RESEARCH STATEMENT

Manjil P. Saikia

My main research interests lie in Enumerative and Algebraic Combinatorics, specially in the enumeration of *alternating sign matrices* and *tilings* of a finite lattice in the plane; and other related problems; as well as in statistics related to *integer partitions* and the use of *q-series identities*.

Alternating Sign Matrices

An alternating sign matrix (ASM) of size n is an $n \times n$ matrix with entries in the set $\{0, \pm 1\}$ such that all row and column sums are equal to 1 and the non-zero entries alternate in each row and column. These matrices, first introduced by Robbins and Rumsey [RR86] in the 1980s, have given rise to a lot of beautiful enumerative results and conjectures. The first such conjecture was given by Mills, Robbins and Rumsey [MRR83] about the number of ASMs of size n (denoted $A(n)$). They conjectured that

$$A(n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

This was later proved by Zeilberger [Zei96a] and Kuperberg [Kup02] independently, using completely different methods.

ASMs, since the time they were introduced have been an object of serious study by combinatorialists, due to their inherent simplicity as well as beauty. Most of the enumeration results about ASMs have beautiful formulas, like the one mentioned in the preceding paragraph, but proving them can be quite difficult. Techniques arising from statistical physics, most prominently the *six-vertex model* have been used quite successfully in solving many of the problems related to ASMs. These matrices also have connections with other branches of mathematics, for instance Okada [Oka06] found connections with representation theory and Schur's functions in his work on ASMs. But almost all of these connections are mysterious and one major direction of study is to understand these connections combinatorially.

Stanley [Sta86] suggested the study of the symmetry classes of ASMs, shortly after these objects were introduced. This led Robbins [Rob00] to conjecture various enumeration formulas for different symmetry classes of ASMs. Kuperberg [Kup96], Okada [Oka06], Razumov-Stroganov [RS06a, RS06b], and Behrend, Fischer and Konvalinka [BFK17] have proved the conjectured formulas for these symmetry classes of ASMs. My current research is focused on the refined enumeration of some of these symmetry classes and their connections with other objects, most notably plane partitions.

It is not difficult to see that, the boundary rows and columns of any ASM contains exactly one entry equal to 1. This observation has given impetus on the refined enumeration of ASMs, where one or more of the boundary rows and columns are fixed i.e. the position of the unique 1 in the boundaries are fixed. Zeilberger [Zei96b] was the first to prove a refined enumeration formula for ASMs, when he gave a simple formula for the number of order n ASMs with the unique 1 in the first row (also, first column by rotation) fixed. This was followed by several other refined enumeration formulas for ASMs as well as for their symmetry classes (see Ayyer and Romik [AR13], Behrend [Beh13], etc. and the references therein).

Among the symmetry classes of ASMs are the ones with vertical symmetry; they are symmetric under reflection along the vertical axis. The stipulation that the first row of any ASM can contain only one 1, forces all VSASMs to be of odd order, and the fixed one in the first row to be fixed in the central column of any VSASMs. Hence, this refinement is of no interest. However, in the case of vertically symmetric ASMs (VSASMs) and vertically and horizontally symmetric ASMs (VHSASMs), it turns out that, the second row of such matrices have exactly two occurrences of 1, and this allows us to ask for a refinement with respect to the second row.

Fischer [Fis09] had conjectured that the number of $(2n+1) \times (2n+1)$ VSASMs, where the first one in the second row is in the ith column is equal to

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!} \left(\prod_{j=1}^{n-1} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!} \right). \tag{1}$$

For VHSASMs no such conjecture was put forth. During my PhD, together with Fischer [FS21], we proved her conjecture and also proved new results on the refined enumeration of VHSASMs in terms of generating functions. We also proved refined enumeration results on several other symmetry classes such as off-diagonally symmetric ASMs (OOSASMs; ASMs with diagonal and off-diagonal symmetry with null diagonals except the central entry), vertically and off-diagonally symmetric ASMs (VOSASMs; OOSASMs with vertical symmetry), vertically and horizontally perverse ASMs (VHSASMs with some restrictions), quarter-turn symmetric ASMs (ASMs invariant under a 90° rotation) and quasi quarter-turn symmetric ASMs (quarter-turn symmetric ASMs of even order with some restrictions). Our results also prove conjectures of Robbins [Rob00] and Duchon [Duc08].

