

NOTES ON ERGODIC THEORY

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ABSTRACT. Notes on the Summer 2025 Reading Group on Ergodic Theory, following [Tse22], organized by Frédéric Kai, Ludovic Rivet, and Zhaoshen Zhai ([website](#)).

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1. LECTURE 1 (SAMY LAHLOU): CRASH COURSE ON MEASURE THEORY, PART I

Given a set X , our goal is to define a map $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ that assigns to each subset $A \subseteq X$ a *measure* $\mu(A) \in [0, \infty]$ that ‘behaves like the volume of A ’. This turns out to be impossible in full generality (and we shall see using ergodic-theoretic methods that this impossibility is for good reason), so we instead restrict to special subsets of X with a nice algebraic (think: ‘constructible’) structure.

Further reading. [Tse23, Lectures 1 to 5] and [Fol99, Chapter 1].

Definition 1.1. Let X be a set. A σ -algebra on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X containing \emptyset and is closed under complements and countable unions. More precisely:

1. (Non-trivial). $\emptyset \in \mathcal{B}$.
2. (Closure under complements). For any $A \in \mathcal{B}$, we have $X \setminus A \in \mathcal{B}$.
3. (Closure under countable unions). For any countable family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$, we have $\bigcup_n A_n \in \mathcal{B}$.

Definition 1.2. If \mathcal{B} is a σ -algebra on a set X , the pair (X, \mathcal{B}) is said to be a *measurable space*.

A useful way to construct a σ -algebra is to start with an arbitrary family $\mathcal{C} \subseteq \mathcal{P}(X)$ and close¹ it under the above three conditions. Abstractly:

Definition 1.3. The σ -algebra *generated* by $\mathcal{C} \subseteq \mathcal{P}(X)$ is $\langle \mathcal{C} \rangle_\sigma := \bigcap \{ \mathcal{B} : \mathcal{B} \supseteq \mathcal{C} \text{ is a } \sigma\text{-algebra on } X \}$.

Note that $\langle \mathcal{C} \rangle_\sigma$ is indeed a σ -algebra on X since the intersection of σ -algebras is again a σ -algebra.

Definition 1.4. Let X be a topological space. The *Borel σ -algebra* on X is $\mathcal{B}(X) := \langle \mathcal{T} \rangle_\sigma$, where \mathcal{T} is the topology on X . The elements of $\mathcal{B}(X)$ are called the *Borel sets* of X .

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¹This ‘closure’ operation can be made precise as follows. Starting with $\mathcal{C}_0 := \mathcal{C}$, throw in all the subsets of X that is necessary to satisfy Definition 1.1 relativized to \mathcal{C}_0 to obtain \mathcal{C}_1 (that is, let \mathcal{C}_1 contain \emptyset and such that if $A \in \mathcal{C}_0$, then $X \setminus A \in \mathcal{C}_1$, and similarly for condition 3). Then, let \mathcal{C}_2 be defined by throwing in all the countable unions and complements of sets in \mathcal{C}_1 . Doing so infinitely-many times and taking the union $\bigcup_\alpha \mathcal{C}_\alpha$ will give us $\langle \mathcal{C} \rangle_\sigma$, but beware that this process must proceed into the transfinite up to $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal; ask your local set theorist why.

Intuitively, for any topological space X , one would like to ‘measure’ the Borel sets. This is justified since if one wants a measure compatible with the topology, then one must be able to measure the open sets, and hence also closed sets, countable unions of closed sets (called F_σ sets), countable intersections of open sets (called G_δ sets), countable intersections of F_σ sets, countable unions of G_δ sets, and so on².

Definition 1.5. A *measure* on a measurable space (X, \mathcal{B}) is a map $\mu : \mathcal{B} \rightarrow [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigsqcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$ for any pairwise disjoint family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$.

The triple (X, \mathcal{B}, μ) is then called a *measure space*. A *Borel measure* is a measure defined on some Borel σ -algebra.

Example 1.6 (Lebesgue). Equip \mathbb{R} with its usual topology. There is³ a unique measure $\lambda : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ on \mathbb{R} , called the *Lebesgue measure*, such that $\lambda([a, b]) = b - a$ for each $a \leq b$.

