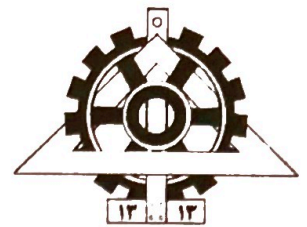


## Homework #2

Stochastic Process – Fall 2024

Instructor: Dr. Ali Olfat

Erfan Panahi (Student Number: 810103084)



• Problem 1.  $X, Y \sim \mathcal{N}(\eta_x, \eta_y, \sigma_x^2, \sigma_y^2, \rho_{xy}) \rightarrow X \sim \mathcal{N}(\eta_x, \sigma_x^2), Y \sim \mathcal{N}(\eta_y, \sigma_y^2)$

■ Part a.

$$f_{x|y}(x|y) = \frac{f_{xy}(x, y)}{f_y(y)} = \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho_{xy}^2)}} \exp\left(\frac{-1}{2(1-\rho_{xy}^2)} \left[ \frac{(x-\eta_x)^2}{\sigma_x^2} - \frac{2\rho_{xy}(x-\eta_x)(y-\eta_y)}{\sigma_x\sigma_y} + \rho_{xy}^2 \frac{(y-\eta_y)^2}{\sigma_y^2} \right]\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho_{xy}^2)}} \exp\left(\frac{-1}{2\sigma_x^2(1-\rho_{xy}^2)} \left[ x - \eta_x - \rho_{xy} \frac{\sigma_x}{\sigma_y} (y - \eta_y) \right]^2\right)$$

Also for  $f_{y|x}(y|x) \Rightarrow f_{y|x}(y|x) = \frac{1}{\sqrt{2\pi\sigma_y^2(1-\rho_{xy}^2)}} \exp\left(\frac{-1}{2\sigma_y^2(1-\rho_{xy}^2)} \left[ y - \eta_y - \rho_{xy} \frac{\sigma_y}{\sigma_x} (x - \eta_x) \right]^2\right)$

■ Part b. Considering Part a. we can model  $X|Y$  and  $Y|X$  distributions with a gaussian distribution.

$$X|Y \sim \mathcal{N}\left(\eta_x + \rho_{xy} \frac{\sigma_x}{\sigma_y} (y - \eta_y), \sigma_x^2(1 - \rho_{xy}^2)\right)$$

$$E_{x|y}(X|Y=y) = \eta_x + \rho_{xy} \frac{\sigma_x}{\sigma_y} (y - \eta_y)$$

$$\sigma_{x|y}^2 = E_{x|y}(X^2|Y=y) - E_{x|y}^2(X|Y=y) = \sigma_x^2(1 - \rho_{xy}^2)$$

■ Part c.  $\star \rightarrow X$  and  $Y$  are jointly normal when for each  $a$ , and  $b$ ,  $Z = aX + bY$  is normal.

$$\begin{cases} Z = aX + bY \\ W = cX + dY \end{cases} \Rightarrow V = eZ + fW = e(ax + by) + f(cx + dy) = \underbrace{(ae + fc)}_{a'} X + \underbrace{(eb + fd)}_{b'} Y$$

$$\Rightarrow V = a'X + b'Y \Rightarrow V \text{ is normal}$$

Z and W are also jointly normal  $\leftarrow$

• Problem 2.  $f_{xy}(x, y) = \begin{cases} xe^{-x(y+1)} & , x > 0, y > 0 \\ 0 & , \text{otherwise} \end{cases}$

■ Part a.  $E(XY) = 1$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{xy}(x, y) dx dy = \int_0^{\infty} x^2 e^{-x} \int_0^{\infty} y e^{-xy} dy dx = \int_0^{\infty} x^2 e^{-x} \cdot \frac{1}{x^2} dx = -e^{-x} \Big|_0^{\infty} = 1$$

■ Part b.  $E_{x|y}(X|Y=y)$ ; First we should drive the conditional PDF for  $X|Y$ .

