No :
Date:

فرایندهای اتفاقی - تمرین سری ۷   عرفان پناهی - ۸۱۰۱۰۳۰۸۴

★ سوال ۱ :

$$A_k \in \{-1,+1\}\ (\text{i.i.d}) \quad ; \quad X(t)=\sum_{k=-\infty}^{\infty} A_k P(t-kT)\ \ (P(t) \text{ is a given pulse}) \quad ; \quad Pr\{A_k=+1\}=Pr\{A_k=-1\}=\frac{1}{2}$$

★ Part a) PSD of $X(t)$:

$$X(t)=\sum_{k=-\infty}^{\infty} A_k P(t-kT) = \underbrace{\sum_{k=-\infty}^{\infty} A_k \delta(t-kT)}_{Y(t)} * P(t)$$

$$\Rightarrow S_X(f) = S_Y(f)\,|P(f)|^2 \quad ; \quad P(f)=\mathcal{F}\{P(t)\}$$

در نتیجه ابتدا $S_Y(f)$ را بدست می‌آوریم $\Leftarrow$ برای اینکار ابتدا باید $R_Y(\tau)$ را بدست آوریم:

$$R_Y(t+\tau,t) = E\{Y(t+\tau)Y^*(t)\} = E\left\{\sum_{k=-\infty}^{\infty} A_k\delta(t+\tau-kT)\sum_{\ell=-\infty}^{\infty} A_\ell\,\delta(t-\ell T)\right\}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} E\{A_kA_\ell\}\,\delta(t+\tau-kT)\,\delta(t-\ell T) \overset{A_k:\,\text{i.i.d}}{=} \sum_{k=-\infty}^{\infty} \overbrace{E\{A_k^2\}}^{\frac{1}{2}\times(-1)^2+\frac{1}{2}(1)^2=1}\,\delta(t+\tau-kT)\,\delta(t-kT)$$

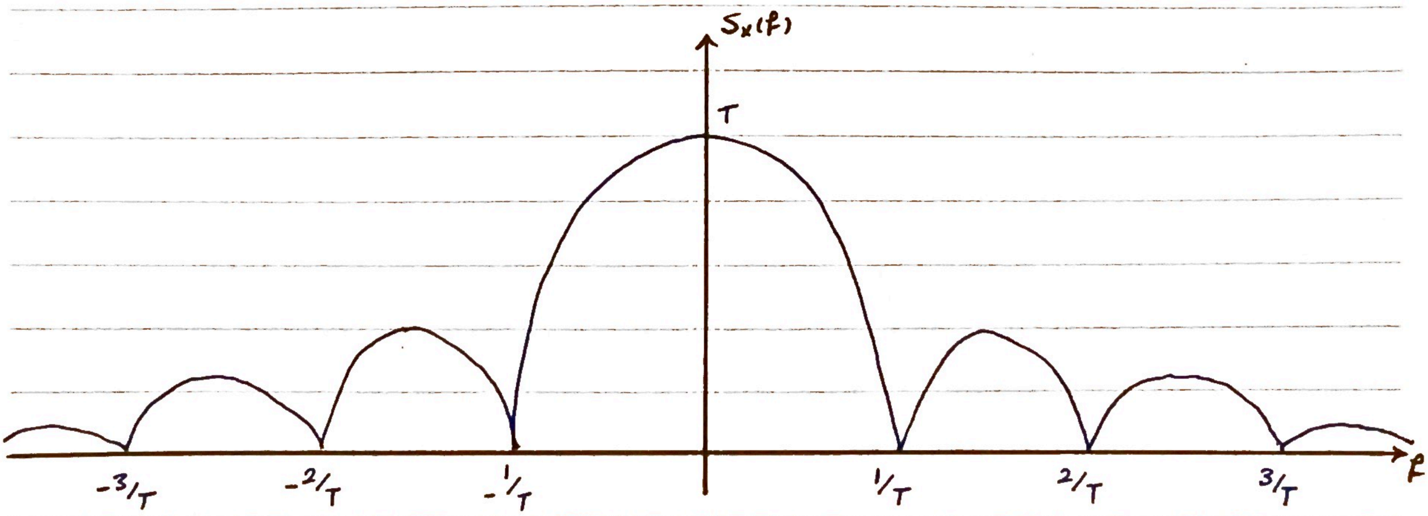
$$= \sum_{k=-\infty}^{\infty} \delta(t+\tau-kT)\,\delta(t-kT) = \delta(\tau)\underbrace{\boxed{\sum_{k=-\infty}^{\infty}\delta(t-kT)}}_{\text{میانگین: } \frac{1}{T}} \Rightarrow \bar{R}_Y(\tau) = \frac{\delta(\tau)}{T}$$

