NOTES ON ERGODIC THEORY

ZHAOSHEN ZHAI

ABSTRACT. Notes on the Summer 2025 Reading Group on Ergodic Theory, following [Tse22], organized by Frédéric Kai, Ludovic Rivet, and Zhaoshen Zhai (website).

CONTENTS

1. LECTURE 1 (SAMY LAHLOU): CRASH COURSE ON MEASURE THEORY, PART I

Given a set X, our goal is to define a map $\mu : \mathcal{P}(X) \to [0, \infty]$ that assigns to each subset $A \subseteq X$ a *measure* $\mu(A) \in [0, \infty]$ that 'behaves like the volume of A'. This turns out to be impossible in full generality (and we shall see using ergodic-theoretic methods that this impossibility is for good reason), so we instead restrict to special subsets of X with a nice algebraic (think: 'constructible') structure.

Further reading. [Tse23, Lectures 1 to 5] and [Fol99, Chapter 1].

Definition 1.1. Let X be a set. A *σ-algebra on X* is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X containing $\varnothing$ and is closed under complements and countable unions. More precisely:

1. (Non-trivial). $\varnothing \in \mathcal{B}$.
2. (Closure under complements). For any $A \in \mathcal{B}$, we have $X \setminus A \in \mathcal{B}$.
3. (Closure under countable unions). For any countable family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$, we have $\bigcup_n A_n \in \mathcal{B}$.

Definition 1.2. If $\mathcal{B}$ is a σ-algebra on a set X, the pair $(X, \mathcal{B})$ is said to be a *measurable space*.

A useful way to construct a σ-algebra is to start with an arbitrary family $\mathcal{C} \subseteq \mathcal{P}(X)$ and close[1] it under the above three conditions. Abstractly:

Definition 1.3. The σ-algebra *generated* by $\mathcal{C} \subseteq \mathcal{P}(X)$ is $\langle \mathcal{C} \rangle_\sigma := \bigcap \{\mathcal{B} : \mathcal{B} \supseteq \mathcal{C} \text{ is a } \sigma\text{-algebra on } X\}$.

Note that $\langle \mathcal{C} \rangle_\sigma$ is indeed a σ-algebra on X since the intersection of σ-algebras is again a σ-algebra.

Definition 1.4. Let X be a topological space. The *Borel σ-algebra* on X is $\mathcal{B}(X) := \langle \mathcal{T} \rangle_\sigma$, where $\mathcal{T}$ is the topology on X. The elements of $\mathcal{B}(X)$ are called the *Borel sets* of X.

Date: May 10, 2025.

[1] This 'closure' operation can be made precise as follows. Starting with $\mathcal{C}_0 := \mathcal{C}$, throw in all the subsets of X that is necessary to satisfy Definition 1.1 relativized to $\mathcal{C}_0$ to obtain $\mathcal{C}_1$ (that is, let $\mathcal{C}_1$ contain $\varnothing$ and such that if $A \in \mathcal{C}_0$, then $X \setminus A \in \mathcal{C}_1$, and similarly for condition 3). Then, let $\mathcal{C}_2$ be defined by throwing in all the countable unions and complements of sets in $\mathcal{C}_1$. Doing so infinitely-many times and taking the union $\bigcup_\alpha \mathcal{C}_\alpha$ will give us $\langle \mathcal{C} \rangle_\sigma$, but beware that this process must proceed into the transfinite up to $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal; ask your local set theorist why.

Intuitively, for any topological space X, one would like to 'measure' the Borel sets. This is justified since if one wants a measure compatible with the topology, then one must be able to measure the open sets, and hence also closed sets, countable unions of closed sets (called F_σ *sets*), countable intersections of open sets (called G_δ *sets*), countable intersections of F_σ sets, countable unions of G_δ sets, and so on[2].

Definition 1.5. A *measure* on a measurable space $(X,\mathcal{B})$ is a map $\mu:\mathcal{B}\to[0,\infty]$ such that $\mu(\varnothing)=0$ and $\mu(\bigsqcup_{n\in\mathbb{N}} A_n)=\sum_{n\in\mathbb{N}}\mu(A_n)$ for any pairwise disjoint family $\{A_n\in\mathcal{B}:n\in\mathbb{N}\}$.

The triple $(X,\mathcal{B},\mu)$ is then called a *measure space*. A *Borel measure* is a measure defined on some Borel σ-algebra.

