Stochastic Processes

University of Tehran

Instructor: Dr. Ali Olfat — Fall 2024

Homework 3

Due : 1403/8/12

Problem 1

Let X and Y be two random variables with the following joint density.

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & otherwise \end{cases}$$

(a) With the observation of $X = x$, find the best estimate for Y in minimum mean square sense and its MSE.

(b) With the observation of $X = x$, find the best estimate for X^2 in minimum mean square sense.

Problem 2

$X_1, X_2, \ldots, X_n$ form a sequence of i.i.d. random variables with the pdf $f_{X_i}(x) = f(x)$, and $E\{X_i\} = \eta$ and $var(X_i) = \sigma^2$. Define $Y_j = \sum_{k=1}^{j} X_k$
$j = 1, 2, \ldots, n$.

(a) Find the conditional expectation $E\{Y_n | Y_1, Y_2, \ldots, Y_{n-1}\}$.

(b) Find the conditional expectation $E\{Y_n | Y_1, Y_2, \ldots, Y_{n-2}\}$.

(c) Find the conditional variance $var(Y_n | Y_1, Y_2, \ldots, Y_{n-1})$.

(d) Find the joint pdf $f_{\underline{Y}}(y_1, y_2, \ldots, y_n)$.

Problem 3

$X_1, X_2, \dots, X_n$ are a sequence of i.i.d. continuous random variables with pdf $f_{X_i}(x) = f(x)$. If we arrange the sequence in decreasing order $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(n)}$, i.e. $X_{(1)}$ is the largest element of the vector $\underline{X} = [X_1, X_2, \dots, X_n]^T$ and so on, the vector $\underline{X}^{OS} = [X_1, X_2, \dots, X_n]^T$ is called the order statistics of the vector $\underline{X}$.

(a) Find the pdf of $f_{\underline{X}^{OS}}(x_1, x_2, \dots, x_n)$.

(b) Find the pdf of $f_{X_i}(x_i)$ for $i = 1, 2, \dots n$.

(c) Find the joint pdf of $f_{X_{(i)}X_{(j)}}(x_i, x_j)$ for $i < j$.

(d) For the special case where $X_1, X_2, \dots, X_n$ are i.i.d uniform random variables on the interval $[0, 1]$, find $E\{X_{(i)}\}$ and $E\{X_{(1)}|X_{(n)} = x\}$.

Problem 4

The random vector $\underline{X} = [X_1, X_2, X_3]^T$ has the mean of $\underline{m}_X = [1, 0, -2]^T$ and the covariance matrix

$$C_X = \begin{bmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{bmatrix}$$

Is there a linear relation between the elements of $\underline{X}$? If yes, find the relation and if the answer is no, explain it.

Problem 5

The random vector $\underline{X} = [X_1, X_2, X_3]^T$ has the mean of $\underline{m}_X = [5, -5, 6]^T$ and the covariance matrix of:

$$C_X = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

Find a linear transformation $\underline{Y} = A\underline{X} + \underline{b}$ so that $\underline{Y}$ is a white normalized vector.

Problem 6

Suppose that X_1 and X_2 are two jointly random variables with the following pdf.

$$f_{X_1X_2}(x_1, x_2) = \frac{2}{\pi\sqrt{7}} exp\left\{-\frac{8}{7}\left(x_1^2 + \frac{3}{2}x_1x_2 + x_2^2\right)\right\}$$

Find matrix A so that $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, and that Y_1 and Y_2 are independent.