This note is to solve a model with one-period debt and no sunspot. It's a starting point. I'll just write the model equations and then describe a solution algorithm. I'm using the same notation as in the JPE paper.

1 Model

Value of no default.

$$v^{nd}(b,g,\epsilon) = \max_{c,n} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta g^{1-\gamma} \mathbb{E}\left[\max\left\{ v^{nd}(b',g',\epsilon'), v^d(b',g',\epsilon') \right\} |g \right] \right\} \tag{1}$$

$$\begin{aligned} c + b &= y + n \\ gb' &= R(n,g)n \\ y &= ge^{\sigma\epsilon} \\ n &\leq \bar{n}(g) \end{aligned}$$

Value of default

$$\begin{aligned} v^d(b,g,\epsilon) &= \frac{c^{1-\gamma}}{1-\gamma} + \beta g^{1-\gamma} \mathbb{E}\Big[\theta v^d(b',g',\epsilon') \qquad (2) \\ &\quad + (1-\theta) \max\left\{ v^{nd}(\kappa b',g',\epsilon'), v^d(b',g',\epsilon') \right\} |g \Big] \\ c &= \phi(g) y \\ gb' &= b \\ y &= ge^{\sigma\epsilon} \end{aligned}$$

Schedule

$$1 = \frac{R(n,g)}{1+r^*} \mathbb{E}[Q(b',g',\epsilon')|g] \tag{3}$$

$$gb' = R(n,g)n \tag{4}$$

Prices

$$Q(b,g,\epsilon) = [1-\mathbf{d}(b,g,\epsilon)] + \mathbf{d}(b,g,\epsilon)X(b,g,\epsilon) \tag{5}$$

$$\begin{aligned} X(b,g,\epsilon) = \frac{1}{1+r^*}\mathbb{E}\Big[&\theta X(b',g',\epsilon') + (1-\theta)\times \\ &\{[1-\mathbf{e}(b',g',\epsilon')]\,X(b',g',\epsilon') + \mathbf{e}(b',g',\epsilon')\kappa Q(\kappa b',g',\epsilon')\}\,|g\Big] \end{aligned} \tag{6}$$

with $b' = b/g$ in equation (6).

The default and re-entry decisions, $\mathbf{d}(b,g,\epsilon)$ and $\mathbf{e}(b,g,\epsilon)$, are given as

$$\mathbf{d}(b,g,\epsilon) = \begin{cases} 0, & \text{if } v^{nd}(b,g,\epsilon) \geq v^d(b,g\epsilon), \\ 1, & \text{otherwise;} \end{cases} \tag{7}$$

$$\mathbf{e}(b,g,\epsilon) = \begin{cases} 1, & \text{if } v^{nd}(b,g,\epsilon) \geq v^d(b,g,\epsilon), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

1.1 Solution algorithm

Step 0: Set grids. Grid for debt, $\vec{b} = \{b_1, b_2, \ldots, b_{N_b}\}$, with $b_1 < 0$ (saving) and $b_{N_b} > 0$ (borrowing). Grid for endowment shocks, $\vec{\epsilon} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N_\epsilon}\}$.

The values for b_1 and b_{N_b} it's trial and error. We want them large enough to not restrict the problem, but not more than needed. Having a simulation to text this may be a god idea. N_b should be large because we restrict choices to be in $\vec{b}$. We used $N_b = 7350$ in the paper. We can probably do less with one period debt, but should still be in the thousands.

For $\vec{\epsilon}$ let's do the same as in the paper: $N_\epsilon = 17$ and use Tauchen method.

We also need $\vec{g} = \{g_L, g_H\}$, the transition matrix $P_g(g'|g)$, and the variance σ_ϵ. All these numbers come from the estimation (see Table).

Step 1: Guess $v^{nd}(b,g,\epsilon)$.

Step 2: Given $v^{nd}(b,g,\epsilon)$, solve for $v^d(b,g,\epsilon)$. Note that, given $v^{nd}(b,g,\epsilon)$, equation (2) is a fixed point in $v^d(b,g,\epsilon)$. Iterate on equation (2) until $v^d(\cdot)$ convergences.