Example 1.6 (Lebesgue)**.** Equip $\mathbb{R}$ with its usual topology. There is[3] a unique measure $\lambda:\mathcal{B}(\mathbb{R})\to[0,\infty]$ on $\mathbb{R}$, called the *Lebesgue measure*, such that $\lambda([a,b])=b-a$ for each $a\le b$.

Example 1.7 (Bernoulli)**.** Equip $2=\{0,1\}$ with the discrete topology and consider the product topology on $2^{\mathbb{N}}$. For each $p\in[0,1]$, is a unique measure $\mu_p:\mathcal{B}(2^{\mathbb{N}})\to[0,\infty]$ on $2^{\mathbb{N}}$, called the *Bernoulli* (p) *measure*, such that for each word $w\in 2^{<\mathbb{N}}$, we have $\mu_p([w])=p^{n_1}(1-p)^{n_0}$ where n_i is the number of $i\in\{0,1\}$ in w and $[w]$ is the set of all sequences in $2^{\mathbb{N}}$ containing w as a prefix.

If $p=0$ (similarly if $p=1$), then $\mu_p(\xi)\in\{0,1\}$, and we have $\mu_p(\xi)=1$ iff $(p,p,p,\dots)\in\xi$. Thus, all of the measure is concentrated at $(p,p,p,\dots)$. Measures in which this occurs are called *Dirac measures*.

Example 1.8 (Dirac)**.** Let X be a set and fix $x\in X$. The *Dirac measure concentrated* at x is the measure $\delta_x:\mathcal{P}(X)\to\{0,1\}$ defined by $\delta_x(A):=1$ iff $x\in A$, and $\delta_x(A):=0$ iff $x\notin A$.

Definition 1.9. A measure μ on $(X,\mathcal{B})$ is said to be *finite* if $\mu(X)<\infty$, a *probability measure* if $\mu(X)=1$, and σ*-finite* if there is a partition $X=\bigsqcup_{n\in\mathbb{N}}X_n$ such that $X_n\in\mathcal{B}$ and $\mu(X_n)<\infty$ for all $n\in\mathbb{N}$.

Unless otherwise stated, all measures are assumed to be σ-finite. In fact, we will usually only deal with probability measures, since we can also normalize a finite measure to a probability measure by $\mu\mapsto\mu/\mu(X)$.

Lastly, even though μ is only defined on the σ-algebra $\mathcal{B}$, we can slightly extend μ to a larger σ-algebra.

Definition 1.10. Let $(X,\mathcal{B},\mu)$ be a measure space. A subset $Z\subseteq X$ is said to be μ*-null* if there exists some $Z'\in\mathcal{B}$ such that $Z\subseteq Z'$ and $\mu(Z')=0$. We write Null_μ for the set of all μ-null subsets of X. A subset $A\subseteq X$ is said to be μ*-conull* if $X\setminus A$ is μ-null.

Definition 1.11. Let $(X,\mathcal{B},\mu)$ be a measure space. A subset $A\subseteq X$ is μ*-measurable*[4] if there exists some $B\in\mathcal{B}$ and some μ-null set Z such that $A=B\cup Z$. We write Meas_μ for the set of all μ-measurable sets.

It is an exercise that $\mathrm{Meas}_\mu=\langle\mathcal{B}\cup\mathrm{Null}_\mu\rangle_\sigma$. Moreover, μ admits a unique extension to a map $\overline{\mu}:\mathrm{Meas}_\mu\to[0,\infty]$, called the *completion* of μ, and this measure satisfies $\mathrm{Meas}_{\overline{\mu}}=\mathrm{Meas}_\mu$. HINT: $\overline{\mu}(B\cup Z):=\mu(B)$.

Definition 1.12. A measure μ is *complete* if $\overline{\mu}=\mu$.

For convenience, we will always assume that measures are complete. Neither measures λ nor μ_p in Examples 1.6 and 1.7 are complete, so we tacitly extend them.

We end with some easy exercises on measures; please read/prove them, as they will be used freely in the future; they are roughly ranked by difficulty. Throughout, let $(X,\mathcal{B},\mu)$ be a measure space and let $A_n\in\mathcal{B}$.

Exercise 1.13 (Monotonicity)**.** If $A_1\subseteq A_2$, then $\mu(A_1)\le\mu(A_2)$.

Deduce that if μ is finite, then μ is a bounded function. (Are σ-finite measures bounded?)

Exercise 1.14 (Inclusion-exclusion)**.** For any $A_1,A_2\in\mathcal{B}$, we have $\mu(A_1\cup A_2)+\mu(A_1\cap A_2)=\mu(A_1)+\mu(A_2)$.

