This note describes the problem with a lender of last resort (LOLR). We assume one-period debt. We start the case with no sunspot. Notation is very similar to the previous note.

1 Model

Value of no default.

$$
\begin{aligned}
v^{nd}(b,\ell,g,\epsilon) &= \max_{c,n,n^\ell} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta g^{1-\gamma} \mathbb{E}\left[\max\left\{ v^{nd}(b',\ell',g',\epsilon'), v^{d}(b',\ell',g',\epsilon') \right\} |g \right] \right\} \qquad (1) \\
c + b + \ell &= y + n + n^\ell \\
y &= g e^{\sigma\epsilon} \\
g b' &= R(n,\ell',g) n, \quad g\ell' = R^\ell_{nd} n^\ell \\
\ell' &\leq \bar{\ell}_{nd}, \quad n \leq \bar{n}(g)
\end{aligned}
$$

where ℓ is the amount of debt outstanding with the LOLR, and n^ℓ is the new issuance with the LOLR. We assume the LOLR charges a given rate R^ℓ_{nd}. There is a limit $\bar{\ell}_{nd}$ on how much the sovereign can borrow from the LOLR.

Value of default

$$
\begin{aligned}
v^{d}(b,\ell,g,\epsilon) = \max_{c,n^\ell} \Bigg\{ & \frac{c^{1-\gamma}}{1-\gamma} + \beta g^{1-\gamma} \mathbb{E}\Big[\theta v^{d}(b',\ell',g',\epsilon') \qquad (2) \\
& + (1-\theta) \max\left\{ v^{nd}(\kappa b',\ell',g',\epsilon'), v^{d}(b',\ell',g',\epsilon') \right\} |g \Big] \Bigg\} \\
c + \ell &= \phi(g) y + n^\ell \\
y &= g e^{\sigma\epsilon} \\
g b' &= b, \quad g\ell' = R^\ell_{d} n^\ell \\
\ell' &\leq \bar{\ell}_{d}
\end{aligned}
$$

The assumption is that the sovereign cannot default on the LOLR. We further assume that the sovereign can also keep on borrowing from the LOLR during periods of default/market exclusion. However, the LOLR can set a rate R^ℓ_d and a borrowing limit $\bar{\ell}_d$ that are different during periods of default/market exclusion.

Let $\ell'_d(b,\ell,g,\epsilon)$ be the borrowing policy from the sovereign during default.

Schedule

$$
1 = \frac{R(n,\ell',g)}{1+r^*} \mathbb{E}[Q(b',\ell',g',\epsilon')|g] \qquad (3)
$$

$$
g b' = R(n,\ell',g) n \qquad (4)
$$

Prices

$$Q(b,\ell,g,\epsilon) = [1-\mathbf{d}(b,\ell,g,\epsilon)] + \mathbf{d}(b,\ell,g,\epsilon)X(b,\ell,g,\epsilon) \tag{5}$$

$$\begin{aligned} X(b,\ell,g,\epsilon) = \frac{1}{1+r^*}\mathbb{E}\Big[&\theta X(b',\boldsymbol{\ell}'(h),g',\epsilon') + (1-\theta)\times \\ &\left\{\left[1-\mathbf{e}(b',\boldsymbol{\ell}'_d(h),g',\epsilon')\right]X(b',\boldsymbol{\ell}'(h),g',\epsilon') + \mathbf{e}(b',\boldsymbol{\ell}'(h),g',\epsilon')\kappa Q(\kappa b',\boldsymbol{\ell}'(h),g',\epsilon')\right\}|g\Big] \end{aligned} \tag{6}$$

with $h=(b,\ell,g,\epsilon)$ and $b'=b/g$ in equation (6). IMPORTANT: in equation (6) we need to use the sovereign policy $\ell' = \boldsymbol{\ell}'_d(b,\ell,g,\epsilon)$.

