SUMMER 2025 READING GROUP ON ERGODIC THEORY

Exercise Sheet 6 (Ludovic Rivet): The density of sets of integers

Definition 0.1. Recall the upper and lower densities of a set of integers:

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{1}{n}|A \cap \{0, 1, ..., n\}|$$

$$\bar{d}(A) = \liminf_{n\to\infty} \frac{1}{n}|A \cap \{0, 1, ..., n\}|$$

If these numbers are equal, they are called the natural density of A.

Problem 0.1. *(i) Find a set $A \subset \mathbb{N}$ of positive density.*

(ii) Show that the set of prime numbers has density 0.

(iii) Show that the density of the set of square-free integers[1] *is $\frac{6}{\pi^2}$.*

Problem 0.2. *Let B be the set of integers with an odd number of digits. Compute its lower and upper density.*

Theorem 0.1 (Polynomial Recurrence)**.** *For all measure preserving system and set A of positive measure, there is an n with*

$$\mu(A \cap T^{n^2} A) > 0$$

Problem 0.3. *Prove the Furstenberg-Sarkozy theorem assuming the polynomial recurrence:*

Let A have positive upper density. Then, there are $x, y \in A$ so that $x - y$ is a square.

[1] An integer is *square-free* if it is not divisible by any squares