Even Unimodular Lattices in dimension 8, 16 and 24

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Contents

1 Lattices, Root Systems and Coxeter Numbers

The goal of this project is to classify even unimodular lattices in dimension 8, 16 and 24. Theorem 2.2 shows that even unimodular lattices can only exist in dimension divisible by 8; hence our interest for these dimensions. We will find there is exactly 1 in dimension 8, 2 in dimension 16, and 24 in dimensions 24. Interestingly, the number explodes if we continue: there are over 80,000,000 in dimension 32 (A consequence of the Smith-Minkowski-Siegel mass formula).
In doing so, we will encounter many fundamental objects in a variety of subjects. We will cross paths with Coxeter numbers and Modular forms, and see how they play a role in our context. We will also get quite familiar with root systems, and how they relate to lattices. Two cases will stand out: the E_8 root system, and the *Leech Lattice* as it is not generated by a root system.

1.1 What is a Lattice

Definition 1.1. A **lattice** is the $\mathbb{Z}$-span of vectors in $\mathbb{R}^n$. That is, a lattice is isomorphic to $L = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \cdots \oplus \mathbb{Z}v_n$ for $v_i \in \mathbb{R}^n$ a collection of linearly independent vectors.

We almost always think of lattices geometrically: an infinite set of point arranged in a given patern. See Figure 1 for examples. A standard way to concisely describe a lattice is to arrange the

basis vectors in a matrix C ($n \times n$). The *Gram Matrix* $A = CC^t$ captures information about the lattice. Another way to write it is $a_{ij} = (e_i, e_j)$ for e_i the basis of the lattice.
Notice that a lattice could also be looked at as a collection of parallelotopes (n-dimensional parallelograms). Their volume is computed as $\mathrm{Vol}(\Lambda) = |\det(C)| = \sqrt{\det(A)}$.

Since lattices are subsets of $\mathbb{R}^n$, we can steal some structure from linear algebra. For example, the usual dot product between vectors gives an inner product on Λ:

$$(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Call $\sqrt{(x,x)}$ the length of the vector x; we will always work with the squared length (x, x). Write $\Lambda(n) = \{x \in \Lambda : (x, x) = n\}$. Next, we can define the dual of a lattice.

Definition 1.2. Define the dual lattice $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) = \{x \in \mathbb{R}^n : \forall y \in \Lambda, \quad (x, y) \in \mathbb{Z}\}$. An exercise is to show the Gram matrix of Λ^* is A_Λ^{-1}.

Definition 1.3. The direct sum of lattices is the set $\Lambda_1 \oplus \Lambda_2 \subset \mathbb{R}^n \oplus \mathbb{R}^m$ generated by the bases of Λ_1 and Λ_2 and such that $(v, w) = 0$ for $v \in \Lambda_1$ and $w \in \Lambda_2$.

We can now define our two special kind of lattices. The first condition gives something about length, the second about volume. Together, they turn out to be very restrictive, and understanding their wedding will be the goal of this paper.

- A lattice is *even* if every vector has even squared length: $(x, x) \in 2\mathbb{Z}$.
- A lattice is *unimodular* if $\Lambda = \Lambda^*$, which is equivalent to $\mathrm{Vol}(\Lambda) = 1$.

For example, $2\mathbb{Z} \subset \mathbb{R}$ is an even lattice, but not unimodular ($\mathrm{Vol}(2\mathbb{Z}) = \det(2) = 2$). $\mathbb{Z}^2 \subset \mathbb{R}^2$ is unimodular, but not even (for $x = (1, 0)$, $(x, x) = 1 \cdot 1 + 0 \cdot 0 = 1 \notin 2\mathbb{Z}$). The reader might feel already the difficulty for a lattice to be both even and unimodular. It turns out to be impossible in dimension 2 or 3, so visualization is limited (see Theorem2.2).

1.2 Root Systems

We will first define root systems abstractly, and then give a way to produce them concretely. These objects allow for simple description of otherwise complex structures. An example is in the context of Lie Theory, where Root Systems allow the classification of a special kind of Lie Group. The theory revolves around a myriad of definitions and facts, which we shall not prove for lack of space. For a summary of the theory, see the set of notes by P. Etingoff [Eti20].

Definition 1.4. A *root system* $R = (E, \Phi)$ is a finite collection of nonzero vectors (called roots) $\Phi \in \mathbb{R}^n \cong E$ such that:

- R spans $\mathbb{R}^n$
- For $\alpha \in R$ and $r \in \mathbb{R}$, $r\alpha \in R \iff r \in \{\pm 1\}$
- For $\alpha, \beta \in R$, the number $\alpha^\vee(\beta) = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ in an integer
- If $\alpha, \beta \in R$, then $s_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in R$.

