SUMMER 2025 READING GROUP ON ERGODIC THEORY

Exercise Sheet 1 (Samy Lahlou): Crash course on Measure Theory, Part I

Throughout, let $(X, \mathcal{B}, \mu)$ be a measure space and let $A_n \in \mathcal{B}$.

Exercise 1 (Monotonicity)**.** If $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$.

Deduce that if μ is finite, then μ is a bounded function. (Are σ-finite measures bounded?)

Exercise 2 (Inclusion-exclusion)**.** For any $A_1, A_2 \in \mathcal{B}$, we have $\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)$.

Exercise 3 (Continuity $\nearrow$)**.** If $(A_n)_{n\in\mathbb{N}}$ is increasing, then $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu(A_n)$.

Exercise 4 (Continuity $\searrow$)**.** If $(A_n)_{n\in\mathbb{N}}$ is decreasing and $\mu(A_1) < \infty$, then $\mu(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu(A_n)$.

Exercise 5. Show that $\lambda(\mathbb{Q}) = 0$. Hint: What is the Lebesgue measure of singletons?

Let P be a property of some points in X. We say that P *holds μ-almost everywhere* (or *μ-almost surely*) if $\{x \in X : x \text{ satisfies } P\}$ is μ-conull.

Exercise 6 (Borel-Cantelli Lemmas)**.** Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of μ-measurable sets.

1. If $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then μ-almost every $x \in X$ lives in at-most finitely-many A_n.
2. (Measure Compactness). If $\mu(X) < \infty$ and there exists $\varepsilon > 0$ such that $\mu(A_n) \geq \varepsilon$ for all $n \in \mathbb{N}$, then at least an ε-measure set of $x \in X$ lives in infinitely-many A_n's.

For measurable spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$, define $\mathcal{B}_1 \otimes \mathcal{B}_2 := \langle B_1 \times B_2 : B_i \in \mathcal{B}_i \rangle_\sigma$.

Exercise 7. Show that if X_i are second-countable topological spaces, then $\mathcal{B}(X_1 \times X_2) = \mathcal{B}(X_1) \otimes \mathcal{B}(X_2)$.

Exercise 8. Let X be a topological space. A *Cantor set* is a subset $C \subseteq X$ homeomorphic to $2^{\mathbb{N}}$.

1. Show that the 'middle-thirds Cantor set' $C \subseteq [0, 1]$ is a Cantor set as in the above definition. Moreover, show that $\lambda(C) = 0$. Hint: Recall the construction $C = \bigcap_{n\in\mathbb{N}} C_n$ and use continuity.
2. Define a Cantor set $C \subseteq [0, 1]$ with positive Lebesgue measure. Hint: fatten the standard construction.

A measurable set $A \subseteq X$ is said to be an *atom* if there is no subset $B \subseteq A$ with $0 < \mu(B) < \mu(A)$. For example, singletons $\{x\}$ are atoms under the Dirac measure δ_x. More generally:

Exercise 9. If $(X, \mathcal{B}, \mu)$ is a σ-finite measure space, $\mathcal{B}$ is *countably generated* (i.e., $\mathcal{B} = \langle \mathcal{B}_0 \rangle$ for some countable $\mathcal{B}_0 \subseteq \mathcal{P}(X)$), and *separates points* (i.e., if $x \neq y$, then there exists $B \in \mathcal{B}$ such that $x \in B \not\ni y$.), then every atom $A \in \mathcal{B}$ is a singleton.

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