SUMMER 2025 READING GROUP ON ERGODIC THEORY

Lecture 2 (Samy Lahlou): Crash course on Measure Theory, Part II

Let (X, μ)[1] be a measure space. Our goal is to define the *Lebesgue integral* $\int f \, \mathrm{d}\mu$ for a function $f : X \to \mathbb{R}$. Again, this is not possible in full generality, so we restrict ourselves to the so-called *measurable functions*.

Further reading. [Tse23, Lectures 9 to 13, 17 to 21] and [Fol99, Chapters 2 and 3].

Definition 1. A *simple function* is an $\mathbb{R}$-linear combination of characteristic functions on μ-measurable sets, i.e., if $(E_i)_{i \le n}$ is a collection of pairwise-disjoint μ-measurable sets and $(a_i)_{i \le n}$ are distinct reals, then $\varphi \coloneqq \sum_{i \le n} a_i \chi_{E_i}$ is a said to be a *simple function.* Define its *(Lebesgue) integral* as $\int \varphi \, \mathrm{d}\mu \coloneqq \sum_{i \le n} a_i \mu(E_i)$.

For a (bounded) positive function $f : X \to \mathbb{R}_{\ge 0}$, we might define $\int f \, \mathrm{d}\mu$ by approximating f by simple functions from below, say by an increasing sequence (φ_n) of simple functions such that $f = \lim_n \varphi_n$ uniformly. However, not all functions f admit such an approximation.

To see this, let us attempt to construct such a sequence (φ_n). For each n, we will approximate the cutoff of f at 2^n, i.e., the function $\min(f, 2^n)$. We do so by partitioning the codomain $[0, 2^n]$ into intervals of length 2^{-n}, for a total of $k_n \coloneqq 2^n/2^{-n} = 2^{2n}$ intervals. Set $E_k \coloneqq f^{-1}([2^{-n}k, \infty))$ for each $k \in \{1, \dots, k_n\}$, and let $\varphi_n \coloneqq \sum_{k \le k_n} 2^{-n} \chi_{E_k}$. One easily checks that $f = \lim_n \varphi_n$ uniformly.

However, E_k is not guaranteed to be μ-measurable! To fix this, we simply define the issue away.

Definition 2. A function $f : X \to Y$ between measurable spaces $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces is said to be $(\mathcal{B}, \mathcal{C})$*-measurable* if $f^{-1}(C) \in \mathcal{B}$ for all $C \in \mathcal{C}$.

A function $f : X \to Y$ between topological spaces is said to be *Borel* if it is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable. A *Borel isomorphism* is a bijection $f : X \to Y$ such that both f and f^{-1} are Borel.

Exercise 3. Continuous maps are Borel. Hint: Define a σ-algebra containing open sets in the codomain.

So far we only dealt with measurable spaces. Let us now bring a measure into the picture.

Definition 4. Let (X, μ) be a measure space and Y be a topological space. A function $f : X \to Y$ is said to be μ*-measurable* if it is $(\mathrm{Meas}_\mu, \mathcal{B}(Y))$-measurable.

Remark 5. Compositions of μ-measurable functions *need not* be μ-measurable.

The following exercise is one of the main reasons why μ-measurable functions are introduced, and ultimately also why the Lebesgue integral is superior compared to the Riemann integral.

Exercise 6. In separable metric spaces, pointwise limits of μ-measurable functions are μ-measurable, i.e., if (f_n) is a sequence of μ-measurable maps $f_n : X \to Y$ from a measure space (X, μ) to a separable space Y, and $f \coloneqq \lim_n f_n$ (pointwise), then $f : X \to Y$ is μ-measurable.

Hint: Let $\mathcal{C} \coloneqq \{B \in \mathcal{B}(Y) : f^{-1}(B) \in \mathrm{Meas}_\mu\}$. Show that $\mathcal{C}$ is a σ-algebra containing all open set in Y, so $\mathcal{C} = \mathcal{B}(Y)$, as desired. For each $U \subseteq Y$ open, use separability to write $U = \bigcup_{n \in \mathbb{N}} B_n$, where each B_n is a ball whose closure is contained in U, and show that $f^{-1}(U) \in \mathrm{Meas}_\mu$.

Exercise 7. If $f_1, f_2 : (X, \mu) \to \mathbb{R}$ are μ-measurable and $g : \mathbb{R}^2 \to \mathbb{R}$ is Borel, then $g(f_1, f_2) : X \to \mathbb{R}$ is also μ-measurable. In particular, $f_1 + f_2$ and $f_1 \cdot f_2$ are μ-measurable.