Example 1.7 (Bernoulli). Equip $2 = \{0, 1\}$ with the discrete topology and consider the product topology on $2^{\mathbb{N}}$. For each $p \in [0, 1]$, is a unique measure $\mu_p : \mathcal{B}(2^{\mathbb{N}}) \rightarrow [0, \infty]$ on $2^{\mathbb{N}}$, called the *Bernoulli (p) measure*, such that for each word $w \in 2^{<\mathbb{N}}$, we have $\mu_p([w]) = p^{n_1}(1-p)^{n_0}$ where n_i is the number of $i \in \{0, 1\}$ in w and $[w]$ is the set of all sequences in $2^{\mathbb{N}}$ containing w as a prefix.

If $p = 0$ (similarly if $p = 1$), then $\mu_p(\xi) \in \{0, 1\}$, and we have $\mu_p(\xi) = 1$ iff $(p, p, p, \dots) \in \xi$. Thus, all of the measure is concentrated at (p, p, p, \dots) . Measures in which this occurs are called *Dirac measures*.

Example 1.8 (Dirac). Let X be a set and fix $x \in X$. The *Dirac measure concentrated at x* is the measure $\delta_x : \mathcal{P}(X) \rightarrow \{0, 1\}$ defined by $\delta_x(A) := 1$ iff $x \in A$, and $\delta_x(A) := 0$ iff $x \notin A$.

Definition 1.9. A measure μ on (X, \mathcal{B}) is said to be *finite* if $\mu(X) < \infty$, a *probability measure* if $\mu(X) = 1$, and *σ -finite* if there is a partition $X = \bigsqcup_{n \in \mathbb{N}} X_n$ such that $X_n \in \mathcal{B}$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$.

Unless otherwise stated, all measures are assumed to be σ -finite. In fact, we will usually only deal with probability measures, since we can also normalize a finite measure to a probability measure by $\mu \mapsto \mu/\mu(X)$.

Lastly, even though μ is only defined on the σ -algebra \mathcal{B} , we can slightly extend μ to a larger σ -algebra.

Definition 1.10. Let (X, \mathcal{B}, μ) be a measure space. A subset $Z \subseteq X$ is said to be *μ -null* if there exists some $Z' \in \mathcal{B}$ such that $Z \subseteq Z'$ and $\mu(Z') = 0$. We write Null_μ for the set of all μ -null subsets of X . A subset $A \subseteq X$ is said to be *μ -conull* if $X \setminus A$ is μ -null.

Definition 1.11. Let (X, \mathcal{B}, μ) be a measure space. A subset $A \subseteq X$ is *μ -measurable*⁴ if there exists some $B \in \mathcal{B}$ and some μ -null set Z such that $A = B \cup Z$. We write Meas_μ for the set of all μ -measurable sets.

It is an exercise that $\text{Meas}_\mu = \langle \mathcal{B} \cup \text{Null}_\mu \rangle_\sigma$. Moreover, μ admits a unique extension to a map $\bar{\mu} : \text{Meas}_\mu \rightarrow [0, \infty]$, called the *completion* of μ , and this measure satisfies $\text{Meas}_{\bar{\mu}} = \text{Meas}_\mu$. HINT: $\bar{\mu}(B \cup Z) := \mu(B)$.

Definition 1.12. A measure μ is *complete* if $\bar{\mu} = \mu$.

For convenience, we will always assume that measures are complete. Neither measures λ nor μ_p in Examples 1.6 and 1.7 are complete, so we tacitly extend them.

We end with some easy exercises on measures; please read/prove them, as they will be used freely in the future; they are roughly ranked by difficulty. Throughout, let (X, \mathcal{B}, μ) be a measure space and let $A_n \in \mathcal{B}$.

Exercise 1.13 (Monotonicity). If $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$.

Deduce that if μ is finite, then μ is a bounded function. (Are σ -finite measures bounded?)

Exercise 1.14 (Inclusion-exclusion). For any $A_1, A_2 \in \mathcal{B}$, we have $\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)$.

²This goes up the *Borel hierarchy*, studied in *Descriptive Set Theory*; see [Kec95].

³We will not prove this fact, but it is an application of Carathéodory’s Extension Theorem; see [Tse23, Lecture 4].

