SUMMER 2025 READING GROUP ON ERGODIC THEORY

Exercise Sheet 9 (Zhaoshen Zhai): Weak mixing and Almost-periodic Functions

The purpose of this exercise sheet is to give another proof of the following theorem.

Theorem A. *If a measure-preserving dynamical system is not weak-mixing, then there exists an almost periodic function that is not constant a.e.*

First, a technical lemma needed for Exercise 2.

Exercise 1. Let $x_0, x_1, \ldots \in \mathbb{R}$. If $\text{C-lim}\, x_n = x$ and $\text{C-lim}\, x_n^2 = x^2$ for some $x \in \mathbb{R}$, then $\text{D-lim}\, x_n = x$.
Hint: Show that $\text{C-lim}\, |x_n - x|^2 = 0$. Why does this imply that $\text{D-lim}\, x_n = x$?

The *product* of systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ is the system $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \times \nu, T \times S)$, where $\mathcal{B} \otimes \mathcal{C}$ is the σ-algebra generated by $B \times C$ for $B \in \mathcal{B}$ and $C \in \mathcal{C}$, and $\mu \times \nu$ is the (unique) measure on $\mathcal{B} \otimes \mathcal{C}$ such that $(\mu \times \nu)(B \times C) = \mu(B)\nu(C)$ for all $B \in \mathcal{B}$ and $C \in \mathcal{C}$. You can use the following facts.

Fact. For $f \in L^2(X)$ and $g \in L^2(Y)$, let $f \otimes g : X \times Y \to \mathbb{R}$ by defined by $(f \otimes g)(x, y) := f(x)g(y)$. Then

$$\{f \otimes g : f \in L^2(X), g \in L^2(Y)\}$$

linearly span a dense subset of $L^2(X \times Y)$.

Fact (Fubini-Tonelli)**.** For $f \in L^2(X)$ and $g \in L^2(Y)$, we have $\int_{X \times Y} f \otimes g \, \mathrm{d}(\mu \times \nu) = \int_X f \, \mathrm{d}\mu \int_Y g \, \mathrm{d}\nu$.

Exercise 2. The following are equivalent for a measure-preserving dynamical system X.

1. X is weak mixing.
2. $X \times X$ is weak mixing.
3. $X \times X$ is ergodic.

Hint: Compute $\langle (T \times T)^n (f_1 \otimes f_2), g_1 \otimes g_2 \rangle$ and use $\mathbb{E}(f_1 \otimes f_2) = \mathbb{E}(f_1)\mathbb{E}(f_2)$.

Exercise 3. Follow the steps below to prove Theorem A: if a measure-preserving dynamical system X is not weak-mixing, then there exists an almost periodic function that is not constant a.e.

1. Note that we can assume that X is ergodic (no need for ergodic decomposition).
2. Show that there is a non-constant $(T \times T)$-invariant function $K \in L^2(X \times X)$. Assume that $\mathbb{E}(K) = 0$.
3. Show that the Hilbert-Schmidt operator Φ_K on $L^2(X)$ given by $\Phi_K f(y) := \int_X K(x, y) f(x) \, \mathrm{d}\mu(x)$ (we say that Φ_K has *kernel* K) commutes with T. Thus $\Phi_K f \in \mathcal{AP}(X)$ for every $f \in L^2(X)$, so it suffices to find one such that $\Phi_K f \neq 0$ (since $\mathbb{E}(\Phi_K f) = 0$).
4. Suppose that there is no $f \in L^2(X)$ such that $\Phi_K f \neq 0$. Show that K is orthogonal to $f \otimes g$ for all $f, g \in L^2(X)$, hence $K = 0$, a contradiction. Hint: The map $x \mapsto \int_X K(x, y) \mathrm{d}\mu(y)$ is T-invariant.

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