

# CS5100 Foundations of Artificial Intelligence

## **Module 6 Lesson 10**

### **Machine Learning 2**

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# Overview

Classification

Classification  
using Naïve  
Bayes

Generalization &  
Overfitting

Parameter  
Estimation &  
Smoothing





# Machine Learning

Machine Learning: how to acquire a model based on data or experience

- Learning parameters (e.g., probabilities)
- Learning structure
- Learning hidden concepts (e.g., clustering)

# Classification Example

## Spam Filter ..1

Input: one or more email(s)

Output: spam/ham label(s)

Setup:

- Get a large collection of example emails, each labeled “spam” or “ham”
- Note: someone has to hand label all this data!
- Want to learn to predict labels of new, future emails



Dear Sir.

First, I must solicit your confidence in this transaction, this is by virtue of its nature as being utterly confidential and top secret. ...



TO BE REMOVED FROM FUTURE MAILINGS, SIMPLY REPLY TO THIS MESSAGE AND PUT "REMOVE" IN THE SUBJECT.

99 MILLION EMAIL ADDRESSES  
FOR ONLY \$99



Ok, I know this is blatantly OT but I'm beginning to go insane. Had an old Dell Dimension XPS sitting in the corner and decided to put it to use, I know it was working pre being stuck in the corner, but when I plugged it in, hit the power nothing happened.

# Classification Example

## Spam Filter ..2

Features: Attributes used to make the ham / spam decision

- Words: FREE!, confidential, ...
- Text Patterns: \$nn, CAPS
- Non-text: SenderInContacts?, TimeOfDay ...
- ....



Dear Sir.

First, I must solicit your confidence in this transaction, this is by virtue of its nature as being utterly confidential and top secret. ...



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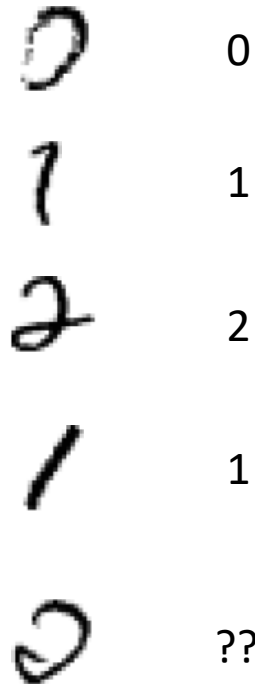


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# Classification Example

## Digit Recognition ..1

- Input: images / pixel grids
- Output: a digit 0-9
- Setup:
  - Get a large collection of example images, each labeled with a digit
  - Note: someone has to hand label all this data!
  - Want to learn to predict labels of new, future digit images





# Classification Example

## Digit Recognition ..2

Features: attributes used to decide on digit

- Pixels: (6,8)=ON
- Shape Patterns: NumComponents, AspectRatio, NumLoops ...
- ...

0	0
1	1
2	2
1	1
0	??

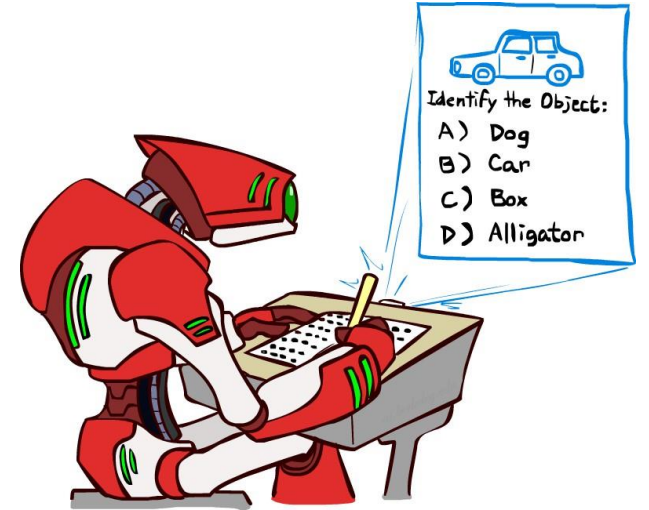
# Other Classification Tasks

**Classification:** *given inputs  $x$ ,  
predict labels (classes)  $y$*

**Examples:**

- News classification (input: news stories, classes: politics, sports, business...)
- OCR (input: images, classes: characters)
- Medical diagnosis (input: symptoms, classes: diseases)
- Automatic essay grading (input: document, classes: grades)
- Fraud detection (input: account activity, classes: fraud / no fraud)
- Customer service email routing  
plus many more

Classification: important commercial technology!





# Important Concepts ..1

Data: labeled instances,  
e.g. emails marked spam/ham

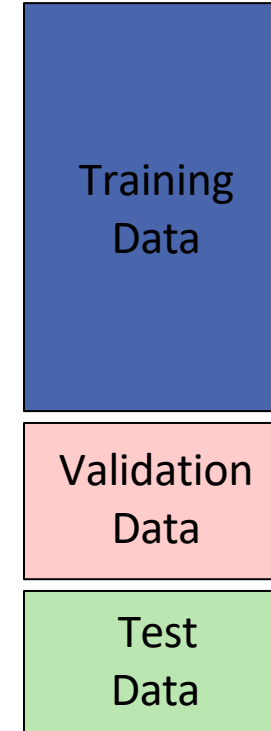
- Training set, Validation set, Test set

Features:

- attribute-value pairs which characterize each  $x$

Experimentation cycle

- Learn parameters (e.g. model probabilities) on training set
- Tune hyperparameters on validation set
- Compute accuracy of test set
- Important: never “peek” at the test set!



