University of Tehran,
College of Engineering

Homework #1

Stochastic Process – Fall 2024
Instructor: **Dr. Ali Olfat**

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Problem 1. $A_1 \cap A_2 \cap \dots \cap A_n = \emptyset$, $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$, X is a random variable defined on Ω

■ Part a. Considering that A_i $(i=1,\dots,n)$ partition the sample space (Ω), and X is defined on sample space (Ω), we can write:

$$X = \{X \cap A_1\} \cup \{X \cap A_2\} \cup \dots \cup \{X \cap A_n\} \longrightarrow \{X<x\} = \{X<x \cap A_1\} \cup \dots \cup \{X<x \cap A_n\}$$

$$\longrightarrow P(X<x) = P(X<x, A_1) + \dots + P(X<x, A_n)$$

Also, we can write: $P(X<x, A_i) = P(X<x \mid A_i) P(A_i) = F_X(x \mid A_i) P(A_i)$

$$\longrightarrow P(X<x) = F_X(x) = F_X(x, A_1) + \dots + F_X(x, A_n) = F_X(x|A_1)P(A_1) + \dots + F_X(x|A_n)P(A_n)$$

If we take the derivative of both sides of the equation respect to x, then:

$$\xrightarrow{\partial/\partial x} f_X(x) = f_X(x|A_1)P(A_1) + \dots + f_X(x|A_n)P(A_n)$$

★ **Part a (another way):** $\{X=x\} = \lim_{\Delta\to 0} \{x-\frac{\Delta}{2} < X \leq x+\frac{\Delta}{2}\}$

Considering that A_i $(i=1,\dots,n)$ partition the sample space Ω, and X is defined on Ω, we can write:

$$\{X=x\} = \lim_{\Delta\to 0}\{x-\tfrac{\Delta}{2} < X \leq x+\tfrac{\Delta}{2}\} = \lim_{\Delta\to 0}\left[[\{x-\tfrac{\Delta}{2}<X\leq x+\tfrac{\Delta}{2}\} \cap A_1] \cup \dots \cup [\{x-\tfrac{\Delta}{2}<X\leq x+\tfrac{\Delta}{2}\}]\right]$$

Also we have: $P(\{x-\frac{\Delta}{2}<X\leq x+\frac{\Delta}{2}\} \cap A_i) = \underbrace{P(\{x-\frac{\Delta}{2}<X\leq x+\frac{\Delta}{2}\} \mid A_i)}_{F_X(x+\frac{\Delta}{2}|A_i) - F_X(x-\frac{\Delta}{2}|A_i)} P(A_i)$

$$\Rightarrow \lim_{\Delta\to 0} \overbrace{P(x-\tfrac{\Delta}{2}<X\leq x+\tfrac{\Delta}{2})}^{F_X(x+\frac{\Delta}{2}) - F_X(x-\frac{\Delta}{2})} = \lim_{\Delta\to 0}\sum_{i=1}^{n}\left[F_X(x+\tfrac{\Delta}{2}|A_i) - F_X(x-\tfrac{\Delta}{2}|A_i)\right]P(A_i)$$

By dividing both sides of above equation by Δ and using the definition of derivative, we have:

$$\Rightarrow \underbrace{\lim_{\Delta\to 0}\frac{F_X(x+\frac{\Delta}{2}) - F_X(x-\frac{\Delta}{2})}{\Delta}}_{\partial F_X(x)/\partial x = f_X(x)} = \sum_{i=1}^{n}\underbrace{\lim_{\Delta\to 0}\frac{F_X(x+\frac{\Delta}{2}|A_i) - F_X(x-\frac{\Delta}{2}|A_i)}{\Delta}}_{\partial F_X(x|A_i)/\partial x = f_X(x|A_i)} P(A_i)$$

$$\longrightarrow f_X(x) = \sum_{i=1}^{n} f_X(x|A_i)P(A_i) = f_X(x|A_1)P(A_1) + \dots + f_X(x|A_n)P(A_n)$$

