Stochastic Processes

University of Tehran

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Homework 7

Due : 1403/10/6

Problem 1

Let $X(t) = \sum_{k=-\infty}^{+\infty} A_k P(t-kT)$ where $A_k \in \{-1, +1\}$ is a sequence of i.i.d. random variables with $Pr\{A_k = +1\} = Pr\{A_k = -1\} = 0.5$ and $P(t)$ is a given (Energy Signal) pulse.

(a) Find the power spectral density (PSD) of $X(t)$.

(b) For the special case where $P(t) = \prod\left(\frac{t-0.5T}{T}\right)$, plot the PSD of $X(t)$.

Problem 2

Let $x(t)$ be a zero-mean stationary Gaussian random process with PSD $S_x(f) = \Lambda(f)$. Let $x^{'}(t) = \frac{d}{dt}x(t)$.

(a) Find the pdf of vector $\underline{X} = \left[x(t), x^{'}(t), x(t-1)\right]^T$.

(b) Find the PSD of $x^{'}(t)$ and cross power spectrum of $x(t)$ and $x^{'}(t)$.

(c) Can τ be found such that $x^{'}(t+\tau)$ and $x^{'}(t)$ be independent random variables for all t?

Problem 3

Let $X(t) = A\cos(2\pi Yt + \Theta)$, where $A > 0$ is a known constant, Y is a random variable with pdf $f_Y(y)$, and Θ is a uniform random variable on $[0, 2\pi]$. Y and Θ are independent. Find the power spectrum of $X(t)$.

Problem 4

Let $X(t)$ be a stationary random process with:

$$S_X(f) = \frac{4(\pi^2 f^2 + 1)}{(4\pi^2 f^2 + 1)(4\pi^2 f^2 + 9)}$$

(a) Find the innovation process of $X(t)$.

(b) Find a causal filter with impulse response $h(t)$, so that when $X(t)$ is passed through it the output $Y(t)$ has the autocorrelation function $R_Y(\tau) = e^{-|\tau|}$.

Problem 5

Let $X_1(t)$ and $X_2(t)$ be two random processes with cross-correlation function $R_{X_1X_2}(t_1, t_2)$. Let $Y_1(t)$ and $Y_2(t)$ denote the outputs of two deterministic systems with known impulse responses $h_1(t)$ and $h_2(t)$ to the inputs $X_1(t)$ and $X_2(t)$ respectively, as depicted in Figure 1.

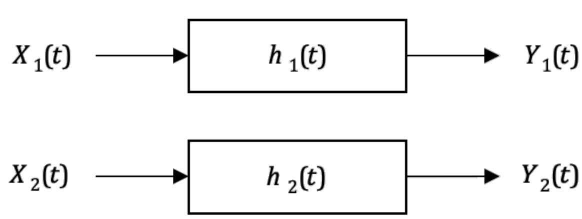

Figure 1

(a) Find $R_{X_1Y_2}(t_1, t_2)$, the cross-correlation function, and $S_{X_1Y_2}(f)$, the cross-spectral density $X_1(t)$ and $Y_2(t)$. Simplify your answer.

(b) Find $R_{Y_1Y_2}(t_1, t_2)$ and $S_{Y_1Y_2}(f)$. Simplify your answer.

(c) Show that if $X_1(t)$ and $X_2(t)$ are jointly stationary then $Y_1(t)$ and $Y_2(t)$ are also jointly stationary.

(d) Assume that $\forall f, S_{Y_1Y_2}(f) = 0$, where $h_1(t)$ and $h_2(t)$ are unknown. What can be inferred about $h_1(t)$ and $h_2(t)$? Articulate.

Problem 6

Let X be a random variable uniformly distributed on the interval $[-2, 3]$. Let the random process $X(t) = e^{-X}\cos(2\pi X t)$ undergo an ideal low-pass filter with the cutoff frequency $f_0 = 1$. Denote the output process as $Y(t)$.

(a) Find the power spectral density of $X(t)$.

(b) Find the power of $Y(t)$.

Problem 7

Let $X(t)$ be a zero-mean stationary random process with $R_X(\tau) = e^{-|\tau|}$. Define:

$$Y(t) = \int_0^2 X(t-s)ds$$

Find the power spectrum of $Y(t)$.