We think of $\alpha^\vee$ as an element of the dual space E^* and call it a coroot. We think of the roots as vectors in E, and it is instructive to convince oneself that s_α is a symetry of E along $\alpha^\perp$, the axis orthogonal to α. We call the rank of a root system the dimension of the space it spans: $\mathrm{rk}(R) = \dim(E)$.
Some vectors stand out in root systems. To be more precise, we can choose a root v, and test every the other roots x against it by looking at the sign of (v, x). Those who test to a positive number form the set of positive roots Φ^+. From these positive roots, let Π be the set of those which can't be written as the sum of two positive roots. Π is called the set of simple roots. Note R was a spanning set; we trimmed many vectors out of it and are left with the following:

Proposition 1.1. *Π is a basis for $\mathbb{R}^n$.*

Root systems can be added together: define $R_1 + R_2 = (E_1 \oplus E_2, \Phi_1 \cup \Phi_2)$. A root system that cannot be written as the sum of two others is called *irreducible*. The classification of irreducible root systems is a fascinating subject, for which we again refer the reader to [Eti20]. For definitions, see Table 1. We collect the following results:

Theorem 1.1. *The rank 2 root systems are $A_1 \times A_1$, A_2, B_2, G_2.*

Theorem 1.2. *The irreducible root systems come in two kinds:*

- *infinite families: A_n, B_n, C_n, D_n*
- *exceptional ones: F_4, G_2, E_6, E_7, E_8*

Of course, the first theorem is a special case of the second. In practice, we are often interested by how two roots in a bigger root system relate to each other. These two roots generate a rank 2 root system, and their relative position is classified by the first theorem.

To conclude this section, we relate root systems to lattices. We can sometimes collect a root system from a lattice.

Proposition 1.2. *For Λ an even lattice with $\mathbb{Z}\Lambda(2) = \Lambda$, $\Lambda(2)$ is a root system.*

Proof. We verify each axiom in the definition of an abstract room system. Throughout, α, β are roots.
First, recall the $\mathbb{R}$ span of a lattice is the whole vector space. So $\mathbb{R}\Lambda(2) = \mathbb{R}^n$ using the assumption.
If $r\alpha$ is a root, then $(r\alpha, r\alpha) = r^2(\alpha, \alpha) = 2r$ implies $r \in \pm 1$.
Since α is a root, $(\alpha, \alpha) = 2$, so $\alpha^\vee(\beta) = (\alpha, \beta)$ is an integer.
Finally, $(s_\alpha(\beta), s_\alpha(\beta)) = (\beta, \beta) - 2(\beta, (\alpha, \beta)\alpha) + ((\alpha, \beta)\alpha, (\alpha, \beta)\alpha) = 2 - 2(\alpha, \beta)^2 + 2(\alpha, \beta)^2 = 2$.
This concludes the proof. $\square$

In this situation, call Λ a root lattice. We will see with Proposition 3.1 that in dimension $8, 16, 24$, either $\Lambda(2)$ is empty or it has rank 24. Notice that all roots in the root system of an even root lattice have same length. That is, we look only at those root systems where $(\alpha, \alpha) = 2$ for all roots. This is *only* the case for root systems that are linear combinations of those of type A, D and E, which we call *of type ADE* (see Table 1). That is, root systems of the form $\sum_i \alpha_i A_i + \sum_j \beta_j D_j + \sum_{k \in \{6,7,8\}} \gamma_k E_k$ for $\alpha_i, \beta_j, \gamma_k$ the number of copies of each irreducible root lattices.

Theorem 1.3. *Let $R = (E, \Phi)$ a root system of type ADE. Then, the $\mathbb{Z}$-span of Φ is a lattice. Moreover, the lattice is even.*

Proof. One can easily prove from their respective definition that it is the case for each irreducible systems of type A_n, D_n and E_k. The combination of root systems $R_1 + R_2$ yields the direct sum of lattices $\Lambda_{R_1} \oplus \Lambda_{R_2}$, so the result follows for every root system of type ADE. □

Note that the lattice is not necessarily uniquely determined by the root system $\Lambda(2)$. Showing that $\Lambda(2)$ is a given root system R does not ensure that Λ itself is the lattice generated by R; only that it contains it.

1.3 Coxeter Number

In this section, we associate to a root system a number, called the Coxeter number. Note that this theory exists in a larger context, where we look at Coxeter groups. The goal of this section will be for the most part to help prove Proposition 3.2. We start with two definitions:

Definition 1.5. Let $R = (E, \Phi)$, and let $\Phi = \{\alpha_1, \cdots, \alpha_n\}$. Then, the group generated by the s_α is called the Weyl group of R, $W(R) =< s_{\alpha_i} : \alpha_i \in \Phi >$

Definition 1.6. The Coxeter number of R, denoted by h is defined to be the order of the element $s_{\alpha_1} s_{\alpha-2} \cdots s_{\alpha_n}$ in $W(R)$. It does not depend on the ordering of the α_i.