⁴Very confusing terminology. One might think that elements of \mathcal{B} are the ‘measurable’ ones, but this removes μ from the picture. In general, there are much more μ -measurable sets than there are sets in \mathcal{B} . Indeed, there are 2^{\aleph_0} -many Borel sets on \mathbb{R} , but there are $2^{2^{\aleph_0}}$ -many λ -measurable sets!

Exercise 1.15 (Continuity \nearrow). If $(A_n)_{n \in \mathbb{N}}$ is increasing, then $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \lim_n \mu(A_n)$.

Exercise 1.16 (Continuity \searrow). If $(A_n)_{n \in \mathbb{N}}$ is decreasing and $\mu(A_1) < \infty$, then $\mu(\bigcap_{n \in \mathbb{N}} A_n) = \lim_n \mu(A_n)$.

Exercise 1.17. Show that $\lambda(\mathbb{Q}) = 0$. HINT: What is the Lebesgue measure of singletons?

Let P be a property of some points in X . We say that P holds μ -almost everywhere (or μ -almost surely) if $\{x \in X : x \text{ satisfies } P\}$ is μ -conull.

Exercise 1.18 (Borel-Cantelli Lemmas). Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of μ -measurable sets.

1. If $\sum_{n \in \mathbb{N}} \mu(A_n) < \infty$, then μ -almost every $x \in X$ lives in at-most finitely-many A_n .
2. (Measure Compactness). If $\mu(X) < \infty$ and there exists $\varepsilon > 0$ such that $\mu(A_n) \geq \varepsilon$ for all $n \in \mathbb{N}$, then at least an ε -measure set of $x \in X$ lives in infinitely-many A_n 's.

For measurable spaces (X_1, \mathcal{B}_1) and (X_2, \mathcal{B}_2) , define $\mathcal{B}_1 \otimes \mathcal{B}_2 := \langle B_1 \times B_2 : B_i \in \mathcal{B}_i \rangle_\sigma$.

Exercise 1.19. Show that if X_i are second-countable topological spaces, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$.

Exercise 1.20. Let X be a topological space. A *Cantor set* is a subset $C \subseteq X$ homeomorphic to $2^{\mathbb{N}}$.

1. Show that the ‘middle-thirds Cantor set’ $C \subseteq [0, 1]$ is a Cantor set as in the above definition. Moreover, show that $\lambda(C) = 0$. HINT: Recall the construction $C = \bigcap_{n \in \mathbb{N}} C_n$ and use continuity.
2. Define a Cantor set $C \subseteq [0, 1]$ with positive Lebesgue measure. HINT: fatten the standard construction.

A measurable set $A \subseteq X$ is said to be an *atom* if there is no subset $B \subseteq A$ with $0 < \mu(B) < \mu(A)$. For example, singletons $\{x\}$ are atoms under the Dirac measure δ_x . More generally:

Exercise 1.21 (Atomic Decomposition). If (X, \mathcal{B}, μ) is a σ -finite measure space, \mathcal{B} is *countably generated* (i.e., $\mathcal{B} = \langle \mathcal{B}_0 \rangle$ for some countable $\mathcal{B}_0 \subseteq \mathcal{P}(X)$), and *separates points* (i.e., if $x \neq y$, then there exists $B \in \mathcal{B}$ such that $x \in B \not\ni y$), then every atom $A \in \mathcal{B}$ is a singleton. Moreover, if $\{x_\alpha\}$ are all the atoms (how many can there be?), then $\mu = \mu_0 + \sum_\alpha a_\alpha \delta_{x_\alpha}$ for some atomless measure μ_0 and some $a_\alpha \geq 0$.

2. LECTURE 2 (SAMY LAHLOU): CRASH COURSE ON MEASURE THEORY, PART II

Let (X, \mathcal{B}, μ) be a measure space. Our goal is to define the *Lebesgue integral* $\int f d\mu$ for a function $f : X \rightarrow \mathbb{R}$. Again, this is not possible in full generality, so we restrict ourselves to the so-called *measurable functions*.

Further reading. [Tse23, Lectures 9 to 13, 17 to 21] and [Fol99, Chapters 2 and 3].

