Uncertainty-Aware Multidimensional Scaling - Supplemental Material

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Abstract—In this supplemental material, we provide the detailed mathematical derivation of the uncertainty-aware multidimensional scaling (UAMDS) model for normal distributions that complements the main document: The paper on "Uncertainty-Aware Multidimensional Scaling" published in IEEE Transactions on Visualization and Computer Graphics [3]. Specifically, we evaluate the stress term resulting from our general UAMDS model, where we assume that the random vectors are normally distributed and independent, and that the projection operators are affine linear. In addition to the mathematical derivation, we provide additional visualizations for UAMDS that complement the techniques from the main document.

1 Normal Distribution Model: Basic Evaluation

Given random vectors $P_1,\dots,P_K$ with realizations in the high-dimensional space $\mathbb{R}^N$ and associated probability density functions $f_{P_1},\dots,f_{P_K}:\mathbb{R}^N\to\mathbb{R}$, the goal is to find low-dimensional random vectors $X_1,\dots,X_K$ via local projections $\Phi_1,\dots,\Phi_K:\mathbb{R}^N\to\mathbb{R}^n$ and $X_k=\Phi_k\circ P_k$ with realizations in a desired low-dimensional space $\mathbb{R}^n$ and associated probability density functions $f_{X_1},\dots,f_{X_K}:\mathbb{R}^n\to\mathbb{R}$ such that the following stress is minimized (compare Section 3 from the main document):

$$\tilde{\mathcal{S}}(\Phi_1,\dots,\Phi_K)=\sum_{i,j}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_{P_i,P_j}(v,w)\cdot\left(\|v-w\|_2^2-\left\|\Phi_i(v)-\Phi_j(w)\right\|_2^2\right)^2\,dv\,dw, \tag{1}$$

where f_{P_i,P_j} is the joint probability density function of f_{P_i} and f_{P_j}.

1.1 Basic Evaluation: Normal Distribution Assumption

Given the generic model of UAMDS (Equation 1), the normal distribution model can be evaluated. To this end, the following assumption is made.

Assumption 1: The random vectors $P_1,\dots,P_K$ are independent and normally distributed with $P_k\sim\mathcal{N}(\mu^{P_k},\Sigma^{P_k})$, $\mu^{P_k}\in\mathbb{R}^N$, $\Sigma^{P_k}\in\mathbb{R}^{N\times N}$, and the singular value decompositions $\Sigma^{P_k}=U^{P_k}S^{P_k}{U^{P_k}}^\top$ with $S^{P_k}=\operatorname{diag}(s_1^{P_k},\dots,s_N^{P_k})$ and $U^{P_k}=\begin{bmatrix}u_1^{P_k} & \dots & u_N^{P_k}\end{bmatrix}$.

Due to Assumption 1, the joint probability density functions $f_{P_i,P_j}(v,w)$ are $f_{P_i}(v,w)\cdot f_{P_j}(v,w)$ and the underlying probability density functions can be expressed in terms of the expected values μ^{P_k} and covariance matrices Σ^{P_k}. To compute the stress $\tilde{\mathcal{S}}$, we have to distinguish between the degenerate case and the non-degenerate case. In the degenerate case, where singular values of the covariance matrix Σ^{P_k} can be zero, we need to express the integrals via a limit $\Sigma_*^{P_k}\to\Sigma^{P_k}$, where $\Sigma_*^{P_k}$ is a sequence of symmetric positive definite matrices. In the non-degenerate case, i.e., the covariance matrix Σ^{P_k} has only positive singular values (the matrix is symmetric positive definite), there are no singularities in the upcoming formulation (thus, the sequence can be defined as $\Sigma_*^{P_k}=\Sigma^{P_k}$). With these considerations, we can compute the stress $\tilde{\mathcal{S}}$ as follows:

$$\begin{aligned}\tilde{\mathcal{S}}(\Phi_1,\dots,\Phi_K)&=\sum_{i,j}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_{P_i,P_j}(v,w)\cdot\left(\|v-w\|_2^2-\left\|\Phi_i(v)-\Phi_j(w)\right\|_2^2\right)^2\,dv\,dw \\ &=\sum_{i,j}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_{P_i}(v,w)\cdot f_{P_j}(v,w)\cdot\left(\|v-w\|_2^2-\left\|\Phi_i(v)-\Phi_j(w)\right\|_2^2\right)^2\,dv\,dw \\ &=\sum_{i,j}\lim_{\Sigma_*^{P_j}\to\Sigma^{P_j}}\int_{\mathbb{R}^N}\lim_{\Sigma_*^{P_i}\to\Sigma^{P_i}}\int_{\mathbb{R}^N}\frac{e^{-\frac{1}{2}(v-\mu^{P_i})^\top{\Sigma_*^{P_i}}^{-1}(v-\mu^{P_i})}}{\sqrt{(2\pi)^N\cdot\det(\Sigma_*^{P_i})}}\frac{e^{-\frac{1}{2}(w-\mu^{P_j})^\top{\Sigma_*^{P_j}}^{-1}(w-\mu^{P_j})}}{\sqrt{(2\pi)^N\cdot\det(\Sigma_*^{P_j})}}\cdot\left(\|v-w\|_2^2-\left\|\Phi_i(v)-\Phi_j(w)\right\|_2^2\right)^2\,dv\,dw.\end{aligned} \tag{2}$$

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To describe the limit process more precisely, the covariance matrices $\Sigma^{P_k} = U^{P_k} S^{P_k} U^{P_k \top}$ are approximated via $\Sigma_*^{P_k} = U^{P_k} S_*^{P_k} U^{P_k \top}$, where $S_*^{P_k} = \operatorname{diag}(s_{*1}^{P_k}, \dots, s_{*N}^{P_k})$ with $s_{*i}^{P_k} \downarrow s_i^{P_k} = 0$ in the degenerate case (limit from above) and $s_{*i}^{P_k} = s_i^{P_k}$ in the non-degenerate case (constant sequence). Therefore, it can be observed that the limit process is only necessary for the degenerate case and this definition of a sequence results in a so-called Dirac sequence. The next step is to perform a change of variables with the transformations $\Psi_k : \mathbb{R}^N \to \mathbb{R}^N$, $\Psi_k(v) = U^{P_k} S_*^{P_k \, 1/2} v + \mu^{P_k}$. In fact, the image of Ψ_k is $\Psi_k(\mathbb{R}^N) = \mathbb{R}^N$, the derivative satisfies $\mathrm{D}\Psi_k(v) = U^{P_k} S_*^{P_k \, 1/2}$, and we know that $|\det(\mathrm{D}\Psi_k(v))| = |\det(U^{P_k} S_*^{P_k \, 1/2})| = \det(S_*^{P_k \, 1/2}) = \sqrt{\det(\Sigma_*^{P_k})}$ and, thus, the stress $\tilde{S}$ can be computed in the following way (to get rid of the limit process)

$$
\begin{aligned}
&= \sum_{i,j} \lim_{\Sigma_*^{P_j} \to \Sigma^{P_j}} \int_{\Psi_i(\mathbb{R}^N)} \lim_{\Sigma_*^{P_i} \to \Sigma^{P_i}} \int_{\Psi_j(\mathbb{R}^N)} \frac{e^{-\frac{1}{2}(v-\mu^{P_i})^\top \Sigma_*^{P_i \, -1}(v-\mu^{P_i})}}{\sqrt{(2\pi)^N \cdot \det(\Sigma_*^{P_i})}} \frac{e^{-\frac{1}{2}(w-\mu^{P_j})^\top \Sigma_*^{P_j \, -1}(w-\mu^{P_j})}}{\sqrt{(2\pi)^N \cdot \det(\Sigma_*^{P_j})}} \cdot \left(\|v-w\|_2^2 - \left\|\Phi_i(v) - \Phi_j(w)\right\|_2^2 \right)^2 \, dv \, dw \\
&= \sum_{i,j} \lim_{\Sigma_*^{P_j} \to \Sigma^{P_j}} \int_{\mathbb{R}^N} \lim_{\Sigma_*^{P_i} \to \Sigma^{P_i}} \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{2}v^\top v}}{\sqrt{(2\pi)^N}} \frac{e^{-\frac{1}{2}w^\top w}}{\sqrt{(2\pi)^N}} \cdot \Bigg(\left\| U^{P_i} S_*^{P_i \, 1/2} v + \mu^{P_i} - U^{P_j} S_*^{P_j \, 1/2} w - \mu^{P_j} \right\|_2^2 \\
&\qquad\qquad - \left\| \Phi_i(U^{P_i} S_*^{P_i \, 1/2} v + \mu^{P_i}) - \Phi_j(U^{P_j} S_*^{P_j \, 1/2} w + \mu^{P_j}) \right\|_2^2 \Bigg)^2 \, dv \, dw \\
&= \sum_{i,j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{2}v^\top v}}{\sqrt{(2\pi)^N}} \frac{e^{-\frac{1}{2}w^\top w}}{\sqrt{(2\pi)^N}} \cdot \left(\left\| U^{P_i} S^{P_i \, 1/2} v + \mu^{P_i} - U^{P_j} S^{P_j \, 1/2} w - \mu^{P_j} \right\|_2^2 - \left\| \Phi_i(U^{P_i} S^{P_i \, 1/2} v + \mu^{P_i}) - \Phi_j(U^{P_j} S^{P_j \, 1/2} w + \mu^{P_j}) \right\|_2^2 \right)^2 \, dv \, dw,
\end{aligned}
$$

where we used that the inverse of a symmetric positive definite matrix Σ is given by its singular value decomposition via $\Sigma^{-1} = US^{-1}U^\top$.

1.2 Basic Evaluation: Affine Projection Assumption

The final formulation of the previous subsection has only one model parameter left: the local projections $\Phi_1, \dots, \Phi_K : \mathbb{R}^N \to \mathbb{R}^n$. As motivated in the main document, the normal distributions are projected via adjusted affine transformations. This aspect is captured in the following assumption.

Assumption 2: The local projections $\Phi_1, \dots, \Phi_K : \mathbb{R}^N \to \mathbb{R}^n$ are modeled by adjusted affine transformations $\Phi_k(v) = B^k U^{P_k \top}(v - \mu^{P_k}) + c^k$ with variables $B^k = \begin{bmatrix} b_1^k & \dots & b_N^k \end{bmatrix} \in \mathbb{R}^{n \times N}$ and $c^k \in \mathbb{R}^n$ (the superscript is an index, not the exponentiation).

