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فرایندهای اتفاقی - تمرین ۸

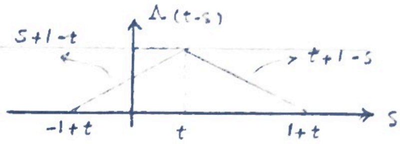
عرفان پناهی - ۸۱۰۱۰۳۰۸۴

★ سوال ۱: $X(t)$: WSS - zere mean - $R_X(\tau) = \Lambda(\tau) \longrightarrow m_X(t) = 0$

★ Find the KL expansion of $X(t)$ for $t \in [0,1]$

kL expansion $\Rightarrow \int_a^b R_X(t,s)\phi_n(s)\,ds = \lambda_n \phi_n(t) \quad a \le t \le b \;;\; \hat{X}(t) = \sum_{n=1}^{\infty} X_n \phi_n(t) \;;\; t \in [a,b]$

$$\circledast \longrightarrow \int_0^1 R_X(t-s)\phi_n(s)\,ds = \lambda_n \phi_n(t) = \int_0^1 \Lambda(t-s)\phi_n(s)\,ds \qquad (t \in [0,1])$$



$$\Rightarrow \int_0^1 \Lambda(t-s)\phi_n(s)\,ds = \int_0^t (s+1-t)\phi_n(s)\,ds + \int_t^1 (t+1-s)\phi_n(s)\,ds$$

$$= \int_0^t \phi_n(s)ds - t\int_0^t \phi_n(s)ds + \int_0^t s\phi_n(s)ds + \int_t^1 \phi_n(s)\,ds + t\int_t^1 \phi_n(s)ds - \int_t^1 s\phi_n(s)ds$$

$$\xrightarrow{\text{دوبار مشتق گیری}} -2\phi_n(t) = \lambda_n \phi_n''(t) \longrightarrow \phi_n(t) = A\sin\left(\sqrt{\tfrac{2}{\lambda_n}}\,t\right) + B\cos\left(\sqrt{\tfrac{2}{\lambda_n}}\,t\right)$$

شرایط مرزی $\Rightarrow \begin{cases} \phi(0)+\phi(1)=0 \\ \phi'(0)+\phi'(1)=0 \end{cases} \Rightarrow \left.\begin{array}{l} B + A\sin\left(\sqrt{\tfrac{2}{\lambda_n}}\right) + B\cos\left(\sqrt{\tfrac{2}{\lambda_n}}\right) = 0 \\ A + A\cos\left(\sqrt{\tfrac{2}{\lambda_n}}\right) - B\sin\left(\sqrt{\tfrac{2}{\lambda_n}}\right) = 0 \end{array}\right\} \Rightarrow A^2 = -B^2 \rightarrow A = \pm jB$

$$\xRightarrow{\text{نرمالایز}} \phi_n(t) = \cos\left(\sqrt{\tfrac{2}{\lambda_n}}\,t\right) + j\sin\left(\sqrt{\tfrac{2}{\lambda_n}}\,t\right) = e^{j\sqrt{\frac{2}{\lambda_n}}\,t} \Rightarrow \boxed{\phi_n(t) = e^{j\sqrt{\frac{2}{\lambda_n}}\,t}}$$

$$\Rightarrow \hat{X} = \sum_{n=1}^{\infty} e^{j\sqrt{\frac{2}{\lambda_n}}\,t} X_n$$

برای اینکه λ_n را بدانیم از شرایط مرزی استفاده می‌کنیم:

$$\phi(0)+\phi(1)=0 \Rightarrow 1 + e^{j\sqrt{\frac{2}{\lambda_n}}} = 0 \longrightarrow \lambda_n \to \infty$$

★ سوال ۲: $x(n)$: i.i.d R.V. $\quad \Pr\{x(n)=1\} = \Pr\{x(n)=-1\} = \frac{1}{2} \Rightarrow y(n) = 0.8y(n-1) + x(n)$

