Properties of CDF: $F(x) = P(X \leq x)$

1) (non-strictly) Increasing

2) (Approaching from) right Continuous.

3) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

Iff a function has all these properties, it's a CDF.

Indep. of R.V's:

X, Y are indep. R.v's if

$$P(X \leq x, Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$$

for all x, y.

Discrete Case:

$$P(X = x, Y = y) = P(X = x)\, P(Y = y)$$

Expected Value of a discrete R.V X

PMF

$$\mathbb{E}(X) = \sum_x \underbrace{x}_{\text{Value}} \cdot \overbrace{P(X=x)}^{\text{PMF}}$$

Consider: $X \sim \text{Bern}(p)$

$$\mathbb{E}(X) = 1 \cdot P(X=1) + 0 \cdot P(X=0) = p$$

If we model an experiment like this:

$$X = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

X is an indicator R.V

then

$$\boxed{\mathbb{E}(X) = P(A)}$$

The Fundamental Bridge

between expected values and Probabilities.

Consider $X \sim \text{Bin}(n, p)$:

Proof #1:

$$\mathbb{E}(X) = \sum_{k=0}^{n} \underbrace{k}_{\text{value}} \overbrace{\binom{n}{k} p^k (1-p)^{n-k}}^{\text{PMF}} = \sum_{k=0}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

Choosing k People out of n with one President

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} = np \overbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}}^{=(p+1-p)^{n-1} = 1}$$

$= np$

(Proof #2 Later in this file)

Linearity of Expectation Values:

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) \quad , \quad \mathbb{E}(cX) = c\,\mathbb{E}(X)$$

Even if X and Y are dependent. if c is a Constant.

Proof #2 of $\mathbb{E}(X)$ given $X \sim Bin(n,p)$

$$X = \sum_{i=1}^{n} X_i \quad \text{where } X_i \sim Bern(p)$$

$$\Rightarrow \mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i) = np$$

By linearity of Expected Values.

Reminder:

Hypergeometric Distribution: Probability of w Successes in n draws, without replacement

from a finite Population of size N that Contains exactly K desired objects.

$$\text{PMF: } P(X=w) = \frac{\binom{w}{K}\binom{N-n}{N-w}}{\binom{N}{n}}$$

$1 \leq j \leq 5$

Consider 5 card hand, $X = (\#\text{aces})$. Let X_j be indicator of j'th card being an ace.

$$\mathbb{E}(X) \underset{\text{indicators}}{=} \mathbb{E}(X_1 + \dots + X_5) \underset{\text{Linearity}}{=} \mathbb{E}(X_1) + \dots + \mathbb{E}(X_5) \underset{\text{Symmetry}}{=} 5\,\mathbb{E}(X_1)$$

$$\underset{\text{Fund. Bridge}}{=} 5 \cdot P(\text{1st card ace}) = 5 \cdot \frac{4}{52} = \frac{5}{13}$$, even though X_j's are dependent.

Intuition: Assume we label the cards 1, ..., 5 LTR. Also, we reveal all cards at once. This way the symmetry becomes more apparent.

Note: This method gives the expected value of any Hypergeometric.

Geometric Distribution.

Geom(p): Given indep. Bern(p) trials, count # failures before 1st success.

Consider $X \sim \text{Geom}(p)$

PMF: $P(X = k) = \underbrace{(1-p)^k}\, p$, $k = \{0, 1, 2, \dots\}$

PMF: $P(X=K) = (1-p)^K p$, $K = 0, 1, 2, \ldots$

All first K fail, with Prob. $(1-p)^K$, then the last one succeeds with Prob. p

PMF is valid since $\sum_{K=0}^{\infty} p(1-p)^K = p \sum_{K=0}^{\infty} (1-p)^K = \frac{p}{1-(1-p)} = 1$

$\sum_{K=0}^{\infty} q^K = \frac{1}{1-q}$: ∞-sum of geometric series.

Expectation Value of $X \sim \text{Geom}(p)$:

Math Proof: Set $q = 1-p$,

$$\mathbb{E}(X) = \sum_{K=0}^{\infty} K p q^K = p \sum_{K=1}^{\infty} K q^K = ?$$

$$\sum_{K=0}^{\infty} q^K = \frac{1}{1-q} \xrightarrow{d/dq} \sum_{K=1}^{\infty} K q^{K-1} = \frac{1}{(1-q)^2} \Longrightarrow \sum_{K=1}^{\infty} K q^K = \frac{q}{(1-q)^2}$$

$$\Longrightarrow \mathbb{E}(X) = p \sum_{K=1}^{\infty} K q^K = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

Story Proof:

Let $c = \mathbb{E}(X)$.

$$c = \underbrace{0 \cdot p}_{\text{first trial succeeds}} + \underbrace{(1+c)\, q}_{\text{first trial fails, we have one fail and problem repeats itself.}} = q + cq \,.$$ Solve for c:

$$c - cq = c(1-q) = cp = q \Rightarrow c = \frac{q}{p}$$