The effect of the affine transformations onto the high-dimensional normally distributioned random vectors $P_k \sim \mathcal{N}(\mu^{P_k}, \Sigma^{P_k})$ is that the low-dimensional distribution is normally distributed as well with $X_k = \Phi_k \circ P_k \sim \mathcal{N}(\mu^{X_k}, \Sigma^{X_k}) = \mathcal{N}(c^k, B^k S^{P_k} B^{k \top})$. The stress $\tilde{S}$ can therefore be rearranged in the following way:

$$
\begin{aligned}
&= \sum_{i,j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{2}v^\top v}}{\sqrt{(2\pi)^N}} \frac{e^{-\frac{1}{2}w^\top w}}{\sqrt{(2\pi)^N}} \cdot \left(\left\| U^{P_i} S^{P_i \, 1/2} v + \mu^{P_i} - U^{P_j} S^{P_j \, 1/2} w - \mu^{P_j} \right\|_2^2 - \left\| B^i S^{P_i \, 1/2} v + c^i - B^j S^{P_j \, 1/2} w - c^j \right\|_2^2 \right)^2 \, dv \, dw \\
&= \sum_{i,j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{2}v^\top v}}{\sqrt{(2\pi)^N}} \frac{e^{-\frac{1}{2}w^\top w}}{\sqrt{(2\pi)^N}} \cdot \left(\left\| \left[U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w \right] + \left[\mu^{P_i} - \mu^{P_j} \right] \right\|_2^2 - \left\| \left[B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w \right] + \left[c^i - c^j \right] \right\|_2^2 \right)^2 \, dv \, dw \\
&= \sum_{i,j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{2}v^\top v}}{\sqrt{(2\pi)^N}} \frac{e^{-\frac{1}{2}w^\top w}}{\sqrt{(2\pi)^N}} \cdot \Bigg(\left\| U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w \right\|_2^2 + \left\langle U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w, \mu^{P_i} - \mu^{P_j} \right\rangle + \left\| \mu^{P_i} - \mu^{P_j} \right\|_2^2 \\
&- \left\| B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w \right\|_2^2 - \left\langle B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w, c^i - c^j \right\rangle - \left\| c^i - c^j \right\|_2^2 \Bigg)^2 \, dv \, dw \\
&= \sum_{i,j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{2}v^\top v}}{\sqrt{(2\pi)^N}} \frac{e^{-\frac{1}{2}w^\top w}}{\sqrt{(2\pi)^N}} \cdot \Bigg(\left[\left\| U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w \right\|_2^2 - \left\| B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w \right\|_2^2 \right] \\
&+ \left[\left\langle U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w, \mu^{P_i} - \mu^{P_j} \right\rangle - \left\langle B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w, c^i - c^j \right\rangle \right] + \left[\left\| \mu^{P_i} - \mu^{P_j} \right\|_2^2 - \left\| c^i - c^j \right\|_2^2 \right] \Bigg)^2 \, dv \, dw \\
&= \sum_{i,j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{e^{-\frac{1}{2}v^\top v}}{\sqrt{(2\pi)^N}} \frac{e^{-\frac{1}{2}w^\top w}}{\sqrt{(2\pi)^N}} \cdot \Bigg(\left(\left\| U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w \right\|_2^2 - \left\| B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w \right\|_2^2 \right)^2 \\
&+ \left(\left\langle U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w, \mu^{P_i} - \mu^{P_j} \right\rangle - \left\langle B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w, c^i - c^j \right\rangle \right)^2 + \left(\left\| \mu^{P_i} - \mu^{P_j} \right\|_2^2 - \left\| c^i - c^j \right\|_2^2 \right)^2 \\
&+ 2 \cdot \left(\left\| U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w \right\|_2^2 - \left\| B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w \right\|_2^2 \right) \left(\left\langle U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w, \mu^{P_i} - \mu^{P_j} \right\rangle - \left\langle B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w, c^i - c^j \right\rangle \right) \\
&+ 2 \cdot \left(\left\| U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w \right\|_2^2 - \left\| B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w \right\|_2^2 \right) \left(\left\| \mu^{P_i} - \mu^{P_j} \right\|_2^2 - \left\| c^i - c^j \right\|_2^2 \right) \\
&+ 2 \cdot \left(\left\langle U^{P_i} S^{P_i \, 1/2} v - U^{P_j} S^{P_j \, 1/2} w, \mu^{P_i} - \mu^{P_j} \right\rangle - \left\langle B^i S^{P_i \, 1/2} v - B^j S^{P_j \, 1/2} w, c^i - c^j \right\rangle \right) \left(\left\| \mu^{P_i} - \mu^{P_j} \right\|_2^2 - \left\| c^i - c^j \right\|_2^2 \right) \Bigg) \, dv \, dw
\end{aligned}
$$

The formulation emphasizes the connection of high-dimensional objects and low-dimensional objects. In fact, the integrand consists of six different terms that describe the difference between mean- and covariance-related properties. The isolation of these terms is crucial to obtain our final formulation. It is derived in the next subsection.

1.3 Basic Evaluation: Resulting Stress Terms

Using the final formulation of the previous subsection, we can split the double integral into the six different terms due to the linearity of the integral. In this way, we can evaluate these terms separately. The evaluation of the terms, which we denote as stress terms, is done in the next section. The stress terms (T1)–(T6) are stated in the following (where the sum $\sum_{i,j}$ over all elements i and j is neglected for the evaluation):

$$\text{(T1)}\quad \frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\left(\left\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\right\|_2^2-\left\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\right\|_2^2\right)^2 dv\,dw$$

$$\text{(T2)}\quad \frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\left(\left\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\right\rangle\right)^2 dv\,dw$$

$$\text{(T3)}\quad \frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\left(\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right)^2 dv\,dw$$

$$\begin{aligned}\text{(T4)}\quad &\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot 2\cdot\left(\left\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\right\|_2^2-\left\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\right\|_2^2\right)\\ &\qquad\left(\left\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\right\rangle^2\right) dv\,dw\end{aligned}$$

$$\begin{aligned}\text{(T5)}\quad &\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot 2\cdot\left(\left\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\right\|_2^2-\left\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\right\|_2^2\right)\\ &\qquad\left(\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right) dv\,dw\end{aligned}$$

$$\begin{aligned}\text{(T6)}\quad &\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot 2\cdot\left(\left\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\right\rangle\right)\\ &\qquad\left(\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right) dv\,dw\end{aligned}$$

2 Normal Distribution Model: Evaluation of Stress Terms

In the previous section, we derived the six stress terms (T1)–(T6) as part of the pairwise sum. This section evaluates these terms via the following two frequently used identities:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\cdot(x^2+q)}(ax^2+bx+c)^2\,dx=\sqrt{2\pi}\cdot e^{-\frac{1}{2}\cdot q}\cdot(3a^2+b^2+2ac+c^2), \tag{3}$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\cdot(x^2+q)}(ax^3+bx^2+cx+d)\,dx=\sqrt{2\pi}\cdot e^{-\frac{1}{2}\cdot q}\cdot(b+d). \tag{4}$$

Both identities can be easily proved. The evaluation of each stress term is performed in the following subsections.

2.1 Stress Term Evaluation: (T1)

To evaluate the stress term (T1), we first rearrange the integrand in a way that the double integral can be solved via the Fubini-Tonelli's theorem, i.e., our goal is to integrate along the canonical axes $v_1,v_2,\ldots,v_N$ and $w_1,w_2,\ldots,w_N$ one after the other (we use δ_{kl} for the Kronecker delta):

$$\begin{aligned}&\left(\left\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\right\|_2^2-\left\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\right\|_2^2\right)^2 \qquad (5)\\
=&\left(v^\top S^{P_i}v+w^\top S^{P_j}w-2v^\top S^{P_i\,1/2}U^{P_i\top}U^{P_j}S^{P_j\,1/2}w-\left(v^\top S^{P_i\,1/2}B^{i\top}B^iS^{P_i\,1/2}v+w^\top S^{P_j\,1/2}B^{j\top}B^jS^{P_j\,1/2}w-2v^\top S^{P_i\,1/2}B^{i\top}B^jS^{P_j\,1/2}w\right)\right)^2\\
=&\left(\sum_{k=1}^N s_k^{P_i}v_k^2+\sum_{k=1}^N s_k^{P_j}w_k^2-2\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_i}s_l^{P_j})^{1/2}u_k^{P_i\top}u_l^{P_j}v_kw_l\right.\\
&\left.-\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_i}s_l^{P_i})^{1/2}b_k^{i\top}b_l^iv_kv_l-\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_j}s_l^{P_j})^{1/2}b_k^{j\top}b_l^jw_kw_l+2\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_i}s_l^{P_j})^{1/2}b_k^{i\top}b_l^jv_kw_l\right)^2\\
=&\left(\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_i}s_l^{P_i})^{1/2}\delta_{kl}v_kv_l+\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_j}s_l^{P_j})^{1/2}\delta_{kl}w_kw_l-2\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_i}s_l^{P_j})^{1/2}u_k^{P_i\top}u_l^{P_j}v_kw_l\right.\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}b_k^{i\top}b_l^i v_k v_l-\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}b_k^{j\top}b_l^j w_k w_l+2\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}b_k^{i\top}b_l^j v_k w_l\Bigg)^2\\
&=\Bigg(\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_k v_l+\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_k w_l-2\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_k w_l\Bigg)^2\\
&=\Bigg(s_1^{P_i}(1-b_1^{i\top}b_1^i)v_1^2+\sum_{l=2}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)v_1 v_l+\sum_{k=2}^{N}(s_k^{P_i}s_1^{P_i})^{1/2}(-b_k^{i\top}b_1^i)v_k v_1-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)v_1 w_l\\
&+\sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_k v_l+\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_k w_l-2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_k w_l\Bigg)^2\\
&=\Bigg(s_1^{P_i}(1-b_1^{i\top}b_1^i)v_1^2+2\sum_{l=2}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)v_l v_1-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)w_l v_1\\
&+\sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_k v_l+\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_k w_l-2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_k w_l\Bigg)^2
\end{aligned}
$$

This formulation emphasizes the direction v_1. In the following, we will integrate along this axis while using the first identity (Equation 3):

$$
\begin{aligned}
&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\Big(\Big\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\Big\|_2^2-\Big\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\Big\|_2^2\Big)^2\,dv\,dw \qquad (6)\\
&=\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}e^{-\frac{1}{2}(v_1^2+\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot\Bigg(s_1^{P_i}(1-b_1^{i\top}b_1^i)v_1^2+2\sum_{l=2}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)v_l v_1-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)w_l v_1\\
&+\sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_k v_l+\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_k w_l-2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_k w_l\Bigg)^2\,dv_1\cdot dv_2\cdots dv_N\,dw\\
&=\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-1}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}}e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot\Bigg(3\big(s_1^{P_i}(1-b_1^{i\top}b_1^i)\big)^2+\Big(2\sum_{l=2}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)v_l-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)w_l\Big)^2\\
&+2\big(s_1^{P_i}(1-b_1^{i\top}b_1^i)\big)\Big(\sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_k v_l+\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_k w_l-2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_k w_l\Big)\\
&+\Big(\sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_k v_l+\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_k w_l-2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_k w_l\Big)^2\Bigg)\,dv_2\cdots dv_N\,dw
\end{aligned}
$$

We see that the integrand now consists of four different terms (not counting the exponential function), where the last term is similar to the previous integrand (except for the components that belong to v_1). Therefore, the evaluation can be done by recursion (mathematical induction). Before we do this, we evaluate the other three terms: the first term can be easily solved via

$$
\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-1}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}}e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot 3\big(s_1^{P_i}(1-b_1^{i\top}b_1^i)\big)^2\,dv_2\cdots dv_N\cdot dw=3\big(s_1^{P_i}(1-b_1^{i\top}b_1^i)\big)^2, \qquad (7)
$$

the second term can be solved with the help of the first identity (Equation 3) and recursion via

$$
\begin{aligned}
&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-1}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}}e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot\Big(2\sum_{l=2}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)v_l-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)w_l\Big)^2\,dv_2\cdots dv_N\cdot dw \qquad (8)\\
&=\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^{N-2}}e^{-\frac{1}{2}(\sum_{k=3}^n v_k^2+\sum_{l=1}^N w_l^2)}\cdot\Bigg(\Big(2(s_1^{P_i}s_2^{P_i})^{1/2}(-b_1^{i\top}b_2^i)\Big)^2\\
&+\Big(2\sum_{l=3}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)v_l-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)w_l\Big)^2\Bigg)\,dv_3\cdots dv_n\cdot dw\\
&=\Big(2(s_1^{P_i}s_2^{P_i})^{1/2}(-b_1^{i\top}b_2^i)\Big)^2\\
&+\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-2}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-2}}e^{-\frac{1}{2}(\sum_{k=3}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot\Big(2\sum_{l=3}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)v_l-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)w_l\Big)^2\,dv_3\cdots dv_n\cdot dw\\
&=\ldots\\
&=\Big(2(s_1^{P_i}s_2^{P_i})^{1/2}(-b_1^{i\top}b_2^i)\Big)^2+\cdots+\Big(2(s_1^{P_i}s_N^{P_i})^{1/2}(-b_1^{i\top}b_N^i)\Big)^2+\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}e^{-\frac{1}{2}(\sum_{l=1}^N w_l^2)}\cdot\Big(-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)w_l\Big)^2\,dw\\
&=4\cdot\sum_{l=2}^{N}\Big((s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)\Big)^2+4\cdot\sum_{l=1}^{N}\Big((s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)\Big)^2,
\end{aligned}
$$

and the third term can be solved with the help of the second identity (Equation 4) via

$$\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-1}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}} e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot 2\Big(s_1^{P_i}(1-b_1^{i\top}b_1^i)\Big) \tag{9}$$
$$\Big(\sum_{k=2}^N\sum_{l=2}^N (s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_kv_l+\sum_{k=1}^N\sum_{l=1}^N (s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_kw_l-2\sum_{k=2}^N\sum_{l=1}^N (s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_kw_l\Big)\,dv_2\cdots dv_n\cdot dw$$
$$=2\Big(s_1^{P_i}(1-b_1^{i\top}b_1^i)\Big)\Big(\sum_{k=2}^N s_k^{P_i}(1-b_k^{i\top}b_k^i)+\sum_{k=1}^N s_k^{P_j}(1-b_k^{j\top}b_k^j)\Big).$$