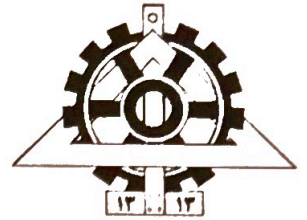
$$\left. \begin{aligned} f_x(x) &= \int_{-\infty}^{\infty} f_{xy}(x, y) dy = xe^{-x} \int_0^{\infty} e^{-xy} dy = xe^{-x} \cdot \frac{1}{x} = e^{-x} \\ f_y(y) &= \int_{-\infty}^{\infty} f_{xy}(x, y) dx = \int_0^{\infty} xe^{-x(y+1)} dx = \frac{1}{(y+1)^2} \end{aligned} \right\} \Rightarrow \begin{aligned} f_{x|y}(x|Y=y) &= x(1+y)^2 e^{-x(y+1)} \\ &\quad x > 0, y > 0 \\ f_{y|x}(y|X=x) &= xe^{-xy} \\ &\quad x > 0, y > 0 \end{aligned}$$

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$$\Rightarrow E_{X|Y}(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|Y=y) dx = \int_0^{\infty} x^2 (1+y)^2 e^{-x(y+1)} dx = (1+y)^2 \frac{2}{(1+y)^3} = \frac{2}{1+y}; y > 0$$

■ Part c.  $E_{Y|X}(Y|X=x)$

$$\Rightarrow E_{Y|X}(Y|X=x) = \int_{-\infty}^{+\infty} y f_{Y|X}(y|X=x) dy = \int_0^{\infty} xy e^{-xy} dy = x \cdot \frac{1}{x^2} = \frac{1}{x}; x > 0$$

■ Part d.  $E(X^2 Y | X=x)$

$$\Rightarrow E(X^2 Y | X=x) = \int_{-\infty}^{+\infty} x^2 y f_{Y|X}(y|X=x) dy = \int_0^{\infty} x^3 y e^{-xy} dy = x^3 \cdot \frac{1}{x^2} = x; x > 0$$

• Problem 4.  $X \sim N(\eta, \sigma^2)$ ,  $Z = \sin(aX)$

Considering the characteristic function of a gaussian distribution, we can write:

$$\Phi_X(\omega) = E(e^{j\omega X}) = \int_{-\infty}^{+\infty} f_X(x) e^{j\omega x} dx = e^{j\omega\eta} e^{-\frac{\omega^2}{2}\sigma^2} = \cos(\omega\eta) e^{-\frac{\omega^2}{2}\sigma^2} + j \sin(\omega\eta) e^{-\frac{\omega^2}{2}\sigma^2}$$

$$\text{Also: } E(e^{j\omega X}) = E(\cos(\omega X) + j \sin(\omega X)) = E(\cos(\omega X)) + j E(\sin(\omega X))$$

$$\text{Equality of the imaginary parts} \Rightarrow E(\sin(\omega X)) = \sin(\omega\eta) e^{-\frac{\omega^2}{2}\sigma^2}$$

$$\omega = a \longrightarrow E(\sin(aX)) = \sin(a\eta) e^{-\frac{a^2}{2}\sigma^2}$$

• Problem 5.  $X \sim \text{Bin}(n; p)$ ,  $E\left(\frac{1}{X+1}\right) \Rightarrow P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$

$$E\left(\frac{1}{X+1}\right) = \sum_{i=0}^n \frac{1}{i+1} \binom{n}{i} p^i (1-p)^{n-i} \quad (I)$$

$$(*) \rightarrow \frac{1}{i+1} \binom{n}{i} = \frac{n!}{(n-i)! i! (i+1)} = \frac{1}{n+1} \frac{(n+1)!}{(n-i)! (i+1)!} = \frac{1}{n+1} \binom{n+1}{i+1}$$

$$(I), (II) \Rightarrow E\left(\frac{1}{X+1}\right) = \sum_{i=0}^n \frac{1}{n+1} \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i} = \frac{1}{p(n+1)} \underbrace{\sum_{i=0}^n \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i}}_{(p+q)^{n+1} - (1-p)^{n+1} = 1 - (1-p)^{n+1}}$$

$$= \frac{1 - (1-p)^{n+1}}{p(n+1)}$$

$$\Rightarrow E\left(\frac{1}{X+1}\right) = \frac{1 - (1-p)^{n+1}}{p(n+1)}$$

• Problem 7.  $\{X_1, \dots, X_n\}$  are i.i.d.

