ON k-LAYERED NUMBERS

FARID JOKAR, PANKAJ JYOTI MAHANTA, AND MANJIL P. SAIKIA

Abstract. A positive integer n is said to be k-layered if its divisors can be partitioned into k sets with equal sum. In this paper, we start the systematic study of these class of numbers. In particular, we state some algorithms to find some even k-layered numbers n such that $2^\alpha n$ is a k-layered number for every positive integer α. We also find the smallest k-layered number for $2 \leq k \leq 8$. Furthermore, we study when $n!$ is a 3-layered and a 4-layered number. In addition, we address an open question concerning the relationship between k-multiperfect numbers and k-layered numbers in some special cases. Moreover, we find an upper bound for the differences of two consecutive k-layered for every positive integer $2 \leq k \leq 5$. Finally, by assuming the smallest k-layered number, we find an upper bound for the difference of two consecutive k-layered numbers.

1. Introduction

A perfect number is a positive integer that is equal to the sum of its proper positive divisors. In 2003, the idea of Zumkeller numbers, which is a generalization of perfect numbers, was first introduced by Zumkeller in the Online Encyclopedia of Integer Sequences (OEIS) [13] (A083207).

Definition 1. A positive integer n is said to be Zumkeller if the set of all positive divisors of n can be partitioned into two subsets with the same sum. A Zumkeller partition for a Zumkeller number n is a partition $\{A_1, A_2\}$ of the set of all positive divisors of n such that A_1 and A_2 sums to the same value.

Shortly thereafter, Clark et al. [11] announced several results and conjectures related to Zumkeller numbers. Peng and Bhaskara Rao [17] found some other results about Zumkeller numbers, they also studied the relationship between practical numbers (to be defined later in the paper) and Zumkeller numbers. They also settled a conjecture from [11]. Moreover, they made substantial contributions towards the second conjecture of Clark et al. [11]. Then a lull followed until recently, when Mahanta, Saikia and Yaqubi [8] revived the study of Zumkeller numbers. Recently, the first author studied a generalization of Zumkeller numbers called k-layered numbers [4] which is the principal object of study in this paper.

Definition 2. A positive integer n is said to be k-layered if the set of all positive divisors of n can be partitioned into k subsets with the same sum. A k-layered partition for a k-layered number n is a partition $\{A_1, A_2, \ldots, A_k\}$ of the set of all positive divisors of n such that for every $1 \leq i \leq k$, each of A_i sums to the same value.

It is clear from this definition that all Zumkeller numbers are 2-layered numbers. Another generalization of perfect numbers which we will consider are multiperfect numbers or k-multiperfect number and near-perfect numbers. A positive integer n is called a multiperfect number or a k-multiperfect number if n satisfies the equation $\sigma(n) = kn$, for some positive integer k. It is easy to check that every perfect number is Zumkeller; this immediately gives us a further motivation to define k-layered numbers as all multiperfect numbers will be k-layered numbers.

This paper is arranged as follows: in Sections 2 and 3, we recall and generalize some results related to k-layered numbers, taking as basis the results already proved for Zumkeller numbers. In Section 4, we find the smallest k-layered number for $2 \leq k \leq 8$. In Section 5, we state some algorithms to find some even k-layered numbers n such that $2^{\alpha}n$ is k-layered for every positive integer α. In addition, we prove that $n!$ is a 3-layered number if and only if $n \geq 5$ and $n \neq 10$. Moreover, we prove that $n!$ is a 4-layered number if and only if $n \geq 9$. In Section 6, we prove that all known 3-multiperfect numbers are 3-layered. Furthermore, by assuming Conjecture 55, we prove that all known 4-multiperfect numbers and all known 5-multiperfect numbers are 4-layered and 5-layered, respectively. In addition, we state some algorithms to find some more k-multiperfect numbers that are k-layered. In Section 7, we find some relationship between Near-perfect numbers and Zumkeller numbers. In Section 8, we find an upper bound for the differences of two consecutive k-layered number for $2 \leq k \leq 5$. Lastly, by assuming the smallest k-layered number, we find an upper bound for the differences of two consecutive k-layered numbers. There is a generous sprinkling of open questions and conjectures in this paper, which we hope subsequent mathematicians will take up.

2. Some preliminary results on k-layered numbers

Let n be a positive integer and $\sigma(n)$ denote the sum of positive divisors of n. We recall that the abundancy of n is defined to be $I(n) = \dfrac{\sigma(n)}{n}$. Also, the number n is said to be abundant, perfect, and deficient if $I(n) > 2$, $I(n) = 2$, and $I(n) < 2$, respectively. Table 1 lists the smallest k-layered number for every positive integer $2 \leq k \leq 8$ (for more details, see Section 4 of this paper).

k	Smallest k-layered number	Number of positive divisors
2	6	4
3	120	16
4	27720	96
5	147026880	896
6	130429015516800	18432
7	1970992304700453905270400	1474560
8	1897544233056092162003806758651798777216000	1245708288

TABLE 1. Smallest k-layered number for $2 \leq k \leq 8$.

In the remainder of this paper, if A is a set of all positive integers, then we define $S(A)$ to be the sum of the integers in A. We do not prove the results in this section because the proofs are exactly similar to the proofs of the corresponding results for Zumkeller numbers in [17]. The following proposition gives a necessary and sufficient condition for an integer n to be k-layered.

Proposition 1. *The number n is k-layered if and only if we can find $k-1$ disjoint subsets $A_1, A_2, \ldots, A_{k-1}$ of positive divisors of n so that for every $1 \le i \le k-1$, A_i sums to the* $\dfrac{\sigma(n)}{k}$.

Proposition 2 from [17] gives some necessary conditions for a Zumkeller number. We can generalize this proposition for k-layered number.

Proposition 2. *If n is a k-layered number, then the following is true:*

(1) $k \mid \sigma(n)$.
(2) *If k is even, then the prime factorization of n must include at least one odd prime to an odd power.*
(3) $\sigma(n) \ge kn$*; this concludes that* $I(n) \ge k$.

Proof. The proof is identical to the proof of Proposition 2 of [17]. □

Remark 3. Let m be the smallest 9-layered number. We can see that the number

$n = 43689243633548208089812102031325136553277817139006272494998568761207040000$

is the smallest number such that $I(n) \ge 9$ (A023199). Therefore, by Proposition 2, $m \ge n$.

Table 1 and Remark 3 lead us to the two following conjecture.

Conjecture 4. Size of the smallest k-layered number grows exponentially with respect to k.

We close this section with two propositions which are a generalization of Corollary 5 and Proposition 6 of [17], respectively.

Proposition 5. *If the integer n is k-layered and w is relatively prime to n, then nw is a k-layered number.*

Proof. The proof is identical to the proof of Corollary 5 of [17]. □

Proposition 6. *Let n be a k-layered number and $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization of n. Then for any positive integers $l_1, l_2, \ldots, l_m$, the number*

$$p_1^{a_1+l_1(a_1+1)} p_2^{a_2+l_2(a_2+1)} \cdots p_1^{a_m+l_m(a_m+1)}$$

is k-layered.

Proof. The proof of the above result is similar to the proof of Proposition 6 in [17]. □

3. Almost practical numbers and k-layered numbers

In this section we discuss the connection between k-layered numbers and almost practical numbers, which are a generalization of practical numbers. The results discussed here will be a generalization of the results connecting Zumkeller numbers and practical numbers that were found by Peng and Bhaskara Rao [17].

First, we recall the definition of practical numbers.

Definition 3. A positive integer n is said to be a practical number if every positive integer less than n can be represented as a sum of distinct positive divisors of n.

Now, we recall two results about practical numbers.

Proposition 7 (Corollary 1, [2])**.** *A positive integer n with the prime factorization $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ and $p_1 < p_2 < \cdots < p_m$ is a practical number if and only if $p_1 = 2$ and $p_{i+1} \leq \sigma(p_1^{k_1} \cdots p_i^{k_i}) + 1$ for $1 \leq i \leq m-1$.*

Proposition 8 (Proposition 8, [17])**.** *A positive integer n is a practical number if and only if every integer less than or equal to $\sigma(n)$ can be written as a sum of distinct divisors of n.*

We now define almost practical numbers, which are a generalization of practical numbers. These numbers were first studied by Stewart [2].

Definition 4. A positive integer n is called an almost practical number if all of the numbers j such that $2 < j < \sigma(n) - 2$, can be written as a sum of distinct divisors of n.

Remark 9. It is clear that every practical number is an almost practical number.

Now, we recall some results related to almost practical numbers from [2].

Proposition 10 (Theorem 3, [2])**.** *Let $n \neq 3$ be an odd positive integer and $1 = d_1 < d_2 < \cdots < d_k = n$ are the divisors of n. We also define $\sigma_i = d_1 + d_2 + \cdots + d_i$. Then, n is an almost practical number if and only if $d_2 = 3, d_3 = 5$ and for $i \geq 3$, at least one of the following is true:*

(1) $d_{i+1} \leq \sigma_i - 2$ *and* $d_{i+1} \neq \sigma_i - 4$,
(2) $d_{i+1} = \sigma_i - 4$ *and* $d_{i+2} = \sigma_i - 2$.

Remark 11. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ is an odd almost practical number where $p_1 < p_2 < \cdots < p_m$ are all prime factors of n, then by Proposition 10, it is clear that $p_3 = 7$.

Theorem 12 (Theorem 4, [2])**.** *Let $n \neq 3$ be an odd almost practical number and p be an odd prime, then pn is an almost practical number if and only if $2p \leq \sigma(n) - 2$ and $2p \neq \sigma(n) - 4$.*

Remark 13. By Propositions 10 and 7, if n is an odd almost practical number, then $2n$ is a practical number.

Proposition 14 (Corollary 4, [2])**.** *Let $n \neq 3$ be an odd almost practical number and p be a prime dividing n, then pn is an almost practical number.*

Example 15. By Proposition 10, one can check directly that $3^3 \times 5 \times 7$ is an almost practical number. Therefore, by Proposition 14, if $n = 3^{\alpha_1} \times 5^{\alpha_2} \times 7^{\alpha_3}$ such that $\alpha_1 \geq 3$, α_2, and α_3 are positive integers, then the number n is an almost practical number.

Example 16. Let $k \geq 4$ be a positive integer. Also, let p_i denotes ith prime number. By Bertrand postulate, for every integer $i \geq 4$, we have

$$2p_i < 4p_{i-1} \leq \sigma(p_2 p_3 \cdots p_{i-1}) - 4.$$

Thus, by Example 15 and Theorem 12, if $m = p_2^{\alpha_1} p_3^{\alpha_2} \cdots p_k^{\alpha_{k-1}}$ such that $\alpha_1 \geq 3$, $\alpha_2, \ldots, \alpha_{k-1}$ are positive integers, then m is an almost practical number.

Now, we are going to investigate the relation between almost practical numbers and Zumkeller numbers. The following proposition is a generalization of Proposition 10 of [17].

