Alternating Sign Matrices and Plane Partitions

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11 October, 2022

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Zeilberger's proved a stronger statement using constant term identities, while Kuperberg exploited a connection with models from statistical mechanics.

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If A is an $n \times n$ matrix, then

$$\det(A)\det(A_{1,n}^{1,n}) = \det(A_1^1)\det(A_n^n) - \det(A_n^1)\det(A_1^n)$$

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This gives us a way of evaluating determinants, in terms of smaller determinants.

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$$\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \frac{1}{a_{2,2}} \times \det \begin{pmatrix} \det \begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} & \det \begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \\ \det \begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \end{pmatrix}.$$

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$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} = \frac{1}{\det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix}}$$

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$$\det_{\lambda} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = a_{1,1}a_{2,2} + \lambda a_{2,1}a_{1,2}.$$

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Using the previous observations, they generalized it to an $n \times n$ determinant.

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Theorem (Robbins-Rumsey)

Let A be an $n \times n$ matrix with entries $a_{i,j}$, $\mathcal{A}_n$ be the set of all ASMs, $\mathcal{I}(B)$ be the inversion number of B and $\mathcal{N}(B)$ be the number of -1's in B. Then

$$\det_\lambda(A) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}.$$

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This was the first appearance of an ASM in the literature.

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Further, if A_n is the number of $n \times n$ ASMs, then $A_{n,1} = A_{n,n} = A_{n-1}$.

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$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}.$$

This means that the $A_{n.k}$'s are uniquely determined by the $A_{n,k-1}$'s when $k > 1$ and by $A_{n,1} = \sum_{k=1}^{n-1} A_{n-1,k}$.

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From here, knowing that $A_n = A_{n+1,1}$ allows one to conjecture the ASM enumeration formula.

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- *Alternating Sign Triangles,*
- *...and a couple more.*

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$$PP \subset \{1, 2, \cdots, a\} \times \{1, 2, \cdots, b\} \times \{1, 2, \cdots, c\}$$

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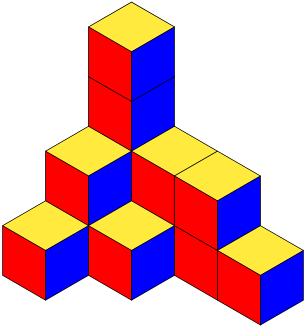

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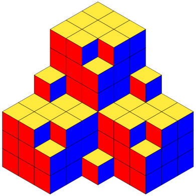

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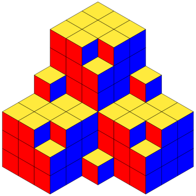

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- Diagonally and Antidiagonally Symmetric ASMs: $a_{i,j} = a_{j,i} = a_{n+1-j,n+1-i}$, n odd (Behrend, Fischer & Konvalinka 2017)
- All symmetries: $a_{i,j} = a_{j,i} = a_{i,n+1-j}$, no 'nice' formula.

Refined Enumeration of ASMs

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- These are called refined enumeration of ASMs.

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Several people have worked on the refined enumration of ASMs as well as their symmetry classes: Behrend, Fischer, Romik, Razumov-Stroganov, Ayyer-Romik, Romik-Karlinsky, etc.

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- Robbins (late 1980s) conjectured formulas for refined enumeration of QTSASMs.
- Duchon (2008) conjectured a formula for the refined enumeration of quasi-QTSASMs.
- We proved these conjectures – and more – in joint work with Ilse Fischer (2020).

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So we can ask for refined enumeration w.r.t. the position of the 1's in the second row.

Fischer's Conjecture

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Razumov and Stroganov (2004) has a formula counting the number of VSASMs with a fixed one in the first column.

$$A_{VC}(2n+1,i) = \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!} \times \sum_{k=1}^{i-1} (-1)^{i+k-1} \frac{(2n+k-2)!(4n-k-1)!}{(4n-2)!(k-1)!(2n-k)!}$$

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$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!} \left(\prod_{j=1}^{n-1} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!} \right).$$

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Theorem (Fischer-S., 2020)

The number of $(2n+1)\times(2n+1)$ *VSASM with a* 1 *in the* i*-th position in it's second row is given by*

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!}\left(\prod_{j=1}^{n-1}\frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!}\right)$$
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A bijective proof of the last relation would be of interest.

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- In some cases the results are in terms of generating functions.