■ Part b. Considering the result from the previous part, if we multiply both sides of the equation by x and the integrate with respect to x, we will have:

$$f_X(x) = f_X(x|A_1)P(A_1) + \dots + f_X(x|A_n)P(A_n) \longrightarrow \int_{-\infty}^{+\infty} x f_X(x)dx = \sum_{i=1}^{n}\int_{-\infty}^{+\infty} x f_X(x|A_i)P(A_i)dx$$

$$\longrightarrow E(X) = \sum_{i=1}^{n} P(A_i)\int_{-\infty}^{+\infty} x f_X(x|A_i)dx = \sum_{i=1}^{n} P(A_i)E(X|A_i) = E(X|A_1)P(A_1) + \dots + E(X|A_n)P(A_n)$$

■ Part c. We can write: $\{X=x\} = \lim_{\Delta\to 0}\{x-\frac{\Delta}{2}<X\leq x+\frac{\Delta}{2}\}$

$$\rightarrow P(B|X=x) = \lim_{\Delta\to 0} P(B \mid x-\tfrac{\Delta}{2}<X\leq x+\tfrac{\Delta}{2}) = \lim_{\Delta\to 0}\frac{P(B, x-\Delta/2<X\leq x+\Delta/2)}{P(x-\Delta/2<X\leq x+\Delta/2)}$$

$$= \lim_{\Delta\to 0}\frac{P(x-\frac{\Delta}{2}<X\leq x+\frac{\Delta}{2} \mid B)P(B)}{P(x-\frac{\Delta}{2}<X\leq x+\frac{\Delta}{2})} = \lim_{\Delta\to 0}\frac{F_X(x+\Delta/2|B) - F_X(x-\Delta/2|B)}{F_X(x+\Delta/2) - F_X(x-\Delta/2)}P(B)$$

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By dividing the numerator and denominator by x and using the definition of derivative, we will have:

$$\longrightarrow P(B \mid X=x) = \lim_{\Delta \to 0} \frac{\dfrac{F_X(x+\frac{\Delta}{2} \mid B) - F_X(x-\frac{\Delta}{2} \mid B)}{\Delta}}{\dfrac{F_X(x+\frac{\Delta}{2}) - F_X(x-\frac{\Delta}{2})}{\Delta}} P(B) = \frac{\dfrac{\partial F_X(x \mid B)}{\partial x}}{\dfrac{\partial F_X(x)}{\partial x}} P(B) = \frac{f_X(x \mid B)}{f_X(x)} P(B)$$

Now, we can write: $\displaystyle\int_{-\infty}^{+\infty} P(B \mid X=x) f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{f_X(x \mid B)}{\cancel{f_X(x)}} P(B) \cdot \cancel{f_X(x)}\,dx = P(B) \overbrace{\int_{-\infty}^{+\infty} f_X(x \mid B)\,dx}^{=1}$

$$\longrightarrow P(B) = \int_{-\infty}^{+\infty} P(B \mid X=x) f_X(x)\,dx = E_X\big(P(B \mid X=x)\big)$$

Problem 2. Find $f_X(x \mid a < X \leqslant b)$

First we drive the conditional CDF, and then by differetiating with respect to x, we obtain the conditional PDF.

$$F_X(x \mid a < X \leqslant b) = P(X < x \mid a < X \leqslant b) = \frac{P(X<x\ ,\ a<X\leqslant b)}{\underbrace{P(a<X\leqslant b)}_{F_X(b)-F_X(a)}}$$

Considering the range of x, we compute the CDF:

if $x<a$: $P(X<x\ ,\ a<X\leqslant b) = 0$

if $a<x<b$: $P(X<x\ ,\ a<X\leqslant b) = P(a<X\leqslant x) = F_X(x) - F_X(a)$

if $b<x$: $P(X<x\ ,\ a<X\leqslant b) = P(a<X\leqslant b)$

So we can write:

$$F_X(x \mid a<X\leqslant b) = \begin{cases} 0 & , \quad x<a \\ \dfrac{F_X(x)-F_X(a)}{F_X(b)-F_X(a)} & , \quad a<x\leqslant b \\ 1 & , \quad b<x \end{cases} \xrightarrow{\partial/\partial x} f_X(x \mid a<x\leqslant b) = \begin{cases} \dfrac{f_X(x)}{F_X(b)-F_X(a)} & , \quad a<x\leqslant b \\ 0 & , \quad o.w. \end{cases}$$