It is also possible to ask for further refined enumeration of ASMs, and indeed several such refinements have already been studied. For instance, there exists formulas for the number of ASMs with two or more of the boundary rows and columns fixed (see [AR13, FR09], etc.) as well as ASMs with fixed number of -1's and fixed boundary statistics (see [Beh13]). In these cases, the formulas invariably become more complicated. Such type of refinements are also possible for some of the symmetry classes of ASMs, which had not received any attention in the literature. For instance, we could ask about the number of vertically symmetric ASMs with one boundary column and the second row fixed or the number of OSASMs (ASMs which are invariant under a diagonal reflection with all diagonal entries equal to 0) with the first row and the first column fixed. In my post doctoral reserach, together with Behrend, we have found the most general generating functions for refined enumeration of VSASMs and OSASMs. These generating functions not only relate the boundary statistics of these ASMs but also takes into account the number of -1 in such ASMs. Our results are reminiscent of results for ordinary ASMs found by Behrend [Beh13], but we use slightly different techniques to obtain the new results. Moreover, our new results gives as corollaries certain results that were obtained during my PhD.

In addition to these results, our work has also brought to light several previously unobserved connections between refined enumeration of symmetry classes of ASMs and different classes of plane partitions (a three dimensional generalization of ordinary integer partitions). Several of these observations are at present only conjectured, but we hope to be able to prove them in the next few months. Our work has also unearthed some previously unstudied sub-classes of ASMs which are equinumerous with certain types of plane partitions. This looks to be a promising direction of future study and gives rise to the first set of research problems that I wish to tackle in the immediate future.

Problem #1: Extend the results of our recent work with Behrend, to find generating functions for refined enumeration of other symmetry classes of ASMs, such as VHSASMs and HTSASMs.

Problem #2: Study the newly unearthed sub-classes of ASMs and connect them to the broader framework of combinatorial properties studied for ASMs and plane partitions.

Aztec Triangles

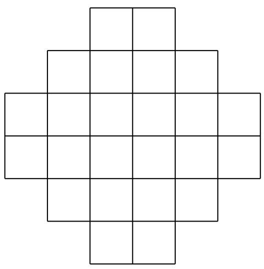

Figure 1: Aztec Diamond of order 3

ASMs have connections with several other combinatorial objects. For instance, while studying some statistics on diagonally and anti-diagonally symmetric ASMs (DASASMs), Ayyer, Behrend and Fischer [ABF20] introduced a new class of objects called alternating sign triangles (ASTs). An AST of size n is a triangular array $a_{i,j}$ with $1 \leq i \leq n$ and $i \leq j \leq 2n - i$ such that the entries are either $1, -1$ or 0, along the columns and rows the non-zero entries alternate, the first non-zero entry from the top is a 1 and the rowsums are equal to 1. Among other properties, they proved that ASMs and ASTs of the same order, are equinumerous, and that the 2-enumeration of ASTs (that is, enumerating them after assigning each -1 with a weight of 2) is given by $2^{n(n-1)/2}$. It might be mentioned here that, ASMs are known to be equinumerous with several families of objects, and finding a natural bijection between these families is one of the most outstanding open problem in enumerative combinatorics.

Earlier, Ciucu [Ciu97] established a bijection between the 2-enumeration of order n ASMs with domino tilings of order n Aztec Diamonds, which were introduced by Elkies, Kuperberg, Larsen and Propp [EKLP92]. The Aztec Diamond of order n (denoted by $\mathrm{AD}(n)$) is the union of all unit squares inside the contour $|x|+|y| = n+1$ (see Figure 1 for an Aztec Diamond of order 3). A domino is the union of any two unit squares sharing an edge, and a domino tiling of a region is a covering of the region by dominoes so that there are no gaps or overlaps. They [EKLP92] considered the problem of counting the number of domino tilings of the order n Aztec Diamond and found them to be equal to $2^{n(n+1)/2}$.

