Stochastic Processes

University of Tehran

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Homework 1

Due : 1403/7/28

Problem 1

Suppose that $A_1, \dots, A_n$ are the events that partition the sample space Ω of a random experiment. Let X be a random variable defined on Ω. Prove the followings.

(a) $f_X(x) = f_X(x|A_1)P\{A_1\} + \cdots + f_X(x|A_n)P\{A_n\}$

(b) $E\{X\} = E\{X|A_1\}P\{A_1\} + \cdots + E\{X|A_n\}P\{A_n\}$

(c) Let B be an arbitrary event. Show that,

$$P\{B\} = \int_{+\infty}^{-\infty} P\{B|X = x\} f_x(x)\, dx = E_X\Big\{P\{B|x = x\}\Big\}$$

Problem 2

If X is a random variable with probability density function $f_X(x)$, then find the conditional pdf $f_X(x|a < X \leq b)$.

Problem 3

Let X be a random variable uniformly distributed on $[0, 2\pi)$

(a) Find the pdf of the random variable $Y = \sin(X)$.

(b) Compute $E\{Y|0 \leq X < \pi\}$.

Problem 4

Let X be a random variable with probability density function $f_X(x) = \frac{1}{2}e^{-|x|}$. Find the pdf of the random variable Y.

$$Y = \begin{cases} \sqrt{X} & X \geq 0 \\ 0 & X < 0 \end{cases}$$

Problem 5

Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin, i.e. their joint probability density function is,

$$f_{XY}(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & otherwise \end{cases}$$

(a) Are X and Y independent? Why?

(b) Find $E\{X|Y = y\}$.

(c) Find the joint probability density function of the polar coordinates, $R = \sqrt{X^2 + Y^2}$ and $\theta = tan^{-1}(\frac{Y}{X})$.

Problem 6

Let X and Y be two randomly chosen points in the intervals $[0, \frac{L}{2}]$ and $[\frac{L}{2}, L]$ respectively. Find the pdf of $Z = Y - X$, and compute $E\{Z\}$.

Problem 7

Let X and Y be two independent and exponentially distributed random variables with parameter λ, i.e.

$$f_X(x) = f_Y(y) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & otherwise \end{cases}$$

(a) Find the pdf of $V = \frac{X}{X+Y}$.

(b) If $U = \min\{X, Y\}$, then find $P\{X = U\}$.

(c) Find the pdf of $Z = \max\{X, Y\} - \min\{X, Y\}$.

(d) Find $P\{X \leq t < X + Y\}$ where $t > 0$.

Problem 8

The number of electrons that leave the cathode of a lamp is a Poisson random variable with parameter λ. The probability that an electron that has left the cathode, hits the anode is p. Find the pdf of the number of the electrons that hit the anode. Compute the mean and the variance of it.

Problem 9

Let X and Y be two random variables with the joint probability density function,

$$f_{XY}(x,y) = \begin{cases} Axy^2 & 0 < 2y \leq x \leq 2 \\ 0 & otherwise \end{cases}$$

(a) Find A.

(b) Are X and Y independent? Why?

(c) Compute $E\{X|Y=y\}$.

(d) Compute $P\{X^2+Y^2 \leq 1|X \geq 0.5\}$.

Problem 10

Let X, Y be independent random variables uniformly distributed over $(0,1)$. Find $E\{X|\min(X,Y)\}$.