{\kappa(T-t)}\right) && (53) \\
&= X \frac{\sinh\left(\kappa(T-t)\right)}{\sinh(\kappa T)} && (54)
\end{aligned}
$$

where $\kappa = \sqrt{\frac{\lambda\sigma^2}{\eta}}$ is referred to as the *urgency parameter*.

The Optimal Execution Problem: Solution VI

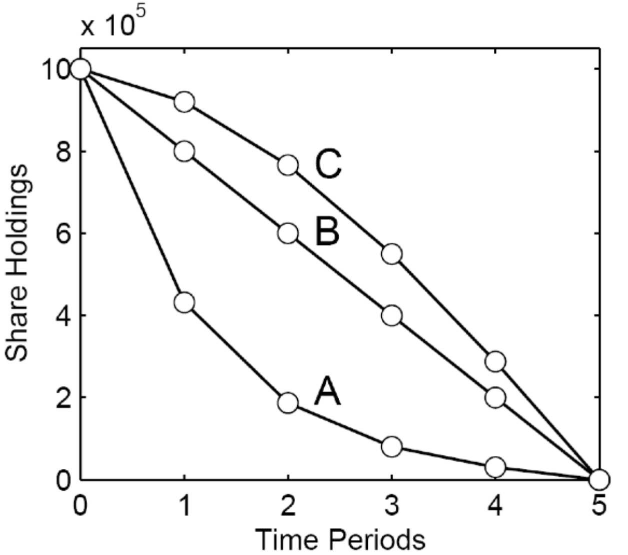
Recall

$$x(t) = X\frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)}, \quad \kappa = \sqrt{\frac{\lambda\sigma^2}{\eta}}. \tag{55}$$

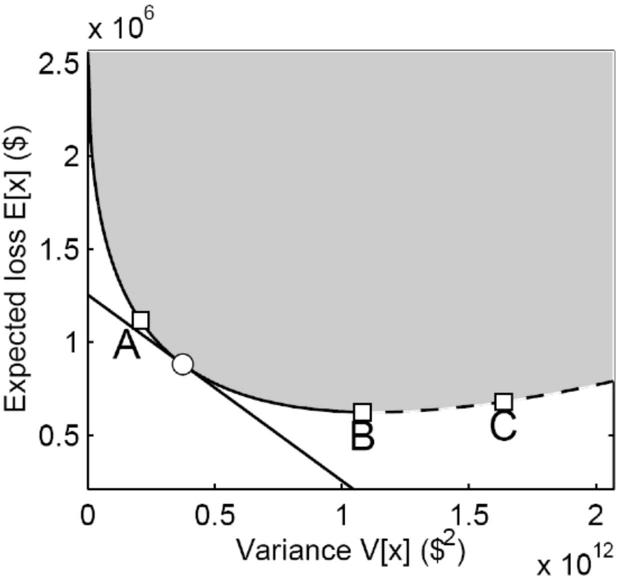
From this, we see that:

- When the risk aversion increases ($\lambda \uparrow$), then the rate of trading increases,
- When the volatility of the security increases ($\sigma \uparrow$), then the rate of trading increases, and
- When the temporary price impact coefficient increases ($\eta \uparrow$), then the rate of trading decreases.

Optimal Trading Trajectories



The Efficient Frontier



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