University of Tehran,
College of Engineering

Homework #2

Stochastic Process – Fall 2024
Instructor: **Dr. Ali Olfat**

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• Problem 1. $X, Y \sim \mathcal{N}(\eta_x, \eta_y, \sigma_x^2, \sigma_y^2, \rho_{xy}) \longrightarrow X \sim \mathcal{N}(\eta_x, \sigma_x^2)\ ,\ Y \sim \mathcal{N}(\eta_y, \sigma_y^2)$

Part a.

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho_{xy}^2)}} \exp\left(\frac{-1}{2(1-\rho_{xy}^2)}\left[\frac{(x-\eta_x)^2}{\sigma_x^2} - \frac{2\rho_{xy}(x-\eta_x)(y-\eta_y)}{\sigma_x\sigma_y} + \rho_{xy}^2\frac{(y-\eta_y)^2}{\sigma_y^2}\right]\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho_{xy}^2)}} \exp\left(\frac{-1}{2\sigma_x^2(1-\rho_{xy}^2)}\left[x-\eta_x-\rho_{xy}\frac{\sigma_x}{\sigma_y}(y-\eta_y)\right]^2\right)$$

Also for $f_{Y|X}(y|x) \Rightarrow f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma_y^2(1-\rho_{xy}^2)}} \exp\left(\frac{-1}{2\sigma_y^2(1-\rho_{xy}^2)}\left[y-\eta_y-\rho_{xy}\frac{\sigma_y}{\sigma_x}(x-\eta_x)\right]^2\right)$

Part b. Considering Part a. we can model $X|Y$ and $Y|X$ distributions with a gaussian distribution.

$$X|Y \sim \mathcal{N}\left(\eta_x + \rho_{xy}\frac{\sigma_x}{\sigma_y}(y-\eta_y)\ ,\ \sigma_x^2(1-\rho_{xy}^2)\right)$$

$$E_{X|Y}(X|Y=y) = \eta_x + \rho_{xy}\frac{\sigma_x}{\sigma_y}(y-\eta_y)$$

$$\sigma_{X|Y}^2 = E_{X|Y}(X^2|Y=y) - E_{X|Y}^2(X|Y=y) = \sigma_x^2(1-\rho_{xy}^2)$$

Part c. $\circledast \to$ X and Y are jointly normal when for each a, and b, $Z = aX + bY$ is normal.

$$\begin{cases} Z = aX + bY \\ W = cX + dY \end{cases} \Longrightarrow V = eZ + fW = e(aX+bY) + f(cX+dY) = \underbrace{(ae+fc)}_{a'}X + \underbrace{(eb+fd)}_{b'}Y$$

$$\Rightarrow V = a'X + b'Y \Rightarrow V \text{ is normal}$$

Z and W are also jointly normal

• Problem 2. $f_{XY}(x,y) = \begin{cases} xe^{-x(y+1)} & ,\ x>0\ ,\ y>0 \\ 0 & ,\ \text{otherwise} \end{cases}$

Part a. $E(XY) = 1$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy = \int_0^\infty x^2e^{-x}\overbrace{\int_0^\infty ye^{-xy}\,dy}^{1/x^2}\,dx = \int_0^\infty \cancel{x^2}e^{-x}\cdot\frac{1}{\cancel{x^2}}\,dx = -e^{-x}\Big|_0^\infty = +1$$

Part b. $E_{X|Y}(X|Y=y)$; First we should drive the conditional PDF for $X|Y$.

$$\left.\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = xe^{-x}\int_0^\infty e^{-xy}\,dy = xe^{-x}\cdot\frac{1}{x} = e^{-x} \\ f_Y(y) &= \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \int_0^\infty xe^{-x(y+1)}\,dx = \frac{1}{(y+1)^2} \end{aligned}\right\} \Rightarrow \begin{aligned} &f_{X|Y}(X|Y=y) = x(1+y)^2e^{-x(y+1)} \quad x>0,\ y>0 \\ &f_{Y|X}(Y|X=x) = xe^{-xy} \quad x>0,\ y>0 \end{aligned}$$

$$\Rightarrow E_{X|Y}(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|Y=y)\,dx = \int_0^{\infty} x^2(1+y)^2 e^{-x(y+1)}\,dx = (1+y)^2\frac{2}{(1+y)^3} = \frac{2}{1+y} \;;\; y>0$$