Now, we can evaluate the first stress term (T1) further by inserting Equations 7, 8, and 9 into Equation 6 and by solving the remaining integrals along the directions $v_2,\ldots,v_N$ inductively (we highlight the recursive representation once again). This leads to the following partial result:

$$\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\Big(\Big\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\Big\|_2^2-\Big\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\Big\|_2^2\Big)^2\,dv\,dw \tag{10}$$
$$=\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\sum_{k=1}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot\Big($$
$$\sum_{k=1}^N\sum_{l=1}^N (s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_kv_l+\sum_{k=1}^N\sum_{l=1}^N (s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_kw_l-2\sum_{k=1}^N\sum_{l=1}^N (s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_kw_l\Big)^2\,dv\,dw$$
$$=3\big(s_1^{P_i}(1-b_1^{i\top}b_1^i)\big)^2+4\cdot\sum_{l=2}^N\Big((s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^i)\Big)^2+4\cdot\sum_{l=1}^N\Big((s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^j)\Big)^2$$
$$+2\Big(s_1^{P_i}(1-b_1^{i\top}b_1^i)\Big)\Big(\sum_{k=2}^N s_k^{P_i}(1-b_k^{i\top}b_k^i)+\sum_{k=1}^N s_k^{P_j}(1-b_k^{j\top}b_k^j)\Big)+\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-1}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}} e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot\Big($$
$$\sum_{k=2}^N\sum_{l=2}^N (s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^i)v_kv_l+\sum_{k=1}^N\sum_{l=1}^N (s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_kw_l-2\sum_{k=2}^N\sum_{l=1}^N (s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)v_kw_l\Big)^2\,dv_2\cdots dv_N\,dw$$
$$=\ldots$$
$$=3\sum_{k=1}^N\big(s_k^{P_i}(1-b_k^{i\top}b_k^i)\big)^2+4\sum_{k=1}^N\sum_{l=k+1}^N\Big((s_k^{P_i}s_l^{P_i})^{1/2}(-b_k^{i\top}b_l^i)\Big)^2+4\sum_{k=1}^N\sum_{l=1}^N\Big((s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)\Big)^2$$
$$+2\sum_{k=1}^N\Big(s_k^{P_i}(1-b_k^{i\top}b_k^i)\Big)\Big(\sum_{l=k+1}^N s_l^{P_i}(1-b_l^{i\top}b_l^i)+\sum_{l=1}^N s_l^{P_j}(1-b_l^{j\top}b_l^j)\Big)+\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_{l=1}^N w_l^2}\cdot\Big(\sum_{k=1}^N\sum_{l=1}^N (s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_kw_l\Big)^2\,dw$$

To evaluate the remaining integral along the directions $w_1,\ldots,v_N$, we can use the same approach. In fact, since this integral is even simpler than the previous one, it can be solved analogously (integrate along w_1, see Equation 5, and then along $w_2,\ldots,w_N$, see Equation 10):

$$\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_{l=1}^N w_l^2}\cdot\Big(\sum_{k=1}^N\sum_{l=1}^N (s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^j)w_kw_l\Big)^2\,dw \tag{11}$$
$$=3\sum_{k=1}^N\big(s_k^{P_j}(1-b_k^{j\top}b_k^j)\big)^2+4\sum_{k=1}^N\sum_{l=k+1}^N\Big((s_k^{P_j}s_l^{P_j})^{1/2}(-b_k^{j\top}b_l^j)\Big)^2+2\sum_{k=1}^N\Big(s_k^{P_j}(1-b_k^{j\top}b_k^j)\Big)\Big(\sum_{l=k+1}^N s_l^{P_j}(1-b_l^{j\top}b_l^j)\Big)$$

In sum, the final result for the first stress term (T1) is obtained by inserting Equation 11 into Equation 10:

$$\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\Big(\Big\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\Big\|_2^2-\Big\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\Big\|_2^2\Big)^2\,dv\,dw$$
$$=3\sum_{k=1}^N\big(s_k^{P_i}(1-b_k^{i\top}b_k^i)\big)^2+4\sum_{k=1}^N\sum_{l=k+1}^N\Big((s_k^{P_i}s_l^{P_i})^{1/2}(-b_k^{i\top}b_l^i)\Big)^2+3\sum_{k=1}^N\big(s_k^{P_j}(1-b_k^{j\top}b_k^j)\big)^2+4\sum_{k=1}^N\sum_{l=k+1}^N\Big((s_k^{P_j}s_l^{P_j})^{1/2}(-b_k^{j\top}b_l^j)\Big)^2$$
$$+4\sum_{k=1}^N\sum_{l=1}^N\Big((s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j)\Big)^2+2\sum_{k=1}^N\Big(s_k^{P_i}(1-b_k^{i\top}b_k^i)\Big)\Big(\sum_{l=k+1}^N s_l^{P_i}(1-b_l^{i\top}b_l^i)+\sum_{l=1}^N s_l^{P_j}(1-b_l^{j\top}b_l^j)\Big)$$
$$+2\sum_{k=1}^N\Big(s_k^{P_j}(1-b_k^{j\top}b_k^j)\Big)\Big(\sum_{l=k+1}^N s_l^{P_j}(1-b_l^{j\top}b_l^j)\Big)$$

$$=\sum_{k=1}^N\Big(s_k^{P_i}(1-b_k^{i\top}b_k^i)\Big)^2+2\sum_{k=1}^N\Big(s_k^{P_i}(1-b_k^{i\top}b_k^i)\Big)^2+4\sum_{k=1}^N\sum_{l=k+1}^N\Big((s_k^{P_i}s_l^{P_i})^{1/2}(-b_k^{i\top}b_l^i)\Big)^2$$
$$+\sum_{k=1}^N\Big(s_k^{P_j}(1-b_k^{j\top}b_k^j)\Big)^2+2\sum_{k=1}^N\Big(s_k^{P_j}(1-b_k^{j\top}b_k^j)\Big)^2+4\sum_{k=1}^N\sum_{l=k+1}^N\Big((s_k^{P_j}s_l^{P_j})^{1/2}(-b_k^{j\top}b_l^j)\Big)^2$$

$$+4\sum_{k=1}^{N}\sum_{l=1}^{N}\left((s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^{j})\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)\left(s_l^{P_j}(1-b_l^{j\top}b_l^{j})\right)$$
$$+2\sum_{k=1}^{N}\sum_{l=k+1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)\left(s_l^{P_i}(1-b_l^{i\top}b_l^{i})\right)+2\sum_{k=1}^{N}\sum_{l=k+1}^{N}\left(s_k^{P_j}(1-b_k^{j\top}b_k^{j})\right)\left(s_l^{P_j}(1-b_l^{j\top}b_l^{j})\right)$$

$$=\sum_{k=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)^2+2\sum_{k=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)^2+2\sum_{k=1}^{N}\sum_{l=k+1}^{N}\left((s_k^{P_i}s_l^{P_i})^{1/2}(-b_k^{i\top}b_l^{i})\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left((s_k^{P_i}s_l^{P_i})^{1/2}(-b_k^{i\top}b_l^{i})\right)^2$$
$$+\sum_{k=1}^{N}\left(s_k^{P_j}(1-b_k^{j\top}b_k^{j})\right)^2+2\sum_{k=1}^{N}\left(s_k^{P_j}(1-b_k^{j\top}b_k^{j})\right)^2+2\sum_{k=1}^{N}\sum_{l=k+1}^{N}\left((s_k^{P_j}s_l^{P_j})^{1/2}(-b_k^{j\top}b_l^{j})\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left((s_k^{P_j}s_l^{P_j})^{1/2}(-b_k^{j\top}b_l^{j})\right)^2$$
$$+4\sum_{k=1}^{N}\sum_{l=1}^{N}\left((s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^{j})\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)\left(s_l^{P_j}(1-b_l^{j\top}b_l^{j})\right)$$
$$+\sum_{k=1}^{N}\sum_{l=k+1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)\left(s_l^{P_i}(1-b_l^{i\top}b_l^{i})\right)+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)\left(s_l^{P_i}(1-b_l^{i\top}b_l^{i})\right)$$
$$+\sum_{k=1}^{N}\sum_{l=k+1}^{N}\left(s_k^{P_j}(1-b_k^{j\top}b_k^{j})\right)\left(s_l^{P_j}(1-b_l^{j\top}b_l^{j})\right)+\sum_{k=1}^{N}\sum_{l=1}^{k-1}\left(s_k^{P_j}(1-b_k^{j\top}b_k^{j})\right)\left(s_l^{P_j}(1-b_l^{j\top}b_l^{j})\right)$$

$$=2\sum_{k=1}^{N}\sum_{l=1}^{N}\left((s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^{i})\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{N}\left((s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^{j})\right)^2+4\sum_{k=1}^{N}\sum_{l=1}^{N}\left((s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^{j})\right)^2$$
$$+2\sum_{k=1}^{N}\sum_{l=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)\left(s_l^{P_j}(1-b_l^{j\top}b_l^{j})\right)+\sum_{k=1}^{N}\sum_{l=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)\left(s_l^{P_i}(1-b_l^{i\top}b_l^{i})\right)+\sum_{k=1}^{N}\sum_{l=1}^{N}\left(s_k^{P_j}(1-b_k^{j\top}b_k^{j})\right)\left(s_l^{P_j}(1-b_l^{j\top}b_l^{j})\right)$$

$$=2\sum_{k=1}^{N}\sum_{l=1}^{N}s_k^{P_i}s_l^{P_i}\left(\delta_{kl}-b_k^{i\top}b_l^{i}\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{N}s_k^{P_j}s_l^{P_j}\left(\delta_{kl}-b_k^{j\top}b_l^{j}\right)^2+4\sum_{k=1}^{N}\sum_{l=1}^{N}s_k^{P_i}s_l^{P_j}\left(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^{j}\right)^2$$
$$+\left(\sum_{k=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^{i})\right)+\sum_{l=1}^{N}\left(s_l^{P_j}(1-b_l^{j\top}b_l^{j})\right)\right)^2$$

2.2 Stress Term Evaluation: (T2)

To evaluate the second stress term (T2), we use the same strategy as for the first stress term (T1) in the previous subsection. This means that the integrand is rearranged with respect to the direction v_1 and (T2) is integrated into this direction:

$$\begin{aligned}
&\left(\left\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\right\rangle\right)^2 \qquad (12)\\
=&\left(\left\langle\sum_{k=1}^{N}\sqrt{s_k^{P_i}}v_ku_k^{P_i}-\sqrt{s_k^{P_j}}w_ku_k^{P_j},\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle\sum_{k=1}^{N}\sqrt{s_k^{P_i}}v_kb_k^{i}-\sqrt{s_k^{P_j}}w_kb_k^{j},c^i-c^j\right\rangle\right)^2\\
=&\left(\sum_{k=1}^{N}\sum_{l=1}^{N}\left(\sqrt{s_k^{P_i}}v_ku_{k,l}^{P_i}-\sqrt{s_k^{P_j}}w_ku_{k,l}^{P_j}\right)\cdot(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{n}\left(\sqrt{s_k^{P_i}}v_kb_{k,l}^{i}-\sqrt{s_k^{P_j}}w_kb_{k,l}^{j}\right)\cdot(c_l^i-c_l^j)\right)^2\\
=&\left(\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_i}}v_ku_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_j}}w_ku_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_i}}v_kb_{k,l}^{i}(c_l^i-c_l^j)+\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_j}}w_kb_{k,l}^{j}(c_l^i-c_l^j)\right)^2\\
=&\left(\sum_{l=1}^{N}\sqrt{s_1^{P_i}}v_1u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{l=1}^{n}\sqrt{s_1^{P_i}}v_1b_{1,l}^{i}(c_l^i-c_l^j)\right.\\
&\left.+\sum_{k=2}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_i}}v_ku_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_j}}w_ku_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_i}}v_kb_{k,l}^{i}(c_l^i-c_l^j)+\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_j}}w_kb_{k,l}^{j}(c_l^i-c_l^j)\right)^2
\end{aligned}$$

