A survey of Amenability and groups with finite Tarski Number

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1 Amenability

In the beginning, one studies algebra and analysis in their own seperate universes. However, one later learns that uniting the two worlds create very beautiful objects of study and fruiteful techniques. Amenability is an analytic property on an algebraic/geometric object. Namely: having a "good" measure.

Definition 1.1. Let X be a set. A *finitely additive probability measure* on X is a map that assigns to any subset of X a positive real number. Precisely, $\mu : \mathcal{P}(X) \to \mathbb{R}^{\geq 0}$ is a f.a.p.m. if:

- $\mu(X) = 1$ (probability measure)
- Given $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$ (finitely additive)

In particular, note $\mu(A) \in [0, 1]$ for any $A \subset X$, so μ is really a map to $[0, 1]$

Note that μ is a measure on a set, there is no algebra involved. Our next definition merges the analytic object with the algebraic structure of a group, in a "good" way.

Definition 1.2. Let Γ be a group acting on a set X. A f.a.p.m. on X is *invariant under* Γ*-action* if $\mu(gA) = \mu(A)$ for all $A \subset X$ and $g \in \Gamma$.

When $\Gamma \curvearrowright X$, we write $gA = \{g \cdot a : a \in A\}$. Even though this definition makes sense for any set X on which Γ can act, we will most often consider $X = \Gamma$, and the action of Γ on itself to be left-multiplication. In this case, we will call μ an invariant f.a.p.m. on Γ.

Example 1.1. *Let Γ a finite group acting on itself by right multiplication, and $\mu(A) = \frac{|A|}{|\Gamma|}$ the scaled cardinality of A. Then, μ is an invariant f.a.p.m. since :*

- $\mu(X) = \frac{|X|}{|X|} = 1$

- *Given $A \cap B = \emptyset$, then $\mu(A \cup B) = \frac{|A \cup B|}{|X|} = \frac{|A|}{|X|} + \frac{|B|}{|X|} = \mu(A) + \mu(B)$*
- *The map $a \to g \cdot a$ is a bijection between A and gA, so $\mu(A) = \mu(gA)$*

We have effectively upgraded groups to measure spaces, in such a way that the two structures dialogue nicely.
The condition that Γ is countable may seem uneeded, and indeed one can remove it. We will however only work in this setting to avoid technical complications from measure theory.

One might ask if this compatibility condition is too soft. Can every group be equipped with such a measure?

Proposition 1.1. *There are no invariant f.a.p.m. on $F_2 = \langle a, b \rangle$, the free group on 2 elements.*

Proof. Define W_g to be all reduced words in F_2 which start by g. Suppose there was such a μ. Now, $a^{-1}W_a \sqcup W_{a^{-1}} = F_2$, Then, by the properties of μ:

$$\begin{aligned} 1 = \mu(F_2) &= \mu(a^{-1}W_a) + \mu(W_{a^{-1}}) \\ &= \mu(W_a) + \mu(W_{a^{-1}}) \\ \implies \frac{1}{2} &= \mu(W_{a^{-1}}) = \mu(W_a)) \end{aligned} \tag{1}$$

The same argument shows $\frac{1}{2} = \mu(W_{b^{-1}}) = \mu(W_b)$.
Now, note

$$F_2 = \{1\} \sqcup W_a \sqcup W_{a^{-1}} \sqcup W(b) \sqcup W_{b^{-1}}$$

$$\begin{aligned} 1 = \mu(F_2) &= 0 + \mu(W_a) + \mu(W_{a^{-1}}) + \mu(W_b) + \mu(W_{b^{-1}}) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\ &= 2 \end{aligned} \tag{2}$$

Contradiction. □

Hence having an invariant f.a.p.m. is a property enjoyed by only certain groups.

Definition 1.3. A group Γ is *amenable* if it admits an f.a.p.m. invariant under the left multiplication action.

Our two examples showed that: (i) all finite groups are amenable and (ii) F_2 is not amenable. In this sense, amenability is sometimes seen as a "generalization of finiteness".
There is a plethora of equivalent characterization of amenability. This property is at the crossroad of many areas of mathematics, which is natural since it's very definition is a combination of group and measure theory. The goal of this paper is to present one of these characterizations that showcases the "niceness" of amenable groups.
Before doing so, we will give more examples of amenable groups.

