University of Tehran,
College of Engineering

Homework #3

Stochastic Process – Fall 2024
Instructor: **Dr. Ali Olfat**

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- **Problem 1.** $f_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & , \quad x^2+y^2 \leqslant 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

Part a. The best estimation for Y (in Minimum square sense with the observation $X=x$)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = \frac{2\sqrt{1-x^2}}{\pi} \quad ; \quad |x| \leqslant 1$$

$$\Rightarrow f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1}{2\sqrt{1-x^2}} \quad ; \quad x^2+y^2 \leqslant 1$$

$$\Rightarrow E(Y|X) = \int_{-\infty}^{+\infty} y\, f_{Y|X}(y|x)\,dy = \frac{1}{2\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \overbrace{y}^{\text{odd}}\,dy = 0 \longrightarrow E(Y|X) = 0$$

$$\hat{Y}_{MMSE} = E(Y|X) = 0 \longrightarrow \hat{Y}_{MMSE} = 0$$

Now, we want to compute Mean Squared Error (MSE): $\quad MSE = E\big(var(Y|X)\big)$

$$var(Y|X) = E(Y^2|X) - \cancel{E^2(Y|X)}^{\,0} = E(Y^2|X)$$

$$E(Y^2|X) = \int_{-\infty}^{+\infty} y^2 f_{Y|X}(y|x)\,dy = \frac{1}{2\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2\,dy = \frac{1-x^2}{3} \quad , \quad |x| \leqslant 1$$

$$\Rightarrow MSE = E(var(Y|X)) = E\left(\frac{1-X^2}{3}\right) = \frac{1}{3} - \frac{1}{3}E(X^2)$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_{-1}^{1} x^2 f_X(x)\,dx = \int_{-1}^{1} \frac{2x^2\sqrt{1-x^2}}{\pi}\,dx$$

$$\begin{aligned} x &= \cos(\theta) \\ dx &= -\sin(\theta)d\theta \end{aligned} \Rightarrow E(X^2) = \int_0^{\pi} \frac{2}{\pi}\cos^2(\theta)\sin^2(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{\pi}\sin^2(2\theta)\,d\theta = \frac{1}{4\pi}\int_0^{\pi}(1-\cos(4\theta))\,d\theta$$

$$= \frac{1}{4\pi}\int_0^{\pi} d\theta \underbrace{- \frac{1}{4\pi}\int_0^{\pi}\cos(4\theta)\,d\theta}_{=0} = \frac{1}{4}$$

$$\Rightarrow MSE = \frac{1}{3} - \frac{1}{3}\times\frac{1}{4} = \frac{1}{4} \qquad \xrightarrow{\text{Part a.}} \begin{cases} \hat{Y}_{MMSE} = 0 \\ MSE = \frac{1}{4} \end{cases}$$

Part b. The best estimation for X^2 (in Minimum mean square sense with the observation $X=x$)

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dx = \frac{2\sqrt{1-y^2}}{\pi} \quad ; \quad |y| \leqslant 1$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{2\sqrt{1-y^2}} \quad ; \quad x^2+y^2 \leqslant 1$$

$$\Rightarrow E(X^2|Y) = \int_{-\infty}^{+\infty} x^2 f_{X|Y}(x|y)\,dx = \frac{1}{2\sqrt{1-y^2}}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2\,dx = \frac{1-y^2}{3} \quad ; \quad \Rightarrow \hat{X}^2_{MMSE} = \frac{1-y^2}{3}$$

$$MSE = E(var(X^2|Y)) \Rightarrow var(X^2|Y) = E(X^4|Y) - E^2(X^2|Y)$$

$$E(X^4|Y) = \frac{1}{2\sqrt{1-y^2}}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^4\,dx = \frac{(1-y^2)^2}{5} \Rightarrow var(X^2|Y) = \frac{(1-y^2)^2}{5} - \frac{(1-y^2)^2}{9} = \frac{4}{45}(1-y^2)^2$$

$$\Rightarrow MSE = E\left(\frac{4}{45}(1-Y^2)^2\right) = \frac{4}{45}E(1-2Y^2+Y^4) = \frac{4}{45}(1-2E(Y^2)+E(Y^4))$$

$$\left.\begin{array}{l} E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_{-1}^{1}\frac{2y^2\sqrt{1-y^2}}{\pi}\,dy = \frac{1}{4} \\ E(Y^4) = \int_{-\infty}^{\infty} y^4 f_Y(y)\,dy = \int_{-1}^{1}\frac{2y^4\sqrt{1-y^2}}{\pi}\,dy = \frac{1}{8} \end{array}\right\} \Rightarrow MSE = \frac{1}{18} \Rightarrow \left\{\begin{array}{l} \hat{X}^2_{MMSE} = \frac{1-y^2}{3} \\ MSE = \frac{1}{18} \end{array}\right.$$

