Gambler's Ruin:

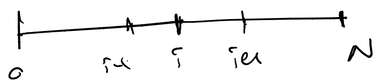
Two gamblers A and B, playing a sequence of rounds, where in each round they bet \$1. $P = P(A \text{ wins a round})$. $q = 1 - p$

Find the probability that A wins entire game. (i.e. B is ruined)

Assuming A starts with \$i, B starts with \$(N − i).

$P =$ Probability of going right

You can think of it as a Random Walk.



Strategy: Condition on first round. (Note that the problem has a recursive structure)

Let $P_i = P(A \text{ wins game} \mid A \text{ starts at } \$i)$

Then by the LOTP:

$$\boxed{P_i = pP_{i+1} + qP_{i-1}} \qquad 1 \leq i \leq N-1 \qquad \begin{cases} P_0 = 0 \\ P_N = 1 \end{cases}$$

This is a difference equation, a discrete version of a differential equation.

Just like ODE's, we make a guess about the Solution.

$$P_i = p\,P_{i+1} + q\,P_{i-1}$$

Guess. $P_i = x^i$, So: $\quad (q = 1-p)$

$$x^i = p x^{i+1} + q\, x^{i-1} \implies p x^2 - x + q = 0$$

$$\implies x = \frac{1 \pm \sqrt{1-4pq}}{\sqrt{2p}} = \frac{1 \pm (2p-1)}{\sqrt{2p}} \in \left\{ 1, \frac{q}{p} = \frac{1-p}{p} \right\}$$

(writing the General Solution) $\implies P_i = A\, 1^i + B\left(\frac{q}{p}\right)^i$, assuming $p \neq q$

$$= A + B\left(\frac{q}{p}\right)^i$$

(checking the Boundary Conditions) $\implies P_0 = 0 = A + B \implies B = -A$

$$P_N = 1 = A\left(1 - \left(\frac{q}{p}\right)^N\right)$$

$$\Rightarrow P_i = \begin{cases} \dfrac{P_i}{1} = \dfrac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} & \text{if } p \neq q \\[2em] \dfrac{i}{N} & \text{if } p = q \end{cases}$$

Proof. Using a CV : $x = \frac{q}{p}$, and knowing that $\frac{1-(1)^i}{1-(1)^N}$ is $\frac{0}{0}$, we write :

$$\lim_{x\to 1} \frac{1-x^i}{1-x^N} \underset{\text{L'Hopital}}{=} \frac{-i x^{i-1}}{-N x^{N-1}} = \frac{i}{N}$$

Takeaway : Since Casinos have more money than you, they'll always win. even if the game is just a little rigged in their favor.

Now, (the same thing happens in the unfair case) if we add the probabilites of A and B winning, we get $\frac{i}{N} + \frac{N-i}{N} = \frac{N}{N} = 1$. $\Rightarrow$ Probability of the game going on forever with no one winning is 0.

Random Variables.

A Random Variable is a function from the Sample Space S to $\mathbb{R}$.

Think of it as a numerical "Summary" of an aspect of the experiment.

Source of Randomness

Essentially, mapping outputs of an experiment to Real numbers.

Defn. (Bernoulli) A r.v. X is said to have Bern (p) distribution

if X has only 2 possible values 0 and 1, where

$$P(\underbrace{X=1}_{\text{event}}) = p \ , \ P(X=0) = 1-p \quad *$$

event : $\{ s : X(s) = 1 \}$: all outputs which are assigned to 1 by the r.v

Defn. $\boxed{\text{Binomial } (n,p)}$: The distribution of # Successes in n independent Bern (p)

trials is called Bin (n,p), its distribution is given by :

$$* \quad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for } 0 \leq k \leq n.$$

k integer

Proof: 111 000 $\Longrightarrow$ Probability of this is $p^k (1-p)^{n-k}$, but all permutations with

k number of 1's are equivalent for this purpose.

Defn. Probability Mass Function (PMF): Defines the probabilities for the events.

Examples marked with * above.

Another (obvious) thing: Given $X \sim Bin(n, p)$, $Y \sim Bin(m, p)$

Then $X + Y \sim Bin(n+m, p)$