Now, we insert Equation 12 into the stress term (T2) and integrate along the direction v_1 using the first identity (Equation 3):

$$\begin{aligned}
&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\left(\left\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\right\rangle\right)^2dv\,dw\\
=&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}e^{-\frac{1}{2}(v_1^2+\sum_{k=2}^{N}v_k^2+\sum_{l=1}^{N}w_l^2)}\cdot\left(\sum_{l=1}^{N}\sqrt{s_1^{P_i}}v_1u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{l=1}^{n}\sqrt{s_1^{P_i}}v_1b_{1,l}^{i}(c_l^i-c_l^j)\right.
\end{aligned}$$

$$+\sum_{k=2}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_i}}v_k u_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_j}}w_k u_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_i}}v_k b_{k,l}^{i}(c_l^{i}-c_l^{j})+\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_j}}w_k b_{k,l}^{j}(c_l^{i}-c_l^{j})\Bigg)^2$$

$$=\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-1}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}}e^{-\frac{1}{2}(\sum_{k=2}^{N}v_k^2+\sum_{l=1}^{N}w_l^2)}\cdot\Bigg(\Big(\sum_{l=1}^{N}\sqrt{s_1^{P_i}}u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{l=1}^{n}\sqrt{s_1^{P_i}}b_{1,l}^{i}(c_l^{i}-c_l^{j})\Big)^2$$

$$+\Big(\sum_{k=2}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_i}}v_k u_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_j}}w_k u_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_i}}v_k b_{k,l}^{i}(c_l^{i}-c_l^{j})+\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_j}}w_k b_{k,l}^{j}(c_l^{i}-c_l^{j})\Big)^2\Bigg)$$

$$=\Big(\sum_{l=1}^{N}\sqrt{s_1^{P_i}}u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{l=1}^{n}\sqrt{s_1^{P_i}}b_{1,l}^{i}(c_l^{i}-c_l^{j})\Big)^2+\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-1}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}}e^{-\frac{1}{2}(\sum_{k=2}^{N}v_k^2+\sum_{l=1}^{N}w_l^2)}\cdot\Bigg($$

$$\sum_{k=2}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_i}}v_k u_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_j}}w_k u_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_i}}v_k b_{k,l}^{i}(c_l^{i}-c_l^{j})+\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_j}}w_k b_{k,l}^{j}(c_l^{i}-c_l^{j})\Bigg)^2.$$

Again, we observe a recursive pattern. Therefore, we can solve the stress term (T2) inductively, which leads to the following final result:

$$\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\Big(\Big\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\Big\rangle-\Big\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\Big\rangle\Big)^2\,dv\,dw$$

$$=\sum_{k=1}^{N}\Big(\sum_{l=1}^{N}\sqrt{s_k^{P_i}}u_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{l=1}^{n}\sqrt{s_k^{P_i}}b_{k,l}^{i}(c_l^{i}-c_l^{j})\Big)^2+\sum_{k=1}^{N}\Big(\sum_{l=1}^{N}\sqrt{s_k^{P_j}}u_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{l=1}^{n}\sqrt{s_k^{P_j}}b_{k,l}^{j}(c_l^{i}-c_l^{j})\Big)^2$$

$$=\sum_{k=1}^{N}\Big(\Big\langle\sqrt{s_k^{P_i}}u_k^{P_i},\mu^{P_i}-\mu^{P_j}\Big\rangle-\Big\langle\sqrt{s_k^{P_i}}b_k^{i},c^i-c^j\Big\rangle\Big)^2+\sum_{k=1}^{N}\Big(\Big\langle\sqrt{s_k^{P_j}}u_k^{P_j},\mu^{P_i}-\mu^{P_j}\Big\rangle-\Big\langle\sqrt{s_k^{P_j}}b_k^{j},c^i-c^j\Big\rangle\Big)^2$$

$$=\sum_{k=1}^{N}s_k^{P_i}\Big(\Big\langle u_k^{P_i},\mu^{P_i}-\mu^{P_j}\Big\rangle-\Big\langle b_k^{i},c^i-c^j\Big\rangle\Big)^2+\sum_{k=1}^{N}s_k^{P_j}\Big(\Big\langle u_k^{P_j},\mu^{P_i}-\mu^{P_j}\Big\rangle-\Big\langle b_k^{j},c^i-c^j\Big\rangle\Big)^2$$

2.3 Stress Term Evaluation: (T3)

The third stress term (T3) can be easily solved due to the linearity of the integral, leading to the following result:

$$\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot\Big(\Big\|\mu^{P_i}-\mu^{P_j}\Big\|_2^2-\Big\|c^i-c^j\Big\|_2^2\Big)^2\,dv\,dw=\Big(\Big\|\mu^{P_i}-\mu^{P_j}\Big\|_2^2-\Big\|c^i-c^j\Big\|_2^2\Big)^2$$

2.4 Stress Term Evaluation: (T4)

To evaluate the fourth stress term (T4), we use partial results of the stress terms (T1) and (T2) as (T4) is a combination of these. Again, for the evaluation the same strategy is applied, i.e., we emphasize the direction v_1 by using Equation 5 and Equation 12 and integrate into this direction:

$$2\Big(\Big\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\Big\|_2^2-\Big\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\Big\|_2^2\Big)\Big(\Big\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\Big\rangle-\Big\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\Big\rangle\Big)$$

$$=2\Bigg(\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^{i})v_kv_l+\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^{j})w_kw_l-2\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^{j})v_kw_l\Bigg)$$

$$\cdot\Bigg(\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_i}}v_k u_{k,l}^{P_i}\cdot(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_j}}w_k u_{k,l}^{P_j}\cdot(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_i}}v_k b_{k,l}^{i}\cdot(c_l^{i}-c_l^{j})+\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_j}}w_k b_{k,l}^{j}\cdot(c_l^{i}-c_l^{j})\Bigg)$$

$$=2\Bigg(\Big[s_1^{P_i}(1-b_1^{i\top}b_1^{i})\Big]v_1^2+\Big[2\cdot\sum_{l=2}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\top}b_l^{i})v_l-2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\top}u_l^{P_j}-b_1^{i\top}b_l^{j})w_l\Big]v_1$$

$$+\sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\top}b_l^{i})v_kv_l+\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\top}b_l^{j})w_kw_l-2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^{j})v_kw_l\Bigg)$$

$$\cdot\Bigg(\Big[\sum_{l=1}^{N}\sqrt{s_1^{P_i}}u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})+\sum_{l=1}^{n}\sqrt{s_1^{P_i}}b_{1,l}^{i}(c_l^{i}-c_l^{j})\Big]v_1$$

$$+\sum_{k=2}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_i}}v_k u_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_j}}w_k u_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_i}}v_k b_{k,l}^{i}(c_l^{i}-c_l^{j})+\sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_j}}w_k b_{k,l}^{j}(c_l^{i}-c_l^{j})\Bigg)$$

Before we integrate, we consider the following: since the first bracket has a quadratic structure "$av_1^2+bv_1+c$" and the second one a linear structure "av_1+b", the multiplication leads to a cubic structure "$av_1^3+bv_1^2+cv_1+d$". According to the second identity (Equation 4), the integration along v_1 will eliminate the terms with an odd exponent (i.e., "av_1^3" and "bv_1" will vanish). With this consideration, we obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot 2\cdot\Big(\left\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\right\|_2^2-\left\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\right\|_2^2\Big)\\
&\cdot\Big(\left\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\right\rangle^2\Big)\,dv_1\cdots dv_N\cdot dw\\
=&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}(v_1^2+\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot 2\cdot\Bigg(\Big[s_1^{P_i}(1-b_1^{i\,\top}b_1^i)\Big]v_1^2+\Big[2\cdot\sum_{l=2}^N(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\,\top}b_l^i)v_l-2\sum_{l=1}^N(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\,\top}u_l^{P_j}-b_1^{i\,\top}b_l^j)w_l\Big]v_1\\
&+\sum_{k=2}^N\sum_{l=2}^N(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\,\top}b_l^i)v_kv_l+\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\,\top}b_l^j)w_kw_l-2\sum_{k=2}^N\sum_{l=1}^N(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\,\top}u_l^{P_j}-b_k^{i\,\top}b_l^j)v_kw_l\Bigg)\\
&\cdot\Bigg(\Big[\sum_{l=1}^N\sqrt{s_1^{P_i}}u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})+\sum_{l=1}^n\sqrt{s_1^{P_i}}b_{1,l}^i(c_l^i-c_l^j)\Big]v_1\\
&+\sum_{k=2}^N\sum_{l=1}^N\sqrt{s_k^{P_i}}v_ku_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^N\sum_{l=1}^N\sqrt{s_k^{P_j}}w_ku_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^N\sum_{l=1}^n\sqrt{s_k^{P_i}}v_kb_{k,l}^i(c_l^i-c_l^j)+\sum_{k=1}^N\sum_{l=1}^n\sqrt{s_k^{P_j}}w_kb_{k,l}^j(c_l^i-c_l^j)\Bigg)\,dv_1\cdots dv_N\cdot dw\\
=&\frac{1}{(\sqrt{2\pi})^N}\frac{1}{(\sqrt{2\pi})^{N-1}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}} e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot 2\cdot\Bigg(\\
&\Big[s_1^{P_i}(1-b_1^{i\,\top}b_1^i)\Big]\cdot\Big[\sum_{k=2}^N\sum_{l=1}^N\sqrt{s_k^{P_i}}v_ku_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^N\sum_{l=1}^N\sqrt{s_k^{P_j}}w_ku_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^N\sum_{l=1}^n\sqrt{s_k^{P_i}}v_kb_{k,l}^i(c_l^i-c_l^j)+\sum_{k=1}^N\sum_{l=1}^n\sqrt{s_k^{P_j}}w_kb_{k,l}^j(c_l^i-c_l^j)\Big]\\
&+\Big[2\cdot\sum_{l=2}^N(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\,\top}b_l^i)v_l-2\sum_{l=1}^N(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\,\top}u_l^{P_j}-b_1^{i\,\top}b_l^j)w_l\Big]\cdot\Big[\sum_{l=1}^N\sqrt{s_1^{P_i}}u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})+\sum_{l=1}^n\sqrt{s_1^{P_i}}b_{1,l}^i(c_l^i-c_l^j)\Big]\\
&+\Big[\sum_{k=2}^N\sum_{l=2}^N(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl}-b_k^{i\,\top}b_l^i)v_kv_l+\sum_{k=1}^N\sum_{l=1}^N(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl}-b_k^{j\,\top}b_l^j)w_kw_l-2\sum_{k=2}^N\sum_{l=1}^N(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\,\top}u_l^{P_j}-b_k^{i\,\top}b_l^j)v_kw_l\Big]\\
&\cdot\Big[\sum_{k=2}^N\sum_{l=1}^N\sqrt{s_k^{P_i}}v_ku_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^N\sum_{l=1}^N\sqrt{s_k^{P_j}}w_ku_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^N\sum_{l=1}^n\sqrt{s_k^{P_i}}v_kb_{k,l}^i(c_l^i-c_l^j)+\sum_{k=1}^N\sum_{l=1}^n\sqrt{s_k^{P_j}}w_kb_{k,l}^j(c_l^i-c_l^j)\Big]\Bigg)\,dv_2\cdots dv_N\cdot dw
\end{aligned}
$$