Proposition 17. *Let $n \neq 3$ be an almost practical number. Then, n is Zumkeller if and only if $\sigma(n)$ is even.*

Proof. The proof is similar to the proof of Proposition 10 of [17]. $\square$

Example 18. Let $k \geq 4$ be a positive integer, and also let p_i denotes ith prime number. Now, let $m = p_2^{\alpha_1} p_3^{\alpha_2} \cdots p_k^{\alpha_{k-1}}$ such that $\alpha_1 \geq 3$, $\alpha_2, \ldots$ and α_{k-1} are positive integers. Thus, by Example 16 and Proposition 17, m is a Zumkeller number if and only if we can find an integer $1 \leq i \leq k-1$ such that α_i is an odd number. In other words, the number m is Zumkeller if and only if m is not square.

Now, we state a proposition which is a generalization of Proposition 13 of [17].

Proposition 19. *Let α be a positive integer. Also, let n be non-k-layered number and p be a prime number with $\gcd(n, p) = 1$. If np^α is a k-layered number, then $p \leq \sigma(n)$. Moreover, if q is prime factor of k such that $q \nmid \sigma(n)$, then $q \mid \sigma(p^\alpha)$.*

Proof. The proof is identical to the proof of Proposition 13 of [17]. $\square$

In Proposition 20 of [17], the authors by a simple method proved that every odd Zumkeller number possesses at least three distinct prime factors. On the other hand, we know that for every positive integer t, the number 2^t is a deficient number. Therefore, by Propositions 2 and 19, if p is a prime number such that $\sigma(2^t) < p$, then the number $2^t p^\alpha$ fails to be a Zumkeller number for every positive integer α. Also, by Proposition 17, if $p \leq \sigma(2^t)$ and α is an odd number, then the number $2^t p^\alpha$ is Zumkeller. This was also proved in [8] (see Theorem 2.6) by a different method. Thus, we have the following corollary.

Corollary 20. *Let p_1 and p_2 be prime numbers. Also, let α_1 and α_2 be positive integers. The number $p_1^{\alpha_1} p_2^{\alpha_2}$ is Zumkeller if and only if $p_1 = 2$, $p_2 \leq \sigma(2^{\alpha_1})$ and α_2 is odd.*

Remark 21. Let p be an odd prime number. Also. let α and β be positive integers. One can check that the number $2^\alpha p^\beta$ is abundant if and only if $p \leq \sigma(2^\alpha)$ (for more details, see the proof of Theorem 2.6 of [8]). Thus, if n is an abundant number that possesses two factors, then n is Zumkeller if and only if n satisfies the conditions of Proposition 2.

4. The smallest k-layered number for $2 \leq k \leq 8$

In this short section we find the smallest k-layered numbers for $k \leq 8$. We recall that the number n is said to be superabundant if $I(n) > I(k)$ for all positive integers $k < n$. Also, we have the following lemma.

Lemma 22. *Let $m_1 < m_2$ be two consecutive superabundant numbers. For every positive integer $t < m_2$, $I(t) \leq I(m_1)$, and the equality holds when $t = m_1$.*

Proof. If $t \leq m_1$, then $I(t) \leq I(m_1)$. So, we assume $m_1 < t_1 < m_2$, such that $I(t_1) > I(m_1)$. Since m_1 and m_2 are consecutive superabundant numbers, therefore t_1 is not a superabundant number. So, there exists $t_2 < t_1$ such that $I(t_2) > I(t_1)$. Hence $I(t_2) > I(t_1) > I(m_1)$, which implies $m_1 < t_2 < t_1$. We once again conclude that there exists an integer t_3 where $m_1 < t_3 < t_2 < t_1$ such that $I(m_1) < I(t_1) < I(t_2) < I(t_3)$. Thus, for every positive integer $r \geq 3$, by this algorithm, inductively, we can find distinct positive integers $t_1, t_2, \ldots,$ and t_r such that $m_1 < t_r < t_{r-1} < \cdots < t_1 < m_2$ and

$I(m_2) > I(t_r) > I(t_{r-1}) > \cdots > I(t_1) > I(m_1)$; this contradicts the finiteness of the set $\{a : a \in \mathbb{N}, m_1 < a < m_2\}$. $\square$

A list of all superabundant numbers less than 10^{1200} can be found in [12]. Let n be a positive integer. In the remainder of this paper, we define D_n as the set all positive divisors of n. Now, we find the smallest k-layered numbers for integers $2 \leq k \leq 8$. We investigate five different cases for k.

Case 1: $k = 2$. One can check directly that 6 is the smallest Zumkeller number.

Case 2: $k = 3$. One can check directly that $n = 120$ is the first superabundant number such that $I(n) \geq 3$ (see [12]). Thus, by Lemma 22 and Proposition 2, there is no 3-layered number smaller than 120. Moreover, if we define $A_1 = \{20, \quad 40, \quad 60\}$, $A_2 = \{1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 8, \quad 10, \quad 12, \quad 15, \quad 24, \quad 30\}$, and $A_3 = \{120\}$, then $\{A_1, A_2, A_3\}$ is a 3-layered partition for n

Case 3: $k = 4$. By the same method that we used for finding the smallest 3-layered number, we can see that there is no 4-layered number smaller than $n = 27720 = 2^3 \times 3^2 \times 5 \times 7 \times 11$. We define

$$A_1 = \{ 2^3 \times 3^2 \times 5, \quad 2^3 \times 3^2 \times 5 \times 7 \times 11\},$$

$$\begin{aligned} A_2 = \{ & 2, \quad 2^3 \times 3, \quad 2^2 \times 3 \times 5, \quad 2 \times 5 \times 7, \quad 2^2 \times 3 \times 5 \times 11, \quad 2 \times 5 \times 7 \times 11, \\ & 2^2 \times 3^2 \times 5 \times 7, \quad 2 \times 3^2 \times 7 \times 11, \quad 2^2 \times 5 \times 7 \times 11, \quad 2^3 \times 3 \times 7 \times 11, \\ & 2^2 \times 3^2 \times 5 \times 11, \quad 2^2 \times 3 \times 5 \times 71, \quad 2^2 \times 3^2 \times 5 \times 7 \times 11\}, \end{aligned}$$

$$\begin{aligned} A_3 = \{ & 1, \quad 2 \times 3^2, \quad 2 \times 3 \times 5 \times 7 \times 11, \quad 2^2 \times 3^2 \times 7 \times 11, \quad 2^3 \times 5 \times 7 \times 11, \\ & 3^2 \times 5 \times 7 \times 11, \quad 2^3 \times 3^2 \times 5 \times 11, \quad 2^3 \times 3^2 \times 7 \times 11, \quad 2 \times 3^2 \times 5 \times 7 \times 11\}, \end{aligned}$$

and $A_4 = D_n \setminus (A_1 \cup A_2 \cup A_3)$, then one can check directly that $\{A_1, A_2, A_3, A_4\}$ is a 4-layered partition for n.

Case 4: $k = 5$. One can check directly that $t = 122522400 = 2^5 \times 3^2 \times 5^2 \times 7 \times 11 \times 13 \times 17$ is the first superabundant number such that $I(t) \geq 5$. But, $5 \nmid \sigma(t)$. By a computational software like Magma, it is easy to check that the number $n = 147026880 = 2^6 \times 3^3 \times 5 \times 11 \times 13 \times 17$ is the smallest positive integer larger than t such that $I(n) \geq 5$ and $5 \mid \sigma(n)$. Thus, by Lemma 22 and Proposition 2, there is no 5-layered number smaller than $n = 2^6 \times 3^3 \times 5 \times 11 \times 13 \times 17$. We define

$$\begin{aligned} A_1 = \{ & 3, \quad 2^6, \quad 2^6 \times 3, \quad 3^2 \times 5 \times 17, \quad 2^6 \times 11 \times 13, \quad 2^6 \times 3 \times 5 \times 7 \times 17, \\ & 2^6 \times 3^3 \times 11 \times 17, \quad 2^6 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 17\}, \end{aligned}$$

$$\begin{aligned} A_2 = \{ & 2^2 \times 17, \quad 2^6 \times 7 \times 13, \quad 2^2 \times 3^3 \times 5 \times 7 \times 11 \times 17, \quad 2^6 \times 3 \times 5 \times 7 \times 13 \times 17, \\ & 2^5 \times 3^3 \times 5 \times 11 \times 13 \times 17, \quad 2^5 \times 3^2 \times 5 \times 7 \times 11 \times 13 \times 17, \\ & 2^4 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 17, \quad 2^5 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 17\}, \end{aligned}$$

$$\begin{aligned} A_3 = \{ & 5, \quad 2^4, \quad 3^2 \times 5 \times 7, \quad 2^6 \times 5, \quad 2^6 \times 3^2, \quad 2^3 \times 3 \times 5 \times 7, \quad 2^6 \times 3^3, \\ & 2^4 \times 3 \times 7 \times 11 \times 17, \quad 2^6 \times 3^3 \times 5 \times 13, \quad 2^6 \times 7 \times 11 \times 13 \times 17, \\ & 2^4 \times 3^2 \times 7 \times 11 \times 13 \times 17, \quad 2^6 \times 3^2 \times 5 \times 11 \times 13 \times 17, \\ & 2^6 \times 3^3 \times 5 \times 7 \times 11 \times 17, \quad 2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13 \times 17, \\ & 2^6 \times 3^3 \times 5 \times 7 \times 13 \times 17, \quad 2^5 \times 3^3 \times 7 \times 11 \times 13 \times 17, \\ & 2^6 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17, \quad 2^3 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 17, \\ & 2^6 \times 3^3 \times 5 \times 11 \times 13 \times 17, \quad 2^6 \times 3^3 \times 7 \times 11 \times 13 \times 17\}, \end{aligned}$$

$$\begin{aligned} A_4 = \{ & 2^2,\quad 2^4\times 3\times 11,\quad 2\times 3^3\times 5\times 7\times 17,\quad 2^6\times 3^3\times 5\times 13\times 17,\\ & 2^3\times 3^3\times 7\times 11\times 13\times 17,\quad 2^6\times 3^2\times 5\times 7\times 11\times 17,\\ & 2^4\times 3\times 5\times 7\times 11\times 13\times 17,\quad 2^6\times 3^3\times 11\times 13\times 17,\quad 2^5\times 3^3\times 5\times 7\times 11\times 13,\\ & 2^6\times 3^2\times 5\times 7\times 13\times 17,\quad 2\times 3^3\times 5\times 7\times 11\times 13\times 17,\\ & 2^5\times 3^2\times 7\times 11\times 13\times 17,\quad 2^4\times 3^3\times 5\times 11\times 13\times 17,\quad 2^6\times 5\times 7\times 11\times 13\times 17,\\ & 2^5\times 3^3\times 5\times 7\times 11\times 17,\quad 2^3\times 3^2\times 5\times 7\times 11\times 13\times 17,\\ & 2^5\times 3^3\times 5\times 7\times 13\times 17,\quad 2^4\times 3^3\times 7\times 11\times 13\times 17,\\ & 2^5\times 3\times 5\times 7\times 11\times 13\times 17,\quad 2^6\times 3^3\times 5\times 7\times 11\times 13,\\ & 2^2\times 3^3\times 5\times 7\times 11\times 13\times 17,\quad 2^6\times 3^2\times 5\times 7\times 11\times 13\times 17\}, \end{aligned}$$

and $A_5 = D_n \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$, then one can check directly that $\{A_1, A_2, A_3, A_4, A_5\}$ is a 5-layered partition for n.