In Table 1, we include the Coxeter number of each. It is calculated by the formula $h = \frac{|\Phi|}{\text{rk}(R)}$. For a proof of this formula, see Theorem 1 in Chapter 6 of [Bou68].
Write $R_\alpha = \{\beta \in R : (\alpha, \beta) \neq 0\}$ the set of roots not orthogonal to α. For the next lemma, we use that $|R_\alpha| = hl$, which is proven in the same theorem in [Bou68].

Lemma 1.1. *Let R an irreducible root system of type AED. Then, $\sum_{\beta \in R_\alpha} (\alpha, \beta)^2 = h(\alpha, \alpha)^2$.*

Proof. Let $f(\alpha) = \sum_{\beta \in R_\alpha} (\alpha, \beta)^2$. One can show f is a positive W invariant quadratic form. One can show there is $\lambda \in \mathbb{R}$ so that $f(\alpha) = \lambda(\alpha, \alpha)$ (see [Kam24]).
Let $\{e_i\}_0^n$ an orthonormal basis for E (the space spanned by the roots). Note $f(e_i) = 1$. Then,

$$\sum_i f(e_i) = \lambda n = \sum_i \sum_{\beta \in R_\alpha} (e_i, \beta)^2 = \sum_\beta 1 = |R_\alpha| = hl \qquad \Longrightarrow \qquad \lambda = h$$

□

Proposition 1.3. *Let R an irreducible root system of type AED, and. Then, $|R_\alpha| = 4h - 6$.*

Proof. Using the previous proposition, $\sum_{\beta \in R_\alpha} (\alpha, \beta)^2 = h(\alpha, \alpha)^2$. Consider the root system generated by α and β: it has rank 2. Since both roots have same length and are not colinear, the classification of rank 2 systems gives $(\alpha, \beta) = \pm\frac{1}{2}(\alpha, \alpha)$. Hence, $\sum_{\beta \in R_\alpha} (\alpha, \beta)^2 = 2(\alpha, \alpha)^2 + (|R_\alpha| - 2)\left(\frac{1}{4}(\alpha, \alpha)^2\right) = \frac{1}{4}(|R_\alpha| + 6)(\alpha, \alpha)^2$. Combining, $|R_\alpha| = 4h - 6$. □

2 Even Unimodular Lattices and Theta functions

2.1 Modular forms

We begin by briefly mentioning facts related to modular forms that shall be of use later. Modular forms are complex functions satisfying a sort of invariance under the modular group $SL_2(\mathbb{Z})$. We denote the upper half plane by $\mathcal{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. Define the following action of the modular group on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

Definition 2.1. A funtion $f : \mathcal{H} \to \mathbb{C}$ is modular of weight k if

- f is holomorphic on $\mathcal{H} \cup \{\infty\}$
- for any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, f satisfies $f(\gamma z) = (cz+d)^k f(z)$

Modular forms are fundamental objects in complex analysis and number theory. Their appearance in our context will be justified in section 3. We write M_k the set of modular forms of weight k. We briefly note that the product of modular forms of weights k and l is a modular form of weight $k+l$, so $M = \bigoplus_{k\in\mathbb{Z}} M_k$ forms a graded ring. For a proper introduction to modular forms, we refer the reader to [FD05].
We define two modular forms explicitly, the so called (normalized) Eisenstein series. The following theorem will show they are the only ones we need.

Definition 2.2. The functions E_4 and E_6 are modular forms:

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} \qquad E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}$$

Theorem 2.1. *$M_4 = \mathbb{C}E_4$. More generally, $M = \mathbb{C}[E_4, E_6]$*

2.2 The Theta Series

Definition 2.3. Let $q = e^{2\pi i\tau}$. Given a lattice Λ, we define it's theta function to be

$$\Theta_\Lambda(\tau) = \sum_{x\in\Lambda}^{\infty} q^{\frac{1}{2}(x,x)}$$

Thus, the theta function is the series in q where the nth coefficient is the number of vectors of squared length n. In particular, we will be interested in vectors of squared length 2: their number is the coefficient of q. Since we look at unimodular lattices, Θ_Λ is actually a modular form of weight $rk(\Lambda)/2$. We will prove this later.

