Problem Set 1

Following Samy's talk

Problem 0.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Given A, B be measurable sets, show that*

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

Problem 0.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that μ is* monotone*:*

$$A \subset B \implies \mu(A) \leq \mu(B)$$

This problem shows that if μ is finite (i.e. $\mu(X) < \infty$), every set has finite measure.

Problem 0.3. *Show that μ is continuous. That is, if $A_1 \subset ... \subset A_n \subset ...$ is a countable chain, then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

Similarily, if $A_1 \supset ... \supset A_n \supset ...$ is a countable chain and $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

Problem 0.4. *Show that the following sets have Lebesgue measure 0:*

(i) $\mathbb{Q}$

(ii) The Cantor set. (Hint: Continuity)

Problem 0.5. *Let F be a finite set. Verify that the counting measure*

$$\mu(A) = |A|$$

is indeed a measure on $(F, \mathcal{P}(F))$. ($|A|$ denotes the cardinality of the set).