SUMMER 2025 READING GROUP ON ERGODIC THEORY

Lecture 1 (Samy Lahlou): Crash course on Measure Theory, Part I

Given a set X, our goal is to define a map $\mu : \mathcal{P}(X) \to [0, \infty]$ that assigns to each subset $A \subseteq X$ a *measure* $\mu(A) \in [0, \infty]$ that 'behaves like the volume of A'. This turns out to be impossible in full generality (and we shall see using ergodic-theoretic methods that this impossibility is for good reason), so we instead restrict to special subsets of X with a nice algebraic (think: 'constructible') structure.

Further reading. [Anu23, Lectures 1 to 5] and [Fol99, Chapter 1].

Definition 1. Let X be a set. A *σ-algebra on X* is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X containing $\varnothing$ and is closed under complements and countable unions. More precisely:

1. (Non-trivial). $\varnothing \in \mathcal{B}$.
2. (Closure under complements). For any $A \in \mathcal{B}$, we have $X \setminus A \in \mathcal{B}$.
3. (Closure under countable unions). For any countable family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$, we have $\bigcup_n A_n \in \mathcal{B}$.

Definition 2. If $\mathcal{B}$ is a σ-algebra on a set X, the pair $(X, \mathcal{B})$ is said to be a *measurable space*.

A useful way to construct a σ-algebra is to start with an arbitrary family $\mathcal{C} \subseteq \mathcal{P}(X)$ and close[1] it under the above three conditions. Abstractly:

Definition 3. The σ-algebra *generated* by $\mathcal{C} \subseteq \mathcal{P}(X)$ is $\langle \mathcal{C} \rangle_\sigma \coloneqq \bigcap \{\mathcal{B} : \mathcal{B} \supseteq \mathcal{C} \text{ is a } \sigma\text{-algebra on } X\}$.

Note that $\langle \mathcal{C} \rangle_\sigma$ is indeed a σ-algebra on X since the intersection of σ-algebras is again a σ-algebra.

Definition 4. Let X be a topological space. The *Borel σ-algebra* on X is $\mathcal{B}(X) \coloneqq \langle \mathcal{T} \rangle_\sigma$, where $\mathcal{T}$ is the topology on X. The elements of $\mathcal{B}(X)$ are called the *Borel sets* of X.

Intuitively, for any topological space X, one would like to 'measure' the Borel sets. This is justified since if one wants a measure compatible with the topology, then one must be able to measure the open sets, and hence also closed sets, countable unions of closed sets (called *F_σ sets*), countable intersections of open sets (called *G_δ sets*), countable intersections of F_σ sets, countable unions of G_δ sets, and so on[2].

Definition 5. A *measure* on a measurable space $(X, \mathcal{B})$ is a map $\mu : \mathcal{B} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and $\mu(\bigsqcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$ for any pairwise disjoint family $\{A_n \in \mathcal{B} : n \in \mathbb{N}\}$.

The triple $(X, \mathcal{B}, \mu)$ is then called a *measure space*. A *Borel measure* is a measure defined on some Borel σ-algebra.

Example 6 (Lebesgue)**.** Equip $\mathbb{R}$ with its usual topology. There is[3] a unique measure $\lambda : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ on $\mathbb{R}$, called the *Lebesgue measure*, such that $\lambda([a, b]) = b - a$ for each $a \leq b$.

Example 7 (Bernoulli)**.** Equip $2 = \{0, 1\}$ with the discrete topology and consider the product topology on $2^{\mathbb{N}}$. For each $p \in [0, 1]$, is a unique measure $\mu_p : \mathcal{B}(2^{\mathbb{N}}) \to [0, \infty]$ on $2^{\mathbb{N}}$, called the *Bernoulli (p) measure*, such that for each word $w \in 2^{<\mathbb{N}}$, we have $\mu_p([w]) = p^{n_1}(1 - p)^{n_0}$ where n_i is the number of $i \in \{0, 1\}$ in w and $[w]$ is the set of all sequences in $2^{\mathbb{N}}$ containing w as a prefix.