Ayyer, Behrend and Fischer [ABF20] have also found an analogous bijection for the 2-enumeration of ASTs with tilings of another region which we call *Aztec Triangles*, denoted by $\mathcal{Q}_n$ (which we do not define here). There is a very well-known connection between tilings of certain regions in the finite lattice (for instance, domino tilings of Aztec Diamonds) with perfect matchings of certain bipartite graphs. Since, their proof of this result is not graph-theoretic, this motivates us to look for such a proof. We also, notice that the formulas appearing in the 2-enumeration of ASMs and ASTs are the same upto a factor of 2. This suggests a possible connection between these two objects and hence the 2-enumeration of ASMs and ASTs. One of my immediate goals is to study the following research problems.

Problem #3: Explore the connection between ASTs and ASMs, and find a graph-theoretic proof of the 2-enumeration of ASTs.

Problem #4: Is it possible to relate other enumeration results for Aztec Diamonds, with similar results for $\mathcal{Q}_n$?

Tilings of regions in the Plane

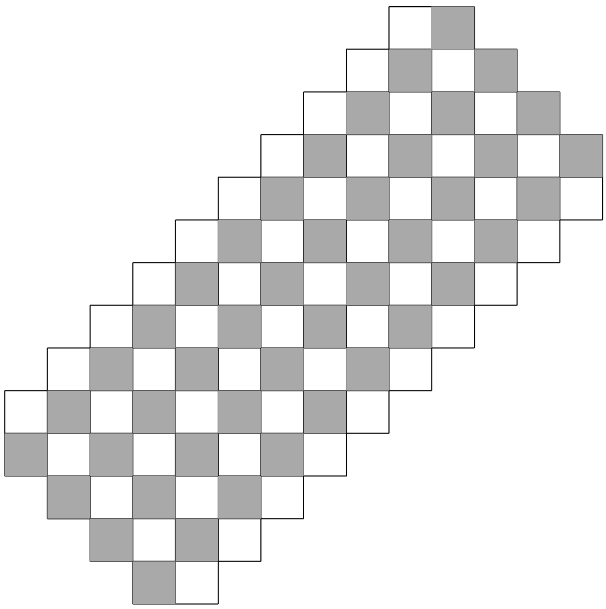

Figure 2: Checkerboard representation of an Aztec Rectangle with $a = 4, b = 10$

Another interest of mine are tiling problems in the square lattice. Since the publication of the first proof for the number of domino tilings of the Aztec Diamond (see Figure 1), several extensions of it have also been studied. The most natural, among these extensions are Aztec rectangles (see Figure 2). We denote by $\mathcal{AR}_{a,b}$, the Aztec rectangle which has a unit squares on the southwestern side and b unit squares on the northwestern side ($b \geq a$). For $a < b$, $\mathcal{AR}_{a,b}$ does not have any tiling by dominoes, this can be seen easily from the checkerboard representation of $\mathcal{AR}_{a,b}$, where we see that the number of white squares are more than the number of black squares. However, if we remove $b-a$ unit squares (called *defects*) from the southeastern side then we have a simple product formula for the number of domino tilings of these regions which was found by Mills, Robbins and Rumsey [MRR83].

Enumeration formulas for domino tilings of Aztec rectangles with holes or defects in them, which are not restricted to just one of the boundary sides have been studied by several other people, for instance by Ciucu [Ciu96], Lai [Lai15], etc. In 2016, I proved several enumeration results for Aztec Rectangles and Aztec Diamonds with the most general class of boundary holes that are possible. These results give Pfaffian formulas for the number of domino tilings of $\mathcal{AR}_{a,b}$ with arbitrary defects on three of the boundary sides (except one side of smaller length) and the number of domino tilings of $\mathcal{AR}_{a,a} = \mathrm{AD}(a)$ with arbitrary defects on all the boundary sides (see [Sai17]). These extend previous work by several authors including those cited above.

There are several other extensions of Aztec Rectangles that have been studied in the literature, most notably by Lai [Lai14, Lai16, Lai17]. My future plan is to study these regions, in particular the regions called Trimmed Aztec Rectangles and Double Aztec Rectangles. So far, no enumeration formula for these regions with defects has been found, but it seems that at least the Double Aztec Rectangle with boundary holes would yield to a nice formula.

Problem #5: Find a formula for the number of domino tilings of Trimmed and Double Aztec Rectangles of suitable dimensions, with defects on their boundaries.