Problem 5. $f_{xy}(x,y) = \begin{cases} \frac{1}{\pi} , & x^2+y^2 \leqslant 1 \\ 0 , & o.w. \end{cases}$ $\longrightarrow |x| \leqslant \sqrt{1-y^2}\ ,\ |y| \leqslant \sqrt{1-x^2}$

Part a. First we drive the marginal PDF of x and y, then we check if X and Y are independent.

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{dy}{\pi} = \frac{2\sqrt{1-x^2}}{\pi} \xrightarrow{\text{symmetry}} f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi} \qquad |x|\leqslant 1\ ,\ |y| \leqslant 1$$

$$f_{xy}(x,y) \neq f_X(x) \cdot f_Y(y) \Rightarrow X \text{ and } Y \text{ aren't independent}$$

Part b. $\displaystyle E(X \mid Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x \mid Y=y)\,dx = \int_{-\infty}^{+\infty} x \frac{f_{XY}(x,y)}{f_Y(y)}\,dx = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{\overset{\text{odd}}{x}}{2\sqrt{1-y^2}}\,dx = 0$

Part c. $R = g(X,Y) = \sqrt{X^2+Y^2}$, $\Theta = h(X,Y) = \tan^{-1}\left(\frac{Y}{X}\right)$, $\displaystyle f_{R\Theta}(r,\theta) = \sum_i \frac{f_{XY}(x_i,y_i)}{|J(x_i,y_i)|}$

roots: $x_1 = r\cos\theta$, $y_1 = r\sin\theta$

$$\frac{\partial g}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\ ,\ \frac{\partial g}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}\ ,\ \frac{\partial h}{\partial x} = \frac{-y}{x^2+y^2}\ ,\ \frac{\partial h}{\partial y} = \frac{x}{x^2+y^2} \longrightarrow J = \begin{vmatrix} \partial g/\partial x & \partial g/\partial y \\ \partial h/\partial x & \partial h/\partial y \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}} \rightarrow J(x_1,y_1) = \frac{1}{r}$$

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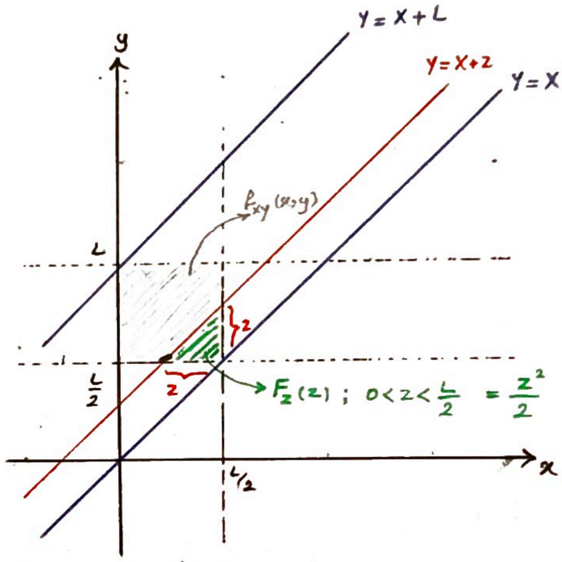
$\Rightarrow f_{R\Theta}(r,\theta) = \frac{1/\pi}{1/r} = \frac{r}{\pi}$ (The joint pdf of the polar coordinates) $0<r<1$, $0<\theta<2\pi$

$\rightarrow f_R(r) = \int_0^{2\pi} f_{R\Theta}(r,\theta)\,d\theta = \int_0^{2\pi} \frac{r}{\pi}\,d\theta = 2r$, $0 \leq r \leq 1$

