University of Tehran,
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Homework #5

Stochastic Process – Fall 2024
Instructor: **Dr. Ali Olfat**

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Problem 1. $X(t)$: Markov stochastic process $\quad t_1 \leq t_2 \leq \cdots \leq t_n$

Prove that: $f_{X_1}(x_1; t_1 \mid X(t_2) = x_2, \ldots, X(t_n) = x_n) = f_{X_1}(x_1; t_1 \mid X(t_2) = x_2)$

$$f_{X_1}(x_1; t_1 \mid X(t_2) = x_2, \ldots, X(t_n) = x_n) = \frac{f_{X_1 \cdots X_n}(X(t_1) = x_1, \cdots, X(t_n) = x_n)}{f_{X_2 \cdots X_n}(X(t_2) = x_2, \ldots, X(t_n) = x_n)} \qquad \text{I}$$

Also we know: $f_{X_1 \cdots X_n}(X(t_1) = x_1, \ldots, X(t_n) = x_n) = f_{X_1}(x_1; t_1)\, f_{X_2}(x_2, t_2 \mid X(t_1) = x_1) \cdots$
$f_{X_n}(x_n, t_n \mid X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1})$

$f_{X_2 \cdots X_n}(X(t_2) = x_2, \ldots, X(t_n) = x_n) = f_{X_2}(x_2; t_2)\, f_{X_3}(x_3, t_3 \mid X(t_2) = x_2) \cdots$
$f_{X_n}(x_n, t_n \mid X(t_2) = x_2, \ldots, X(t_{n-1}) = x_{n-1})$

Markov SP Property
$$\begin{cases} f_{X_1 \cdots X_n}(X(t_1) = x_1, \cdots, X(t_n) = x_n) = f_{X_1}(x_1; t_1)\, f_{X_2}(x_2; t_2 \mid X(t_1) = x_1) \cdots f_{X_n}(x_n, t_n \mid X(t_{n-1}) = x_{n-1}) \\ f_{X_2 \cdots X_n}(X(t_2) = x_2, \cdots, X(t_n) = x_n) = f_{X_2}(x_2; t_2)\, f_{X_3}(x_3; t_3 \mid X(t_2) = x_2) \cdots f_{X_n}(x_n, t_n \mid X(t_{n-1}) = x_{n-1}) \end{cases} \qquad \text{II}$$

According to I and II we can write:

$$f_{X_1}(x_1; t_1 \mid X(t_2) = x_2, \cdots, X(t_n) = x_n) = \frac{f_{X_1}(x_1; t_1)\, f_{X_2}(x_2; t_2 \mid X(t_1) = x_1)}{f_{X_2}(x_2; t_2)}$$

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$$= \frac{f_{X_1 X_2}(X(t_1) = x_1, X(t_2) = x_2)}{f_{X_2}(X(t_2) = x_2)} = f_{X_1}(x_1; t_1 \mid X(t_2) = x_2)$$

Problem 2. $X(t)$: Poisson process with uniform density λ

$\hookrightarrow$ jumps at Poisson points $t_i \geq 0$ $(i = 1, 2, \ldots)$; $T_n = t_{i+n} - t_i \quad (n \geq 1)$

Poisson process $\Rightarrow P(X(t) = k) = e^{-\lambda t} \dfrac{(\lambda t)^k}{k!}$

$$F_{T_n}(t) = P(T_n \leq t) = 1 - P(T_n > t) = 1 - P\left\{ \begin{matrix} \text{zero points} \\ \text{in period} \\ (t_i, t_{i+n}) \end{matrix} \right\} = 1 - P(X(t) = k-1)$$

$$\longrightarrow F_{T_n}(t) = 1 - P\{X(t) = k-1\} = 1 - e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}$$

$$\xrightarrow{\partial/\partial t} f_{T_n}(t) = \frac{\partial F_{T_n}(t)}{\partial t} = \lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!} - (k-1)e^{-\lambda t}\frac{\lambda^{k-1}t^{k-2}}{(k-1)!}$$

$$= e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}\left[\lambda - \frac{k-1}{t}\right]$$

Problem 4. $W(t)$: Wiener process ; $W(t) = \int_0^t X(\alpha)\,d\alpha \longrightarrow \begin{cases} X(t): \text{zero-mean Stationary white Gaussian process} \\ R_X(\tau) = N_0\,\delta(\tau) \end{cases}$

