NOTES ON ERGODIC THEORY

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Abstract. Notes on the Summer 2025 Reading Group on Ergodic Theory, following [Anu22], organized by Frédéric Kai, Ludovic Rivet, and Zhaoshen Zhai (website).

Contents

1. Lecture 1 (Samy Lahlou): Crash course on Measure Theory, Part I

Given a set X, our goal is to define a map $\mu : \mathcal{P}(X) \to [0, \infty]$ that assigns to each subset $A \subseteq X$ a *measure* $\mu(A) \in [0, \infty]$ that 'behaves like the volume of A'. This turns out to be impossible in full generality (and we shall see using ergodic-theoretic methods that this impossibility is for good reason), so we instead restrict to special subsets of X with a nice algebraic (think: 'constructible') structure.

Further reading. [Anu23, Lectures 1 to 5] and [Fol99, Chapter 1].

Definition 1.1. Let X be a set. A *σ-algebra on X* is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X containing $\varnothing$ and is closed under complements and countable unions. More precisely:

1. (Non-trivial). $\varnothing \in \mathcal{B}$.
2. (Closure under complements). For any $A \in \mathcal{B}$, we have $X \setminus A \in \mathcal{B}$.
3. (Closure under countable unions). For any countable family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$, we have $\bigcup_n A_n \in \mathcal{B}$.

Definition 1.2. If $\mathcal{B}$ is a σ-algebra on a set X, the pair $(X, \mathcal{B})$ is said to be a *measurable space*.

A useful way to construct a σ-algebra is to start with an arbitrary family $\mathcal{C} \subseteq \mathcal{P}(X)$ and close[1] it under the above three conditions. Abstractly:

Definition 1.3. The σ-algebra *generated* by $\mathcal{C} \subseteq \mathcal{P}(X)$ is $\langle \mathcal{C} \rangle_\sigma := \bigcap \{\mathcal{B} : \mathcal{B} \supseteq \mathcal{C} \text{ is a } \sigma\text{-algebra on } X\}$.

Note that $\langle \mathcal{C} \rangle_\sigma$ is indeed a σ-algebra on X since the intersection of σ-algebras is again a σ-algebra.

Definition 1.4. Let X be a topological space. The *Borel σ-algebra* on X is $\mathcal{B}(X) := \langle \mathcal{T} \rangle_\sigma$, where $\mathcal{T}$ is the topology on X. The elements of $\mathcal{B}(X)$ are called the *Borel sets* of X.

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[1] This 'closure' operation can be made precise as follows. Starting with $\mathcal{C}_0 := \mathcal{C}$, throw in all the subsets of X that is necessary to satisfy Definition 1.1 relativized to $\mathcal{C}_0$ to obtain $\mathcal{C}_1$ (that is, let $\mathcal{C}_1$ contain $\varnothing$ and such that if $A \in \mathcal{C}_0$, then $X \setminus A \in \mathcal{C}_1$, and similarly for condition 3). Then, let $\mathcal{C}_2$ be defined by throwing in all the countable unions and complements of sets in $\mathcal{C}_1$. Doing so infinitely-many times and taking the union $\bigcup_\alpha \mathcal{C}_\alpha$ will give us $\langle \mathcal{C} \rangle_\sigma$, but beware that this process must proceed into the transfinite up to $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal; ask your local set theorist why.

Intuitively, for any topological space X, one would like to 'measure' the Borel sets. This is justified since if one wants a measure compatible with the topology, then one must be able to measure the open sets, and hence also closed sets, countable unions of closed sets (called *F_σ sets*), countable intersections of open sets (called *G_δ sets*), countable intersections of F_σ sets, countable unions of G_δ sets, and so on[2].

Definition 1.5. A *measure* on a measurable space $(X, \mathcal{B})$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and $\mu(\bigsqcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$ for any pairwise disjoint family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$.

The triple $(X, \mathcal{B}, \mu)$ is then called a *measure space*. A *Borel measure* is a measure defined on some Borel σ-algebra.

Example 1.6 (Lebesgue)**.** Equip $\mathbb{R}$ with its usual topology. There is[3] a unique measure $\lambda : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ on $\mathbb{R}$, called the *Lebesgue measure*, such that $\lambda([a, b]) = b - a$ for each $a \le b$.