$\rightarrow f_\Theta(\theta) = \int_0^1 f_{R\Theta}(r,\theta)\,dr = \int_0^1 \frac{r}{\pi}\,dr = \frac{1}{2\pi}$, $0 \leq \theta < 2\pi \longrightarrow \Theta \sim \text{Uniform}[0,2\pi)$

Problem 6. $X \sim U[0,\frac{L}{2}]$; $Y \sim U[\frac{L}{2},L]$; $Z = Y - X$

$$X \perp\!\!\!\perp Y \Rightarrow f_{xy}(x,y) = f_x(x) f_y(y) = \begin{cases} \frac{4}{L^2} & , \ 0 \leq x \leq \frac{L}{2} , \ \frac{L}{2} \leq y \leq L \\ 0 & , \ \text{otherwise} \end{cases}$$



First, we drive the CDF of Z ($F_z(z)$):

$z < 0 : F_z(z) = 0$

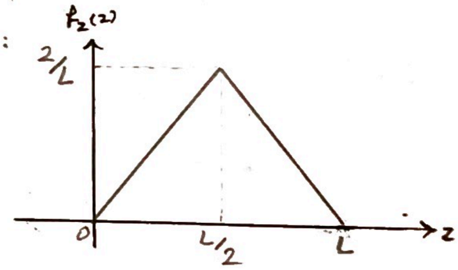
$0 \leq z \leq \frac{L}{2} : F_z(z) = \frac{4}{L^2} \cdot \frac{z^2}{2} = \frac{2z^2}{L^2}$

$\frac{L}{2} \leq z \leq L : F_z(z) = \frac{4}{L^2}\left(\frac{L^2}{4} - \frac{(4-z)^2}{2}\right) = 1 - 2\left(1-\frac{z}{L}\right)^2$

$z \geq 1 : F_z(z) = 1$

Now, by differtiating with respect to z, we can reach $f_z(z)$:

$$\Rightarrow f_z(z) = \frac{\partial F_z(z)}{\partial z} = \begin{cases} \frac{4z}{L^2} & ; \ 0 \leq z \leq \frac{L}{2} \\ \frac{4}{L}\left(1-\frac{z}{L}\right) & ; \ \frac{L}{2} \leq z \leq L, \\ 0 & ; \ \text{otherwise} \end{cases} \Rightarrow$$



Finaly, we can drive the E(Z):

$$E(Z) = \int_{-\infty}^{+\infty} z f_z(z)\,dz = \int_0^{L/2} \frac{4z^2}{L^2}\,dz + \underbrace{\int_{L/2}^{L} \frac{4z}{L}\left(1-\frac{z}{L}\right)dz}^{z'=z-L/2} = \int_0^{L/2} \frac{4z^2}{L^2}\,dz + \int_0^{L/2} \left(\frac{2z}{L} - \frac{4z^2}{L^2}\right)dz = \int_0^{L/2} \frac{2z}{L}\,dz = \frac{L}{2}$$

another way $\Rightarrow E(Z) = E(Y-X) = E(Y) - E(X) = \frac{3L}{4} - \frac{L}{4} = \frac{L}{2}$

Problem 3. $X \sim U[0,2\pi)$, $Y = \sin(X)$

■ **Part a.** $f_Y(y) = ?$ $Y = g(X) \longrightarrow g'(X) = \cos(X)$; root of $Y = g(X) \Rightarrow \begin{cases} x_1 = \sin^{-1}(y) \ ; \ 0<y<1 \\ x_2 = -\sin^{-1}(y) \ ; \ -1<y<0 \end{cases}$

$$\Rightarrow f_Y(y) = \sum_i \frac{f_X(x_i)}{|g'(x_i)|} = \frac{1/2\pi}{|\cos(\sin^{-1}(y))|} + \frac{1/2\pi}{|\cos(-\sin^{-1}(y))|} = \frac{1}{\pi\sqrt{1-y^2}} \ ; \ |y| < 1$$