$$\bar{R}_Y(\tau) = \frac{\delta(\tau)}{T} \xrightarrow{\mathcal{F}} S_Y(f) = \frac{1}{T} \Rightarrow S_X(f) = \frac{|P(f)|^2}{T}$$

★ Part b)

$$P(t) = \Pi\left(\frac{t-0.5T}{T}\right) \xrightarrow{\mathcal{F}} P(f) = T\,\text{sinc}(Tf)\,e^{-j\pi fT} \longrightarrow |P(f)| = T\,\text{sinc}(Tf)$$

$$\Rightarrow S_X(f) = \frac{T^2\,\text{sinc}^2(Tf)}{T} = T\,\text{sinc}^2(Tf)$$



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$X(t)$: zero-mean stationary random process ; $R_X(\tau) = e^{-|\tau|}$ ★سوال ۷:

$Y(t) = \int_0^2 X(t-s)\,ds$; Power spectrum of $Y(t)$?

$H(f) = 2\,\mathrm{sinc}(2f)\,e^{-j2\pi f}$

$$\Rightarrow Y(t) = \int_0^2 X(t-s)\,ds = \int_{-\infty}^{\infty} X(t-s) \underbrace{\Pi\left(\frac{s-1}{2}\right)}_{h(s)} ds \Rightarrow h(t) = \Pi\left(\frac{t-1}{2}\right)$$

$$R_X(\tau) = e^{-|\tau|} \xrightarrow{\mathcal{F}} S_X(f) = \frac{2}{1+(2\pi f)^2} \Rightarrow S_Y(f) = S_X(f)\cdot|H(f)|^2$$

$$\Rightarrow S_Y(f) = \frac{2}{1+(2\pi f)^2}\,4\,\mathrm{sinc}^2(f) \Rightarrow S_Y(f) = \frac{8\,\mathrm{sinc}^2(f)}{1+(2\pi f)^2}$$

$X(t) = A\cos(2\pi Y t + \Theta)$; $\Theta \sim U[0, 2\pi]$; $\Theta \perp Y$; $f_Y(y)$ ★سوال ۳:

$$R_X(t+\tau, t) = E\{X(t+\tau)X^*(t)\} = E\{A\cos(2\pi Y(t+\tau)+\Theta)\cdot A^*\cos(2\pi Y t+\Theta)\}$$

$$= \frac{A^2}{2} E\{\cos(2\pi Y\tau) + \overset{\text{صفر}}{\cos(2\pi Y(2t+\tau)+2\Theta)}\} = \frac{A^2}{2} E\{\cos(2\pi Y\tau)\}$$

$$\Rightarrow S_X(f) = \mathcal{F}\left\{\frac{A^2}{2} E\{\cos(2\pi Y\tau)\}\right\} = \frac{A^2}{2} E\{\mathcal{F}\{\cos(2\pi Y\tau)\}\}$$

$$= \frac{A^2}{2} E\left\{\frac{1}{2}\delta(f-y) + \frac{1}{2}\delta(f+y)\right\} = \frac{A^2}{4} E\{\delta(f-y) + \delta(f+y)\}$$

$$= \frac{A^2}{4}\int_{-\infty}^{+\infty} (\delta(f-y) + \delta(f+y))\, f_Y(y)\,dy = \frac{A^2}{4}(f_Y(f) + f_Y(-f))$$

$$\Rightarrow S_X(f) = \frac{A^2}{4}\left(f_Y(f) + f_Y(-f)\right)$$

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★ سوال ٤ :