The techniques that are used to study these tiling problems come from a variety of sources. However, one of the most powerful methods used is called Kuo's Condensation Method, after Eric Kuo [Kuo04]. This method was generalized by Ciucu [Ciu15]; however his result did not generalize all of Kuo's original results. In 2016, while studying the tilings of Aztec rectangles, I further gave a uniform generalization of both Kuo's and Ciucu's results (see [Sai17]). This result has so far, not been used in enumerating tilings. I would like to see if some regions might be more amenable to this result than the usual versions of Kuo's condensation.

It is also possible to study tilings of the regions described above with other sets of tiles. Recently, in collaboration with Akagi, Gaona, Mendoza and Villagra [AGM$^+$20], we studied tilings of regions in the square lattice with L-shaped trominoes. In general, deciding the existence of a tiling with L-trominoes for an arbitrary region is NP-complete. However, we identified restrictions to the problem where it has a polynomial-time algorithm. We have characterized the complexity of deciding the existence of an L-tromino tiling of Aztec Rectangles with no holes as well as with some number of fixed holes. We also studied tilings of arbitrary regions where only 180° rotations of L-trominoes are available and characterized the instances when such a region has a polynomial-time algorithm and when it remains NP-complete.

This work has naturally lead to many different avenues for further research, most notably about when a polynomial-time algorithm is available to tile an Aztec rectangle or some variant of the region (with an unknown number of holes) by L-trominoes. We have only studied several sub cases of this problem so far, and in the future I would like to continue working on the following research problem.

Problem #6: Find the classes of holes that admit a polynomial-time algorithm to tile an Aztec rectangle (and other similar regions) with L-trominoes.

Another aspect of this problem is to find enumeration results. But, at the moment, it seems difficult to find a closed form formula for the number of L-tromino tilings of Aztec Rectangles or Aztec Diamonds. I have been able to prove several bounds on the number of such tilings which I strongly believe can be strengthened further, and in the future I would like to work towards improving them.

Statistics on Integer Partitions

A recent interest of mine are (integer) partitions. We define a partition λ of a non-negative integer n to be an integer sequence $(\lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. We say that λ is a partition of n, denoted by $\lambda \vdash n$ and $\sum_{i=1}^{\ell} \lambda_i = n$. Partitions appear in a variety of different context in algebraic combinatorics and have been studied extensively by generations of mathematicians, starting with Euler. I am interested in finding combinatorial connections between various different partition statistics and also in finding combinatorial proofs of existing analytical results in this area.

In collaboration with several different authors, I have recently studied statistics on the parity of parts in partitions [BBD$^+$22, MSS22] as well on the multiplicity of parts in partitions [MS22]. These projects have naturally led to several different possibilities for future research. In particular, the following problems seems to be the most accesible at the moment.

Problem #7: Extend the results obtained for the statistics on the parity of parts to other classes of integer partitions.

Problem #8: Find combinatorial proofs and techniques for the type of results discussed in our recent paper [MS22].

Problem #9: Study the mex-statistic of integer partitions and find refined results relating this statistic with 2-color partitions.

Other Interests

Another side interest, at the moment is that of classifying the various types of generalized perfect numbers that have been defined, which I explain briefly. A natural number n is called a perfect number if the sum of all the proper divisors of n (denoted by $\sigma(n)$) is n (i.e., if $2n = \sigma(n)$). One of the most important problems in number theory is to determine whether there are any odd perfect numbers. In trying to solve this problem, various extensions of the definition of perfect numbers have been put forth, including some that were introduced in collaboration with Laugier and Sarmah [LSS16].

One extension of a perfect number is that of a deficient perfect number. A number n is called a deficient perfect number if the equation $2n - d = \sigma(n)$ is satisfied for some proper divisor d of n. The problem to determine all odd deficient perfect numbers is still open. Recently, in collaboration with Dutta [DS20] and also independently [Sai18], I have attempted to resolve the first case for which the question is still open. We have conjectured [DS20], that there are only finitely many of them and I would like to study this conjecture further; it seems possible to extend some of the results that I have obtained so far [Sai18]. Apart from this, a joint work with Mahanta and Yaqubi [MSY20] on another generalization of perfect numbers have also raised several different questions and avenues of research which I would like to pick up at some point.

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