Case 5: $6 \le k \le 8$. By the same method that we used for finding the smallest 3-layered number and 4-layered number, we can see that there are no 6-layered number, 7-layered number, and 8-layered number smaller than $n_6 = 130429015516800 = 2^7 \times 3^3 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29, n_7 = 1970992304700453905270400 = 2^7 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53$, and $n_8 = 18975442330560921620038067586517987772160000 = 2^{10} \times 3^5 \times 5^3 \times 7^2 \times 11^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53 \times 59 \times 61 \times 67 \times 71 \times 73 \times 79 \times 83 \times 89$, respectively. In [5], we can find 6-layered partition, 7-layered partition, and 8-layered partition for numbers n_6, n_7 and n_8, respectively.

Remark 23. Let $2 \le k \le 8$ be a positive integer. If n be the smallest such that $k \mid \sigma(n)$ and $I(n) \ge k$, then n is a k-layered number.

The Remark 23 leads us to the following conjecture.

Conjecture 24. For every positive integer $k \ne 1$, if the positive integer n is the smallest number such that $k \mid \sigma(n)$ and $I(n) \ge k$, then n is the smallest k-layered number.

Remark 25. Let $k \ne 1$ be a positive integer. If the positive integer n is a k-layered number, then by Proposition 2, $k \mid \sigma(n)$ and $I(n) \ge k$. But, the converse is not true. Let n be an abundant number or a perfect number, then by Proposition 2, the number n^2 is a non-Zumkeller number. Let p be a prime number such that $\sigma(n^2) < p$. By Proposition 19, the number $m = n^2 p$ fails to be Zumkeller. But, $2 \mid \sigma(m)$ and $I(m) \ge 2$. For instance, take $n = 6$ and $k = 2$. Then $m = 6^2 \times 97$ is not Zumkeller but $2 \mid \sigma(m)$ and $I(m) \ge 2$. One can check that the number m is the smallest non-Zumkeller number which satisfies conditions of Proposition 2 for $k = 2$; we omit the details here.

5. Some more results about k-layered numbers

Mahanta, Saikia and Yaqubi [8], proved that if the number n is Zumkeller, then the number $2^\alpha n$ is Zumkeller for every positive integer α. This does not generalize to all k-layered numbers. For instance, the number $m = 2^5 \times 3 \times 7$ is a 3-layered number because if we define $A_1 = \{2^5 \times 3 \times 7\}$, $A_2 = \{2^4 \times 3 \times 7, 2^3 \times 3 \times 7, 2^2 \times 3 \times 7, 2 \times 3 \times 7, 3 \times 7, 2 \times 7, 2 \times 3, 1\}$, then one can check directly that $A_1 = A_2 = \dfrac{\sigma(m)}{3}$. This concludes that the number m is 3-layered. However, the number $2m$ fails to be 3-layered since $\sigma(2m)$ is not divisible by 3. By Proposition 5, we know that if n is an odd k-layered number, then for positive integer α the number $2^\alpha n$ is a k-layered number. This leads us to the following definition.

Definition 5. The even k-layered number n is said to be a good k-layered number if the number $2^\alpha n$ is k-layered for every positive integer α.

The following proposition states a necessary condition for the k-layered number n to be a good k-layered.

Proposition 26. *Let α be a positive integer, and let m be an odd positive integer. If $n = 2^\alpha m$ is a good k-layered number, then $k \mid \sigma(m)$.*

Proof. Let n be a good k-layered number such that $k \nmid \sigma(m)$. Then there exists at least one factor p^γ of k such that $p^\gamma \nmid \sigma(m)$ and $p^\gamma \mid \sigma(2^\alpha)$, where p is a prime number and γ is a positive integer. If $p = 2$, then this is a contradiction. If p is odd, then this implies that $p^\gamma \nmid 2^{\alpha+1}$. This concludes that $p^\gamma \nmid \sigma(2^{\alpha+1})$. Thus, $p^\gamma \nmid \sigma(2n)$, which is a contradiction. □

In the following, we state two crucial theorems that help us to find a subset of good k-layered numbers. For better understanding, first, we state a proposition, which is a special case of one of these two theorems.

Proposition 27. *Let n be an odd positive integer and α be a positive integer. Let $m = 2^\alpha n$ be a 3-layered number with 3-layered partition $\{A_1, A_2, A_3\}$. Now, let A_1' be a subset of the set of all positive divisors of n such that A_1' sums to $\dfrac{2\sigma(n)}{3}$. If $A' = \{2^\alpha d : d \in A_1'\}$ and $A' \subset A_1 \cup A_2$, then m is a good 3-layered number.*

Proof. We define $M_1 = A' \cap A_1$ and $M_2 = A' \cap A_2$. Let D be the set of all positive divisors of n. Now, we define

$$M_1' = \{2d : d \in M_1\}, M_2' = \{2d : d \in M_2\}, M_3 = \{2^{\alpha+1} d : d \in (D \setminus A_1')\}.$$

We can see that

$$S((A_1 \setminus M_1) \cup M_1') = S(A_1 \cup M_1).$$

Thus, we see

$$\begin{aligned} S(A_1) + \frac{2^{\alpha+1}\sigma(n)}{3} &= S((A_1 \setminus M_1) \cup M_1' \cup M_2) = S(A_1 \cup M_1 \cup M_2) = S(A_2 \cup M_2 \cup M_1) \\ &= S((A_2 \setminus M_2) \cup M_2' \cup M_1) = S(A_2) + \frac{2^{\alpha+1}\sigma(n)}{3}. \end{aligned}$$

Therefore, by definition of A', one can check directly that

$$\{(A_1 \setminus M_1) \cup M_1' \cup M_2, (A_2 \setminus M_2) \cup M_2' \cup M_1, A_3 \cup M_3\}$$

is a 3-layered partition for $2m$. Thus, by the same method, inductively, we can prove that $\ell = 2^t n$ is a 3-layered number for every integer $t \geq \alpha$. □

By Proposition 27, we have the following corollary.

Corollary 28. *If $n \neq 3$ is an odd almost practical number such that $6 \mid \sigma(n)$, then the number $2n$ is a good 3-layered number.*

Proof. First, we prove that $2n$ is a 3-layered number. Let A_1 is the set of all positive divisors of n. By Proposition 17, n is a Zumkeller number; this concludes that A_1 can be partitioned into two subsets B_1 and B_2 such that each of them sums to $\dfrac{\sigma(n)}{2}$. Now, we define $A_2 = \{2d : d \in B_1\}$ and $A_3 = \{2d : d \in B_2\}$. We know that n is an odd number.

Therefore, for every integer $a \in A_2 \cup A_3$, $a \notin A_1$. Thus, $\{A_1, A_2, A_3\}$ is a 3-layered partition for $2n$. Also, we know that n is an almost practical number such that $6 \mid \sigma(n)$ and $2 \neq \dfrac{2\sigma(n)}{3} \neq \sigma(n) - 2$; this concludes that there exists $A' \subset A_1$ so that A' sums to $\dfrac{2\sigma(n)}{3}$. On the other hand , let $A'' = \{2d : d \in A'\}$. We know that for every integer $d \in A_1$, $2d \in A_2 \cup A_3$. This concludes that $A'' \subset A_2 \cup A_3$. Then, by Proposition 27, $2n$ is a good 3-layered number. $\square$

Now, we present a proposition generalizing Proposition 27 for 4-layered numbers.

Proposition 29. *Let n be an odd Zumkeller number with Zumkeller partition $\{A'_1, A'_2\}$. For positive integer α, we define $A''_1 = \{2^\alpha d : d \in A'_1\}$ and $A''_2 = \{2^\alpha d : d \in A'_2\}$. Now, let $m = 2^\alpha n$ be a 4-layered number with 4-layered partition $\{A_1, A_2, A_3, A_4\}$. If $A''_1 \subset (A_1 \cup A_2)$ and $A''_2 \subset (A_3 \cup A_4)$, then the number m is good 4-layered.*

Proof. We define

$$M_1 = A''_1 \cap A_1, M_2 = A''_1 \cap A_2$$

$$M_3 = A''_2 \cap A_3, M_4 = A''_2 \cap A_4$$

Now, for every integer $1 \leq i \leq 4$, we define $M'_i = \{2d : d \in M_i\}$. One can check directly that

$$\{(A_1 \setminus M_1) \cup M'_1 \cup M_2, (A_2 \setminus M_2) \cup M'_2 \cup M_1, (A_3 \setminus M_3) \cup M'_3 \cup M_4, (A_4 \setminus M_4) \cup M'_4 \cup M_3)\}$$

is a 4-layered partition for m. Thus, by applying the same method like before, inductively, we can prove that $\ell = 2^t n$ is a 4-layered number for every integer $t \geq \alpha$. $\square$

The following two results are a generalization of Propositions 27 and 29, respectively.

Theorem 30. *Let $k \geq 3$ and n be two odd positive integers such that $k \mid \sigma(n)$. Let $A'_1, A'_2, \ldots, A'_{\frac{k-1}{2}}$ be disjoint subsets the set of all positive divisors of n so that A'_i sums to $\dfrac{2\sigma(n)}{k}$ for every integer $1 \leq i \leq \dfrac{k-1}{2}$. Let α be a positive integer and $A''_i = \{2^\alpha d : d \in A'_i\}$ for every integer $1 \leq i \leq \dfrac{k-1}{2}$. Now, let $m = 2^\alpha n$ be a k-layered number with k-layered partition $\{A_1, A_2, \ldots, A_k\}$ such that $A''_i \subset A_{2i-1} \cup A_{2i}$ for every integer $1 \leq i \leq \dfrac{k-1}{2}$. Then, m is a good k-layered number.*

Proof. Let D be the set of all divisors of n. For integers $1 \leq i \leq k-1$, we define

$$M_i = A''_{\lfloor \frac{i+1}{2} \rfloor} \cap A_i, M'_i = \{2d : d \in M_i\},$$

and

$$M = \{2^{\alpha+1} d : d \in (D \setminus A'_1 \cup A'_2 \cup \ldots, A'_{\frac{K-1}{2}})\}.$$

Now, we define

$$B_i = \begin{cases} (A_i \setminus M_i) \cup M_i' \cup M_{i+1}, & \text{if } i \text{ is odd;} \\ (A_i \setminus M_i) \cup M_i' \cup M_{i-1}, & \text{if } i \text{ is even.} \end{cases}$$

One can check directly that $\{B_1, B_2, \ldots, B_{k-1}, A_k \cup M\}$ is a k-layered partition for $2^{\alpha+1}n$. Also, by applying the previous method, inductively, we can prove that $\ell = 2^t n$ is k-layered for every integer $t \geq \alpha$. $\square$