Definition 2.1. A *simple function* is an \mathbb{R} -linear combination of characteristic functions on μ -measurable sets, i.e., if $(E_i)_{i \leq n}$ is a collection of pairwise-disjoint μ -measurable sets and $(a_i)_{i \leq n}$ are distinct reals, then $\varphi := \sum_{i \leq n} a_i \chi_{E_i}$ is said to be a *simple function*. Define its (*Lebesgue*) *integral* as $\int \varphi d\mu := \sum_{i \leq n} a_i \mu(E_i)$.

For a (bounded) positive function $f : X \rightarrow \mathbb{R}_{\geq 0}$, we might define $\int f d\mu$ by approximating f by simple functions from below, say by an increasing sequence (φ_n) of simple functions such that $f = \lim_n \varphi_n$ uniformly. However, not all functions f admit such an approximation.

To see this, let us attempt to construct such a sequence (φ_n) . For each n , we will approximate the cutoff of f at 2^n , i.e., the function $\min(f, 2^n)$. We do so by partitioning the codomain $[0, 2^n]$ into intervals of length 2^{-n} , for a total of $k_n := 2^n / 2^{-n} = 2^{2n}$ intervals. Set $E_k := f^{-1}([2^{-n}k, 2^{-n}(k+1)])$ for each $k \in \{1, \dots, k_n\}$, and let $\varphi_n := \sum_{k \leq k_n} 2^{-n} \chi_{E_k}$. One easily checks that $f = \lim_n \varphi_n$ uniformly.

However, E_k is not guaranteed to be μ -measurable! To fix this, we simply define the issue away.

Definition 2.2. A function $f : X \rightarrow Y$ between measurable spaces (X, \mathcal{B}) and (Y, \mathcal{C}) be measurable spaces is said to be $(\mathcal{B}, \mathcal{C})$ -*measurable* if $f^{-1}(C) \in \mathcal{B}$ for all $C \in \mathcal{C}$.

A function $f : X \rightarrow Y$ between topological spaces is said to be *Borel* if it is $(\mathcal{B}(X), \mathcal{B}(Y))$ -measurable. A *Borel isomorphism* is a bijection $f : X \rightarrow Y$ such that both f and f^{-1} are Borel.

Exercise 2.3. Continuous maps are Borel. HINT: Define a σ -algebra containing open sets in the codomain.

So far we only dealt with measurable spaces. Let us now bring a measure into the picture.

Definition 2.4. Let (X, μ) ⁵ be a measure space and Y be a topological space. A function $f : X \rightarrow Y$ is said to be μ -measurable if it is $(\text{Meas}_\mu, \mathcal{B}(Y))$ -measurable.

Remark 2.5. Compositions of μ -measurable functions *need not* be μ -measurable.

The following exercise is one of the main reasons why μ -measurable functions are introduced, and ultimately also why the Lebesgue integral is superior compared to the Riemann integral.

Exercise 2.6. In separable metric spaces, pointwise limits of μ -measurable functions are μ -measurable, i.e., if (f_n) is a sequence of μ -measurable maps $f_n : X \rightarrow Y$ from a measure space (X, μ) to a separable space Y , and $f := \lim_n f_n$ (pointwise), then $f : X \rightarrow Y$ is μ -measurable.

HINT: Let $\mathcal{C} := \{B \in \mathcal{B}(Y) : f^{-1}(B) \in \text{Meas}_\mu\}$. Show that \mathcal{C} is a σ -algebra containing all open set in Y , so $\mathcal{C} = \mathcal{B}(Y)$, as desired. For each $U \subseteq Y$ open, use separability to write $U = \bigcup_{n \in \mathbb{N}} B_n$, where each B_n is a ball whose closure is contained in U , and show that $f^{-1}(U) \in \text{Meas}_\mu$.

Exercise 2.7. If $f_1, f_2 : (X, \mu) \rightarrow \mathbb{R}$ are μ -measurable and $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is Borel, then $g(f_1, f_2) : X \rightarrow \mathbb{R}$ is also μ -measurable. In particular, $f_1 + f_2$ and $f_1 \cdot f_2$ are μ -measurable.

Exercise 2.8. If (f_n) is a sequence of μ -measurable functions $f_n : X \rightarrow \overline{\mathbb{R}}$, then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, and $\liminf_n f_n$ are also μ -measurable.

Notation 2.9. We write $L(X, \mu)$ for the set of all μ -measurable functions $f : (X, \mu) \rightarrow \overline{\mathbb{R}}$, and $L^+(X, \mu)$ for those which are non-negative.