[2]This goes up the *Borel hierarchy*, studied in *Descriptive Set Theory*; see [Kec95].

[3]We will not prove this fact, but it is an application of Carathéodory's Extension Theorem; see [Tse23, Lecture 4].

[4]Very confusing terminology. One might think that elements of $\mathcal{B}$ are the 'measurable' ones, but this removes μ from the picture. In general, there are much more μ-measurable sets that there are sets in $\mathcal{B}$. Indeed, there are $2^{\aleph_0}$-many Borel sets on $\mathbb{R}$, but there are $2^{2^{\aleph_0}}$-many λ-measurable sets!

Exercise 1.15 (Continuity $\nearrow$)**.** If $(A_n)_{n\in\mathbb{N}}$ is increasing, then $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \lim_n \mu(A_n)$.

Exercise 1.16 (Continuity $\searrow$)**.** If $(A_n)_{n\in\mathbb{N}}$ is decreasing and $\mu(A_1) < \infty$, then $\mu(\bigcap_{n\in\mathbb{N}} A_n) = \lim_n \mu(A_n)$.

Exercise 1.17. Show that $\lambda(\mathbb{Q}) = 0$. Hint: What is the Lebesgue measure of singletons?

Let P be a property of some points in X. We say that P *holds μ-almost everywhere* (or *μ-almost surely*) if $\{x \in X : x \text{ satisfies } P\}$ is μ-conull.

Exercise 1.18 (Borel-Cantelli Lemmas)**.** Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of μ-measurable sets.

1. If $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then μ-almost every $x \in X$ lives in at-most finitely-many A_n.
2. (Measure Compactness). If $\mu(X) < \infty$ and there exists $\varepsilon > 0$ such that $\mu(A_n) \geq \varepsilon$ for all $n \in \mathbb{N}$, then at least an ε-measure set of $x \in X$ lives in infinitely-many A_n's.

For measurable spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$, define $\mathcal{B}_1 \otimes \mathcal{B}_2 := \langle B_1 \times B_2 : B_i \in \mathcal{B}_i \rangle_\sigma$.

Exercise 1.19. Show that if X_i are second-countable topological spaces, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$.

Exercise 1.20. Let X be a topological space. A *Cantor set* is a subset $C \subseteq X$ homeomorphic to $2^{\mathbb{N}}$.

1. Show that the 'middle-thirds Cantor set' $C \subseteq [0,1]$ is a Cantor set as in the above definition. Moreover, show that $\lambda(C) = 0$. Hint: Recall the construction $C = \bigcap_{n\in\mathbb{N}} C_n$ and use continuity.
2. Define a Cantor set $C \subseteq [0,1]$ with positive Lebesgue measure. Hint: fatten the standard construction.

A measurable set $A \subseteq X$ is said to be an *atom* if there is no subset $B \subseteq A$ with $0 < \mu(B) < \mu(A)$. For example, singletons $\{x\}$ are atoms under the Dirac measure δ_x. More generally:

Exercise 1.21 (Atomic Decomposition)**.** If $(X, \mathcal{B}, \mu)$ is a σ-finite measure space, $\mathcal{B}$ is *countably generated* (i.e., $\mathcal{B} = \langle \mathcal{B}_0 \rangle$ for some countable $\mathcal{B}_0 \subseteq \mathcal{P}(X)$), and *separates points* (i.e., if $x \neq y$, then there exists $B \in \mathcal{B}$ such that $x \in B \not\ni y$.), then every atom $A \in \mathcal{B}$ is a singleton. Moreover, if $\{x_\alpha\}$ are all the atoms (how many can there be?), then $\mu = \mu_0 + \sum_\alpha a_\alpha \delta_{x_\alpha}$ for some atomless measure μ_0 and some $a_\alpha \geq 0$.

2. Lecture 2 (Samy Lahlou): Crash course on Measure Theory, Part II

Let $(X, \mathcal{B}, \mu)$ be a measure space. Our goal is to define the *Lebesgue integral* $\int f \, \mathrm{d}\mu$ for a function $f : X \to \mathbb{R}$. Again, this is not possible in full generality, so we restrict ourselves to the so-called *measurable functions*.

Further reading. [Tse23, Lectures 9 to 13, 17 to 21] and [Fol99, Chapters 2 and 3].