We see that the integrand consists of three terms (not counting the exponential function). The last term belongs to the recursive representation, i.e., the integral can be solved inductively. Before we do this, we evaluate the other two terms: Both terms can be solved by using Equation 4 via

$$
\begin{aligned}
&\frac{1}{(\sqrt{2\pi})^N}\frac{1}{(\sqrt{2\pi})^{N-1}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}} e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot 2\cdot\Big[s_1^{P_i}(1-b_1^{i\,\top}b_1^i)\Big]\\
&\cdot\Big[\sum_{k=2}^N\sum_{l=1}^N\sqrt{s_k^{P_i}}v_ku_{k,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=1}^N\sum_{l=1}^N\sqrt{s_k^{P_j}}w_ku_{k,l}^{P_j}(\mu_l^{P_i}-\mu_l^{P_j})-\sum_{k=2}^N\sum_{l=1}^n\sqrt{s_k^{P_i}}v_kb_{k,l}^i(c_l^i-c_l^j)+\sum_{k=1}^N\sum_{l=1}^n\sqrt{s_k^{P_j}}w_kb_{k,l}^j(c_l^i-c_l^j)\Big]\,dv_2\cdots dv_N\cdot dw\\
&=0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(\sqrt{2\pi})^N}\frac{1}{(\sqrt{2\pi})^{N-1}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}} e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot 2\cdot\Big[2\cdot\sum_{l=2}^N(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\,\top}b_l^i)v_l-2\sum_{l=1}^N(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\,\top}u_l^{P_j}-b_1^{i\,\top}b_l^j)w_l\Big]\cdot\\
&\Big[\sum_{l=1}^N\sqrt{s_1^{P_i}}u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})+\sum_{l=1}^n\sqrt{s_1^{P_i}}b_{1,l}^i(c_l^i-c_l^j)\Big]\,dv_2\cdots dv_N\cdot dw\\
&=2\cdot\Big[\sum_{l=1}^N\sqrt{s_1^{P_i}}u_{1,l}^{P_i}(\mu_l^{P_i}-\mu_l^{P_j})+\sum_{l=1}^n\sqrt{s_1^{P_i}}b_{1,l}^i(c_l^i-c_l^j)\Big]\cdot\frac{1}{(\sqrt{2\pi})^N}\frac{1}{(\sqrt{2\pi})^{N-1}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}} e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2+\sum_{l=1}^N w_l^2)}\cdot\\
&\Big[2\cdot\sum_{l=2}^N(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\,\top}b_l^i)v_l-2\sum_{l=1}^N(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\,\top}u_l^{P_j}-b_1^{i\,\top}b_l^j)w_l\Big]\,dv_2\cdots dv_N\cdot dw\\
&=0.
\end{aligned}
$$

We observe that both terms are zero. Therefore, solving the fourth stress term (T4) inductively leads to the following final result:

$$
\begin{aligned}
&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot 2\cdot\Big(\left\|U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w\right\|_2^2-\left\|B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w\right\|_2^2\Big)\\
&\cdot\Big(\left\langle U^{P_i}S^{P_i\,1/2}v-U^{P_j}S^{P_j\,1/2}w,\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle B^iS^{P_i\,1/2}v-B^jS^{P_j\,1/2}w,c^i-c^j\right\rangle^2\Big)\,dv\,dw\\
&=0
\end{aligned}
$$

2.5 Stress Term Evaluation: (T5)

To evaluate the fifth stress term (T5), we use a partial result (Equation 5) of the stress term (T1) to obtain a representation that emphasizes v_1:

$$\begin{aligned}
&\left(\left\|U^{P_i}S^{P_i\,1/2}v - U^{P_j}S^{P_j\,1/2}w\right\|_2^2 - \left\|B_i S^{P_i\,1/2}v - B_j S^{P_j\,1/2}w\right\|_2^2\right) \\
&= \sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl} - b_k^{i\,\top}b_l^i)v_k v_l + \sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl} - b_k^{j\,\top}b_l^j)w_k w_l - 2\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\,\top}u_l^{P_j} - b_k^{i\,\top}b_l^j)v_k w_l \\
&= s_1^{P_i}(1 - b_1^{i\,\top}b_1^i)v_1^2 + 2\sum_{l=2}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\,\top}b_l^i)v_1 v_l - 2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\,\top}u_l^{P_j} - b_1^{i\,\top}b_l^j)v_1 w_l \\
&+ \sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl} - b_k^{i\,\top}b_l^i)v_k v_l + \sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl} - b_k^{j\,\top}b_l^j)w_k w_l - 2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\,\top}u_l^{P_j} - b_k^{i\,\top}b_l^j)v_k w_l
\end{aligned} \tag{13}$$

Now, we insert Equation 13 into the stress term (T5) and integrate along the direction v_1 (and the others) using the second identity (Equation 4):

$$\begin{aligned}
&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot 2\cdot\left(\left\|U^{P_i}S^{P_i\,1/2}v - U^{P_j}S^{P_j\,1/2}w\right\|_2^2 - \left\|B^i S^{P_i\,1/2}v - B^j S^{P_j\,1/2}w\right\|_2^2\right)\left(\left\|\mu^{P_i} - \mu^{P_j}\right\|_2^2 - \left\|c^i - c^j\right\|_2^2\right)dv\,dw \\
&= 2\left(\left\|\mu^{P_i} - \mu^{P_j}\right\|_2^2 - \left\|c^i - c^j\right\|_2^2\right)\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}(v_1^2 + \sum_{k=2}^N v_k^2 + \sum_{l=1}^N w_l^2)}\cdot\Bigg(\\
&s_1^{P_i}(1 - b_1^{i\,\top}b_1^i)v_1^2 + 2\sum_{l=2}^{N}(s_1^{P_i}s_l^{P_i})^{1/2}(-b_1^{i\,\top}b_l^i)v_1 v_l - 2\sum_{l=1}^{N}(s_1^{P_i}s_l^{P_j})^{1/2}(u_1^{P_i\,\top}u_l^{P_j} - b_1^{i\,\top}b_l^j)v_1 w_l \\
&+ \sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl} - b_k^{i\,\top}b_l^i)v_k v_l + \sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl} - b_k^{j\,\top}b_l^j)w_k w_l - 2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\,\top}u_l^{P_j} - b_k^{i\,\top}b_l^j)v_k w_l\Bigg)\,dv_1\cdots dv_N\cdot dw \\
&= 2\left(\left\|\mu^{P_i} - \mu^{P_j}\right\|_2^2 - \left\|c^i - c^j\right\|_2^2\right)\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^{N-1}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^{N-1}} e^{-\frac{1}{2}(\sum_{k=2}^N v_k^2 + \sum_{l=1}^N w_l^2)}\cdot\Bigg(s_1^{P_i}(1 - b_1^{i\,\top}b_1^i) \\
&+ \sum_{k=2}^{N}\sum_{l=2}^{N}(s_k^{P_i}s_l^{P_i})^{1/2}(\delta_{kl} - b_k^{i\,\top}b_l^i)v_k v_l + \sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j})^{1/2}(\delta_{kl} - b_k^{j\,\top}b_l^j)w_k w_l - 2\sum_{k=2}^{N}\sum_{l=1}^{N}(s_k^{P_i}s_l^{P_j})^{1/2}(u_k^{P_i\,\top}u_l^{P_j} - b_k^{i\,\top}b_l^j)v_k w_l\Bigg)\,dv_2\cdots dv_N\cdot dw \\
&= \ldots \\
&= 2\left(\left\|\mu^{P_i} - \mu^{P_j}\right\|_2^2 - \left\|c^i - c^j\right\|_2^2\right)\left(\sum_{k=1}^{N}s_k^{P_i}(1 - b_k^{i\,\top}b_k^i) + \sum_{l=1}^{N}s_l^{P_j}(1 - b_l^{j\,\top}b_l^j)\right)
\end{aligned}$$

2.6 Stress Term Evaluation: (T6)

To evaluate the stress term (T6), we use the partial result Equation 12 of the second stress term (T2) as well as the second identity (Equation 4):

$$\begin{aligned}
&\frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}v^\top v}e^{-\frac{1}{2}w^\top w}\cdot 2\cdot\left(\left\langle U^{P_i}S^{P_i\,1/2}v - U^{P_j}S^{P_j\,1/2}w, \mu^{P_i} - \mu^{P_j}\right\rangle - \left\langle B^i S^{P_i\,1/2}v - B^j S^{P_j\,1/2}w, c^i - c^j\right\rangle\right) \\
&\qquad\left(\left\|\mu^{P_i} - \mu^{P_j}\right\|_2^2 - \left\|c^i - c^j\right\|_2^2\right)dv\,dw \\
&= \frac{1}{\sqrt{(2\pi)^N}}\frac{1}{\sqrt{(2\pi)^N}}\left(\left\|\mu^{P_i} - \mu^{P_j}\right\|_2^2 - \left\|c^i - c^j\right\|_2^2\right)\cdot 2\cdot\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\sum_{k=1}^N v_k^2 + \sum_{l=1}^N w_l^2)}\cdot\Bigg(\\
&\sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_i}}v_k u_{k,l}^{P_i}(\mu_l^{P_i} - \mu_l^{P_j}) - \sum_{k=1}^{N}\sum_{l=1}^{N}\sqrt{s_k^{P_j}}w_k u_{k,l}^{P_j}(\mu_l^{P_i} - \mu_l^{P_j}) - \sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_i}}v_k b_{k,l}^i(c_l^i - c_l^j) + \sum_{k=1}^{N}\sum_{l=1}^{n}\sqrt{s_k^{P_j}}w_k b_{k,l}^j(c_l^i - c_l^j)\Bigg)\,dv\,dw \\
&= 0.
\end{aligned}$$

3 Normal Distribution Model: Composition of Stress

After evaluating the stress terms (T1)–(T6) in the previous section, we can formulate the final stress. In general, the stress $\tilde{\mathcal{S}}$ is a function that depends on the local projections $\Phi_1, \ldots, \Phi_K$ (see Equation 1). Since the projections are characterized by the components $c = (c^1, \ldots, c^K)$ and $B = (B^1, \ldots, B^K)$, we can define the stress equivalently as a function of these. The resulting stress $\mathcal{S} : \mathbb{R}^N \times \cdots \times \mathbb{R}^N \times \mathbb{R}^{N\times N} \times \cdots \times \mathbb{R}^{N\times N} \to \mathbb{R}_{\geq 0}$ can be formulated in two different ways: a notation that is based on the evaluations in the previous sections (scalar notation) or a notation that highlights the matrices and vectors (matrix notation). Both formulations are presented in the following subsections.

3.1 Stress Formulation: Scalar Notation

To compose the stress $\mathcal{S}$, we merge the stress terms (T1)–(T6) into the following formulation:

$$
\begin{aligned}
&\mathcal{S}(c^1,\dots,c^K,B^1,\dots,B^K) \qquad (14)\\
&=\sum_{i,j}\left[2\sum_{k=1}^{N}\sum_{l=1}^{N} s_k^{P_i}s_l^{P_i}\left(\delta_{kl}-b_k^{i\top}b_l^i\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j}\left(\delta_{kl}-b_k^{j\top}b_l^j\right)^2+4\sum_{k=1}^{N}\sum_{l=1}^{N} s_k^{P_i}s_l^{P_j}\left(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j\right)^2\right.\\
&+\left(\sum_{k=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^i)\right)+\sum_{l=1}^{N}\left(s_l^{P_j}(1-b_l^{j\top}b_l^j)\right)\right)^2+\sum_{k=1}^{N}s_k^{P_i}\left(\left\langle u_k^{P_i},\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle b_k^i,c^i-c^j\right\rangle\right)^2+\sum_{k=1}^{N}s_k^{P_j}\left(\left\langle u_k^{P_j},\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle b_k^j,c^i-c^j\right\rangle\right)^2\\
&\left.+\left(\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right)^2+2\left(\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right)\left(\sum_{k=1}^{N}s_k^{P_i}(1-b_k^{i\top}b_k^i)+\sum_{l=1}^{N}s_l^{P_j}(1-b_l^{j\top}b_l^j)\right)\right]\\
&=\sum_{i,j}\left[2\sum_{k=1}^{N}\sum_{l=1}^{N} s_k^{P_i}s_l^{P_i}\left(\delta_{kl}-b_k^{i\top}b_l^i\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j}\left(\delta_{kl}-b_k^{j\top}b_l^j\right)^2+4\sum_{k=1}^{N}\sum_{l=1}^{N} s_k^{P_i}s_l^{P_j}\left(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j\right)^2\right.\\
&+\sum_{k=1}^{N}s_k^{P_i}\left(\left\langle u_k^{P_i},\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle b_k^i,c^i-c^j\right\rangle\right)^2+\sum_{k=1}^{N}s_k^{P_j}\left(\left\langle u_k^{P_j},\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle b_k^j,c^i-c^j\right\rangle\right)^2\\
&\left.+\left(\left[\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right]+\sum_{k=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^i)\right)+\sum_{l=1}^{N}\left(s_l^{P_j}(1-b_l^{j\top}b_l^j)\right)\right)^2\right]
\end{aligned}
$$

We observe that the resulting stress $\mathcal{S}$ consists of three different components, which we will denote as stress components. An interpretation of these is provided in the main document. Since the stress components mainly highlight scalar operations, our next step is to provide a matrix notation that has numerical and algorithmic advantages.

