7.1

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```
In [1]: # Content from Proakis
        # Code ľ 2019, Alexander Kain
        import numpy as np

        from matplotlib import pyplot as plt
        %matplotlib inline
        plt.rcParams['figure.figsize'] = (10.0, 8.0)

        import sympy as sym
        sym.init_printing(use_unicode=True)
```

1 The Discrete Fourier Transform: Its Properties and Applications

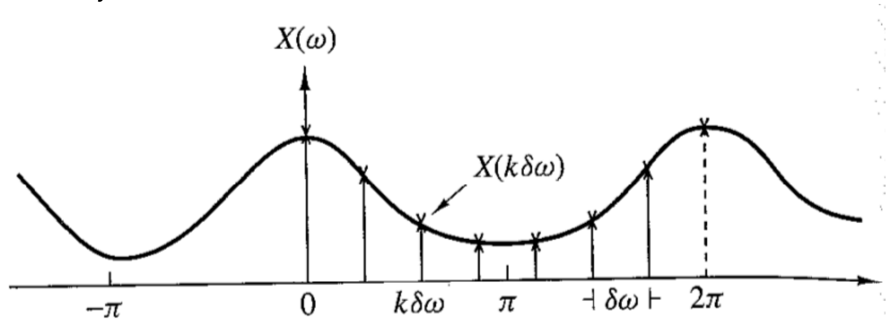
So far, the Fourier Transform $X(\omega)$ of the sequence $x[n]$ was a continuous function of frequency. We now consider a discrete frequency-domain representation, the discrete Fourier Transform (DFT)

1.1 7.1. Frequency-Domain Sampling: The Discrete Fourier Transform

Suppose we sample

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

periodically at $\omega = 2\pi k/N$:



Then

$$X(2\pi k/N) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi kn/N}, k = 0, 1, \ldots, N-1$$

Let us sub-divide the summation into an infinite number of summations containing N terms:

$$X(2\pi k/N) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n]e^{-j2\pi kn/N} =$$

$$\sum_{n=lN}^{lN+N-1} \sum_{l=-\infty}^{\infty} x[n]e^{-j2\pi kn/N} =$$

Let $n \to n - lN$:

$$X(2\pi k/N) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n-lN]e^{-j2\pi k(n-lN)/N} =$$

$$\sum_{n=0}^{N-1} \left(\sum_{l=-\infty}^{\infty} x[n-lN] \right) e^{-j2\pi kn/N}$$

The term in parentheses

$$x_p[n] = \sum_{l=-\infty}^{\infty} x[n-lN]$$

is periodic with fundamental period N. Therefore, it can be written as a Fourier Series:

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, n = 0, 1, \ldots, N-1$$

with Fourier coefficients