★ Part a. PSD of $y(n)$: $\quad y(n) \xrightarrow{Z} Y(z) = 0.8z^{-1}Y(z) + X(z) \longrightarrow H(z) = \frac{1}{1-0.8z^{-1}} \xrightarrow{z^{-1}} h(n) = 0.8^n u[n]$

$$R_x(n_1,n_2) = E\{x(n_1)x^*(n_2)\} = \left.\begin{cases} 1 , & n_1 = n_2 \\ 0 , & n_1 \ne n_2 \end{cases}\right\} \Rightarrow R_x(n+m,n) = \delta[m] \longrightarrow R_x(m) = \delta(m)$$

$$R_x(m) = \delta(m) \xrightarrow{F} S_x(f) = 1 \quad ; \quad H(f) = \frac{1}{1-0.8e^{-j2\pi f}} \longrightarrow |H(f)|^2 = \frac{1}{1.64 - 1.6\cos(2\pi f)}$$

$$\Rightarrow S_y(f) = S_x(f)\cdot|H(f)|^2 = \frac{1}{1.64 - 1.6\cos(2\pi f)}$$

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★ Part b.

ابتدا ثابت می‌کنیم که $y(n-k)$ از $x(n)$ مستقل است:

$$y(n) = 0.8\,y(n-1) + x(n) \longrightarrow y(n) = 0.8^2\,y(n-2) + 0.8\,x(n-1) + x(n)$$

به همین ترتیب $\Rightarrow$ $y(n) = x(n) + 0.8\,x(n-1) + 0.8^2\,x(n-2) + \dots = \sum_{i=0}^{\infty} 0.8^i\,x(n-i)$

★ همانطور مشاهده می‌شود در بسط $y(n-k)$ هیچ ترم $x(n)$ دیده نمی‌شود.

$$\Rightarrow y(n-k) = \sum_{i=0}^{\infty} 0.8^i\,x(n-k-i) \longrightarrow x(n) \perp\!\!\!\perp y(n-k) \qquad k \geq 1$$

حال ثابت می‌کنیم $y(n)$ فرایندی مارکوف است:

$$\Pr\{y(n) = y_n \mid y(n-1) = y_{n-1}, y(n-2) = y_{n-2}, \dots\} = \Pr\{0.8\,y_{n-1} + x(n) = y_n \mid y(n-1) = y_{n-1}, \dots\}$$

استقلال بالا ثابت کردیم:

$$= \Pr\{x(n) = y_n - 0.8\,y_{n-1} \mid y(n-1) = y_{n-1}, \dots\} = \Pr\{x(n) = y_n - 0.8\,y_{n-1} \mid y(n-1) = y_{n-1}\}$$

$$= \Pr\{\underbrace{x(n) - 0.8\,y_{n-1}}_{y(n)} = y_n \mid y(n-1) = y_{n-1}\} = \Pr\{y(n) = y_n \mid y(n-1) = y_{n-1}\}$$

در نتیجه $y(n)$ فرایندی مارکوف است!

★ Part c. PMF of Z : $Z(n) = x(n-1) + x(n)$

$x(n) = 1,\ x(n-1) = 1 \longrightarrow z(n) = 2$; $\Pr\{z(n) = 2\} = \frac{1}{4}$

$x(n) = -1,\ x(n-1) = -1 \longrightarrow z(n) = -2$; $\Pr\{z(n) = -2\} = \frac{1}{4}$

$\left.\begin{array}{l} x(n) = 1,\ x(n-1) = -1 \\ x(n) = -1,\ x(n-1) = 1 \end{array}\right\} \longrightarrow z(n) = 0$; $\Pr\{z(n) = 0\} = \frac{1}{2}$

$$\Pr\{z(n) = k\} = \begin{cases} \frac{1}{4}, & k = 2 \\ \frac{1}{4}, & k = -2 \\ \frac{1}{2}, & k = 0 \end{cases}$$

