

Math 583: Geometric Group Theory

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1 Refreshers from topology

Perhaps surprisingly, we will study groups using results from geometry. In particular, we will look at groups by realizing them as fundamental groups. The following results will be used together repeatedly.

Lemma 1.1. *Topological spaces that are homotopic have the same fundamental group.*

Lemma 1.2. *Let X a CW-complex, and let A be a contractible subcomplex. Then, X/A is homotopic to X . In particular, $\pi_1(X) \cong \pi_1(X/A)$.*

Free groups are easily realized as topological spaces.

Lemma 1.3. *The n -bouquet B_n ($n \leq \infty$), the graph with 1 vertex and n edges, has $\pi_1(B_n) = F_n$.*

Proof. Recall $\pi_1(S^1) = \mathbb{Z}$. The n -bouquet is the wedge product of S^1 with itself n times. That is, gluing of circles along a single point. Since we work only up to homotopy, the choice of point does not matter.

The Van Kampen theorem then yields that $\pi_1(S^1 \# S^1) = \mathbb{Z} * \mathbb{Z}$, and so on. \square

Proposition 1.1. *If Y is a covering space of X , then $\pi_1(Y) < \pi_1(X)$.*

One can ask if every group arises as a topological space. The answer is in the positive; we can build the so called presentation complex. An important tool is the Van Kampen Theorem, which allows to compute a fundamental group from those of a subcover.

Theorem 1.1 (Van Kampen Theorem). *Let $X = X_1 \cup X_2$, and $X_0 = X_1 \cap X_2$. Then, $\pi_1(X) = \pi_1(X_1) *_{\pi_1(X_0)} \pi_1(X_2)$.*

Definition 1.1 (Presentation Complex). Let $G = \langle A | R \rangle$. Then, the complex X_G is defined as the $|A| = n$ -bouquet, with the following faces. If $R_k = a_{i_1} \dots a_{i_l}$ is a relation, think of X_G as a circle with n points around it, all representing x , and edges a_{i_1}, a_{i_2}, \dots etc between them. Then, glue a face from point 1 to l .

Proof. (We define the Amalgamated Product later) Write X_G as a circle with n points around it, and let X_1 be an interior circle and X_2 the outer rim (so X_2 is an annulus). We have X_1 simply connected, $\pi_1(X_2) = F_n$, and $\pi_1(X_0) = \langle\langle R \rangle\rangle$, and the rest follows by definition. \square

Not only can every group be realized as the fundamental group of some space, but so can every homomorphism.

Proposition 1.2. *Every homomorphism $G \rightarrow G'$ arises as the induced map from $f : X \rightarrow X'$ for some X' with fundamental group G' .*

2 Prologue

Definition 2.1 (Free Product). The free product $A * B = (S, \cdot)$ of groups is the following set and product:

- S is the set of all reduced words $a_1 b_1 a_2 b_2 \dots a_n b_n$, where $a_i \in A$ and $b_i \in B$ are all non-identity elements except maybe a_1 and b_n ,
- $w \cdot w'$ is the concatenation of words, followed by reduction (perform the possible operations inside of A or B that might happen where the words meet).

Note that A and B both live as subgroups of their free product.

Definition 2.2 (Free Group). The free group F_n is a free product of \mathbb{Z} with itself n -times, where $n \leq \infty$.

Proposition 2.1. *Every subgroup of F_n is itself free.*

Proof. We have $\pi_1(B_n) = F_n$. So, subgroups of F_n can be determined by looking at covering spaces of B_n .

Every covering space of B_n is a graph Γ . Let S a spanning tree of Γ . Since S is contractible, we have $\pi_1(\Gamma) = \pi_1(\Gamma/S)$. The proof is done since Γ/S has a single vertex, and thus is a bouquet. \square

The following theorem is of great importance, and is named after I. A. Gruško. The proof below is by Stallng.

Theorem 2.1 (Gruško). *Let $\phi : F \rightarrow G$ is a surjective homomorphism where F is free and $G = G_1 * G_2$.*

*Then, $\exists F_1, F_2 < F : \phi(F_i) = G_i$ and $F = F_1 * F_2$.*

Proof. As before, we switch to complexes. Let X_i be complex with $\pi_1(X_i) = G_i$, and let X be the complex obtained by joining X_1 and X_2 by an edge. Place a point x on this edge. Note that $\pi_1(X) = G$ by Van Kampen theorem.

We can find Y compact 2-complex with $\pi_1(Y) = F$ so that $f : Y \rightarrow X$ is a continuous map with $f_* = \phi$. Our goal is to show that the fiber of x is a tree in Y .