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n]e^{-j2\pi kn/N}, k = 0, 1, \ldots, N-1$$

Comparing this to the formula for $X(2\pi k/N)$, we find that

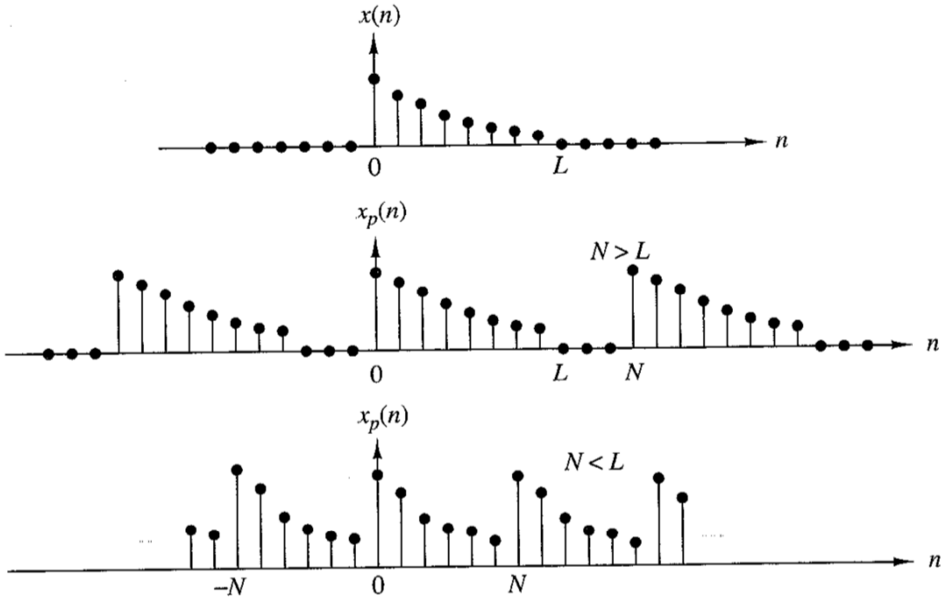
$$c_k = \frac{1}{N} X(2\pi k/N)$$

and

$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(2\pi k/N)e^{j2\pi kn/N}, n = 0, 1, \ldots, N-1$$

This relationship provides the reconstruction of the periodic signal $x_p[n]$ from the samples of the spectrum $X(\omega)$.

However, we would like to also recover $x[n]$ and $X(\omega)$. The former can be recovered from $x_p[n]$ if there is no *time-domain aliasing*, that is, $x[n]$ is time-limited to less than the period N.



If there is no time-domain aliasing, i.e.

$$x[n] = \begin{cases} x_p[n], & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

it is possible to express $X(\omega)$ in terms of its samples. Since $x[n]$ is time-limited we can use appropriate integral boundaries

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(2\pi k/N)e^{j2\pi kn/N}, n = 0, 1, \ldots, N-1$$

Now taking the Fourier Transform

$$X(\omega) = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X(2\pi k/N)e^{j2\pi kn/N}\right)e^{-j\omega n} =$$

$$\sum_{k=0}^{N-1} X(2\pi k/N)\left(\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega-2\pi k/N)n}\right)$$

The inner term in parentheses represents the interpolation function (shifted by $2\pi k/N$ in frequency). Let us define the unshifted version

$$P(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

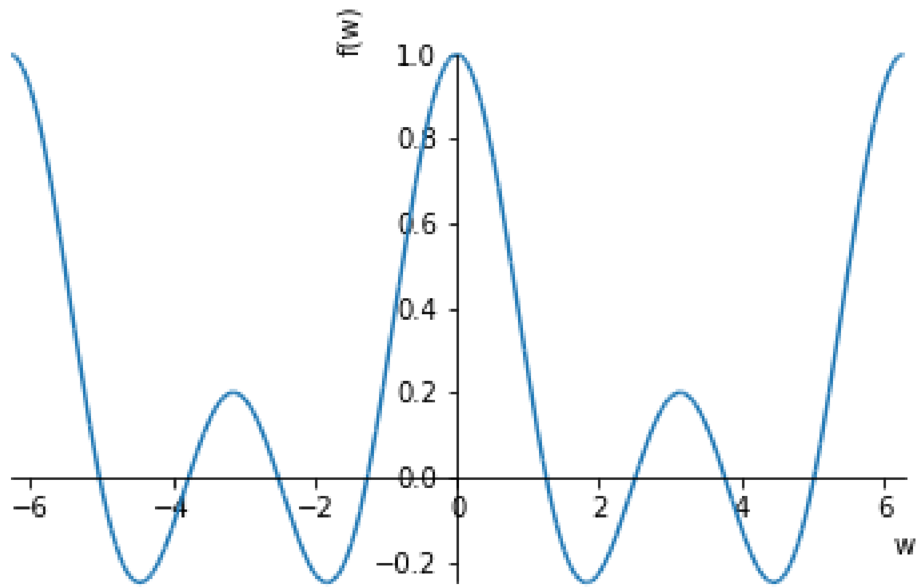
$$= \frac{\sin(\omega N/2)}{N\sin(\omega/2)}e^{-j\omega(N-1)/2}$$

This is the periodic counterpart to the familiar sinc function. Then

$$X(\omega) = \sum_{k=0}^{N-1} X(2\pi k/N)P(\omega - 2\pi k/N)$$