Definition 2.1. A *simple function* is an $\mathbb{R}$-linear combination of characteristic functions on μ-measurable sets, i.e., if $(E_i)_{i\leq n}$ is a collection of pairwise-disjoint μ-measurable sets and $(a_i)_{i\leq n}$ are distinct reals, then $\varphi := \sum_{i\leq n} a_i \chi_{E_i}$ is a said to be a *simple function*. Define its *(Lebesgue) integral* as $\int \varphi \, \mathrm{d}\mu := \sum_{i\leq n} a_i \mu(E_i)$.

For a (bounded) positive function $f : X \to \mathbb{R}_{\geq 0}$, we might define $\int f \, \mathrm{d}\mu$ by approximating f by simple functions from below, say by an increasing sequence (φ_n) of simple functions such that $f = \lim_n \varphi_n$ uniformly. However, not all functions f admit such an approximation.

To see this, let us attempt to construct such a sequence (φ_n). For each n, we will approximate the cutoff of f at 2^n, i.e., the function $\min(f, 2^n)$. We do so by partitioning the codomain $[0, 2^n]$ into intervals of length 2^{-n}, for a total of $k_n := 2^n/2^{-n} = 2^{2n}$ intervals. Set $E_k := f^{-1}([2^{-n}k, \infty))$ for each $k \in \{1, \ldots, k_n\}$, and let $\varphi_n := \sum_{k\leq k_n} 2^{-n} \chi_{E_k}$. One easily checks that $f = \lim_n \varphi_n$ uniformly.

However, E_k is not guaranteed to be μ-measurable! To fix this, we simply define the issue away.

Definition 2.2. A function $f : X \to Y$ between measurable spaces $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces is said to be $(\mathcal{B}, \mathcal{C})$*-measurable* if $f^{-1}(C) \in \mathcal{B}$ for all $C \in \mathcal{C}$.

A function $f : X \to Y$ between topological spaces is said to be *Borel* if it is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable. A *Borel isomorphism* is a bijection $f : X \to Y$ such that both f and f^{-1} are Borel.

Exercise 2.3. Continuous maps are Borel. Hint: Define a σ-algebra containing open sets in the codomain.

So far we only dealt with measurable spaces. Let us now bring a measure into the picture.

Definition 2.4. Let (X, μ)[5] be a measure space and Y be a topological space. A function $f : X \to Y$ is said to be *μ-measurable* if it is $(\mathrm{Meas}_\mu, \mathcal{B}(Y))$-measurable.

Remark 2.5. Compositions of μ-measurable functions *need not* be μ-measurable.

The following exercise is one of the main reasons why μ-measurable functions are introduced, and ultimately also why the Lebesgue integral is superior compared to the Riemann integral.

Exercise 2.6. In separable metric spaces, pointwise limits of μ-measurable functions are μ-measurable, i.e., if (f_n) is a sequence of μ-measurable maps $f_n : X \to Y$ from a measure space (X, μ) to a separable space Y, and $f := \lim_n f_n$ (pointwise), then $f : X \to Y$ is μ-measurable.

Hint: Let $\mathcal{C} := \{B \in \mathcal{B}(Y) : f^{-1}(B) \in \mathrm{Meas}_\mu\}$. Show that $\mathcal{C}$ is a σ-algebra containing all open set in Y, so $\mathcal{C} = \mathcal{B}(Y)$, as desired. For each $U \subseteq Y$ open, use separability to write $U = \bigcup_{n \in \mathbb{N}} B_n$, where each B_n is a ball whose closure is contained in U, and show that $f^{-1}(U) \in \mathrm{Meas}_\mu$.

Exercise 2.7. If $f_1, f_2 : (X, \mu) \to \mathbb{R}$ are μ-measurable and $g : \mathbb{R}^2 \to \mathbb{R}$ is Borel, then $g(f_1, f_2) : X \to \mathbb{R}$ is also μ-measurable. In particular, $f_1 + f_2$ and $f_1 \cdot f_2$ are μ-measurable.

Exercise 2.8. If (f_n) is a sequence of μ-measurable functions $f_n : X \to \overline{\mathbb{R}}$, then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, and $\liminf_n f_n$ are also μ-measurable.

Notation 2.9. We write $L(X, \mu)$ for the set of all μ-measurable functions $f : (X, \mu) \to \overline{\mathbb{R}}$, and $L^+(X, \mu)$ for those which are non-negative.

We are finally ready to define the Lebesgue integral.