■ Part a. pdf of  $Y_1 = \max\{X_1, \dots, X_n\}$

$$F_{Y_1}(y_1) = P(Y_1 \leq y_1) = P(\max\{X_1, \dots, X_n\} \leq y_1) = P(X_1 \leq y_1, X_2 \leq y_1, \dots, X_n \leq y_1)$$

$$= P(X_1 \leq y_1) \dots P(X_n \leq y_1) = F_{X_1}(y_1) \dots F_{X_n}(y_1) \stackrel{\text{i.i.d.}}{=} F_X^n(y_1)$$

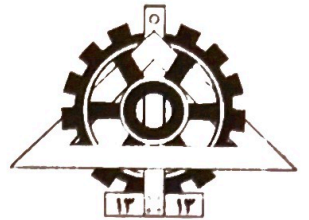


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$$\Rightarrow F_{Y_1}(y_1) = F_X^n(y_1) \xrightarrow{\partial/\partial y_1} f_{Y_1}(y_1) = n f_X(y_1) F_X^{n-1}(y_1)$$

■ Part b. pdf of  $Y_2 = \min\{X_1, \dots, X_n\}$

$$\begin{aligned} F_{Y_2}(y_2) &= P(Y_2 \leq y_2) = 1 - P(Y_2 > y_2) = 1 - P(\min\{X_1, \dots, X_n\} > y_2) \\ &= 1 - P(X_1 > y_2, \dots, X_n > y_2) = 1 - P(X_1 > y_2) P(X_2 > y_2) \dots P(X_n > y_2) \\ &= 1 - (1 - P(X_1 \leq y_2)) (1 - P(X_2 \leq y_2)) \dots (1 - P(X_n \leq y_2)) \\ &\quad \underbrace{F_{X_1}(y_2)} \quad \underbrace{F_{X_2}(y_2)} \quad \underbrace{F_{X_n}(y_2)} \\ &\stackrel{i.i.d}{=} 1 - (1 - F_X(y_2))^n \xrightarrow{\partial/\partial y_2} f_{Y_2}(y_2) = n f_X(y_2) (1 - F_X(y_2))^{n-1} \end{aligned}$$

■ Part c. joint pdf of  $Y_1$  and  $Y_2$

$$F_{Y_1, Y_2}(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2)$$

$$\textcircled{*} \rightarrow P(A) = P(A-B) + P(A \cap B) = P(A \cap B') + P(A \cap B) \Rightarrow P(A \cap B) = P(A) - P(A \cap B')$$

$$\begin{aligned} \Rightarrow F_{Y_1, Y_2}(y_1, y_2) &= P(Y_1 \leq y_1) - P(Y_1 \leq y_1, Y_2 > y_2) \\ &= F_{Y_1}(y_1) - P(\max\{X_1, \dots, X_n\} \leq y_1, \min\{X_1, \dots, X_n\} > y_2) \\ &= F_{Y_1}(y_1) - P(y_2 < X_1 \leq y_1, y_2 < X_2 \leq y_1, \dots, y_2 < X_n \leq y_1) \\ &= F_{Y_1}(y_1) - \underbrace{P(y_2 < X_1 \leq y_1)}_{F_{X_1}(y_1) - F_{X_1}(y_2)} \dots \underbrace{P(y_2 < X_n \leq y_1)}_{F_{X_n}(y_1) - F_{X_n}(y_2)} \\ &\stackrel{i.i.d}{=} F_{Y_1}(y_1) - [F_X(y_1) - F_X(y_2)]^n \end{aligned}$$

$$\begin{aligned} \Rightarrow f_{Y_1, Y_2}(y_1, y_2) &= \frac{\partial}{\partial y_1} \left[ \frac{\partial F_{Y_1, Y_2}}{\partial y_2} \right] = \frac{\partial}{\partial y_1} \left[ n f_X(y_2) (F_X(y_1) - F_X(y_2))^{n-1} \right] \\ &= n(n-1) f_X(y_1) f_X(y_2) (F_X(y_1) - F_X(y_2))^{n-2} \end{aligned}$$

$$\text{when } y_1 \geq y_2 \Rightarrow f_{Y_1, Y_2}(y_1, y_2) = n(n-1) f_X(y_1) f_X(y_2) (F_X(y_1) - F_X(y_2))^{n-2}$$

$$\text{when } y_1 < y_2 \Rightarrow f_{Y_1, Y_2}(y_1, y_2) = 0$$

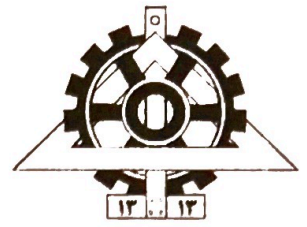
$$\Rightarrow f_{Y_1, Y_2}(y_1, y_2) = n(n-1) f_X(y_1) f_X(y_2) (F_X(y_1) - F_X(y_2))^{n-2} u(y_1 - y_2)$$

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- Problem 6.  $\{X_1, \dots, X_n\}$  are i.i.d,  $P(X_i = k) = -\frac{(1-p)^k}{k \log(p)}$ ;  $k \geq 1$ ,  $0 < p < 1$

