Sympathetic Magic (Common Mistake)

Don't confuse an R.V with its distribution.

Example: $\cancel{PMF(X+Y)} \neq \cancel{PMF(X)} + PMF(Y)$

Remember this: The word is not the thing, the map is not the territory.

R.V $\longleftrightarrow$ Random House, Distribution $\longleftrightarrow$ Blueprint.

Poisson Distribution: $X \sim Pois(\lambda)$

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad k \in \mathbb{N}$$

λ is called the rate parameter. $(\lambda > 0)$

Validity check: $$\sum_{k \geq 0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \underbrace{\sum_{k \geq 0}^{\infty} \frac{\lambda^k}{k!}}_{\text{taylor series for } e^{\lambda}} = e^{-\lambda} e^{\lambda} = 1$$

Expectation Value:

$$\mathbb{E}(X) = e^{-\lambda} \sum_{k \geq 0}^{\infty} k \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k \geq 1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$\Rightarrow \mathbb{E}(X) = \lambda$$

Use Cases:

Counting # Successes where there are a large # trials, each with a small Prob. of success.

Examples:

1. # emails received in an hour. (Many people who could email you, each with a low Prob.)

2. # earthquakes in a year in some region.

Poisson Paradigm (Story of Poisson, Poisson approximation)

Given events $A_1, A_2, \dots, A_n$, $P(A_j) = P_j$ with a large n and small P_j's,

when events are indep. or "weakly dep.", the # of A_j's that do occur

is approximately $\sim \text{Pois}(\lambda)$; $\lambda = \sum_{j=1}^{n} P_j$.

λ is the $\mathbb{E}$(A_j's that do occur), so by linearity we get that

Example: Consider $X \sim \text{Bin}(n, p)$; let $n \to \infty$, $p \to 0$ in a way that $\lambda = np$ is held constant.

Find what happens to $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$. For fixed k.

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$$= \underbrace{\frac{n(n-1)\cdots(n-k+1)}{k!\, n^k}}_{\to \frac{n^k \lambda^k}{n^k k!} \text{ as } n\to\infty} \lambda^k \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda} \text{ as } n\to\infty} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{\to 1 \text{ as } n\to\infty}$$

$$\left(1+\frac{\lambda}{n}\right)^n \to e^{\lambda}$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$ which is $P(X=k)$ for $X \sim Pois(\lambda)$

Example. Have n People, find approximate Prob. that there are 3 People with Same birthday.

Answer: We have $\binom{n}{3}$ triplets of People. Create indicator R.V for each triplet I_{ijk}, $i<j<k$

$$\mathbb{E}(\#\text{ triple matches}) = \binom{n}{3} \frac{1}{365^2} \quad \text{Exact.}$$

But we wanted the Probability, not the expected value. See the difference?

Set $X = \#$ triple matches. It's Approx. $Pois(\lambda)$ with $\lambda = \binom{n}{3}\frac{1}{365^2}$

Justification: Why is $Pois(\lambda)$ a good approximation?

a. Large number of trials: $\binom{n}{3}$

b. Small Probability of each : $\frac{1}{365^2}$

c. e.g. I_{123} and I_{456} are indep. But I_{123} and I_{124} are weakly indep.

$$\Rightarrow P(X=k) \simeq \frac{e^{-\lambda}\lambda^{k}}{k!}$$

$$\lambda = \binom{n}{3}\frac{1}{365^2}$$

$$\Rightarrow P(X \geq 1) = 1 - P(X=0) = 1 - \frac{e^{-\lambda}\lambda^{0}}{0!} = 1 - e^{-\lambda}$$