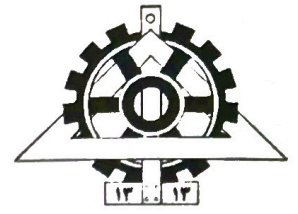


Homework #3

Stochastic Process – Fall 2024

Instructor: Dr. Ali Olfat

Erfan Panahi (Student Number: 810103084)



• Problem 1. $f_{xy}(x, y) = \begin{cases} \frac{1}{\pi} & , \quad x^2 + y^2 \leq 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

■ Part a. The best estimation for Y (in Minimum Square sense with the observation $X=x$)

$$f_x(x) = \int_{-\infty}^{+\infty} f_{xy}(x, y) dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi} \quad ; \quad |x| \leq 1$$

$$\Rightarrow f_{y|x}(y|x) = \frac{f_{xy}(x, y)}{f_x(x)} = \frac{1}{2\sqrt{1-x^2}} \quad ; \quad x^2 + y^2 \leq 1$$

$$\Rightarrow E(Y|X) = \int_{-\infty}^{+\infty} y f_{y|x}(y|x) dy = \frac{1}{2\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y dy = 0 \xrightarrow{\text{odd}} E(Y|X) = 0$$

$$\hat{Y}_{MMSE} = E(Y|X) = 0 \longrightarrow \hat{Y}_{MMSE} = 0$$

Now, we want to compute Mean Squared Error (MSE): $MSE = E(\text{Var}(Y|X))$

$$\text{Var}(Y|X) = E(Y^2|X) - \cancel{E^2(Y|X)} = E(Y^2|X)$$

$$E(Y^2|X) = \int_{-\infty}^{+\infty} y^2 f_{y|x}(y|x) dy = \frac{1}{2\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2 dy = \frac{1-x^2}{3} \quad , \quad |x| \leq 1$$

$$\Rightarrow MSE = E(\text{Var}(Y|X)) = E\left(\frac{1-x^2}{3}\right) = \frac{1}{3} - \frac{1}{3} E(x^2)$$

$$E(x^2) = \int_{-\infty}^{+\infty} x^2 f_x(x) dx = \int_{-1}^1 x^2 f_x(x) dx = \int_{-1}^1 \frac{2x^2 \sqrt{1-x^2}}{\pi} dx$$

$$\begin{aligned} x &= \cos(\theta) \\ dx &= -\sin(\theta) d\theta \end{aligned} \Rightarrow E(x^2) = \int_0^\pi \frac{2}{\pi} \cos^2(\theta) \sin^2(\theta) d\theta = \frac{1}{2\pi} \int_0^\pi \sin^2(2\theta) d\theta = \frac{1}{4\pi} \int_0^\pi (1 - \cos(4\theta)) d\theta$$

$$= \frac{1}{4\pi} \int_0^\pi d\theta - \underbrace{\frac{1}{4\pi} \int_0^\pi \cos(4\theta) d\theta}_{=0} = \frac{1}{4}$$

$$\Rightarrow MSE = \frac{1}{3} - \frac{1}{3} \times \frac{1}{4} = \frac{1}{4}$$

$$\xrightarrow{\text{Part a.}} \begin{cases} \hat{Y}_{MMSE} = 0 \\ MSE = \frac{1}{4} \end{cases}$$

■ Part b. The best estimation for X^2 (in Minimum mean square sense with the observation $X=x$)

$$f_y(y) = \int_{-\infty}^{+\infty} f_{xy}(x, y) dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-y^2}}{\pi} \quad ; \quad |y| \leq 1$$

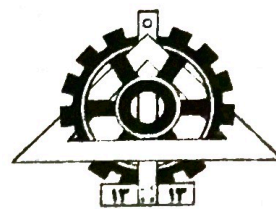
$$\Rightarrow f_{x|y}(x|y) = \frac{f_{xy}(x, y)}{f_y(y)} = \frac{1}{2\sqrt{1-y^2}} \quad ; \quad x^2 + y^2 \leq 1$$