Theorem 31. *Let $k \geq 4$ be an even positive integer. Let the odd positive integer n be a $\frac{k}{2}$-layered number with $\frac{k}{2}$-layered partition $\{A_1', A_2', \ldots, A_{\frac{k}{2}}'\}$. Let α be a positive integer and $A_i'' = \{2^\alpha d : d \in A_i'\}$ for every integer $1 \leq i \leq \frac{k}{2}$. Also, let $m = 2^\alpha n$ be a k-layered number with k-layered partition $\{A_1, A_2, \ldots, A_k\}$ so that $A_i'' \subset A_{2i-1} \cup A_{2i}$ for every integer $1 \leq i \leq \frac{k}{2}$. Then, the number m is a good k-layered number.*

Proof. For integers $1 \leq i \leq k$, we define

$$M_i = A''_{\lfloor \frac{i+1}{2} \rfloor} \cap A_i, M_i' = \{2d : d \in M_i\}.$$

Now, for every $1 \leq i \leq k$, we define

$$B_i = \begin{cases} (A_i \setminus M_i) \cup M_i' \cup M_{i+1}, & \text{if } i \text{ is odd;} \\ (A_i \setminus M_i) \cup M_i' \cup M_{i-1}, & \text{if } i \text{ is even.} \end{cases}$$

It is easy to check that the set $\{B_1, B_2, \ldots, B_k\}$ is k-layered partition for $2^{\alpha+1}n$. Also, by applying the same method, inductively, we can prove the number $2^t n$ is k-layered for every $t \geq \alpha$. $\square$

As a consequence of Theorems 30 and 31, we have the following corollary.

Corollary 32. *For integers $2 \leq k \leq 5$, the smallest k-layered number is a good k-layered number.*

Proof. We investigate the five cases for k.

Case 1: $k = 2$. This was already proved in Proposition 4.12 of [8].

Case 2: $k = 3$. Let $\{A_1, A_2, A_3\}$ be the 3-layered partition for the smallest 3-layered number, 120, which was defined in Section 4. Now, let $n = 120 = 2^3 m$. We know that $\frac{2\sigma(m)}{3} = 16$. If we define $A_1' = \{1, 15\}$, then it is obvious that $A' = \{2^3, 2^3 \times 15\} \subset A_2 \cup A_3$. Thus, by Proposition 27, n is a good 3-layered number.

Case 3: $k = 4$. Let $\{A_1, A_2, A_3, A_4\}$ be the 4-layered partition for the smallest 4-layered $n = 27720$, which was defined in Section 4. Now, let D' be the set of all divisors of $m = 3465 = 3^2 \times 5 \times 7 \times 11$. We define

$A_1' = \{\ 3,\quad 3^2 \times 5,\quad 3 \times 7 \times 11,\quad 3^2 \times 5 \times 7 \times 11\}$,

and $A_2' = D' \setminus A_1'$.

One can check directly that for every integer $1 \leq i \leq 2$, $S(A_i') = \frac{\sigma(m)}{2}$. Therefore, $\{A_1', A_2'\}$ is a Zumkeller partition for m. We know that $n = 2^3 m$. Also, if we define

$A_1'' = \{2^3 d : d \in A_1'\}$ and $A_2'' = \{2^3 d : d \in A_2'\}$, then one can check directly that $A_1'' \subset A_1 \cup A_2$ and $A_2'' \subset A_3 \cup A_4$. Thus, by Proposition 29, n is a good 4-layered number.

Case 4: $k = 5$. Let $\{A_1, A_2, A_3, A_4, A_5\}$ be the 5-layered partition for the smallest 5-layered number $n = 147026880$, which was defined in Section 4. Let $n = 2^6 m$. We define

$$\begin{aligned} A_1' = \{ & 1, \quad 3, \quad 11 \times 13, \quad 3 \times 5 \times 7 \times 17, \quad 3 \times 5 \times 7 \times 13 \times 17, \\ & 2^6 \times 3^3 \times 5 \times 7 \times 11 \times 13 \times 17\}, \end{aligned}$$

and

$$\begin{aligned} A_2' = \{ & 5, \quad 3^2, \quad 3^3, \quad 3^3 \times 5 \times 13, \quad 7 \times 11 \times 13 \times 17, \quad 3^2 \times 5 \times 11 \times 13 \times 17, \\ & 3^3 \times 5 \times 7 \times 11 \times 17, \quad 3 \times 5 \times 7 \times 11 \times 13 \times 17, \quad 3^3 \times 5 \times 11 \times 13 \times 17, \\ & 3^3 \times 7 \times 11 \times 13 \times 17\}. \end{aligned}$$

One can check directly that $S(A_1') = S(A_2') = \dfrac{2\sigma(m)}{5}$. We define $A_i'' = \{2^6 d : d \in A_i'\}$ for $i = 1, 2$. We know $A_1' \subset (A_1 \cup A_2)$ and $A_2' \subset (A_3 \cup A_4')$. Thus, by Theorem 30, n is a good 5-layered. ☐

Corollary 32, leads us naturally to the following open question.

Open question 33. Is the smallest k-layered number a good k-layered number for every positive integer $k \neq 1$?

It appears that the methods used in this paper are not feasible to attack this problem at the moment.

Now, we state a proposition that can be used to construct some k-layered numbers.

Proposition 34. *Let $k, k' \neq 1$ be positive integers. Now, let m be a k-layered number and n be a k'-layered number. If $\gcd(m, n) = 1$, then mn is a kk'-layered number.*

Proof. Let $\{A_1, A_2, \ldots, A_k\}$ be a k-layered partition for m, and $\{B_1, B_2, \ldots, B_{k'}\}$ be a k'-layered partition for n. One can check directly that $\{A_i B_j : 1 \leq i \leq k, 1 \leq j \leq k'\}$ is a kk'-layered partition for mn. ☐

It is clear that Proposition 34 can be generalized.

Corollary 35. *Let $k_1, k_2, \ldots, k_r \neq 1$ be positive integers such that for every integer $1 \leq i \leq r$, m_i is a k_i-layered number. If for every distinct integers $1 \leq i \neq j \leq r$, $\gcd(m_i, m_j) = 1$, then $m_1 m_2 \ldots m_r$ is a $k_1 k_2 \ldots k_r$-layered number.*

The following example shows the power of Proposition 34 for finding a subset of the set of 4-layered numbers.

Example 36. By Example 18, $n_1 = 3^{\alpha_1} \times 5^{\alpha_2} \times 7^{\alpha_3}$, in which $\alpha_1 \geq 3$, α_2, α_3 are positive integers, and at least one of the exponents of its factors is odd, is a Zumkeller number. Let t be a positive integer and p be a prime number such that $p \leq 2^{t+1} - 1$ and $\gcd(p, n_1) = 1$. By Corollary 20, for every odd number α_4, the number $n_2 = 2^t \times p^{\alpha_4}$ is Zumkeller. Therefore, by Proposition 34, $n = n_1 n_2$ is a 4-layered number.

In [17], the authors proved that the number $n!$ is Zumkeller for integers $n \geq 3$ (we can also prove it in a slightly different way, see Remark 45). In the following, we prove that the number $n!$ is 3-layered for every integer $n \geq 5$ and $n \neq 10$, and 4-layered for every integer $n \geq 9$. Before that, we recall a theorem which was proved by Breusch [10]; this theorem is a generalization of Bertrand's postulate .

Theorem 37 (Breusch [10])**.** *For every integer $n \geq 7$, there are primes of the form $3k+1$ and $3k+2$ between n and $2n$.*

Theorem 38. *Let $n \geq 11$ be an integer. Then, the number $n!$ possesses prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ such that $2 = p_1 < p_2 < \cdots < p_k$ and $\alpha_{k-1} = \alpha_k = 1$. In addition, $p_k \leq 2^{\alpha_1}$ and there exists a prime number q such that $q \mid\mid n!$ and $q \equiv 2 \pmod 3$.*

Proof. If $11 \leq n \leq 13$, then one can check directly that $n!$ satisfies the theorem. Now, Let $n \geq 14$. Also, let p' be a prime number. We know that $2p' \leq n$ if and only if $p'^2 \mid n!$. Thus, if p is the largest prime number such that $p^2 \mid n!$, then by definition of n, it is clear that $p \geq 7$. Therefore, by Theorem 37, there exist at least two distinct prime numbers q_1 and q_2 such that $p < q_1, q_2 < 2p$ and for at least one $i \in \{1, 2\}$, $q_i \equiv 2 \pmod 3$. Also, by definition of p, $q_1 \mid\mid n!$ and $q_2 \mid\mid n!$. Furthermore, if $\nu_2(n!)$ denotes the exponent of the largest power of 2 that divides $n!$, then by Legendre's formula, we have

$$p_k \leq n < 2^{\lfloor \frac{n}{2} \rfloor} < 2^{\nu_2(n!)}.$$

□

Now, we state and prove the key theorems of this section.

Theorem 39. *The number $n!$ is a 3-layered number if and only if $n \geq 5$ and $n \neq 10$.*

Proof. On can check directly that for every integer $1 \leq n \leq 4$, $I(n!) < 3$, and $3 \nmid \sigma(10!)$. So, by Proposition 2, $n!$ fails to be 3-layered for every integer $1 \leq n \leq 4$ and $n = 10$.

Now, we investigate the five cases for n:

Case 1: $n = 5$. By Corollary 32, $n! = 120$ is a 3-layered number.

Case 2: $n = 6$. Let $A_1 = \{2 \times 3, 2^4 \times 5, 2^4 \times 3^2 \times 5\}$ and $A_2 = \{2^3, 2^3 \times 3, 2^4 \times 3^2, 2^3 \times 3 \times 5, 2^4 \times 3 \times 5, 2 \times 3^2 \times 5, 2^2 \times 3^2 \times 5\}$. One can check directly that $S(A_1) = S(A_2) = \dfrac{\sigma(n!)}{3}$. Thus, by Proposition 1, the number $n! = 2^4 \times 3^2 \times 5$ is a 3-layered number.

Case 3: $n = 7$. We proved that the number $6! = 2^4 \times 3^2 \times 5$ is a 3-layered number. Thus, by Proposition 5, the number $7! = 2^4 \times 3^2 \times 5 \times 7$ is a 3-layered number.

Case 4: $n = 8$. We define

$A_1 = \{\, 2^3 \times 3 \times 5, \quad 2^3 \times 3^2 \times 5 \times 7, \quad 2^5 \times 3^2 \times 5 \times 7, \quad 2^7 \times 3^2 \times 5 \times 7\}$

$A_2 = \{\, 2^3, \quad 2^3 \times 3 \times 7, \quad 2^7 \times 5 \times 7, \quad 2^6 \times 3 \times 5 \times 7, \quad 2^7 \times 3^2 \times 7, \quad 2^7 \times 3 \times 5 \times 7, 2^6 \times 3^2 \times 5 \times 7\}$

One can check directly that $S(A_1) = S(A_2) = \dfrac{\sigma(n!)}{3}$. Thus, by Proposition 1, the number $n! = 2^7 \times 3^2 \times 5 \times 7$ is a 3-layered number.

