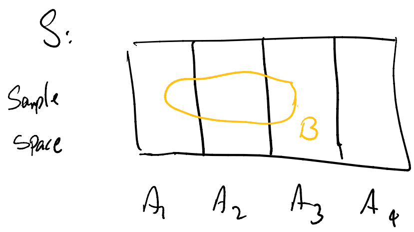
How to Solve a Problem:

1) Try Simple and extreme Cases

2) Break up Problem into Simpler Pieces.



Let $A_1, \ldots, A_n$ be a Partition of B, then

$$P(B) = P(B \cap A_1) + \cdots + P(B \cap A_n)$$

Law of total Probability:

$$= P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)$$

Take Away: Conditional Probability isn't just useful for updating our uncertainty using new evidence, it's also useful for breaking Problems into Simpler Pieces. But you have to be wise in Choosing $A_1, \ldots A_n$, otherwise your Problem will only get harder.

Reminder: A Partition $\{A_1, \ldots, A_n\}$ of a set, is a set of disjoint subsets which when added together, will recreate the original set.

Example. Get random 2-card hand from standard deck.

Find 1. P(both aces | have ace), 2. P(both ace | ace of spades).

1. $$= \frac{P(\text{both aces}, \cancel{\text{have ace}})}{P(\text{have ace})} = \frac{\binom{4}{2} / \binom{52}{2}}{1 - P(\text{no ace})} = \frac{\binom{4}{2} / \binom{52}{2}}{1 - \binom{48}{2} / \binom{52}{2}} = \frac{1}{33}$$

redundant

2. $$\frac{P(\text{both aces} \mid \text{Ace of Spades})}{P(\text{ace of spades})} = \ldots$$

$\boxed{As}$ $\boxed{}$ $$= \frac{3}{51} = \frac{1}{17}$$

one is As

↳ By symmetry, this is equally likely to be any of the other aces

$\Longrightarrow$ Why is the 2nd probability larger than the first?

A: The first has a large prior [P(both aces)] probability space.

Example. Patient gets tested for disease that affects 1% of population, tests positive.

Suppose test is advertised as "95% accurate". Suppose this means that:

D: Patient has disease

T: Patient test Positive

$$\Rightarrow \begin{cases} P(T \mid D) = 0.95 \\ P(T^c \mid D^c) = 0.95 \end{cases}$$

Q: What's the Probability that this Patient actually has the Disease?

Bayes

A: $$P(D \mid T) = \frac{P(T \mid D)\, P(D)}{P(T)} = \frac{0.95 \times 0.01}{\underbrace{P(T \mid D)}_{0.95}\, \underbrace{P(D)}_{0.01} + \underbrace{P(T \mid D^c)}_{1 - P(T^c \mid D^c)}\, \underbrace{P(D^c)}_{0.99}} = 0.16$$

Law of total Probability

$\Longrightarrow$ Here we see the rarity of the disease and the rarity of False test Conteracting.

Coherency of Bayes Rule: Given two evidences, it doesn't matter if you update your Probabilites by using both at once or one at a time.

The results will be the same.

Biohazards:

1. Confusing $P(A|B)$ and $P(B|A)$. [Prosecutor's Fallacy]

2. Confusing $P(A)$ "Prior" with $P(A|B)$ "Posterior".

When we want to compute $P(A|B)$, it doesn't mean that

$P(B)=1$, but rather $P(B|B)=1$. which is always true.

3. Confusing independence with Conditional independence.

Defn. $P(A \cap B | C) = P(A|C)\,P(B|C)$

Question. Does Cond. indep. imply indep.? No.

Example: Chess opponent: Conditionally indep. given opponent's strength,

But not indep. unconditionally.

Question. Does indep. imply Cond. indep.? No.

Example. A: Fire-alarm goes off. It's only caused by either

F: Fire, C: Popcorn. Suppose F and C are indep, but $P(R \mid A, C^c) = 1 \Rightarrow$ Not Cond. indep.

$$P(R \mid A^c, C) = 1$$

Something must have caused the Rise.