Example 1.7 (Bernoulli)**.** Equip $2 = \{0, 1\}$ with the discrete topology and consider the product topology on $2^{\mathbb{N}}$. For each $p \in [0, 1]$, is a unique measure $\mu_p : \mathcal{B}(2^{\mathbb{N}}) \to [0, \infty]$ on $2^{\mathbb{N}}$, called the *Bernoulli* (p) *measure*, such that for each word $w \in 2^{<\mathbb{N}}$, we have $\mu_p([w]) = p^{n_1}(1 - p)^{n_0}$ where n_i is the number of $i \in \{0, 1\}$ in w and $[w]$ is the set of all sequences in $2^{\mathbb{N}}$ containing w as a prefix.

If $p = 0$ (similarly if $p = 1$), then $\mu_p(\xi) \in \{0, 1\}$, and we have $\mu_p(\xi) = 1$ iff $(p, p, p, \ldots) \in \xi$. Thus, all of the measure is concentrated at $(p, p, p, \ldots)$. Measures in which this occurs are called *Dirac measures*.

Example 1.8 (Dirac)**.** Let X be a set and fix $x \in X$. The *Dirac measure concentrated* at x is the measure $\delta_x : \mathcal{P}(X) \to \{0, 1\}$ defined by $\delta_x(A) \coloneqq 1$ iff $x \in A$, and $\delta_x(A) \coloneqq 0$ iff $x \notin A$.

Definition 1.9. A measure μ on $(X, \mathcal{B})$ is said to be *finite* if $\mu(X) < \infty$, a *probability measure* if $\mu(X) = 1$, and *σ-finite* if there is a partition $X = \bigsqcup_{n\in\mathbb{N}} X_n$ such that $X_n \in \mathcal{B}$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$.

Unless otherwise stated, all measures are assumed to be σ-finite. In fact, we will usually only deal with probability measures, since we can also normalize a finite measure to a probability measure by $\mu \mapsto \mu/\mu(X)$.

Lastly, even though μ is only defined on the σ-algebra $\mathcal{B}$, we can slightly extend μ to a larger σ-algebra.

Definition 1.10. Let $(X, \mathcal{B}, \mu)$ be a measure space. A subset $Z \subseteq X$ is said to be *μ-null* if there exists some $Z' \in \mathcal{B}$ such that $Z \subseteq Z'$ and $\mu(Z') = 0$. We write Null_μ for the set of all μ-null subsets of X. A subset $A \subseteq X$ is said to be *μ-conull* if $X \setminus A$ is μ-null.

Definition 1.11. Let $(X, \mathcal{B}, \mu)$ be a measure space. A subset $A \subseteq X$ is *μ-measurable*[4] if there exists some $B \in \mathcal{B}$ and some μ-null set Z such that $A = B \cup Z$. We write Meas_μ for the set of all μ-measurable sets.

It is an exercise that $\mathrm{Meas}_\mu = \langle \mathcal{B} \cup \mathrm{Null}_\mu \rangle_\sigma$. Moreover, μ admits a unique extension to a map $\overline{\mu} : \mathrm{Meas}_\mu \to [0, \infty]$, called the *completion* of μ, and this measure satisfies $\mathrm{Meas}_{\overline{\mu}} = \mathrm{Meas}_\mu$. HINT: $\overline{\mu}(B \cup Z) \coloneqq \mu(B)$.

Definition 1.12. A measure μ is *complete* if $\overline{\mu} = \mu$.

For convenience, we will always assume that measures are complete. Neither measures λ nor μ_p in Examples 1.6 and 1.7 are complete, so we tacitly extend them.

We end with some easy exercises on measures; please read/prove them, as they will be used freely in the future; they are roughly ranked by difficulty. Throughout, let $(X, \mathcal{B}, \mu)$ be a measure space and let $A_n \in \mathcal{B}$.

Exercise 1.13 (Monotonicity)**.** If $A_1 \subseteq A_2$, then $\mu(A_1) \le \mu(A_2)$.

Deduce that if μ is finite, then μ is a bounded function. (Are σ-finite measures bounded?)

Exercise 1.14 (Inclusion-exclusion)**.** For any $A_1, A_2 \in \mathcal{B}$, we have $\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)$.

[2]This goes up the *Borel hierarchy*, studied in *Descriptive Set Theory*; see [Kec95].