■ **Part b.** $E(Y \mid 0 \leq X < \pi) = ?$

First we replace the condition with $Y = \sin(X) \longrightarrow E(Y \mid 0 \leq X < \pi) = E(Y \mid 0 \leq Y \leq 1)$

Considering the result of Problem 2, we can say: $f_Y(Y \mid 0 \leq Y \leq 1) = \frac{f_Y(y)}{F_Y(1) - F_Y(0)} = 2 f_Y(y)$

So we can write:

$$E(Y \mid 0 \leqslant X \leqslant \pi) = \int_{-\infty}^{+\infty} y f_Y(y \mid 0 \leqslant Y \leqslant 1)\, dy = 2\int_0^1 y f_Y(y)\, dy = \frac{2}{\pi}\int_0^1 \frac{y}{\sqrt{1-y^2}}\, dy = \frac{2}{\pi}\sqrt{1-y^2}\Big|_1^0 = \frac{2}{\pi}$$

Problem 4. $f_X(x) = \frac{1}{2}e^{-|x|}$; $Y = \begin{cases} \sqrt{X} & , X \geqslant 0 \\ 0 & , X < 0 \end{cases}$

First, we want to obtain the $F_X(x)$, then we drive the CDF of Y:

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \begin{cases} \frac{1}{2}e^{x} & ; x \leqslant 0 \\ 1 - \frac{1}{2}e^{-x} & ; x \geqslant 0 \end{cases}$$

for $y < 0$: $F_Y(y) = P(Y < y) = 0$

for $y \geqslant 0$: $F_Y(y) = P(Y \leqslant y) = P(Y=0) + P(\sqrt{X} \leqslant y) = \underbrace{P(X \leqslant 0)}_{F_X(0)} + \underbrace{P(0 < X \leqslant y^2)}_{F_X(y^2) - F_X(0)} = F_X(y^2) = 1 - \frac{e^{-y^2}}{2}$

$$\Rightarrow F_Y(y) = \left(1 - \frac{1}{2}e^{-y^2}\right)u(y) \xrightarrow{\partial/\partial y} f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \frac{1}{2}\delta(y) + ye^{-y^2}u(y)$$

$$\Rightarrow f_Y(y) = \frac{1}{2}\delta(y) + ye^{-y^2}u(y) \quad \text{or} \quad f_Y(y) = \begin{cases} \frac{1}{2} & , y = 0 \\ ye^{-y} & , y > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Problem 7. $X \perp\!\!\!\perp Y$, $X, Y \sim Bin(\lambda)$ $\quad f_X(\alpha) = f_Y(\alpha) = \begin{cases} \lambda e^{-\lambda\alpha} & , \alpha > 0 \\ 0 & , \text{o.w.} \end{cases}$

■ Part a. PDF of $V = \frac{X}{X+Y}$

First, we define $W = X$ and then using Jacobian method, we drive the pdf of V.

$$\underset{0<v<1}{V = g(X,Y) = \frac{X}{X+Y}} \;,\; \underset{0<\omega}{W = h(X,Y) = X} \longrightarrow \frac{\partial g}{\partial x} = \frac{y}{(x+y)^2} \;,\; \frac{\partial g}{\partial y} = \frac{-x}{(x+y)^2} \;,\; \frac{\partial h}{\partial x} = 1 \;,\; \frac{\partial h}{\partial y} = 0$$

$$\text{roots: } x_i = w \,,\, y_i = \frac{w}{v} - w \longrightarrow |J| = \begin{vmatrix} \partial g/\partial x & \partial g/\partial y \\ \partial h/\partial x & \partial h/\partial y \end{vmatrix} = \frac{x}{(x+y)^2} = \frac{v^2}{\omega}$$

$$f_{VW}(v,\omega) = \frac{f_{XY}(w, \frac{w}{v} + w)}{v^2/\omega} = \frac{w}{v^2}\lambda e^{-\omega}\cdot\lambda e^{w - \frac{\omega}{v}} = \frac{\lambda^2\omega}{v^2}e^{-\lambda\omega/v} \xrightarrow{\int \cdot\, d\omega} f_V(v)$$

$$\Rightarrow f_V(v) = \int_0^\infty f_{VW}(v,\omega)\, d\omega = \frac{\lambda^2}{v^2}\int_0^\infty \omega e^{-\frac{\lambda\omega}{v}}\, d\omega = \frac{\lambda^2}{v^2}\left(\frac{v^2}{\lambda^2}\right) = 1 \Rightarrow f_V(v) \overset{0<v<1}{=} 1 \Rightarrow V \sim U(0,1)$$

