SUMMER 2025 READING GROUP ON ERGODIC THEORY

Exercise Sheet 5 (Zhaoshen Zhai): Applications and Generalizations of Birkhoff's Pointwise Ergodic Theorem

Throughout, let (X,μ,T) be a measure-preserving dynamical system. The purpose of this exercise sheet is to give some applications and generalizations of the Pointwise Ergodic Theorem, which for convenience, we provide a sketch here. For $f\in L^1(X,\mu)$ and $n\in\mathbb{N}$, let $A_n^Tf:=\frac{1}{n}\sum_{i<n}f\circ T^i$.

Theorem. *If T is ergodic, then for any $f\in L^1(X,\mu)$, we have $\lim_n A_n^Tf=_\mu\int f\,\mathrm{d}\mu$.*

Proof sketch. Assume $\int f\,\mathrm{d}\mu=0$ and recall that $l:=\limsup_n A_n^Tf:X\to\mathbb{R}$ is T-invariant, so l is constant μ-a.e. by ergodicity, say at $l_0\in\mathbb{R}$. Suppose that $f^*:=l_0/2>0$, so for each $x\in X$, there is a minimal $\eta(x)\in\mathbb{N}$ such that $A_{\eta(x)}^Tf(x)\geq f^*$. We are done if there is a uniform $n\in\mathbb{N}$ such that $A_n^Tf\geq f^*/2$. This is not true in general, but after trimming measure-ε parts of X, something like this can be done using:

> **Lemma** (Tiling Lemma). *Let $\eta:X\to\mathbb{N}$ be an arbitrary measurable function. For any $\varepsilon>0$, there exists $n\gg0$ such that for each $x\in X$ except on a measure-ε set, the interval $I_n^T(x)$ can be tiled, up to an ε-fraction, by intervals of the form $I_y:=I_{\eta(y)}^T(y)$ for $y\in X$.* ■

Exercise 1. In the above context, prove that if both f and η are bounded, then $A_n^Tf\geq f^*/2$.
Hint: Prove a stronger Tiling Lemma in this case.

Exercise 2. What is the average value of a given digit $0\leq m\leq 9$, say $m:=7$, to occur in the decimal expansion $0.x_1x_2\dots$ of λ-a.e. $x\in[0,1]$? That is, does $\lim_n\frac{1}{n}|\{i<n:x_i=m\}|$ exist, and what is it?
Hint: Consider the 10-ary Baker's map $b_{10}:[0,1)\to[0,1)$ sending $x\mapsto 10x\ (\mathrm{mod}\ 1)$, which is isomorphic to the shift map on $10^{\mathbb{N}}$.

Exercise 3 (Equidistribution Theorem). A sequence $(x_n)_n$ in S^1 is said to be *equidistributed* if for every interval $I\subseteq S^1$, we have $\lim_n\frac{1}{n}|\{x_i\}_{i<n}\cap I|=\lambda(I)$. Prove that if $x_n=n\alpha$ for some irrational $\alpha\in S^1$, then $(x_n)_n$ is equidistributed. Hint: Don't overthink it.

Exercise 4 (Law of Large Numbers). If you know statistics, prove it! If not, skip it.

Exercise 5 (An ergodic theorem for non-ergodic actions). Intuitively, Birkhoff's Pointwise Ergodic Theorem states that ergodic transformations $T:X\to X$ stir up X so well that they spread any $f\in L^1(X,\mu)$ evenly on X, making it constant at $\int f\,\mathrm{d}\mu$; indeed, '$f\circ T^\infty=\int f\,\mathrm{d}\mu$'.

If T is not ergodic, then there is a non-trivial partition $X=X_1\sqcup X_2$ into T-invariant pieces. The best that one can hope is at after 'enough' partitions $X=\bigsqcup_i X_i$, T still spreads each $f_i:=f\chi_{X_i}$ evenly on X_i. Viewing f from the lens of these T-invariant pieces leads to the *conditional expectation* of f:

> **Definition.** Let $\mathcal{A}\subseteq\mathcal{B}(X)$ be a sub-σ-algebra of $\mathcal{B}(X)$. For each $f\in L^1(X,\mu)$, there is a unique (up to a μ-null set) $\mathcal{A}$-measurable function $f_{\mathcal{A}}$ such that $\int_A f\,\mathrm{d}\mu=\int_A f_{\mathcal{A}}\,\mathrm{d}\mu$ for each $A\in\mathcal{A}$, called the *conditional expectation of f w.r.t. $\mathcal{A}$*. We write $\mathbb{E}(f|\mathcal{A})$ for $f_{\mathcal{A}}$.
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> **Remark.** If $\mathcal{P}\subseteq\mathcal{B}(X)$ is a countable partition of X, then $\mathbb{E}(f|\langle\mathcal{P}\rangle_\sigma)=\sum_{P\in\mathcal{P}}\left(\frac{1}{\mu(P)}\int_P f\,\mathrm{d}\mu\right)\chi_P$.

Prove that for any (not necessarily ergodic) pmp transformation $T:X\to X$ and any $f\in L^1(X,\mu)$, we have $\lim_n A_n^Tf=_\mu\mathbb{E}(f|\mathcal{B}_T)$, where $\mathcal{B}_T\subseteq\mathcal{B}(X)$ is the σ-algebra generated by all T-invariant Borel sets of X.
Hint: Same as the regular proof, only that $f^*:X\to\mathbb{R}$ is not necessarily constant, but just T-invariant.

Exercise 6 (L^p-ergodic theorem). Prove that for any $p\geq1$, we have $A_n^Tf\to_{L^p}\mathbb{E}(f|\mathcal{B}_T)$ for all $f\in L^p(X,\mu)$.
Hint: If f is bounded, then we are done by the DCT. Otherwise, let $f_k\to_{L^p}f$ where each f_k is bounded and triangle-inequality your way through, using that $\|A_n^Tf\|_{L^p}\leq\|f\|_{L^p}$ (prove this too).

Date: May 28, 2025.