Set Theory

Chapter 1

MATH 280 Discrete Mathematical Structures

Sets

- A set is a collection of objects
- We generally use a capital Roman letter to name a set
  - A, B, S, T, etc.
- $x \in A$ means x is an element of A
- $x \notin A$ means x is **not** an element of A
- Sets are unordered
- A set contains only one of each of its objects

Set Enumeration

- $S = \{1,2,3,4,5\}$
- $T = \{1,2,3,4,5,\ldots\}$
- $V = \{1,2,3,4,5,\ldots,99,100\}$
- $\{1,2,3\}$ and $\{2,1,3\}$ are the same sets
  - Sets are unordered
- $\{1,2,3\}$ and $\{2,1,3,1,2\}$ are the same sets
  - A set contains only one of each of its objects

Standard Mathematical Sets

- $\mathbb{N} = \{0,1,2,3,4,5,\ldots\}$
- $\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$
- $\mathbb{Q}$ =the set of rational numbers
- $\mathbb{R}$ =the set of real numbers
- $\mathbb{C}$ =the set of complex numbers
- $\mathbb{P} = \{1,2,3,4,5,\ldots\}$
- $\varnothing = \{\}$
  - Empty set
  - Note: $\{\varnothing\} \neq \{\}$

Set Builder Notations

- $\mathbb{Q} \;=\; \{a/b \mid a,b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{Q} \;=\; \{a/b \;:\; a,b \in \mathbb{Z}, b \neq 0\}$
- $|$, : mean “such that”
- , means “and”
- $\{x \mid x \in \mathbb{R}, x^2 - 5x + 6 = 0\}$
- $\{x \in \mathbb{R} \mid x^2 - 5x + 6 = 0\}$

Real and Complex Numbers

- $\mathbb{R}$ is the set of numbers on the number line
- $\mathbb{C} \;=\; \{a+bi \mid a,b \in \mathbb{R}, i = \sqrt{-1}\}$

Finite and Infinite Sets

- A *finite set* has a finite number of elements
- Any set that is not finite is an *infinite set*
- Finite or infinite? $\{x \mid x \in \mathbb{R}, x^2 - 5x + 6 = 0\}$
- Finite or infinite? $\mathbb{R}$
- Finite or infinite? $\mathbb{Z}$
- Finite or infinite? $\{x \in \mathbb{N} \mid x < 10\}$
- Finite or infinite? $\varnothing$

Cardinality

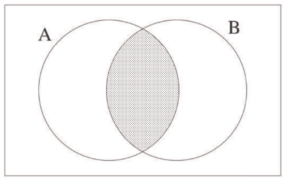
- The cardinality of a finite set equals the number of elements in the set
- The cardinality of an infinite set is more complicated
  - We will explore the cardinality of infinite sets later
- The cardinality of set A is $|A|$
- If $S = \{2, 4, 6, 8\}$, what is $|S|$?
- What is $|\{x \mid x \in \mathbb{R}, x^2 - 5x + 6 = 0\}|$

Subsets

- Let A and B be sets. We say A is a subset of B if and only if every element of A is an element of B.
- $A \subseteq B$
- "if and only if" means "is equivalent to"
- $A = \{2, 4, 6\}, B = \{1, 2, 3, 4, 5, 6, 7\}$
  - $A \subseteq B$
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
- If $A \subseteq B$ and $B \subseteq A$, then $A = B$

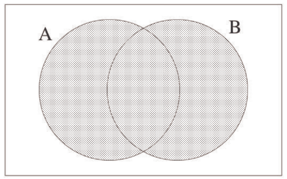
Set Intersection

- $A \cap B$
- Shaded area is intersection
- Rectangle is the *universal set*
- $\{x \mid x \in A \text{ and } x \in B\}$



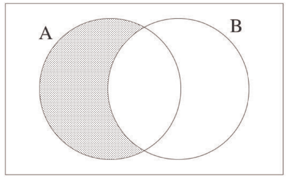
Set Union

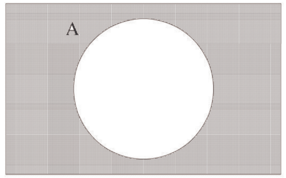
- $A \cup B$
- Shaded area is union
- Rectangle is the *universal set*
- $\{x \mid x \in A \text{ or } x \in B\}$



Set Complement

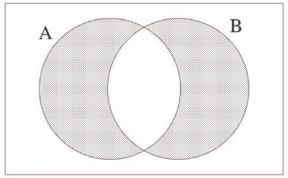
- Let A and B be sets. The complement of B relative to A is the set of elements that are in A and not in B.
- $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
  - Also known as *set difference*
- If U is the universal set, then $U - A = A^C$
- What is U^C?