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$$\Rightarrow E(X^2|Y) = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y) dx = \frac{1}{2\sqrt{1-y^2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 dx = \frac{1-y^2}{3} ; \Rightarrow \hat{X}_{MSE}^2 = \frac{1-y^2}{3}$$

$$MSE = E(\text{var}(X^2|Y)) \Rightarrow \text{var}(X^2|Y) = E(X^4|Y) - E^2(X^2|Y)$$

$$E(X^4|Y) = \frac{1}{2\sqrt{1-y^2}} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^4 dx = \frac{(1-y^2)^2}{5} \Rightarrow \text{var}(X^2|Y) = \frac{(1-y^2)^2}{5} - \frac{(1-y^2)^2}{9} = \frac{4}{45} (1-y^2)^2$$

$$\Rightarrow MSE = E\left(\frac{4}{45} (1-y^2)^2\right) = \frac{4}{45} E(1-2y^2+y^4) = \frac{4}{45} (1-2E(y^2)+E(y^4))$$

$$\left. \begin{aligned} E(y^2) &= \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_{-1}^1 \frac{2y^2 \sqrt{1-y^2}}{\pi} dy = \frac{1}{4} \\ E(y^4) &= \int_{-\infty}^{\infty} y^4 f_Y(y) dy = \int_{-1}^1 \frac{2y^4 \sqrt{1-y^2}}{\pi} dy = \frac{1}{8} \end{aligned} \right\} \Rightarrow MSE = \frac{1}{18} \Rightarrow \left\{ \begin{aligned} \hat{X}_{MSE}^2 &= \frac{1-y^2}{3} \\ MSE &= \frac{1}{18} \end{aligned} \right.$$

* Problem 2. X_1, X_2, \dots, X_n are i.i.d, $f_{X_i}(x) = f(x)$, $E(X_i) = \eta$, $\text{var}(X_i) = \sigma^2$

$$Y_j = \sum_{k=1}^j X_k \quad (j=1, 2, \dots, n)$$

$$\begin{aligned} X_1 &= Y_1 \\ X_2 &= Y_2 - Y_1 \\ &\vdots \\ X_{n-1} &= Y_{n-1} - Y_{n-2} \end{aligned}$$

■ Part a. $E(Y_n | Y_1, Y_2, \dots, Y_{n-1})$

considering that we know Y_1, Y_2, \dots, Y_{n-1} , we can say we know X_1, X_2, \dots, X_{n-1} :

$$\begin{aligned} E(Y_n | Y_1, Y_2, \dots, Y_{n-1}) &= E(Y_n | X_1, X_2, \dots, X_{n-1}) = E(X_n + Y_{n-1} | X_1, X_2, \dots, X_{n-1}) \\ &= \underbrace{E(X_n | X_1, X_2, \dots, X_{n-1})}_{X_i \text{ are iid} = E(X_n) = \eta} + \underbrace{E(Y_{n-1} | X_1, X_2, \dots, X_{n-1})}_{Y_{n-1}} = \eta + y_{n-1} \end{aligned}$$

■ Part b. $E(Y_n | Y_1, Y_2, \dots, Y_{n-2})$

$$\begin{aligned} E(Y_n | Y_1, Y_2, \dots, Y_{n-1}) &= E(X_n + X_{n-1} + Y_{n-2} | X_1, X_2, \dots, X_{n-2}) \\ &= E(X_n) + E(X_{n-1}) + y_{n-2} = 2\eta - y_{n-2} \end{aligned}$$