■ Part c. $E_{Y|X}(Y|X=x)$

$$\Rightarrow E_{Y|X}(Y|X=x) = \int_{-\infty}^{+\infty} y f_{Y|X}(y|X=x)\,dy = \int_0^{\infty} xye^{-xy}\,dy = x\cdot\frac{1}{x^2} = \frac{1}{x} \;;\quad x>0$$

■ Part d. $E(X^2Y\,|\,X=x)$

$$\Rightarrow E(X^2Y|X=x) = \int_{-\infty}^{+\infty} x^2y f_{Y|X}(y|X=x)\,dy = \int_0^{\infty} x^3ye^{-xy}\,dy = x^3\cdot\frac{1}{x^2} = x \;;\quad x>0$$

• Problem 4. $X \sim N(\eta, \sigma^2)$, $Z = \sin(aX)$

Cosidering the characteristic function of a gaussion distribution, we can write:

$$\Phi_X(\omega) = E(e^{j\omega X}) = \int_{-\infty}^{+\infty} f_X(x)e^{j\omega x}\,dx = e^{j\omega\eta}e^{-\frac{\omega^2}{2}\sigma^2} = \cos(\omega\eta)e^{-\frac{\omega^2}{2}\sigma^2} + j\,\underline{\sin(\omega\eta)e^{-\frac{\omega^2}{2}\sigma^2}}$$

Also: $E(e^{j\omega X}) = E(\cos(\omega X) + j\sin(\omega X)) = E(\cos(\omega X)) + j\,\underline{E(\sin(\omega X))}$

Equality of the imaginary parts $\Rightarrow E(\sin(\omega X)) = \sin(\omega\eta)e^{-\frac{\omega^2}{2}\sigma^2}$

$$\omega = a \longrightarrow E(\sin(aX)) = \sin(a\eta)e^{-\frac{a^2}{2}\sigma^2}$$

• Problem 5. $X \sim Bin(n;p)$, $E(\frac{1}{X+1})$ $\Rightarrow P(X=i) = \binom{n}{i}p^i(1-p)^{n-i}$

$$E\left(\frac{1}{X+1}\right) = \sum_{i=0}^{n}\frac{1}{i+1}\binom{n}{i}p^i(1-p)^{n-i} \quad (I)$$

$$(\star) \longrightarrow \frac{1}{i+1}\binom{n}{i} = \frac{n!}{(n-i)!\,i!\,(i+1)} = \frac{1}{n+1}\,\frac{(n+1)!}{(n-i)!\,(i+1)!} = \frac{1}{n+1}\binom{n+1}{i+1}$$

$$(I),(II) \Rightarrow E\left(\frac{1}{X+1}\right) = \sum_{i=0}^{n}\frac{1}{n+1}\binom{n+1}{i+1}p^i(1-p)^{n-i} = \frac{1}{p(n+1)}\underbrace{\sum_{i=0}^{n}\binom{n+1}{i+1}p^{i+1}(\underset{q}{1-p})^{n-i}}_{(p+q)^n - (1-p)^n = 1-(1-p)^n}$$

$$= \frac{1-(1-p)^n}{p(n+1)}$$

$$\Longrightarrow E\left(\frac{1}{X+1}\right) = \frac{1-(1-p)^n}{p(n+1)}$$

• Problem 7. $\{X_1, \dots, X_n\}$ are i.i.d.

■ Part a. pdf of $Y_1 = \max\{X_1, \dots, X_n\}$

$$F_{Y_1}(y_1) = P(Y_1 \leqslant y_1) = P(\max\{X_1,\dots,X_n\} \leqslant y_1) = P(X_1 \leqslant y_1, X_2 \leqslant y_1, \dots, X_n \leqslant y_1)$$

$$= P(X_1 \leqslant y_1)\cdots P(X_n \leqslant y_1) = F_{X_1}(y_1)\cdots F_{X_n}(y_1) \overset{i.i.d}{=} F_X^n(y_1)$$

$$\Rightarrow F_{Y_1}(y_1) = F_X^n(y_1) \xrightarrow{\partial/\partial y_1} f_{Y_1}(y_1) = n f_X(y_1) F_X^{n-1}(y_1)$$

