Lec13
Friday, February 18, 2022 9:31 AM

Universality of the Uniform.

Let F be a Cont. strictly increasing CDF, then

$X = F^{-1}(u) \sim F$ if $u \sim \text{Unif}(0,1)$.

Also, if $X \sim F$, then $F(X) \sim \text{Unif}(0,1)$

$= F(F^{-1}(u))$

Warning: Don't make this mistake.

$F(x) = P(X \leq x)$ Doesn't mean ~~$F(X) = P(X \leq X) = 1$~~ Wrong.

Example: $F(x) = 1 - e^{-x}$ for $x > 0$. $\Rightarrow F(X) = 1 - e^{-X}$

Inversing the Function:

$$u = 1 - e^{-x} \Longrightarrow x = -\ln(1-u) \Rightarrow F^{-1}(u) = -\ln(\underbrace{1-u}) \sim F.$$

$1-u \sim \text{Unif}(0,1)$ if $u \sim \text{Unif}(0,1)$

Scaling:

If $u \sim \text{Unif}(0,1)$, then $\underbrace{a + bu}$ is Unif. on some interval.

$\downarrow$

Linear transform

A non-linear transform on a Uniform $\longrightarrow$ Non-Uniform (e.g. u^2 is non-Uniform)

Independence of Random Variables

Defn. $X_1, \dots, X_n$ indep. if for all $x_1, \dots, x_n$

$$\underbrace{P(X_1 \leq x_1, \dots, X_n \leq x_n)}^{\text{Joint CDF}} = P(X_1 \leq x_1) \cdots P(X_n \leq x_n)$$

Discrete:

$$\underbrace{P(X_1 = x_1, \dots, X_n = x_n)}^{\text{Joint PMF}} = P(X_1 = x_1) \cdots P(X_n = x_n)$$

Note: Remember that for the indep. of 3 events you had to check the triple intersection AND all the Pairwise ones.

In that case, we had to write "finitely many" equations while here.

Since we say: for $\underline{\text{all}}$ $x_1, \dots, x_n \Rightarrow$ infinitely many equations.

Full independence means that knowing any collection of them tells you Nothing about the ones you don't know about.

$\{ 1$ if $X_1 = X_2$

Example: $X_1, X_2 \sim \text{Bern}(\frac{1}{2})$ i.i.d ; $X_3 = \begin{cases} 1 & \text{if } X_1 = X_2 \\ 0 & \text{o.w} \end{cases}$

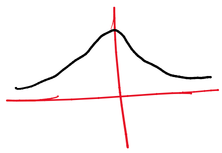
These 3 are $\underbrace{\text{Pairwise indep}}$, but $\underbrace{\text{not indep.}}$

If you know X_1 and X_2, you know X_3

Knowing X_1 gives you nothing about X_2 or X_3, etc

(Gaussian)

Normal Distribution:



Reason of importance:

Central Limit Theorem: Sum of a lot of i.i.d R.V's looks normal.

Consider $Z \sim N(0,1)$ has PDF: $f(z) = \underbrace{c}_{\text{Normalizing Const.}} \cdot e^{-z^2/2}$

$$\int_{-\infty}^{\infty} e^{-z^2/2}\, dz = \text{Gaussian Integral} = \sqrt{2\pi} \Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Expectation Value and Variance:

$Z \sim N(\underset{\text{mean}}{0}, \underset{\text{Variance}}{1})$ Called the standard Normal

$$\mathbb{E}(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{z}_{\text{odd}} \cdot \underbrace{e^{-z^2/2}}_{\text{even}} \, dz = 0$$

integration by Parts

$$\mathbb{E}(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{z^2}_{\text{even}} \cdot \underbrace{e^{-z^2/2}}_{\text{even}} \, dz = \frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} \underbrace{z}_{u} \cdot \underbrace{z \cdot e^{-z^2/2} \, dz}_{dV}$$

$$= \frac{2}{\sqrt{2\pi}} \left(\underbrace{\left(z \cdot -e^{-z^2/2} \right) \Big|_{0}^{+\infty}}_{0} - \overbrace{\int_{0}^{+\infty} - e^{-z^2/2} \, dz}^{\frac{1}{2} \cdot \text{Gaussian}} \right) = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2} = 1$$

$$\text{Var}(Z) = \mathbb{E}(Z^2) - \mathbb{E}^2(Z) = 1 - 0 = 1$$

Notation:

Φ is the Standard Normal CDF:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} \, dt$$

No analytical Solution

Computer-Calculated

$$\Phi(z) = 1 - \Phi(z)$$ by Symmetry.