# Important Concepts ..2

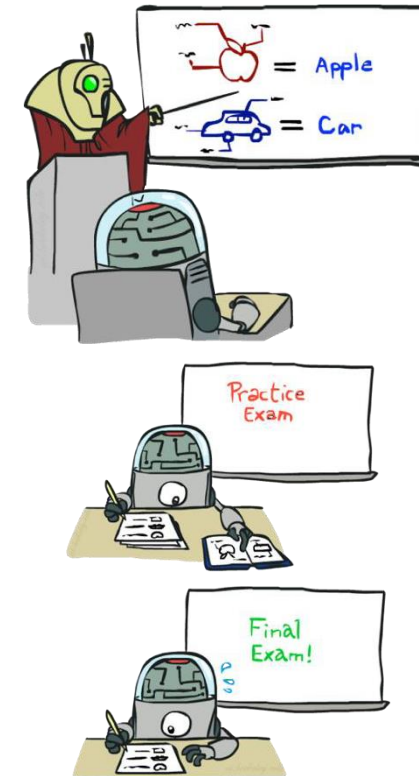
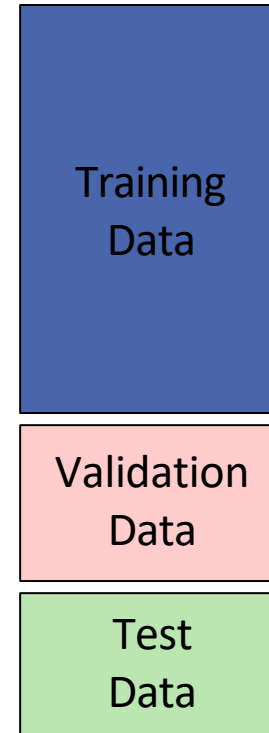
## Evaluation

- Accuracy: fraction of instances predicted correctly

## Overfitting and generalization

- Want a classifier which does well on *test data*
- Overfitting: fitting the training data very closely, but not generalizing well

Example: Chairs!



# Naïve Bayes classification

## Intuition ..1a

Which of these is a **Political** story? Which is a **Sports** story?

- The tech giant Facebook is facing a growing backlash on Capitol Hill as more lawmakers demand that CEO Mark Zuckerberg testify about reports that Cambridge Analytica, a political data consulting firm utilized by the Trump campaign, harvested the information of up to 50 million users.
- Close to quitting tennis after four wrist surgeries in recent years, the Argentine fought to get back to the ATP Tour even as he was reduced to hitting his backhand with one hand instead of his usual two. The struggle paid off Sunday, when Del Potro ... beat top-ranked Roger Federer 6-4, 7-6 (8), 7-6 (2) for the BNP Paribas Open title.

# Naïve Bayes classification

## Intuition ..1b

Which of these is a **Political** story? Which is a **Sports** story?

- The tech giant Facebook is facing a growing backlash on **Capitol Hill** as more **lawmakers** demand that CEO Mark Zuckerberg **testify** about reports that Cambridge Analytica, a **political** data consulting firm utilized by the **Trump campaign**, harvested the information of up to 50 million users.
- Close to **quitting tennis** after four wrist surgeries in recent years, the Argentine fought to get back to the **ATP Tour** even as he was reduced to hitting his **backhand** with one hand instead of his usual two. The struggle paid off Sunday, when **Del Potro** ... **beat top-ranked Roger Federer** 6-4, 7-6 (8), 7-6 (2) for the **BNP Paribas Open** title.

# Naïve Bayes classification

## Intuition ..2

If we have a collection of news stories labeled as Political, Sports, Tech, Business, Crime etc.,  
And if the percentage of each 'class' is known,

We can say:

- Probability of a story being a Political story is related to:
  - $P(\text{Word} \mid \text{Class})$ : Occurrence of words in Politics stories
    - E.g. election (0.1%), Capitol Hill (0.02%), campaign (0.005%) ...
  - $P(\text{Class})$ : Percentage of each class of story in the collection
    - E.g. About 30% Political, 15% Sports, 10% Tech ...

What we want:  $P(\text{Class} \mid \text{Word})$



# Refresher

## Bayes' Rule

Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

Dividing by  $P(y)$ , we get:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

Or in distribution form

$$\mathbf{P(Y|X) = P(X|Y) P(Y) / P(X) = \alpha P(X|Y) P(Y)}$$

$$\mathbf{P(Class|Word) = P(Word | Class) * P(Class) / P(Word)}$$
$$\alpha \mathbf{P(Word | Class) * P(Class)}$$



# General Naïve Bayes ..1

A general Naïve Bayes model:

$$P(Y, F_1 \dots F_n) = P(Y) \prod_i P(F_i | Y)$$

$|Y|$   
parameters

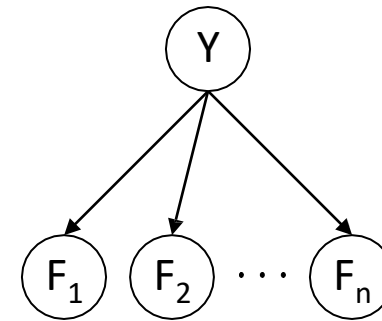
$|Y| \times |F|^n$   
values

$n \times |F| \times |Y|$   
parameters

We only have to specify how each feature depends on the class

Total number of parameters is *linear* in n

Model is very simplistic, but often works anyway



# General Naïve Bayes ..2

What do we need in order to use Naïve Bayes?

- Estimates of local conditional probability tables
  - $P(Y)$ , the priors over labels
  - $P(F_i | Y)$  for each feature (evidence variables)
  - These probabilities are collectively called the *parameters* of the model and denoted by  $\theta$
  - Up until now, we assumed these appeared by magic, but...
  - ... they typically come from training data counts:  
we'll look at this soon


# Naïve Bayes for Digits

Naïve Bayes:

Assume all features are independent

Simple digit recognition version:

- One feature (variable)  $F_{ij}$  for each grid position  $\langle i,j \rangle$ ,  $i,j: 0..15$
- Feature values are on / off, based on underlying image
- Each input maps to a feature vector, e.g.

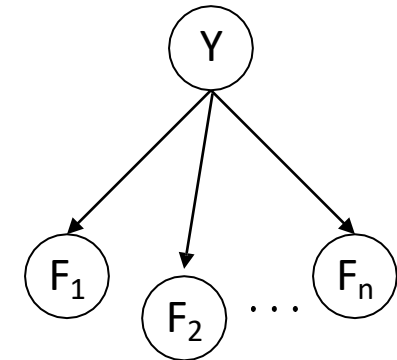
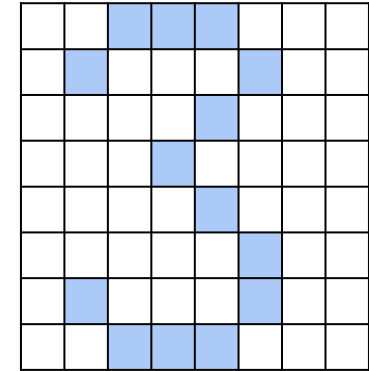
  $\rightarrow \langle F_{0,0} = 0 \ F_{0,1} = 0 \ F_{0,2} = 1 \ F_{0,3} = 1 \ F_{0,4} = 0 \ \dots F_{15,15} = 0 \rangle$

Here: lots of features, each binary-valued

Naïve Bayes model:

$$P(Y|F_{0,0} \dots F_{15,15}) \propto P(Y) \prod_{i,j} P(F_{i,j}|Y)$$

What do we need to learn to classify images as digits?