★ Part d. Mean, Autocorrelation, and the PSD of $z(n)$:

$$E\{z(n)\} = \frac{1}{4} \times 2 + \frac{1}{4} \times (-2) + \frac{1}{2} \times 0 = 0$$

(*) چون حقیقی است * نمی‌گذاریم

$$R_z(n+m, n) = E\{z(n+m)\,z(n)\} = E\{(x(n+m-1) + x(n+m))(x(n-1) + x(n))\}$$

$$= \underbrace{E\{x(n+m-1)\,x(n-1)\}}_{R_x(m) = \delta(m)} + \underbrace{E\{x(n+m-1)\,x(n)\}}_{R_x(m-1) = \delta(m-1)} + \underbrace{E\{x(n+m)\,x(n-1)\}}_{R_x(m+1) = \delta(m+1)} + \underbrace{E\{x(n+m)\,x(n)\}}_{R_x(m) = \delta(m)}$$

$$= 2\delta(m) + \delta(m-1) + \delta(m+1)$$

$$R_z(m) \xrightarrow{\mathcal{F}} S_z(f) = 2 + e^{-j2\pi f} + e^{j2\pi f} = 2\underbrace{(1 + \cos(2\pi f))}_{2\cos^2(\pi f)} = 4\cos^2(\pi f)$$

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$y(n) = 0.3\,y(n-1) + x(n)$; $x(n)$: staitionary white noise with $R_x(m) = \delta(m)$: سوال ۲ ★

$\hookrightarrow S_x(f) = 1$

★ Part a : PSD and Auto correlation of $y(n)$: $-\infty < n < n$

$$y(n) - 0.3\,y(n-1) = x(n) \xrightarrow{z} Y(z) - 0.3 z^{-1} Y(z) = X(z) \longrightarrow H(z) = \frac{1}{1-0.3z^{-1}}$$

$$\longrightarrow H(f) = \frac{1}{1-0.3e^{-j2\pi f}} \xrightarrow{abs^2} |H(f)|^2 = \frac{1}{1.09 + 0.6\cos(2\pi f)}$$

$$\longrightarrow S_y(f) = S_x(f)\,.\,|H(f)|^2 = \frac{1}{1.09+0.6\cos(2\pi f)} \Rightarrow \boxed{S_y(f) = \frac{1}{1.09+0.6\cos(2\pi f)}}$$

$$S_y(f) \xrightarrow{F^{-1}} R_y(f) \Longrightarrow$$ برای اینکار ابتدا سعی می‌کنیم بسط $S_y(f)$ را بنویسیم

$$S_y(f) = \frac{1}{1.09+0.6\cos(2\pi f)} = \frac{1}{1-0.3e^{-j2\pi f}} \cdot \frac{1}{1-0.3e^{j2\pi f}} = \frac{e^{-j2\pi f}}{(e^{-j2\pi f}-0.3)(1-0.3e^{-j2\pi f})}$$

$$= \frac{-10}{3} e^{-j2\pi f} \frac{1}{(1-\frac{10}{3}e^{-j2\pi f})(1-\frac{3}{10}e^{-j2\pi f})} = \frac{100}{91}\left(\frac{1}{1-\frac{3}{10}e^{-j2\pi f}} - \frac{1}{1-\frac{10}{3}e^{-j2\pi f}}\right)$$

$$\xrightarrow{F^{-1}} R_y(m) = \frac{100}{91}\left(\frac{3}{10}\right)^m u[m] + \frac{100}{91}\left(\frac{10}{3}\right)^m u[-m-1]$$

[illegible]

★ Part b. Auto correlation of $y(n)$ if $n \geq 0$ and $y(n) = 0$

$$y(n) = 0.3\,y(n-1) + x(n) \longrightarrow y(n) = x(n) + 0.3\,x(n-1) + 0.3^2 x(n-2) + \dots = \sum_{k=0}^{n} (0.3)^n x(n-k)$$

$$\longrightarrow y(n) = \sum_{k=0}^{n} (0.3)^{n-k} x(k)$$

چون حقیقی است نیازی نیست!