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[1] This 'closure' operation can be made precise as follows. Starting with $\mathcal{C}_0 \coloneqq \mathcal{C}$, throw in all the subsets of X that is necessary to satisfy Definition 1 relativized to $\mathcal{C}_0$ to obtain $\mathcal{C}_1$ (that is, let $\mathcal{C}_1$ contain $\varnothing$ and such that if $A \in \mathcal{C}_0$, then $X \setminus A \in \mathcal{C}_1$, and similarly for condition 3). Then, let $\mathcal{C}_2$ be defined by throwing in all the countable unions and complements of sets in $\mathcal{C}_1$. Doing so infinitely-many times and taking the union $\bigcup_\alpha \mathcal{C}_\alpha$ will give us $\langle \mathcal{C} \rangle_\sigma$, but beware that this process must proceed into the transfinite up to $\alpha < \omega_1$, where ω_1 is the first uncountable ordinal; ask your local set theorist why.

[2] This goes up the *Borel hierarchy*, studied in *Descriptive Set Theory*; see [Kec95].

[3] We will not prove this fact, but it is an application of Carathéodory's Extension Theorem; see [Anu23, Lecture 4].

If $p = 0$ (similarly if $p = 1$), then $\mu_p(\xi) \in \{0, 1\}$, and we have $\mu_p(\xi) = 1$ iff $(p, p, p, \ldots) \in \xi$. Thus, all of the measure is concentrated at $(p, p, p, \ldots)$. Measures in which this occurs are called *Dirac measures*.

Example 8 (Dirac)**.** Let X be a set and fix $x \in X$. The *Dirac measure concentrated* at x is the measure $\delta_x : \mathcal{P}(X) \to \{0, 1\}$ defined by $\delta_x(A) \coloneqq 1$ iff $x \in A$, and $\delta_x(A) \coloneqq 0$ iff $x \notin A$.

Definition 9. A measure μ on $(X, \mathcal{B})$ is said to be *finite* if $\mu(X) < \infty$, a *probability measure* if $\mu(X) = 1$, and *σ-finite* if there is a partition $X = \bigsqcup_{n\in\mathbb{N}} X_n$ such that $X_n \in \mathcal{B}$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$.

Unless otherwise stated, all measures are assumed to be σ-finite. In fact, we will usually only deal with probability measures, since we can also normalize a finite measure to a probability measure by $\mu \mapsto \mu/\mu(X)$.

Lastly, even though μ is only defined on the σ-algebra $\mathcal{B}$, we can slightly extend μ to a larger σ-algebra.

Definition 10. Let $(X, \mathcal{B}, \mu)$ be a measure space. A subset $Z \subseteq X$ is said to be *μ-null* if there exists some $Z' \in \mathcal{B}$ such that $Z \subseteq Z'$ and $\mu(Z') = 0$. We write Null_μ for the set of all μ-null subsets of X. A subset $A \subseteq X$ is said to be *μ-conull* if $X \setminus A$ is μ-null.

Definition 11. Let $(X, \mathcal{B}, \mu)$ be a measure space. A subset $A \subseteq X$ is *μ-measurable*[4] if there exists some $B \in \mathcal{B}$ and some μ-null set Z such that $A = B \cup Z$. We write Meas_μ for the set of all μ-measurable sets.

It is an exercise that $\mathrm{Meas}_\mu = \langle \mathcal{B} \cup \mathrm{Null}_\mu \rangle_\sigma$. Moreover, μ admits a unique extension to a map $\overline{\mu} : \mathrm{Meas}_\mu \to [0, \infty]$, called the *completion* of μ, and this measure satisfies $\mathrm{Meas}_{\overline{\mu}} = \mathrm{Meas}_\mu$. HINT: $\overline{\mu}(B \cup Z) \coloneqq \mu(B)$.

Definition 12. A measure μ is *complete* if $\overline{\mu} = \mu$.

For convenience, we will always assume that measures are complete. Neither measures λ nor μ_p in Examples 6 and 7 are complete, so we tacitly extend them.

References

[Anu23] Anush Tserunyan, *Advanced Real Analysis 1*, available at `https://www.math.mcgill.ca/atserunyan/Courses/2023F.Math564.Analysis1/`.

[Fol99] Gerald B. Folland, *Real Analysis*: *Modern Techniques and Their Applications*, 2nd ed., Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts, John Wiley & Sons, 2013, 1999.

[Kec95] Alexander S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, Springer New York, NY, 1995.

[4] Very confusing terminology. One might think that elements of $\mathcal{B}$ are the 'measurable' ones, but this removes μ from the picture. In general, there are much more μ-measurable sets that there are sets in $\mathcal{B}$. Indeed, there are $2^{\aleph_0}$-many Borel sets on $\mathbb{R}$, but there are $2^{2^{\aleph_0}}$-many λ-measurable sets!