Combinatorics

Chapter 2

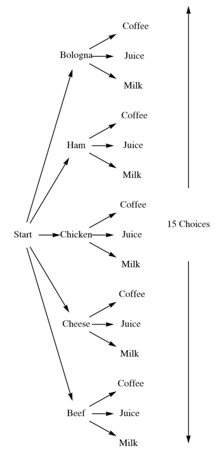
MATH 280 Discrete Mathematical Structures

Combinatorics

- Combinatorics is the study of counting
- The proper application of combinatorial principles enable us to count a large numbers of objects quickly
- Example: A snack bar serves five different sandwiches and three different beverages. How many different lunches can a person order?

Ordering Lunch

- A snack bar serves five different sandwiches and three different beverages. How many different lunches can a person order?
- $S = \{\text{Bologna}, \text{Ham}, \text{Chicken}, \text{Cheese}, \text{Beef}\}$
  $B = \{\text{coffee}, \text{milk}, \text{juice}\}$
- Build a *decision tree*



Rule of Products

- Cardinality of cartesian product
- Compute $|A \times B|$, where A and B are finite sets.
- Given two finite sets A and B, $|A| = m$ and $|B| = n$, the number of ways to select exactly one element from A and one element from B is $m \cdot n$.
- Also known as the *multiplication principle*
- Extended rule of products: If there are p_1 ways to make the first selection, p_2 ways to make the second selection, $\ldots$, p_n ways to make the n^{th} selection, and all the selections are independent (one selections does not affect another—for example, selecting two elements from the same set), then the number of ways to make the total selections is

$$p_1 \cdot p_2 \cdots p_n = \prod_{i=1}^{n} p_i$$

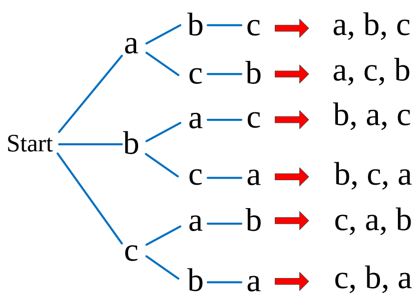
Cardinality of the Power Set

- Let A be a finite set
- Show that $|\mathcal{P}(A)| = 2^{|A|}$
- We need to count the number of subsets of A.
- Let $n = |A|$, and $S \in \mathcal{P}(A)$ (that is, $S \subseteq A$). In how many ways can we make S?
- Let $n = |A|$, and $S \in \mathcal{P}(A)$. For each element $x \in A$, either $x \in S$ or $x \notin S$. Thus, there are exactly two choices for each element in A. Since there are n elements in A, the rule of products dictates that there are $2 \cdot 2 \cdot 2 \cdots 2 = 2^n$ different subsets of A. Therefore $|\mathcal{P}(A)| = 2^{|A|}$ ■

Permutations

- Specific application of the rule of products
- How many ways can we order the elements of the set $A = \{a, b, c\}$?
- There are three ways to choose the first element. That leaves two elements remaining from which to choose.
- There are two ways to choose the second element, since one element already has been chosen.
- With only one element remaining after the first two selections, there is only one way to choose the last element.
- $3 \cdot 2 \cdot 1 = 6$ ways.
- $\langle a,b,c\rangle$ $\langle a,c,b\rangle$ $\langle b,a,c\rangle$ $\langle b,c,a\rangle$ $\langle c,a,b\rangle$ $\langle c,b,a\rangle$

$\{a,b,c\}$ Selection Tree



Factorial

$$
\begin{aligned}
0! &= 1, \text{ by definition} \\
1! &= 1 \\
2! &= 2 \cdot 1 = 2 \\
3! &= 3 \cdot 2 \cdot 1 = 6 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\
\vdots & \quad \vdots \\
n! &= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = \prod_{i=1}^{n} i
\end{aligned}
$$

- Factorial grows fast.
  - $10! = 3,628,800$
  - $20! = 2,432,902,008,176,640,000$