For an ASM or quasi-ASM A in a class XASM, let

$$\begin{aligned}
\Theta_{\mathrm{XASM}}(A) &= \text{number of } -1\text{'s in the fundamental region of } A, \\
\Psi^{i}_{\mathrm{T}}(A) &= (\text{position of the 1 in the top row of } A) - i, \\
\overline{\Psi}^{i}_{\mathrm{T}}(A) &= (\text{position of the first 1 in the second-top row of } A) - i, \\
\Psi^{i}_{\mathrm{L}}(A) &= (\text{position of the 1 in the leftmost column of } A) - i,
\end{aligned}$$

Generating Functions

Generating Functions

The definitions of certain ASM generating functions are then

$$Z^{\mathrm{VSASM}}_{2n+1}(x, z_1, z_2) = \sum_{A \in \mathrm{VSASM}(2n+1)} x^{\Theta_{\mathrm{VSASM}}(A)}\, z_1^{\Psi^2_{\mathrm{L}}(A)}\, z_2^{\overline{\Psi}^1_{\mathrm{T}}(A)},$$

$$Z^{\mathrm{HVSASM}}_{2n+1}(x, z) = \sum_{A \in \mathrm{HVSASM}(2n+1)} x^{\Theta_{\mathrm{HVSASM}}(A)}\, z^{\overline{\Psi}^2_{\mathrm{T}}(A)},$$

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We have not yet defined some classes of ASMs.

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We have for all $n \geq 1$,

$$(1+z)\, Z_{2n+1}^{\mathrm{VSASM}}(1,z,1) = Z_{2n+1}^{\mathrm{VSASM}}(1,1,z) + z^{2n-1} Z_{2n+1}^{\mathrm{VSASM}}(1,1,\bar{z}).$$

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Theorem (Behrend-S., 2022)

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Vertically and Horizontally Symmetric ASMs

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$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

VHSASMs

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We have for all $n \geq 1$,

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and

$$Z^{\mathrm{HVSASM}}_{4n+3}(1,z) + z^{4n-1} Z^{\mathrm{HVSASM}}_{4n+3}(1,\bar{z}) = (1+z)\, \widetilde{Z}^{\mathrm{CSTCPP}}_{2n+2}(1,z)\, Z^{\mathrm{VSASM}}_{2n+1}(1,z,1).$$

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Again, we have not yet defined one generating function in the above. We have generalized the above results, but we need some more background on plane partitions.

Vertically and Horizontally Perverse ASM

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$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
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We have generalized the above results.

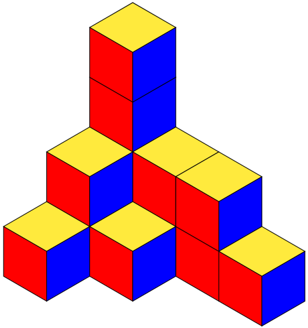
Plane Partitions

Plane Partitions

A *plane partition* in an $a \times b \times c$ box is a subset

$$PP \subset \{1, 2, \cdots, a\} \times \{1, 2, \cdots, b\} \times \{1, 2, \cdots, c\}$$

with $(i', j', k') \in PP$ if $(i, j, k) \in PP$ and $(i', j', k') \leq (i, j, k)$.



Cyclically Symmetric Plane Partitions

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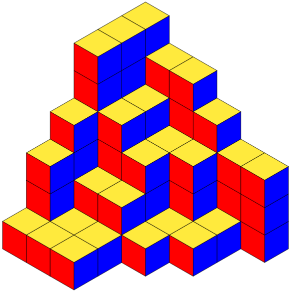

Cyclically Symmetric Transpose Complementary Plane Partitions

Cyclically Symmetric Transpose Complementary Plane Partitions

The complement of a plane partition π in a $r \times s \times t$ box is defined by the plane partition

$$\pi^c := \{(r+1-i, s+1-j, t+1-k) : (i,j,k) \notin \pi\}.$$

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Mills, Robbins and Rumsey in 1987 proved that the number of cyclically symmetric transpose-complementary plane partitions in an $2n \times 2n \times 2n$ box is

$$\prod_{i=0}^{n-1} \frac{(2i+1)\,(6i+2)!}{(6i+1)\,(2n+2i)!}.$$

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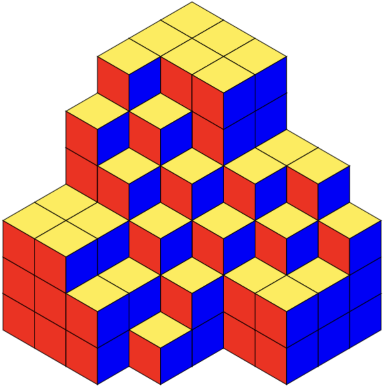

