Lec3

Wednesday, January 5, 2022 7:09 PM

Birthday Problem:

We have K People, and we want to find the Probability that 2 of them have the same birthday.

Exclude Feb 29th, other 365 Days are equally likely.

Assume independence of Births. [no twins, etc.]

Extreme Case: If $K > 365$, Prob. is 1. [Pigeonhole Principle]

Solution:

Calculate the Complement of the event.

$$P(\text{no match}) = \frac{365 \cdot 364 \cdot 363 \cdots (365-K+1)}{365^K}$$ → should be K terms

$= 1$ if $K = 1$ [no two People for there to be a match]

$$P(\text{Match}) = 1 - P(\text{no match})$$

Note: The important quantity here is not K, but $\binom{K}{2}$, Since we have that many pairs of people.

Properties (derived from axioms of Probability, defined in Previous lecture)

① $P(A^c) = 1 - P(A)$

Proof: $P(S) = P(A \cup A^c) = P(A) + P(A^c) = 1$

② If $A \subseteq B$, then $P(A) \leq P(B)$

If A occurs, then B occurs.

Proof. $B = A \cup (B \cap A^c) \Rightarrow P(B) = P(A) + P(B \cap A^c)$

③ $P(A \cup B) = P(A) + P(B) - \overbrace{P(A \cap B)}^{\text{was double counted}}$

Proof. $P(A \cup B) = P(A \cup (B \cap A^c)) = P(A) + P(B \cap A^c)$

Strategy. wishful thinking $\Rightarrow \stackrel{?}{=} P(A) + P(B) - P(A \cap B)$

which is equivalent to: $P(B \cap A) + P(B \cap A^c) = P(B)$

This is true! Since $A \cap B$ and $A^c \cap B$ are disjoint

and their union is B.

④ General Inclusion-Exclusion:

$$P\left(\bigcup_{a=1}^{n} A_a\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$- \cdots + (-1)^{n+1} P\left(\bigcup_{b=1}^{n} A_b\right)$$

de Montmort's Problem (Matching Problem):

Deck of n shuffled Cards labeled $1, 2, \dots n$, Probability of the n-th Card's

Place in the Deck matching its label. [at least 1 Card like this]

Solution:

Inc-Exc is the easiest method.

Let A_j be event that j-th Card matches, we want to know

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$$P(A_1 \cup A_2 \cup \cdots A_n).$$

$P(A_j) = \frac{1}{n}$. Since all Positions are equally likely for Card labeled j.

(This doesn't depend on j, and that's what Saves us.)

$$P(A_1 \cap A_2) = \frac{1 \cdot 1 \cdot (n-2)!}{n!} = \frac{1}{n(n-1)}$$

(1 is first, 2 is second, rest can be any where.)

$$P(A_1 \cap \cdots \cap A_k) = \frac{(n-k)!}{n!}$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = n \times \frac{1}{n} - \binom{n}{2} \frac{1}{n(n-1)} + \binom{n}{3} \frac{1}{n(n-1)(n-2)} - \cdots$$

$$= 1 - \frac{n(n-1)}{2!} \times \frac{1}{n(n-1)} + \frac{n(n-1)(n-2)}{3!} \times \frac{1}{n(n-1)(n-2)} - \cdots$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \cdots + (-1)^{n} \frac{1}{n!}$$

(Taylor Series for e^x around 0) $\approx 1 - \frac{1}{e}$

Reminder (Taylor Series):

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x)$$

for e^x at 0, $\Longrightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} \approx e^x$

$$e^{-1} \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} = \frac{1}{2!} - \frac{1}{3!} + \cdots$$