```
In [2]: # visualize part of interpolation function P
        w = sym.symbols('w', real=True)
        N, k = sym.symbols('N k', integer=True)
        P = sym.sin(w * N / 2) / (N * sym.sin(w / 2))

        sym.plot(P.subs({N: 5}), (w, -2 * sym.pi, 2 * sym.pi));
```



$P(\omega)$ has the property that $P(0) = 1$ and

$$P(\frac{2\pi k}{N}) = 0, k = 1, 2, \ldots, N-1$$

Consequently, the interpolation function gives exactly the sample values $X(2\pi j/N)$ for $\omega = 2\pi k/N$; for all other frequencies the result is a linear combination of the original spectral samples.

1.1.1 7.1.2. The Discrete Fourier Transform

In general, the equally-spaced frequency samples $X(2\pi k/N), k = 0, 1, \ldots, N-1$ do *not* uniquely represent the original sequence $x[n]$; instead they correspond to a periodic sequence $x_p[n]$ of period N where $x_p[n]$ may be an aliased version of $x[n]$

$$x_p[n] = \sum_{l=-\infty}^{\infty} x[n - lN]$$

When the sequence $x[n]$ has a finite duration of length $L \leq N$, then $x_p[n]$ is simply a periodic repetition of $x[n]$, where

$$x_p[n] = \begin{cases} x[n], & 0 \leq n \leq L-1 \\ 0, & L \leq n \leq N-1 \end{cases}$$

In this case the frequency samples *uniquely* represent the finite-duration sequence $x[n]$, and the Fourier Transform is

$$X(\omega) = \sum_{n=0}^{L-1} x[n] e^{-j\omega n}$$

When we sample this we obtain

$$X[k] \equiv X(2\pi k/N) = \sum_{n=0}^{L-1} x[n] e^{-j2\pi kn/N}$$

and without loss of generality we can increase the upper index in the sum to $N-1$:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, k = 0, 1, 2, \ldots, N-1$$

which is the *discrete Fourier Transform* (DFT). The inverse DFT (IDFT) is

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, n = 0, 1, \ldots, N-1$$

Example 7.1.2 A finite-duration sequence of length L is given as

$$x[n] = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the DFT of this sequence for $N \geq L$:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, k = 0, 1, 2, \ldots, N-1$$

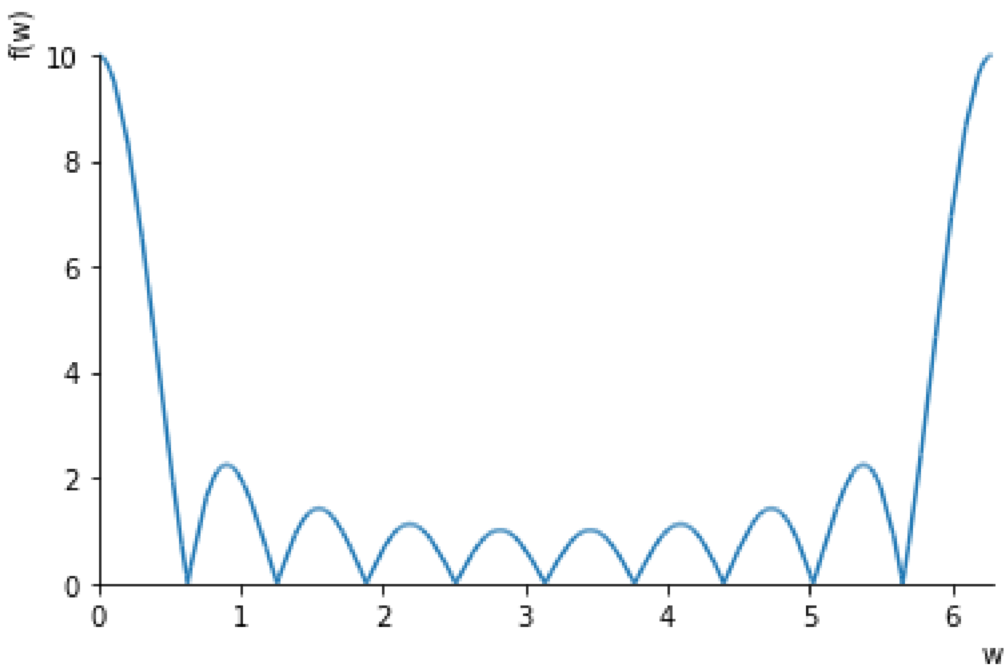
The Fourier Transform is

$$X(\omega) = \sum_{n=0}^{L-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}}$$

In [3]:
```
# code here
w = sym.symbols('w', real=True)
L, N, k = sym.symbols('L N k', integer=True)
X = (1 - sym.exp(-1j * w * L)) / (1 - sym.exp(-1j * w))

