A FAMILY OF LACUNARY RECURRENCES FOR LUCAS NUMBERS

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ABSTRACT. We prove an infinite family of lacunary recurrences for the Lucas numbers using combinatorial means.

1. INTRODUCTION

A recurrence relation involving only terms of a given sequence with indices in arithmetic progression is called a *lacunary recurrence*. The *gap* of such a lacunary recurrence is the common difference in the indices in arithmetic progression. Several such lacunary recurrences are known for sequences including but not limited to Bernoulli numbers, Euler numbers, k-Fibonacci numbers, etc. We refer the reader to the recent paper of Ballantine and Merca [BM19] for relevant references and other examples.

Ballantine and Merca [BM19] proved an infinite family of lacunary recurrences for Fibonacci numbers. They closed the paper by asking the natural question of whether such an infinite family of lacunary recurrences can be found for the Lucas numbers. The aim of this article is to prove such an infinite family of lacunary recurrences. Before stating and proving our result, let us recall some definitions and relations.

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

with $F_0 = 0$ and $F_1 = 1$. We use the convention $F_n = 0$ when $n < 0$. Similarly, the Lucas sequence $\{L_n\}_{n\geq 0}$ defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2},$$

with $L_0 = 2$ and $L_1 = 1$. We use the convention $L_n = 0$ when $n < 0$. These two sequences are related by the identity

$$L_n = F_{n-1} + F_{n+1}. \tag{1}$$

Several interesting relationships between Fibonacci numbers are known, two of them, which are relevant for this paper are

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n, \tag{2}$$

and

$$(-1)^n F_{m-n} = F_m F_{n+1} - F_{m+1} F_n. \tag{3}$$

For these and many other identities we refer the reader to Honsberger's book [Hon85, Chapter 8] and to the more recent book by Koshy [Kos18, Chapter 5]. The second identity (3) is called d'Ocagne's identity.

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Lucas in 1876 proved a lacunary recurrence of gap 2 for the Fibonacci numbers in the following equivalent form

$$F_n = \frac{1+(-1)^n}{2} + F_{n-2} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} F_{n-2k}.$$

This was generalized by Ballantine and Merca [BM19] to the following

Theorem 1.1. [BM19, Theorem 1] *Given a positive integer $N \geq 2$, we have*

$$F_n = F_N \cdot F_{N-1}^{\lfloor \frac{n-1}{N} \rfloor + 1} \cdot F_{(n-1) \bmod N} + F_{N+1} \cdot F_{n-N} + F_N^2 \cdot \sum_{k=2}^{\lfloor \frac{n-1}{N} \rfloor} F_{N-1}^{k-2} \cdot F_{n-kN},$$

for all $n \geq N$.

It is quite natural to ask, as did Ballantine and Merca [BM19] if a similar result holds for the Lucas numbers? We now present such a result in the following theorem.

Theorem 1.2. *Given a positive integer $N \geq 2$, we have*

$$L_n = L_N \sum_{i=1}^{d} (-1)^{(N+1)(i+1)} L_{n-(2i-1)N} + (-1)^{(N+1)(d+2)} L_{n-2dN}, \tag{4}$$

where $d = \left\lfloor \left\lfloor \frac{n}{N} \right\rfloor /2 \right\rfloor$ and $\frac{n}{2} \geq N \geq 0$.

A simple consequence of the above theorem is the following congruence.

Corollary 1.3. *For a given integer $N \geq 2$ we have*

$$L_n - (-1)^{(N+1)(d+2)} L_{n-2dN} \equiv 0 \pmod{L_N},$$

where $d = \left\lfloor \left\lfloor \frac{n}{N} \right\rfloor /2 \right\rfloor$ and $\frac{n}{2} \geq N \geq 0$.

2. A Combinatorial Proof of Theorem 1.2

It is well-known that the Fibonacci numbers can be interpreted as tilings of an $1 \times n$ board with squares and dominoes. We call such a board an n-board. If the number of such tilings is f_n, then it can be proved that $F_{n+1} = f_n$ (see for instance, the book by Benjamin and Quinn [BQ03]). With this notation, equations (2) and (3) now becomes

$$f_m f_n + f_{m-1} f_{n-1} = f_{m+n} \tag{5}$$

and

$$f_{m-1} f_n - f_m f_{n-1} = (-1)^n f_{m-n-1} \tag{6}$$

Both these identities can be easily proven using the combinatorial interpretation of f_k.