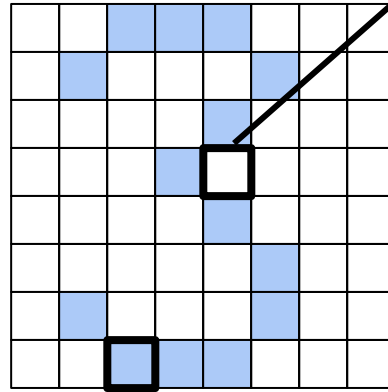


# Example

## Conditional Probabilities

$P(Y)$

1	0.1
2	0.1
3	0.1
4	0.1
5	0.1
6	0.1
7	0.1
8	0.1
9	0.1
0	0.1



$P(F_{5,5} = on|Y)$

1	0.05
2	0.01
3	0.90
4	0.80
5	0.90
6	0.90
7	0.25
8	0.85
9	0.60
0	0.80

$P(F_{3,1} = on|Y)$

1	0.01
2	0.05
3	0.05
4	0.30
5	0.80
6	0.90
7	0.05
8	0.60
9	0.50
0	0.80

# Parameter Estimation

Estimating the distribution of a random variables like  $X$  or  $X \mid Y$

*Elicitation*: ask a human

- Need domain experts, sophisticated ways of eliciting probabilities

*Empirically*: use training data (learning!)

- E.g.: for each outcome  $x$ , look at the *empirical rate* of that value:

$$P_{\text{ML}}(x) = \frac{\text{count}(x)}{\text{total samples}}$$

- This is the estimate that *maximizes the likelihood of the data*



$$P_{\text{ML}}(\textcolor{red}{r}) = 2/3$$



# Naïve Bayes for Text ..1

Bag-of-words Naïve Bayes:

- Features:  $W_i$  is the word at position  $i$
- As before: predict label (spam vs. ham) conditioned on feature variables
- As before: assume features are conditionally independent given label
- New: each  $W_i$  is identically distributed

Generative model:

$$P(Y, W_1 \dots W_n) = P(Y) \prod_i P(W_i | Y)$$

↓  
*Word at position  
 $i$ , not  $i^{\text{th}}$  word in  
the dictionary!*



# Naïve Bayes for Text ..2

## Bag-of-words model

Usually, each variable gets its own conditional probability distribution  $P(F|Y)$

In a bag-of-words model

- Each position is identically distributed
- All positions share the same conditional probabilities  $P(W|Y)$
- Why make this assumption?
  - Called “bag-of-words” because model not dependent on word order or reordering

$$P(Y, W_1 \dots W_n) = P(Y) \prod_i P(W_i|Y)$$

# Example Spam Filtering

Model:

What are the parameters?

$$P(Y, W_1 \dots W_n) = P(Y) \prod_i P(W_i|Y)$$

$P(Y)$

ham	: 0.66
spam	: 0.33

$P(W|\text{spam})$

the	:	0.0156
to	:	0.0153
and	:	0.0115
of	:	0.0095
you	:	0.0093
a	:	0.0086
with:		0.0080
from:		0.0075
...		

$P(W|\text{ham})$

the	:	0.0210
to	:	0.0133
of	:	0.0119
2002:		0.0110
with:		0.0108
from:		0.0107
and	:	0.0105
a	:	0.0100
...		

Where do these tables  
come from?

Counts from examples!

# Spam Example

$$P(Y, W_1 \dots W_n) = P(Y) \prod_i P(W_i | Y)$$

$$= P(\text{Outcome}) * \prod_i P(\text{Event}_i | \text{Outcome})$$

Word	P(w spam)	P(w ham)
Category prior	0.33333	0.66666
Gary	0.00002	0.00021
would	0.00069	0.00084
you	0.00881	0.00304
like	0.00086	0.00083
to	0.01517	0.01339
lose	0.00008	0.00002
weight	0.00016	0.00002
while	0.00027	0.00027
you	0.00881	0.00304
sleep	0.00006	0.00001
Product	9.659E-34	1.3045E-35

# Naïve Bayes Quick Check ..1

Training Data

chills	runny nose	headache	fever	flu?
Y	N	Mild	Y	N
Y	Y	No	N	Y
Y	N	Strong	Y	Y
N	Y	Mild	Y	Y
N	N	No	N	N
N	Y	Strong	Y	Y
N	Y	Strong	N	N
Y	Y	Mild	Y	Y

Test Data

chills	runny nose	headache	fever	flu?
Y	N	Mild	N	?

$P(\text{Outcome} \mid \text{Events}) =$

$$P(\text{Outcome}) * \frac{\prod_i P(\text{Event}_i \mid \text{Outcome})}{\prod_i P(\text{Event}_i)}$$

# Naïve Bayes Quick Check ..2

Training Data

chills	runny nose	headache	fever	flu?
Y	N	Mild	Y	N
Y	Y	No	N	Y
Y	N	Strong	Y	Y
N	Y	Mild	Y	Y
N	N	No	N	N
N	Y	Strong	Y	Y
N	Y	Strong	N	N
Y	Y	Mild	Y	Y

Test Data

chills	runny nose	headache	fever	flu?
Y	N	Mild	N	?

$P(\text{Outcome} \mid \text{Events}) =$

$$P(\text{Outcome}) * \frac{\prod_i P(\text{Event}_i \mid \text{Outcome})}{\prod_i P(\text{Event}_i)}$$

- $P(\text{flu} = Y) = 5/8 = 0.625$
- $P(\text{chills}=Y \mid \text{flu}) = 3/5 = 0.6$
- $P(\text{runnynose}=N \mid \text{flu}) = 1/5 = 0.2$
- $P(\text{headache}=mild \mid \text{flu}) = 2/5 = 0.4$
- $P(\text{fever}=N \mid \text{flu}) = 1/5 = 0.2$
- $P(\text{chills}=Y) = 4/8 = 0.5$
- $P(\text{runnynose}=N) = 3/8 = 0.375$
- $P(\text{headache}=mild) = 3/8 = 0.375$
- $P(\text{fever}=N) = 3/8 = 0.375$

- $P(\text{flu} = Y \mid \text{Events}) =$

$$\frac{0.625 * 0.6 * 0.2 * 0.4 * 0.2}{0.5 * 0.375 * 0.375 * 0.375} = 0.228$$



# Naïve Bayes Quick Check ..3

Training Data

chills	runny nose	headache	fever	flu?
Y	N	Mild	Y	N
Y	Y	No	N	Y
Y	N	Strong	Y	Y
N	Y	Mild	Y	Y
N	N	No	N	N
N	Y	Strong	Y	Y
N	Y	Strong	N	N
Y	Y	Mild	Y	Y

Test Data

chills	runny nose	headache	fever	flu?
Y	N	Mild	N	?