sym.plot(sym.Abs(X.subs({L: 10})), (w, 0, 2 * sym.pi));
```



Clearly, at $X(0) = L$.

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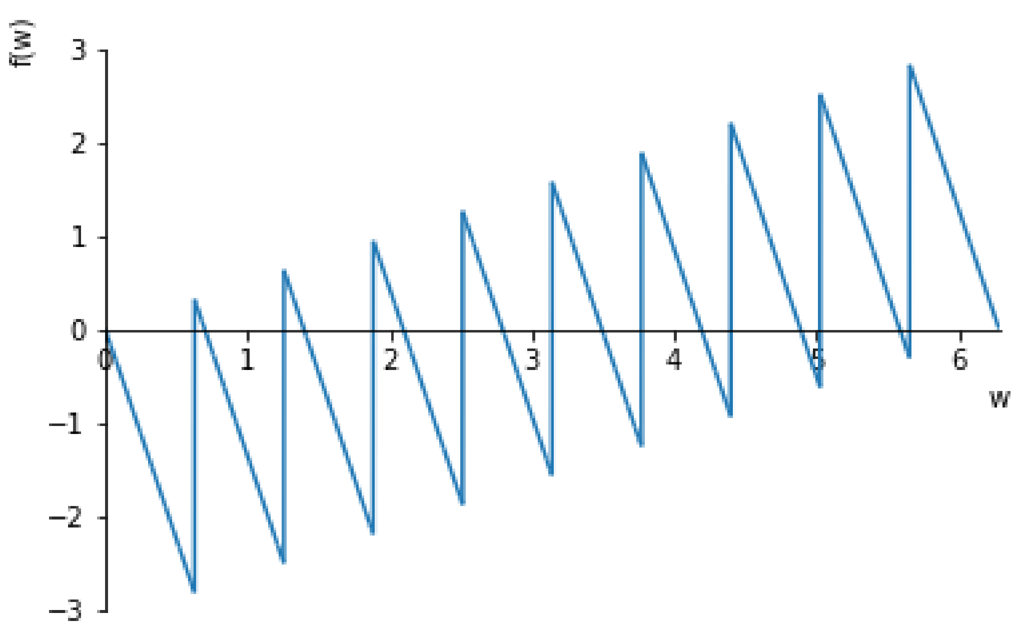
In [4]: sym.plot(sym.arg(X.subs({L: 10})), (w, 0, 2 * sym.pi));
```



The DFT is $X(\omega)$ evaluated at $\omega = 2\pi k/N$:

In [5]:
```python
X.subs({w: 2 * sym.pi * k / N})
```

Out[5]:

$$\frac{1 - e^{-\frac{2.0 i \pi L k}{N}}}{1 - e^{-\frac{2.0 i \pi k}{N}}}$$

If $N = L$ then:

In [6]:
```python
X.subs({w: 2 * sym.pi * k / L})
```

Out[6]:

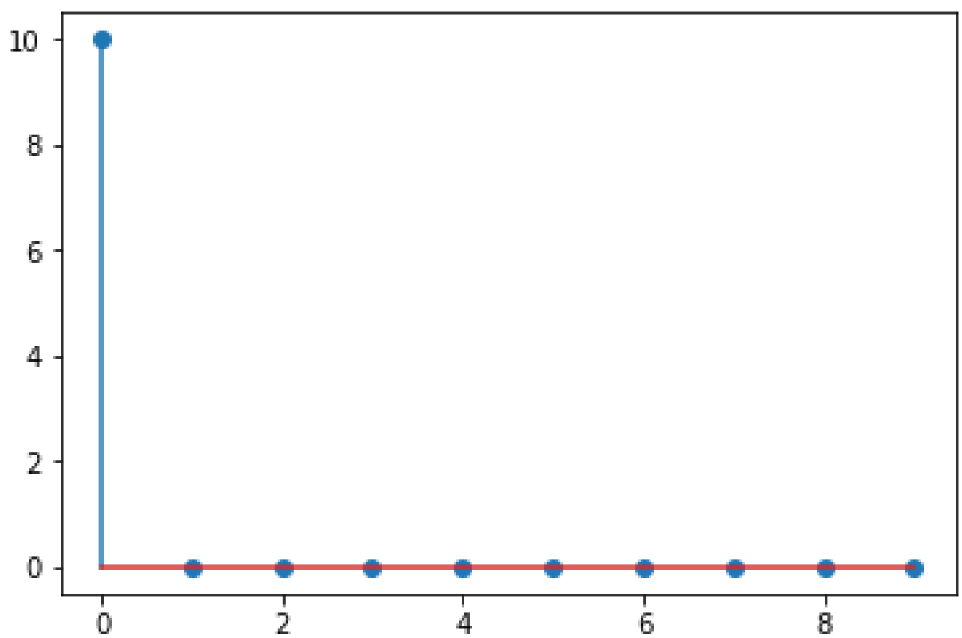
$$\frac{1 - e^{-2.0 i \pi k}}{1 - e^{-\frac{2.0 i \pi k}{L}}}$$

which is zero for any $k = 1, 2, \ldots, L-1$, in other words

$$X[k] = \begin{cases} L, & k = 0 \\ 0, & k = 1, 2, \ldots, L-1 \end{cases}$$

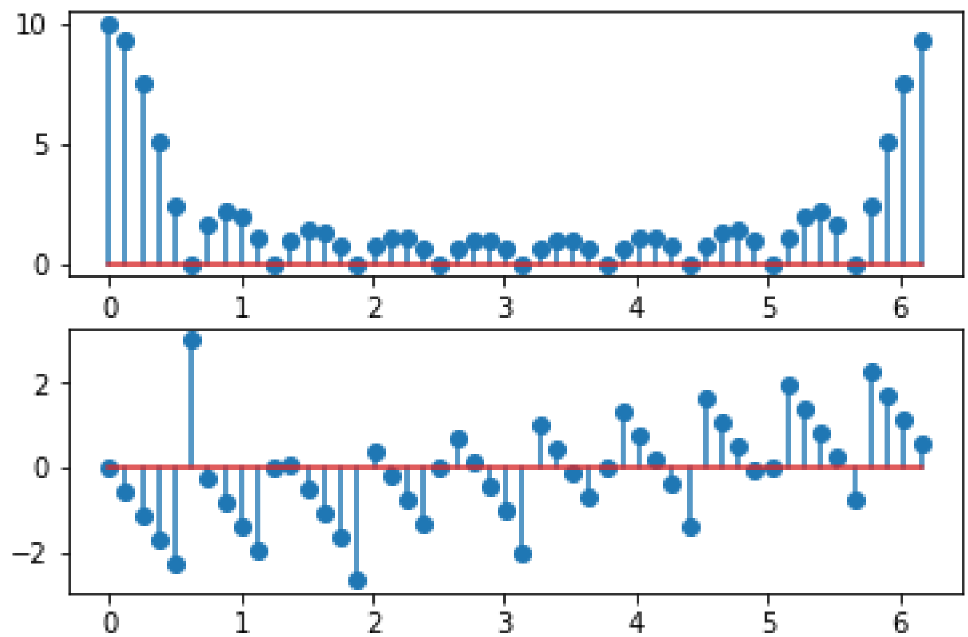
In [7]:
```python
# calculate this numerically as opposed to analytically
from numpy.fft import fft  # fft is a fast *implementation* of the dft

L = 10
x = np.ones(L)
plt.stem(abs(fft(x)));  # N = L
```