It is also known (see Chapter 2 of the book by Benjamin and Quinn [BQ03]) that the number l_n of ways to tile a circular board composed of n labelled cells with curved squares and dominoes is equal to L_n. We call such a tiling of the circular n-board to be an n-bracelet. There are two types of bracelets, an *in-phase* or an *out-of-phase*. A bracelet is out-of-phase if a domino covers the cells numbered n and 1, and it is called in-phase if squares cover the cells numbered n and 1. An example of an out-of-phase 4-bracelet and an in-phase 4-bracelet is shown in Figure 1, where dominoes are coloured black and squares are white. We note that an in-phase tiling of an n-bracelet can be made

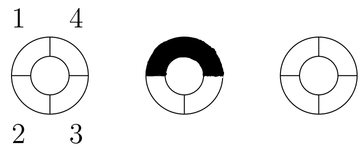

FIGURE 1. Examples of bracelets.

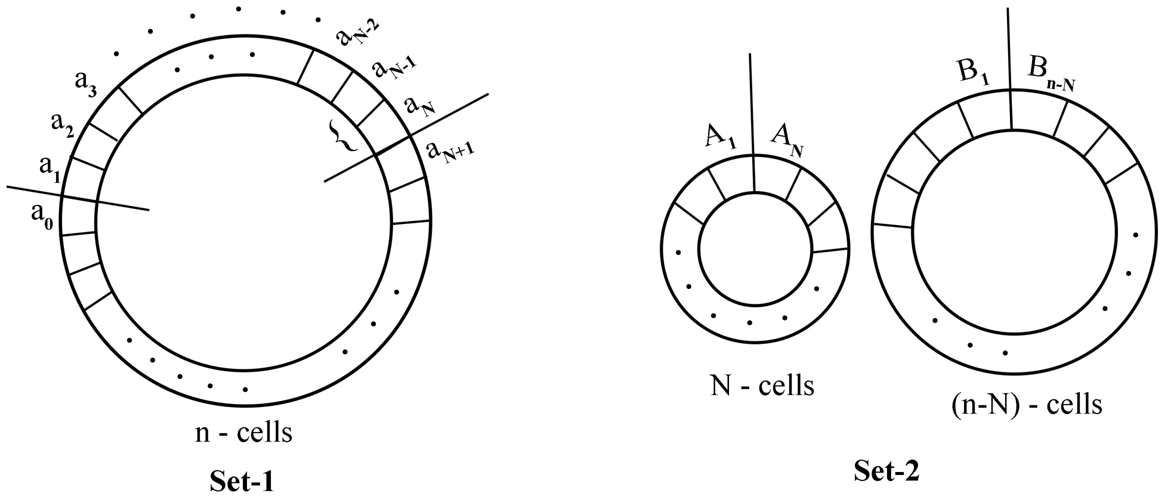

FIGURE 2. The two sets of bracelets considered in the proof of Theorem 1.2.

into a tiling of an n-board. From this observation it is easy to see the validity of equation (1). We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let us draw two sets of circular boards as shown in Figure 2, and call them Set 1 and Set 2. We mark the cells as shown in Figure 2. The number of bracelets of Set-1 is L_n and of Set-2 is $L_N \times L_{n-N}$, where $\frac{n}{2} \geq N \geq 0$.

We can break the tilings of Set-2 in the following four parts:

(a) $f_N \times f_{n-N}$. (Here both the N-bracelet and $(n-N)$-bracelets are in-phase.)
(b) $f_{N-2} \times f_{n-N}$. (Here only the N-bracelet is out-of-phase.)
(c) $f_N \times f_{n-N-2}$. (Here only the $(n-N)$-bracelet is out-of-phase.)
(d) $f_{N-2} \times f_{n-N-2}$. (Here both are out-of-phase.)

Observe that the tilings of (a) can be made into tilings of the n-bracelet in such a way that the N-board covers the cells of the n-bracelet from a_1 to a_N. And hence the $(n-N)$-board covers the remaining cells of the n-bracelet. In these tilings of the n-bracelet, there is no domino which cover the cells a_0 and a_1 or a_N and a_{N+1}.

Observe that the tilings of (b) can be made into tilings of the n-bracelet in such a way that a domino covers the cells a_0 and a_1, and the N-board covers from the cell a_0 to the cell a_{N-1}. So, the $(n-N)$-board covers the remaining cells of the n-bracelet. In these tilings no domino covers the cells a_{N-1} and a_N.

There are only two types of tilings that remains in the set of all tilings of the n-bracelet, apart from the ones discussed above:

(1) Tilings where one domino covers the cells a_0 and a_1 and another domino covers the cells a_{N-1} and a_N. The total number of such tilings is $f_{N-3}f_{n-N-1}$.
(2) Tilings where one domino covers the cells a_N and a_{N+1}, but no domino covers the cells a_0 and a_1. The total number of such tilings is $f_{n-2} - f_{N-2}f_{n-N-2}$.