$X \sim U[-2, 3] \; ; \; X(t) = e^{-X}\cos(2\pi X t)$

$$X(t) \longrightarrow \boxed{\begin{matrix}\text{Low-pass}\\ f_0 = 1\end{matrix}} \longrightarrow Y(t)$$

★ Part a) PSD of $X(t)$:

$$R_X(t+\tau, t) = E\{e^{-X}\cos(2\pi X(t+\tau))\, e^{-X}\cos(2\pi X t)\}$$

$$= \frac{1}{2} E\{e^{-2X}(\cos(2\pi X\tau) + \cos(2\pi X(2t+\tau))\}$$

$$\Rightarrow \overline{R_X(\tau)} = \frac{1}{2} E\{e^{-2X}\cos(2\pi X\tau)\}$$

$$\longrightarrow S_X(f) = \mathcal{F}\{R_X(\tau)\} = \frac{1}{2} E\{e^{-2X}\mathcal{F}\{\cos(2\pi X\tau)\}\}$$

$$= \frac{1}{4} E\{e^{-2X}(\delta(f-X) + \delta(f+X))\}$$

$$= \frac{1}{4}\int_{-2}^{3} \frac{1}{5} e^{-2x}(\delta(f-x) + \delta(f+x))\, dx$$

$x \in [-2, 3] \leftarrow \Pi\left(\frac{f-0.5}{5}\right)$

$$= \frac{1}{20}\int_{-2}^{3} e^{-2x}\delta(f+x)\, dx + \frac{1}{20}\int_{-3}^{2} e^{+2x}\delta(f+x)\, dx$$

$\rightarrow x \in [-3, 2] \Rightarrow \Pi\left(\frac{f+0.5}{5}\right)$

$$= \frac{1}{20} e^{2f}\,\Pi\left(\frac{f-0.5}{5}\right) + \frac{1}{20} e^{-2f}\,\Pi\left(\frac{f+0.5}{5}\right)$$

★ Part b) $S_Y(f) = S_X(f) \cdot |H(f)|^2 \quad ; \quad H(f) = \Pi\left(\frac{f}{2}\right)$

$$\Rightarrow S_Y(f) = \frac{1}{20}\underbrace{(e^{2f} + e^{-2f})}_{2\cosh(2f)}\,\Pi\left(\frac{f}{2}\right) = \frac{1}{10}\cosh(2f)\,\Pi\left(\frac{f}{2}\right)$$

Power of $Y(t)$: $P_Y = \int_{-\infty}^{+\infty} S_Y(f)\, df = \int_{-1}^{1} \frac{1}{10}\cosh(2f)\, df = \frac{1}{20}\sinh(2f)\Big|_{-1}^{1}$

$$\Rightarrow P_Y = \frac{1}{20}[\sinh(2) \underbrace{- \sinh(-2)}_{= \sinh(2)}] = \frac{1}{10}\sinh(2)$$

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★سوال ۴ :

$$S_X(f) = \frac{4(\pi^2 f^2+1)}{(4\pi^2 f^2+1)(4\pi^2 f^2+9)} \xrightarrow{s=j2\pi f} S_X(s) = \frac{4(1-\frac{s^2}{4})}{(1-s^2)(9-s^2)} = \frac{(2-s)(2+s)}{(1-s)(1+s)(3-s)(s+3)}$$

★ Part a) innovation process of $X(t)$:

innovation

$$S_X(s) = L(s).L^*(-s) \Rightarrow L(s) = \frac{2+s}{(1+s)(3+s)} = \frac{1}{3+s} + \frac{1}{(1+s)(3+s)} = \frac{1}{3+s} + \frac{1}{2}\frac{1}{1+s} - \frac{1}{2}\frac{1}{3+s}$$