■ Part b. $U = \min\{X, Y\}$, $P\{X = U\}$ $\longrightarrow$ Region of Integration: $0 < x < y < \infty$

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \lambda^2 e^{-\lambda(x+y)} \Rightarrow P(X = U) = \iint_{R_{xy}} f_{XY}(x,y)\, dx\, dy$$

$$\Rightarrow P(X = U) = \int_0^\infty \int_0^y \lambda^2 e^{-\lambda(x+y)}\, dx\, dy = \lambda\int_0^\infty e^{-\lambda y}\left(-e^{-\lambda x}\right)\Big|_0^y dy = \lambda\int_0^\infty e^{-\lambda y}(1 - e^{-\lambda y})\, dy$$

$$= \left(-e^{-\lambda y} + \frac{1}{2}e^{-2\lambda y}\right)\Big|_0^\infty = 1 - \frac{1}{2} = \frac{1}{2} \quad \text{(Because of the symmetry of } X \text{ and } Y\text{)} \checkmark$$

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■ Part c. $Z = \max\{X,Y\} - \min\{X,Y\} \longrightarrow Z = |X-Y| \longrightarrow 0<z<\infty$

if $X \geq Y \Rightarrow Z = X - Y$, else $\Rightarrow Z = Y - X$

$$\Rightarrow F_Z(z) = P(Z \leq z) = P(\overbrace{X-Y \leq z}^{X \leq z+Y}, X>Y) + P(\overbrace{Y-X \leq z}^{Y \leq z+X}, X<Y)$$

$$= \int_0^\infty \int_y^{z+y} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_x^{x+z} f_{XY}(x,y)\,dy\,dx$$

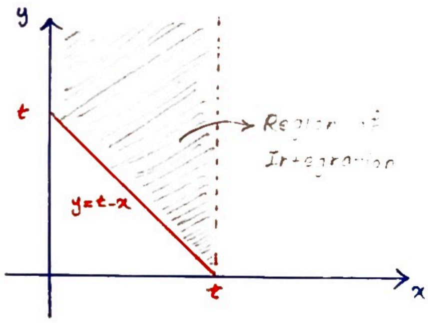
$$= \lambda^2 \int_0^\infty e^{-\lambda y} \int_y^{y+z} e^{-x\lambda}\,dx\,dy + \lambda^2 \int_0^\infty e^{-\lambda x} \int_x^{x+z} e^{-\lambda y}\,dy\,dx$$

$$= \lambda(1-e^{-\lambda z}) \underbrace{\int_0^\infty e^{-2\lambda y}\,dy}_{1/2\lambda} + \lambda(1-e^{-\lambda z}) \underbrace{\int_0^\infty e^{-2\lambda x}\,dx}_{1/2\lambda} = 1 - e^{-\lambda z}$$

$$\Rightarrow F_Z(z) = 1 - e^{-\lambda z} \longrightarrow f_Z(z) = \frac{\partial F_Z(z)}{\partial z} = \lambda e^{-\lambda z} \longrightarrow f_Z(z) = \lambda e^{-\lambda z}, \quad 0<z<\infty$$