Cyclically Symmetric Transpose Complementary Rhombus Tilings of a Hexagon

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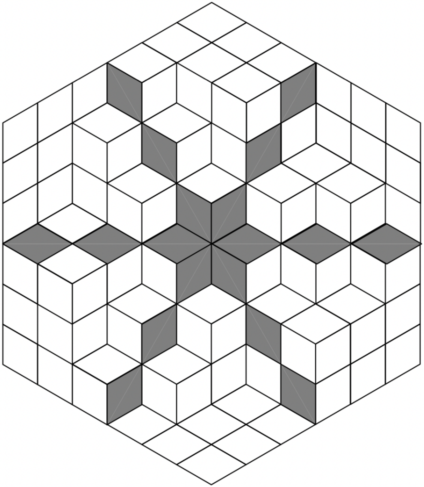

Plane Partitions to Non Intersecting Lattice Paths

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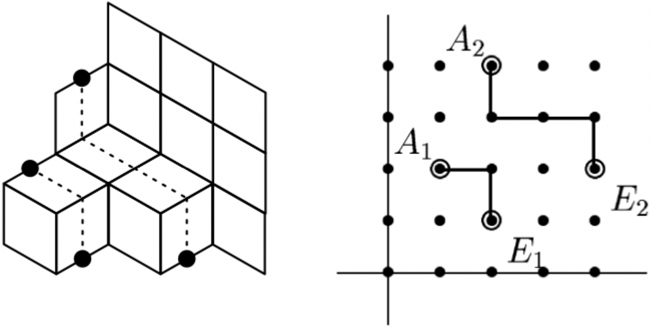

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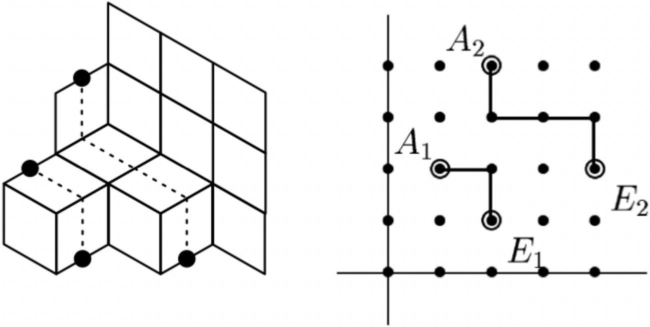

The starting points are $A_i(i, 2i)$ and the ending points are $E_j(2j, j)$.

Generating Functions

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$$Z_{2n}^{\mathrm{CSTCPP}}(x,z) = \sum_{P} x^{\text{special parts in } P} z^{\#(n)}$$

$$\widetilde{Z}_{2n}^{\mathrm{CSTCPP}}(x,z) = \sum_{P} x^{\text{quasi special parts in } P} z^{n+1+\#(n)_1-\#(n-1)_1},$$

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- $\#(i)_j$ = number of i's in row j.

Related Results

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Theorem (Behrend-S., 2022)

We have, for all $n \geq 1$

$$\widetilde{Z}_{2n}^{\mathrm{CSTCPP}}(x,z) = \det_{1 \leq i,j \leq n-1} \left(\begin{cases} \displaystyle\sum_{k=0}^{\min(2i,2j)} \binom{i}{k-i}\binom{j}{k-j} x^{2i-k}, & j \leq n-2 \\ \displaystyle\sum_{k=0}^{2i} \sum_{m=n-1}^{k} \sum_{l=n-1}^{m} \binom{2n-4-m}{k-m}\binom{i}{k-i} x^{2i-k} z^{2l-m}, & j = n-1 \end{cases} \right).$$

Related Results contd.

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- We call them quasi-VSASMs. The above result can also be written in terms of quasi-VSASMs.

Theorem (Behrend-S., 2022)

The number of quasi-VSASMs of order $2n$ *is equal to*

$$\prod_{i=0}^{n-1} \frac{(2i+1)\,(6i+2)!}{(6i+1)\,(2n+2i)!}.$$

Generalized Results

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We have, for all $n \geq 1$

$$Z_{4n+1}^{\mathrm{HVSASM}}(x,z) + z^{4n-3}\, Z_{4n+1}^{\mathrm{HVSASM}}(x,\bar{z}) \\ = (1+z)\, \widetilde{Z}_{2n}^{\mathrm{CSTCPP}}(x,z)\, Z_{2n+1}^{\mathrm{VSASM}}(x,z,1),$$

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Thank you for your attention!