Case 5: $n \geq 9$. Let $n = 9$. Then, $n! = 2^7 m$, where $m = 3^4 \times 5 \times 7$. By Example 15, m is an odd almost practical number such that $6 \mid \sigma(m)$. Therefore, by Corollary 28, $n!$ is a 3-layered number. Now, let $n > 9$ and $n! = 2^{\alpha_1} m'$ with $\gcd(2^{\alpha_1}, m') = 1$ such that m' is an odd almost practical number. Also, let $p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ be the prime factorization of m'. We want to prove that $(n+1)! = 2^{\beta_1} m''$ with $\gcd(2^{\beta_1}, m'') = 1$ such that m'' is an odd almost practical number. First, let $n+1$ be a composite number. Then, if p is a prime number such that $p \mid n+1$, then there exists a positive integer $d \neq 1$ such that $n+1 = pd$. This concludes that $p < n$. Therefore, $p \mid n!$. Thus, $(n+1)! = 2^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where $\beta_i \geq \alpha_i$ for every integer $1 \leq i \leq k$. Therefore, by Proposition 14, $m'' = p_2^{\beta_2} \cdots p_k^{\beta_k}$ is an odd almost practical number. Moreover, if $n+1 = q$ is a prime number, then by Bertrand postulate, we have

$$2q < 4p_k < \sigma(p_2 p_3 \cdots p_k) - 4.$$

By Theorem 12, we once again conclude that m'' is an odd almost practical number. Thus, for every integer $n \geq 9$, $n! = 2^{\alpha_1} t$, where t is an odd almost practical number. Also, by Theorem 38, there exists a prime number q such that $q \mid\mid n!$ and $q \equiv 2 \pmod 3$ for all $n \geq 11$. Therefore, there exists a positive integer t' such that $t = qt'$ and $\gcd(q, t') = 1$; this concludes that $6 \mid \sigma(q)\sigma(t') = \sigma(t)$. Thus, by Corollary 28, for every integer $n \geq 9$ except $n = 10$, n! is a 3-layered number. $\square$

Remark 40. In fact, in case 5, by Corollary 28, we proved that the number $n!$ is a good 3-layered number for $n \geq 11$. One can check that $n!$ is a good 3-layered number for $5 \leq n \leq 9$, we omit the details here. For instance, by Corollary 32 the number 5! is a good 3-layered number. Thus, we can say that the number $n!$ is 3-layered if and only if $n!$ is a good 3-layered number.

Remark 41. Let $n \geq 9$ and $n \neq 10$. Using the same arguments of Case 5 above, we get the number $\dfrac{n!}{2^t}$ is a 3-layered number for every integer $0 \leq t \leq \nu_2(n!) - 1$. Thus, by the previous remark, if $r = \nu_2(n!) - 1$, then the number $\dfrac{n!}{2^r}$ is a good 3-layered number for every integer $n \geq 9$ and $n \neq 10$.

Theorem 42. *The number $n!$ is 4-layered number if and only if $n \geq 9$.*

Proof. On can check directly that for every integer $1 \leq n \leq 8$, $I(n!) < 4$. Then, by Proposition 2, $n!$ fails to be 4-layered for every integer $1 \leq n \leq 8$. Now, we want to prove that for every integer $n \geq 9$ the number $n!$ is 4-layered. We investigate the three cases for n:

Case 1: $n = 9$. We define

$A_1 = \{\, 2^2 \times 3,\ \ 2^3 \times 3^4,\ \ 2^6 \times 3 \times 5 \times 7,\ \ 2^7 \times 3^4 \times 5 \times 7\}$

$A_2 = \{\, 2^2,\ \ 2^7 \times 7,\ \ 2^4 \times 3^4 \times 5,\ \ 2^6 \times 3^3 \times 5 \times 7,\ \ 2^7 \times 3^3 \times 5 \times 7,\ \ 2^6 \times 3^4 \times 5 \times 7\}$

$A_3 = \{\, 2^2 \times 3^2,\ \ 2^6 \times 3^2 \times 5,\ \ 2^5 \times 3^3 \times 5 \times 7,\ \ 2^6 \times 3^4 \times 7,\ \ 2^7 \times 3^2 \times 5 \times 7,\ \ 2^4 \times 3^4 \times 5 \times 7,\ \ 2^7 \times 3^4 \times 5,\ \ 2^7 \times 3^4 \times 7,\ \ 2^5 \times 3^4 \times 5 \times 7\}$

One can check directly that for every integer $1 \leq i \leq 3$, $S(A_i) = \dfrac{\sigma(n!)}{4}$. Thus, by Proposition 1, the number $n! = 362880 = 2^7 \times 3^4 \times 5 \times 7$ is a 4-layered number.

Case 2: $n = 10$. We define

$A_1 = \{2,\ 2^5 \times 3,\ \ 2^4 \times 3^3 \times 7,\ \ 2^7 \times 3^2 \times 5^2 \times 7,\ \ 2^8 \times 3^4 \times 5^2 \times 7\}$

$A_2 = \{1,\ \ 2^4,\ \ 3^2 \times 5^2,\ \ 2^6 \times 3^2 \times 5,\ \ 2^8 \times 3^2 \times 5 \times 7,\ \ 2^8 \times 3^4 \times 5 \times 7,\ \ 2^8 \times 3^3 \times 5^2 \times 7,\ \ 2^7 \times 3^4 \times 5^2 \times 7\}$

$A_3 = \{2 \times 3^2,\ \ 2^5 \times 7,\ \ 2^5 \times 3^3 \times 5^2,\ \ 2^7 \times 3^4 \times 5^2,\ 2^6 \times 3^3 \times 5^2 \times 7,\ \ 2^7 \times 3^4 \times 5 \times 7,\ \ 2^8 \times 3^2 \times 5^2 \times 7,\ \ 2^5 \times 3^4 \times 5^2 \times 7,\ \ 2^8 \times 3^4 \times 5^2,\ \ 2^7 \times 3^3 \times 5^2 \times 7,\ \ 2^6 \times 3^4 \times 5^2 \times 7\}$.

One can check directly that for every integer $1 \leq i \leq 3$, $S(A_i) = \dfrac{\sigma(n!)}{4}$. Thus, by Proposition 1, the number $n! = 3628800 = 2^8 \times 3^4 \times 5^2 \times 7$ is a 4-layered number.

Case 3: $n \geq 11$. Let $n \geq 11$ and $2^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k}$ be a prime factorization of $n!$. Now, let $t = (p_k - 1)!$. By the definition of n, $(p_k - 1) \geq 10$. Thus, there exists an odd almost practical number m' such that $t = 2^{\alpha}m'$ (see the proof of Corollary 39 for Case 5). Also, we know that $m' = p_2^{\beta_2}p_3^{\beta_3}\cdots p_{k-1}^{\beta_{k-1}}$, where $1 \leq \beta_i \leq \alpha_i$ for every integer $1 \leq i \leq k-1$. Therefore, by Proposition 14, the number $m = p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}}$ is an odd almost practical number. Also, by Theorem 38, $\alpha_{k-1} = 1$; this concludes that $2 \mid \sigma(m')$. Then, by Proposition 17, m is an odd Zumkeller number. Moreover, by Theorem 38, $p_k < \sigma(2^{\alpha_1})$. Therefore, by Corollary 20, the number $2^{\alpha_1}p_k$ is a Zumkeller number. Thus, by Proposition 34, $n! = 2^{\alpha}p_k m$ is a 4-layered number. □

Remark 43. In Case 3 above, we proved that for every integer $n \geq 11$ there exist Zumkeller numbers m_1 and m_2 such that $\gcd(m_1, m_2) = 1$ and $n! = m_1m_2$. Without loss of generality, we let m_2 be odd. By Theorem 4.12 of [8], the number $2^{\ell}m_1$ is a Zumkeller number for every positive integer ℓ. Thus, by Proposition 5, the number $n!$ is a good 4-layered number for $n \geq 11$. Also, by Proposition 52, the number 9! is a good 4-layered number. In addition, one can check directly that the numbers 10! is good 4-layered; we omit the details here. Therefore, we can say that the number $n!$ is 4-layered if and only if $n!$ is a good 4-layered number.

Remark 44. By Theorem 38 and Corollary 20, $2^{\lfloor \frac{n}{2} \rfloor}p_k$ is also a Zumkeller number. Thus, using the same arguments of Case 3 of the above proof, we get $\dfrac{n!}{2^t}$ is a 4-layered number for $n \geq 11$, where $1 \leq t \leq \sum_{i=2}^{\infty}\left\lfloor \dfrac{n}{2^i} \right\rfloor$. Therefore, by the previous remark, if $r = \sum_{i=2}^{\infty}\left\lfloor \dfrac{n}{2^i} \right\rfloor$, then the number $\dfrac{n!}{2^r}$ is a good 4-layered number for $n \geq 11$.

Remark 45. By Bertrand postulate and Proposition 7, every primorial number [1] $n \neq 2$ is a practical number. Also, by Proposition 7, if n is a practical number and $d \mid n$, then nd is also a practical number. Then, By Proposition 17, if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{\ell}^{\alpha_{\ell}}$, where p_i be the ith prime, and $\alpha_1, \alpha_2, \ldots, \alpha_{\ell}$ are positive integers for every positive $1 \leq i \leq \ell$, then the number n is a Zumkeller number if and only if $2 \mid \sigma(n)$. In addition, by Bertrand's postulate and the same method that we used to prove Theorem 38, we can prove that for integers $n \geq 3$, there exists a prime number p such that $p \mid\mid n!$. Thus, $2 \mid \sigma(n!)$ for integers $n \geq 3$. Therefore, by Proposition 17, $n!$ is a Zumkeller number for integers $n \geq 3$.

Remark 46. Let n be a practical number. One can check directly that if $n \leq 10^7$, $k \mid \sigma(n)$, and $I(n) \geq k$, then n is k-layered.

Proposition 7, Theorems 39 and 42, Remarks 40, 41, 43, 44, and 46 leads us to the following conjecture.

Conjecture 47. Let n be a practical number. If $k \mid \sigma(n)$ and $I(n) \geq k$, then n is k-layered.

Now, we state a well-known proposition which recalls some elementary properties of the abundancy function I.

Proposition 48. *The abundancy function I possesses following properties.*

[1]For the nth prime number p_n, the primorial $p_n\#$ is defined as the product of the first n primes.

(i) *Let p and q be primes such that $p < q$. Let α and β be positive integers such that $\beta \le \alpha$. Then, $I(q^\beta) < I(p^\alpha)$.*

(ii) *Let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of the positive integer n. Then*

$$I(n) = \prod_{i=1}^{k} I(p_i^{\alpha_i}) < \prod_{i=1}^{k} \frac{p_i}{p_i - 1}.$$

By the prime number theorem for arithmetic progressions, we can state the following theorem, which is a generalization of Theorem 37.

