Homework 1 | Question 1

$$p(m|\mu) = \binom{m}{m+l}(1-\mu)^l \mu^m$$

$$\boxed{\mu_{ML} = \frac{m}{m+l}}$$

$$p(\mu) = \text{Beta}(a,b)$$

$$\mathbb{E}[\mu] = \frac{a}{a+b}$$

$$\text{Posterior } \theta_{ML} = \frac{m+a-1}{m+l+a+b-2}$$

$$(\lambda)\frac{a}{a+b} + (1-\lambda)\frac{m}{m+l} = \frac{m+a-1}{m+l+a+b-2}$$

?

Book 2.12

$\mathcal{U}(x|a,b) = \frac{1}{b-a}$, $a \le x \le b$

- Proof of normalization: If the distribution has been normalized, then the integral over the distribution with respect to x will equal 1.

$$\int_x \mathcal{U}(x|a,b)dx = \int_a^b (b-a)^{-1}dx$$
$$= (b-a)^{-1}[x]_a^b$$
$$= (b-a)^{-1}(b-a)$$

$$\boxed{\int_x \mathcal{U}(x|a,b)dx = 1}$$

- Finding the mean: The mean will be the expectation of x.

$$E[x|a,b] = \int_x p(x|a,b)f(x)dx = \int_a^b (b-a)^{-1}x\,dx$$
$$= (2(b-a))^{-1}[x^2]_a^b$$
$$= (2(b-a))^{-1}(b^2-a^2)$$
$$= (2\cancel{(b-a)})^{-1}\cancel{(b-a)}(b+a)$$

$$\boxed{E[x|a,b] = \frac{b+a}{2}}$$

- Finding the variance: The variance will be a function of expected values.

$$Var[x|a,b] = E[x^2|a,b] - E[x|a,b^2]$$

$$E[x^2|a,b] = \int_a^b (b-a)^{-1}x^2dx = (3(b-a))^{-1}(b^3-a^3)$$

$$E[x|a,b]^2 = \left(\frac{b+a}{2}\right)^2 = (b+a)^2\left(\frac{1}{4}\right)$$

$$Var[x|a,b] = (12(b-a))^{-1}\left[(4(b^3-a^3)) - (3(b-a)(b+a)^2)\right]$$

$$= \frac{4\cancel{(b-a)}(b^2+ab+a^2) - 3\cancel{(b-a)}(b^2+ba+a^2)}{12\cancel{(b-a)}}$$

$$= \frac{(4-3)(b^2+ba+a^2)}{12}$$

$$\boxed{Var[x|a,b] = \frac{(b+a)^2}{12}}$$

- Sum of probabilities is 1

$$p(x|\lambda) = e^{-\lambda}\lambda^{x}(x!)^{-1}$$

$$\Sigma_x p(x|\lambda) = \Sigma_{x=0}^{\infty} e^{-\lambda}\lambda^{x}(x!)^{-1}$$

$$= e^{-\lambda}\Sigma_{x=0}^{\infty}\lambda^{x}(x!)^{-1}$$

$$= e^{-\lambda}e^{\lambda} \qquad \bullet\ e^{\lambda} = \Sigma_{n=0}^{\infty}\lambda^{n}(n!)^{-1}$$

a $\boxed{\Sigma_x p(x|\lambda) = 1}$

- The mean will be the expected value.

$$E[x|\lambda] = \Sigma_x (x) p(x|\lambda) = \Sigma_{x=0}^{\infty}(x)e^{-\lambda}\lambda^{x}(x!)^{-1}$$

- The (x) term will force $x=0$ to 0
  So we can change the sum to $\Sigma_{x=1}^{\infty}$.

$$E[x|\lambda] = \Sigma_{x=1}^{\infty}(x)e^{-\lambda}\lambda^{x}(x!)^{-1}$$

$$= e^{-\lambda}\Sigma_{x=1}^{\infty}(x)\lambda^{x}(x!)^{-1}$$

$$= e^{-\lambda}\Sigma_{x=1}^{\infty}\lambda^{x}((x-1)!)^{-1}$$

$$= e^{-\lambda}\lambda\Sigma_{x=1}^{\infty}\lambda^{x-1}((x-1)!)^{-1}$$

$$= e^{-\lambda}\lambda\Sigma_{g=0}^{\infty}\lambda^{g}(g!)^{-1} \qquad \bullet\ g = x-1$$

$$= e^{-\lambda}e^{\lambda}\lambda \qquad \bullet\ e^{\lambda} = \Sigma_g \lambda^{g}(g!)^{-1}$$

b

$$\boxed{E[x|\lambda] = \lambda}$$

- Maximum Likelihood: Get likelihood function, set derivative to 0.