$P(\text{Outcome} \mid \text{Events}) =$

$$P(\text{Outcome}) * \frac{\prod_i P(\text{Event}_i \mid \text{Outcome})}{\prod_i P(\text{Event}_i)}$$

- $P(\text{flu} = \text{N}) = 3/8 = 0.375$
- $P(\text{chills}=\text{Y} \mid \text{flu}=\text{N}) = 0.33$
- $P(\text{runnynose}=\text{N} \mid \text{flu}=\text{N}) = 0.66$
- $P(\text{headache}=\text{mild} \mid \text{flu}=\text{N}) = 0.33$
- $P(\text{fever}=\text{N} \mid \text{flu}=\text{N}) = 0.66$
- $P(\text{chills}=\text{Y}) = 4/8 = 0.5$
- $P(\text{runnynose}=\text{N}) = 3/8 = 0.375$
- $P(\text{headache}=\text{mild}) = 3/8 = 0.375$
- $P(\text{fever}=\text{N}) = 3/8 = 0.375$

- $P(\text{flu} = \text{N} \mid \text{Events}) =$

$$\frac{0.375 * 0.33 * 0.66 * 0.33 * 0.66}{0.5 * 0.375 * 0.375 * 0.375} = 0.675$$



# A simplification

Denominator  $d = \sum P(\text{Event}_i)$  is common to both (all)

Usually we're trying to get relative probabilities of outcomes

So denominator may be dropped

$$P(\text{flu} = Y \mid \text{Events}) = 0.006 / d$$

$$P(\text{flu} = N \mid \text{Events}) = 0.0178 / d$$

So: No flu!



# Naïve Bayes: Advantages

Simple conceptual model

Works well in a lot of cases

Fast and easy to compute:

- Determine probabilities  
Perform some multiplications
- But not great at estimating probabilities

Easy to retrain (update) when you get additional training data

# Naïve Bayes & scikit-learn

Three variants

based on different methods to compute  $P(E_i|O)$

## **Gaussian NB**

features assumed to be in normal distribution

## **Multinomial NB**

used for text problems, where data is represented as word count or tf-idf vectors; incorporates smoothing\*

## **Bernoulli NB**

may be multiple features, but all features assumed to be binary-valued

# Example Overfitting

Raw probabilities alone don't affect the posteriors; relative probabilities (odds ratios) do:

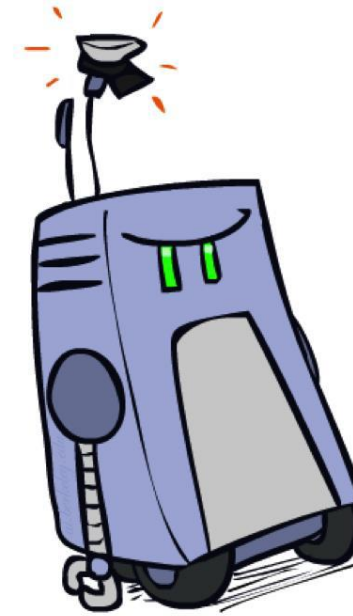
$$\frac{P(W|\text{ham})}{P(W|\text{spam})}$$

south-west	:	inf
nation	:	inf
morally	:	inf
nicely	:	inf
extent	:	inf
seriously	:	inf
...		

$$\frac{P(W|\text{spam})}{P(W|\text{ham})}$$

screens	:	inf
minute	:	inf
guaranteed	:	inf
\$205.00	:	inf
delivery	:	inf
signature	:	inf
...		

*What went wrong here?*





# Generalization and Overfitting ..1

Relative frequency parameters will **overfit** the training data!

- Just because we never saw a 3 with pixel (15,15) **on** during training doesn't mean we won't see it at test time
- Unlikely that every occurrence of "minute" is 100% spam
- Unlikely that every occurrence of "seriously" is 100% ham
- What about all the words that don't occur in the training set at all?
- In general, we can't go around giving unseen events zero probability





# Generalization and Overfitting ..2

- As an extreme case, imagine using the entire email as the only feature
  - Would get the training data perfect (if deterministic labeling)
  - Wouldn't *generalize* at all
  - Just making the bag-of-words assumption gives us some generalization, but is not enough
- To generalize better: we need to **smooth** or **regularize** the estimates



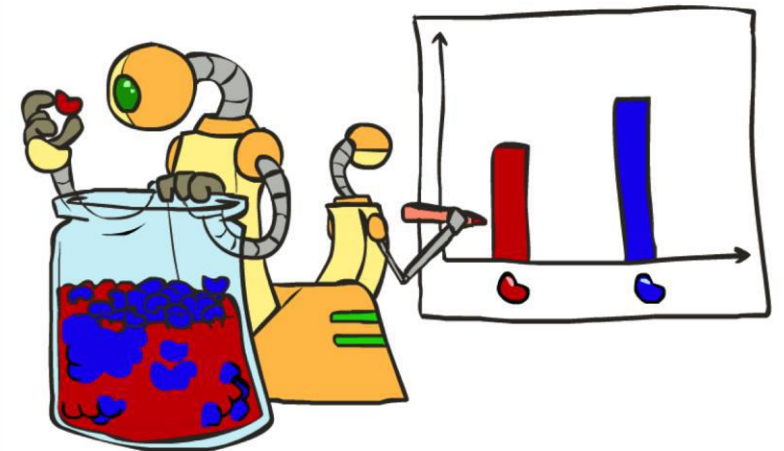
# Estimation & Smoothing

Problems with maximum likelihood estimates:

- If I flip a coin once, and it comes up heads, what is the estimate for  $P(\text{heads})$ ?
- What if we have 8 heads after 10 flips?
- What if we have 8 million heads after 10 million flips?