$Y(t) = W^2(t)$

a. PDF of $Y(t)$: $(f_Y(y;t))$

$$f_{W(t)}(w;t) = \frac{1}{\sqrt{2\pi N_0 t}}\exp\left(\frac{-w^2}{2N_0 t}\right) \quad ; \quad Y = W^2 \xrightarrow{Y>0} \begin{cases} W_1 = -\sqrt{Y} \\ W_2 = \sqrt{Y} \end{cases} \quad ; \quad f_Y(y;t) = \frac{\sum_i f_W(w_i;t)}{|J|}$$

$$|J| = |2W| = 2\sqrt{y} \implies f_Y(y;t) = \frac{f_W(\sqrt{y};t)}{2\sqrt{y}} + \frac{f_W(-\sqrt{y};t)}{2\sqrt{y}}$$

$$\longrightarrow f_Y(y;t) = \frac{1}{\sqrt{2\pi N_0 t y}}\exp\left(\frac{-y}{2N_0 t}\right) \quad ; \quad y > 0 \ , \ t > 0$$

b. Is $Y(t)$ an IIP? IIP Def. $t_1 \leq t_2 \leq \cdots \leq t_n$: $Y(t_1) \sqcup Y(t_2) - Y(t_1) \sqcup \cdots \sqcup Y(t_n) - Y(t_{n-1})$

$$f_{Y(t_1)Y(t_2)}(Y_1, Y_2; t_1, t_2) = \frac{1}{4\sqrt{y_1 y_2}}\Big[f_W(\sqrt{y_1}, \sqrt{y_2}; t_1, t_2) + f_W(\sqrt{y_1}, -\sqrt{y_2}; t_1, t_2) + f_W(-\sqrt{y_1}, \sqrt{y_2}; t_1, t_2) + f_W(-\sqrt{y_1}, -\sqrt{y_2}; t_1, t_2)\Big]$$

$$= \frac{1}{4\sqrt{y_1 y_2}}\Big[f_{W_1}(\sqrt{y_1}) f_{W_2-W_1}(\sqrt{y_2} + \sqrt{y_1}) + f_{W_1}(\sqrt{y_1}) f_{W_2-W_1}(\sqrt{y_1}, -\sqrt{y_2}) + f_{W_1}(-\sqrt{y_1}) f_{W_2-W_1}(\sqrt{y_2} - \sqrt{y_1}) + f_{W_1}(-\sqrt{y_1}) f_{W_2-W_1}(-\sqrt{y_1}, -\sqrt{y_2})\Big]$$

$$= \frac{\exp\left(\frac{-y_1}{2N_0 t_1}\right)}{4\pi N_0\sqrt{y_1 y_2 t_1(t_2 - t_1)}}\left[\exp\left(-\frac{(\sqrt{y_1} + \sqrt{y_2})^2}{2N_0(t_2 - t_1)}\right) + \exp\left(-\frac{(\sqrt{y_1} - \sqrt{y_2})^2}{2N_0(t_2 - t_1)}\right)\right]$$

$$\implies f_Y(y_1, y_2; t_1, t_2) \neq f_Y(y_2 - y_1; t_1, t_2)\, f_Y(y_1, t_1) \implies \underline{Y(t) \text{ is NOT IIP!}}$$

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Problem 5. $X(t) = A\cos(2\pi Ft + \theta)$

A: constant ; $F \sim U[2,4]$; $\theta \sim U[0,2\pi]$; $F \perp\!\!\!\perp \theta$

- **Mean of $X(t)$:**

$$E\{X(t)\} = E\{A\cos(2\pi Ft+\theta)\} = AE\{\cos(2\pi Ft)\cos(\theta) - \sin(2\pi Ft)\sin(\theta)\}$$

$$= AE\{\cos(2\pi Ft)\}\underbrace{\cancel{E\{\cos(\theta)\}}}_{0} - AE\{\sin(2\pi Ft)\}\underbrace{\cancel{E\{\sin(\theta)\}}}_{0} = 0 \quad \text{I}$$

- **Autocorrelation function of $X(t)$:**

$$R_x(t_1, t_2) = E\{X(t_1)X^*(t_2)\} = E\{A^2\cos(2\pi Ft_1+\theta)\cos(2\pi Ft_2+\theta)\}$$

$$= \frac{A^2}{2}E\{\cos(2\pi F(t_1+t_2)+2\theta)\} + \frac{A^2}{2}E\{\cos(2\pi F(t_1-t_2))\}$$

$$= \frac{A^2}{2}E\{\cos(2\pi F(t_1+t_2))\}\underbrace{\cancel{E\{\cos(2\theta)\}}}_{0} - \frac{A^2}{2}E\{\cos(2\pi F(t_1+t_2))\}\underbrace{\cancel{E\{\sin(2\theta)\}}}_{0}$$

$$+ \frac{A^2}{2}E\{\cos(2\pi F(t_1-t_2))\} = \frac{A^2}{2}\int_2^4 \frac{1}{2}\cos(2\pi f(t_1-t_2))\,df$$

