Math 583: Geometric Group Theory

Problem Sets

1 PS1

Problem 1.1. *Let $\mu(G)$ the min cardinality of a generating set of G. Show $\mu(G_1 * G_2) = \mu(G_1) + \mu(G_2)$.*

Solution 1.1. *Let $X = \{g_i\}$, $Y = \{h_i\}$ be generating sets of minimal size. Then, $X \cup Y$ is a generating set for $G_1 * G_2$ by definition. $|X \cup Y| = |X| + |Y|$ so we proved $\leq$.*
Now, let $F \twoheadrightarrow G_1 * G_2$ be a presentation with F the free group of rank $\mu(G_1 * G_2)$. By Gruško, this surjection decomposes as $F_i \twoheadrightarrow G_i$, so $\mu(G_i) \leq rk(F_i)$. But $F_1 * F_2 = F$, so $\mu(G_1) + \mu(G_2) \leq rk(F_1) + rk(F_2) = rk(F) = \mu(G_1 * G_2)$. This is the second inequality, so equality has been shown.

Problem 1.2. *Let G a f.g. group. Show that for some n, $G = G_1 * ... * G_n$ for G_i indecomposables.*

Solution 1.2. *Suppose G is indecomposable. Then we are done. Suppose not, so $G = A * B$. Repeat the argument on A and B, and so on. The algorithm is bound to terminate since, by problem 1, the rank of the groups strictly decrease as we go down the tree.*

Problem 1.3. *Kuroš theorem: if $H < G_1 * G_2$, then H is the free product of free groups and conjugates of subgroups of G_i.*

Solution 1.3. *Let X_i presentation complexes of G_i, X the dumbell with X_i. By van Kampen, $\pi_1(X) = G$. Covering spaces of X give subrgoups of G. Let $p : (Y, y) \to (X, x)$ a covering space, with $p_*(\pi(Y, y)) = H$.*

Decompose $p^{-1}(X_i) = \cup Y_{ij}$ a disjoint union of sheets. Then, we claim there exists Z, T 1-subcomplexes of Y such that: $y \in Z, T$, T is a tree such that $\pi_1(Y, y) \cong \pi_1(Z, y) * *\pi_1(Y_{ij} \cup T, y)$. Then, Z is a graph so $\pi_1(Z, y)$ is free. Note that $p_*(\pi_1(Y_{ij} \cup T, y)) = p_*(\pi_1(Y_{ij}, y))$ are conjugate subgroup of G.

Construct T in the following way. Choose spanning trees T_{ij} of Y_{ij}, $Z = p^{-1}(bar) \cup \cup T_{ij}$. Let T a spanning tree of Z so that T contains all the T_{ij}. This can be done by quotienting Z by all T_ij and taking that quotient's spanning tree. The vertices are pulled back to the tree itself. Note that y is in both Z and T.

Apply Van Kampen to $\{Z, Y_{ij} \cup T\}$. For that we need all intersections to be trees, which is easy to check. Thus we get the free decomposition of $\pi_1(Y, y)$, and so the proof is complete.

Problem 1.4. *Let $G = G_1 * G_2$. If $[g, h] \in G_1$ is nontrivial, then $g, h \in G_1$.*

Solution 1.4. *Consider $H = \langle g, h \rangle$, which by Kuroš is $H = (H \cap G_1) * (\text{conjugates of subgroups and free groups})$ where the first term is nontrivial since it contains $[g, h]$. If H is not inside G_1, then C is not trivial. By Gruško, the factors are both 1-generated. Then $H \cap G_1$ and C are cyclic (maybe infinite).*

Consider $H \to G \cap G_1$ mapping $C \mapsto 0$. But $H \cap G_1$ is abelian, so $[g,h] \mapsto 0$. But $[g,h]$ was a nontrivial element of $H \cap G_1$, so it should map to something non zero. We have contradicted the assumption that C was non trivial. Thus $H = H \cap G_1$, so we are done.

Problem 1.5. *Show that each indecomposable subgroup of $G_1 * G_2$ is either $\mathbb{Z}$ or contained in a conjugate of G_i.*

Solution 1.5. *This follows from Kuroš theorem.*

Problem 1.6. *Show that if $G = G_1 * G_2$ and for $w \in G$, $w^{-1}G_1w \cap G_i$ is not empty, then $i = 1$ and $w \in G_1$.*

Solution 1.6. *Let $w^{-1}gw \in G_i$, for some nontrivial $g \in G_1$. Since $w \in G_1 * G_2$, we can write $w = w_0\widetilde{w}$ for $w_0 \in G_1$, and $\widetilde{w}$ a reduced word starting with an element of G_2. Then, $w^{-1}gw = \widetilde{w}^{-1}g'\widetilde{w}$ for $g' = w_0^{-1}gw_0$. In particular, g' is still in G_1.*

But $\widetilde{w}^{-1}g'\widetilde{w}$ is an already reduced word inside G_1, so no elements of G_2 can appear in the expression. Thus, $\widetilde{w} = 1$. Hence, $w = w_0 \in G_1$, and so $w^{-1}G_1w = G_1$. Since $G_1 \cap G_2$ is empty, $w^{-1}G_1w \cap G_i = G_1 \cap G_i$ nontrivial forces $i = 1$.

Problem 1.7. *Let G be finitely generated. If $G = G_1 * ... * G_n = H_1 * ... * H_m$, then $m = n$ and G_i is isomorphic to a conjugate of H_i up to permutation.*

Solution 1.7. *If all G_i are $\mathbb{Z}$, then G is free and so the subgroups H_j are also $\mathbb{Z}$. By equality of rank, we obtain the result.*

Suppose, after reordering, that G_i is not $\mathbb{Z}$ for $i \leq k$, and H_j is not $\mathbb{Z}$ for $j \leq k'$. Then, G_1 is a (not free) subgroup of the free product of the H_i, so $G_1 < wH_1w^{-1}$ for some $w \in G$ (we choose H_1 without loss of generality). Similarily, $H_1 < vG_iv^{-1}$ for some i, so we have. Notice $wvG_i(wv)^{-1} \cap G_1$ is nontrivial, so by Q6, we have $wv \in G_1$. Now,

$$G_1 < wvG_i(wv)^{-1} < wH_1w^{-1} < G_1$$

Hence, G_1 and H_1 are conjugate, thus isomorphic. Apply the same idea to every G_i (up to reordering the H_i) for $1 \leq i \leq k$, so that every G_i is isomorphic to some H_j. We need to check that no two G_i go to the same H_i. This is automatic because if it's the case, then G_i is conjugate to G_j and Q6 again, $i = j$.

Note now that $G_1 * ... * G_k \cong H_1 * ...H_k$. Letting G' be the normal closure of $G_1 * ... * G_k$ and H' similarily, we have $G/G' = G/H' = G_{k+1} * ... * G_n = H_{k+1} * ... * H_m$. This forces H_i to be $\mathbb{Z}$ for $k+1 \leq i \leq m$, and by a rank argument, $n - k = m - k$, so $n = m$.