DRP 2025

1 Refreshers from Lie Theory

Definition 1.1. A representation of a Lie Algebra is a vector space V with a morphism $\rho : \mathfrak{g} \to End(V)$ such that the bracket is preserved: $\rho([X,Y]) = [\rho(X), \rho(Y)]$.

Definition 1.2. Given representations V and W, the tensor representation $V \otimes W$ is given by

$$X(v \otimes w) = (Xv) \otimes w + v \otimes (Xw)$$

2 Overview

Our objects of study are Lie Algebras. These are vector spaces (usually over $\mathbb{C}$), equipped with a bracket. They have been fully classified in some context.

Theorem 2.1. *Finite dimensional simple complex Lie algebras are classified by Dynkin Diagrams:* $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.

We want to understand the representation theory of these Lie Algebra. The theory is simpler in type ADE (simply laced).

For example, the Lie algebra of type A_n is $sl_{n+1}(\mathbb{C}) = \{X \in M_{n+1}(\mathbb{C}) : tr(X) = 0\}$. The bracket is given by $[X,Y] = XY - YX$.
An easy representation is given by the natural action $sl_{n+1}(\mathbb{C}) \curvearrowright \mathbb{C}^{n+1}$ given by matrix operation on vectors $\rho_X(v) = Xv$. This is indeed a representation since $\rho([X,Y]) = [\rho(X), \rho(Y)]$. We can go further

Proposition 2.1. *The Lie algebra $sl_{n+1}(\mathbb{C})$ acts on $\wedge^k \mathbb{C}^{n+1}$ for all k. Moreover, any irreducible representation arises as a subrepresentation of some tensor products of these.*

Let's look precisely at $\wedge^2\mathbb{C}^4$. First, $sl_4(\mathbb{C})$ has dimension $4^2 - 1 = 15$, and a Chevalley basis given by:

$$E_i = \begin{bmatrix} 1 & \delta_{1,i} & 0 & 0 \\ 0 & 1 & \delta_{2,i} & 0 \\ 0 & 0 & 1 & \delta_{2,i} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F_i = E_i^t \quad H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad H_2, H_3 \quad [E_i, E_j] \quad [F_i, F_j]$$

The Cartan subalgebra is $\mathfrak{h} = span_{\mathbb{C}}(H_1, H_2, H_3)$. Note that it is commutative.
$\wedge^2\mathbb{C}^4$ has dimension 6. If e_1, e_2, e_3, e_4 is the standard basis for $\mathbb{C}^4$, then $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, $e_2 \wedge e_3$, $e_2 \wedge e_4$, $e_3 \wedge e_4$ forms a basis. How do the H_i act on these?

$$H_1 e_1 \wedge e_2 = H_1 e_1 \wedge e_2 + e_1 \wedge H_1 e_2 = 0 \quad H_2 e_1 \wedge e_2 = e_1 \wedge e_2 \quad H_3 e_1 \wedge e_2 = 0$$

Notice that H_i acts as a scalar. Hence we make the following definition

Definition 2.1. Let $\lambda : \mathbb{C}H_1 \oplus \mathbb{C}H_2 \oplus \mathbb{C}H_3 \to \mathbb{C}$ be such that $He_1 \wedge e_2 = \lambda(H)e_1 \wedge e_2$.

Note $\lambda \in \mathfrak{h}^*$. In our case, we found out that λ is 1 only for H_2 and 0 otherwise, which we write as $\lambda = \varpi_2$. In genral, $\varpi_i(H_j) = \delta_{i,j}$.

3 The Clifford Algebra

Definition 3.1. The tensor algebra $T^\bullet V = \bigoplus T^k V = \bigoplus V^{\otimes k}$ has ring structure from the maps $T^k V \otimes T^{k'} V \to T^{k+k'} V$.
Given a vector space V and a quadratic bilinear form Q on V, we define $Cliff(V, Q)$ to be the quotient of $T^\bullet V$ by the ideal generated by $v \otimes v - Q(v, v)1$.

This algebra arises as the universal object of the following diagram:

Given $j : V \to E$ where E is any associate algebra and $i : V \to C(V, Q)$ such that $i(v) \otimes i(v) = Q(v, v)1$,

there exists a unique homomorphism of algebras ϕ such that $\phi(i(v)) = j(v)$ for all $v \in V$.