$$R_y(n+m, n) = E\{y(n+m)\,y(n)\} = E\left\{\sum_{k=0}^{n+m} 0.3^{n+m-k} x(k) \sum_{k=0}^{n} 0.3^{n-k} x(k)\right\}$$

اگر $m > 0$: $$R_y(n+m, n) = E\left\{\left(\sum_{k=0}^{n} 0.3^{n+m-k} x(k) + \sum_{k=n+1}^{n+m} 0.3^{n+m-k} x(k)\right)\sum_{k=0}^{n} 0.3^{n-k} x(k)\right\}$$

امید ضرب این دو به خاطر استقلال صفر خواهد شد.

$E\{x(k)x(k')\} = 0$, $k \neq k'$

$$= E\left\{0.3^m \sum_{k=0}^{n} 0.3^{n-k} x(k) \sum_{k=0}^{n} 0.3^{n-k} x(k)\right\} = 0.3^m E\left\{\sum_{k=0}^{n} (0.3)^{2n-2k} x^2(k)\right\}$$

$$= 0.3^m \sum_{k=0}^{n} (0.3)^{2n-2k} \underbrace{E\{x^2(k)\}}_{R_x(0)=1} = 0.3^m \frac{1-0.3^{2n+2}}{1-0.09} = \frac{100}{91}(0.3)^m\left(1-0.3^{2n+2}\right)$$

برای $m \geq 0$

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$\times \left(\frac{10}{3}\right)^m \times (0.3)^m$

$-n \leqslant m < 0$ اگر : $R_Y(n+m,n) = E\left\{ \sum_{k=0}^{n+m} 0.3^{n+m-k}\, x(k) \cdot \left(\sum_{k=0}^{n+m} 0.3^{n-k}\, x(k) + \sum_{k=n+m+1}^{n} 0.3^{n-k}\, x(k) \right) \right\}$

مشابه قسمت قبل

$= E\left\{ \left(\frac{10}{3}\right)^m \sum_{k=0}^{n+m} 0.3^{2(n+m-k)}\, x^2(k) \right\} = \left(\frac{10}{3}\right)^m \sum_{k=0}^{n+m} 0.3^{2(n+m-k)} \overset{1}{R_x(0)} = \left(\frac{10}{3}\right)^m \frac{1-0.3^{2(n+m-1)}}{1-0.09}$

$= \frac{100}{91} \left(\frac{10}{3}\right)^m \left(1-0.3^{2(n+m-1)}\right)$

$m < -n$ اگر : $R_Y(n+m,n) = 0 \longrightarrow$ بخاطر مستقل شدن عناصر هر دو سری

$$\Rightarrow R_Y(n+m,n) = \begin{cases} \frac{100}{91}(0.3)^m \left(1-0.3^{2(n+1)}\right) & ; \quad 0 \leqslant m \\ \frac{100}{91}\left(\frac{10}{3}\right)^m \left(1-0.3^{2(n+m-1)}\right) & ; \quad -n \leqslant m < 0 \\ 0 & ; \quad m \leqslant -n \end{cases}$$

★ سوال ۶ :

$x(t)$: zero-mean WSS process ; $S_x(f) = \begin{cases} 1+\cos(20\pi f) & , |f| \leqslant \frac{1}{20} \\ 0 & , |f| > \frac{1}{20} \end{cases}$

$x(t) \longrightarrow \boxed{C/D} \longrightarrow y(n) = x(nT)$, with T as the sampling input.

$f_s = \frac{1}{10} \longrightarrow f_s = \frac{1}{10} = f_0 \times 2 = 2 \times \frac{1}{20}$ ✓

★ Part a. $T = 10$ sec , $C_Y(n,m)$, $S_Y(f)$

در نتیجه با داشتن $y(n)$ می توان $x(n)$ را باز یابی کرد.