Definition 2.10. Let (X, μ) be a measure space. The *(Lebesgue) integral* of $f \in L^+(X, \mu)$ is

$$\int f \,\mathrm{d}\mu := \sup \left\{ \int \varphi \,\mathrm{d}\mu : 0 \le \varphi \le f \text{ simple function} \right\}.$$

In general, if $f \in L(X, \mu)$, we decompose $f = f^+ - f^-$ where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$. The *(Lebesgue) integral* of f is $\int f \,\mathrm{d}\mu := \int f^+ \,\mathrm{d}\mu - \int f^- \mathrm{d}\mu$, provided that one of the terms is finite.

If $\int f \,\mathrm{d}\mu < \infty$, we say that f is *μ-integrable*, in which case we write $f \in L^1(X, \mu)$. More generally,

Definition 2.11. Take $p \in [1, \infty]$ and let $L^p(X, \mu)$ be the set of all μ-measurable functions $f : X \to \overline{\mathbb{R}}$ such that $\|f\|_p < \infty$, where $\|f\|_p := (\int |f|^p \,\mathrm{d}\mu)^{1/p}$ if $p < \infty$ and $\|f\|_\infty := \operatorname{ess-sup} |f| := \inf\{c \ge 0 : |f| \le c \ \mu\text{-a.e.}\}$.

Exercise 2.12. Let $f, g \in L^p(X, \mu)$. If $f \le g$, then $\|f\|_p \le \|g\|_p$.

Since a μ-measurable function $f : X \to \overline{\mathbb{R}}$ can be approximated from below by simple functions (φ_n), we should be able to calculate $\int f \,\mathrm{d}\mu$ as the limit of $\int \varphi_n \,\mathrm{d}\mu$. Indeed,

Theorem 2.13 (Monotone Convergence Theorem)**.** *If $(f_n) \in L^+(X, \mu)$ and $f_n \nearrow f$, then $\int f_n \,\mathrm{d}\mu \nearrow \int f \,\mathrm{d}\mu$.*

Corollary 2.14. *If $(f_n) \in L^+(X, \mu)$, then $\sum_n \int f_n \,\mathrm{d}\mu = \int \sum_n f_n \,\mathrm{d}\mu$.*

Exercise 2.15. For any $f, g \in L^1(X, \mu)$ and $a, b \in \mathbb{R}$, we have $\int (af + bg) \,\mathrm{d}\mu = a \int f \,\mathrm{d}\mu + b \int g \,\mathrm{d}\mu$.

Hint: Simple $\rightsquigarrow_{\mathrm{MCT}} L^+ \rightsquigarrow L^1$.

Exercise 2.16. Let $f, g \in L^1(X, \mu)$. If $f = g$ μ-a.e., then $\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$. Hint: Consider $\int (f - g) \,\mathrm{d}\mu$.

We list two more convergence theorems that will be useful later on.

Theorem 2.17 (Fatou's Lemma)**.** *If $(f_n) \in L^+(X, \mu)$, then $\int \liminf_n f_n \,\mathrm{d}\mu \le \liminf_n \int f_n \mathrm{d}\mu$.*

Theorem 2.18 (Dominated Convergence Theorem)**.** *Let $(f_n) \in L^1(X, \mu)$. If $f_n \to f$ μ-a.e. and $|f_n| \le g$ for some $g \in L^1(X, \mu)$, then $\lim_n \int f_n \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu$.*

---

[5] Whenever the σ-algebra is not stated, we assume that μ is defined on Meas_μ. In particular, we assume that μ is complete.

Let us now discuss differentiation of functions $f : X \to \mathbb{R}$; for convenience, we assume[6] that $f \in L^+(X, \mu)$. For these functions, we can define a new measure ν on $\mathcal{B}$ by $\nu(B) := \int_B f \, \mathrm{d}\mu := \int f \cdot \chi_B \, \mathrm{d}\mu$, which measures the 'area under the curve'. Note that for each $B \in \mathcal{B}$, we have B is ν-null whenever B is μ-null.

It turns out that the 'correct' setting to discuss differentiation is between two measures μ and ν which satisfy the above condition.

Definition 2.19. If μ, ν are measures on a measurable space $(X, \mathcal{B})$ and B is ν-null whenever B is μ-null for each $B \in \mathcal{B}$, we say that ν is *absolutely continuous w.r.t* μ, and write $\nu \ll \mu$.