Let us now compute the difference (say A) in the total tilings of (1) and (2) with that of the total tilings in (c) and (d)

$$\begin{aligned}A := (f_{N-3}f_{n-N-1} + f_{n-2} - f_{N-2}f_{n-N-2}) - (f_N f_{n-N-2} + f_{N-2}f_{n-N-2})\\ = f_{N-3}f_{n-N-1} + f_{(N-1)+(n-N-1)} - 2f_{N-2}f_{n-N-2} - f_N f_{n-N-2}.\end{aligned}$$

Using equation (5) in the above we get

$$\begin{aligned}A =& f_{N-3}f_{n-N-1} + f_{N-1}f_{n-N-1} + f_{N-2}f_{n-N-2} - 2f_{N-2}f_{n-N-2} - f_N f_{n-N-2}\\ =& -(f_{n-N-2}f_{N-2} - f_{n-N-1}f_{N-3}) - (f_{n-N-2}f_N - f_{n-N-1}f_{N-1}).\end{aligned}$$

Using equation (6) in the above we get

$$\begin{aligned}A =& -(-1)^{N-2}f_{(n-N-1)-(N-2)-1} - (-1)^N f_{(n-N-1)-N-1}\\ =&(-1)^{N-1}f_{n-2N} + (-1)^{N+1}f_{n-2N-2}\\ =&(-1)^{N+1}L_{n-2N}.\end{aligned}$$

In the last step we used equation (1). Hence

$$f_{N-3}f_{n-N-1} + f_{n-2} - f_{N-2}f_{n-N-2} = f_N f_{n-N-2} + f_{N-2}f_{n-N-2} + (-1)^{N+1}L_{n-2N}.$$

Finally, adding the total number of the other tilings (namely those in (a) and (b)) in both sides of the above we get

$$L_n = L_N \times L_{n-N} + (-1)^{N+1}L_{n-2N}. \tag{7}$$

The left hand side follows because the number of tilings in (1), (2), (a) and (b) is L_n, while the right hand side follows because the number of tilings in (a)–(d) is $L_N \times L_{n-N}$. Replacing n by $n-2N$ in the above, we get

$$L_{n-2N} = L_N L_{n-3N} + (-1)^{N+1}L_{n-4N}. \tag{8}$$

Therefore, from equations (7) and (8), we get

$$L_n = L_N L_{n-N} + (-1)^{N+1}L_N L_{n-3N} + L_{n-4N}.$$

Again,

$$L_{n-4N} = L_N L_{n-5N} + (-1)^{N+1}L_{n-6N}.$$

So, we have

$$\begin{aligned}L_n =& L_N L_{n-N} + (-1)^{N+1}L_N L_{n-3N} + L_N L_{n-5N} + (-1)^{N+1}L_{n-6N}\\ =& L_N L_{n-N} + (-1)^{N+1}L_N L_{n-3N} + L_N L_{n-5N} + (-1)^{N+1}L_N L_{n-7N} + L_{n-8N}\\ =& L_N\left(L_{n-N} + (-1)^{N+1}L_{n-3N} + L_{n-5N} + (-1)^{N+1}L_{n-7N}\right) + L_{n-8N}.\end{aligned}$$

This gives us,

$$L_n = L_N \Big((-1)^{(N+1)(1+1)} L_{n-N} + (-1)^{(N+1)(2+1)} L_{n-3N} \\ +(-1)^{(N+1)(3+1)} L_{n-5N} + (-1)^{(N+1)(4+1)} L_{n-7N} \Big) + (-1)^{(N+1)(5+1)} L_{n-8N}.$$

We can proceed in this way up to the $\left\lfloor \left\lfloor \dfrac{n}{N} \right\rfloor /2 \right\rfloor$-th step. This proves equation (4). $\square$

3. Concluding Remarks

We can combine several known identities involving Lucas and Fibonacci numbers with Theorems 1.1 and 1.2 to give several new results involving more complicated sums. We do not explore this here.

A generalization of the Fibonacci sequence, called the Gibonacci sequence $\{G_n\}_{n\geq 0}$ is given by the same recurrence

$$G_n = G_{n-1} + G_{n-2},$$

for all $n \geq 2$. Changing the initial conditions for G_0 and G_1 gives rise to different sequences, two of which are the Fibonacci and Lucas sequences. There exist combinatorial interpretations for such a Gibonacci sequence, which is similar to the interpretation for the Lucas sequence. It would seem that by tweaking our proofs, a more general lacunary recurrence could be found for the Gibonacci sequence. We leave this as an open problem.

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