$$= \frac{A^2}{8\pi(t_1-t_2)}\left[\sin(8\pi(t_1-t_2)) - \sin(4\pi(t_1-t_2))\right]$$

$$\Rightarrow \quad R_x(\tau) = \frac{A^2}{8\pi\tau}\left[\sin(8\pi\tau) - \sin(4\pi\tau)\right] \quad \text{II}$$

$$\text{I}, \text{II} \Longrightarrow X(t) \text{ is a WSS Process}$$

Problem 6. $X(t)$: zero-mean stationary Gaussian process $(R_x(\tau))$

$Y = Ae^{jX(t)}$; A is a Poisson RV : $A \sim \text{Poisson}(a)$; $A \perp\!\!\!\perp X(t)$

- **Mean of $Y(t)$:**

$$E\{Y(t)\} = E\{Ae^{jX(t)}\} = E\{A\}E\{e^{jX(t)}\} = aE\{e^{jX(t)}\} = a\Phi_{X(t)}(1)$$

- **Normal RV** $\Longrightarrow \Phi_x(\omega) = e^{j\omega m_x}e^{-\frac{1}{2}\sigma_x^2\omega^2} \longrightarrow m_x = 0, \quad \sigma_x^2 = R_x(0)$

$$\Rightarrow E\{Y(t)\} = a e^{-\frac{R_x(0)}{2}} \quad \text{I}$$

- Autocorrelation function of $Y(t)$:

$$R_Y(t_1,t_2) = E\{Y(t_1)Y^*(t_2)\} = E\{Ae^{jX(t_1)} \cdot Ae^{-jX(t_2)}\} = \overbrace{E\{A^2\}}^{a^2+a} E\{e^{j\overbrace{(X(t_1)-X(t_2))}^{U}}\}$$

$$= (a+a^2)\, E\{e^{jU}\} = (a+a^2)\, \Phi_U(1)$$

$$U \sim \mathcal{N}(m_U, \sigma_U^2) \quad ; \quad \Phi_U(\omega) = e^{j\omega m_U} \cdot e^{-\frac{1}{2}\sigma_U^2\omega^2} \longrightarrow \Phi_U(1) = e^{jm_U} e^{-\frac{1}{2}\sigma_U^2}$$

$$m_U = E\{U\} = E\{X(t_1)-X(t_2)\} = \cancel{E\{X(t_1)\}}^{0} - \cancel{E\{X(t_2)\}}^{0} = 0$$

$$\sigma_U^2 = E\{U^2\} = E\{(X(t_1)-X(t_2))^2\} = E\{X^2(t_1) - 2X(t_1)X(t_2) + X^2(t_2)\}$$

$$= \underbrace{E\{X^2(t_1)\}}_{R_X(0)} - 2\underbrace{E\{X(t_1)X(t_2)\}}_{R_X(t_1-t_2)} + \underbrace{E\{X^2(t_2)\}}_{R_X(0)} = 2R_X(0) - 2R_X(t_1-t_2)$$

$$\Rightarrow R_Y(t_1,t_2) = (a+a^2)\, e^{-(R_X(0)-R_X(t_1-t_2))}$$

$$\longrightarrow R_Y(\tau) = (a+a^2)\, e^{R_X(\tau)-R_X(0)} \quad \text{II}$$

$$\text{I}, \text{II} \Rightarrow Y(t) \text{ is a WSS process}$$

Problem 3. $W(t) = \int_0^t X(\alpha)\,d\alpha$; $X(t)$: WSS Gaussian process ; $R_X(\tau) = N_0\,\delta(\tau)$

- We want to estimate $W(2)$ given $W(1)$ by MMSE criterion.

$$f_{W(t)}(w;t) = \frac{1}{\sqrt{2\pi N_0 t}} e^{-\frac{w^2}{2N_0 t}} \Rightarrow \begin{cases} w(1): f_{W(1)}(w) = \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{w^2}{2N_0}\right) \\ w(2): f_{W(2)}(w) = \frac{1}{\sqrt{2\pi 2N_0}} \exp\left(\frac{-w^2}{4N_0}\right) \end{cases}$$

$$W(1), W(2) \Rightarrow \text{Jointly Normal}\,(\mu_{W(1)} = \mu_{W(2)} = 0 \;,\; \sigma^2_{(W_1)} = N_0 \;,\; \sigma^2_{(W_2)} = 2N_0 \;,\; \rho_{W_1W_2})$$

