SUMMER 2025 READING GROUP ON ERGODIC THEORY

Exercise Sheet 8 (Zhaoshen Zhai): Correspondence principles;
The finite-infinite bridge; Baby steps towards the Structure Theorem.

The first three exercises explore further correspondences between finite Ramsey theory, infinite Ramsey theory, and dynamical systems; you don't need to do all of them. The next few exercises are general lemmas used to study the structure of weak mixing and compact systems.

Throughout, we let $(X, \mathcal{B}, \mu, T)$ be an invertible measure-preserving dynamical system, and recall that a *topological dynamical* system is a pair (X, T) consisting of a compact metrizable space X and a homeomorphism $T : X \to X$. The invertibility conditions are not necessary, but are here for convenience.

Exercise 1. Show that the following are equivalent.

> **Theorem** (Simple recurrence in open covers). *Let $(U_\alpha)_\alpha$ be an open cover of a topological dynamical system (X, T). There exists α such that $U_\alpha \cap T^{-n}U_\alpha \neq \varnothing$ for infinitely many $n \in \mathbb{N}$.*

> **Theorem** (Infinite pigeonhole-principle). *For any $c \geq 1$, any c-colouring of $\mathbb{Z}$ always contains a colour class with infinitely-many elements.*

> **Theorem** (Finite pigeonhole-principle). *For any $c, k \geq 1$, there exists $N(c, k)$ such that if $n \geq N(c, k)$ and we colour $\{1, \dots, n\}$ by c colours, then there is a colour class of at least k elements.*

Exercise 2. Show that the following are equivalent.

> **Theorem** (Multiple recurrence in open covers). *Let $(U_\alpha)_\alpha$ be an open cover of a topological dynamical system (X, T). There exists α such that for each $k \geq 1$, we have $\bigcap_{i<k} T^{-in}U_\alpha \neq \varnothing$ for some $n \in \mathbb{N}$.*

> **Theorem** (Infinitary van der Waerden). *For any $c \geq 1$, any c-colouring of $\mathbb{Z}$ always contains a colour with arbitrarily long arithmetic progressions.*

> **Theorem** (Finitary van der Waerden). *For any $c, k \geq 1$, there exists $N(c, k)$ such that if $n \geq N(c, k)$ and we colour $\{1, \dots, n\}$ with c colours, then there is a monochromatic k-term arithmetic progression.*

Exercise 3. Show that the following are equivalent.

> **Theorem** (Furstenberg's multiple recurrence; v1). *For any $k \geq 1$ and any set $A \in \mathcal{B}$ with positive measure, there exists $n \geq 1$ such that $\mu(\bigcap_{i<k} T^{-in}A) > 0$.*

> **Theorem** (Furstenberg's multiple recurrence; v2). *For any $k \geq 1$ and any set $A \in \mathcal{B}$ with positive measure, we have $\liminf_{N\to\infty} \frac{1}{N}\sum_{n<N} \mu(\bigcap_{i<k} T^{-in}A) > 0$*

> **Theorem** (Furstenberg's multiple recurrence; v3). *For any $k \geq 1$ and any $f \in L^\infty(X, \mu)$ with $f \geq 0$ and $\int f \mathrm{d}\mu > 0$, we have $\liminf_{N\to\infty} \frac{1}{N}\sum_{n<N} \int \prod_{i<k} T^{-in}f \,\mathrm{d}\mu > 0$.*

> **Theorem** (Infinitary Szemerédi). *Any subset of $\mathbb{Z}$ with positive upper density contains arbitrarily long arithmetic progressions.*

> **Theorem** (Finitary Szemerédi). *For any $\delta > 0$ and $k \geq 1$, there exists $N(\delta, k)$ such that if $n \geq N(\delta, k)$, then any subset of $\{1, \dots, n\}$ with at least δn elements contains a k-term arithmetic progression.*

Remark. Your proofs of 'infinitary $\Leftrightarrow$ finitary' probably use some sort of compactness-and-contradiction argument. One can instead use the Compactness Theorem in first-order logic to establish these results.

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Definition. Let (v_n) be a sequence in a Hilbert space H and let $v \in H$.