■ Part b. pdf of $Y_2 = \min\{X_1, \dots, X_n\}$

$$F_{Y_2}(y_2) = P(Y_2 \leqslant y_2) = 1 - P(Y_2 > y_2) = 1 - P(\min\{X_1, \dots, X_n\} > y_2)$$

$$= 1 - P(X_1 > y_2, \dots, X_n > y_2) = 1 - P(X_1 > y_2) P(X_2 > y_2) \cdots P(X_n > y_2)$$

$$= 1 - (1 - \underbrace{P(X_1 \leqslant y_2)}_{F_{X_1}(y_2)})(1 - \underbrace{P(X_2 \leqslant y_2)}_{F_{X_2}(y_2)}) \cdots (1 - \underbrace{P(X_n \leqslant y_2)}_{F_{X_n}(y_2)})$$

$$\overset{i.i.d}{=} 1 - (1 - F_X(y_2))^n \xrightarrow{\partial/\partial y_2} f_{Y_2}(y_2) = n f_X(y_2) \left(1 - F_X(y_2)\right)^{n-1}$$

■ Part c. joint pdf of Y_1 and Y_2

$$F_{Y_1 Y_2}(y_1, y_2) = P(Y_1 \leqslant y_1, Y_2 \leqslant y_2)$$

$$\circledast \longrightarrow P(A) = P(A - B) + P(A \cap B) = P(A \cap B') + P(A \cap B) \Longrightarrow P(A \cap B) = P(A) - P(A \cap B')$$

$$\Rightarrow F_{Y_1 Y_2}(y_1, y_2) = P(Y_1 \leqslant y_1) - P(Y_1 \leqslant y_1, Y_2 > y_2)$$

$$= F_{Y_1}(y_1) - P(\max\{X_1, \dots, X_n\} \leqslant y_1, \min\{X_1, \dots, X_n\} > y_2)$$

$$= F_{Y_1}(y_1) - P(y_2 < X_1 \leqslant y_1, y_2 < X_2 \leqslant y_1, \dots, y_2 < X_n \leqslant y_1)$$

$$= F_{Y_1}(y_1) - \underbrace{P(y_2 < X_1 \leqslant y_1)}_{F_{X_1}(y_1) - F_{X_1}(y_2)} \cdots \underbrace{P(y_2 < X_n \leqslant y_1)}_{F_{X_n}(y_1) - F_{X_n}(y_2)}$$

$$\overset{i.i.d}{=} F_{Y_1}(y_1) - \left[F_X(y_1) - F_X(y_2)\right]^n$$

$$\Rightarrow f_{Y_1 Y_2}(y_1, y_2) = \frac{\partial}{\partial y_1}\left[\frac{\partial F_{Y_1 Y_2}}{\partial y_2}\right] = \frac{\partial}{\partial y_1}\left[n f_X(y_2)\left(F_X(y_1) - F_X(y_2)\right)^{n-1}\right]$$

$$= n(n-1) f_X(y_1) f_X(y_2) \left(F_X(y_1) - F_X(y_2)\right)^{n-2}$$

when $y_1 \geqslant y_2 \Rightarrow f_{Y_1 Y_2}(y_1, y_2) = n(n-1) f_X(y_1) f_X(y_2) \left(F_X(y_1) - F_X(y_2)\right)^{n-2}$

when $y_1 < y_2 \Rightarrow f_{Y_1 Y_2}(y_1, y_2) = 0$

$$\Rightarrow f_{Y_1 Y_2}(y_1, y_2) = n(n-1) f_X(y_1) f_X(y_2) \left(F_X(y_1) - F_X(y_2)\right)^{n-2} u(y_1 - y_2)$$

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- Problem 6. $\{X_1, \dots, X_n\}$ are i.i.d , $P(X_i = k) = -\dfrac{(1-p)^k}{k\log(p)}$; $k \geq 1$, $0<p<1$