Theorem 49. *Let m be a positive integer and p be a prime number. There exists a positive integer s such that for integers $n \ge s$, there exist distinct primes $p_1, p_2, \ldots, p_m$ between n and $2n$, where $p_i \equiv p - 1 \pmod{p}$ for every integer $1 \le i \le m$.*

By using the previous theorem and the same method that we used in the proof of Theorem 38, we can conclude the following theorem.

Theorem 50. *Let m be a positive integer, p be a prime number, and the positive integer n be large enough. Then there exist distinct primes $p_1, p_2, \ldots, p_m$ such that $p_i \equiv p - 1 \pmod{p}$ and $p_i \,||\, n!$ for integers $1 \le i \le m$.*

Thus, by the previous theorem and some elementary properties of abundancy function, for every positive integer k, there exists a positive integer s such $k \mid \sigma(n!)$ and $I(n!) \ge k$ for every integer $n \ge s$. Then we have the following corollary.

Corollary 51. *Let $k \ge 2$ be a positive integer. If Conjecture 47 holds, then there exists a positive integer s such that for integers $n \ge s$, the number $n!$ is a good k-layered number.*

We close this section by the following theorem, which find all 4-layered numbers of the form $2^\alpha p^3 qr$, where p, q, and r are primes.

Theorem 52. *Let α be a positive integer. Let p, q, and r be primes. The number $2^\alpha p^3 qr$ is 4-layered if and only if $\alpha \ge 5$, $p = 3$, $q = 5$, and $r = 7$.*

Proof. By property (ii) of the abunancy function I in proposition 48, $I(2^\alpha) < \dfrac{2}{2-1} = 2$ for every positive integer α. Now, let $n = 2^\alpha p^3 qr$ be 4-layered such that $3^3 \nmid n$ or $5 \nmid n$ or $7 \nmid n$. If $5 \nmid n$, by some properties of the abunancy function we get, $I(n) \le I(2^\alpha \times 3^3 \times 7 \times 7) < 4$, which contradicts the assumption that n is 4-layered. If $7 \nmid n$, we get $I(n) \le I(2^\alpha \times 3^3 \times 5 \times 11) < 4$. Similarly in each possible case we get $I(n) < 4$, a contradiction. Also, one can check directly that $I(2^\alpha \times 3^3 \times 5 \times 7) < 4$ for all $\alpha < 4$.

Thus, for completing the proof, it is sufficient to proof that the number $n = 2^5 \times 3^3 \times 5 \times 7$ is a good 4-layered number. Let

$$A_1 = \{\, 2^5 \times 3^3 \times 5 \times 7\},$$

$$\begin{aligned} A_2 = \{\, & 2 \times 3,\quad 2^4 \times 3^2,\quad 2^5 \times 3 \times 5,\quad 2 \times 3^3 \times 5 \times 7,\quad 2^4 \times 3^2 \times 5 \times 7,\quad 2^3 \times 3^3 \times 5 \times 7, \\ & 2^4 \times 3^3 \times 5 \times 7\}, \end{aligned}$$

$$\begin{aligned} A_3 = \{\, & 3,\quad 2^4 \times 3,\quad 3^3 \times 5 \times 7,\quad 2^4 \times 3^3 \times 5,\quad 2^3 \times 3^2 \times 5 \times 7,\quad 2^4 \times 3^3 \times 7, \\ & 2^5 \times 3 \times 5 \times 7,\quad 2^2 \times 3^3 \times 5 \times 7,\quad 2^5 \times 3^3 \times 5,\quad 2^5 \times 3^2 \times 5 \times 7\}, \end{aligned}$$

and $A_4 = D\setminus(A_1\cup A_2\cup A_3)$. One can check directly that that $\{A_1, A_2, A_3, A_4\}$ is a 4-layered partition for n. Let $n = 2^5m$. Let D' be the set of all positive divisors of m. Let $A'_1 = \{3^3\times 5\times 7,\ \ 3\times 5\}$ and $A'_2 = D\setminus A'_1$. One can check directly that $\{A'_1, A'_2\}$ is a Zumkeller partition for m. Also, if we define $A''_1 = \{2^5d : d \in A'_1\}$ and $A''_2 = \{2^5d : d \in A'_2\}$, then one can check directly that $A''_1 \subset (A_1 \cup A_2)$ and $A''_2 \subset (A_3 \cup A_4)$. Thus, by Proposition 29, n is a good 4-layered number. □

6. k-multiperfect numbers and k-layered numbers

We recall that the number n is said to be k-multiperfect if $\sigma(n) = kn$ for some integer $k \geq 2$. It is clear that if n is 2-multiperfect, then n is said to be perfect. Since every perfect number is Zumkeller, this suggests us the following open question.

Open question 53. For every positive integer $k \neq 1$, is every k-multiperfect number k-layered?

In the following, we will address this open question in some special cases. Before that, we recall that the number n is said to be semiperfect if n is equal to the sum of all or some of its proper divisors. Also, the abundant number n which is not semiperfect is called weird. The existence of odd weird numbers is still an open question.

Remark 54. 70 is the smallest weird number. If n is weird and p is a prime number such that $\sigma(n) < p$, then np is a weird number (for more details, see [3]). This shows that there exist infinity many weird numbers. Nevertheless, it is not yet known whether there exists infinitely many primitive weird numbers[2] or not. Recently, a wide range of number theorists have tried to introduce some optimal algorithms to find new primitive weird numbers. (For instance, see [6, 7].)

Fang and Beckert proved, using parallel tree search, that there are no odd weird numbers up to 10^{21}([14]). This leads us to the following conjecture.

Conjecture 55. Every odd abundant number is semiperfect.

Now, we recall two well-known propositions about semiperfect numbers.

Proposition 56. *Let m be a positive integer, and let p be a prime number such that $p \leq 2^{m+1} - 1$. Then, the number $n = 2^m p$ is a semiperfect number.*

Proposition 57. *If the positive integer m is divisible by the semiperfect number n, then m is semiperfect.*

Remark 58. Let α and β be positive integers. Also, let p_1 and p_2 be primes. By Remark 21, the number $n = p_1^{\alpha_1}p_2^{\alpha_2}$ is an abundant number if and only if $p_1 = 2$ and $3 \leq p_2 \leq 2^{\alpha+1}-1$. Also, by Propositions 56 and 57, all numbers of this form are semiperfect. This concludes that all abundant number which possess exactly two distinct prime factors are semiperfect.

It is believed that all k-multiperfect number of abunancy $3, 4, 5, 6$ and 7 are known. [1] contains the list of all known k-multiperfect numbers. According to this list, one can directly obtain the following proposition.

[2]A weird number n that is not multiple of other weird number is said to be primitive weird number.

Proposition 59. (i) *All known k-multiperfect numbers are practical*

(ii) *All the* 6 *known* 3*-multiperfect numbers are divisible by* 3 *or* 5*. Also, all* 3*-multiperfect numbers are divisible by* 2^3*.*

(iii) *All* 36 *known* 4*-multiperfect numbers are divisible by* 2^2*. In addition, if* $n = 2^\alpha t$ *is one the known* 4*-multiperfect, then* $\frac{n}{2}$ *satisfies the conditions of Proposition 7. This concludes that* $\frac{n}{2}$ *is a practical number.*

(iv) *Let n be one of the known* 5*-multiperfect numbers. Then,* $2^7 \mid n$ *and* $5 \mid n$ *or* $7 \mid n$*.*

As a consequence of part (i) of the above proposition, we have the following corollary.

Corollary 60. *If Conjecture 47 holds, then every known k-multiperfect number is k-layered.*

Thus, we have the following theorem.

Theorem 61. *Every known* 3*-multiperfect number is* 3*-layered.*

Proof. Let n be one of the known 3-multiperfect numbers. Also, Let D be the set of all proper divisors of n. By Propositions 56 and 57, and part (ii) of Proposition 59, the number n is semiperfect. This concludes that there exists $A_1 \subset D$ such that $S(A_1) = n = \frac{\sigma(n)}{3}$. Now, we define $A_2 = \{n\}$ and $A_3 = D \setminus (A_1 \cup A_2)$. Thus, by the definition of n, $\{A_1, A_2, A_3\}$ is a 3-layered partition for n. □

Proposition 62. *Let* $n = 2^\alpha t$ *is an even positive integer with t odd and* $I(n) \geq 4$*. Then,* $I(t) > 2$*.*

Proof. We know that $I(2^\alpha) = \frac{2^{\alpha+1} - 1}{2^\alpha} < 2$. This concludes that $I(t) > 2$. □

Now, by assuming Conjecture 55, we state a theorem similar to Theorem 61 about all known 4-multiperfect numbers.

Theorem 63. *If Conjecture 55 holds, then every known* 4*-multiperfect numbers are* 4*-layered.*

Proof. Let $n = 2^\alpha m$ be a known 4-multiperfect number with $\gcd(2^\alpha, m) = 1$. Also, let D_1, D_2, and D_3 be the set of all positive divisors of $\frac{n}{2}, n$, and m, respectively. By part (iii) of Proposition 59, the number $\frac{n}{2}$ is a practical number. Also, by Proposition 62, $\sigma(\frac{n}{2}) > n$. Thus, by definition of practical numbers, there exists $A_1 \subset D_1$ such that $S(A_1) = n = \frac{\sigma(n)}{4}$. Moreover, by assuming Conjecture 55, there exists $A \subset D_3 \setminus \{m\}$ such that $S(A) = m$. Now, we define $A_2 = \{2^\alpha d : d \in A\}$. It is clear that $S(A_2) = n = \frac{\sigma(n)}{4}$. Furthermore, we define $A_3 = \{n\}$. By definition of A_1, A_2, A_3 for every integers $1 \leq i \neq j \leq 3$, $A_i \cap A_j = \emptyset$. Then, by Proposition 1, n is 4-layered. □

Now, we proof a theorem similar to Theorem 63 about 5-multiperfect numbers.

Theorem 64. *By assuming Conjecture 55, every known* 5*-multiperfect numbers are* 5*-layered.*

Proof. Let n be one of the known 5-multiperfect numbers. By part (iv) of Proposition 59, we know there exist a prime number $p = 5$ or $p = 7$ where $p \mid n$. Now, let α and β be a positive integers such that $2^\alpha \mid\mid n$ and $p^\beta \mid\mid n$. Let $n = 2^\alpha p^\beta t$. we have $I(2^\alpha p^\beta) < \dfrac{2}{2-1} \times \dfrac{5}{5-1} = 2.5$.

We know $I(n) = I(2^\alpha p^\beta)I(t) = 5$; this concludes that $I(t) > 2$. Now, let D_1 and D_2 be the set of all positive divisors of $2^\alpha p^\beta = m$ and t, respectively. By Proposition 56, there exists $A \subset D_1$ where $S(A) = m$. Also, by assuming Conjecture 55, there exists $A' \subset D_2$ where $S(A') = t$. Now, we define $A_1 = \{n\}, A_2 = \{dt : d \in A\}, A_3 = \{dm : d \in A'\}, A_4 = \{d_1d_2 : d_1 \in A \wedge d_2 \in A'\}$. One can check directly that for every integer $1 \leq i \leq 4$, $S(A_i) = \dfrac{\sigma(n)}{5} = n$. Thus, by Proposition 1, n is 5-layered. □

Now, we state an essential proposition.