Proposition 2.1. *Let Λ a lattice or rank n. Then,*

$$\Theta_\Lambda(-\frac{1}{z}) = \left(\frac{z}{i}\right)^{n/2} \det(\Lambda)^{-1/2} \Theta_{\Lambda^*}(z)$$

Proof. The proof is a simple computation using the Poisson summation formula, which we will assume: Letting $\hat{f}$ be the Fourier transform of f, then $\det(L)^{1/2} \sum_{x\in\Lambda} f(x) = \sum_{y\in\Lambda^*} \hat{f}(y)$
We only show equality for pure imaginary numbers it ($t > 0$), since both sides are holomorphic.

$$\Theta_\Lambda\left(-\frac{1}{it}\right) = \sum_{x\in\Lambda} \exp\left(-\frac{\pi}{t}(x,x)\right) = \det(\Lambda)^{-1/2} \sum_{y\in\Lambda^*} \widehat{\exp\left(-\frac{\pi}{t}(y,y)\right)}$$

$$= \det(\Lambda)^{-1/2} \sum_{y\in\Lambda^*} \sqrt{t}^n \exp\left(-\pi t(y,y)\right) = t^{n/2} \det(\Lambda)^{-1/2} \Theta_{\Lambda^*}(it)$$

□

We will deduce two things from this formula. First we will show even unimodular lattices only exist in certain dimensions. Then, we will show that the theta series for these lattices are actually modular forms.

Theorem 2.2. *Let Λ an even unimodular lattice or rank n. Then, n is a multiple of 8.*

Proof. Let, towards a contradiction, Λ be an even unimodular lattice of rank not congruent to 0 modulo 8. Then, either Λ, $\Lambda \oplus \Lambda$ or $\Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$ has rank congruent to 4 modulo 8. So we can assume $rk(\Lambda) \equiv 4 \pmod 8$.
Let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. These matrices generate the modular group. Note Θ_Λ is T invariant, since $e^{i\pi(x,x)} = 1$ by $(x,x) \in 2\mathbb{Z}$. On the other hand,

$$\Theta_\Lambda(Sz) = \Theta_\Lambda(-\frac{1}{z}) = \left(\frac{z}{i}\right)^{n/2} \det(\Lambda)^{-1/2}\Theta_{\Lambda^*}(z) = -z^{n/2}\Theta_\Lambda(z)$$

since Λ is unimodular. Also, $\left(\frac{1}{i}\right)^{n/2} = -1$, since $\frac{n}{2} \equiv 2 \pmod 4$.
Now, notice $(TS)z = -\frac{1}{z} + 1$, $(TS)^2 z = -\frac{1}{z-1}$ and $(TS)^3 z = z$. Compare $\Theta_\Lambda(z)$ and $\Theta_\Lambda((TS)^3 z)$

$$\begin{aligned}
\Theta_\Lambda(z) &= \Theta_\Lambda((TS)^3 z)) = \Theta_\Lambda((TS(TS(TSz)))) \\
&= \left(-(TS^2 z)^{n/2}\right)\left(-(TSz)^{n/2}\right)\left(-(z)^{n/2}\right)\Theta_\Lambda(z) \\
&= -\left(\frac{1}{z-1}\right)^{n/2} - \left(\frac{z-1}{z}\right)^{n/2}\left(-z^{n/2}\right)\Theta_\Lambda(z) \\
&= -\Theta_\Lambda(z)
\end{aligned}$$

We have reached a contradiction. Hence there exists no even unimodular lattice of rank $\not\equiv 0 \pmod 8$ $\square$

Note that in our proof, we showed $\Theta_\Lambda(z+1) = \Theta_\Lambda(z)$, and $\Theta_\Lambda(-\frac{1}{z}) = (-z+0)^{n/2}\Theta_\Lambda(z)$. Since S and T generate $SL_2(\mathbb{Z})$, we collect the following result.

Theorem 2.3. *Let Λ an even unimodular lattice or rank n. Then, Θ_Λ is a modular form of weight $n/2$.*

Missing is the fact that theta functions are holomorphic, which we will not prove.

3 Even Unimodular Lattices of Rank 8, 16, and 24

In this section, we classify even unimodular lattices of rank 8, 16, and 24. Doing so will require everything we have discussed before. The concepts come together in a perhaps surprising and very elegant way. The cases for 8 and 16 are quite similar, and do not require a lot of work given what we already have. The case for 24 is more involved.
The results were proved in turn by Mordell in 1938 ([Mor38]), Witt in 1941 ([Wit41]), and Niemeier in 1973 ([Nie73]). The 24 even unimodular lattices of rank 24 are known as the Niemeier lattices, in honor of his work. It is interesting to note that Mordell and Witt did not do their works in the context of lattices. A fact which we did not mention is that lattices are in bijection with quadratic forms. (see 2.2 in [JHC99])

Mordell showed "there are two quadratic forms of determinant unity in 8 variables". The "determinant unity" condition corresponds precisely that of unimodularity, and only one of the two forms Mordell gets corresponds to an even Lattice. Witt worked in the context of modular forms. The reason his work translates to the world of lattices will be clear after the proof of theorems 3.1 and 3.2. The proofs we follow can be found in chapter 2.3 of [GC19] or chapter 3 of [Ebe13].
Niemeier's work does revolve around Lattices. However, his argument is rather long and tedious. We follow instead a paper by Venkov, which is chapter 18 of [JHC99].