The default and re-entry decisions, $\mathbf{d}(b,\ell,g,\epsilon)$ and $\mathbf{e}(b,\ell,g,\epsilon)$, are given as

$$\mathbf{d}(b,\ell,g,\epsilon) = \begin{cases} 0, & \text{if } v^{nd}(b,\ell,g,\epsilon) \geq v^d(b,\ell,g,\epsilon), \\ 1, & \text{otherwise;} \end{cases} \tag{7}$$

$$\mathbf{e}(b,\ell,g,\epsilon) = \begin{cases} 1, & \text{if } v^{nd}(b,\ell,g,\epsilon) \geq v^d(b,\ell,g,\epsilon), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

1.1 Implementation Comments

We need to set the borrowing limits $\{\bar{\ell}_{nd}, \bar{\ell}_d\}$ and the rates $\{R^\ell_{nd}, R^\ell_d\}$. Let's start assuming the borrowing limit and the rates do not depend on the market status: $\bar{\ell}_{nd} = \bar{\ell}_d = \bar{\ell}$ and $R^\ell_{nd} = R^\ell_{nd} = R^\ell$.

For the borrowing limit. Let $\bar{g} = \mathbb{E}[g]$ be the average growth rate (on the ergodic distribution). I would set $\bar{\ell} = \Delta\bar{g}$. Start with small numbers, moving from 1% to 5%. That is $\Delta \in \{0.01,\ 0.02,\ 0.03,\ 0.04,\ 0.05\}$.

For the interest rate. Take the risk-free rate and add a few points. Say from 4 p.p. to 10 p.p. So we wold have $R^\ell = 1 + r^* + \sigma^\ell$ and $\sigma^\ell = \{\frac{4}{100},\ \frac{6}{100},\ \frac{8}{100},\ \frac{10}{100}\}$. We can explore more values for σ^ℓ alter n. Ideally, we want to make sure σ^ℓ is large enough so that it's not subsidizing borrowing.

1.2 Solution algorithm

Step 0: Set grids. Grid for debt, $\vec{b} = \{b_1, b_2, \ldots, b_{N_b}\}$, with $b_1 < 0$ (saving) and $b_{N_b} > 0$ (borrowing). Grid for LOLR lending $\vec{\ell} = \{\ell_1, \ell_2, \ldots, \ell_{N_\ell}\}$ Grid for endowment shocks, $\vec{\epsilon} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N_\epsilon}\}$.

I would start with small values for N_b and N_ℓ. Probably around 1000 for each. Once it's running, we will want to increase the grid sizes.

For $\vec{\epsilon}$ let's do the same as in the paper: $N_\epsilon = 17$ and use Tauchen method. (We should move to Rouwenhorst method. See here: https://www.karenkopecky.net/RouwenhorstPaper.pdf).

We also need $\vec{g} = \{g_L, g_H\}$, the transition matrix $P_g(g'|g)$, and the variance σ_ϵ. All these numbers come from the estimation (see Table).

Step 1: Guess $v^{nd}(b,\ell,g,\epsilon)$.

Step 2: Given $v^{nd}(b,\ell,g,\epsilon)$, solve for $v^d(b,\ell,g,\epsilon)$. Note that, given $v^{nd}(b,\ell,g,\epsilon)$, equation (2) is a fixed point in $v^d(b,\ell,g,\epsilon)$. Iterate on equation (2) until $v^d(\cdot)$ convergences.

From this step, save $\boldsymbol{\ell}'_d(b,\ell,g,\epsilon)$

Step 3: Given $v^{nd}(b,\ell,g,\epsilon)$ and $v^d(b,\ell,g,\epsilon)$, compute $\mathbf{d}(b,\ell,g,\epsilon)$ and $\mathbf{e}(b,\ell,g,\epsilon)$ as in equations (7)-(8).

Step 4: Given the borroing policy $\boldsymbol{\ell}'_d(b,\ell,g,\epsilon)$, the default decision $\mathbf{d}(b,\ell,g,\epsilon)$, and the re-entry decision $\mathbf{e}(b,\ell,g,\epsilon)$, we solve for the prices $Q(b,\ell,g,s)$ and $X(b,\ell,g,\epsilon)$.