■ Part c. $\text{var}(Y_n | Y_1, Y_2, \dots, Y_{n-2})$

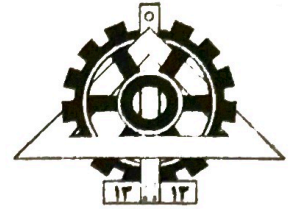
$$\begin{aligned} \text{var}(Y_n | Y_1, Y_2, \dots, Y_{n-1}) &= E(Y_n^2 | Y_1, Y_2, \dots, Y_{n-1}) - E^2(Y_n | Y_1, Y_2, \dots, Y_{n-1}) \\ &= E((X_n + Y_{n-1})^2 | X_1, X_2, \dots, X_{n-1}) - (\eta + y_{n-1})^2 \\ &= \underbrace{E(X_n^2)}_{\sigma^2 + \eta^2} + 2 \underbrace{E(X_n)}_{\eta} y_{n-1} + y_{n-1}^2 - (\eta + y_{n-1})^2 = \sigma^2 + (\eta + y_{n-1})^2 - (\eta + y_{n-1})^2 \\ &= \sigma^2 \end{aligned}$$

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Part c. $f_Y(y_1, y_2, \dots, y_n)$

We know that: $f_X(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(x_n) = f(x_1) f(x_2) \dots f(x_n)$

$$f_Y(y_1, y_2, \dots, y_n) = \sum_i \frac{f_X(x_{1i}, x_{2i}, \dots, x_{ni})}{|J(x_{1i}, \dots, x_{ni})|}$$

$$Y_1 = X_1 \longrightarrow x_1 = y_1,$$

$$Y_2 = X_1 + X_2 \longrightarrow x_2 = y_2 - y_1,$$

...

$$Y_n = X_1 + \dots + X_n \longrightarrow x_n = y_n - y_{n-1},$$

$$J = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & \dots & \dots & 1 \end{vmatrix} = 1$$

$$\Rightarrow f_Y(y_1, y_2, \dots, y_n) = \frac{f_X(y_1, y_2 - y_1, \dots, y_n - y_{n-1})}{1} = f(y_1) f(y_2 - y_1) \dots f(y_n - y_{n-1})$$

$$\Rightarrow f_Y(y_1, y_2, \dots, y_n) = f(y_1) f(y_2 - y_1) \dots f(y_n - y_{n-1})$$

• Problem 4. $C_X = \begin{bmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{bmatrix}$, $\underline{X} = [X_1, X_2, X_3]^T$, $\underline{m}_X = [1, 0, -2]^T$

We examine that if $|C_X| = 0$, there is a linear relation between the elements of \underline{X} .

$$|C_X| = \begin{vmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{vmatrix} = 10 \begin{vmatrix} 10 & -5 \\ -5 & 10 \end{vmatrix} - (-5) \begin{vmatrix} -5 & -5 \\ -5 & 10 \end{vmatrix} - 5 \begin{vmatrix} -5 & 10 \\ -5 & -5 \end{vmatrix} = 750 - 375 - 375 = 0$$

→ There is a linear relation between the elements of \underline{X} .

Now we derive the linear relation between the \underline{X} 's elements.

$$\underline{C}_X \underline{q} = \underline{0} \longrightarrow \begin{bmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} 10 & -5 & -5 \\ -15 & 15 & 0 \\ 15 & 0 & -15 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underline{0} \Rightarrow \begin{matrix} 15q_1 = 15q_2 \\ 15q_1 = 15q_3 \end{matrix}$$

$$\Rightarrow q_1 = q_2 = q_3 \Rightarrow \underline{q} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

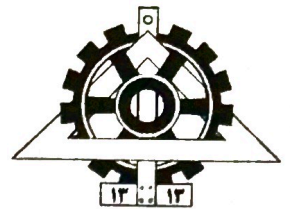
$$\underline{q}^H \underline{\bar{X}} = 0 \longrightarrow \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} X_1 - 1 \\ X_2 + 0 \\ X_3 - 2 \end{bmatrix} = 0 \longrightarrow \boxed{X_1 + X_2 + X_3 = -1}$$

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- Problem 5. $\underline{X} = [x_1, x_2, x_3]^T$; $\underline{m}_X = [5, -5, 6]^T$; $\underline{Y} = \underline{A}\underline{X}$ (\underline{Y} is a white normalized vec)