صفرها و قطب‌ها در سمت چپ

$$= \frac{1}{2}\left(\frac{1}{1+s} + \frac{1}{3+s}\right)$$

$$L(s) = \frac{1}{2}\left(\frac{1}{1+s} + \frac{1}{3+s}\right) \xrightarrow{\mathcal{L}^{-1}} \underline{L(t) = \frac{1}{2}\left(e^{-3t} + e^{-t}\right)u(t)}$$

★ Part b) $R_Y(\tau) = e^{-|\tau|} \longrightarrow S_Y(f) = \frac{2}{1+4\pi^2 f^2}$; $|H(f)|^2 = \frac{S_Y(f)}{S_X(f)} = \frac{4\pi^2 f^2+9}{2(\pi^2 f^2+1)}$

$$\Rightarrow |H(f)|^2 = \frac{(3+j2\pi f)(3-j2\pi f)}{2(1-j\pi f)(1+j\pi f)} = H(f)H^*(f) \Rightarrow H(f) = \frac{3+j2\pi f}{\sqrt{2}(1+j\pi f)}$$

$$H(f) \xrightarrow{\mathcal{F}^{-1}} h(t) = 3\sqrt{2}\, e^{-2t}u(t) + \frac{\partial}{\partial t}\left(\sqrt{2}\, e^{-2t}u(t)\right) = \sqrt{2}\, e^{-2t}u(t) + \sqrt{2}\,\delta(t)$$

$$\Rightarrow \underline{h(t) = \sqrt{2}\left(\delta(t) + e^{-2t}u(t)\right)}$$

★سوال ۵ :

$$X_1(t) \longrightarrow \boxed{h_1(t)} \longrightarrow Y_1(t) \qquad X_2(t) \longrightarrow \boxed{h_2(t)} \longrightarrow Y_2(t)$$

★ Part a) Find $R_{X_1Y_2}(t_1,t_2)$ and $S_{X_1Y_2}(f)$

$$R_{X_1Y_2}(t_1,t_2) = E\left(X_1(t_1)Y_2^*(t_2)\right) = E\left(X_1(t_1)\left(X_2^*(t_2) * h_2^*(t_2)\right)\right) = \underbrace{E\left(X_1(t_1)X_2^*(t_2)\right)}_{R_{X_1X_2}(t_1,t_2)} * h_2^*(t_2)$$

$$\longrightarrow R_{X_1Y_2}(t_1,t_2) = R_{X_1X_2}(t_1,t_2) * h_2^*(t_2) \longrightarrow \overline{R_{X_1Y_2}(\tau)} = \overline{R_{X_1X_2}(\tau)} * h_2^*(-\tau)$$

$$S_{X_1Y_2}(f) = \mathcal{F}\{R_{X_1Y_2}(\tau)\} = \mathcal{F}\{R_{X_1X_2}(\tau)\}\,\mathcal{F}\{h_2^*(-\tau)\} = S_{X_1X_2}(f).H_2^*(f)$$

★ Part b) $R_{Y_1Y_2}(t_1, t_2)$ and $S_{Y_1Y_2}(f)$

$$R_{Y_1Y_2}(t_1,t_2) = E\{Y_1(t_1)Y_2^*(t_2)\} = E\{X_1(t_1) * h_1(t_1) \cdot X_2^*(t_2) * h_2^*(t_2)\}$$

$$= \underbrace{E\{X_1(t_1)X_2^*(t_2)\}}_{R_{X_1X_2}(t_1,t_2)} * h_1(t_1) * h_2^*(t_2) = R_{X_1X_2}(t_1,t_2) * h_1(t_1) * h_2^*(t_2)$$

$$\Rightarrow R_{Y_1Y_2}(\tau) = R_{X_1X_2}(\tau) * h_1(\tau) * h_2^*(-\tau)$$

$$\xrightarrow{\mathcal{F}} S_{Y_1Y_2}(f) = S_{X_1X_2}(f) \cdot H_1(f) \cdot H_2^*(f)$$