[3]We will not prove this fact, but it is an application of Carathéodory's Extension Theorem; see [Anu23, Lecture 4].

[4]Very confusing terminology. One might think that elements of $\mathcal{B}$ are the 'measurable' ones, but this removes μ from the picture. In general, there are much more μ-measurable sets that there are sets in $\mathcal{B}$. Indeed, there are $2^{\aleph_0}$-many Borel sets on $\mathbb{R}$, but there are $2^{2^{\aleph_0}}$-many λ-measurable sets!

Exercise 1.15 (Continuity $\nearrow$)**.** If $(A_n)_{n\in\mathbb{N}}$ is increasing, then $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu(A_n)$.

Exercise 1.16 (Continuity $\searrow$)**.** If $(A_n)_{n\in\mathbb{N}}$ is decreasing and $\mu(A_1) < \infty$, then $\mu(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu(A_n)$.

Exercise 1.17. Show that $\lambda(\mathbb{Q}) = 0$. Hint: What is the Lebesgue measure of singletons?

Let P be a property of some points in X. We say that P *holds μ-almost everywhere* (or *μ-almost surely*) if $\{x \in X : x \text{ satisfies } P\}$ is μ-conull.

Exercise 1.18 (Borel-Cantelli Lemmas)**.** Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of μ-measurable sets.

1. If $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then μ-almost every $x \in X$ lives in at-most finitely-many A_n.
2. (Measure Compactness). If $\mu(X) < \infty$ and there exists $\varepsilon > 0$ such that $\mu(A_n) \geq \varepsilon$ for all $n \in \mathbb{N}$, then at least an ε-measure set of $x \in X$ lives in infinitely-many A_n's.

For measurable spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$, define $\mathcal{B}_1 \otimes \mathcal{B}_2 \coloneqq \langle B_1 \times B_2 : B_i \in \mathcal{B}_i \rangle_\sigma$.

Exercise 1.19. Show that if X_i are second-countable topological spaces, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$.

Exercise 1.20. Let X be a topological space. A *Cantor set* is a subset $C \subseteq X$ homeomorphic to $2^{\mathbb{N}}$.

1. Show that the 'middle-thirds Cantor set' $C \subseteq [0, 1]$ is a Cantor set as in the above definition. Moreover, show that $\lambda(C) = 0$. Hint: Recall the construction $C = \bigcap_{n\in\mathbb{N}} C_n$ and use continuity.
2. Define a Cantor set $C \subseteq [0, 1]$ with positive Lebesgue measure. Hint: fatten the standard construction.

A measurable set $A \subseteq X$ is said to be an *atom* if there is no subset $B \subseteq A$ with $0 < \mu(B) < \mu(A)$. For example, singletons $\{x\}$ are atoms under the Dirac measure δ_x. More generally:

Exercise 1.21. If $(X, \mathcal{B}, \mu)$ is a σ-finite measure space, $\mathcal{B}$ is *countably generated* (i.e., $\mathcal{B} = \langle \mathcal{B}_0 \rangle$ for some countable $\mathcal{B}_0 \subseteq \mathcal{P}(X)$), and *separates points* (i.e., if $x \neq y$, then there exists $B \in \mathcal{B}$ such that $x \in B \not\ni y$.), then every atom $A \in \mathcal{B}$ is a singleton.

2. Lecture 2 (Samy Lahlou): Crash course on Measure Theory, Part II

TODO: Lebesgue integral, L^p, measurable functions, simple functions

Further reading. [Anu23, Lectures 9 to **TODO**] and [Fol99, Chapters 2 and 3].

3. Lecture 3 (Peng Bo): Introduction to Ergodic Theory

TODO: intro

Further reading. None!

References

[Anu22] Anush Tserunyan, *Topics in Ergodic Theory and Measured Group Theory*, available at `https://www.math.mcgill.ca/atserunyan/Courses/2022_W.Math594.Erg&MsGrp/`.

[Anu23] ———, *Advanced Real Analysis 1*, available at `https://www.math.mcgill.ca/atserunyan/Courses/2023F.Math564.Analysis1/`.

[Fol99] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts, John Wiley & Sons, 2013, 1999.

[Kec95] Alexander S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, Springer New York, NY, 1995.

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