Factorial Algorithm

```
def factorial(n):
    if n < 2:
        return 1
    else:
        return n * factorial(n - 1)
```

```
def factorial(n):
    product = 1
    while n > 1:
        product *= n
        n -= 1
    return product
```

Permutation

- An ordered arrangement of k elements selected from a set with n elements, $0 < k \leq n$, where no two elements of the arrangement are the same, is called a permutation of n things taken k at a time.
- $P(n,k)$

$$
\begin{aligned}
P(n,k) &= n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) \\
&= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) \cdot \cancel{(n-k)} \cdot \cancel{(n-k-1)} \cdots \cancel{2} \cdot 1}{\cancel{(n-k)} \cdot \cancel{(n-k-1)} \cdots \cancel{2} \cdot 1} \\
&= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) \cdot (n-k) \cdot (n-k-1) \cdots 2 \cdot 1}{(n-k) \cdot (n-k-1) \cdots 2 \cdot 1} \\
&= \frac{n!}{(n-k)!}
\end{aligned}
$$

Permutation Examples

A club has 10 members eligible to serve as president, vice-president, and treasurer. In how many ways can officers be chosen?

- 10 ways to choose the president
- 9 ways to choose the vice-president
- 8 ways to choose the treasurer
- $10 \cdot 9 \cdot 8 = 720$

$$
\begin{aligned}
P(10,3) &= \frac{10!}{(10-3)!} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} \\
&= 10 \cdot 9 \cdot 8 \\
&= 720
\end{aligned}
$$

Permutation Examples

How many four digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, if no repetition of digits is allowed?

$$
\begin{aligned}
P(10,4) &= \frac{10!}{(10-4)!} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}} \\
&= 10 \cdot 9 \cdot 8 \cdot 7 \\
&= 5,040
\end{aligned}
$$

If repetition is allowed, it is the simple rule of products:

$$10 \cdot 10 \cdot 10 \cdot 10 = 10,000$$

When to Use a Permutation

- We need to arrange the elements of a single set
- Each element appears exactly once in the list (permutation)

Set Partitions

- A partition of set A is a set of non-empty subsets of A, $A_1, A_2, \ldots, A_k$ such that:
  - All together they form the complete set A: $A_1 \cup A_2 \cup \ldots \cup A_k = A$
  - The subsets are mutually disjoint: $A_i \cap A_j = \varnothing$, for all $1 \le i, j \le k$, $i \ne j$.
- The subsets that make up a partition are also known as *blocks*
- What is the maximum number of blocks in finite set A?
  - $|A|$

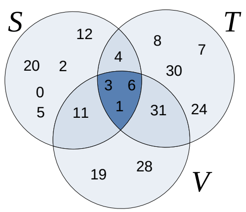
Laws of Addition

- Basic law of addition: If A is a finite set where $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$, and $A_i \cap A_j = \varnothing$, for all $1 \le i, j \le k$, $i \ne j$, then $|A| = |A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = |A_1| + |A_2| + |A_3| + \ldots + |A_n|$
- Can we count the number of students in a class by adding the number of freshmen, sophomores, juniors, seniors, and those who belong to none of these categories?
- Can we count the number of students in a class by adding the number of computer science majors, mathematics majors, business majors, and those who have some other major?
  - Student class standing partitions the class
  - Student major does **not** partition the class

Laws of Addition

- If A_1 and A_2 are finite sets, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- If A_1, A_2, and A_3 are finite sets, $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- Also known as principle of inclusion-exclusion

S = {0, 1, 2, 3, 4, 5, 6, 11, 12, 20}
T = {1, 4, 6, 7, 8, 24, 31}
V = {1, 3, 6, 11, 19, 28, 31}



Principle of Inclusion-exclusion

- If A_1, A_2, and A_3 are finite sets, $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- A vendor sells only pencils, erasers, and lead. On one day the vendor handled 207 transactions. 152 transactions involved pencils, 25 involved erasers, and 114 involved lead. 64 transactions involved pencils and lead, 12 involved pencils and erasers, and 9 transactions involved all three. How many transactions involved erasers and lead?
  - Let P = the set of transactions involving pencils
  - Let L = the set of transactions involving lead
  - Let E = the set of transactions involving erasers