* **Problem 2.** $X_1, X_2, \dots, X_n$ are i.i.d, $f_{X_i}(x) = f(x)$, $E(X_i) = \eta$, $var(X_i) = \sigma^2$

$$Y_j = \sum_{k=1}^{j} X_k \quad (j = 1, 2, \dots, n)$$

$X_1 = Y_1$
$X_2 = Y_2 - Y_1$
$\dots$
$X_{n-1} = Y_{n-1} - Y_{n-2}$

■ **Part a.** $E(Y_n | Y_1, Y_2, \dots, Y_{n-1})$

considering that we know $Y_1, Y_2, \dots, Y_{n-1}$, we can say we know $X_1, X_2, \dots, X_{n-1}$:

$$E(Y_n | Y_1, Y_2, \dots, Y_{n-1}) = E(Y_n | X_1, X_2, \dots, X_{n-1}) = E(X_n + Y_{n-1} | X_1, X_2, \dots, X_{n-1})$$

$$= \underbrace{E(X_n | X_1, X_2, \dots, X_{n-1})}_{X_i \text{ are iid } = E(X_n) = \eta} + \underbrace{E(Y_{n-1} | X_1, X_2, \dots, X_{n-1})}_{y_{n-1}} = \eta + y_{n-1}$$

■ **Part b.** $E(Y_n | Y_1, Y_2, \dots, Y_{n-2})$

$$E(Y_n | Y_1, Y_2, \dots, Y_{n-1}) = E(X_n + X_{n-1} + Y_{n-2} | X_1, X_2, \dots, X_{n-2})$$

$$= E(X_n) + E(X_{n-1}) + y_{n-2} = 2\eta - y_{n-2}$$

■ **Part c.** $var(Y_n | Y_1, Y_2, \dots, Y_{n-1})$

$$var(Y_n | Y_1, Y_2, \dots, Y_{n-1}) = E(Y_n^2 | \overbrace{Y_1, Y_2, \dots, Y_{n-1}}^{X_1, \dots, X_{n-1}}) - \overbrace{E^2(Y_n | Y_1, Y_2, \dots, Y_{n-1})}^{(\eta + y_{n-1})^2}$$

$$= E((X_n + Y_{n-1})^2 | X_1, X_2, \dots, X_{n-1}) - (\eta + y_{n-1})^2$$

$$= \underbrace{E(X_n^2)}_{\sigma^2 + \eta^2} + 2\underbrace{E(X_n)}_{\eta} y_{n-1} + y_{n-1}^2 - (\eta + y_{n-1})^2 = \sigma^2 + (\eta + y_{n-1})^2 - (\eta + y_{n-1})^2$$

$$= \sigma^2$$

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- Part c. $f_Y(y_1, y_2, \dots, y_n)$

We know that: $f_{\underline{X}}(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \dots f_{X_n}(x_n) = f(x_1) f(x_2) \dots f(x_n)$

$$f_{\underline{Y}}(y_1, y_2, \dots, y_n) = \sum_i \frac{f_{\underline{X}}(x_{1_i}, x_{2_i}, \dots, x_{n_i})}{|J(x_{1_i}, \dots, x_{n_i})|}$$

$Y_1 = X_1 \longrightarrow x_1 = y_1$

$Y_2 = X_1 + X_2 \longrightarrow x_2 = y_2 - y_1$; $J = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{vmatrix} = 1$

$\dots$

$Y_n = X_1 + \dots + X_n \longrightarrow x_n = y_n - y_{n-1}$

$$\Rightarrow f_{\underline{Y}}(y_1, y_2, \dots, y_n) = \frac{f_{\underline{X}}(y_1, y_2 - y_1, \dots, y_n - y_{n-1})}{1} = f(y_1) f(y_2 - y_1) \dots f(y_n - y_{n-1})$$

$$\Rightarrow f_{\underline{Y}}(y_1, y_2, \dots, y_n) = f(y_1) f(y_2 - y_1) \dots f(y_n - y_{n-1})$$

- Problem 4. $C_x = \begin{bmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{bmatrix}$, $\underline{X} = [X_1, X_2, X_3]^T$, $\underline{m}_x = [1, 0, -2]^T$

We examin that if $|C_x| = 0$, there is a linear relation between the elements of $\underline{X}$.

$$|C_x| = \begin{vmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{vmatrix} = 10 \underbrace{\begin{vmatrix} 10 & -5 \\ -5 & 10 \end{vmatrix}}_{75} - (-5) \underbrace{\begin{vmatrix} -5 & -5 \\ -5 & 10 \end{vmatrix}}_{-75} - 5 \underbrace{\begin{vmatrix} -5 & 10 \\ -5 & -5 \end{vmatrix}}_{75} = 750 - 375 - 375 = 0$$

$\longrightarrow$ There is a linear relation between the elements of $\underline{X}$.