■ Part a. PGF of  $X_i$

$$\Gamma_X(z) = E(z^X) = \sum_{k=1}^{\infty} z^k \frac{-(1-p)^k}{k \log(p)} = -\frac{1}{\log(p)} \sum_{k=1}^{\infty} \frac{((1-p)z)^k}{k}$$

$$\textcircled{*} \rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \Rightarrow -\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\Rightarrow \Gamma_X(z) = \frac{-1}{\log(p)} \sum_{k=1}^{\infty} \frac{((1-p)z)^k}{k} = \frac{\log(1 - z(1-p))}{\log(p)} \quad \textcircled{*}$$

■ Part b. PGF of  $Y = \sum_{k=1}^N X_k$  where  $N \sim \text{Poisson}(\lambda)$ ,  $N \perp\!\!\!\perp X_i(s)$

\* Iterational Expectation Theorem:  $E(X) = E_Z(E_{X|Z}(X|Z))$

$$\Rightarrow E(Z^Y) = E_N(E(Z^Y | N)) \longrightarrow \Gamma_Y(z) = E_N(E(Z^Y | N))$$

First we should compute the  $E(Z^Y | N=n)$ :

$$E(Z^Y | N) = E\left(Z^{\sum_{k=1}^N X_k}\right) = E(Z^{X_1} \dots Z^{X_N}) = E(Z^{X_1}) \dots E(Z^{X_N})$$

$$X_i(s) \text{ are i.i.d} \Rightarrow E(Z^Y | N) = (E(Z^{X_1}))^N = \Gamma_X^N(z)$$

Considering the PGF of a poisson distribution we can drive the  $\Gamma_Y(z)$ :

$$\Gamma_Y(z) = E_N(\Gamma_X^N(z)) = \Gamma_N(\Gamma_X(z))$$

$$\textcircled{*} \rightarrow \text{Poisson Dist. PGF: } \Gamma_N(z) = \exp(\lambda(z-1))$$

$$\Rightarrow \Gamma_Y(z) = \Gamma_N(\Gamma_X(z)) = \exp(\lambda(\Gamma_X(z) - 1))$$

$$= \exp(-\lambda) \left[ \exp\left(\frac{\lambda \log(1 - z(1-p))}{\log(p)}\right) \right]$$

$$= \exp(-\lambda) (1 - z(1-p))^{\frac{\lambda}{\log(p)}}$$

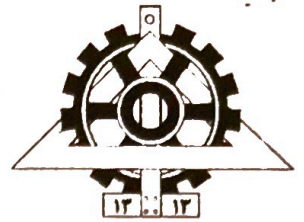


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• Problem 3.  $\{X_1, X_2, \dots, X_N, \dots\}$  are i.i.d  $N$ : integer random variable,  $X_i \perp N$

$$Z = \sum_{k=1}^N X_k$$

■ Part a. characteristic function of  $Z$  in terms of  $\Phi_x(\omega)$

$$E(Z) = E_N(E(Z|N)) \rightarrow \Phi_Z(\omega) = E_N(E(e^{j\omega Z} | N))$$

$$A = E(e^{j\omega Z} | N) = E(e^{j\omega \sum_{k=1}^N X_k}) \stackrel{iid}{=} \underbrace{E(e^{j\omega X_1})}_{\Phi_{X_1}(\omega)} \dots \underbrace{E(e^{j\omega X_N})}_{\Phi_{X_N}(\omega)} \stackrel{iid}{=} \Phi_x^N(\omega)$$

$$\text{We also know that } \Rightarrow E_N(Z^N) = \Gamma_N(Z) \Rightarrow \Phi_Z(\omega) = E_N(\Phi_x^N(\omega)) = \Gamma_N(\Phi_x(\omega))$$

$$\Rightarrow \Phi_Z(\omega) = \Gamma_N(\Phi_x(\omega))$$

■ Part b. The mean and variance of  $Z$

$$E(e^{j\omega Z}) = \Phi_Z(\omega) \Rightarrow \frac{\partial}{\partial \omega} \Phi_Z(\omega) \Big|_{\omega=0} = jE(Z) \Rightarrow E(Z) = -j \frac{\partial}{\partial \omega} \Phi_Z(\omega) \Big|_{\omega=0}$$

$$\Phi_Z(\omega) = \Gamma_N(\Phi_x(\omega)) = E_N(\Phi_x^N(\omega)) = \sum_{n=1}^N P(N=n) \Phi_x^n(\omega)$$

$$\rightarrow \frac{\partial}{\partial \omega} \Phi_Z(\omega) = \sum_{n=1}^N n P(N=n) \Phi_x^{n-1}(\omega) \Phi'_x(\omega)$$