Although the L-point DFT is sufficient to uniquely represent the sequence $x[n]$ in the frequency domain, one can see that it does not provide sufficient detail to give a good picture of the underlying spectral characteristics of $x[n]$. We can interpolate $X(\omega)$ at more closely-spaced frequencies by increasing N, which is equivalent to appending $N - L$ zeros to the sequence, also known as *zero padding*:

```
In [11]: N = 50  # try 50 or 100
         w = np.arange(N) * 2 * np.pi / N
         plt.subplot(2,1,1)
         plt.stem(w, abs(fft(x, N)));  # second argument specifies the desired # of points
         plt.subplot(2,1,2)
         plt.stem(w, np.angle(fft(x, N)));  # note what happens when the magnitude is zero
```



1.1.2 7.1.4. Relationship of the DFT to Other Transforms

Relationship to the Fourier series coefficients of a period sequence A periodic sequence can be represented in the form

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N}$$

where the Fourier coefficients are

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{-j2\pi nk/N}, k = 0, 1, \ldots, N-1$$

In comparison, we see that this has the identical form of a DFT; if we let $x[n] = x_p[n], n = 0, 1, \ldots, N-1$, then the DFT is

$$X[k] = Nc_k$$

In conclusion, the N-point DFT provides the line spectrum of the periodic extension of a finite signal $x[n]$.

Relationship fo the Fourier transform of an aperiodic sequence We have already shown that given a periodic finite sequence $x[n]$ with Fourier transform $X(\omega)$, if we sample the spectrum

$$X[k] = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi nk/N}, k = 0, 1, \ldots, N-1$$

then the $X[k]$ are the DFT coefficients of the periodic sequence

$$x_p[n] = \sum_{l=-\infty}^{\infty} x[n - lN]$$

The finite-duration sequence

$$\hat{x}[n] = \begin{cases} x_p[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

is distinct from the original sequence $x[n]$, unless $x[n]$ has finite duration of length $L \leq N$, in which case $\hat{x}[n] = x[n]$, and the IDFT of $X[k]$ will return the original sequence $x[n]$.

Relationship to the z-transform Consider a sequence $x[n]$ with z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

with a ROC that includes the unit circle. If $X(z)$ is sampled we obtain

$$X[k] = X(z)\big|_{z=e^{j2\pi nk/N}}, k = 0, 1, \ldots, N-1$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi nk/N}$$

If the sequence $x[n]$ has finite duration of length $\leq N$, the sequence can be recovered from its N-point DFT; thus its z-transform is uniquely determined by its N-point DFT as follows:

$$X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \right) z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{j2\pi kn/N} z^{-n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} z^{-1}\right)^n$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{1 - \left(e^{j2\pi k}\right)^{N-1} z^{-N}}{1 - e^{j2\pi k/N} z^{-1}}$$

$$= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X[k]}{1 - e^{j2\pi k/N} z^{-1}}$$

When this expression is evaluated on the unit circle:

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X[k]}{1 - e^{j2\pi k/N} e^{-j\omega}}$$

$$= \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X[k]}{1 - e^{-j(\omega - 2\pi k/N)}}$$

it gives an expression for the Fourier Transform in terms of a polynomial (Lagrange) interpolation formula; this is reducable to the interpolation formula seen earlier.