To obtain matrices \underline{A} and \underline{B} for the linear transformation, we need to compute the matrix \underline{L} that comes from the Cholesky decomposition of \underline{C}_X . So, we first check whether the given matrix \underline{C}_X is positive definite.

$$|\underline{C}_X - \lambda \underline{I}| = 0 \rightarrow \begin{vmatrix} 5-\lambda & 2 & -1 \\ 2 & 5-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{vmatrix} = 0 \rightarrow (5-\lambda)^2(4-\lambda) - 4(4-\lambda) - (5-\lambda) = 0$$

$$\rightarrow \lambda^3 - 14\lambda^2 + 60\lambda - 79 = 0 \rightarrow \lambda > 0 \rightarrow \underline{C}_X: p.d.$$

$$\rightarrow \underline{Y} = \underline{A}\underline{X} + \underline{B} = \underline{L}^{-1}\underline{X} - \underline{L}^{-1}\underline{m}_X ; \underline{C}_X = \underline{L}\underline{L}^H$$

$$L_{jj} = \sqrt{C_{X_{jj}} - \sum_{k=1}^{j-1} L_{jk}^* L_{jk}} ; L_{ij} = \frac{1}{L_{jj}} \left(C_{X_{ij}} - \sum_{k=1}^{j-1} L_{jk}^* L_{ik} \right)$$

$$L_{11} = \sqrt{C_{X_{11}}} = \sqrt{5} ; L_{21} = \frac{1}{L_{11}} (C_{X_{21}}) = \frac{2}{\sqrt{5}} ; L_{31} = \frac{1}{L_{11}} (C_{X_{31}}) = \frac{-1}{\sqrt{5}}$$

$$L_{22} = \sqrt{C_{X_{22}} - L_{21}^* L_{21}} = \sqrt{5 - \frac{4}{5}} = \sqrt{\frac{21}{5}} ; L_{32} = \frac{1}{L_{22}} (C_{X_{32}} - L_{21}^* L_{31}) = \frac{2}{\sqrt{105}}$$

$$L_{33} = \sqrt{C_{X_{33}} - L_{31}^* L_{31} - L_{32}^* L_{32}} = \sqrt{\frac{79}{21}}$$

$$\Rightarrow \underline{L} = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 2/\sqrt{5} & \sqrt{21/5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{105} & \sqrt{79/21} \end{bmatrix} \rightarrow \underline{L}^{-1} = \begin{bmatrix} 0.4472 & 0 & 0 \\ -0.1952 & 0.488 & 0 \\ 0.1228 & -0.0491 & 0.5156 \end{bmatrix} = \underline{A} \quad \checkmark$$

$$\underline{B} = -\underline{L}^{-1}\underline{m}_X = -\underline{L}^{-1} \begin{bmatrix} 5 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2.2361 \\ 3.4157 \\ -3.9528 \end{bmatrix}$$

- Problem 6. $f_{X_1, X_2}(x_1, x_2) = \frac{2}{\pi\sqrt{7}} \exp\left(-\frac{8}{7}(x_1^2 + \frac{3}{2}x_1x_2 + x_2^2)\right)$

If the mean vector of \underline{X} (\underline{m}_X) is zero, we can determine \underline{A} in such a way that \underline{Y} is white, and consequently Y_1 and Y_2 will be independent.

Therefore, we first check whether the mean vector is zero or not.