1.1 Groups that *are* amenable

Denote by *Amen* the class of all amenable groups, *Fin* the class of finite groups. Example 1.1 shows *Fin* $\subset$ *Amen*.
In this section, we discuss some closeness properties of the class *Amen*. Most of these fall under the hood of the next proposition.

Proposition 1.2. *Let G and H be amenable. Then, let K a group such that the sequence is exact. Say K is an extension of G by H. Then, K also is amenable.*

For example, finite products and semidirect products arise in this way.
For now, we don't know any infinite amenable groups. It turns out that $\mathbb{Z}$ is amenable, and an explicit measure is given in (TODO) Kechris, but the construction is technical so we avoid it.
Assuming this, we can show that every abelian group is amenable. (Rememeber that we only consider finitely generated groups). In this sense, we can also think of amenable groups are "generalizing abelian groups".

Proposition 1.3. *All abelian groups are amenable.*

Proof. By the fundamental theorem of abelian groups, Γ is a direct sum of finite cyclic group and $\mathbb{Z}^d$ for some number d. Since $\mathbb{Z}$ and finite groups are amenable, a direct product of these will also be. □

2 Paradoxical Decomposition and The Tarski Number

The classical example of a non amenable groups is F_2. The weird fact that created the contradiction in our proof was that F_2 could be obtained by gluing 2 sets of same measure, but at the same time by 4 sets of this same measure. This property is in fact so weird that we call it paradoxical.

Definition 2.1. Let Γ be a group. An *(n,m)-paradoxical decomposition* of Γ is a collection of subsets $A_1, ..., A_n$ and $B_1, ..., B_m$ along with group elements $g_1, ..., g_n$ and $h_1, ..., h_m$ such that :

- the A_i and B_j are all pairwise disjoint

- $\Gamma = \bigsqcup_{i=1}^n A_i \sqcup \bigsqcup_{j=1}^m B_j$

- $\Gamma = \bigsqcup_{i=1}^n g_i A_i = \bigsqcup_{j=1}^m h_j B_j$

The sets $S_1 = \{g_1, ..., g_n\}$ and $S_2 = \{h_1, ..., h_m\}$ are called the *translating sets* of the decomposition. A group which admits a (n, m)-decomposition is called (n, m)-paradoxical.

While this definition emphasizes the "paradoxicality" of the decomposition, it is not very practical to deal with. In practice, one would rather not have to show that all the sets are disjoint. The following lemma makes our lives easier.

Lemma 2.1. *Let Γ a group. The following are equivalent, and thus all serve as definitions of paradoxical decomposition. The numbers n and m are the same in all definitions, and thus depend only on the decomposition.*

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$$\begin{aligned}\Gamma &= \bigsqcup_{i=1}^n A_i \sqcup \bigsqcup_{j=1}^m B_j \\ &= \bigsqcup_{i=1}^n g_i A_i = \bigsqcup_{j=1}^m h_j B_j\end{aligned} \tag{3}$$

- $\Gamma = \bigsqcup g_i A_i = \bigsqcup h_j B_j$
- $\Gamma = \cup g_i A_i = \cup h_j B_j$

For example, our proof of the non-amenability of F_2 relied in essence on the paradoxical decomposition given by $W_a, W_{a^{-1}}$ and $W_b, W_{b^{-1}}$, with translating sets $S_1 = \{1, a^{-1}\}$ and $S_2 = \{1, b^{-1}\}$. The first line in the lemma is not respected by this decomposition, since the union of the W sets does not contain 1. One might then ask if there is a "simpler" paradoxical decomposition of Γ; that is one that involves less sets. By the lemma, $F_2 = W_a \sqcup aW_{a^{-1}} = W_b \sqcup bWb^{-1}$ implies that F_2 is (2,2)-paradoxical.