$$likelihood(\lambda) = \Pi_{i=1}^{n} P(x_i|\lambda) = \Pi_{i=1}^{n}\lambda^{x_i}e^{-\lambda}(x_i!)^{-1}$$

* log likelihood: $\ell(\lambda) = \Sigma_{i=1}^{n}(x_i \ln(\lambda) - \lambda - \ln(x_i!))$

$$= \ln(\lambda)\Sigma_{i=1}^{n}x_i - n\lambda - \Sigma_{i=1}^{n}(x_i!)$$

* derivative $\frac{\partial}{\partial\lambda}\ell(\lambda) = \lambda^{-1}\Sigma_{i=1}^{n}x_i - n = 0$ · Solve for solution

$$\boxed{\lambda_{ML} = \frac{\Sigma_{i=1}^{n}x_i}{n}}$$ c

- Gamma form of the posterior

To prove that posterior takes the form of the of the prior we will prove that the prior multiplied into the likelihood is proportional to the prior form.

- Let the prior be the gamma distribution:

$$p(\lambda) = b^{-a} (\Gamma a)^{-1} \lambda^{a-1} e^{-\lambda/b}$$

- and the likelihood

$$p(x|\lambda) = \lambda^{x} e^{-\lambda} (x!)^{-1}$$

- Therefore the posterior

$$\pi(\lambda|x) = \lambda^{x+a-1} e^{-\lambda(1+1/b)} (b^{a}\Gamma(a))^{-1} (x!)^{-1}$$

- The last two terms are constant with respect to λ, and can be ignored.

We can now say that the posterior is proportionaly

d

$$\boxed{\begin{gathered}\pi(\lambda|x) \propto \lambda^{x+a-1} e^{-\lambda\left(\frac{b+1}{b}\right)} \\ \text{gamma}\left(x+a, \frac{b}{b+1}\right)\end{gathered}}$$

Homework1

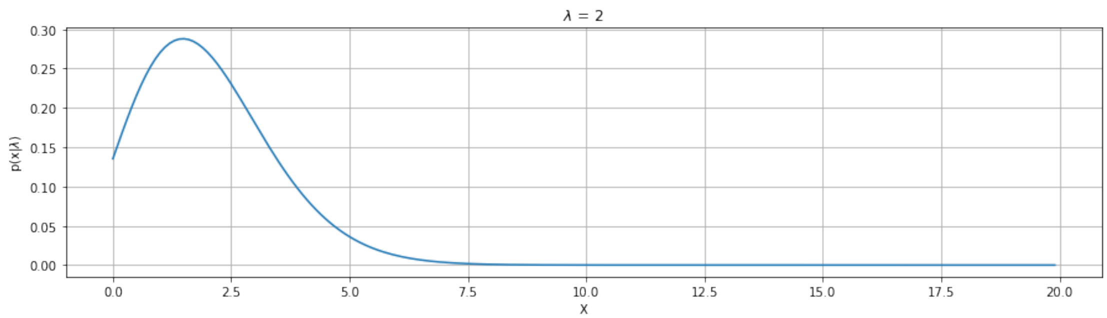
April 24, 2019

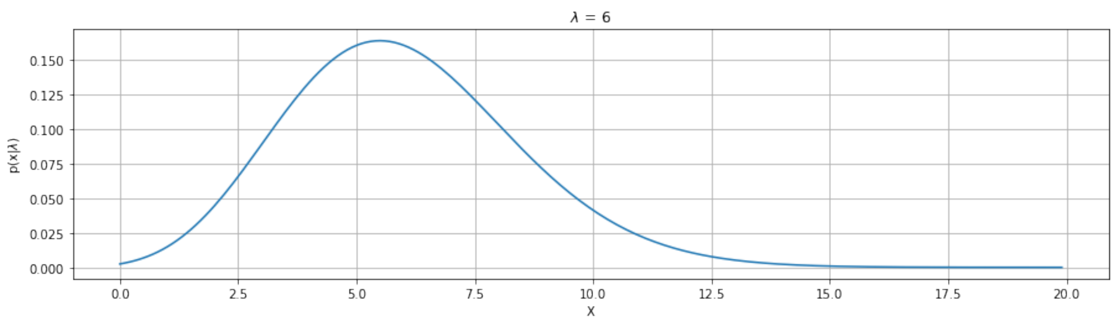
```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        from scipy.special import factorial
        #from scipy.stats import gamma

        %matplotlib inline
```