⇒ As can be inferred from the joint density, the density corresponds to a jointly normal distribution of two variables, Thus we have:

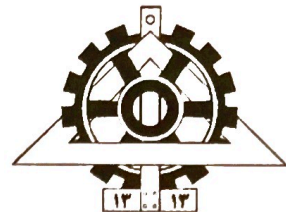
$$\rho_{X_1, X_2} = -\frac{3}{4}, \sigma_{X_1} = \sigma_{X_2} = 1, m_{X_1} = m_{X_2} = 0$$

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$$C_x = E((\underline{X} - \underline{m}_x)(\underline{X} - \underline{m}_x)^H) = E(\underline{X}\underline{X}^H) = \begin{bmatrix} E(X_1^2) & E(X_1 X_2) \\ E(X_2 X_1) & E(X_2^2) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho_{x_1 x_2} \\ \rho_{x_2 x_1} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix}$$

Now, we compute the eigen values and vectors of C_x :

$$|C_x - \lambda I| = (1 - \lambda)^2 - \frac{9}{16} = 0 \rightarrow (1 - \lambda)^2 = \frac{9}{16} \rightarrow 1 - \lambda = \pm \frac{3}{4} \rightarrow \begin{cases} \lambda = \frac{7}{4} \\ \lambda = \frac{1}{4} \end{cases} \Rightarrow C_x: \text{p.d.}$$

$$\begin{aligned} C_x \underline{q} = \lambda \underline{q} &\rightarrow \begin{cases} \lambda = \frac{7}{4} \left\{ \begin{aligned} q_1 - \frac{3}{4} q_2 &= \frac{7}{4} q_1 \\ -\frac{3}{4} q_1 + q_2 &= \frac{7}{4} q_2 \end{aligned} \right\} \Rightarrow q_1 = -q_2 \rightarrow \underline{q} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} \\ \lambda = \frac{1}{4} \left\{ \begin{aligned} q_1 - \frac{3}{4} q_2 &= \frac{1}{4} q_1 \\ -\frac{3}{4} q_1 + q_2 &= \frac{1}{4} q_2 \end{aligned} \right\} \Rightarrow q_1 = q_2 \rightarrow \underline{q} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \end{cases} \\ &\Rightarrow \underline{Q} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

$$\Rightarrow L = \underline{Q} \underline{\Lambda}^{1/2} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & -\sqrt{2}/4 \end{bmatrix} = \frac{\sqrt{2}}{4} \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix}$$

$$\Rightarrow L^{-1} = \sqrt{\frac{2}{7}} \times \frac{4}{\sqrt{2}} \begin{bmatrix} -\sqrt{7} & -\sqrt{7} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -\frac{4}{\sqrt{7}} & \frac{4}{\sqrt{7}} \end{bmatrix} = A \rightarrow \underline{y} = A \underline{x}$$

$$\rightarrow \begin{cases} y_1 = -4x_1 - 4x_2 \sim x_1 + x_2 \\ y_2 = -\frac{4}{\sqrt{7}}x_1 + \frac{4}{\sqrt{7}}x_2 \sim x_1 - x_2 \end{cases}$$

• Problem 3. $x_1, x_2, \dots, x_n \Rightarrow$ are i.i.d ; $f_{x_i}(x) = f(x)$; $x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(n)} \Rightarrow \underline{x}^{os} = [x_{(1)}, x_{(2)}, \dots, x_{(n)}]^T$

■ Part a. $f_{\underline{x}^{os}}(x_1, x_2, \dots, x_n)$

We know that: $f_{\underline{x}}(x_1, x_2, \dots, x_n) = f_{x_1}(x_1) f_{x_2}(x_2) \dots f_{x_n}(x_n) = f(x_1) f(x_2) \dots f(x_n)$

$$f_{\underline{y}}(y_1, y_2, \dots, y_n) = \sum_i \frac{f_{\underline{x}}(x_{1i}, x_{2i}, \dots, x_{ni})}{|J(x_{1i}, \dots, x_{ni})|}$$

$$\begin{aligned} x_1 &= x_{(1)} = x_i \\ x_2 &= x_{(2)} = x_j \quad (j \neq i) \Rightarrow |J| = 1 \Rightarrow f_{\underline{x}^{os}}(\underline{x}) = \frac{f(x_1) f(x_2) \dots f(x_n)}{1} = f(x_1) f(x_2) \dots f(x_n) \\ &\dots \end{aligned}$$