We are finally ready to define the Lebesgue integral.

Definition 2.10. Let (X, μ) be a measure space. The (Lebesgue) integral of $f \in L^+(X, \mu)$ is

$$\int f \, d\mu := \sup \left\{ \int \varphi \, d\mu : 0 \leq \varphi \leq f \text{ simple function} \right\}.$$

In general, if $f \in L(X, \mu)$, we decompose $f = f^+ - f^-$ where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$. The (Lebesgue) integral of f is $\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu$, provided that one of the terms is finite.

If $\int f \, d\mu < \infty$, we say that f is μ -integrable, in which case we write $f \in L^1(X, \mu)$. More generally,

Definition 2.11. Take $p \in [1, \infty]$ and let $L^p(X, \mu)$ be the set of all μ -measurable functions $f : X \rightarrow \overline{\mathbb{R}}$ such that $\|f\|_p < \infty$, where $\|f\|_p := (\int |f|^p \, d\mu)^{1/p}$ if $p < \infty$ and $\|f\|_\infty := \text{ess-sup } |f| := \inf \{c \geq 0 : |f| \leq c \text{ } \mu\text{-a.e.}\}$.

Exercise 2.12. Let $f, g \in L^p(X, \mu)$. If $f \leq g$, then $\|f\|_p \leq \|g\|_p$.

Since a μ -measurable function $f : X \rightarrow \overline{\mathbb{R}}$ can be approximated from below by simple functions (φ_n) , we should be able to calculate $\int f \, d\mu$ as the limit of $\int \varphi_n \, d\mu$. Indeed,

Theorem 2.13 (Monotone Convergence Theorem). *If $(f_n) \in L^+(X, \mu)$ and $f_n \nearrow f$, then $\int f_n \, d\mu \nearrow \int f \, d\mu$.*

Corollary 2.14. *If $(f_n) \in L^+(X, \mu)$, then $\sum_n \int f_n \, d\mu = \int \sum_n f_n \, d\mu$.*

Exercise 2.15. For any $f, g \in L^1(X, \mu)$ and $a, b \in \mathbb{R}$, we have $\int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu$.

HINT: Simple $\rightsquigarrow_{\text{MCT}} L^+ \rightsquigarrow L^1$.

Exercise 2.16. Let $f, g \in L^1(X, \mu)$. If $f = g$ μ -a.e., then $\int f \, d\mu = \int g \, d\mu$. HINT: Consider $\int (f - g) \, d\mu$.

We list two more convergence theorems that will be useful later on.

Theorem 2.17 (Fatou's Lemma). *If $(f_n) \in L^+(X, \mu)$, then $\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu$.*

Theorem 2.18 (Dominated Convergence Theorem). *Let $(f_n) \in L^1(X, \mu)$. If $f_n \rightarrow f$ μ -a.e. and $|f_n| \leq g$ for some $g \in L^1(X, \mu)$, then $\lim_n \int f_n \, d\mu = \int f \, d\mu$.*

⁵Whenever the σ -algebra is not stated, we assume that μ is defined on Meas_μ . In particular, we assume that μ is complete.

Let us now discuss differentiation of functions $f : X \rightarrow \mathbb{R}$; for convenience, we assume⁶ that $f \in L^+(X, \mu)$. For these functions, we can define a new measure ν on \mathcal{B} by $\nu(B) := \int_B f d\mu := \int f \cdot \chi_B d\mu$, which measures the ‘area under the curve’. Note that for each $B \in \mathcal{B}$, we have B is ν -null whenever B is μ -null.

It turns out that the ‘correct’ setting to discuss differentiation is between two measures μ and ν which satisfy the above condition.

Definition 2.19. If μ, ν are measures on a measurable space (X, \mathcal{B}) and B is ν -null whenever B is μ -null for each $B \in \mathcal{B}$, we say that ν is *absolutely continuous w.r.t μ* , and write $\nu \ll \mu$.