We start by a lemma, the proof of which relies on machinery we have not defined. The missing parts however are mostly simple definitions, so the reader should not feel cheated. We will give a proof assuming prior acquaintance with cusp forms and harmonic functions. For a complete proof written in our context, see section 3.2 in the text [Ebe13].

Lemma 3.1. *Let Λ be an even unimodular lattice of rank $n =$ 8, 16 or 24. Then, for all $y \in \mathbb{R}^n$,*

$$\sum_{\alpha \in \Lambda(2)} (y, \alpha)^2 = \frac{2}{n}(y, y)|\Lambda(2)|$$

Proof. Define $f(x) = (x, y)^2 - \frac{(x,x)(y,y)}{n}$. Then, f is harmonic so $F(\tau) = \sum_{x \in \Lambda} f(x) q^{\frac{x^2}{2}}$ is a cusp form in τ for $q = e^{2\pi i \tau}$. It is of weight $n/2 + 2$ which is either 6, 10, or 14. There are, however, no nonzero cusp forms of these weights weight, so every coefficient c_r in $F(\tau) = \sum_r c_r q^r$ is 0.
If the reader is ready to assume these facts, of which only the nonexistence of cusps of weight 14 is nontrivial after definitions, the proof is almost complete. Indeed, $c_r = \sum_{x \in \Lambda, x^2 = 2r} f(x) = \sum_{x \in \Lambda(2)} (x, y)^2 - \frac{2}{n}(y, y)|\Lambda(2)| = 0$ is now 0 for all values of y. This concludes the proof. ☐

We can now show a surprising property for the lattices that interest us. In the language of roots, we show that $\Lambda(2)$ is a root system iff $\Lambda(2) \neq \emptyset$.

Proposition 3.1. *For Λ an even unimodular lattice of rank 8, 16 or 24, either $\Lambda(2) = \emptyset$, or $\Lambda(2)$ spans $\mathbb{R}^n$.*

Proof. Suppose that $\Lambda(2)$ to does not span $\mathbb{R}^n$. We can then choose a nonzero $y \in \mathbb{R}^n$ orthogonal to all vectors in $\Lambda(2)$. Then, the LHS of the equality in the lemma becomes 0. Since y is nonzero, $(y, y) \neq 0$, so the equality forces $|\Lambda(2)| = 0$. Hence if $\Lambda(2)$ does not span $\mathbb{R}^n$, $\Lambda(2) = \emptyset$. ☐

There is room for confusion after this proposition. The statement is *not* that if $\Lambda(2) \neq \emptyset$, then Λ is the root lattice with root system $\Lambda(2)$. Indeed, $\Lambda(2)$ is a root system, but the root lattice it generates might be a sublattice of Λ. In fact, we will see during the classification that showing what root lattice Λ might be is easy, but showing that the root lattice *is* actually Λ is more challenging.

Proposition 3.2. *All irreducible components of $\Lambda(2)$ have same Coxeter number.*

Proof. Let $\beta \in \Lambda(2)$, $n = \mathrm{rk}(\lambda)$. The lemma applied to β gives $\sum_{\alpha \in \Lambda(2)} (\beta, \alpha)^2 = \frac{4}{n}|\Lambda(2)|$. Recall roots in different irreducible components are orthogonal. Recall also that for R_α the set of roots such that $(\alpha, \alpha') = 1$, we have $|R_\alpha| = 4h(R) - 6$ (see Proposition 1.3). The number of roots such that $(\alpha, \alpha') = 1$ is thus $2h(R) - 4$. Combining, $\sum_{\alpha \in \Lambda(2)} (\beta, \alpha)^2 = 2 \cdot 2^2 + 2(2h(R) - 4)$. Combining again, $\frac{4}{n}|\Lambda(2)| = 8 + 2(2h(R) - 4)$, so $|\Lambda(2)| = nh(R)$. In particular, h does not depend on the choice of R. ☐

3.1 rank 8 and 16

We begin by classifying even unimodular lattices of rank 8 and 16. The arguments are very similar, and much simpler than that of the rank 24 case. Roughly, given a lattice, we determine explicitly it's theta series and use it to find what root system $\Lambda(2)$ must be. It will remain to show that the root lattice generated by this root system is in fact the whole Λ. This second part is more challenging, and we shall not pay too close attention to it. More will be said in 3.2.2.