Basic idea behind solution:

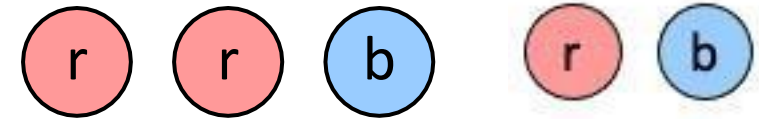
- We have a prior expectation about parameters – e.g.  $P(\text{heads})$  here
- Given only little evidence, we should skew towards our prior expectation
- Given a lot of evidence, we should listen to the data



# Laplace Smoothing ..1

Laplace's estimate:

Pretend you saw every outcome once more than you actually did



$$\begin{aligned} P_{LAP}(x) &= \frac{c(x) + 1}{\sum_x [c(x) + 1]} \\ &= \frac{c(x) + 1}{N + |X|} \end{aligned}$$

$$P_{ML}(X) =$$

$$P_{LAP}(X) =$$

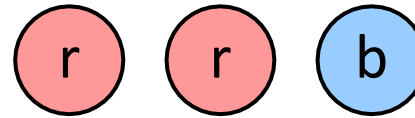
# Laplace Smoothing ..2

Laplace's estimate (extended):

Pretend you saw every outcome  $k$  extra times

$$P_{LAP,k}(x) = \frac{c(x) + k}{N + k|X|}$$

What's Laplace with  $k = 0$ ?



$$P_{LAP,0}(X) =$$

$$P_{LAP,1}(X) =$$

$$P_{LAP,100}(X) =$$

# Real NB Smoothing

For real classification problems, smoothing is critical

New odds ratios:

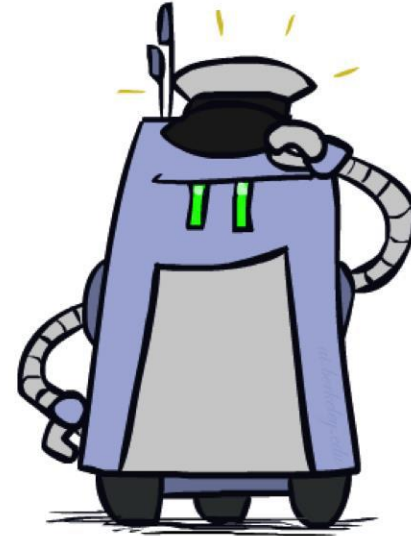
$$\frac{P(W|\text{ham})}{P(W|\text{spam})}$$

helvetica	:	11.4
seems	:	10.8
group	:	10.2
ago	:	8.4
areas	:	8.3
...		

$$\frac{P(W|\text{spam})}{P(W|\text{ham})}$$

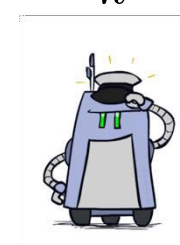
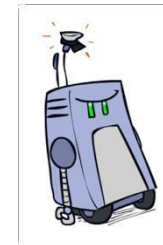
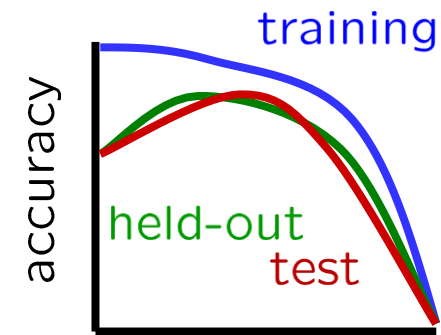
verdana	:	28.8
Credit	:	28.4
ORDER	:	27.2
<FONT>	:	26.9
money	:	26.5
...		

*Do these make more sense?*



# Tuning on Held-Out Data

- Now we've got two kinds of unknowns
  - Parameters: the probabilities  $P(X|Y)$ ,  $P(Y)$
  - Hyperparameters: e.g. the amount / type of smoothing to do,  $k$ ,  $\alpha$
- What should we learn where?
  - Learn parameters from training data
  - Tune hyperparameters on different data
    - Why?
  - For each value of the hyperparameters, train and test on the held-out data
  - Choose the best value and do a final test on the test data





# Baselines

First step: get a **baseline**

- Baselines are very simple “straw man” procedures
- Help determine how hard the task is
- Help know what a “good” accuracy is

Weak baseline: most frequent label classifier

- Gives all test instances whatever label was most common in the training set
- E.g. for spam filtering, might label everything as ham
- Accuracy might be very high if the problem is skewed
- E.g. calling everything “ham” gets 66%, so a classifier that gets 70% isn’t very good...

For real research, usually use previous work as a (strong) baseline

# Examples of Ham Looking like Spam

Dear GlobalSCAPE Customer, GlobalSCAPE has partnered with ScanSoft to offer you the latest version of OmniPage Pro, for just \$99.99\* - the regular list price is \$499! The most common question we've received about this offer is - Is this genuine? We would like to assure you that this offer is authorized by ScanSoft, is genuine and valid. You can get the . . . . .

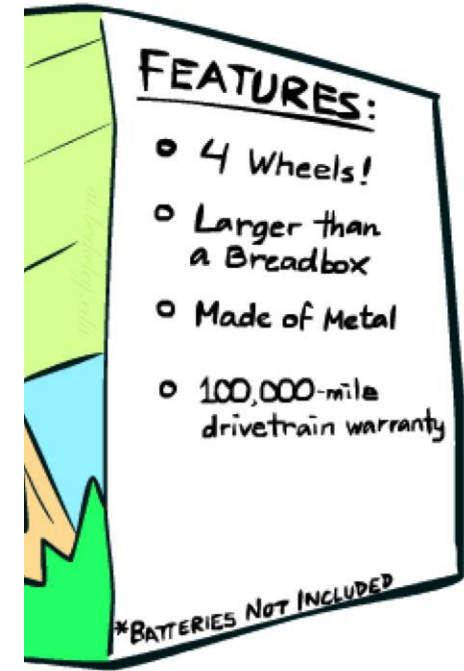
To receive your \$30 Amazon.com promotional certificate, click through to <http://www.amazon.com/apparel> and see the prominent link for the \$30 offer. All details are there. We hope you enjoyed receiving this message. However, if you'd rather not receive future e-mails announcing new store launches, please click . . .

