PROOF OF TWO CONJECTURED CONGRUENCES OF GOSWAMI AND JHA

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Abstract. We prove two conjectured congruences of Goswami and Jha using an algorithm of Radu (implemented by Smoot for Mathematica).

1. Introduction

Recently, Goswami and Jha [2] studied a few partition functions which were first studied by Andrews [1] and Uncu [7]. They proved several congruences for these partition functions, which we define now. Let $\mathcal{EO}(n)$ denote the integer partitions of n in which each even part is less than each odd part. Further, let $\overline{\mathcal{EO}}(n)$ denote the number of partitions of n counted by $\mathcal{EO}(n)$ in which only the largest part appears an odd number of times. Andrews [1, Corollary 3.2] proved that the generating function of $\overline{\mathcal{EO}}(n)$ is given by

$$\sum_{n\geq 0}\overline{\mathcal{EO}}(n)q^n=\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2}, \tag{1}$$

where

$$(a;q)_\infty=\prod_{i\geq 0}(1-aq^i), \quad |q|<1.$$

By looking at a different subset of $\mathcal{EO}(n)$, Uncu [7] considered partitions $\mathcal{EO}_u(n)$ whose generating function is given by

$$\sum_{n\geq 0}\mathcal{EO}_u(n)q^n=\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2}. \tag{2}$$

Goswami and Jha [2] proved several congruences for the functions $\mathcal{EO}(n)$, $\overline{\mathcal{EO}}(n)$ and $\mathcal{EO}_u(n)$ using the theory of modular forms and 2-dissection formulas from Ramanujan's notebook. For all $n\geq 1$, some sample congruences are

$$\overline{\mathcal{EO}}(4n+2)\equiv 0 \pmod 2, \tag{3}$$

and

$$\mathcal{EO}_u(4n+2)\equiv 0 \pmod 2. \tag{4}$$

At the end of the paper, they conjectured the following congruences.

Conjecture 1. *For all $n\geq 1$, the following are true*

$$\overline{\mathcal{EO}}(10n+2)\equiv\overline{\mathcal{EO}}(10n+4)\equiv 0 \pmod 2, \tag{5}$$

and

$$\mathcal{EO}_u(10n+2)\equiv\mathcal{EO}_u(10n+6)\equiv 0 \pmod 2. \tag{6}$$

We prove this conjecture here.

Theorem 1. *Conjecture 1 is true.*

2. Proof of Theorem 1

To prove Theorem 1, we will use Smoot's [6] implementation of an algorithm of Radu [4] which we describe now. Radu's algorithm can be used to prove Ramanujan type congruences of the form stated in the previous section. The algorithm takes as an input the generating function

$$\sum_{n\geq 0}a_r(n)q^n=\prod_{\delta|M}\prod_{n\geq 1}(1-q^{\delta n})^{r_\delta},$$

2020 *Mathematics Subject Classification.* 11P81, 05A17.

Key words and phrases. integer partitions, restricted integer partitions.

and positive integers m and N, with M another positive integer and $(r_\delta)_{\delta|M}$ is a sequence indexed by the positive divisors δ of M. With this input, Radu's algorithm tries to produce a set $P_{m,j}(j) \subseteq \{0,1,\dots,m-1\}$ which contains j and is uniquely defined by $m,(r_\delta)_{\delta|M}$ and j. Then, it decides if there exists a sequence $(s_\delta)_{\delta|N}$ such that

$$q^\alpha \prod_{\delta|M}\prod_{n\geq 1}(1-q^{\delta n})^{s_\delta}\cdot \prod_{j'\in P_{m,j}(j)}\sum_{n\geq 0} a(mn+j')q^n,$$

is a modular function with certain restrictions on its behaviour on the boundary of $\mathbb{H}$.

Smoot [6] implemented this algorithm in Mathematica and we will use his `RaduRK` package which requires the software packaage `4ti2`. Documentation on how to intall and use these packages are available from Smoot [6]. Now we will use this implemented `RaduRK` algorithm to prove Theorem 1.

Proof of Theorem 1. Since the proof of both equations (5) and (6) are similar, we only show the proof of (5) in details.

The generating function of $\overline{\mathcal{EO}}(n)$ given in (1) can be described by setting $M=4$ and $r=\{0,-2,3\}$. We now take $m=10$ and guess $N=10$ and take $j=2$. Then inputing into Mathematica we get the following.