Theorem 3.1 (Mordell)**.** *Let Λ be even unimodular of rank 8. Then, $\Lambda \cong E_8$.*

Proof. By Theorem 2.4, the Theta series of Λ is a modular form of weight $8/2 = 4$, and by Theorem 2.1, $\Theta_\Lambda \in \mathbb{C}E_4$. Since there is a single vector or norm 0, $\Theta_\Lambda(q) = 1 + q(\cdots)$. Since the constant factor of E_4 is 1, it follows that $\Theta_\Lambda = E_4$.
Hence, $\Theta_\Lambda(q) = 1 + 240q + \cdots$. It follows that $|\Lambda(2)| = 240$, so the Λ is a root lattice whose root system has 240 elements. This system could be reducible, but the only possible choice for a root system of rank 8 with 240 elements is E_8. Note the lattice $\mathbb{Z}E_8$ formed by the $\mathbb{Z}$ span of the roots in E_8 is unimodular (check on a set of simple roots). Hence, Λ is isomorphic to the E_8 lattice. □

Theorem 3.2 (Witt)**.** *Let Λ be even unimodular of rank 16. Then, Λ is either D_{16} or $E_8 \oplus E_8$.*

Proof. By Theorem 2.4, the Theta series of Λ is a modular form of weight $16/2 = 8$, and by Theorem 2.1, $\Theta_\Lambda \in \mathbb{C}[E_4, E_6]$. The only modular forms of weight 8 are thus scalar multiples of E_4^2 (since the weights add as we multiple forms), and $\Theta_\Lambda = E_4^2$ since it has constant factor 1.
$\Theta_\Lambda(q) = E_4^2(q) = (1 + 240q + \cdots)^2 = 1 + 480q + \cdots$, so $|\Lambda(2)| = 480$. We are thus looking at a (possibly reducible) root system of rank 16 with 480 elements. There are 2 possibilities: $E_8 + E_8$, and D_{16}.
In the $E_8 + E_8$ case, the same argument as in the previous theorem gives that $\Lambda = E_8 \oplus E_8$. In the D_{16} case, we have that the D_{16} root lattice embeds in Λ. Showing equality requires more tool: we shall simply assume it. □

3.2 rank 24

We now classify the possible even unimodular lattices of rank 24. There are 24, called the Niemeier lattices, each corresponding to one of the following root systems (the set of which we call $\mathfrak{N}$):

(i) $\emptyset$

(ii) $24A_1, 12A_2, 8A_3, 6A_4, 4A_6, 3A_8, 2A_{12}, A_{24}$

(iii) $6D_4, 4D_6, 3D_8, 2D_{12}, D_{24}$

(iv) $4E_6, 3E_8$

(v) $4A_5 + D_4, 2A_7 + 2D_5, 2A_9 + D_6, A_{15} + D_9, E_8 + D_{16}, 2E_7 + D_{10}, E_7 + A_{17}$

(vi) $E_6 + D_7 + A_{11}$

First, we will show why an even unimodular lattices of rank 24 must have $\Lambda(2)$ be one of these root systems. We will then show lattices with the following root systems exist are unique. The careful reader will have noticed that we include $\emptyset$ in our list. This is not a mistake: there is indeed a lattice with no vectors of length 2. This will turn out to be the Leech lattice, which we shall treat separately in the last section. It is surprising that every "possible" root system in the sense of the next section is actually realized as a unique lattice.

3.2.1 Possible Root systems

We consider Λ an even unimodular lattice that has a vector of length 2. By the last proposition, Λ is a root lattice, and we show in this section that it's root system $\Lambda(2)$ is a member of $\mathfrak{N}\backslash\{\emptyset\}$, listed above. It will remain to show that each system in $\mathfrak{N}$ generates one and only one even unimodular lattice of rank 24.

Theorem 3.3. *Suppose Λ is a root lattice of rank 24, and that every irreducible component of $\Lambda(2)$ has the same Coxeter number h. Then, $\Lambda(2) \in \mathfrak{N}\backslash\{\emptyset\}$.*

Proof. Since $\Lambda(2)$ is the root system of a lattice, it is of type ADE. Hence, we have to determine the possible numbers $\alpha_i, \beta_j, \gamma_k$ in

$$\Lambda(2) = \sum_i \alpha_i A_i + \sum_j \beta_j D_j + \sum_{k\in\{6,7,8\}} \gamma_k E_k$$

First, recall $\mathrm{rk}(A_i) = i$, $\mathrm{rk}(D_j) = j$, $\mathrm{rk}(E_k) = k$. Thus, $\mathrm{rk}(\Lambda(2)) = 24 = \sum_i \alpha_i i + \sum_j \beta_j j + \sum_{k\in\{6,7,8\}} \gamma_k k$.
Also, the Coxeter number of each irreducible root system must be equal to h. Thus, we cannot have different root systems of the same types, since they would not have the same Coxeter number. We can now remove every $\sum$ in the form of $\Lambda(2)$. Suppose that $\Lambda(2) = \alpha_i A_i$ for some i. Then, $\mathrm{rk}(\Lambda(2)) = 24$ forces $\alpha_i i = 24$, so $\Lambda(2)$ is either $24A_1, 12A_2, 8A_3, 6A_4, 4A_6, 3A_8, 2A_12$ or A_24. Similarly for type D and E, we conclude that if $\Lambda(2)$ has no irreducible components of different type, then $\Lambda(2)$ must be one of the systems listed in (ii),(iii),(iv).
Next, we look more closely at the condition on the Coxeter number. Suppose $\Lambda(2) = \alpha_i A_i + \beta_j D_j$. Then, we have $i + 1 = h(A_i) = h(D_j) = 2j - 2$, so $i = 2j - 3$. The rank condition determines α_i and β_j. Solving the system, and repeating the process for all pairs of types shows $\Lambda(2)$ must be in those listed as (v).
Finally, suppose $\Lambda(2) = \alpha_i A_i + \beta_j D_j + \gamma_k E_k$. Then, we have to solve simultaneously $\alpha_i i + \beta_j j + \gamma_k k = 24$ and $i + 1 = 2j - 2 = h(E_k)$. Doing so results in a single solution: $\Lambda(2) = A_{11} + D_7 + E_6$. We have exhausted all possible forms in $\mathfrak{N}$, so the proof is done. □