Proposition 2.1. *If S_1, S_2 are translating sets, then gS_1 and hS_2 are also translating sets for any $g, h \in \Gamma$.*

Proof. $G = \bigsqcup g_i A_i$, so $g\Gamma = g \bigsqcup g_i A_i = \bigsqcup g(g_i A_i) = \bigsqcup (gg_i)A_i$. □

Thus, one can always assume that S_i both contain the identity element. If S_1 contained only the identity, then $\Gamma = \sqcup_{g \in S_1} A_i = 1A_1 = A_1$, so there cannot be B_i's disjoint from A_1. Thus, $|S_i| \geq 2$.
The previous discussion has led us to seek minimal decompositions. We have however seen that a decomposition must contain at least 4 pieces.

Definition 2.2. The *Tarski number* of a group Γ is defined as $\tau(\Gamma) = \inf\{n + m\}$, where the infimum is taken over all $(n, m)-$ paradoxical decompositon of Γ.

In this new language, $\tau(F_2) = 4$, and $\tau(\Gamma) \geq 4$ for all Γ.
When we defined invariant f.a.p.m., we asked if every group could be equiped with such a measure. Now, we have seen groups that admit paradoxical decompositions, so we ask if this is the case for all groups. The answer, perhaps surprisingly, is that amenability is *exactly* the property that prevents paradoxicality.

Theorem 2.1 (Tarski Alternative). *A group is amenable if and only if it does not admit a paradoxical decomposition.*

Proof. First, assume a group has both a decomposition A_i, B_j and an invariant f.a.p.m. (it is amenable. Then:

$$\begin{aligned} 1 = \mu(\Gamma) &= \mu\left(\bigsqcup_{i=1}^{n} A_i \sqcup \bigsqcup_{j=1}^{m} B_j\right) \\ &= \sum_{i=1}^{n} \mu(A_i) + \sum_{j=1}^{m} \mu(B_j) \\ &= \sum_{i=1}^{n} \mu(g_i A_i) + \sum_{j=1}^{m} \mu(h_j B_j) \\ &= \mu\left(\bigsqcup_{i=1}^{n} g_i A_i\right) + \mu\left(\bigsqcup_{j=1}^{m} h_j B_j\right) \\ &= \mu(\Gamma) + \mu(\Gamma) = 2 \end{aligned} \tag{4}$$

The other direction involves Hall's marriage lemma, a graph theoretical result, and we refer the reader to (TODO). □

The careful reader will have noticed that we defined the Tarski number with an infimum. This is because, for Γ amenable, we define $\tau(\Gamma) = \infty$.

We now center our attention to the study of this Tarski number. We showed $\tau(\Gamma) \geq 4$, but a natural question to ask is if for all $n \geq 4$, there exist a group with Tarski number equal to n. It turns out this question is still open, but we have a number of partial results.
We will first show some fundamental results, before specifying to more involved problems.

2.1 General results

We begin by a fact we will use over and over in our proofs. Computing the Tarski number is often very hard, and we proceed with bounds.

Proposition 2.2. *Let H be a subgroup of G. Then $\tau(G) \leq \tau(H)$.*

Proof. If H is amenable, the claim is empty. So let A_i and B_j be a decomposition of H with transalting sets g_i and h_j.
Now, there exists elements $t_k \in G$ ($[G : H] \in \mathbb{N} \cup \infty$ of them) so that $G = \bigcup t_k H$.
Let $A'_i = \bigcup_k t_k A_i$ and $B'_j = \bigcup_k t_k B_j$. Then, $G = \bigcup t_k H = \bigcup t_k(\bigcup g_i A_i) = \bigcup g_i A'_i$, and similarly for B'_j. Hence, the same translating sets yield a decomposition of G. Thus $\tau(G) \leq n + m = \tau(H)$. □

We presented F_2 as the prime example of a non-amenable group. It turns that, in the sense of the Tarski number, it is the "most" paradoxical group. That is, the paradox is very obvious; it involves the least amount of pieces.

Proposition 2.3. *A group has tarski number 4 if and only if it contains a subgroup isomorphic to F_2.*

Proof. Let Γ contain a copy of F_2. Then, by the previous proposition, $\tau(\Gamma) \leq 4$. But we know $\tau(G) \geq 4$ for any groups.
The other direction is a direct application of the ping pong lemma. We refer the reader to (TODO).
□

(TODO VON NEUMAN?) For a long time, it was conjectured that all non-amenable group contained a copy of F_2. If true, this would make the Tarski number a very weak measure, since it would always be equal to 4 or infinity.
Finding a paradoxical group whose Tarski number isn't 4 is hard. However, Golod and Shafarevich found one with Tarski number 6. The following proposition gives one side of the proof.