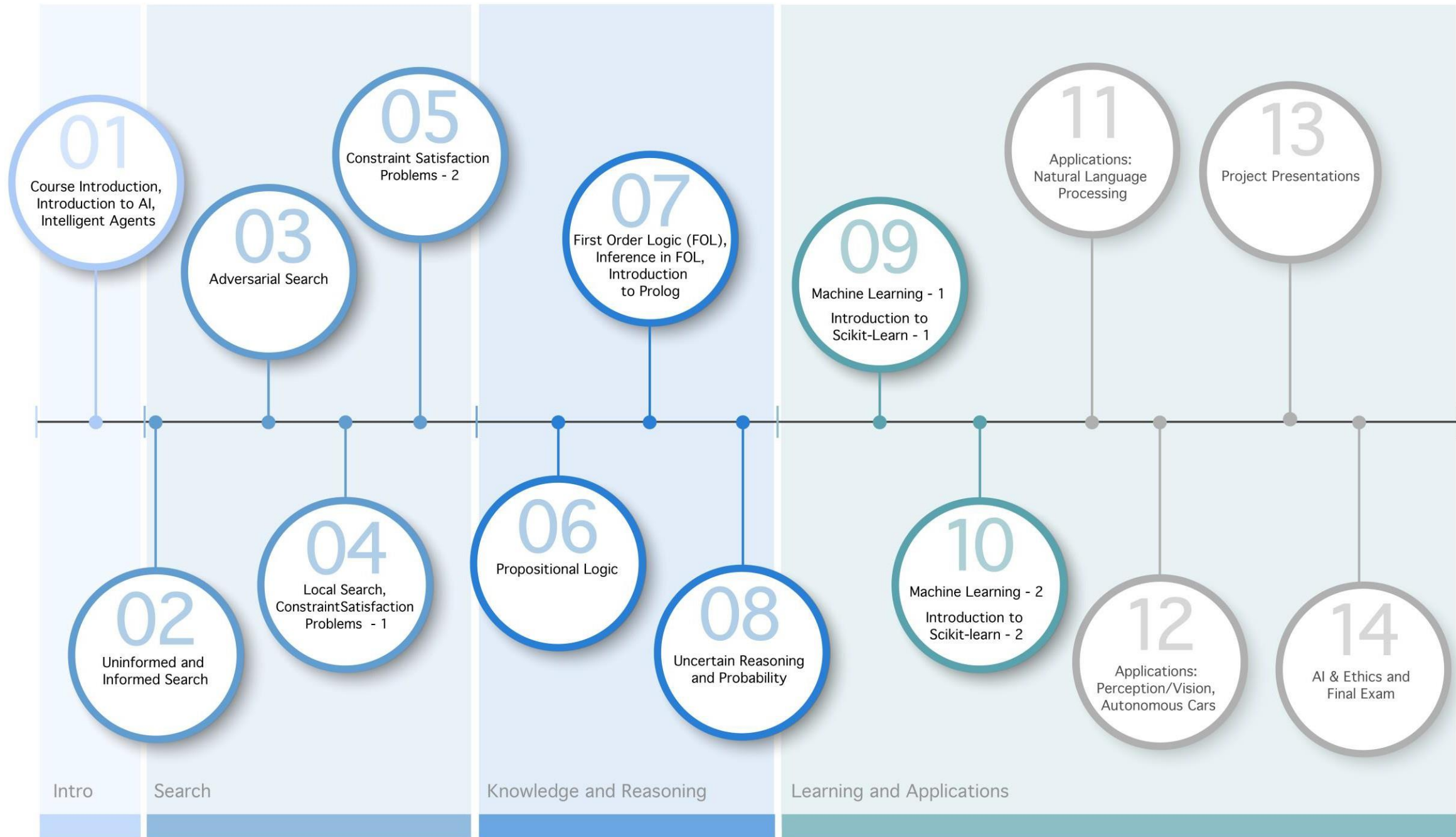
# What to Do About Errors?

Need more features— words aren't enough!

- Have you emailed the sender before?
- Have 1K other people just gotten the same email?
- Is the sending information consistent?
- Is the email in ALL CAPS?
- Do inline URLs point where they say they point?
- Does the email address you by (your) name?

Can add these information sources as new variables in the NB model









## Extra: ML & Chairs

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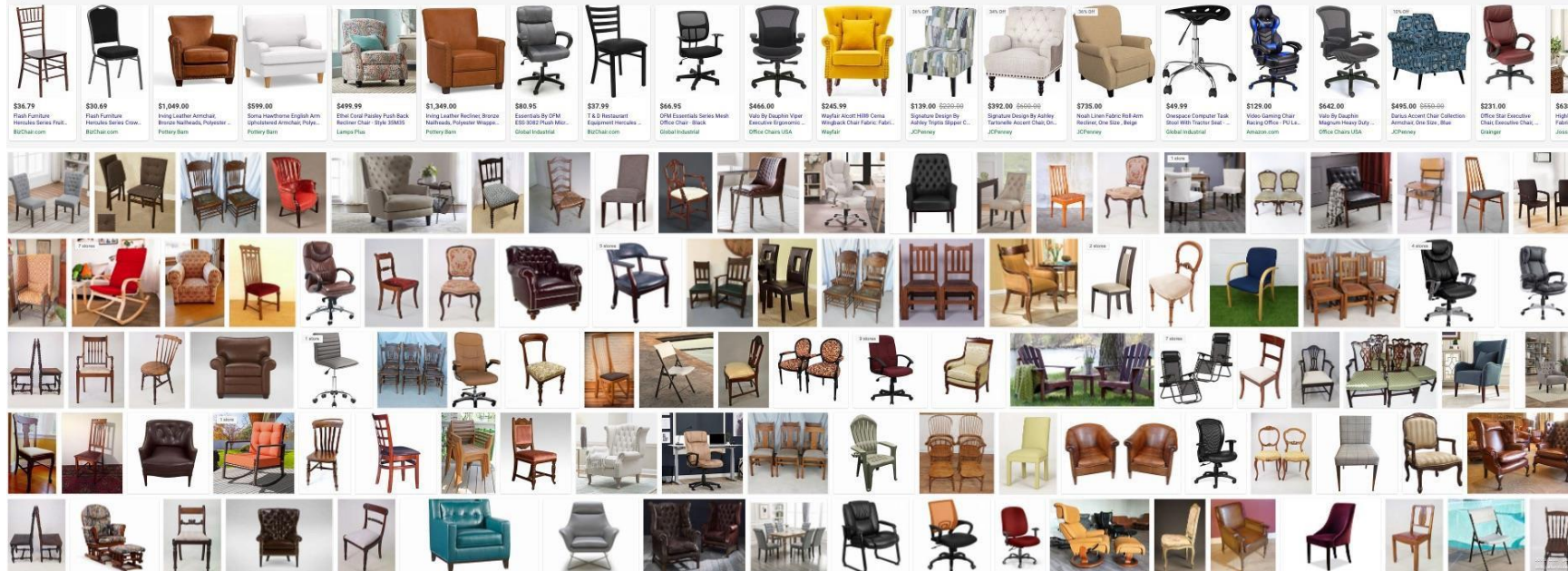