3.2 Stress Formulation: Matrix Notation

To obtain a matrix notation of our derived stress $\mathcal{S}$, we have to rearrange the three stress components. For the first stress component, the matrix notation is given by

$$
\begin{aligned}
&2\sum_{k=1}^{N}\sum_{l=1}^{N} s_k^{P_i}s_l^{P_i}\left(\delta_{kl}-b_k^{i\top}b_l^i\right)^2+2\sum_{k=1}^{N}\sum_{l=1}^{N}(s_k^{P_j}s_l^{P_j}\left(\delta_{kl}-b_k^{j\top}b_l^j\right)^2+4\sum_{k=1}^{N}\sum_{l=1}^{N} s_k^{P_i}s_l^{P_j}\left(u_k^{P_i\top}u_l^{P_j}-b_k^{i\top}b_l^j\right)^2\\
&=2\cdot\left\|S^{P_i}-S^{P_i\,1/2}B^{i\top}B^iS^{P_i\,1/2}\right\|_F^2+2\cdot\left\|S^{P_j}-S^{P_j\,1/2}B^{j\top}B^jS^{P_j\,1/2}\right\|_F^2+4\cdot\left\|S^{P_i\,1/2}U^{P_i\top}U^{P_j}S^{P_j\,1/2}-S^{P_i\,1/2}B^{i\top}B^jS^{P_j\,1/2}\right\|_F^2,
\end{aligned}
$$

and, analogously, for the second stress component

$$
\begin{aligned}
&\sum_{k=1}^{N}s_k^{P_i}\left(\left\langle u_k^{P_i},\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle b_k^i,c^i-c^j\right\rangle\right)^2+\sum_{k=1}^{N}s_k^{P_j}\left(\left\langle u_k^{P_j},\mu^{P_i}-\mu^{P_j}\right\rangle-\left\langle b_k^j,c^i-c^j\right\rangle\right)^2\\
&=\left\|S^{P_i\,1/2}U^{P_i\top}(\mu^{P_i}-\mu^{P_j})-S^{P_i\,1/2}B^{i\top}(c^i-c^j)\right\|_2^2+\left\|S^{P_j\,1/2}U^{P_j\top}(\mu^{P_i}-\mu^{P_j})-S^{P_j\,1/2}B^{j\top}(c^i-c^j)\right\|_2^2.
\end{aligned}
$$

The third stress component has a different matrix notation that makes use of the trace:

$$
\begin{aligned}
&\left(\left[\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right]+\sum_{k=1}^{N}\left(s_k^{P_i}(1-b_k^{i\top}b_k^i)\right)+\sum_{l=1}^{N}\left(s_l^{P_j}(1-b_l^{j\top}b_l^j)\right)\right)^2\\
&=\left(\left[\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right]+\left[\operatorname{trace}(S^{P_i})-\operatorname{trace}(B^iS^{P_i}B^{i\top})+\operatorname{trace}(S^{P_j})-\operatorname{trace}(B^jS^{P_j}B^{j\top})\right]\right)^2.
\end{aligned}
$$

Hence, the final formulation in matrix notation is given by

$$
\begin{aligned}
&\mathcal{S}(c^1,\dots,c^K,B^1,\dots,B^K) \qquad (15)\\
&=2\cdot\left\|S^{P_i}-S^{P_i\,1/2}B^{i\top}B^iS^{P_i\,1/2}\right\|_F^2+2\cdot\left\|S^{P_j}-S^{P_j\,1/2}B^{j\top}B^jS^{P_j\,1/2}\right\|_F^2+4\cdot\left\|S^{P_i\,1/2}U^{P_i\top}U^{P_j}S^{P_j\,1/2}-S^{P_i\,1/2}B^{i\top}B^jS^{P_j\,1/2}\right\|_F^2\\
&+\left\|S^{P_i\,1/2}U^{P_i\top}(\mu^{P_i}-\mu^{P_j})-S^{P_i\,1/2}B^{i\top}(c^i-c^j)\right\|_2^2+\left\|S^{P_j\,1/2}U^{P_j\top}(\mu^{P_i}-\mu^{P_j})-S^{P_j\,1/2}B^{j\top}(c^i-c^j)\right\|_2^2\\
&+\left(\left[\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right]+\left[\operatorname{trace}(S^{P_i})-\operatorname{trace}(B^iS^{P_i}B^{i\top})+\operatorname{trace}(S^{P_j})-\operatorname{trace}(B^jS^{P_j}B^{j\top})\right]\right)^2.
\end{aligned}
$$

4 Normal Distribution Model: Derivative of Stress

For the derivative of the stress, we use the matrix notation, which facilitates the computation. Therefore, we can differentiate with respect to the variables c^k and B^k. For the sake of transparency, we split the formulation into the three stress components. For the first stress component, we obtain the derivatives via well-known Frobenius norm rules

$$\frac{\partial}{\partial c^k}\sum_{i,j}\left[2\cdot\left\|S^{P_i}-S^{P_i\,1/2}B^{i\top}B^iS^{P_i\,1/2}\right\|_F^2+2\cdot\left\|S^{P_j}-S^{P_j\,1/2}B^{j\top}B^jS^{P_j\,1/2}\right\|_F^2+4\cdot\left\|S^{P_i\,1/2}U^{P_i\top}U^{P_j}S^{P_j\,1/2}-S^{P_i\,1/2}B^{i\top}B^jS^{P_j\,1/2}\right\|_F^2\right]=0,$$

$$\begin{aligned}
&\frac{\partial}{\partial B^k}\sum_{i,j}\left[2\cdot\left\|S^{P_i}-S^{P_i\,1/2}B^{i\top}B^iS^{P_i\,1/2}\right\|_F^2+2\cdot\left\|S^{P_j}-S^{P_j\,1/2}B^{j\top}B^jS^{P_j\,1/2}\right\|_F^2+4\cdot\left\|S^{P_i\,1/2}U^{P_i\top}U^{P_j}S^{P_j\,1/2}-S^{P_i\,1/2}B^{i\top}B^jS^{P_j\,1/2}\right\|_F^2\right]\\
&=\left[8\cdot B^kS^{P_k}B^{k\top}B^kS^{P_k}-8B^kS^{P_k\,2}\right]+\left[8\cdot B^kS^{P_k}B^{k\top}B^kS^{P_k}-8B^kS^{P_k\,2}\right]+\left[16\cdot B^kS^{P_k}B^{k\top}B^kS^{P_k}-16B^kS^{P_k\,2}\right]\\
&+\sum_{l\neq k}^{N}\left[16\cdot B^kS^{P_k}B^{k\top}B^lS^{P_l}-16B^lS^{P_l}U^{l\top}U^kS^{P_k}\right]\\
&=\left[16\cdot B^kS^{P_k}B^{k\top}B^kS^{P_k}-16B^kS^{P_k\,2}\right]+\sum_{l=1}^{N}\left[16\cdot B^kS^{P_k}B^{k\top}B^lS^{P_l}-16B^lS^{P_l}U^{l\top}U^kS^{P_k}\right].
\end{aligned}$$

Analogously, we obtain the derivative for the second stress component as follows:

$$\begin{aligned}
&\frac{\partial}{\partial c^k}\sum_{i,j}\left[\left\|S^{P_i\,1/2}U^{P_i\top}(\mu^{P_i}-\mu^{P_j})-S^{P_i\,1/2}B^{i\top}(c^i-c^j)\right\|_2^2+\left\|S^{P_j\,1/2}U^{P_j\top}(\mu^{P_i}-\mu^{P_j})-S^{P_j\,1/2}B^{j\top}(c^i-c^j)\right\|_2^2\right]\\
&=\sum_{l\neq k}^{N}-2B^kS^{P_k\,1/2}\left(U^{P_k\top}(\mu^{P_k}-\mu^{P_l})-B^{k\top}(c^k-c^l)\right)-2B^lS^{P_l\,1/2}\left(U^{P_l\top}(\mu^{P_l}-\mu^{P_k})-B^{l\top}(c^l-c^k)\right)\\
&-\sum_{l\neq k}^{N}-2B^lS^{P_l\,1/2}\left(U^{P_l\top}(\mu^{P_l}-\mu^{P_k})-B^{l\top}(c^l-c^k)\right)-2B^kS^{P_k\,1/2}\left(U^{P_k\top}(\mu^{P_k}-\mu^{P_l})-B^{k\top}(c^k-c^l)\right)=\ 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial B^k}\sum_{i,j}\left[\left\|S^{P_i\,1/2}U^{P_i\top}(\mu^{P_i}-\mu^{P_j})-S^{P_i\,1/2}B^{i\top}(c^i-c^j)\right\|_2^2+\left\|S^{P_j\,1/2}U^{P_j\top}(\mu^{P_i}-\mu^{P_j})-S^{P_j\,1/2}B^{j\top}(c^i-c^j)\right\|_2^2\right]\\
&=\sum_{l\neq k}^{N}\left[2(c^k-c^l)(c^k-c^l)^\top B^kS^{P_k\,1/2}-2(c^k-c^l)(\mu^{P_k}-\mu^{P_l})^\top U^{P_k}S^{P_k\,1/2}\right]\\
&+\sum_{l\neq k}^{N}\left[2(c^l-c^k)(c^l-c^k)^\top B^kS^{P_k\,1/2}-2(c^l-c^k)(\mu^{P_l}-\mu^{P_k})^\top U^{P_k}S^{P_k\,1/2}\right]\\
&=\sum_{l\neq k}^{N}\left[4(c^k-c^l)(c^k-c^l)^\top B^kS^{P_k\,1/2}-4(c^k-c^l)(\mu^{P_k}-\mu^{P_l})^\top U^{P_k}S^{P_k\,1/2}\right].
\end{aligned}$$