Laws of Addition

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  - Let P = the set of transactions involving pencils
  - Let L = the set of transactions involving lead
  - Let E = the set of transactions involving erasers
  - $|P \cup L \cup E| = 207$
  - $|L| = 114$
  - $|P| = 152$
  - $|E| = 25$
  - $|P \cap L| = 64$
  - $|P \cap E| = 12$
  - $|P \cap L \cap E| = 9$

Laws of Addition

- If A_1, A_2, and A_3 are finite sets, $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- Solution:
  - $|P \cup L \cup E| = 207$
  - $|L| = 114$
  - $|P| = 152$
  - $|E| = 25$
  - $|P \cap L| = 64$
  - $|P \cap E| = 12$
  - $|P \cap L \cap E| = 9$
- $|P \cup L \cup E| = |L| + |P| + |E| - |P \cap L| - |P \cap E| - |L \cap E| + |P \cap L \cap E|$
- $|P \cup L \cup E| + |L \cap E| = |L| + |P| + |E| - |P \cap L| - |P \cap E| + |P \cap L \cap E|$
- $|L \cap E| = |L| + |P| + |E| - |P \cap L| - |P \cap E| + |P \cap L \cap E| - |P \cup L \cup E|$
- $|L \cap E| = 114 + 152 + 25 - 64 - 12 + 9 - 207 = 17$

Combinations

- How many ways to permute three elements from $\{a,b,c,d\}$?
  - $P(4,3) = \frac{4!}{(4-3)!} = 24$
  - $\langle a,b,c\rangle$, $\langle a,b,d\rangle$, $\langle a,c,b\rangle$, $\langle a,c,d\rangle$, $\langle a,d,b\rangle$, $\langle a,d,c\rangle$, $\langle b,a,c\rangle$, $\langle b,a,d\rangle$, $\langle b,c,a\rangle$, $\langle b,c,d\rangle$, $\langle b,d,a\rangle$, $\langle b,d,c\rangle$, $\langle c,a,b\rangle$, $\langle c,a,d\rangle$, $\langle c,b,a\rangle$, $\langle c,b,d\rangle$, $\langle c,d,a\rangle$, $\langle c,d,b\rangle$, $\langle d,a,b\rangle$, $\langle d,a,c\rangle$, $\langle d,b,a\rangle$, $\langle d,b,c\rangle$, $\langle d,c,a\rangle$, $\langle d,c,b\rangle$

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- How many ways are there to choose three elements from $\{a,b,c,d\}$?
- Order does not matter
- $C(n,k)$ or $\binom{n}{k}$ represents the number of combinations of n objects taken k at a time.
- Read $\binom{n}{k}$ as "n choose k"
- $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Combination Example

A club has 10 members eligible to serve on a committee. In how many ways can a committee of three be formed?

- Order of choice does not matter; what matters is being chosen or not

$$
\begin{aligned}
\binom{10}{3} &= \frac{10!}{(10-3)! \cdot 3!} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7! \cdot 3!} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!} \cdot 3 \cdot 2} \\
&= \frac{10 \cdot \cancel{3} \cdot 3 \cdot \cancel{2} \cdot 4}{\cancel{3} \cdot \cancel{2}} \\
&= 10 \cdot 3 \cdot 4 \\
&= 120
\end{aligned}
$$

Counting subsets

- How many subsets of size k exist for finite set A?
- $\binom{|A|}{k} = \frac{|A|!}{(|A|-k)!k!}$
- Combination is ideal because sets are unordered

Permutations vs. Combinations

- When you need to determine how many ways you can order elements selected from a set, use a permutation
- When you need to determine how many ways you can select a collection of elements from a set, use a combination
- If order matters, use a permutation
- If order does not matter, use a combination