# The Chair Example

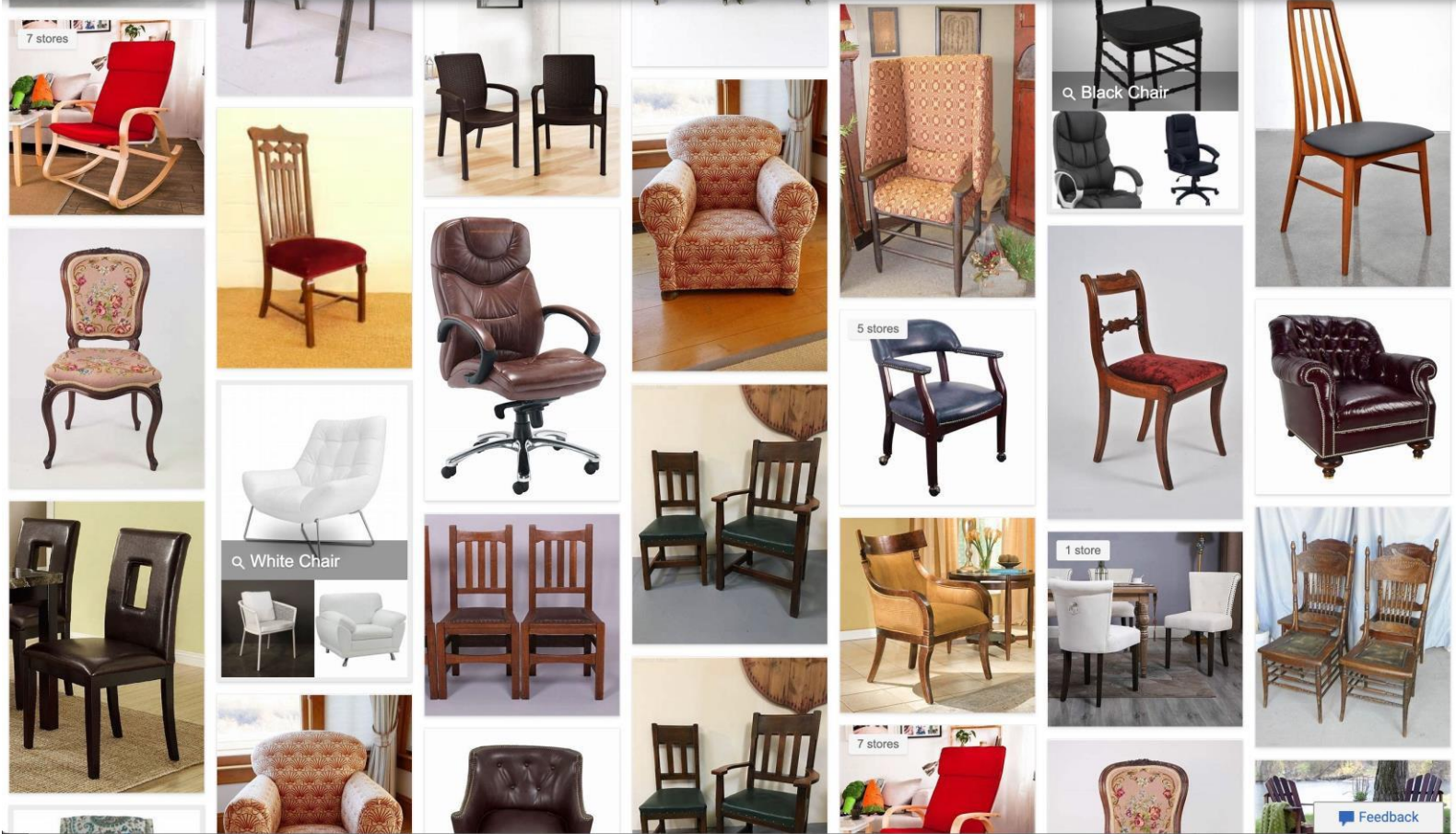


<https://www.westelm.com/products/windsor-dining-Chair> -h5047

# What “is” a Chair ?

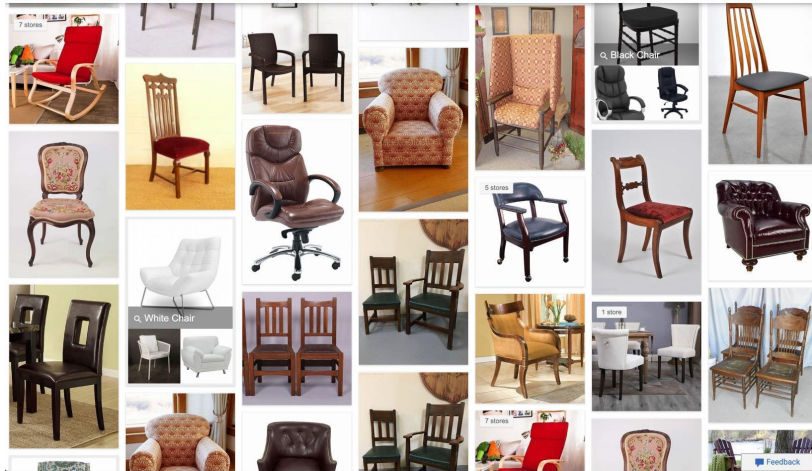


# How would you describe a Chair ? .. 1





How would you **model** a chair ? .. 2



## Chair 0



All images of Chairs from bing.com searches on  
Chairs, stools, loveseats, beanbags,...



Chair 1



Chair 2



## Chair 3



# Chair 4





# Chair 5





## Chair 6



Chair ?7



# Chair 8



# Chair 9



Chair ?10





Chair ?11



Define something using positive,  
negative, and near-miss examples.