Note that we need to evaluate $v^d(b,g,\epsilon)$ for values $b' = b/g$ and $b' = \kappa b/g$. These values may not be in the grid. Take the closest value in the grid. A good example of why we need a large N_b is so that this approximation is never too off.

Step 3: Given $v^{nd}(b, g, \epsilon)$ and $v^d(b, g, \epsilon)$, compute $\mathbf{d}(b, g, \epsilon)$ and $\mathbf{e}(b, g, \epsilon)$ as in equations (7)-(8).

Step 4: Given $\mathbf{d}(b, g, \epsilon)$ and $\mathbf{e}(b, g, \epsilon)$, we solve for the prices $Q(b, g, s)$ and $X(b, g, \epsilon)$.

In particular, guess $X(b, g, \epsilon)$ and compute $Q(b, g, \epsilon)$ using equation (5). Then compute the implied $X(b, g, \epsilon)$ using equation (6). Iterate until $X(\cdot)$ converges.

As in *Step 2*, we will have to evaluate $Q(b, g, \epsilon)$ for values $b' = b/g$ and $b' = \kappa b/g$ likely outside the grid. As before, take the closest grid point available.

Step 5: We need to compute the schedule $R(n, g)$. This is the hardest step, even if we don't currently have the sunspot. The computations below is for a given g.

Fix a debt value for next period $b' \in \vec{b}$. Say it's the i^{th} position, so that $b' = b_i$. Next, compute the expected price tomorrow associated with $b' = b_i$. That is: $Q_i^{\mathbb{E}}(g) = \mathbb{E}[Q(b_i, g', \epsilon')|g]$.

Next, we use equations (3)-(4) to compute the schedule. Given $Q_i^{\mathbb{E}}(g)$, we obtain $R_i(g) = \frac{1+r^*}{Q_i^{\mathbb{E}}(g)}$. Then, we compute the implied issuance $n_i(g) = \frac{gb_i}{R_i(g)}$. Then, $(R_i(g), n_i(g))$ is a point in the schedule for current growth level g.

We do this for $i = 1, \ldots, N_b$ and obtain the schedule $\mathcal{S}(g) = \{R_i(g), n_i(g)\}_{i=1}^{N_b}$. Note that, for a given R_i we obtain a unique n_i. But may find the same (or similar) n_i in $\mathcal{S}$. Hence the multiplicity.

Compute expected default probability $p_i(g) = \mathbb{E}[d(b_i, g, \epsilon)|g]$. We will impose an upper bound p_{ub} on schedule choices based on this default probability.

Step 6: Now we update $v^{nd}(b, g, \epsilon)$. For given current states $\{b, g, \epsilon\}$,we do as follows.

Given our current guess $v^{nd}(\cdot)$, compute the expected value $w(b, g, \epsilon) = \max\left\{v^{nd}(b, g, \epsilon), v^d(b, g, \epsilon)\right\}$, and the expected value $w^{\mathbb{E}}(b, g) = \mathbb{E}\left[w(b, g', \epsilon')|g\right]$.

Take the schedule of *Step 5*, $\mathcal{S}(g)$. Evaluate all values in the schedule such that default probabilities are below p_{ub}. That is, for each $b' = b_i \in \vec{b}$, evaluate the value of the policy

$$\hat{v}_i^{nd}(b, g, \epsilon) = \frac{[ge^{\sigma\epsilon} + n_i - b]^{1-\gamma}}{1-\gamma} + \beta g^{1-\gamma} w^{\mathbb{E}}(b_i, g) \tag{9}$$

Then, the implied $v^{nd}(\cdot)$ is the maximum over b_i such that default probability is below p_{ub}

$$\hat{v}^{nd}(b, g, \epsilon) = \max_i \left\{ \hat{v}_i^{nd}(b, g, \epsilon) \text{s. to } p_i \leq p_{ub} \right\} \tag{10}$$

Compare $\hat{v}^{nd}(b, g, \epsilon)$ to the initial guess $v^{nd}(b, g, s)$. If it's close, we are done. Otherwise, update $v^{nd}(b, g, s)$ and go back to *Step 2*.