1. We say that v_n *converges to* v *in density*, and write $\text{D-lim}_n v_n = v$, if $\{n \in \mathbb{N} : \|v_n - v\| > \varepsilon\}$ has zero upper density for each $\varepsilon > 0$.
2. We say that v_n *converges to* v *in the Cesàro sense*, and write $\text{C-lim}_n v_n = v$, if $\lim_N \frac{1}{N}\sum_{n<N} v_n = v$.
3. The *Cesàro supremum* of v_n is $\text{C-sup}\, v_n \coloneqq \limsup_N \left\|\frac{1}{N}\sum_{n<N} v_n\right\|$.

Exercise 4. Let $(v_n)_n$ be a bounded sequence in a Hilbert space H and let $v \in H$.

a) Prove that $\text{C-lim}_n v_n = 0$ iff $\text{C-sup}_n v_n = 0$.
b) Prove that $\lim_n v_n = v$ implies $\text{D-lim}_n v_n = v$, which in turn implies $\text{C-lim}\, v_n = v$.
c) Prove that $\text{C-lim}_n \|v_n - v\| = 0$ iff $\text{D-lim}_n v_n = v$.
d) What happens if (v_n) is unbounded?

Definition. A measure-preserving dynamical system (X, μ, T) is said to be

1. *mixing* if $\lim_n \langle T^{-n} f, g\rangle = \mathbb{E}(f)\mathbb{E}(g)$ for every $f, g \in L^2(X)$.
2. *weak mixing* if $\text{D-lim}_n \langle T^{-n} f, g\rangle = \mathbb{E}(f)\mathbb{E}(g)$ for every $f, g \in L^2(X)$.

Exercise 5. Prove that (X, μ, T) is ergodic iff $\text{C-lim}_n \langle T^{-n} f, g\rangle = \mathbb{E}(f)\mathbb{E}(g)$ for all $f, g \in L^2(X)$.

Exercise 6 (van der Corput Lemma)**.** Let $(v_n)_n$ be a bounded sequence in a Hilbert space H. Prove that if $\text{C-sup}_h \text{C-sup}_n \langle v_n, v_{n+h}\rangle = 0$, then $\text{C-lim}_n v_n = 0$.

Sketch: Normalize (v_n) and prove, via a telescoping estimate and averaging over $\{0, \dots, H-1\}$, that

$$\left\|\frac{1}{N}\sum_{n<N} v_n\right\|^2 \leq O\left(\frac{1}{H^2}\sum_{h,h'<H}\frac{1}{N}\sum_{n<N}\langle v_{n+h}, v_{n+h'}\rangle\right) + O\left(\frac{H^2}{N^2}\right)$$

for any $N, H \geq 1$. Use another telescoping argument to show that

$$\text{C-sup}_n \langle v_{n+h}, v_{n+h'}\rangle = \text{C-sup}_n \langle v_{n+|h-h'|}, v_n\rangle$$

for any $h, h' < H$, and use this to eliminate h'.

Definition. A function $f \in L^2(X)$ is *weak mixing* if $\text{D-lim}_n \langle T^{-n} f, f\rangle = 0$.

Exercise 7. Show that for any weak mixing $f \in L^2(X)$, we have $\text{D-lim}_n \langle T^{-n} f, g\rangle = 0$ for every $g \in L^2(X)$. Deduce that X is weak mixing iff every $f \in L^2(X)$ with mean zero is weak mixing.

Hint: Use Exercises 4 and 6, and Cauchy-Schwarz.

Exercise 8. For any $f \in L^2(X)$, prove that $\{T^n f : n \in \mathbb{Z}\}$ has compact closure in $L^2(X)$ iff for each $\varepsilon > 0$, the set $\{n \in \mathbb{Z} : \|f - T^n f\| < \varepsilon\}$ is syndetic. If any of these hold, we say that f is *almost periodic.*

Exercise 9. Prove that if f is weak mixing and g is almost periodic, then $\langle f, g\rangle = 0$.

Exercise 10. Prove that the set $\mathcal{AP}(X)$ of almost periodic functions in $L^2(X)$ is closed.