■ Part a. PGF of X_i

$$\Gamma_x(z) = E(z^x) = \sum_{k=1}^{\infty} z^k \frac{-(1-p)^k}{k\log(p)} = -\frac{1}{\log(p)} \sum_{k=1}^{\infty} \frac{((1-p)z)^k}{k}$$

$$\circledast \longrightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \Rightarrow -\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\Rightarrow \Gamma_x(z) = \frac{-1}{\log(p)} \sum_{k=1}^{\infty} \frac{((1-p)z)^k}{k} = \frac{\log(1-z(1-p))}{\log(p)} \quad \circledast$$

■ Part b. PGF of $Y = \sum_{k=1}^{N} X_k$ where $N \sim \text{Poisson}(\lambda)$, $N \perp\!\!\!\perp X_i(s)$

★ Iteratial Expectation Theorem: $E(X) = E_Z(E_{X|Z}(X|Z))$

$$\Rightarrow E(z^Y) = E_N(E(z^Y|N)) \longrightarrow \Gamma_Y(z) = E_N(E(z^Y|N))$$

First we should compute the $E(Z^Y|N=n)$:

$$E(Z^Y|N) = E(Z^{\sum_{k=1}^{N} X_i}) = E(Z^{X_1} \cdots Z^{X_N}) = E(Z^{X_1}) \dots E(Z^{X_N})$$

$X_i(s)$ are i.i.d $\Rightarrow E(Z^Y|N) = (E(Z^X))^N = \Gamma_X^N(z)$

Cosidering the PGF of a poisson distribution we can drive the $\Gamma_Y(z)$:

$$\Gamma_Y(z) = E_N(\Gamma_X^N(z)) = \Gamma_N(\Gamma_X(z))$$

$$\circledast \longrightarrow \text{Poisson Dist. PGF: } \Gamma_N(z) = \exp(\lambda(z-1))$$

$$\Rightarrow \Gamma_Y(z) = \Gamma_N(\Gamma_X(z)) = \exp(\lambda(\Gamma_X(z) - 1))$$

$$= \exp(-\lambda) \left[\exp\left(\frac{\lambda \log(1-z(1-p))}{\log(p)} \right) \right]$$

$$= \exp(-\lambda) (1 - z(1-p))^{\lambda/\log(p)}$$

- Problem 3. $\{X_1, X_2, \dots, X_n, \dots\}$ are i.i.d N: integer random variable , $X_i \perp\!\!\!\perp N$

$$Z = \sum_{k=1}^{N} X_k$$

- Part a. characteristic function of Z in terms of $\Phi_X(\omega)$

$$E(Z) = E_N(E(Z|N)) \longrightarrow \Phi_Z(\omega) = E_N(\underbrace{E(e^{j\omega Z}|N)}_{A})$$

$$A = E(e^{j\omega Z}|N) = E(e^{j\omega \sum_{k=1}^{N} X_k}) \overset{iid}{=} \underbrace{E(e^{j\omega X_1})}_{\Phi_{X_1}(\omega_1)} \dots \underbrace{E(e^{j\omega X_n})}_{\Phi_{X_2}(\omega)} \overset{iid}{=} \Phi_X^N(\omega)$$

We also know that $\Rightarrow E_N(Z^N) = \Gamma_N(Z) \Rightarrow \Phi_Z(\omega) = E_N(\Phi_X^N(\omega)) = \Gamma_N(\Phi_X(\omega))$

$$\Rightarrow \Phi_Z(\omega) = \Gamma_N(\Phi_X(\omega))$$

- Part b. The mean and variance of Z

$$E(e^{j\omega Z}) = \Phi_Z(\omega) \Rightarrow \frac{\partial}{\partial\omega}\Phi_Z(\omega)\Big|_{\omega=0} = jE(Z) \Rightarrow E(Z) = -j\frac{\partial}{\partial\omega}\Phi_Z(\omega)\Big|_{\omega=0}$$

$$\Phi_Z(\omega) = \Gamma_N(\Phi_X(\omega)) = E_N(\Phi_X^n(\omega)) = \sum_{n=1}^{N} P(N=n)\,\Phi_X^n(\omega)$$