0.1 e) Plots of Poisson for $\lambda = 2$ and $\lambda = 6$

```
In [2]: def pois(lam, x_axis):
            return np.exp(-lam)*np.power(lam, x_axis)/factorial(x_axis)
```

```
In [3]: x = np.arange(0, 20, 0.1)

        plt.subplot(2, 1, 1)
        plt.plot(x, pois(2,x))

        plt.xlabel('X')
        plt.ylabel('p(x|$\lambda$)')
        plt.title('$\lambda$ = 2')
        plt.grid(True)

        plt.subplot(2, 1, 2)
        plt.plot(x, pois(6,x))

        plt.xlabel('X')
        plt.ylabel('p(x|$\lambda$)')
        plt.title('$\lambda$ = 6')
        plt.grid(True)

        fig = plt.gcf()
        fig.subplots_adjust(hspace=.7)
        fig.set_size_inches((15.0,10))

        plt.show()
```





0.2 f) Maximum likelihood for 'poisson.txt'

```
In [4]: data = np.loadtxt('poisson.txt')
        ML = sum(data)/len(data)
        ML
```

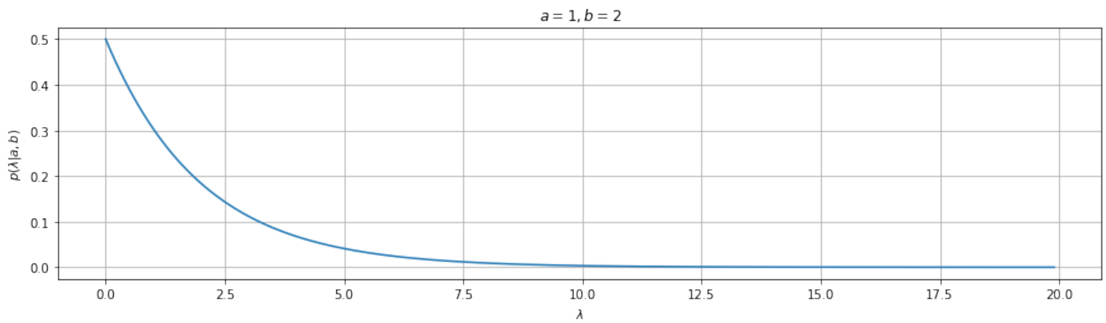
```
Out[4]: 5.24
```

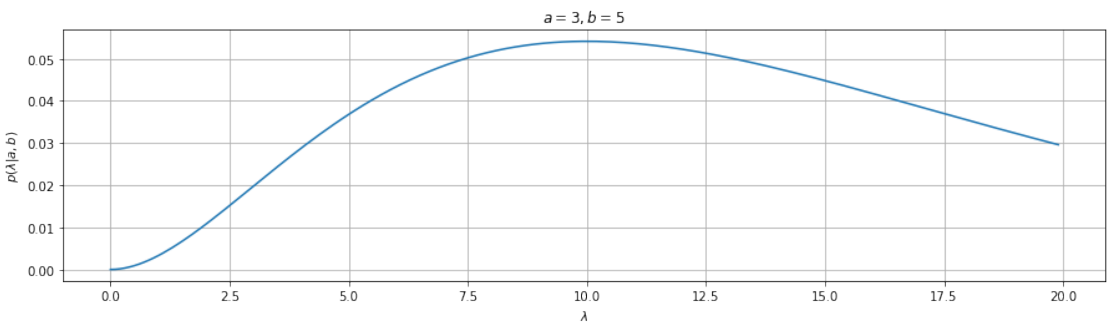
0.3 g) Gamma Distribution plot for $(a = 1, b = 2)$ and $(a = 3, b = 5)$.

```
In [89]: def gamma(a,b,x_axis):
             return (x_axis**(a-1)*np.exp(-x_axis/b))/(b**a*factorial(a-1))
```

```
In [90]: plt.subplot(2, 1, 1)
         plt.plot(x, gamma(1,2,x))

         plt.xlabel('$\lambda$')
         plt.ylabel('$p(\lambda|a,b)$')
         plt.title('$a=1, b=2$')
         plt.grid(True)

         plt.subplot(2, 1, 2)
         plt.plot(x, gamma(3,5,x))
```

```
plt.xlabel('$\lambda$')
plt.ylabel('$p(\lambda|a,b)$')
plt.title('$a=3, b=5$')
plt.grid(True)

fig = plt.gcf()
fig.subplots_adjust(hspace=.7)
fig.set_size_inches((15.0,10))

plt.show()
```