Proposition 65. *Let n and m be positive integers with gcd(n, m)=1. If n is k-layered and $\dfrac{\sigma(m)}{k+1}$ is a sum of some subset of the set of all divisors of m, then nm is $(k+1)$-layered.*

Proof. Let $\{A_1, A_2, \dots, A_k\}$ be a k-layered partition for n. Also, let A' be a subset of the set of all divisors of n such that $S(A') = \dfrac{\sigma(m)}{k+1}$. Now, let D be the set of all divisors of m. We define $A'' = D \setminus A'$. One can check directly that for every integer $1 \leq i \leq k$, $S(A_iA'') = \dfrac{\sigma(nm)}{k+1}$. Thus, by Proposition 1, nm is $(k+1)$-layered. □

As a consequence of Proposition 65, we have the following corollary.

Corollary 66. *Let p be a prime number and n be a p-layered number. If* $\gcd(n, p) = 1$, *then pn is a $(p + 1)$-layered*

Remark 67. Let p be a prime number and n be a p-multiperfect number. One can check directly that if $\gcd(n, p) = 1$, then pn is a $(p + 1)$-multiperfect number.

By Corollary 66 and Remark 67, we have the following theorem.

Theorem 68. *Let p be a prime number and n be a positive integer such that $p \nmid n$. If n be a p-multiperfect number and also a p-layered number, then pn is a $(p+1)$-multiperfect and $(p + 1)$-layered number.*

Example 69. One can check that the number $n = 2^{14} \times 5 \times 7 \times 19 \times 31 \times 151$ is a 3-multiperfect number. By Theorem 61, n is 3-layered. Thus, by Theorem 68, $3n$ is 4-multiperfect and 4-layered

Example 70. One can check that the number $n = 2^{29} \times 3^{10} \times 7^5 \times 11^2 \times 13 \times 19^3 \times 23 \times 31 \times 43 \times 83 \times 107 \times 151 \times 181 \times 331 \times 3851$ is a 5-multiperfect number. By Remark 67, $5n$ is 6-multiperfect. If conjecture 55 holds, then by Theorem 63, n is 5-layered. Also, if n is 5-layered, then by Corollary 66, $5n$ is 6-layered.

Now, we state a proposition which can find more k-multiperfect numbers which are k-layered.

Proposition 71. *Let n and m be a k-multiperfect and $(k + 1)$-multiperfect numbers, respectively such that $n \mid m$. If n is a k-layered number, then m is $(k+1)$-layered number.*

Proof. Let $\{A_1, A_2, \dots, A_k\}$ be a k-layered partition for the k-multiperfect number n. Now, let $dn = m$. For every integer $1 \le i \le k$, we define $B_i = \{ad : a \in A_i\}$. Then, for every integer $1 \le i \le k$, $\sigma(B_i) = m = \dfrac{\sigma(m)}{k+1}$. Thus, by Proposition 1, m is $(k+1)$-layered. □

Remark 72. One can check directly that exactly half of known 4-multiperfect numbers are divisible by at least a 3-multiperfect number. Then, by Propositions 61 and 71, at least half of known 4-multiperfect are 4-layered.

We close this section by an example that implies an application of Proposition 71.

Example 73. Let $a_1 = 2 \times 3, a_2 = 2^3 \times 3 \times 5, a_3 = 2^5 \times 3^3 \times 5 \times 7, a_4 = 2^{11} \times 3^3 \times 5^2 \times 7^2 \times 13 \times 19 \times 31, a_5 = 2^{19} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13^2 \times 19^2 \times 31^2 \times 37 \times 41 \times 61 \times 127, a_6 = 2^{39} \times 3^{11} \times 5^7 \times 7^3 \times 11 \times 13^2 \times 17 \times 19^2 \times 29 \times 31^2 \times 37 \times 41 \times 61 \times 73 \times 79 \times 83 \times 127 \times 157 \times 313 \times 331 \times 2203 \times 30841 \times 61681$. One can check that for every integer $1 \le i \le 6$, the number a_i is a $(i+1)$-perfect number. Also, it is easy to see that for every integer $1 \le i \le 5$, $a_i \mid a_{i+1}$ and 6 is a Zumkeller number. Thus, by Proposition 71, for every integer $1 \le i \le 6$, a_i is a $(i+1)$-layered number.

7. Near-perfect numbers and Zumkeller numbers

We recall that the number n is said to be a near-perfect number if n is the sum of all of its proper divisors, except for one of them, which we term the redundant divisor [9]. By definition of near-perfect numbers, we have the following well-known proposition.

Proposition 74. *The number n is a near-perfect number with redundant divisor d if and only if d is a proper divisor of n, and $\sigma(n) = 2n + d$*

Remark 75. By Proposition 74, if the number n be an odd near-perfect number, then $\sigma(n)$ is odd. Thus, by Proposition 2, n fails to be Zumkeller.

Now, we state a proposition similar to Proposition 17, about near-perfect numbers.

Proposition 76. *Let n be a near-perfect number. Then, n is Zumkeller if and only if $\sigma(n)$ is even.*

Proof. By Proposition 2, if n is a Zumkeller number, then $\sigma(n)$ is even. Now, let $\sigma(n)$ is even. By the definition of n and Proposition 74, n possesses a divisor d such that $2d$ is a proper divisor of n and $\sigma(n) = 2n + 2d$. Let D be the set of all positive divisors of n. We define $A = \{n, d\}$. One can check directly that $\{A, D \setminus A\}$ is a Zumkeller partition for n. □

The following proposition was stated by Ren and Chen [15]; this proposition classifies all near-perfect numbers with two distinct prime factors.

Proposition 77. *A near-perfect number which has two distinct prime factor is of the form*

(i) $2^{t-1}(2^t - 2^k - 1)$, *where $2^t - 2^k - 1$ is prime.*
(ii) $2^{2p-1}(2^p - 1)$, *where $2^p - 1$ is a Mersenne prime.*
(iii) $2^{p-1}(2^p - 1)^2$, *where $2^p - 1$ is a Mersenne prime.*
(iv) 40.

As a direct consequence of Propositions 76 and 77, we have the following corollary.

Corollary 78. *Let n be a near-perfect number with two distinct prime factors. The number n is non-Zumkeller if and only if there exists a Mersenne prime p such that $n = 2^{p-1}(2^p - 1)^2$.*

Now, we recall a proposition of [16]. This proposition classifies all near-perfect numbers in the form of $2^\alpha p_1 p_2$, where p_1 and p_2 are odd primes with $p_1 < p_2$.

Theorem 79. *Let α be a positive integer. Let p_1 and p_2 be odd primes with $p_1 < p_2$. Then, the number $n = 2^\alpha p_1 p_2$ is near-perfect if and only if n satisfies one the following conditions.*

(i) $p_1 = \dfrac{2^{\alpha+1} - 1 + k}{2^\beta + 1 - k}$, *where* $k = \dfrac{2^{\alpha+1} - 1}{p_2}$ *and* $1 \leq \beta \leq \alpha - 1$.

(ii) $p_1 = 2^{\alpha+1} - 1 + \dfrac{2^\alpha - 2^{\beta-1}}{k}$, *where k is determined by equation*

$$p_2 = (2^{\alpha+1} - 1)(2k + 1) - 2^\beta, 1 \leq \beta \leq \alpha.$$

(iii) $p_2 = 2^{\alpha+1} - 1 + \dfrac{2^{2\alpha+1} - 2^\alpha - 2^{\beta-1}}{k}$, *where* $k = \dfrac{p_1 - (2^{\alpha+1} - 1)}{2}$ *and* $1 \leq \beta \leq \alpha$.

Remark 80. Let $n = 2^\alpha p_1 p_2$ be a number satisfying case (i) of Theorem 79. One can check directly that $p_1 < 2^{\alpha+1} - 1$. This concludes that $2^\alpha p_1$ is a practical number. Thus, without knowing Propositions 76 and 79, by Propositions 7 and 17, we know that all the numbers satisfying case (i) of Theorem 79 are Zumekeller, but by case (ii) and case (iii) of Theorem 79, we can construct a new subset of Zumkeller numbers.

In [16], the authors checked that there exists exactly 8 near-perfect numbers satisfying case (i) of Theorem 79 for $\alpha < 1000$, $\beta = \dfrac{\alpha + 5}{2}$, and $k = 2^{\beta-2} + 1$; the number $2^9 \times 11 \times 31$ is the smallest of such numbers. Also, for case (ii) of Theorem 79, They checked that there exist exactly 289 near-perfect number; the number $2^3 \times 17 \times 101$ is the smallest such number. Furthermore, for case (iii) of Theorem 79, they checked that there exist exactly 248 near-perfect number for $\alpha < 100$ and $k \leq 100$; the number $2^3 \times 17 \times 131$ is the smallest such number.

Now, we recall the definition of half-Zumkeller numbers.

Definition 6. A positive integer n is said to be a half-Zumkeller number if the proper positive divisors of n can be partitioned into two disjoint non-empty subsets of equal sum. A half-Zumkeller partition for a half-Zumkeller number n is a partition $\{A_1, A_2\}$ of the set of proper positive divisors of n such that $S(A_1) = S(A_2)$.

In [11], the authors raised the following conjecture.

Conjecture 81. If n is an even and Zumkeller number, then n is half-Zumkeller.

In [17], the authors verified that the conjecture is true in some cases. They proved that if n is an even Zumkeller number such that $\sigma(n) < 3n$, then n is a half-Zumkeller. Thus, by Remark 75, we have the following proposition.

Proposition 82. *Let n be a near-perfect number. If n is a Zumkeller number, then n is half-Zumkeller.*

8. The difference of two consecutive k-layered numbers

In this section, we let $k \neq 1, m, a, b, s, r,$ and z be positive integers. We also let $p_1, p_2, \cdots, p_m$ be distinct primes such that $p_1 < p_2 < \cdots < p_m$. We start this section with two definitions.

Definition 7. The ascending chain $\ell = (a+t)_{t=0}^{z}$ is said to be a $(p_1, p_2, \ldots, p_m)$-gcd chain if for every integer $0 \leq j \leq z$, $\gcd(a+j, p_1 p_2 \cdots p_m) \neq 1$.

Definition 8. The chain $\ell = (a+t)_{t=0}^{z}$ is said to be a $(p_1, p_2, \ldots, p_m)$-gcd-max chain if ℓ satisfies the following conditions:
(i) ℓ is a $(p_1, p_2, \ldots, p_m)$-gcd chain,
(ii) If $\ell' = (a+t)_{t=0}^{s}$ is a $(p_1, p_2, \ldots, p_m)$-gcd chain, then $z \geq s$.
We also define $L_{max}(p_1, p_2, \ldots, p_m)$ as the cardinality of ℓ.