3.2.2 Every root system yields a unique lattice

We have shown that an even unimodular lattice is either such that $\Lambda(2) = \emptyset$, or it is a root lattice with $\Lambda(2) \in \mathfrak{N}$. We have two more things to prove. First, that each root system in $\mathfrak{N}$ generate a unique lattice. Second, we have to show that $\Lambda(2) = \emptyset$ completely determines the lattice. The former will be done in the next section, the latter now.

A complete proof is given in Venkov's paper (Chapter 18 of [JHC99]). It uses the theory of codes, which we do have the space to develop for the scope of this project. We can however give an idea of how it works. The proof hinges on the following proposition from code theory

Proposition 3.3. *Suppose R a root system of rank n. Even Unimodular lattices of rank n whose root system is R are in bijection with the orbits of even self-dual codes over R.*

Hence, given a root system, showing there is one and only one lattice with $\Lambda(2) = R$ is equivalent to showing existence and uniqueness of the code. Venkov gives such a code for every root system in $\mathfrak{N}$, along with an argument for uniqueness.

3.2.3 The Leech Lattice

Up to now, we have a complete classification of the (even unimodular) root lattices of rank 24. We know that an even unimodular lattice of rank 24 is either a root lattice or $\Lambda(2) = \emptyset$. We now investigate this second possibility.
The conclusion will be that there is a unique lattice with this property, up to isomorphism, called the Leech lattice. This is probably one of the most famous lattice, given it's many secrets. It is for example at the heart of the Monstruous Moonshine, a phenomenon linking modular forms to representation theory in a very mysterious way. There are many constructions for the Leech lattice, in varying levels of complexity. Note the conclusion of this section is in itself a definition: the Leech lattice is the unique even unimodular lattice in $\mathbb{R}^{24}$ with no vector of length 2.

A complete proof is given in chapter 12 of [JHC99]. We will not go in depth, but rather focus on the comment Conway makes about the impossibility to extend this proof to larger dimensions (and construction for the Leech lattice):

We conjecture that any such extended argument must be considerably more complicated, since it is very doubtful that the extreme "tightness" which the reader will observe so often in our proof can ever occur again.

Indeed, the proof Conway presents hinges on many mind-blowing equalities. We will gather some of them. We start in the same way as usual: what can we say about Θ_Λ?

Proposition 3.4. *The theta series of the a lattice for which* $\Lambda(0) = 1$ *and* $\Lambda(2) = 0$ *is*

$$\Theta_\Lambda(\tau) = 1 + 0q^2 + 196,560q^4 + 16,773,120q^6 + 398,034,000q^8 + \cdots$$

This follows from the same strategies as before: Θ_Λ is modular so Theorem 2.1 tells us what it looks like, and the conditions of the proposition fix a specific form. The heart of argument is showing existence and unicity of a lattice with these properties.
A first "tight" equality required in the proof is that $\frac{|\Lambda(0)|}{1} + \frac{|\Lambda(4)|}{2} + \frac{|\Lambda(6)|}{2} + \frac{|\Lambda(8)|}{48}$ is precisely equal to 2^{24}.
A second miracle happens when counting vectors of length 4. There are $196,560$ by the Theta series. Conway studies vectors with some property, and gather there are at most $759 \cdot 2^7 + 24 \cdot 2^{12} + \binom{24}{2} \cdot 2^2$ such vectors. This number is precisely $\Lambda(4)$, so inequalities become equalities.

We mention that the paper by Venkov (Chapter 18 of [JHC99]) also gives a proof of this characterization of the Leech lattice. It is much shorter, but it assumes the classification of root lattices in rank 24, and it uses code theory. The proof of Conway also has the advantage to showcase how special the Leech lattice is, and how it's existence is most surprising.