Now we drive the linear relation between the $\underline{X}$'s elements.

$$\underline{C_x}\,\underline{q} = \underline{0} \longrightarrow \begin{bmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} 10 & -5 & -5 \\ -15 & 15 & 0 \\ 15 & 0 & -15 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underline{0} \Rightarrow \begin{matrix} 15q_1 = 15q_2 \\ 15q_1 = 15q_3 \end{matrix}$$

$$\Rightarrow q_1 = q_2 = q_3 \Rightarrow \underline{q} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{q}^H \tilde{\underline{X}} = 0 \longrightarrow \left[\frac{\sqrt{3}}{3} \;\; \frac{\sqrt{3}}{3} \;\; \frac{\sqrt{3}}{3}\right] \begin{bmatrix} X_1 - 1 \\ X_2 + 0 \\ X_3 + 2 \end{bmatrix} = 0 \longrightarrow \boxed{X_1 + X_2 + X_3 = -1}$$

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• Problem 5. $\underline{X} = [x_1, x_2, x_3]^T$; $\underline{m_X} = [5, -5, 6]^T$; $\underline{Y} = A\underline{X}$ ($\underline{Y}$ is a white normalized vec)

To obtain matrices A and B for the linear transformation, we need to compute the matrix L that comes from the Cholesky decomposition of C_X. So, we first check whether the given matrix C_X is positive definite.

$$|C_X - \lambda I| = 0 \longrightarrow \begin{vmatrix} 5-\lambda & 2 & -1 \\ 2 & 5-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{vmatrix} = 0 \longrightarrow \underbrace{(5-\lambda)^2(4-\lambda) - 4(4-\lambda) - (5-\lambda)}^{-16+4\lambda-5+\lambda = 5\lambda-21} = 0$$

$$\longrightarrow \lambda^3 - 14\lambda^2 + 60\lambda - 79 = 0 \longrightarrow \lambda > 0 \rightarrow C_X : p.d.$$

$$\longrightarrow \underline{Y} = A\underline{X} + \underline{B} = \underline{L}^{-1}\underline{X} - \underline{L}^{-1}\underline{m_X} \quad ; \quad C_X = LL^H$$

$$L_{jj} = \sqrt{C_{x_{jj}} - \sum_{k=1}^{j-1} L_{jk}^* L_{jk}} \quad ; \quad \underset{i>j}{L_{ij}} = \frac{1}{L_{jj}}\left(C_{x_{ij}} - \sum_{k=1}^{j-1} L_{jk}^* L_{ik}\right)$$

$$L_{11} = \sqrt{C_{x_{11}}} = \sqrt{5} \quad , \quad L_{21} = \frac{1}{L_{11}}(C_{x_{21}}) = \frac{2}{\sqrt{5}} \quad , \quad L_{31} = \frac{1}{L_{11}}(C_{x_{31}}) = \frac{-1}{\sqrt{5}}$$

$$L_{22} = \sqrt{C_{x_{22}} - L_{21}^* L_{21}} = \sqrt{5 - \frac{4}{5}} = \sqrt{\frac{21}{5}} \quad ; \quad L_{32} = \frac{1}{L_{22}}(C_{x_{32}} - L_{21}^* L_{31}) = \frac{2}{\sqrt{105}}$$

$$L_{33} = \sqrt{C_{x_{33}} - L_{31}^* L_{31} - L_{32}^* L_{32}} = \sqrt{\frac{79}{21}}$$

$$\Rightarrow L = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 2/\sqrt{5} & \sqrt{21/5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{105} & \sqrt{79/21} \end{bmatrix} \longrightarrow L^{-1} = \begin{bmatrix} 0.4472 & 0 & 0 \\ -0.1952 & 0.488 & 0 \\ 0.1228 & -0.0491 & 0.5156 \end{bmatrix} = A \quad \checkmark$$

$$B = -\underline{L}^{-1}\underline{m_X} = -L^{-1}\begin{bmatrix} 5 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2.2361 \\ 3.4157 \\ -3.9528 \end{bmatrix}$$

• Problem 6. $f_{X_1X_2}(x_1, x_2) = \frac{2}{\pi\sqrt{7}} \exp\left(-\frac{8}{7}\left(x_1^2 + \frac{3}{2}x_1x_2 + x_2^2\right)\right)$

If the mean vector of $\underline{X}$ ($\underline{m_X}$) is zero, we can determine A in such a way that Y is white, and consequently Y_1 and Y_2 will be independent.