Set Symmetric Difference

- Set of elements in A and B but not both
- $A \oplus B = (A \cup B) - (A \cap B)$



Cartesian Product

- $A = \{1,2,3\}, \quad B = \{a,b\}$
- $A \times B \;=\; \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

	a	b
1	$(1,a)$	$(1,b)$
2	$(2,a)$	$(2,b)$
3	$(3,a)$	$(3,b)$

Power set

- The *power set* of a set is the set of all its subsets.
- If $A = \{1,2,3\}$,
  $\mathcal{P}(A) = \{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$
- $\mathcal{P}(A) = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},A\}$
- $|A| = 3$,
  $|\mathcal{P}(A)| = 8$
- For arbitrary finite set S,
  $|\mathcal{P}(S)| = 2^{|S|}$

Cardinality of Power set

- Use a bitstring to represent set membership
- Use three bits for three elements

Subset	c	b	a
$\{\}$	0	0	0
$\{a\}$	0	0	1
$\{b\}$	0	1	0
$\{a,b\}$	0	1	1
$\{c\}$	1	0	0
$\{a,c\}$	1	0	1
$\{b,c\}$	1	1	0
$\{a,b,c\}$	1	1	1

- How many bits are needed to represent the subsets of a set with cardinality 8? How subsets would 8 bits admit?
- How many bits are needed to represent the subsets of a set with cardinality n? How subsets would n bits admit?

Decimal vs. Binary Number System

$473,406$

$\boxed{4}$	$\boxed{7}$	$\boxed{3}$	$\boxed{4}$	$\boxed{0}$	$\boxed{6}$
10^5	10^4	10^3	10^2	10^1	10^0
100,000	10,000	1,000	100	10	1

$$\begin{aligned} 473,406 &= 4\times 10^5 + 7\times 10^4 + 3\times 10^3 + 4\times 10^2 + 0\times 10^1 + 6\times 10^0 \\ &= 400,000 + 70,000 + 3,000 + 400 + 0 + 6 \\ &= 473,406 \end{aligned}$$

100111_2

$\boxed{1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$
2^5	2^4	2^3	2^2	2^1	2^0
32	16	8	4	2	1

$$\begin{aligned} 100111_2 &= 1\times 2^5 + 0\times 2^4 + 0\times 2^3 + 1\times 2^2 + 1\times 2^1 + 1\times 2^0 \\ &= 32 + 0 + 0 + 4 + 2 + 1 \\ &= 39 \end{aligned}$$

Algorithm for Decimal to Binary Conversion

```python
def decimal_to_binary(n):
    '''  n is a positive integer represented in base 10.
         Returns a the string containing the base 2
         representation of n.  The least significant bit
         is on the right.   '''
    k = n              #  Initialize k
    bitstring = ''     #  Initialize empty bitstring
    while k > 0:
        q = k // 2     #  Compute quotient of k / 2
        r = k % 2      #  Compute remainder of k / 2
        # Add r to the front of bitstring
        bitstring = str(r) + bitstring
        k = q          #  Reassign k
    return bitstring
```

Summations

$$\begin{aligned}\sum_{i=1}^{5} i &= 1+2+3+4+5 \\ &= 15\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{5}(2i+1) &= 2(1)+1+2(2)+1+2(3)+1+2(4)+1+2(5)+1 \\ &= 2+1+4+1+6+1+8+1+10+1 \\ &= 35\end{aligned}$$

$$\sum_{i=1}^{5} f(i) = f(1)+f(2)+f(3)+f(4)+f(5)$$

Summations

$$\sum_{i=1}^{n} f(i)$$

```
//  C++
int sum(int n, int (*f)(int)) {
    int s = 0;
    for (int i = 1; i <= n; i++)
        s += f(i);
    return s;
}
```

Summations

$$\sum_{i=1}^{n} f(i)$$

```
#  Python
def sum(n, f):
    s = 0
    for i in range(1, n + 1):
        s += f(i)
    return s
```

Decomposing Sums

- We can split a single summation into multiple summations as we see fit:

$$\begin{aligned}\sum_{i=1}^{7} i &= 1+2+3+4+5+6+7 \\ &= (1+2+3+4)+(5+6+7) \\ &= \sum_{i=1}^{4} i + \sum_{i=5}^{7} i\end{aligned}$$

- This will come in handy later

Product

- $\prod_{i=1}^{n} f(i) = f(1)\cdot f(2)\cdot f(3)\cdots f(n)$

Generalized Set Operations

- $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$
- $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$
- $\bigtimes_{i=1}^{n} A_i = A_1 \times A_2 \times A_3 \times \ldots \times A_n$