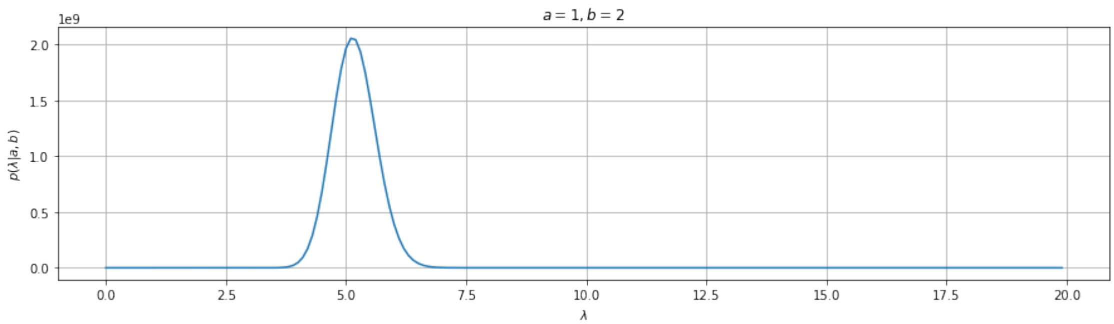
0.4 h) Posterior

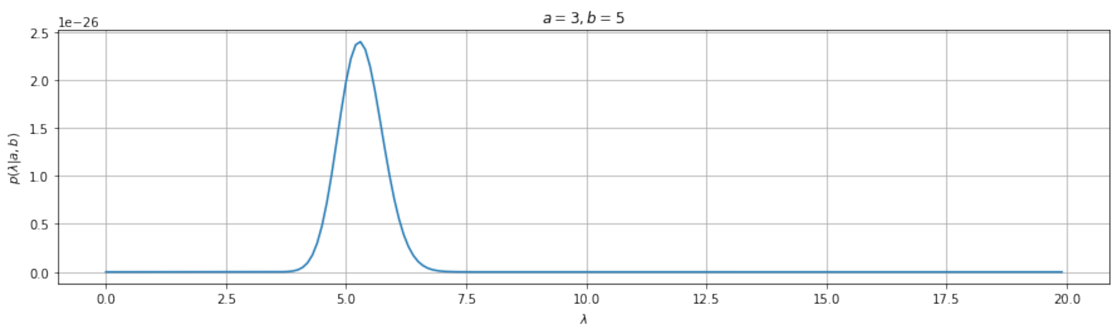
0.4.1 Version 1: no input

0.4.2 Version 2: Input based

```
In [91]: def likelihood(lam, x_data):
             return np.array([np.prod(((l**x_data)*(np.exp(-l)))/(factorial(x_data)) for l in

In [92]: plt.subplot(2, 1, 1)
         plt.plot(x, (likelihood(x, data)*gamma(1,2,x))/((1/len(data))**len(data)))

         plt.xlabel('$\lambda$')
```

```
plt.ylabel('$p(\lambda|a,b)$')
plt.title('$a=1, b=2$')
plt.grid(True)

plt.subplot(2, 1, 2)
plt.plot(x,  likelihood(x, data)*gamma(3,5,x))

plt.xlabel('$\lambda$')
plt.ylabel('$p(\lambda|a,b)$')
plt.title('$a=3, b=5$')
plt.grid(True)

fig = plt.gcf()
fig.subplots_adjust(hspace=.7)
fig.set_size_inches((15.0,10))

plt.show()
```





We can see that applying the input data to the gamma priors has really pinched them in close together. A small difference is still noticable. The (a=1, b=2) is more to the left and more thin than the (a=3, b=5) version.

- To prove that the Possion distribution

$$p(x|\lambda) = e^{-\lambda}\lambda^{x}(x!)^{-1}$$

can be expressed in terms of the exponential family distribution form.

$$f(x|\eta) = (Z(\eta))^{-1} h(x) \exp\{\eta^T t(x)\}$$

$$\longrightarrow \quad p(x|\lambda) = e^{-\lambda}\lambda^{x}(x!)^{-1}$$

$$= (e^{\lambda})^{-1}((x!)^{-1}) \exp\{\log(\lambda^{x})\}$$

$$= \underbrace{(e^{\lambda})^{-1}}_{Z(\eta)}\underbrace{((x!)^{-1})}_{h(x)} \exp\{\underbrace{\log(\lambda)}_{\eta}\underbrace{x}_{t(x)}\}$$

$$\boxed{\bullet\ \eta = \log(\lambda) \quad \bullet\ t(x) = x \quad \bullet\ h(x) = (x!)^{-1} \quad \bullet\ Z(\eta) = e^{\lambda}}$$

- To prove $Z(\eta)$ is proper we have the equation

$$Z(\eta) = \sum_{x=0}^{\infty} h(x) \exp\{\eta^T t(x)\}$$

- Substituting our values above

$$e^{\lambda} = \sum_{x=0}^{\infty} (x!)^{-1} \exp\{\log(\lambda) x\}$$

$$e^{\lambda} = \sum_{x=0}^{\infty} (x!)^{-1} \lambda^{x}$$

$$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}$$

$$\boxed{e^{\lambda} = e^{\lambda}} \quad \bullet \text{ by definition}$$