$$\rho_{W_1W_2} = \frac{\text{cov}(W_1,W_2)}{\sigma_{W_1}\sigma_{W_2}} = \frac{E\{W_{(1)}W_{(2)}\}}{\sigma_{W(1)}\sigma_{W(2)}} = \frac{R_W(1,2)}{1\times\sqrt{2}} = \frac{\sqrt{2}}{2}N_0$$

according to the property of Jointly Normal dist. $\Rightarrow E\{W_2|W_1\} = \cancel{m_{W_2}}^{0} + \underbrace{\rho_{W_1W_2}}_{\frac{\sqrt{2}}{2}N_0} \frac{\overbrace{\sigma_{W_2}}^{\sqrt{2N_0}}}{\underbrace{\sigma_{W_1}}_{\sqrt{N_0}}} (W_1 - \cancel{m_{W_1}})$

$$\Rightarrow \hat{W}(2)_{MMSE} = E\{W(2)|W(1)\} = \frac{2}{2}N_0 \cdot \frac{\sqrt{2N_0}}{\sqrt{N_0}}\, W(1) = W(1)$$

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$$MSE = E\{(w(2)-\hat{w}(2))^2\} = E\{(w(2)-w(1))^2\} = E\{w^2(2) - 2w(1)w(2) + w^2(1)\}$$

$$= \underbrace{E\{w^2(2)\}}_{2N_0=\sigma^2_{w(2)}} - 2\underbrace{E\{w(1)w(2)\}}_{N_0=R_w(1,2)} + \underbrace{E\{w^2(1)\}}_{N_0=\sigma^2_{w(1)}} = 2N_0 - 2N_0 + N_0 = N_0$$

$$\text{Estimation} \Rightarrow \hat{w}(2)_{MMSE} = w(1) \quad ; \quad MSE = N_0$$

Problem 8. $U(n)$: iid sequence of Gaussian RVs $\longrightarrow \mathcal{N}(0,1)$

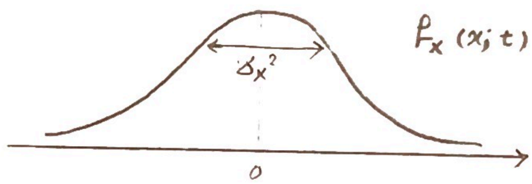
$X(t) \Rightarrow$ obtained by linearly interpolating between U's $\rightarrow X(t) = U(t)$ for $t = n \in \mathbb{Z}$

$\hookrightarrow X(t) = (1-a)\,U(t) + a\,U(t+1) \quad ; \quad t = n$

Part a. $f_X(x;t)$

$$\longmapsto X(t) = \underbrace{(1-a)\,U(n)}_{U_1} + \underbrace{a\,U(n+1)}_{U_2} \quad ; \quad t=n \Rightarrow \sigma_x^2 = (1-a)^2 + a^2 = 2a^2 - 2a + 1$$

$$\longrightarrow f_X(x;t) = \underbrace{f_{U_1}(u_1) * f_{U_2}(u_2)}$$



$f_X(x;t)$

$f_{U_1}(u_1)$ and $f_{U_2}(u_2)$ both have a bell shape. So the convolution of them ($f_X(x;t)$) will have a bell shape too!

$$t = n + a \longrightarrow a = t - n \longrightarrow \sigma_x^2 = 2(n+1-t)^2 + 2t - 2n - 1 = (n-t)^2$$

$$\Rightarrow f_X(x;t,n) = \frac{1}{\sqrt{2\pi(t-n)}}\exp\left(\frac{-x^2}{2(n-t)^2}\right)$$

Part b. Is the $X(t)$ wide sense stationary?

- **Mean :** $E\{X(t)\} = 0$

$(1-a_1)U(n_1) + a_1U(n_1+1)$

$(1-a_2)U(n_1) + a_2U(n_1+1)$

- **Auto-correlation :** $R_X(t_1,t_2) = E\{X(t_1)X^*(t_2)\} = (1-a_1)(1-a_2)\underbrace{E\{U^2(n_1)\}}_{1} + a_1a_2\underbrace{E\{U^2(n_1+1)\}}_{1}$

$$\left.\begin{array}{l} a_1 = t_1 - n_1 \\ a_2 = t_2 - n_2 \end{array}\right\} \Rightarrow R_X(t_1,t_2) = (n_1+1-t_1)(n_1+1-t_2) + (t_1-n_1)(t_2-n_1)$$

$$= 2n_1^2 - 2n_1t_1 - 2n_1t_2 + 2t_1t_2 - t_1 - t_2 + 2n_1 + 1$$

$$\longrightarrow R_X(t_1,t_2) \neq R_X(t_1-t_2) \longrightarrow \underline{X(t) \text{ is not WSS!}}$$