Proposition 2.4. *If Γ is torsion (i.e. all elements have finite order), then $\tau(\Gamma) \geq 6$.*

Proof. We show that $\tau(\Gamma)$ is not 4 nor 5.
If $\tau(\Gamma) = 4$, then it contains F_2 and hence is not torsion.
If $\tau(\Gamma) = 5$, let X_1, X_2 and Y_1, Y_2, Y_3 be the pieces with respective translating sets $1, g$ and $1, h_1, h_2$.
Assume that $\Gamma = X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup Y_3$.
Since Γ is torsion, let $|g| = n$. Now,

$$X_1 = \Gamma \backslash g X_2 = g(\Gamma \backslash X_2) = g(X_1 \cup Y_1 \cup Y_2 \cup Y_3)$$

In particular, $gX_1 \subset X_1$ and $g(Y_1 \cup Y_2 \cup Y_3) \subset X_1$. Repeating, we get $g^k X_1 \subset X_1$ for all k. Now, $X_1 \supset g^{n-1} X_1 \supset g^{n-1}(g(Y_1 \cup Y_2 \cup Y_3)) = 1(Y_1 \cup Y_2 \cup Y_3)$.
The last inclusion is a contradiction since X_1 is disjoint from all Y_i. Hence, $\tau(\Gamma) \neq 5$. □

Golod and Shafarevich found groups who admitted a paradoxical decomposition with 6 pieces, and were torsion. By this proposition, these groups thus have Tarski number 6.

We finish this section by stating some results that we will not explore deeper, but lead to intresting questions.

Proposition 2.5. *The following results hold:*

- *If $H \lhd G$, then $\tau(G/H) = \tau(G)$.*
- *If $G = H \times K$, then $min(\tau(H), \tau(K)) \leq 2(\tau(G) - 1)^2$*
- *For $G \times G$, we have $\tau(G \times G) \leq \tau(G)^2$*

For the last bullet, it is conjectured that $\tau(G \times G) = \tau(G)$.

The next result gives a bound on the growth of the Tarski number relative to index. Note the similitude with the Nielson-Schraier formula for the rank of a free group.

Proposition 2.6. *If H is a finite index subgroup of G, the $\tau(H) - 2 \leq [G : H](\tau(G) - 2)$.*

We then show how involved the computation of the number can get. Gili Golan (TODO) has showed existence of a group with Tarski number 5, using techniques from descriptive set theory (cost) and the L^2 Betti numbers. Here is the key result:

Proposition 2.7. *Let $G = \langle a, b, c | R \rangle$, a quotient of F_3 where a has infinite order. If $cost(G) \geq 2.5$, then G has a paradoxical decomposition with translating sets $\{1, a\}$ and $\{1, b, c\}$.*

Thus, a group with this presentation who does not contain F_2 and who's cost is greater than 2.5 has Tarski number 5.

We finish this section with a result that shows how little is known in this topic. Let T the set of number who arise as the Tarski number of some group. It is known that T is infinite, and that $4, 5, 6$ are in T. To the author's knowledge, no other numbers are known to be in T.

2.2 The Banach-Tarski Paradox

The Banach-Tarski paradox is perhaps one of the most notorious paradox in mathematics. In layman's terms, it states that one can cut a sphere in such a way that rearranging the pieces yields two times the original sphere, effectively creating volume. Unfortunately for us humans, this trick only exists in the realm of mathematics, as it involves the axiom of choice: the pieces are not constructible with a kitchen knife.
The idea of creating volume is very similar to that of paradoxicality in a group. In fact, the heart of the paradox lies on the group of $3D$ rotations $SO(3, \mathbb{R})$ being paradoxical. We begin by proving this result, and then show the construction of the Banach-Tarski paradox. The reader can harmlessly skip the lemma.