Therfore, we first check wether the mean vector is zero or not.

$\Rightarrow$ As can be inferred from the joint density, the density corresponds to a jointly normal distribution of two variables, Thus we have: $\rho_{X_1X_2} = -\frac{3}{4}$, $\sigma_{X_1} = \sigma_{X_2} = 1$, $m_{X_1} = m_{X_2} = 0$

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$$C_x = E((\underline{X}-\underline{m}_x)(\underline{X}-\underline{m}_x)^H) = E(\underline{X}\underline{X}^H) = \begin{bmatrix} E(X_1^2) & E(X_1X_2) \\ E(X_2X_1) & E(X_2^2) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho_{X_1X_2} \\ \rho_{X_1X_2} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix}$$

Now, we compute the eigen values and vectors of C_x:

$$|C_x - \lambda I| = (1-\lambda)^2 - \frac{9}{16} = 0 \longrightarrow (1-\lambda)^2 = \frac{9}{16} \longrightarrow 1-\lambda = \pm\frac{3}{4} \longrightarrow \begin{cases} \lambda = \frac{7}{4} \\ \lambda = \frac{1}{4} \end{cases} \Rightarrow C_x : \text{p.d.}$$

$$\underline{C_x}\,\underline{q} = \lambda \underline{q} \xrightarrow{\lambda=\frac{7}{4}} \left.\begin{cases} q_1 - \frac{3}{4}q_2 = \frac{7}{4}q_1 \\ -\frac{3}{4}q_1 + q_2 = \frac{7}{4}q_2 \end{cases}\right\} \Rightarrow q_1 = -q_2 \rightarrow \underline{q} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$\xrightarrow{\lambda=\frac{1}{4}} \left.\begin{cases} q_1 - \frac{3}{4}q_2 = \frac{1}{4}q_1 \\ -\frac{3}{4}q_1 + q_2 = \frac{1}{4}q_2 \end{cases}\right\} \Rightarrow q_1 = q_2 \rightarrow \underline{q} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\Longrightarrow Q = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow L = Q\Lambda^{1/2} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & \sqrt{7}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/4 & \sqrt{14}/4 \\ \sqrt{2}/4 & -\sqrt{14}/4 \end{bmatrix} = \frac{\sqrt{2}}{4}\begin{bmatrix} 1 & \sqrt{7} \\ 1 & -\sqrt{7} \end{bmatrix}$$

$$\Rightarrow L^{-1} = \sqrt{\frac{2}{7}} \times \frac{4}{\sqrt{2}}\begin{bmatrix} -\sqrt{7} & -\sqrt{7} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -\frac{4}{\sqrt{7}} & \frac{4}{\sqrt{7}} \end{bmatrix} = A \longrightarrow \underline{Y} = A\underline{X}$$

$$\longrightarrow \begin{cases} Y_1 = -4X_1 - 4X_2 \sim X_1 + X_2 \\ Y_2 = -\frac{4}{\sqrt{7}}X_1 + \frac{4}{\sqrt{7}}X_2 \sim X_1 - X_2 \end{cases}$$

- **Problem 3.** $X_1, X_2, \dots, X_n \Rightarrow$ are i.i.d ; $f_{X_i}(x) = f(x)$; $X_{(1)} \geqslant X_{(2)} \geqslant \dots \geqslant X_{(n)} \Rightarrow \underline{X}^{os} = [X_{(1)}, X_{(2)}, \dots, X_{(n)}]^T$

- Part a. $f_{\underline{X}^{os}}(x_1, x_2, \dots, x_n)$

We know that: $f_{\underline{X}}(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n) = f(x_1) f(x_2) \dots f(x_n)$

$$f_{\underline{Y}}(y_1, y_2, \dots, y_n) = \sum_i \frac{f_{\underline{X}}(x_{1_i}, x_{2_i}, \dots, x_{n_i})}{|J(x_{1_i}, \dots, x_{n_i})|}$$

$X_1 = X_{(1)} = X_i$
$X_2 = X_{(2)} = X_j \quad (i \neq i)$
...

$$\Rightarrow |J| = 1 \Rightarrow f_{\underline{X}^{os}}(\underline{x}) = \frac{f(x_1) f(x_2) \dots f(x_n)}{1} = f(x_1) f(x_2) \dots f(x_n)$$