Exercise 8. If (f_n) is a sequence of μ-measurable functions $f_n : X \to \overline{\mathbb{R}}$, then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, and $\liminf_n f_n$ are also μ-measurable.

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[1] Whenever the σ-algebra is not stated, we assume that μ is defined on Meas_μ. In particular, we assume that μ is complete.

Notation 9. We write $L(X,\mu)$ for the set of all μ-measurable functions $f:(X,\mu)\to\overline{\mathbb{R}}$, and $L^+(X,\mu)$ for those which are non-negative.

We are finally ready to define the Lebesgue integral.

Definition 10. Let (X,μ) be a measure space. The *(Lebesgue) integral* of $f\in L^+(X,\mu)$ is

$$\int f\,\mathrm{d}\mu \coloneqq \sup\left\{\int \varphi\,\mathrm{d}\mu : 0\le\varphi\le f \text{ simple function}\right\}.$$

In general, if $f\in L(X,\mu)$, we decompose $f=f^+-f^-$ where $f^+\coloneqq\max\{f,0\}$ and $f^-\coloneqq\max\{-f,0\}$. The *(Lebesgue) integral* of f is $\int f\,\mathrm{d}\mu\coloneqq\int f^+\,\mathrm{d}\mu-\int f^-\mathrm{d}\mu$, provided that one of the terms is finite.

If $\int f\,\mathrm{d}\mu<\infty$, we say that f is *μ-integrable*, in which case we write $f\in L^1(X,\mu)$. More generally,

Definition 11. Take $p\in[1,\infty]$ and let $L^p(X,\mu)$ be the set of all μ-measurable functions $f:X\to\overline{\mathbb{R}}$ such that $\|f\|_p<\infty$, where $\|f\|_p\coloneqq(\int|f|^p\,\mathrm{d}\mu)^{1/p}$ if $p<\infty$ and $\|f\|_\infty\coloneqq\text{ess-sup}\,|f|\coloneqq\inf\{c\ge0:|f|\le c\ \mu\text{-a.e.}\}$.

Exercise 12. Let $f,g\in L^p(X,\mu)$. If $f\le g$, then $\|f\|_p\le\|g\|_p$.

Since a μ-measurable function $f:X\to\overline{\mathbb{R}}$ can be approximated from below by simple functions (φ_n), we should be able to calculate $\int f\,\mathrm{d}\mu$ as the limit of $\int\varphi_n\,\mathrm{d}\mu$. Indeed,

Theorem 13 (Monotone Convergence Theorem). *If $(f_n)\in L^+(X,\mu)$ and $f_n\nearrow f$, then $\int f_n\,\mathrm{d}\mu\nearrow\int f\,\mathrm{d}\mu$.*

Corollary 14. *If $(f_n)\in L^+(X,\mu)$, then $\sum_n\int f_n\,\mathrm{d}\mu=\int\sum_n f_n\,\mathrm{d}\mu$.*

Exercise 15. For any $f,g\in L^1(X,\mu)$ and $a,b\in\mathbb{R}$, we have $\int(af+bg)\,\mathrm{d}\mu=a\int f\,\mathrm{d}\mu+b\int g\,\mathrm{d}\mu$.
HINT: Simple $\rightsquigarrow_{\mathrm{MCT}}L^+\rightsquigarrow L^1$.

Exercise 16. Let $f,g\in L^1(X,\mu)$. If $f=g$ μ-a.e., then $\int f\,\mathrm{d}\mu=\int g\,\mathrm{d}\mu$. HINT: Consider $\int(f-g)\,\mathrm{d}\mu$.

We list two more convergence theorems that will be useful later on.

Theorem 17 (Fatou's Lemma). *If $(f_n)\in L^+(X,\mu)$, then $\int\liminf_n f_n\,\mathrm{d}\mu\le\liminf_n\int f_n\mathrm{d}\mu$.*

Theorem 18 (Dominated Convergence Theorem). *Let $(f_n)\in L^1(X,\mu)$. If $f_n\to f$ μ-a.e. and $|f_n|\le g$ for some $g\in L^1(X,\mu)$, then $\lim_n\int f_n\,\mathrm{d}\mu=\int f\,\mathrm{d}\mu$.*

Let us now discuss differentiation of functions $f:X\to\mathbb{R}$; for convenience, we assume[2] that $f\in L^+(X,\mu)$. For these functions, we can define a new measure ν on $\mathcal{B}$ by $\nu(B)\coloneqq\int_B f\,\mathrm{d}\mu\coloneqq\int f\cdot\chi_B\,\mathrm{d}\mu$, which measures the 'area under the curve'. Note that for each $B\in\mathcal{B}$, we have B is ν-null whenever B is μ-null.