Lemma 2.2. *The Tarski number of $SO(3, \mathbb{R})$ is 4.*

Proof. Note that any rotation of space can be given by fixing a point on the unit circle and rotation around the axis generated by this point and the origin by an angle of ϕ. Hence, all elements in $SO(3,\mathbb{R})$ are conjugate to one of the form

$$\begin{pmatrix} cos(\phi) & sin(\phi) & 0 \\ sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can check that the elements

$$A = \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

generate a subgroup isomorphic to F_2. Hence, $\tau(SO(3,\mathbb{R})) = 4$ □

Note that $SO(2,\mathbb{R})$ is abelian, hence amenable. Note also that the group of translation in any dimension is abelian. Thus, the paradox is not possible in 2 dimensions, and the "weirdness" happens when using rotation.
Here is the statement of the Banach-Tarski paradox.

Theorem 2.2. *The sphere S^2 has subsets $U_1, ..., U_4$ such that $S^2 = \bigcup U_i = U_1 \cup gU_2 = U_3 \cup hU_4$ for some elements $g, h \in SO(3,\mathbb{R})$.*

Proof. By the lemma, $SO(2,\mathbb{R})$ has a subgroup isomorphic to the free group on 2 elements, which we will denote $\hat{F_2} = \langle A, B \rangle$.
Let W_g be the subset of $\hat{F_2}$ consisting of all those words which start with the letter g. Then,

$$\begin{aligned} \hat{F_2} &= W_A \cup W_{A^{-1}} \cup W_B \cup W_{B^{-1}} \cup \{1\} \\ &= W_A \cup AW_{A^{-1}} = W_B \cup BW_{B^{-1}} \end{aligned}$$

Now, let $U = \{p \in S^2 : \exists C \in SO(3,\mathbb{R}) \text{ s.t. } Cp = p\}$, the set of points which are fixed by some element in $\hat{F_2}$. Note that for all rotation, there are exactly 2 fixed points, namely those on the axis. Thus, since $\hat{F_2}$ is countable, so is U.
Since S^2 is uncountable, there is a point in S^2 not in U. Let $\mathbf{Y} = \{\hat{F_2}z : z \in Z\}$ be the set of orbits, and let Z be a set of representatives. Note that defining Z requires the axiom of choice. This is the step in the proof that makes the construction impossible with a kitchen knife.
Now, $S^2 = U \cup \bigcup \hat{F_2}z = U \cup \hat{F_2}Z = U \cup W_A Z \cup W_{A^{-1}}Z \cup W_B Z \cup W_{B^{-1}}Z \cup Z$. Also,

$$\begin{aligned} S^2 \backslash U &= W_A Z \cup W_{A^{-1}}Z \cup W_B Z \cup W_{B^{-1}}Z \cup Z \\ &= W_A Z \cup AW_{A^{-1}}Z = W_B Z \cup BW_{B^{-1}}Z \end{aligned}$$

At this point, we almost have our decomposition. Let $\hat{W_A} = W_A Z$ etc... That is, W_A is a collection in $\hat{F_2}$, but $\hat{W_A}$ is a collection of points in S^2.Indeed, the pieces are $\hat{W_A}, \hat{W_{A^{-1}}}$ and same with B, translating elements A and B. All we are missing are the points in U, a countable collection. The rest of the proof is simply putting these missing points in the correct piece.

Let $x \in U$, and consider it's orbit $O = \hat{F_2}x$. (TODO)

□

2.3 The Burnside Groups

Up to now, all our results gave lower bounds for the Tarski number. In this section, we develop an upper bound, which can be computed using the cogrowth.
We will need a lot of new technology from geometric group theory. We fix a group G, given as a quotient $\pi : F_n \to G$ with kernel K.

Definition 2.3. Let $l : F_n \to \mathbb{N}$ be the wordlength function. Let $\sigma_t = \sigma_t(K) = |\{w \in K : l(w) = t\}$. The cogrowth of a group is $\alpha_K = \limsup_{t\to\infty} (\sigma_t)^{\frac{1}{t}}$.