Bijection between ASMs and Six Vertex Model

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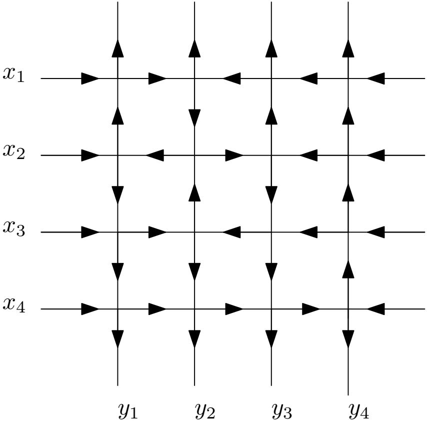
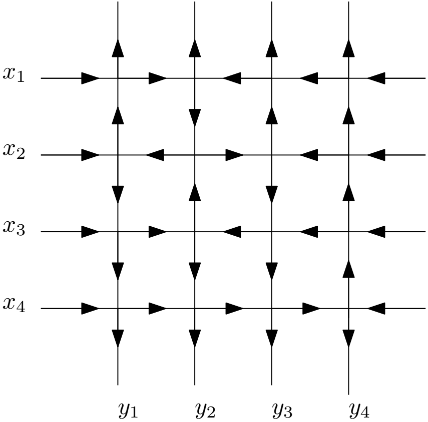


Figure: Six Vertex Model with Domain Wall Boundary Condition.

Bijection between ASMs and Six Vertex Model

A state of a corresponding six-vertex model is an orientation on the edges of this graph, such that both the in-degree and the out-degree of each vertex with degree 4 is 2.



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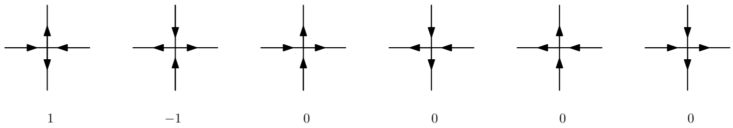

Figure: The corresponding states of the six-vertex model and the entries of an ASM.

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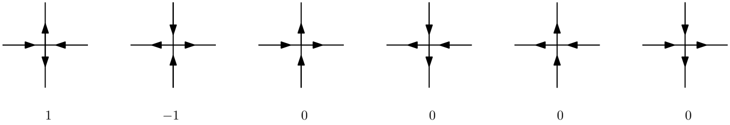

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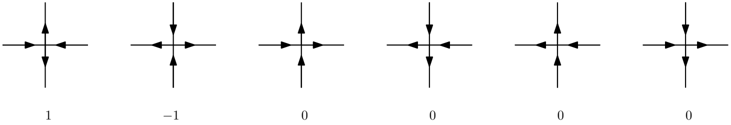

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Such a matrix will be an ASM, and we get a bijection between ASMs and states of the six-vertex model.

Example

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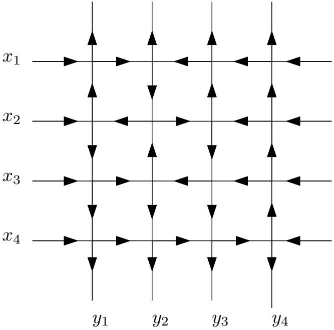

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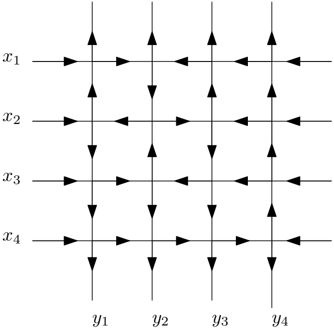

Figure: Six Vertex Model with Domain Wall Boundary Condition.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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- Specializing the parameters in Z_n, we get enumeration results.

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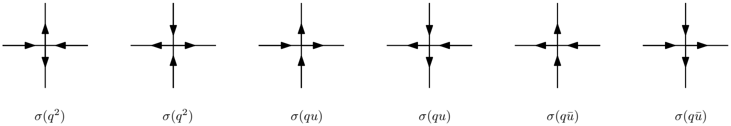

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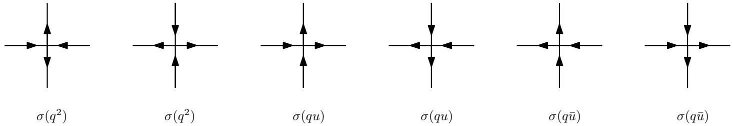

Figure: The weights of the vertices of an ASM with spectral parameter u.