$\underbrace{E\{x(nT)\}}_{0}$ $= E\{y(mT)\}$

$C_Y(n,m) = R_Y(n,m) - E\{y(n)\}E\{y(m)\}$

حقیقی

$= E\{y(n)\, y^*(m)\} = E\{x(nT)\, x(mT)\} = R_x((n-m)T)$

$R_x(\tau) = \mathcal{F}^{-1}\{S_x(f)\} = \mathcal{F}^{-1}\{1\} + \mathcal{F}^{-1}\{\cos(20\pi f)\} = \delta(\tau) + \frac{1}{2}\delta(\tau-10) + \frac{1}{2}\delta(\tau+10)$

$\longrightarrow C_Y(n,m) = R_x((n-m)T) = R_x(10(n-m)) = \delta(10(n-m)) + \frac{1}{2}\delta(10(n-m)-10)$
$+ \frac{1}{2}\delta(10(n-m)+10)$

$= \frac{1}{10}\delta(n-m) + \frac{1}{20}\delta(n-m-1) + \frac{1}{20}\delta(n-m+1)$

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$$\Rightarrow R_y(n+m, n) = \frac{1}{10}\delta(m) + \frac{1}{20}\delta(m-1) + \frac{1}{20}\delta(m+1) = R_y(m)$$

$$\xrightarrow{\mathcal{F}} S_y(f) = \frac{1}{10} + \frac{1}{20}e^{-j2\pi f} + \frac{1}{20}e^{j2\pi f} = \frac{1}{10}\left(1+\cos(2\pi f)\right) = \frac{1}{5}\cos^2(\pi f)$$

$$\Rightarrow \boxed{S_y(f) = \frac{1}{5}\cos^2(\pi f)}$$

برای بازیابی کافی است از یک فیلتر ایده‌آل کوتاه‌گذر با فرکانس قطع $f_c = \frac{1}{T} = \frac{1}{10}$ استفاده کنیم. $H(f) = 10\,\Pi\left(\frac{f}{20}\right)$

★ Part b. $T = 20$ sec , $C_y(n,m)$, $S_y(f) = ?$

$$C_y(n,m) = R_y(n,m) = R_x((n-m)T) = R_x(20(n-m))$$

$$= \delta(20(n-m)) + \delta(20(n-m)-10) + \delta(20(n-m)+10)$$

$$= \frac{1}{20}\delta(n-m) + \cancelto{0}{\frac{1}{20}\delta\left(n-m-\frac{1}{2}\right)} + \cancelto{0}{\frac{1}{20}\delta\left(n-m+\frac{1}{2}\right)} = \boxed{\frac{1}{20}\delta(n-m)}$$

$$\rightarrow R_y(m) = \frac{1}{20}\delta(m) \xrightarrow{\mathcal{F}} \boxed{S_y(f) = \frac{1}{20}}$$

★ سوال ۶ :

$x(n)$: discrete stationary random process $\quad S_x(f) = \dfrac{4}{5 - 4\cos(2\pi f)}$

★ Part a. $S_x(f) = \dfrac{4}{5 - 2e^{-j2\pi f} - 2e^{j2\pi f}} = \dfrac{4}{(2 - e^{j2\pi f})(2 - e^{-j2\pi f})} \longrightarrow S_x(z) = \dfrac{4}{(2-z)(2-z^{-1})}$

$$\rightarrow S_x(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z\right)} = L_x(z)\,L_x(z^{-1}) \longrightarrow L(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$\rightarrow \Gamma_x(z) = \frac{1}{L(z)} = 1 - \frac{1}{2}z^{-1} \longrightarrow \gamma(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

$$w(n) = x(n) * \gamma(n) = x(n) - \frac{1}{2}x(n-1) \longrightarrow \text{Innovation process}$$