■ Part d. $P\{X \leq t < X+Y\}$ where $t>0$

$$\{X \leq t < X+Y\} \Rightarrow \begin{cases} Y > t-x \\ 0 < X \leq t \end{cases} \Rightarrow P\{X \leq t < X+Y\} = \int_0^t \int_{t-x}^\infty f_{XY}(x,y)\,dy\,dx$$



$$\Rightarrow P\{X \leq t < X+Y\} = \int_0^t \int_{t-x}^\infty \lambda^2 e^{-\lambda(x+y)}\,dy\,dx = \lambda^2 \int_0^t e^{-\lambda x} \int_{t-x}^\infty e^{-\lambda y}\,dy\,dx$$

$$= \lambda \int_0^t e^{-\lambda x} (e^{\lambda x - \lambda t})\,dx = \lambda e^{-\lambda t} \int_0^t dx = \lambda t e^{-\lambda t}$$

$$\Rightarrow P\{X \leq t < X+Y\} = \lambda t e^{-\lambda t}$$

Problem 8. The number of electrons that leave the cathods: $X \sim Poisson(\lambda)$
The probability that an electron hits the anode is p: $Y \sim Bernollie(p)$

$$f_X(x) = \lambda^x \frac{e^{-\lambda}}{x!}, \quad x = 1,2,\dots \quad , \quad f_Y(y) = p^y(1-p)^y, \quad y = 0,1$$

⊛→ The number of electrons that hits the anode :

$$E(X) = \lambda, \quad var(X) = \lambda, \quad E(Y) = p, \quad var(Y) = p(1-p)$$

Iteratrial Expectation Theorem: $P(A) = E_X(P(A \mid X = x))$

$$P(Z=k) = E_X(P(Z \mid X=n)) = \sum_{n=0}^\infty \binom{n}{k} p^k (1-p)^{n-k} \lambda^n \frac{e^{-\lambda}}{n!} = \left(\frac{p}{1-p}\right)^k e^{-\lambda} \sum_{n=k}^\infty \binom{n}{k} \frac{(1-p)^n \lambda^n}{n!}$$

$$= \left(\frac{p}{1-p}\right)^k e^{-\lambda} \sum_{n=k}^\infty \frac{n!(1-p)^n \lambda^n}{(n-k)!\,k!\,n!} = \left(\frac{p}{1-p}\right)^k \frac{e^{-\lambda}}{k!} \sum_{n=k}^\infty \frac{\lambda^n (1-p)^n}{\underbrace{(n-k)}_{n'}!} = (p\lambda)^k \frac{e^{-\lambda}}{k!} \sum_{n'=0}^\infty \frac{\lambda^{n'}(1-p)^{n'}}{n'!}$$

$$\circledast \rightarrow e^{x} = 1 + x + \cdots + \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow P(Z=k) = (\lambda p)^k \frac{e^{-\lambda}}{k!} \sum_{n=0}^{\infty} \frac{[(1-p)\lambda]^n}{n!} = (\lambda p)^k \frac{e^{-\lambda}}{k!} e^{\lambda(1-p)} = (\lambda p)^k \frac{e^{-\lambda p}}{k!} \Rightarrow Z \sim Poisson(\lambda p)$$

$$\Rightarrow f_Z(z) = (\lambda p)^z \frac{e^{-\lambda p}}{z!} \quad , \quad z = 1, 2, \cdots$$

$$Z \sim Poisson(\lambda p) \Rightarrow E(Z) = var(Z) = \lambda p$$

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Problem 9. $f_{XY}(x,y) = \begin{cases} Axy^2 & , \quad 0 < 2y \leqslant x \leqslant 2 \\ 0 & , \quad \text{otherwise} \end{cases}$

Part a. $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = \int_0^2\int_0^{x/2} Axy^2\,dy\,dx = A\int_0^2 x\int_0^{x/2} y^2 dy\,dx = \frac{A}{24}\int_0^2 x^4 dx = \frac{32}{120}A = 1 \longrightarrow \boxed{A = \frac{15}{4}}$

Part b. First we determine the morginal PDF of X and Y, then we check if X and Y are independent.

$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \int_0^{x/2}\frac{15}{4}xy^2\,dy = \frac{5}{32}x^4 \;,\; 0<x<2 \qquad , \qquad f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \int_{2y}^{2}\frac{15}{4}xy^2\,dx = \frac{15}{2}y^2(1-y^2) \;\; 0<y<1$

$f_{XY}(x,y) \neq f_X(x)f_Y(y) \longrightarrow$ X and Y aren't independent.