$$\longrightarrow \frac{\partial}{\partial\omega}\Phi_Z(\omega) = \sum_{n=1}^{N} nP(N=n)\,\Phi_X^{n-1}(\omega)\,\Phi_X'(\omega)$$

$$\Rightarrow E(Z) = -j\frac{\partial}{\partial\omega}\Phi_Z(\omega)\Big|_{\omega=0} = \underbrace{\sum_{n=1}^{N} nP(N=n)}_{E(N)}\,\underbrace{\Phi_X^{n-1}(0)}_{1}\,\underbrace{\left[-j\Phi_X'(0)\right]}_{E(X)} = E(X)\cdot E(N)$$

$$\frac{\partial^2}{\partial\omega^2}\Phi_Z(\omega)\Big|_{\omega=0} = -E(Z^2) \Rightarrow E(Z^2) = -\frac{\partial^2}{\partial\omega^2}\Phi(\omega)\Big|_{\omega=0}$$

$$\longrightarrow \frac{\partial^2}{\partial\omega^2}\Phi_Z(\omega) = \frac{\partial}{\partial\omega}\left[\sum_{n=1}^{N} nP(N=n)\,\Phi_X^{n-1}(\omega)\,\frac{\partial\Phi_X(\omega)}{\partial\omega}\right]$$

$$= \sum_{n=1}^{N} nP(N=n)\left[(n-1)\Phi_X^{n-2}(\omega)\,\Phi_X'^2(\omega) + \Phi_X^{n-1}(\omega)\,\Phi_X''(\omega)\right]$$

$$\longrightarrow E(Z^2) = -\sum_{n=1}^{N} nP(N=n)\left[(n-1)\underbrace{\Phi_X'^2(0)}_{-E^2(X)} + \underbrace{\Phi_X''(0)}_{-E(X^2)}\right] = +\sum_{n=1}^{N} n\,P(N=n)\left[nE^2(X) + \underbrace{E(X^2) - E^2(X)}_{var(X)}\right]$$

$$= \underbrace{\sum_{n=1}^{N} n^2 P(N=n)}_{E(N^2)}\,E^2(X) + \underbrace{\sum_{n=1}^{N} n\,P(N=n)}_{E(N)}\,var(X) = E(N^2)E^2(X) + E(N)\,var(X)$$

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Part c. X_i(s) are Normal RV's, N is a Geometric RV $\longrightarrow P(N=i) = p(1-p)^{i-1}$

$X_i \sim \mathcal{N}(\eta, \sigma^2)$ $\qquad N \sim Geo(p)$

First we try to compute the PGF of a Geometric Random Variable with parameter p.

$$\Gamma_N(z) = \sum_{i=1}^{\infty} z^i P(N=i) = \sum_{i=1}^{\infty} p(1-p)^{i-1} z^i = \frac{p}{1-p}\sum_{i=1}^{\infty}(z(1-p))^i = \frac{p}{\cancel{1-p}} \cdot \frac{z\cancel{(1-p)}}{1-z(1-p)} = \frac{pz}{1-z(1-p)}$$

Also, we have the characteristic function of a Normal Random Variables:

$$\Phi_X(\omega) = e^{j\omega\eta} e^{-\frac{\sigma^2\omega^2}{2}} \quad ; \quad X \sim \mathcal{N}(\eta, \sigma^2)$$

$$\Rightarrow \Phi_Z(\omega) = \Gamma_N(\Phi_X(\omega)) = \frac{p e^{j\omega\eta} e^{-\frac{\sigma^2\omega^2}{2}}}{1-(1-p)e^{j\omega\eta}e^{-\frac{\sigma^2\omega^2}{2}}} = \frac{p}{e^{-j\omega\eta}e^{\frac{\sigma^2\omega^2}{2}} - (1-p)}$$

★ Now, considering the results of "Part b", we can comput $m_Z = E(Z)$ and $var(Z)$:

$$m_Z = E(Z) = E(X)E(N) = \eta \cdot \frac{1}{p} = \frac{\eta}{p}$$

$$var(Z) = E(N^2)E^2(X) + var(X)E(N) = \frac{2-p}{p^2}\eta^2 + \sigma^2\frac{1}{p} = \frac{2\eta^2}{p^2} + \frac{\sigma^2-\eta^2}{p}$$