The derivative of the third stress component can be calculated as follows:

$$\begin{aligned}
&\frac{\partial}{\partial c^k}\sum_{i,j}\left(\left[\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right]+\left[\mathrm{trace}(S^{P_i})-\mathrm{trace}(B^iS^{P_i}B^{i\top})+\mathrm{trace}(S^{P_j})-\mathrm{trace}(B^jS^{P_j}B^{j\top})\right]\right)^2\\
&=\sum_{l\neq k}^{N}\left(\left[\left\|\mu^{P_k}-\mu^{P_l}\right\|_2^2-\left\|c^k-c^l\right\|_2^2\right]+\left[\mathrm{trace}(S^{P_k})-\mathrm{trace}(B^kS^{P_k}B^{k\top})+\mathrm{trace}(S^{P_l})-\mathrm{trace}(B^lS^{P_l}B^{l\top})\right]\right)^2\cdot\left(-2(c^k-c^l)\right)\\
&-\sum_{l\neq k}^{N}\left(\left[\left\|\mu^{P_l}-\mu^{P_k}\right\|_2^2-\left\|c^l-c^k\right\|_2^2\right]+\left[\mathrm{trace}(S^{P_l})-\mathrm{trace}(B^lS^{P_l}B^{l\top})+\mathrm{trace}(S^{P_k})-\mathrm{trace}(B^kS^{P_k}B^{k\top})\right]\right)^2\cdot\left(-2(c^l-c^k)\right)\\
&=\sum_{l\neq k}^{N}\left(\left[\left\|\mu^{P_k}-\mu^{P_l}\right\|_2^2-\left\|c^k-c^l\right\|_2^2\right]+\left[\mathrm{trace}(S^{P_k})-\mathrm{trace}(B^kS^{P_k}B^{k\top})+\mathrm{trace}(S^{P_l})-\mathrm{trace}(B^lS^{P_l}B^{l\top})\right]\right)^2\cdot\left(-4(c^k-c^l)\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial B^k}\sum_{i,j}\left(\left[\left\|\mu^{P_i}-\mu^{P_j}\right\|_2^2-\left\|c^i-c^j\right\|_2^2\right]+\left[\mathrm{trace}(S^{P_i})-\mathrm{trace}(B^iS^{P_i}B^{i\top})+\mathrm{trace}(S^{P_j})-\mathrm{trace}(B^jS^{P_j}B^{j\top})\right]\right)^2\\
&=\sum_{l=1}^{N}\left(\left[\left\|\mu^{P_k}-\mu^{P_l}\right\|_2^2-\left\|c^k-c^l\right\|_2^2\right]+\left[\mathrm{trace}(S^{P_k})-\mathrm{trace}(B^kS^{P_k}B^{k\top})+\mathrm{trace}(S^{P_l})-\mathrm{trace}(B^lS^{P_l}B^{l\top})\right]\right)^2\cdot\left(-2B^kS^{P_k}\right)\\
&+\sum_{l=1}^{N}\left(\left[\left\|\mu^{P_l}-\mu^{P_k}\right\|_2^2-\left\|c^l-c^k\right\|_2^2\right]+\left[\mathrm{trace}(S^{P_l})-\mathrm{trace}(B^lS^{P_l}B^{l\top})+\mathrm{trace}(S^{P_k})-\mathrm{trace}(B^kS^{P_k}B^{k\top})\right]\right)^2\cdot\left(-2B^kS^{P_k}\right)\\
&=\sum_{l=1}^{N}\left(\left[\left\|\mu^{P_k}-\mu^{P_l}\right\|_2^2-\left\|c^k-c^l\right\|_2^2\right]+\left[\mathrm{trace}(S^{P_k})-\mathrm{trace}(B^kS^{P_k}B^{k\top})+\mathrm{trace}(S^{P_l})-\mathrm{trace}(B^lS^{P_l}B^{l\top})\right]\right)^2\cdot\left(-4B^kS^{P_k}\right).
\end{aligned}$$

5 ADDITIONAL VISUALIZATIONS

In this section, we provide additional visualizations that are not part of the main paper. Furthermore, figures that appeared to be small in the paper, are shown as full frame versions for better readability.

5.1 Student Grades Data Set

Fig. 1 shows UA-PCA [2] and UAMDS projections of the student grades data set with our second modeling of textual grade descriptions. It also shows the Shepard diagrams for covariance trace and anisotropy as well as the pairwise stress components for UA-PCA (middle row) and UAMDS (bottom row).

It can be seen that the trace of the covariances (cf. Subfigures (d) and (h)) is preserved better with UAMDS than with UA-PCA. The anisotropy preservation (cf. Subfigures (e) and (i)) is similar between UA-PCA and UAMDS, most points are close to the diagonal. The outliers, e.g., blue and brown that correspond to distributions of David and Jack, are represented more isotropic than in high-dimensional space. From the stress component scatter plots, it can be seen that UAMDS improves on the volume of Tom's projected distribution as indicated by the red dot in Subfigures (f) and (j). This means that the larger spread of the distribution in the UAMDS projection is more faithful to the data.

5.2 Pokemon Data Set

Fig. 2 is a copy of Figure 9 from the main paper but spanning over a larger area for better readability.
Fig. 3 shows UA-PCA and UAMDS projections of the first 151 Pokemon (generation 1) of the Pokemon stats data set. It also shows the Shepard diagrams for covariance trace and anisotropy as well as the pairwise stress components for UA-PCA (middle row) and UAMDS (bottom row). UA-PCA was used as initialization for UAMDS. In this way, the projections are easier to compare, since it makes UAMDS converge to a solution close to the UA-PCA projection and also eliminates possible rotated or reflected UAMDS variants.

It can be seen that covariance traces are all underestimated by UA-PCA. UAMDS manages to preserve some of the covariance traces. The anisotropy is overestimated by UAMDS for unusual (outlier) Pokemon (such as Chansey or Jigglypuff), and pairings with these. UA-PCA manages to preserve anisotropy better in this case, but has a tendency to underestimate. We believe that UA-PCA provides a more trustworthy projection in this case due to the nature of the data set. When looking at the data distributions in the first plot, it can be seen that variability is very similar for each Pokemon and each dimension. All covariances are rather isotropic, so that UAMDS' ability to project each distribution differently has less use. This is different for the student grades data set shown in Fig. 1, where distributions are more anisotropic.

5.3 MNIST Data Set

For the curious reader, we provide a UAMDS projection of the MNIST data set [1] consisting of images of written digits. While the different digit classes (0 to 9) are not normally distributed, we estimate their distribution as such nonetheless. The resulting UAMDS projection of the distributions and the individual data points are shown in Fig. 4. From the highlighted data points for class 0, 1, and 5, it can be seen that the projected data is not normally distributed, still we can get a sense of spread and density of the data. The shown similarity of the distribution barycenters are convincing, e.g., similarly shaped digits three, eight, and five are close.

REFERENCES

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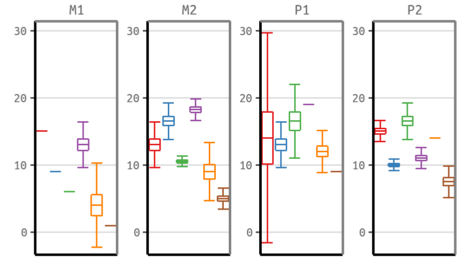
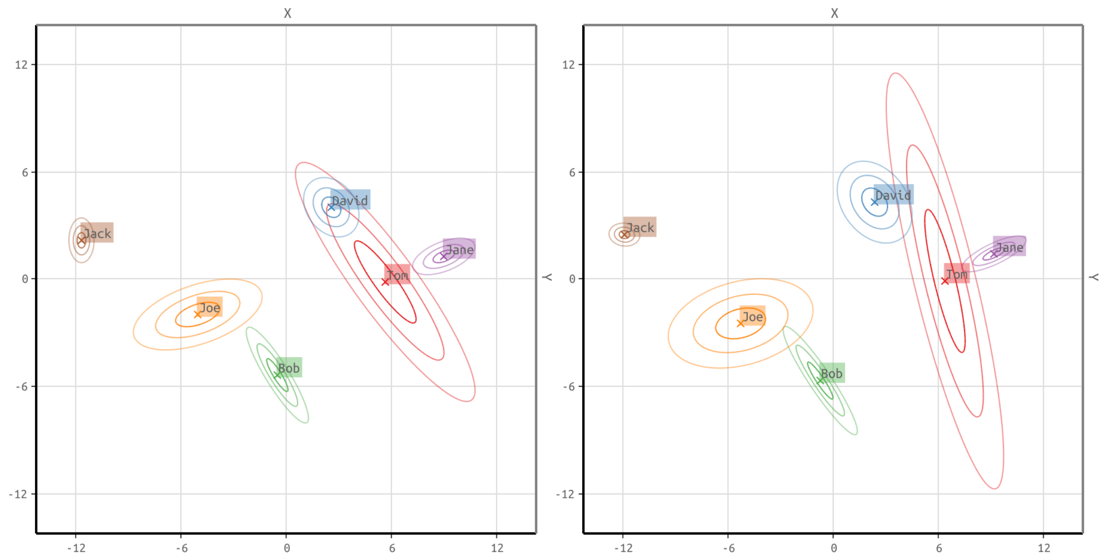
[2] J. Görtler, T. Spinner, D. Streeb, D. Weiskopf, and O. Deussen. Uncertainty-aware principal component analysis. *IEEE Transactions on Visualization and Computer Graphics*, 26(1):822–831, 2020. doi: 10.1109/TVCG.2019.2934812

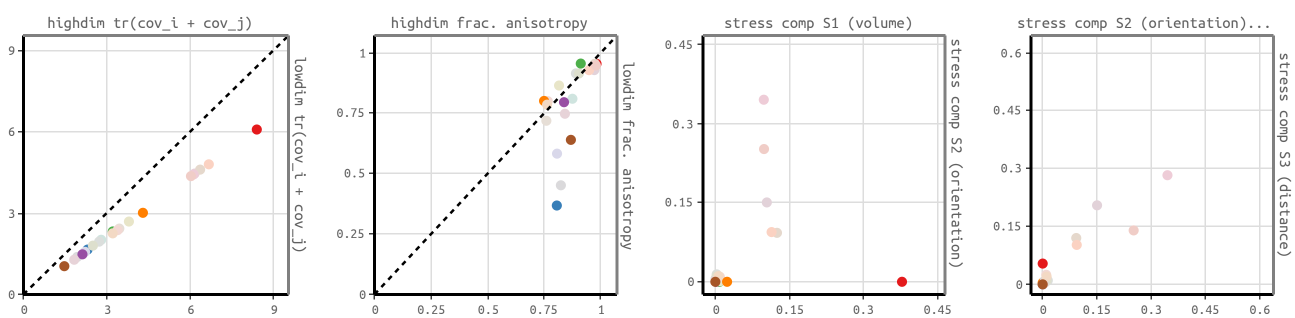
[3] D. Hägele, T. Krake, and D. Weiskopf. Uncertainty-aware multidimensional scaling. *IEEE Transactions on Visualization and Computer Graphics*, 29(1), 2023. doi: 10.1109/TVCG.2022.3209420



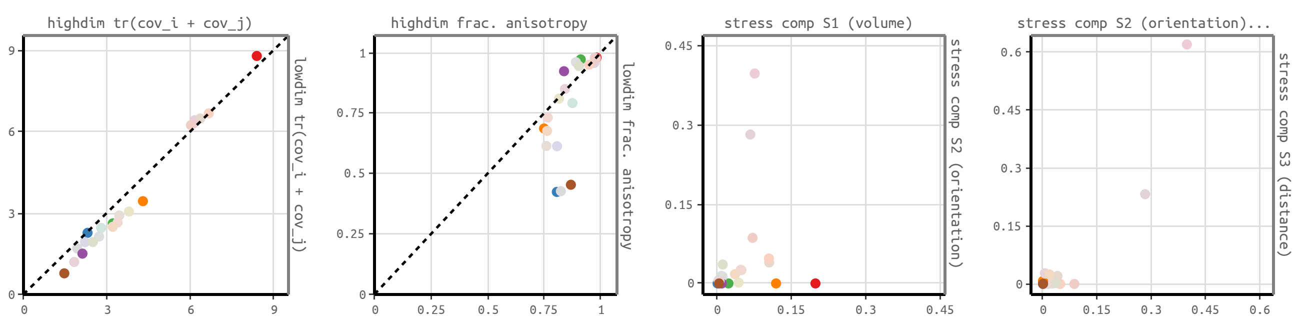
(a) data



(b) UA-PCA (c) UAMDS



(d) Shepard trace (e) Shepard anisotropy (f) stress 1,2 (g) stress 2,3



(h) Shepard trace (i) Shepard anisotropy (j) stress 1,2 (k) stress 2,3

Fig. 1. Student grades data set projections and quality assessment. Middle row corresponds to UA-PCA, bottom row to UAMDS. *Subfigures (b) and (c) are copies of Figure 8 (right-hand side) from the main document [3] licensed under a Creative Commons Attribution License (CC BY 4.0).*