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★ Part b. Has $x(n)$ an AR model? $X(z) \longrightarrow \boxed{H_{Ap}(z)} \longrightarrow V(z)$

$$\Gamma_x(z) = 1 - \frac{1}{2} z^{-1} = H_{AR}(z) = \frac{V(z)}{X(z)} \longrightarrow X(z) = \frac{1}{2} z^{-1} X(z) + V(z)$$

$$\xrightarrow{z^{-1}} x(n) = \frac{1}{2} x(n-1) + v(n) \longrightarrow AR(1) \text{ model}$$

★ Part c. Has $x(n)$ a MA model?   یک فرایند $AR(1)$ معادل $MA(\infty)$ خواهد بود !

$$L_x(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} = H_{MA}(z) = \frac{X(z)}{V(z)} \longrightarrow X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \cdot V(z)$$

$$\xrightarrow{z^{-1}} x(n) = (\tfrac{1}{2})^n u(n) * v(n) = \sum_{k=0}^{\infty} (\tfrac{1}{2})^k v(n-k) \longrightarrow MA(\infty) \text{ model}$$

★سوال ۴ :

$v(n)$: WSS white process with unit variance

$$v(n) = \sum_{k=0}^{\infty} (k+1)^2 \, 3^{-k} x(n-k) \longrightarrow$$ فرایند $AR(\infty)$ است !

★ Part a. ARMA (N, M)

$$v(n) \xrightarrow{z} V(z) = \sum_{k=0}^{\infty} (k+1)^2 (3z)^{-k} X(z) \longrightarrow \frac{V(z)}{X(z)} = \sum_{k=0}^{\infty} (k+1)^2 (3z)^{-k}$$

★ می‌دانیم $$\sum_{k=0}^{\infty} (k+1)^2 x^k = \frac{1+x}{(1-x)^3} \longrightarrow \frac{V(z)}{X(z)} = \frac{1 + \frac{1}{3} z^{-1}}{(1 - \frac{1}{3} z^{-1})^3}$$

ARMA مشخصه $$= \frac{(1 - \frac{1}{3} z^{-1})^3}{(1 + \frac{1}{3} z^{-1})} = \frac{X(z)}{V(z)} = \frac{27 - 9z^{-1} + 3z^{-2} - z^{-3}}{27 + 9z^{-1}}$$

$$\longrightarrow 27 X(z) + 9z^{-1} X(z) = 27 V(z) - 9z^{-1} V(z) + 3z^{-2} V(z) - z^{-3} V(z)$$

$$\longrightarrow x(n) = \underbrace{-\frac{1}{3} x(n-1)}_{N=1} + \underbrace{v(n) - \frac{1}{3} v(n-1) + \frac{1}{9} v(n-2) - \frac{1}{27} v(n-3)}_{M=3}$$

$$\longrightarrow X(n) \equiv ARMA(1, 3)$$

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★ Part b. Does $x(n)$ have an $MA(\infty)$ model?

$$\frac{X(z)}{V(z)} = \frac{(1-\frac{1}{3}z^{-1})^3}{1+\frac{1}{3}z^{-1}} = \frac{72}{1+\frac{1}{3}z^{-1}} - z^{-2} + 12z^{-1} - 63 \qquad |z| > \frac{1}{3}$$

$$\longrightarrow h[n] = 72\left(-\frac{1}{3}\right)^n u(n) - \delta(n-2) + 12\,\delta(n-1) - 63\,\delta(n)$$

$$\longrightarrow x(n) = v(n) * h(n) = \sum_{m=0}^{\infty} 72\left(-\frac{1}{3}\right)^n v(n-m) - 63v(n) + 12v(n-1) - v(n-2)$$

$$= 9v(n) + 4v(n-1) + 7v(n-2) + 72\sum_{m=3}^{\infty}\left(-\frac{1}{3}\right) v(n-m) : \ MA(\infty) \text{ model}$$