★ Part c) Show that: $R_{X_1X_2}(t+\tau, t) = R_{X_1X_2}(\tau) \longrightarrow R_{Y_1Y_2}(t+\tau; t) = R_{Y_1Y_2}(\tau)$

$$R_Y(t+\tau,t) = E\{Y(t+\tau)Y^*(t)\} = E\{(X_1(t+\tau) * h_1(t+\tau))(X_2^*(t) * h_2(t))\}$$

$$= E\left\{\int_{-\infty}^{\infty} X_1(t+\tau-\alpha)h_1(\alpha)\,d\alpha \int_{-\infty}^{\infty} X_2^*(t-\beta)\,h_2^*(\beta)\,d\beta\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left(X_1(t+\tau-\alpha)X_2^*(t-\beta)\right)h_1(\alpha)h_2^*(\beta)\,d\alpha\,d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{X_1X_2}(t+\tau-\alpha, t-\beta)\,h_1(\alpha)\,h_2^*(\beta)\,d\alpha\,d\beta$$

$$= \int_{-\infty}^{\infty} \left(R_{X_1X_2}(\tau+\beta) * h_1(\tau)\right) h_2^*(\beta)\,d\beta$$

$$= R_{X_1X_2}(\tau) * h_1(\tau) * h_2^*(-\tau) = R_{Y_1Y_2}(\tau)$$

همچنین می‌دانیم هر یک از Y_1، Y_2 WSS هستند $\Leftarrow$ Y_1 و Y_2 $\overbrace{\text{توأماً ایستان به مفهوم وسیع}}^{\text{jointly WSS}}$ خواهند بود

★ Part d) $\forall f: S_{Y_1Y_2}(f) = 0 \Rightarrow h_1(t)$ and $h_2(t)$: ?

$$S_{Y_1Y_2}(f) = S_{X_1X_2}(f)\,H_1(f)\,H_2^*(f) \longrightarrow \begin{cases} S_{X_1X_2}(f) = 0 \ \checkmark \\ \text{یا} \\ H_1(f)H_2(f) = 0 \end{cases} \longrightarrow$$

طیف‌های $h_1(t)$ ، $h_2(t)$ نباید همپوشانی داشته باشند!

$f \in f_1 : H_1(f) \neq 0 \longrightarrow H_2(f) = 0$

$f \in f_2 : H_2(f) \neq 0 \rightarrow H_1(f) = 0$

$\longrightarrow$ $h_1(t)$ و $h_2(t)$ دو فیلتر متعامد هستند.

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$x(t)$: zero-mean staitionary Gaussian r.p $\quad S_x(f) = \Lambda(f)$; $x'(t) = \frac{d}{dt} x(t)$: ۱ سوال ★

★ Part a) pdf of vector $\underline{X} = [x(t),\ x'(t),\ x(t-1)]^T$

★ Part b) PSD of $x'(t)$ and cross power spectrum of $x(t)$ and $x'(t)$

$$x(t) \longrightarrow \boxed{\delta'(t)} \longrightarrow x'(t) \Rightarrow R_{x'}(\tau) = R_x(\tau) * \delta'(\tau) * \delta'(-\tau) = -\frac{d^2}{d\tau^2} R_x(\tau)$$

$$\longrightarrow R_{x'}(\tau) = -\frac{d^2}{d\tau^2} R_x(\tau) \xrightarrow{\mathcal{F}} \underline{S_{x'}(f) = -(j2\pi f)^2 S_x(f) = 4\pi^2 f^2 \Lambda(f)}$$

$$R_{xx'}(\tau) = R_x(\tau) * \delta'(-\tau) = -\frac{d}{d\tau} R_x(\tau) \xrightarrow{\mathcal{F}} S_{xx'}(f) = -j2\pi f\, S_x(f)$$

$$\longrightarrow \underline{S_{xx'}(f) = -j2\pi f\, \Lambda(f)}$$