Part c. $E(X \mid Y=y)$

$$E(X\mid Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x\mid Y=y)\,dx = \int_{-\infty}^{+\infty} x\frac{f_{XY}(x,y)}{f_Y(y)}\,dx = \int_{2y}^{2}\frac{\frac{15}{4}x^2y^2}{\frac{15}{2}y^2(1-y^2)}\,dx = \frac{x^3}{6(1-y^2)}\Bigg|_{2y}^{2} = \frac{8-8y^3}{6(1-y^2)}$$

$$= \frac{4}{3}\,\frac{1+y+y^2}{1+y} \longrightarrow E(X\mid Y=y) = \frac{4}{3}\,\frac{1+y+y^2}{1+y} \quad , \quad 0<y\leqslant 1$$

Part d. $P(X^2+Y^2 \leqslant 1 \mid X \geqslant 0.5) = ?$

$$P(X^2+Y^2\leqslant 1 \mid X\geqslant 0.5) = \frac{P(X^2+Y^2\leqslant 1 \;,\; X\geqslant 0.5)}{P(X\geqslant 0.5)} \quad \longrightarrow y\in(-\sqrt{1-x^2}\,,\sqrt{1+x^2}) \;,\; x(0.5,1)$$

$$\Rightarrow P(X^2+Y^2\leqslant 1 \;,\; X\geqslant 0.5) = \int_{0.5}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{15}{4}xy^2\,dy\,dx = \int_{0.5}^{1}\frac{5}{2}x(1-x^2)^{3/2}\,dx = I$$

$$u = 1-x^2 \longrightarrow du = -2x\,dx \longrightarrow dx = \frac{du}{-2x} \Rightarrow I = -\frac{5}{4}\int_{3/4}^{0} u^{3/2}\,du = u^{5/2}\Big|_0^{3/4} = \frac{1}{2}\left(\frac{3}{4}\right)^{5/2} = \frac{9\sqrt{3}}{64}$$

$$\Rightarrow P(X\geqslant 0.5) = \int_{0.5}^{2}\frac{5}{32}x^4\,dx = \frac{1}{32}x^5\Big|_{0.5}^{2} = \frac{1}{32}\left(32-\frac{1}{32}\right) = 1-\frac{1}{(32)^2}$$

$$\Rightarrow P(X^2+Y^2\leqslant 1 \mid X\geqslant 0.5) = \frac{9\sqrt{3}/32}{1-1/(32)^2} = \frac{4.5\sqrt{3}}{31\frac{31}{32}} = \frac{7.7942}{31.96875} \simeq 0.244$$

Problem 10. $X, Y \sim \text{Uniform}[0,1] \;,\; X \perp\!\!\!\perp Y$

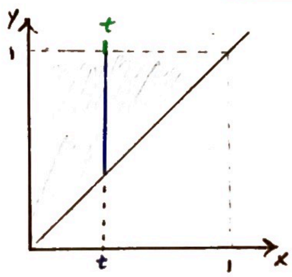
Suppose that $t = \min(X,Y)$:

if $X>Y \longrightarrow Y=t \longrightarrow E(X\mid \min(X,Y)=t=Y) = \frac{1+t}{2}$

if $X<Y \longrightarrow X=t \longrightarrow E(X\mid \min(X,Y)=t=X) = t$

$P(X>Y) = P(X<Y) = \frac{1}{2}$

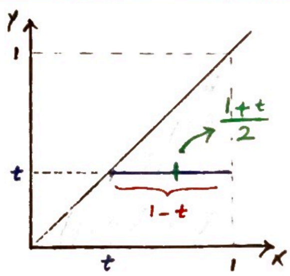
$$\Rightarrow E(X\mid \min(X,Y)=t) = \frac{t}{2}+\frac{1+t}{4} = \boxed{\frac{3t+1}{4}}$$



for $Y>X$

$t=\min(X,Y)=X$

$X=t$



for $X>Y$

$t=\min(X,Y)=Y$

$Y=t$