Theorem 2.20 (Lebesgue-Radon-Nikodym Theorem)**.** *If $\nu \ll \mu$ are σ-finite measures on a measurable space $(X, \mathcal{B})$, then there exists a $\mathcal{B}$-measurable map $f : X \to \mathbb{R}_{\geq 0}$ such that $\nu(B) = \int_B f \, \mathrm{d}\mu$ for all $B \in \mathcal{B}$.*

Such a function $f : X \to \mathbb{R}_{\geq 0}$ is unique μ-a.e., and is called the *Radon-Nikodym derivative of ν w.r.t. μ*, denoted $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Thus, we have $\nu(B) = \int_B \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\mu$ for all $B \in \mathcal{B}$.

Corollary 2.21. *In the above setting, we have $\int g \, \mathrm{d}\mu = \int g \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \mathrm{d}\nu$ for all $g \in L^1(X, \mu)$.*

To relate $\mathrm{d}\nu/\mathrm{d}\mu$ to derivatives in calculus (say on $\mathbb{R}^n$), we let $\mu := \lambda$ be Lebesgue measure on $\mathbb{R}^n$.

Theorem 2.22 (Lebesgue Differentiation Theorem)**.** *For any locally-integrable function $f : \mathbb{R}^n \to \mathbb{R}$ (i.e. if $f \cdot \chi_K \in L^1(\mathbb{R}^n, \lambda)$ for every compact $K \subseteq \mathbb{R}^n$) and for λ-a.e. $x \in \mathbb{R}^n$, we have*

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{\lambda(B_\varepsilon(x))} \int_{B_\varepsilon(x)} f \, \mathrm{d}\lambda.$$

Corollary 2.23. *For any locally-finite Borel measure $\mu \ll \lambda$ on $\mathbb{R}^n$ and for λ-a.e. $x \in \mathbb{R}^n$, we have*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{\lambda(B_\varepsilon(x))}.$$

We end by briefly mentioning the 'Isomorphism Theorem'. These justify why we only gave three examples in Lecture 1, and allows us to work in concrete spaces like $[0, 1]$ or $2^{\mathbb{N}}$.

Definition 2.24. A measurable space $(X, \mathcal{B})$ is said to be *standard Borel* if $\mathcal{B}$ is the Borel σ-algebra of some Polish (i.e. separable and completely metrizable) topology on X.

A probability space $(X, \mathcal{B}, \mu)$ is *standard* if $(X, \mathcal{B})$ is standard Borel.

Theorem 2.25 (Borel Isomorphism Theorem)**.** *Any two uncountable standard Borel spaces are Borel isomorphic. In particular, they all have cardinality continuum and are Borel isomorphic to $2^{\mathbb{N}}$.*

Definition 2.26. Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces. If $f : X \to Y$ is $(\mathcal{B}, \mathcal{C})$-measurable and μ is a measure on $\mathcal{B}$, the *pushforward measure of μ by f* is the measure $f_*\mu$ on $\mathcal{C}$ defined by $f_*\mu(C) := \mu(f^{-1}(C))$.

Definition 2.27. A function $f : X \to Y$ between measure spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ is said to be a *measure isomorphism* if $f_*\mu = \nu$, and if there exist a μ-conull set $X_0 \subseteq X$ and a ν-conull set $Y_0 \subseteq Y$ such that $f|_{X_0} : X_0 \to Y_0$ is a bijection and $f|_{X_0}$ (resp. $f^{-1}|_{Y_0}$) is μ-measurable (resp. ν-measurable).

Theorem 2.28 (Measure Isomorphism Theorem)**.** *Any two atomless standard probability spaces are measure isomorphic. In particular, they are all measure isomorphic to $([0, 1], \lambda)$.*

3. Lecture 3 (Peng Bo): Introduction to Ergodic Theory

TODO: intro

Further reading. None!

[6] Otherwise, we will need to discuss 'signed measures'.

References

[Fol99] Gerald B. Folland. *Real Analysis. Modern Techniques and Their Applications*. 2nd ed. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. John Wiley & Sons, 1999.

[Kec95] Alexander S. Kechris. *Classical Descriptive Set Theory*. Graduate Texts in Mathematics. Springer New York, NY, 1995.

[Tse23] Anush Tserunyan. *MATH564: Advanced Real Analysis 1*. URL: https://www.math.mcgill.ca/atserunyan/Courses/2023F.Math564.Analysis%201/.

[Tse22] Anush Tserunyan. *MATH594: Topics in Ergodic Theory and Measured Group Theory*. URL: https://www.math.mcgill.ca/atserunyan/Courses/2022_W.Math594.Erg&MsGrp/.

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, QC, H3A 0B9, Canada

Email address: zhaoshen.zhai@mail.mcgill.ca