$$\Rightarrow E(Z) = -j \frac{\partial}{\partial \omega} \Phi_Z(\omega) \Big|_{\omega=0} = \underbrace{\sum_{n=1}^N n P(N=n)}_{E(N)} \underbrace{\Phi_x^{n-1}(0)}_1 \underbrace{[-j \Phi'_x(0)]}_{E(X)} = E(X) \cdot E(N)$$

$$\frac{\partial^2}{\partial \omega^2} \Phi_Z(\omega) \Big|_{\omega=0} = -E(Z^2) \Rightarrow E(Z^2) = -\frac{\partial^2}{\partial \omega^2} \Phi_Z(\omega) \Big|_{\omega=0}$$

$$\begin{aligned} \rightarrow \frac{\partial^2}{\partial \omega^2} \Phi_Z(\omega) &= \frac{\partial}{\partial \omega} \left[ \sum_{n=1}^N n P(N=n) \Phi_x^{n-1}(\omega) \frac{\partial \Phi_x(\omega)}{\partial \omega} \right] \\ &= \sum_{n=1}^N n P(N=n) \left[ (n-1) \Phi_x^{n-2}(\omega) \Phi'_x{}^2(\omega) + \Phi_x^{n-1}(\omega) \Phi''_x(\omega) \right] \end{aligned}$$

$$\rightarrow E(Z^2) = -\sum_{n=1}^N n P(N=n) \left[ \underbrace{(n-1) \Phi_x^{n-2}(0) \Phi'_x{}^2(0)}_{-E^2(X)} + \underbrace{\Phi_x^{n-1}(0) \Phi''_x(0)}_{-E(X^2)} \right] = \sum_{n=1}^N n P(N=n) \left[ n E^2(X) + \underbrace{E(X^2) - E^2(X)}_{\text{var}(X)} \right]$$

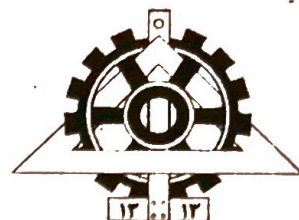
$$\begin{aligned} &= \underbrace{\sum_{n=1}^N n^2 P(N=n) E^2(X)}_{E(N^2)} + \underbrace{\sum_{n=1}^N n P(N=n)}_{E(N)} \text{var}(X) = E(N^2) E^2(X) + E(N) \text{var}(X) \end{aligned}$$

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Part c.  $X_i(s)$  are Normal RV's,  $N$  is a Geometric RV  $\rightarrow P(N=i) = p(1-p)^{i-1}$   
 $X_i \sim \mathcal{N}(\eta, \sigma^2)$   $N \sim \text{Geo}(p)$

First we try to compute the PGF of a Geometric Random Variable with parameter  $p$ .

$$\Gamma_N(z) = \sum_{i=1}^{\infty} z^i P(N=i) = \sum_{i=1}^{\infty} p(1-p)^{i-1} z^i = \frac{p}{1-p} \sum_{i=1}^{\infty} (z(1-p))^{i-1} = \frac{p}{1-p} \cdot \frac{z(1-p)}{1-z(1-p)} = \frac{pz}{1-z(1-p)}$$

Also, we have the characteristic function of a Normal Random Variables:

$$\Phi_X(\omega) = e^{j\omega\eta} e^{-\frac{\sigma^2\omega^2}{2}} ; X \sim \mathcal{N}(\eta, \sigma^2)$$

$$\Rightarrow \Phi_Z(\omega) = \Gamma_N(\Phi_X(\omega)) = \frac{p e^{j\omega\eta} e^{-\frac{\sigma^2\omega^2}{2}}}{1 - (1-p) e^{j\omega\eta} e^{-\frac{\sigma^2\omega^2}{2}}} = \frac{p}{e^{-j\omega\eta} e^{\frac{\sigma^2\omega^2}{2}} - (1-p)}$$

\* Now, considering the results of 'Part b', we can compute  $m_z = E(Z)$  and  $\text{var}(Z)$ :

$$m_z = E(Z) = E(X)E(N) = \eta \cdot \frac{1}{p} = \frac{\eta}{p}$$

$$\text{Var}(Z) = E(N^2)E^2(X) + \text{var}(X)E(N) = \frac{2-p}{p^2} \eta^2 + \sigma^2 \frac{1}{p} = \frac{2\eta^2}{p^2} + \frac{\sigma^2 - \eta^2}{p}$$