Proposition 2.8. *For G a quotient of F_m, we have $\sqrt{2m-1} \leq \alpha_K \leq 2m-1$*

Proof. Note that if $K' \leq K$, then $\sigma_t(K') \leq \sigma_t(K)$. So, $\alpha_1 \leq \alpha_K \leq \alpha_{F_m}$. Hence, we compute both to get a lower and upper bound on the cogrowth.
A computation shows $\sigma_t(F_m) = |\{w \in F_m : l(w) = t\}| = \frac{m(2m-1)^t-1}{m-1}$ (look at the Cayley graph). So:

$$\begin{aligned}
\alpha_{F_m} &= \limsup_{t\to\infty} \left(\frac{m(2m-1)^t-1}{m-1}\right)^{\frac{1}{t}} \\
&= \limsup_{t\to\infty} \left(\frac{m}{2m-1}\right)^{\frac{1}{t}} (2m-1) - \left(\frac{1}{m-1}\right)^{\frac{1}{t}} \\
&= 2m-1
\end{aligned}$$

Similarly, one finds that $\alpha_1 = \sqrt{2m-1}$, yielding the result. ∎

We used that the ball of radius K in F_m has size $b_m(K) = \frac{m(2m-1)^K-1}{m-1}$.
There is a way to tell if a group is amenable from it's cogrowth, due to Kesten.

Proposition 2.9. *Define $\rho = \bar{\rho}_m(\alpha) = \frac{\sqrt{2m-1}}{2m}\left(\frac{\sqrt{2m-1}}{\alpha} + \frac{\alpha}{\sqrt{2m-1}}\right)$.
Kesten's criterion states that G is amenable if and only if $\rho = 1$.*

(TODO) The value ρ is called the spectral radius, and we will give a better definition later. The criterion can be restated since $\rho = 1$ if and only if $\alpha = 2m-1$.

Next, we define the isoperimetric constant.

Definition 2.4. The perimeter of a subset of a graph is the set of vertices exactly one edge away from the set: $\partial F = N_1(F) \backslash F = \{x \in V(G) : d(x, F) = 1\}$.
Then, the isoperimetric constant of a graph X is $i(X) = inf \frac{|\partial F|}{F}$ where the infimum is taken over all nonempty finite sets $F \subset V(X)$.

Note already that if X has a finite connected component, take it to be F and get $i(X) = 0$. We can get a lower bound on $i(X)$ by using the spectral radius.

Definition 2.5. Let $\rho^n(x, y)$ be the probability of walking from x to y after taking n random steps. The spectral radius is defined to be $\rho(X) = \limsup_{n\to\infty} (\rho^n(x, y))^{\frac{1}{n}}$ (it is independent from x, y).

For example, if X is the infinite regular graph of degree $d \geq 2$, $\rho(X) \geq 2\frac{\sqrt{d-1}}{d}$.

Proposition 2.10. $i(X) \geq 4\frac{2-\rho(X)}{\rho(X)}$. *Hence, define $\bar{i}(\rho) = 4\frac{1-p}{p}$ so that $i \geq \bar{i}$.*

Now, we can give equivalent conditions for paradoxicality in terms of these new tools.

Theorem 2.3. *Let X be a connected graph of bounded degree. The following are equivalent:*

- *X is paradoxical*

- $i(X) > 0$

- $\rho(X) < 1$

We have one more thing to define before reaching the Tarski number upper bound.

Definition 2.6. Call K a doubling characteristic distance if $|B_K(F)| \geq 2|F|$ for all finite subsets $F \subset V(X)$.

It turns out that if $i(X)$ is strictly positive, the constant $K = \left\lceil \frac{ln(2)}{ln(1+i(X))} \right\rceil$ is a doubling characteristic constant.

We are finally ready to state the theorem.