We can now finish the proof. Let $F_i = f^{-1}(X_i)$, and apply Van Kampen Theorem to $Y/f^{-1}(x) = f^{-1}(X_1) \cup f^{-1}(X_2)$, noting that their intersection is $f^{-1}(x)$, a tree. We get

$$F = \pi_1(Y) = \pi_1(Y/f^{-1}(x)) = \pi_1(f^{-1}(X_1)) * \pi_1(f^{-1}(X_2)) = F_1 * F_2$$

So the F_i are subgroups of F , thus free groups by the previous theorem, and we get the advertized free product equality.

Note also that $f_*(F_i) = \phi(F_i) < G_i$. But since f_* is surjective we must have equality.

To find the objects in our goal, we do the following. In general, $f^{-1}(x)$ is a forest. We decrease the number of components by finding a path $l \in Y^{(1)}$ connecting different trees, and so that $f(l)$ falls to a trivial element of $\pi_1(X_i)$. If we can do that, then attach an edge e to the ends of l and a cell D delimited by $e \cup l$. Since $f(l)$ is trivial, we can extend f so that $f^{-1}(x)$ now contains e .

To find l , start with an edge path L joining trees in $f^{-1}(x)$. By surjectivity of $\phi = f_*$, we can find γ a closed edge-path in Y based at the starting point of L with $[f(\gamma)] = [f(L)]$ in $\pi_1(X)$. Let $l = \gamma^{-1}L$. This is a path with $f^{-1}(l)$ trivial in $\pi_1(X)$. Partition $l = l_1 l_2 \dots l_n$ where all the endpoints are in $f^{-1}(x)$, and l_i is mapped alternatingly in X_1 and X_2 . Let $g_j = [f(l_j)]$. If $g_j = 1$, but then endpoints of l_j lie on the same tree in $f^{-1}(x)$, we can replace l_j by a path in $f^{-1}(x)$ which we merge with l_{j-1} and l_{j+1} . At this point, $[f(l)] = 1 = g_1 \dots g_n$ lying alternatingly in G_1 and G_2 . So at least one g_j is trivial, and l_j is the desired path. \square

3 The Amalgamated Product

In what follows, we have spaces X_1 and X_2 , an open cover for X , with fundamental groups G_1 and G_2 , and $X_0 = X_1 \cap X_2$ with fundamental group G_0 .

Definition 3.1 (Amalgamated product). The amalgamated product of groups $G_1 *_{G_0} G_2$ is a quotient of the free product. The data are injections $\phi_i : G_0 \hookrightarrow G_i$. Then,

$$G_1 *_{G_0} G_2 = G_1 * G_2 / \ll \phi_1(g)\phi_2(g)^{-1} : \forall g \in G_0 \gg$$

Suppose we only know $\phi_i : G_0 \hookrightarrow G_i$, and want to build X whose fundamental group is the amalgamated product. The next corollary follows directly from Van Kampen's theorem. Recall that every ϕ is induced by some f with appropriate fundamental groups, so let $\phi_i = (f_i)_*$.

Corollary 3.1. *The space $X = X_1 \cup X_0 \times [0, 1] \cup X_2 / \sim$, with the relation $(x, 0) \sim f_1(x)$ and $(x, 1) \sim f_2(x)$.*

To understand the structure of the amalgamated product, we want to understand its subgroups. To do so, we look at the covers. A good place to start is with the universal cover. We first look at one side of the dumbbell.

Lemma 3.1. *The universal cover of $Y_1 = X_1 \cup X_0 \times [0, 1] / \sim$ is $\widetilde{Y}_1 = \widetilde{X}_1 \cup \bigsqcup \widetilde{X}_0 \times [0, 1] / \sim$.*

Proof. Let $p : \widetilde{Y}_1 \rightarrow Y_1$ be the universal cover. First, we show $\widetilde{X}_0 \times [0, 1]$ are the components of $p^{-1}(X_0 \times [0, 1])$. If not, then there is α some loop in X_0 giving a nontrivial element in $\pi_1(X_0)$, and it lifts to a loop $\tilde{\alpha}$ in $\widetilde{X}_0 \times [0, 1]$. Since ϕ^i is injective (by the data), $\phi^i(\alpha)$ is a non-trivial element of $\pi_1(X_1)$, so its lift *alpha* must have disjoint endpoints, contradicting that $\tilde{\alpha}$ was a loop.

Then, \widetilde{Y}_1 is a copy of \widetilde{X}_1 with copys of $\widetilde{X}_0 \times [0, 1]$ glued via the lifts $\widetilde{X}_0 \rightarrow \widetilde{X}_i$ of f^i . □

Proposition 3.1. *The universal cover of $X = X_1 \cup X_0 \times [0, 1] \cup X_2 / \sim$ is copies of \widetilde{Y}_1 and \widetilde{Y}_2 glued in a tree like fashion by cylinders \widetilde{X}_0 .*

Proof. This construction is simply connected. □