Matching Problem (Contd.):

$$P(A_1 \cap A_2 \cap \dots \cap A_k) = \frac{(n-k)!}{n!}$$

[Any subset of size k, in fact.
It's the same for all of them by symmetry.]

There are $\binom{n}{k}$ subsets of size k.

$$P(\text{no match}) = P\left(\bigcap_{j=1}^{n} A_j^c\right) \approx \frac{1}{e} = 1 - P(\text{a match})$$

Definition of Independence:

- Events A and B are indep. if $P(A \cap B) = P(A) \cap P(B)$

Note: This is completely different than disjointness.

Disjointness: If A occured, B can't occur.

- Events $A, B, C, \dots$ are independent if
    - All are pairwise indep. and
    - $P(A, B, C, \dots) = P(A)\,P(B)\,P(C) \dots$

Note: One doesn't imply the other.

Newton-Pepys Problem:

Assuming Fair Dice, which is the most likely?

A) at least one 6's with 6 dice

B) " " two 6's " 12 dice

C) " " three 6's " 18 dice.

$$P(A) = 1 - P(\text{no } 6) = 1 - \left(\frac{5}{6}\right)^6 \simeq 0.665$$

$$P(B) = 1 - P(\text{no } 6) - P(\text{one } 6) = 1 - \left(\frac{5}{6}\right)^{12} - 12 \times \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11} \simeq 0.619$$

$$P(C) = 1 - \sum_{K=0}^{2} \underbrace{\binom{18}{K}\left(\frac{1}{6}\right)^K\left(\frac{5}{6}\right)^{18-K}}_{\text{Binomial Coefficient.}} \simeq 0.597$$

Conditional Probability:

How should you update Probs / uncertainty based on new evidence?

Definition. $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Intuition: Conditioning on B, we assume that B has happened, so we have to look at the subspace in which B has happened. (If any other event like C happens as well, it'll only make the subspace smaller since we'll have to take its intersection with B.)

Couple of theorems:

1) $P(A \cap B) = P(B)\,P(A \mid B) = P(B \cap A) = P(A)\,P(B \mid A)$

2) $P(A_1, A_2, \dots, A_n) = P(A_1)\,P(A_2 \mid A_1)\,P(A_3 \mid A_1, A_2) \cdots P(A_n \mid A_1, \dots A_{n-1})$

3) $$P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}$$ Bayes' Rule!

3) $P(A \mid B) = \dfrac{P(B \mid A)\, P(A)}{P(B)}$ Bayes' Rule!