Theorem 2.4. *Let α_G be the cogrowth of $\pi : F_m \to G$, and X the Cayley graph of G. For all $\alpha \in [\alpha_G, 2m-1]$, we have $\tau(G) \leq 2b_m(K(i(\rho_m(\alpha))))$.*

Proof. We have $i(x) \geq 4\frac{1-\rho(x)}{\rho(x)} = \bar{i}(\rho_m(\alpha))$. Then $K_X = \left\lceil \frac{\ln(2)}{\ln(1+\rho(x))} \right\rceil \leq K(i(\rho_m(\alpha)))$.
Now, let $\phi_i : X_i \to V(X)$ for X_1, X_2 a partition of V(X). Define a decomposition in the following way:

$$X_1 = \bigsqcup A_{1,g} \text{ for } A_{1,g} = \{x \in X_1 : \phi_1(x) = gx\}$$

$$X_2 = \bigsqcup A_{2,g} \text{ for } A_{2,g} = \{x \in X_2 : \phi_2(x) = gx\}$$

where the unions are taken over all g in the unit ball around the identity vertex.
Now, $|B^G_{K_X}(1)| \leq |B^{F_m}_{K_X}(1)| = b_m(k)$, so there are at most $2b_m(k)$ pieces in the decompositions. Then,

$$\tau(G) \leq 2b_m(K_X) \leq 2b_m(K(i(\rho_m(\alpha))))$$

To check that the A_i, g indeed form a paradoxical decompositions, we check 3 things. First, their union is $X_1 \sqcup X_2 = V(X)$. Then, $A_{1,g}$ is necessarily disjoint from $A_{2,h}$ since $X_1 \cap X_2 = \emptyset$. Also, if $x \in A_{1,g} \cap A_{1,h}$, then $\phi_1(x) = gx = hx$ so $g = h$. Lastly, $\bigsqcup gA_{1,g} = \phi_1(X_1) = V(X)$. Hence the sets are indeed a paradoxical decomposition. ☐

2.4 Group Actions

The study of Tarski numbers of groups is very difficult, and most questions remain without answer. However, if we change our definition a little, the results become way more general. That is, we define the Tarski number of a group action.

Definition 2.7. Let G act on X. A paradoxical decomposition is a partition $P_1, ..., P_m$ and $Q_1, ..., Q_m$ of X and translating sets g_i and h_j such that $X = \bigcup g_i P_i = \bigcup h_j Q_j$. The *Tarski number of a group action is $\tau(G \curvearrowright X) = \inf\{n + m\}$ where the infimum runs over all paradoxical decompositions.*

The same kind of fundamental result as before hold.

Proposition 2.11.
- *If $H \leq G$, then $\tau(G \curvearrowright X) \leq \tau(H \curvearrowright X)$*
- *If $f : X \twoheadrightarrow Y$ is a G-equivariant map, then $\tau(G \curvearrowright X) \leq \tau(G \curvearrowright Y)$*

Proof.
- Let S_1, S_2 translating sets for $H \curvearrowright X$. These sets are also in G, so they work as translating sets for $G \curvearrowright X$.
- Let P_i and Q_j, with translating sets S_1, S_2 be the paradoxical decomposition for $G \curvearrowright Y$. Since f is surjective, we can take preimages. Hence, let $P_i' = f^{-1}(P_i)$ and $Q_j' = f^{-1}(Q_j)$. We claim these sets form a paradoxical decomposition.
  Indeed, $P_i' \cap P_j' = f^{-1}(P_i) \cap f^{-1}(P_j) = f^{-1}(P_i \cap P_j) = f^{-1}(\emptyset) = \emptyset$, and $\bigcup g_i P_i' = \bigcup g_i f^{-1}(P_i) = f^{-1}(\bigcup g_i P_i) = f^{-1}(Y) = X$, where we used G-equivariance.

$\square$

In this setting, the question of what numbers can be realized as the Tarski number of a number has an answer.

Theorem 2.5. *For all $n \geq 4$, there is a group action $G \curvearrowright X$ such that $\tau(G \curvearrowright X) = n$.*

The proof is an explicit construction for each n, due to G. Golan (TODO). Even tho this offers a complete answer, there is still interest in studying this version of things, as it can give results in the original group setting. For example:

Proposition 2.12. *Let $H \triangleleft G$. If $G \curvearrowright G/H$ is paradoxical as an action, then G/H is paradoxical as a group.*

Proof. Let P_i and Q_j, with translating sets S_1, S_2 be the paradoxical decomposition for $G \curvearrowright G/H$. Let $\pi : G \curvearrowright G/H$ be the natural quotient map. Let $S_i' = \pi(S_i)$.
Then, the P_i and Q_j are a paradoxical decomposition of G/H with translating sets S_1' and S_2'. $\square$