We normalize further by $\sigma(q^2)$ so that we have all entries 1 and -1 to have weight 1.

How are weights assigned?

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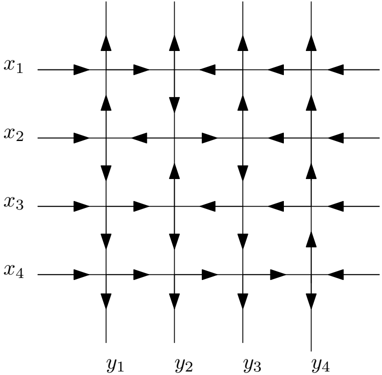

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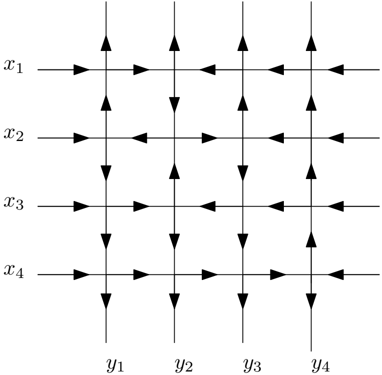

A vertex lying at the intersection of a vertical line with parameter y_j and a horizontal line with parameter x_i is assigned the label $\frac{x_i}{y_j}$.

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We will explain this connection shortly.

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$$\begin{matrix} 0 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{matrix}$$

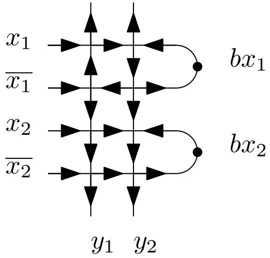

Figure: An U-turn ASM with the corresponding six-vertex state.

New weights

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This gives rise to two new type of vertices whose corresponding weights are given below.

$\sigma(bu)$ $\sigma(b\overline{u})$

Figure: Weights of the new vertices.

Partition Function

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$$Z_U(n;\mathrm{x},\mathrm{y}) = \frac{\sigma(q^2)^n \prod_i(\sigma(b\overline{y_i})\sigma(q^2x_i^2)) \prod_{i,j}(\sigma'(x_i\overline{y_j})\sigma'(x_iy_j))}{\prod_{i<j}(\sigma(\overline{x_i}x_j)\sigma(y_i\overline{y}_j)) \prod_{i\leq j}(\sigma(\overline{x_i x_j})\sigma(y_iy_j))} \times \det M_U(n;\mathrm{x},\mathrm{y}), \tag{1}$$

where $\sigma'(x) = \sigma(qx)\sigma(q\overline{x})$ and M_U is an $n \times n$ matrix defined as

$$M_U(n;\mathrm{x},\mathrm{y})_{i,j} = \frac{1}{\sigma'(x_i\overline{y}_j)} - \frac{1}{\sigma'(x_iy_j)}.$$

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- We need only the first $n+1$ columns of the VSASM to know the full matrix.
- The middle column is an alternating row with 1 and -1.
- So, n columns are sufficient to know the whole matrix.
- Moreover, the first and last rows are always the same.

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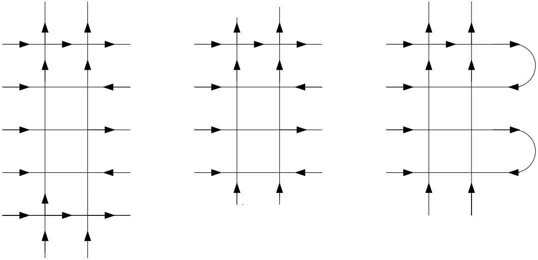

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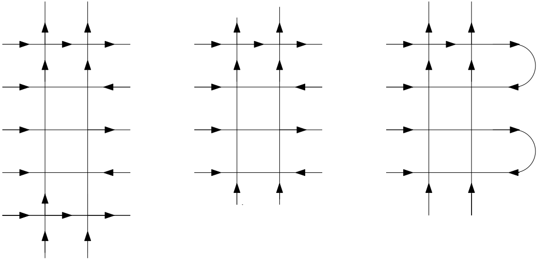

Figure: Transformation of a VSASM into an UASM.

Notice that all U-turns are downward pointing.

Thank you for your attention!