4 List of Root Systems

Recall that we are only interested in root systems with the property that all vectors have the same length; so called of type ADE. Only root systems of type A, D or E have this property (hence the name). We shall only define those root systems.
Recall that $h = \frac{|R|}{\mathrm{rk}(R)}$. The Coxeter number could be computed from scratch in each case, but this makes life easier. We write e_i the standard basis vector in $\mathbb{R}^n$, all 0 except 1 at coordinate i.
We only give the simple roots. The reader can collect the set of all roots by applying the s_α and scaling by -1. The root lattice of type R is the $\mathbb{Z}$-span of the simple roots. The reader can verify that the number of many simple roots is the rank of the root system.

Figure 1: Rank 2 Root Systems and Root Lattices (Drawing by my girlfriend, Camila De O. Ortiz)

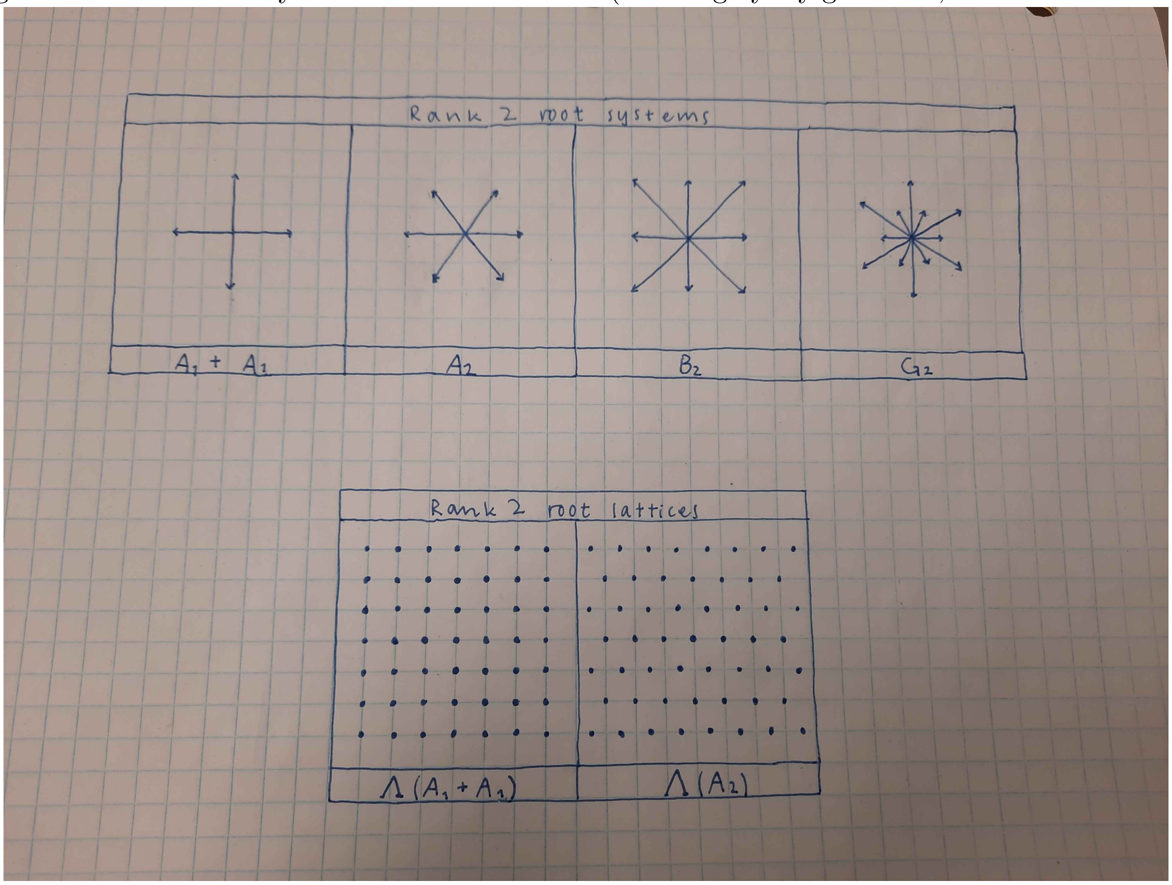

Table 1: Irreducible Root Systems of type ADE

Root system	Simple Roots	$\lvert R\rvert$	rk(R)	h
A_n	$\{e_i - e_{i+1} : 0 \leq i \leq n-1\}$	$n(n+1)$	n	$n+1$
D_n	$\{e_i - e_{i+1} : 0 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$	$2n(n-1)$	n	$2(n-1)$
E_6	$\{e_i - e_{i+1} : 0 \leq i \leq 4\} \cup \{e_4 + e_5\}$ $\cup\{(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2})\}$	72	6	12
E_7	$\{e_i - e_{i+1} : 0 \leq i \leq 5\} \cup \{e_5 + e_6\}$ $\cup\{(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2})\}$	126	7	18
E_8	$\{e_i - e_{i+1} : 0 \leq i \leq 6\} \cup \{e_6 + e_7\}$ $\cup\{(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$	240	8	30

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