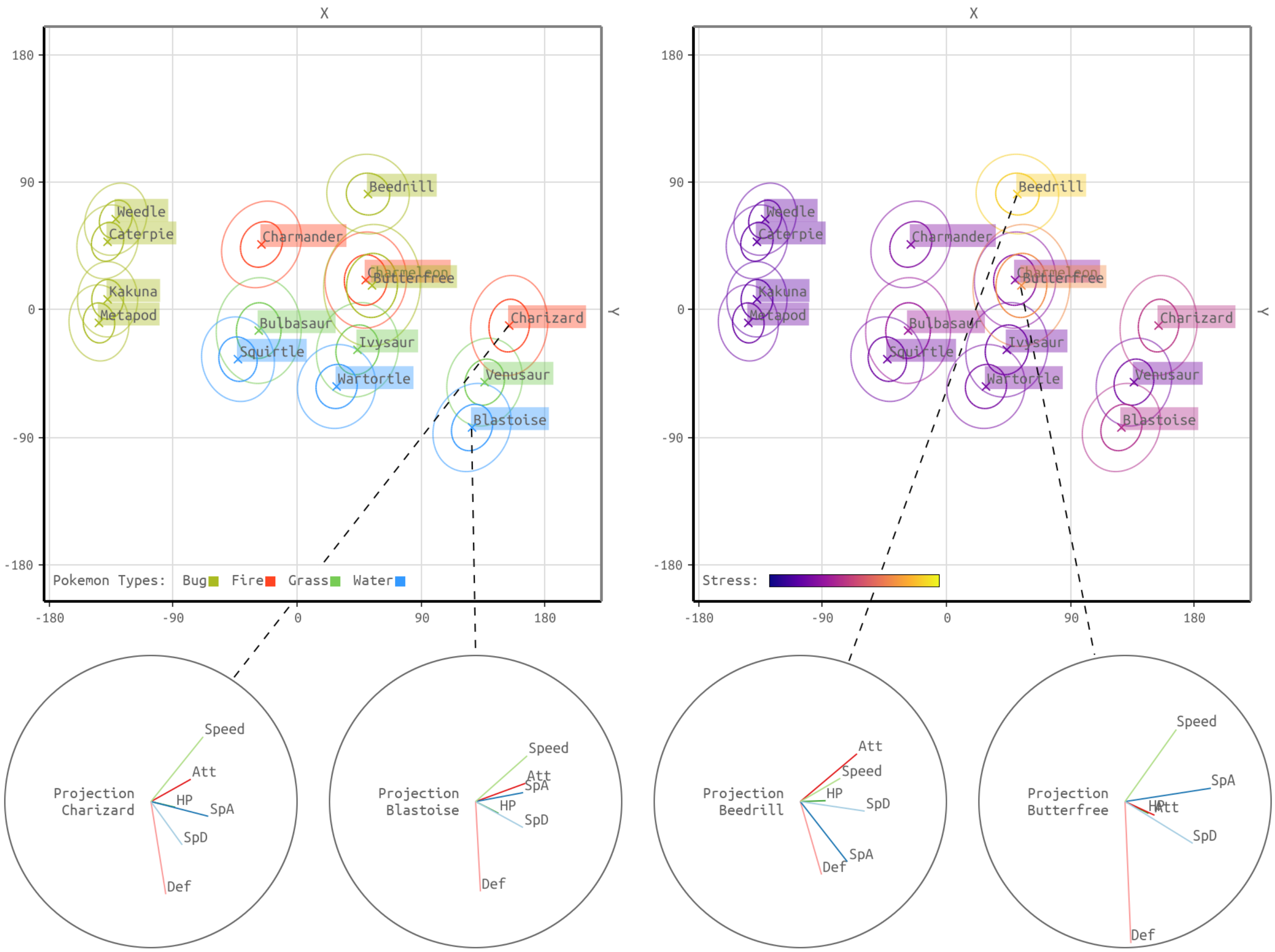
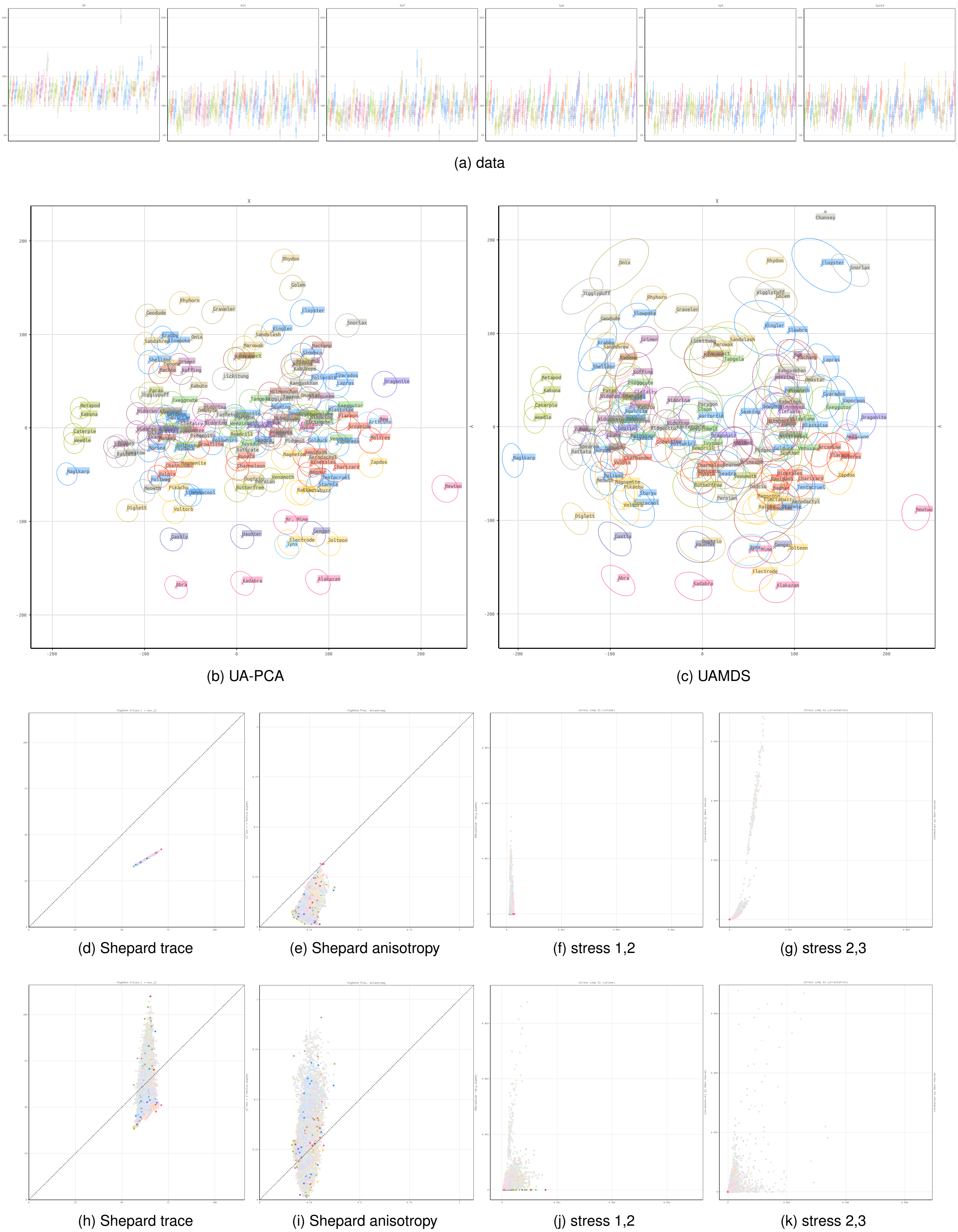


Fig. 2. UAMDS projection of the first 15 Pokemon from the Pokemon data set. Left uses coloring by type of Pokemon (bug, fire, grass, water) to show relationships among Pokemon, e.g., all fire Pokemon belong to the Charmander evolution family. Right side color codes the stress of projected distributions. Visualized on the bottom are the projection matrices of the affine transformations for the individual Pokemon, where each column of the matrix is depicted as a direction into which the corresponding stat it is projected. *Copy of Figure 9 from the main document [3] licensed under a Creative Commons Attribution License (CC BY 4.0).*



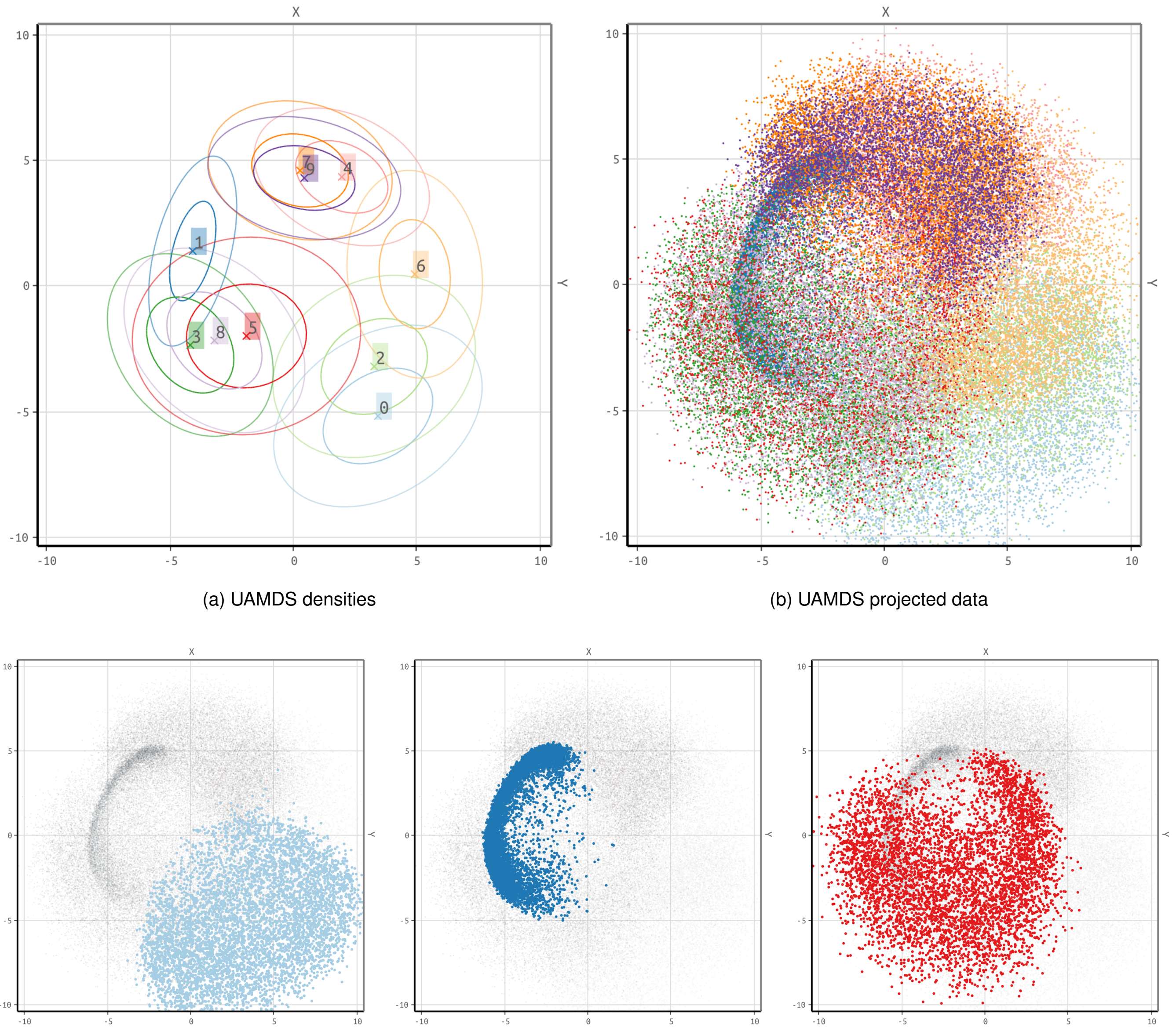
(a) data

(b) UA-PCA (c) UAMDS

(d) Shepard trace (e) Shepard anisotropy (f) stress 1,2 (g) stress 2,3

(h) Shepard trace (i) Shepard anisotropy (j) stress 1,2 (k) stress 2,3

Fig. 3. Pokemon stats data set projections and quality assessment. Middle row corresponds to UA-PCA, bottom row to UAMDS.



(a) UAMDS densities

(b) UAMDS projected data

(c) class 0

(d) class 1

(e) class 5

Fig. 4. UAMDS projection of MNIST data set. (a) shows the projected probability densities for each digit assuming data is normally distributed. The other plots show the projection of the actual data points.