Chair ?12



# Chair ?13

## Magis Spun Chair

by Thomas Heatherwick for Magis

★★★★★ 5.0 (6) [Write a review](#)

\$995.00 + **FREE SHIPPING**

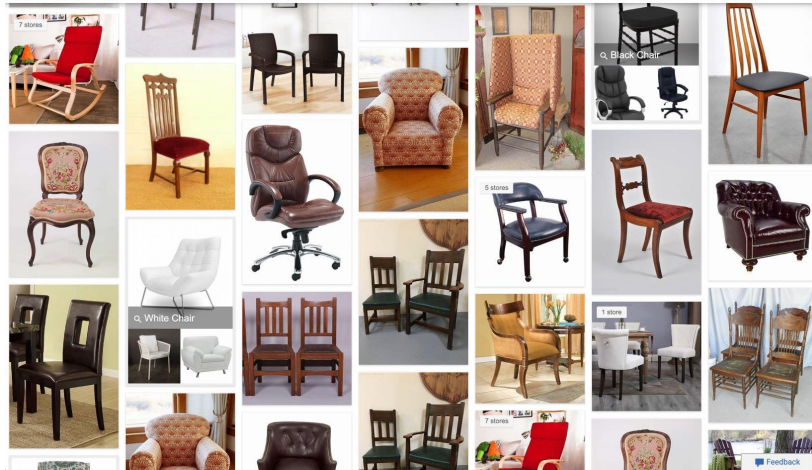


Chair ?14



Ergonomic Kneeling Chair Posture Rocking Knee Stool For Home Office Meditation

# How would you **model** a chair ? .. 3

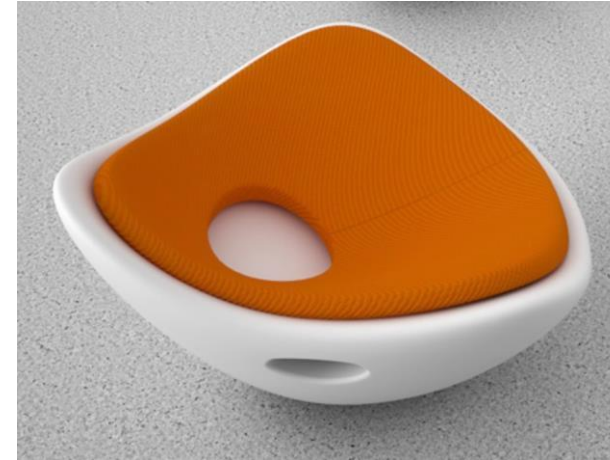


- Sitting surface?
- Back?
- Kind of back?
- Legs?
- #Legs?
- Arms?
- Height adjustment?
- Back adjustment?
- Material? ...



## Introspection: Let's analyze what we said/thought

- The more precise we are in our description, the fewer chairs we describe and/or the longer the description
- Relaxing some 'conditions' (e.g., chairs have 4 legs → chairs have legs) is useful; relaxing it even more is better (e.g., chairs have legs → chairs have some support) !



[Balance Chair on Behance](#)

Task: How do we get a good-enough description to include (most) chairs but exclude (most) non-chairs?

Think of models as compressed descriptions!

# Optimization & Generalization

## Tension between Optimization & Generalization

**Optimization:** adjusting model to get best perf on training data ('learning' from data)

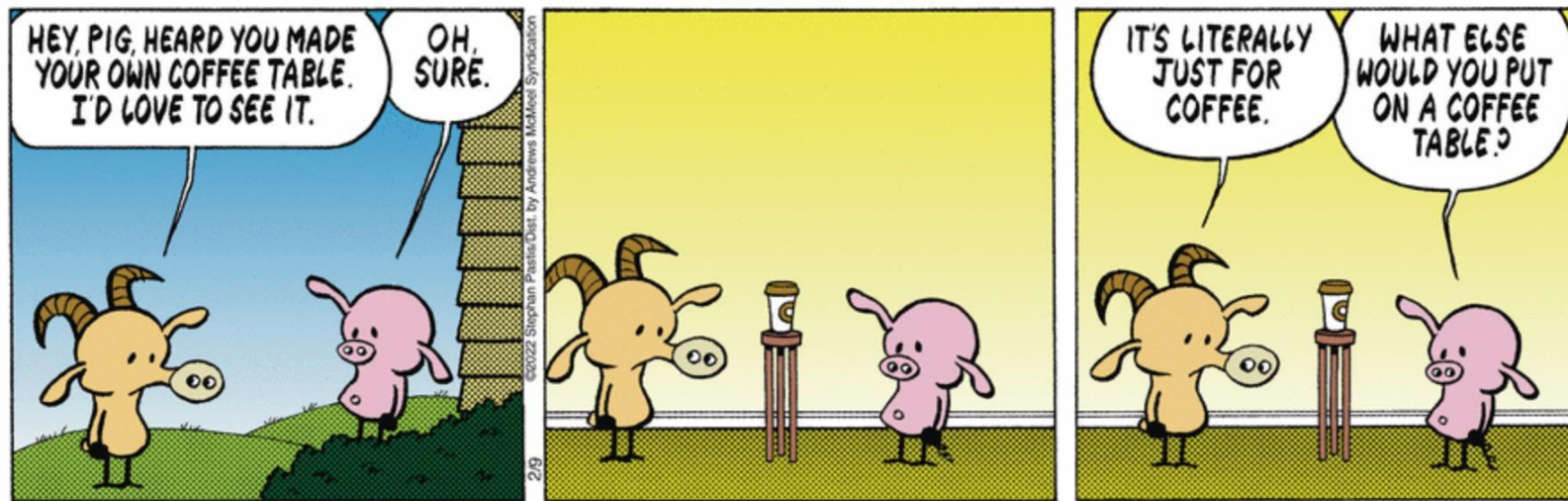
**Generalization:** how this model behaves on unseen data

Goal: Good generalization. But:

- Only model fitting is under our control
- If we fit too well, we get overfitting, and generalization suffers

From a recent newspaper...

Pearls Before Swine by Stephan Pastis for February 09, 2022



$$\begin{aligned}
p(C_k, x_1, \dots, x_n) &= p(x_1, \dots, x_n, C_k) \\
&= p(x_1 \mid x_2, \dots, x_n, C_k) p(x_2, \dots, x_n, C_k) \\
&= p(x_1 \mid x_2, \dots, x_n, C_k) p(x_2 \mid x_3, \dots, x_n, C_k) p(x_3, \dots, x_n, C_k) \\
&= \dots \\
&= p(x_1 \mid x_2, \dots, x_n, C_k) p(x_2 \mid x_3, \dots, x_n, C_k) \cdots p(x_{n-1} \mid x_n, C_k) p(x_n \mid C_k) p(C_k)
\end{aligned}$$

$$p(x_i \mid x_{i+1}, \dots, x_n, C_k) = p(x_i \mid C_k)$$

$$\begin{aligned}
p(C_k \mid x_1, \dots, x_n) &\propto p(C_k, x_1, \dots, x_n) \\
&= p(C_k) p(x_1 \mid C_k) p(x_2 \mid C_k) p(x_3 \mid C_k) \cdots \\
&= p(C_k) \prod_{i=1}^n p(x_i \mid C_k),
\end{aligned}$$