SUMMER 2025 READING GROUP ON ERGODIC THEORY

EXERCISE SHEET 8 (ZHAOSHEN ZHAI): CORRESPONDENCE PRINCIPLES

Exercise 1. Show that the following are equivalent.

Theorem (Simple recurrence in open covers). *Let $(U_\alpha)_\alpha$ be an open cover of a topological dynamical system*[a] *(X,T). There exists α such that $U_\alpha \cap T^{-n}U_\alpha \neq \varnothing$ for infinitely many $n \in \mathbb{N}$.*

Theorem (Infinite pigeonhole-principle). *For any $c \geq 1$, any c-colouring of $\mathbb{Z}$ always contains a colour class with infinitely-many elements.*

Theorem (Finite pigeonhole-principle). *For any $c, k \geq 1$, there exists $N(c,k)$ such that if $n \geq N(c,k)$ and we colour $\{1,\dots,n\}$ by c colours, then there is a colour class of at least k elements.*

[a] Recall that a *topological dynamical system* is a pair (X,T) consisting of a compact metrizable topological space X and a continuous map $T : X \to X$.

Exercise 2. Show that the following are equivalent.

Theorem (Multiple recurrence in open covers). *Let $(U_\alpha)_\alpha$ be an open cover of a topological dynamical system (X,T). There exists α such that for each $k \geq 1$, we have $\bigcap_{i<k} T^{-in}U_\alpha \neq \varnothing$ for some $n \in \mathbb{N}$.*

Theorem (Infinitary van der Waerden). *For any $c \geq 1$, any c-colouring of $\mathbb{Z}$ always contains a colour with arbitrarily long arithmetic progressions.*

Theorem (Finitary van der Waerden). *For any $c, k \geq 1$, there exists $N(c,k)$ such that if $n \geq N(c,k)$ and we colour $\{1,\dots,n\}$ with c colours, then there is a monochromatic k-term arithmetic progression.*

Exercise 3. Show that the following are equivalent.

Theorem (Furstenberg's multiple recurrence; v1). *Let $(X,\mathcal{B},\mu,T)$ be a measure-preserving dynamical system. For any $k \geq 1$ and any set $A \in \mathcal{B}$ with positive measure, there exists $n \geq 1$ such that*

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-(k-1)n}A) > 0.$$

Theorem (Furstenberg's multiple recurrence; v2). *Let $(X,\mathcal{B},\mu,T)$ be a measure-preserving dynamical system. For any $k \geq 1$ and any set $A \in \mathcal{B}$ with positive measure, we have*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n<N} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-(k-1)n}A) > 0.$$

Theorem (Furstenberg's multiple recurrence; v3). *Let $(X,\mathcal{B},\mu,T)$ be a measure-preserving dynamical system. For any $k \geq 1$ and any $f \in L^\infty(X,\mu)$ with $f \geq 0$ and $\int f \,\mathrm{d}\mu > 0$, we have*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n<N} \int f(T^{-n}f)(T^{-2n}f)\cdots(T^{-(k-1)n}f)\,\mathrm{d}\mu > 0.$$

Theorem (Infinitary Szemerédi). *Any subset of $\mathbb{Z}$ with positive upper density contains arbitrarily long arithmetic progressions.*

Theorem (Finitary Szemerédi). *For any $\delta > 0$ and $k \geq 1$, there exists $N(\delta,k)$ such that if $n \geq N(\delta,k)$, then any subset of $\{1,\dots,n\}$ with at least δn elements contains a k-term arithmetic progression.*

Date: June 18, 2025.