It turns out that the 'correct' setting to discuss differentiation is between two measures μ and ν which satisfy the above condition.

Definition 19. If μ,ν are measures on a measurable space $(X,\mathcal{B})$ and B is ν-null whenever B is μ-null for each $B\in\mathcal{B}$, we say that ν is *absolutely continuous w.r.t* μ, and write $\nu\ll\mu$.

Theorem 20 (Lebesgue-Radon-Nikodym Theorem). *If $\nu\ll\mu$ are σ-finite measures on a measurable space $(X,\mathcal{B})$, then there exists a $\mathcal{B}$-measurable map $f:X\to\mathbb{R}_{\ge0}$ such that $\nu(B)=\int_B f\,\mathrm{d}\mu$ for all $B\in\mathcal{B}$.*

Such a function $f:X\to\mathbb{R}_{\ge0}$ is unique μ-a.e., and is called the *Radon-Nikodym derivative of ν w.r.t. μ*, denoted $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Thus, we have $\nu(B)=\int_B\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mathrm{d}\mu$ for all $B\in\mathcal{B}$.

Corollary 21. *In the above setting, we have $\int g\,\mathrm{d}\mu=\int g\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\mathrm{d}\nu$ for all $g\in L^1(X,\mu)$.*

To relate $\mathrm{d}\nu/\mathrm{d}\mu$ to derivatives in calculus (say on $\mathbb{R}^n$), we let $\mu\coloneqq\lambda$ be Lebesgue measure on $\mathbb{R}^n$.

Theorem 22 (Lebesgue Differentiation Theorem). *For any locally-integrable function $f:\mathbb{R}^n\to\mathbb{R}$ (i.e. if $f\cdot\chi_K\in L^1(\mathbb{R}^n,\lambda)$ for every compact $K\subseteq\mathbb{R}^n$) and for λ-a.e. $x\in\mathbb{R}^n$, we have*

$$f(x)=\lim_{\varepsilon\to0}\frac{1}{\lambda(B_\varepsilon(x))}\int_{B_\varepsilon(x)}f\,\mathrm{d}\lambda.$$

[2]Otherwise, we will need to discuss 'signed measures'.

Corollary 23. *For any locally-finite Borel measure $\mu \ll \lambda$ on $\mathbb{R}^n$ and for λ-a.e. $x \in \mathbb{R}^n$, we have*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{\lambda(B_\varepsilon(x))}.$$

We end by briefly mentioning the 'Isomorphism Theorems'. These justify why we only gave three examples in Lecture 1, and allows us to work in concrete spaces like $[0,1]$ or $2^{\mathbb{N}}$.

Definition 24. A measurable space $(X, \mathcal{B})$ is said to be *standard Borel* if $\mathcal{B}$ is the Borel σ-algebra of some Polish (i.e. separable and completely metrizable) topology on X.

A probability space $(X, \mathcal{B}, \mu)$ is *standard* if $(X, \mathcal{B})$ is standard Borel.

Theorem 25 (Borel Isomorphism Theorem)**.** *Any two uncountable standard Borel spaces are Borel isomorphic. In particular, they all have cardinality continuum and are Borel isomorphic to $2^{\mathbb{N}}$.*

Definition 26. Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces. If $f : X \to Y$ is $(\mathcal{B}, \mathcal{C})$-measurable and μ is a measure on $\mathcal{B}$, the *pushforward measure of μ by f* is the measure $f_*\mu$ on $\mathcal{C}$ defined by $f_*\mu(C) \coloneqq \mu(f^{-1}(C))$.

Definition 27. A function $f : X \to Y$ between measure spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ is said to be a *measure isomorphism* if $f_*\mu = \nu$, and if there exist a μ-conull set $X_0 \subseteq X$ and a ν-conull set $Y_0 \subseteq Y$ such that $f|_{X_0} : X_0 \to Y_0$ is a bijection and $f|_{X_0}$ (resp. $f^{-1}|_{Y_0}$) is μ-measurable (resp. ν-measurable).

Theorem 28 (Measure Isomorphism Theorem)**.** *Any two atomless standard probability spaces are measure isomorphic. In particular, they are all measure isomorphic to $([0,1], \lambda)$.*

References

[Fol99] Gerald B. Folland. *Real Analysis. Modern Techniques and Their Applications.* 2nd ed. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. John Wiley & Sons, 1999.

[Tse23] Anush Tserunyan. *MATH564: Advanced Real Analysis 1.* URL: https://www.math.mcgill.ca/atserunyan/Courses/2023F.Math564.Analysis%201/.