Theorem 2.20 (Lebesgue-Radon-Nikodym Theorem). *If $\nu \ll \mu$ are σ -finite measures on a measurable space (X, \mathcal{B}) , then there exists a \mathcal{B} -measurable map $f : X \rightarrow \mathbb{R}_{\geq 0}$ such that $\nu(B) = \int_B f d\mu$ for all $B \in \mathcal{B}$.*

Such a function $f : X \rightarrow \mathbb{R}_{\geq 0}$ is unique μ -a.e., and is called the *Radon-Nikodym derivative of ν w.r.t. μ* , denoted $\frac{d\nu}{d\mu}$. Thus, we have $\nu(B) = \int_B \frac{d\nu}{d\mu} d\mu$ for all $B \in \mathcal{B}$.

Corollary 2.21. *In the above setting, we have $\int g d\mu = \int g \frac{d\mu}{d\nu} d\nu$ for all $g \in L^1(X, \mu)$.*

To relate $d\nu/d\mu$ to derivatives in calculus (say on \mathbb{R}^n), we let $\mu := \lambda$ be Lebesgue measure on \mathbb{R}^n .

Theorem 2.22 (Lebesgue Differentiation Theorem). *For any locally-integrable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ (i.e. if $f \cdot \chi_K \in L^1(\mathbb{R}^n, \lambda)$ for every compact $K \subseteq \mathbb{R}^n$) and for λ -a.e. $x \in \mathbb{R}^n$, we have*

$$f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\lambda(B_\varepsilon(x))} \int_{B_\varepsilon(x)} f d\lambda.$$

Corollary 2.23. *For any locally-finite Borel measure $\mu \ll \lambda$ on \mathbb{R}^n and for λ -a.e. $x \in \mathbb{R}^n$, we have*

$$\frac{d\mu}{d\lambda}(x) = \lim_{\varepsilon \rightarrow 0} \frac{\mu(B_\varepsilon(x))}{\lambda(B_\varepsilon(x))}.$$

We end by briefly mentioning the ‘Isomorphism Theorems’. These justify why we only gave three examples in Lecture 1, and allows us to work in concrete spaces like $[0, 1]$ or $2^{\mathbb{N}}$.

Definition 2.24. A measurable space (X, \mathcal{B}) is said to be *standard Borel* if \mathcal{B} is the Borel σ -algebra of some Polish (i.e. separable and completely metrizable) topology on X .

A probability space (X, \mathcal{B}, μ) is *standard* if (X, \mathcal{B}) is standard Borel.

Theorem 2.25 (Borel Isomorphism Theorem). *Any two uncountable standard Borel spaces are Borel isomorphic. In particular, they all have cardinality continuum and are Borel isomorphic to $2^{\mathbb{N}}$.*

Definition 2.26. Let (X, \mathcal{B}) and (Y, \mathcal{C}) be measurable spaces. If $f : X \rightarrow Y$ is $(\mathcal{B}, \mathcal{C})$ -measurable and μ is a measure on \mathcal{B} , the *pushforward measure of μ by f* is the measure $f_*\mu$ on \mathcal{C} defined by $f_*\mu(C) := \mu(f^{-1}(C))$.

Definition 2.27. Two measure spaces (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) are said to be a *measure isomorphic* if there is a *measure-preserving transformation* $f : X \rightarrow Y$, i.e., a map $f : X \rightarrow Y$ such that $f_*\mu = \nu$, and such that there is a μ -conull set $X_0 \subseteq X$ and a ν -conull set $Y_0 \subseteq Y$ on which f and f^{-1} restrict to Borel isomorphisms.

Theorem 2.28 (Measure Isomorphism Theorem). *Any two atomless standard probability spaces are measure isomorphic. In particular, they are all measure isomorphic to $([0, 1], \lambda)$.*

3. LECTURE 3 (PENG BO): AN INTRODUCTION TO ERGODIC THEORY

Generally speaking, a *dynamical system* is just a group action. For instance, actions of \mathbb{Z} (resp. \mathbb{R}) on some space X lead to *discrete* (resp. *continuous*) dynamical systems, and different actions on different spaces are studied from different point of views:

1. Continuous actions on topological spaces lead to *topological dynamics*.
2. Measure preserving actions on measure spaces and lead to *measured group theory* and *ergodic theory*.

⁶Otherwise, we will need to discuss ‘signed measures’.

3. Geometric (proper and cocompact) actions on geodesic metric spaces lead to *geometric group theory*.
4. Linear actions on Banach spaces (or TVSs) lead to *functional analysis* and C^* -algebras.

