SUMMER 2025 READING GROUP ON ERGODIC THEORY

EXERCISE SHEET 4 (ZHAOSHEN ZHAI): ERGODIC ACTIONS AND ITS CONSEQUENCES

Throughout, let λ denote the Lebesgue measure on $\mathbb{R}$ (or on $[0,1]$) and let μ denote the Bernoulli(1/2)-measure on $2^{\mathbb{N}}$. The purpose of this exercise sheet is two-fold.

1. To introduce and give examples of ergodic *actions* of a group $G \curvearrowright (X, \mu)$.
2. To show how ergodicity gives rise to non-measurable sets.

Definition. Let G be a group and let (X, μ) be a standard measure space. An action $\varphi : G \curvearrowright X$ is

- *Borel* if for each $g \in G$, the map $x \mapsto gx$ is Borel.
- *measure-preserving* if it is Borel and $\mu(gB) = \mu(B)$ for each $g \in G$ and each Borel $B \subseteq X$.
- *ergodic* if it is measure-preserving and the orbit equivalence relation $\mathbb{E}_\varphi$ of φ is ergodic.

Exercise 1. Let (X, μ) be an atomless standard probability space and let $\varphi : G \curvearrowright (X, \mu)$ be a pmp action. Prove that if φ is ergodic, then every transversal of $\mathbb{E}_\varphi$ is non-measurable.

HINT: Let $T \subseteq X$ be a measurable transversal, so $X = \bigsqcup_{g \in G} gT$. Observe that $\mu(T) > 0$, and use that (X, μ) is atomless to partition $T = S_1 \sqcup S_2$ non-trivially. What can you say about the $\mathbb{E}_\varphi$-saturations of S_i?

Exercise 2. Consider the translation action $\varphi : \mathbb{Q} \curvearrowright (\mathbb{R}, \lambda)$, whose orbit equivalence relation is given by $x \mathbb{E}_{\mathbb{Q}} y$ iff $x - y \in \mathbb{Q}$. Use the 99% Lemma for λ to show that φ is ergodic.

Remark. Transversals for $\mathbb{E}_{\mathbb{Q}}$ (restricted to $[0,1]$), called *Vitali sets*, are non-measurable by Exercise 1.

Lemma (99% Lemma for μ). *For any measurable $A \subseteq 2^{\mathbb{N}}$, there exists a cylinder $[w] \subseteq 2^{\mathbb{N}}$ such that at-least 99% of $[w]$ is covered by A, i.e. $\mu(A \cap [w])/\mu([w]) \geq 0.99$.*

Exercise 3. For each $n \in \mathbb{N}$, let $\sigma_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the n^{th}*-bit flip* map, defined by flipping x_n to $1 - x_n$ and fixing all other coordinates. Let $G \coloneqq \langle \sigma_n \rangle_{n \in \mathbb{N}} \cong \bigoplus_n \mathbb{Z}/2\mathbb{Z}$, which naturally acts on $2^{\mathbb{N}}$.

1. Show that the orbit equivalence relation $\mathbb{E}_\varphi$ is given by *eventual equality* (denoted $\mathbb{E}_0$), where $x \mathbb{E}_0 y$ iff there exists $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq N$.
2. Observe that φ is a pmp action (skip this, if you want, as it is just measure theory).
3. Use the 99% Lemma for μ to show that φ is ergodic.

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