We recall a standard well-known proposition, which is a generalization of the Chinese reminder theorem.

Proposition 83. *Let $n_1, n_2, \ldots, n_k, a_1, a_2, \ldots,$ and a_k be positive integers. Then, the simultaneous set of congruences $x \equiv a_1 \pmod{n_1}$, $x \equiv a_2 \pmod{n_2}$, $\ldots$, and $x \equiv a_k \pmod{n_k}$ has a solution if and only if for every integers $1 \leq i \neq j \leq k$, $\gcd(n_i, n_j) \mid a_i - a_j$.*

The following proposition finds a lower bound for $L_{max}(p_1, p_2, \ldots, p_m)$.

Proposition 84. *$L_{max}(p_1, p_2, \ldots, p_m) \geq m$.*

Proof. By Proposition 83, we can find a positive integer a such that $a \equiv 0 \pmod{p_1}$, $a \equiv -1 \pmod{p_2}$, $\ldots$, and $a \equiv -(m-1) \pmod{p_m}$. Then, the chain $\ell = (a+t)_{t=0}^{m-1}$ is a $(p_1, p_2, \ldots, p_m)$-gcd chain. □

We know that L_{max} can be known as a function defined on the set of all positive square-free integers. The following proposition state another property of this function.

Proposition 85. *$L_{max}(p_1, p_2, \ldots, p_m) < L_{max}(p_1, p_2, \ldots, p_m, p_{m+1})$.*

Proof. Let $\ell = (a+t)_{t=0}^{s}$ be a $(p_1, p_2, \ldots, p_m)$-gcd chain. This concludes that for every integer $0 \leq t \leq s$, there exists positive integer $1 \leq i_t \leq m$ such that $p_{i_t} \mid a+t$. Thus, by Proposition 83, if $0 \leq t \neq t' \leq s$, then $\gcd(p_{i_t}, p_{i_{t'}}) \mid t - t'$. Therefore, by the definition of p_{m+1} and Proposition 83, there exists a positive integer b such that $b \equiv 0 \pmod{p_{i_0}}$, $b \equiv -1 \pmod{p_{i_1}}$, $\ldots$, $b \equiv -s \pmod{p_{i_s}}$, and $b \equiv -(s+1) \pmod{p_{m+1}}$. Thus, $\ell' = (b+t)_{t=0}^{s+1}$ is a $(p_1, p_2, \ldots, p_m, p_{m+1})$-gcd chain. This completes the proof. □

In remainder of this note, we define ϕ to be the Euler's totient function. Let n be a positive integer with prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$. We recall that $\phi(n) = n \prod_{i=1}^{m} (1 - \dfrac{1}{p_i})$. In Proposition 84, we found a lower bound for $L_{max}(p_1, p_2, \ldots, p_m)$. Now, we find an upper bound for $L_{max}(p_1, p_2, \ldots, p_m)$.

Proposition 86. *$L_{max}(p_1, p_2, \ldots, p_m) \leq p_1 p_2 \cdots p_m - (p_1 - 1)(p_2 - 1) \cdots (p_m - 1)$.*

Proof. Let $t = p_1 p_2 \cdots p_m$. Let r be a non-negative integer such that $0 \leq r < t$. We know that for every positive integer j, $\gcd(jt + r, t) = 1$ if and only if $\gcd(r, t) = 1$. Therefore, between 0 and t there exist exactly $\phi(t)$ distinct integers $r_1, r_2, \ldots,$ and $r_{\phi(t)}$ such that $1 = r_1 < r_2 < \cdots < r_{\phi(t)} < t$ and $\gcd(jt + r_i, t) = 1$ for every integers $1 \leq i \leq \phi(t)$ and

$j \geq 0$. This concludes that between t and $2t$ there exist exactly $\phi(t)$ positive integers $a_1 < a_2 < \cdots < a_{\phi(t)}$ such that $\gcd(a_i, t) = 1$ for every integer $1 \leq i \leq \phi(t)$. Thus, $L_{max}(p_1, p_2, \ldots, p_m) \leq t - \phi(t)$. This completes the proof. □

The following proposition calculates the function L_{max} at some value.

Proposition 87. (i) $L_{max}(3, 5, 7, 11) = 6$.
(ii) $L_{max}(3, 5, 7, 11, 13, 17) = 12$.

Proof. (i) By the Chinese remainder theorem, there exists a positive integer a such that $a \equiv 0 \pmod 5$, $a \equiv -1 \pmod 3$, $a \equiv -2 \pmod 7$, and $a \equiv -3 \pmod{11}$. So, $a \equiv -4 \pmod 3$ and $a \equiv -5 \pmod 5$. Therefore, $\ell = (a+t)_{t=0}^{5}$ is a $(3, 5, 7, 11)$ - gcd chain. Now, it is sufficient to prove that for every positive integer a', there exists an integer $0 \leq i \leq 6$ such that $\gcd(a'+i, 3 \times 5 \times 7 \times 11) = 1$. Let $\ell' = (a'+t)_{t=0}^{6}$ be a chain of positive integers, and let $D = \{3, 5, 7, 11\}$. For every integer $d \in D$, We define A_d as the set of all numbers b of chain ℓ' such that $d \mid b$. Then, $|A_d| \leq \left\lceil \dfrac{7}{d} \right\rceil$. We investigate the two cases for a':

Case 1: $3 \mid a'$. We know that $|A_5| \leq 2$. Also, one can check directly that in this case, if $|A_5| = 2$, then $|A_5 \cap A_3| = 1$. This concludes that $|A_3 \cup A_5| \leq 4$. Thus, $|A_3 \cup A_5 \cup A_7 \cup A_{11}| \leq 6$.

Case 2: $3 \nmid a'$. In this case, $|A_3| \leq 2$. Thus, one can check directly that $|A_3 \cup A_5 \cup A_7 \cup A_{11}| \leq 6$.

This completes the proof.

(ii) By Chinese remainder theorem, there exists a positive integer a such that $a \equiv 0 \pmod{11}$, $a \equiv -1 \pmod 3$, $a \equiv -2 \pmod 7$, $a \equiv -3 \pmod 5$, $a \equiv -5 \pmod{13}$, and $a \equiv -6 \pmod{17}$. One can check directly that $\ell = (a+t)_{t=0}^{11}$ is a $(3, 5, 7, 11, 13, 17)$ - gcd chain. Now, it is sufficient to prove that for every positive integer a', there exists an integer $0 \leq i \leq 12$ such that $\gcd(a'+i, 3 \times 5 \times 7 \times 11 \times 13 \times 17) = 1$. Let $\ell' = (a'+t)_{t=0}^{12}$ be a chain of positive integers, and let $D = \{3, 5, 7, 11, 13, 17\}$. For every integer $d \in D$, We define A_d as the set of all numbers b of chain ℓ' such that $d \mid b$. Then, $|A_d| \leq \left\lceil \dfrac{13}{d} \right\rceil$. We investigate the two cases for a':

Case 1: $3 \mid a'$. We know that $|A_{11}| \leq 2$. Also, one can check directly that in this case, if $|A_{11}| = 2$, then $|A_{11} \cap A_3| = 1$. Then, $|A_3 \cup A_{11}| \leq 6$. Also, we know that $|A_5| \leq 3$, and if $|A_5| = 3$, then $|A_3 \cap A_5| = 1$. This concludes that $|A_3 \cup A_5 \cup A_{11}| \leq 8$. Thus, $|A_3 \cup A_5 \cup A_7 \cup A_{11} \cup A_{13} \cup A_{17}| \leq 12$.

Case 2: $3 \nmid a'$. By the definition of a', $|A_3| \leq 4$. Once again, we know that if $|A_5| = 3$, then $|A_3 \cap A_5| = 1$. Therefore, $|A_3 \cup A_5| \leq 6$. Thus, $|A_3 \cup A_5 \cup A_7 \cup A_{11} \cup A_{13} \cup A_{17}| \leq 12$.

This completes the proof.

□

We can find some more results about the function L_{max}. But, we do not pursue this here. Now, we state a proposition that finds a lower density for the set of k-layered numbers by having the smallest k-layered number.

Proposition 88. *Let $k \neq 1$ be a positive integer, and n be the smallest k-layered number with prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where $p_1 < p_2 < \cdots < p_m$. Then $(L_{max}(p_1, p_2, \ldots, p_m)+1)n$ is an upper bound for the difference of two consecutive k-layered numbers.*

Proof. Let a and b be two consecutive k-layered numbers. By the division algorithm, there exist non-negative integers $s \neq 0$ and r such that $a = sn + r$ and $0 \leq r < s$. By the definition of L_{max} and Proposition 5, we know that there exists an integer ℓ with $s+1 \leq \ell \leq s+1+L_{max}(p_1, p_2, \ldots, p_m)$ such that ℓn is k-layered. Then $b-a \leq \ell n - sn \leq (L_{max}(p_1, p_2, \ldots, p_m) + 1)n$. $\square$

Corollary 89. *Let the number n be the smallest k-layered number. If n with prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ is a good k-layered number, then $b-a \leq (L_{max}(p_2, p_3, \ldots, p_m)+1)n$.*

By Propositions 86 and 88, we have the following corollary.

Corollary 90. *Let $k \neq 1$ be a positive integer, and n be the smallest k-layered number. Let a and b be two consecutive k-layered numbers. Then,*

$$|b-a| \leq (p_1 p_2 \cdots p_m - (p_1 - 1)(p_2 - 1) \cdots (p_m - 1) + 1)n.$$

Remark 91. Let $k \neq 1$ be a positive integer. Corollary 90 gives a lower density for the set of k-layered numbers by having the smallest k-layered number.

Let k be a positive integer such that $2 \leq k \leq 5$, and n be the smallest k-layered number. In Section 4, we showed that n possesses a prime factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where p_i is the ith prime number for integers $1 \leq i \leq m$. Then, by Corollaries 32 and 89, we conclude that $(L_{max}(p_2, \ldots, p_m) + 1)n$ is an upper bound for the difference of two consecutive k-layered number. Thus, by Proposition 87, we have the following table.

k	The smallest k-layered number	An upper bound for the difference of two consecutive k-layered numbers
2	6	12
3	120	360
4	27720	194040
5	147026880	1911349440

TABLE 2. An upper bound for the difference of two consecutive k-layered for $2 \leq k \leq 5$.

Remark 92. The numbers 282 and 294 are two consecutive Zumkeller numbers. Thus, by Table 2, the number 12 is a sharp upper bound for difference of two consecutive Zumkeller numbers.

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Mathematics Section, Department of Basic Sciences, Shahid Rajaee Teacher Training University, Tehran 16785-136, Iran
Email address: jokar.farid@gmail.com

Gonit Sora, Dhalpur, Assam 784165, India
Email address: pankaj@gonitsora.com

School of Mathematics, Cardiff University, Cardiff, CF24 4AG, UK
Email address: manjil@saikia.in