Here, we will be interested in measure preserving actions on measure spaces. To motivate these actions, we will also mention the theory of continuous actions on topological spaces, which was the original motivation for von-Neumann to introduce measure preserving actions in the first place. For simplicity, we will focus on actions of \mathbb{Z} (so it suffices to provide a single generator) on compact (resp. probability) spaces.

Definition 3.1. A *topological dynamical system* is a pair (X, f) , consisting of a compact topological space X and a homeomorphism $f : X \rightarrow X$.

Definition 3.2. A *measure-preserving dynamical system* is a tuple (X, μ, T) consisting of a standard probability space (X, μ) and a measure-preserving transformation $T \in \text{MPT}(X, \mu)$.

Definition 3.3. A topological system (X, f) is *minimal* if there is no proper subsystem, i.e. if there is no f -invariant compact subspace $Y \subseteq X$.

Exercise 3.4. A topological system (X, f) is minimal if every f -orbit in X is dense. HINT: If $\{f^n(x)\}_{n \in \mathbb{Z}}$ is not dense, then its closure is a proper subsystem.

The measure-theoretic analogue of minimal system is an *ergodic* system. As we shall see, these are much more amenable than their topological-counterparts.

Definition 3.5. A measure-preserving system (X, μ, T) is *ergodic* if every T -invariant Borel subset is either μ -null or μ -conull.

Example 3.6. Consider the circle $S^1 := \mathbb{R}/\mathbb{Z}$ and an irrational angle $\theta \in S^1$. Then the (*irrational*) *rotation* $r_\theta : S^1 \rightarrow S^1$ given by $x \mapsto x + \theta$ is both minimal and ergodic.

We will show that r_θ is ergodic in **TODO**. For minimality, it suffices to show that the orbit $\{n\theta\}_{n \in \mathbb{Z}}$ approaches 0. Indeed, there exists some $n \in \mathbb{Z}$ such that $[n\theta, (n+1)\theta]$ contains 0 in the quotient. Without loss of generality, we can assume that $d(0, (n+1)\theta) \leq d(0, n\theta)$, so $\theta_1 := (n+1)\theta \leq \theta/2$ in the quotient and is in the orbit of θ . Replacing θ by θ_1 and repeating furnishes a sequence $\theta_n \rightarrow 0$.

In contrast to the above example, every homeomorphism on S^2 has a fixed point by Brouwer's fixed point theorem. Thus, there is no minimal system on S^2 . More generally, there is no minimal system in S^{2n} .

Open Question 3.7. Characterize all manifolds supporting a minimal system. REMARK: Negative results include the unit interval and S^{2n} . This question is still open, even in dimension three.

Of course, any such characterization is modulo the isomorphism relation.

Definition 3.8. Two topological dynamical systems (X, f) and (Y, g) are *isomorphic* if there exists a homeomorphism $h : X \rightarrow Y$ such that $g = h \circ f \circ h^{-1}$.

The Measure Isomorphism Theorem (Theorem 2.28) asserts that any two atomless standard probability spaces are measure isomorphic. This allows us to reduce the classification of (atomless) measure-preserving dynamical systems to just measure-preserving transformations.

Open Question 3.9. Classify ergodic measure-preserving transformations up to isomorphism.

Here is a celebrated positive result.

Definition 3.10. A *Bernoulli shift* is a measure-preserving dynamical system $(\Sigma^{\mathbb{Z}}, \pi^{\mathbb{Z}}, S)$ where Σ is a finite set, π is a probability measure on Σ , and S is the *left-shift* map $S(x)(n) := x(n+1)$.

The *entropy* of a Bernoulli shift is $-\sum_{i \in \Sigma} \pi(i) \log(\pi(i))$.

Theorem 3.11 (Ornstein). *Two Bernoulli shifts are isomorphic iff their entropies coincide.*

Lastly, we end with a recurrence question: given a μ -measurable subset $U \subseteq X$ and a point $x \in X$, how many points in the orbit of x are in U ?

Theorem 3.12 (Pointwise Ergodic Theorem). *Let (X, μ, T) be an ergodic system and let $U \subseteq X$ be a μ -measurable subset. The sequence $|\{n \leq m : T^n x \in U\}|/m$ converges to $\mu(U)$ as $m \rightarrow \infty$ for μ -a.e. $x \in X$.*

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