In particular, guess $X(b,g,\epsilon)$ and compute $Q(b,g,\epsilon)$ using equation (5). Then compute the implied $X(b,g,\epsilon)$ using equation (6). Iterate until $X(\cdot)$ converges.

For this step, note that we need to use the sovereign borrowing policy during default: $\boldsymbol{\ell}'(b,\ell,g,\epsilon)$

Step 5: We need to compute the schedule $R(n,\ell',g)$. This is the hardest step, and even more now with two debt choices. The computations below is for a given g.

Consider debt levels for next period: $b' = b_i$ and $\ell' = \ell_j$. That is, private debt is the i^{th} position of the grid $\vec{b}$ and LOLR debt is the j^{th} position of the grid $\vec{\ell}$.

Compute the expected price tomorrow associated with these decisions $b' = b_i$ and $\ell' = \ell_j$. That is: $Q^{\mathbb{E}}_{ij}(g) = \mathbb{E}[Q(b_i,\ell_j,g',\epsilon')|g]$. The implied rate then is $R_{ij}(g) = \frac{1+r^*}{Q^{\mathbb{E}}_{ij}(g)}$. The issuance then is $n_{ij}(g) = \frac{gb_i}{R_{ij}(g)}$

We can do this $\forall i = 1,\ldots,N_b$ and $\forall j = 1,\ldots,N_\ell$. The implied schedule is $\mathcal{S}(g) = \{R_{ij}(g), n_{ij}(g)\}_{i,j}$.

Compute expected default probability $p_{ij}(g) = \mathbb{E}[\mathbf{d}(b_i,\ell_j,g,\epsilon)|g]$. We will impose an upper bound p_{ub} on schedule choices based on this default probability.

Step 6: Now we update $v^{nd}(b,\ell,g,\epsilon)$. For given current states $\{b,\ell,g,\epsilon\}$,we do as follows.

Given our current guess $v^{nd}(\cdot)$, compute the expected value $w(b,\ell,g,\epsilon) = \max\left\{v^{nd}(b,\ell,g,\epsilon), v^d(b,\ell,g,\epsilon)\right\}$, and the expected value $w^{\mathbb{E}}(b,\ell,g) = \mathbb{E}\left[w(b,\ell,g',\epsilon')|g\right]$.

Take the schedule of *Step 5*, $\mathcal{S}(g)$. Evaluate all values in the schedule such that default probabilities are below p_{ub}. That is, for each $b' = b_i \in \vec{b}$ and $\ell' = \ell_j \in \vec{\ell}$, evaluate the value of the policy

$$\hat{v}^{nd}_{ij}(b,\ell,g,\epsilon) = \frac{c_{ij}^{1-\gamma}}{1-\gamma} + \beta g^{1-\gamma} w^{\mathbb{E}}(b_i,\ell_j,g) \tag{9}$$
$$c_{ij} = ge^{\sigma\epsilon} + n_{ij}(g) + n^{\ell}_{ij} - b - \ell$$
$$n^{\ell}_{ij} = \frac{g\ell_j}{R^{\ell}}$$

Then, the implied $v^{nd}(\cdot)$ is the maximum over $\{b_i,\ell_j\}$ such that $\ell_j \leq \bar{\ell}$ and default probability is below p_{ub}

$$\hat{v}^{nd}(b,\ell,g,\epsilon) = \max_{i,j}\left\{\hat{v}^{nd}_{ij}(b,\ell,g,\epsilon) \;\; \text{s. to} \;\; \ell_j \leq \bar{\ell} \;\; \text{and} \;\; p_{ij} \leq p_{ub}\right\} \tag{10}$$

Compare $\hat{v}^{nd}(b,\ell,g,\epsilon)$ to the initial guess $v^{nd}(b,,\ell,g,s)$. If it's close, we are done. Otherwise, update $v^{nd}(b,,\ell,g,s)$ and go back to *Step 2*.