NOTES ON ERGODIC THEORY

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Abstract. **TODO**

Contents

1. Lecture 1 (Samy Lahlou): Crash course on Measure Theory, Part I

Given a set X, our goal is to define a map $\mu : \mathcal{P}(X) \to [0, \infty]$ that assigns to each subset $A \subseteq X$ a *measure* $\mu(A) \in [0, \infty]$ that 'behaves like the volume of A'. This turns out to be impossible in full generality (and we shall see using ergodic-theoretic methods that this impossibility is for good reason), so we instead restrict to special subsets of X with a nice algebraic (think: 'constructible') structure.

TODO: add references to Anush's notes and Folland.

Definition 1.1. Let X be a set. A *σ-algebra on X* is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X containing $\varnothing$ and is closed under complements and countable unions. More precisely:

1. $\varnothing \in \mathcal{B}$.
2. For any $A \in \mathcal{B}$, we have $X \setminus A \in \mathcal{B}$.
3. For any countable family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$, we have $\bigcup_n A_n \in \mathcal{B}$.

Definition 1.2. If $\mathcal{B}$ is a σ-algebra on a set X, the pair $(X, \mathcal{B})$ is said to be a *measurable space*.

A useful way to construct a σ-algebra is to start with a collection $\mathcal{C} \subseteq \mathcal{P}(X)$ and close[1] it under the above three conditions. Abstractly:

Definition 1.3. The σ-algebra *generated* by $\mathcal{C} \subseteq \mathcal{P}(X)$ is $\langle \mathcal{C} \rangle_\sigma \coloneqq \bigcap \{\mathcal{B} : \mathcal{B} \supseteq \mathcal{C} \text{ is a } \sigma\text{-algebra on } X\}$.

Note that $\langle \mathcal{C} \rangle_\sigma$ is indeed a σ-algebra on X since the intersection of σ-algebras is again a σ-algebra.

Definition 1.4. Let X be a topological space. The *Borel σ-algebra* on X is the σ-algebra $\mathcal{B}(X) \coloneqq \langle \mathcal{T} \rangle_\sigma$, where $\mathcal{T}$ is the topology on X. The elements of $\mathcal{B}(X)$ are called the *Borel sets* of X.

Intuitively, for any topological space X, one would like to measure open sets, closed sets, countable unions of closed sets (called *F_σ sets*), countable intersections of open sets (called *G_δ sets*), countable intersections of F_σ sets, countable unions of G_δ sets, and so on[2].

Definition 1.5. A *measure* on a measurable space $(X, \mathcal{B})$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and $\mu(\bigsqcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$ for any pairwise disjoint family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$.

The triple $(X, \mathcal{B}, \mu)$ is then called a *measure space*.

Date: May 6, 2025.

Notes for the summer 2025 reading group on Ergodic Theory, organized by Frédéric Kai, Ludovic Rivet, and Zhaoshen Zhai (website).

[1] This 'closure' operation can be made precise as follows. Starting with $\mathcal{C}_0 \coloneqq \mathcal{C}$, throw in all the subsets of X that is necessary to satisfy Definition 1.1 relativized to $\mathcal{C}_0$ to obtain $\mathcal{C}_1$ (that is, let $\mathcal{C}_1$ contain $\varnothing$ and such that if $A \in \mathcal{C}_0$, then $X \setminus A \in \mathcal{C}_1$, and similarly for condition 3). Then, let $\mathcal{C}_2$ be defined by throwing in all the countable unions and complements of sets in $\mathcal{C}_1$. Doing so infinitely-many times and taking the union $\bigcup_\alpha \mathcal{C}_\alpha$ will give us $\langle \mathcal{C} \rangle_\sigma$, but *beware* that this process must proceed into the transfinite up to $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal; ask your local set theorist why.

[2] This goes up the *Borel hierarchy*.

Definition 1.6. A measure μ on $(X,\mathcal{B})$ is said to be *finite* if $\mu(X)<\infty$, a *probability measure* if $\mu(X)=1$, and σ*-finite* if there is a partition $X=\bigsqcup_{n\in\mathbb{N}}X_n$ such that $X_n\in\mathcal{B}$ and $\mu(X_n)<\infty$ for all $n\in\mathbb{N}$.

Example 1.7. There is[3] a unique measure $\lambda:\mathcal{B}(\mathbb{R})\to[0,\infty]$ on $\mathbb{R}$ such that $\lambda([a,b])=b-a$ for each $a\leq b$, called the *Lebesgue measure*. This measure will be the default measure on $\mathbb{R}$ unless stated otherwise.

Exercise 1.8. For each $p\in[0,1]$, is a unique measure $\mu_p:\mathcal{B}(2^{\mathbb{N}})\to[0,\infty]$ on $2^{\mathbb{N}}$ such that for each word $w\in 2^{<\mathbb{N}}$, we have $\mu_p([w])=p^{n_1}(1-p)^{n_0}$, where n_i is the number of $i\in\{0,1\}$ in w and $[w]$ is the set of all sequences in $2^{\mathbb{N}}$ containing w as a prefix, called the *Bernoulli*(p) *measure*.

TODO: add intuition about densities; ask Samy

Definition 1.9. Let $(X,\mathcal{B},\mu)$ be a measure space. A subset $Z\subseteq X$ is said to be μ*-null* if there exists some $Z'\in\mathcal{B}$ such that $Z\subseteq Z'$ and $\mu(Z')=0$. We write Null_μ for the set of all μ-null subsets of X.

Definition 1.10. Let $(X,\mathcal{B},\mu)$ be a measure space. A subset $A\subseteq X$ is μ*-measurable*[4] if there exists some $B\in\mathcal{B}$ and some μ-null set Z such that $A=B\cup Z$. We write Meas_μ for the set of all μ-measurable sets.

Fact 1.11. For any measure space $(X,\mathcal{B},\mu)$, we have $\mathrm{Meas}_\mu=\langle\mathcal{B}\cup\mathrm{Null}_\mu\rangle_\sigma$. Moreover, μ admits a unique extension to a map $\overline{\mu}:\mathrm{Meas}_\mu\to[0,\infty]$, called the *completion* of μ, and this measure satisfies $\mathrm{Meas}_{\overline{\mu}}=\mathrm{Meas}_\mu$.

We end with some exercises. Throughout, let $(X,\mathcal{B},\mu)$ be a measure space.

Exercise 1.12 (Monotonicity)**.** For any $A,B\in\mathcal{B}$ with $A\subseteq B$, we have $\mu(A)\leq\mu(B)$. Deduce that if μ is finite, then μ is a bounded function.

Exercise 1.13 (Inclusion-exclusion)**.** For any $A,B\in\mathcal{B}$, we have $\mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B)$.

Exercise 1.14 (Continuity $\nearrow$)**.** If $(A_n)_{n\in\mathbb{N}}$ is increasing, then $\mu(\bigcup_{n\in\mathbb{N}}A_n)=\lim_{n\to\infty}\mu(A_n)$.

Exercise 1.15 (Continuity $\searrow$)**.** If $(A_n)_{n\in\mathbb{N}}$ is decreasing and $\mu(A_1)<\infty$, then $\mu(\bigcap_{n\in\mathbb{N}}A_n)=\lim_{n\to\infty}\mu(A_n)$.

References

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[3] We will not prove this fact, but it is an application of Carathéodory's Extension Theorem.

[4] Very confusing terminology. One might think that elements of $\mathcal{B}$ are the 'measurable' ones, but this removes μ from the picture. In general, there are much more μ-measurable sets that there are sets in $\mathcal{B}$. Indeed, **TODO**