Problem Set 1

Following Samy's talk

Problem 0.1. Let (X, \mathcal{M}, μ) be a measure space. Given A, B be measurable sets, show that

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

Problem 0.2. Let (X, \mathcal{M}, μ) be a measure space. Show that μ is monotone:

$$A \subset B \implies \mu(A) \le \mu(B)$$

This problem shows that if μ is finite (i.e. $\mu(X) < \infty$), every set has finite measure.

Problem 0.3. Show that μ is continuous. That is, if $A_1 \subset ... \subset A_n \subset ...$ is a countable chain, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

Similarly, if $A_1 \supset ... \supset A_n \supset ...$ is a countable chain and $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

Problem 0.4. Show that the following sets have Lebesgue measure 0:

- $(i) \mathbb{Q}$
- (ii) The Cantor set. (Hint: Continuity)

Problem 0.5. Let F be a finite set. Verify that the counting measure

$$\mu(A) = |A|$$

is indeed a measure on $(F, \mathcal{P}(F))$. (A) denotes the cardinality of the set).