`In[1] := RK[10,4,{0,-2,3},10,2]`

$$\prod_{\delta|\mathtt{M}}(\mathtt{q}^\delta;\mathtt{q}^\delta)_\infty^{\mathtt{r}_\delta} = \sum_{\mathtt{n}=0}^{\infty}\mathtt{a}(\mathtt{n})\,\mathtt{q}^{\mathtt{n}}$$

$$\boxed{\mathtt{f}_1(\mathtt{q})\cdot \prod_{\mathtt{j}'\in \mathtt{P}_{\mathtt{m},\mathtt{r}}(\mathtt{j})}\sum_{\mathtt{n}=0}^{\infty}\mathtt{a}(\mathtt{mn}+\mathtt{j}')\,\mathtt{q}^{\mathtt{n}} = \sum_{\mathtt{g}\in \mathtt{AB}}\mathtt{g}\cdot \mathtt{p}_{\mathtt{g}}(\mathtt{t})}$$

`Modular Curve:` $\mathtt{X}_0(\mathtt{N})$

`Out[2] =`

N:	10
{M,$(r_\delta)_{\delta\vert M}$}:	$\{4,\{0,-2,3\}\}$
m:	10
$P_{m,r}$(j):	$\{2,4\}$
f_1(q):	$\frac{(q;q)_\infty^6\left(q^5;q^5\right)_\infty^6}{q^3(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^{12}}$
t:	$\frac{\left(q^2;q^2\right)_\infty\left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty(q^{10};q^{10})_\infty^5}$
AB:	$\{1\}$
$\{p_g$(t): g∈AB$\}$	$\left\{4t^3+8t^2+4t\right\}$
Common Factor:	4

This gives us

$$f_1(q)\cdot\left(\sum_{n\geq 0}\overline{\mathcal{EO}}(10n+2)q^n\right)\left(\sum_{n\geq 0}\overline{\mathcal{EO}}(10n+4)q^n\right) = 4t^3+8t^2+4t,$$

where $f_1(q) = \dfrac{(q;q)_\infty^6\left(q^5;q^5\right)_\infty^6}{q^3\left(q^2;q^2\right)_\infty^2\left(q^{10};q^{10}\right)_\infty^{12}}$ and $t = \dfrac{\left(q^2;q^2\right)_\infty\left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty\left(q^{10};q^{10}\right)_\infty^5}$. It is not difficult to see that

$$\sum_{n\geq 0}\overline{\mathcal{EO}}(10n+2)q^n \not\equiv 0 \pmod 4$$

and

$$\sum_{n\geq 0} \overline{\mathcal{EO}}(10n+4)q^n \not\equiv 0 \pmod 4.$$

For instance $\overline{\mathcal{EO}}(12) = 10$ and $\overline{\mathcal{EO}}(34) = 110$. This now proves equation (5).

In a similar way we can prove equation (6) by using the following

```
RK[10,4,{0,-2,2},10,2].
```

□

3. Concluding Remarks

As we can see, we have used considerably less machinary than Goswami and Jha to prove Theorem 1, although there is a lot hidden in the implementation of `RaduRK`. We can prove several other congruences proved by Goswami and Jha in a similar manner. For instance, equation (3) can be proved by using the following

```
RK[4, 4, {0, -2, 3}, 10, 2],
```

while equation (4) can be proved by using the following

```
RK[4, 4, {0, -2, 2}, 10, 2].
```

Andrews [1, Eq. (1.6)] proved, for all $n \geq 1$ we have $\overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod 5$. This follows immediately by using

```
RK[10, 4, {0, -2, 3}, 10, 8].
```

Ray and Barman [5, Theorem 1.3] using an algorithm of Radu [3] further proved that for all $n \geq 1$, we have

$$\overline{\mathcal{EO}}(50n+18) \equiv \overline{\mathcal{EO}}(50n+28) \equiv \overline{\mathcal{EO}}(50n+38) \equiv \overline{\mathcal{EO}}(50n+48) \equiv 0 \pmod{20}.$$

This is not easily proven using `RaduRK` due to the limits on the computation time. Some missed congruences which can be found using `RaduRK` are

$$\overline{\mathcal{EO}}(6n+2) \equiv \overline{\mathcal{EO}}(6n+4) \equiv 0 \pmod 2.$$

As a consequence of Theorem 1, we now have the following congruences via Goswami and Jha [2, Theorem 5.4]

$$\overline{\mathcal{EO}}(250n+58) \equiv \overline{\mathcal{EO}}(250n+108) \equiv 0 \pmod 2,$$

and

$$\mathcal{EO}_u(250n+154) \equiv \mathcal{EO}_u(250n+54) \equiv 0 \pmod 2$$

via Goswami and Jha [2, Theorem 6.9]. It would be interesting to find independent justifications of these congruences.

Acknowledgements

The author thanks Dr. Ankush Goswami for sending his paper [2] and Dr. Hirakjyoti Das for helpful comments. The author is partially supported by the Leverhulme Trust Research Project Grant RPG-2019-083.

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