1) Definition of Expectation

$$E[X] = \sum_{i=1}^{N} x_i p_i \text{ where } p_i \text{ is } P(x_i)$$

Assuming equal probability for every x_i

$$E[X] = (1/N) \sum_{i=1}^{N} x_i$$
$$= \mu$$

Definition of Variance

$$Var(X) = E[(X-\mu)^2]$$
$$= E[(X - E[X])^2] \text{ using the above}$$
$$= E[X^2 - 2XE[X] + E^2[X]]$$
$$= E[X^2] - 2E[X]E[X] + E^2[X]$$

By the commnitave property

$$= E[X^2] - 2E^2[X] + E^2[X]$$

$$\boxed{Var(X) = E[X^2] - E^2[X]}$$

We know that

$$E_{x|y}(x) = \int_x f(x)\,p(x|y)\,dx$$

Wraping the E_y term around this expression

$$E_y(E_{x|y}(x)) = \int_y \int_x f(x)\,p(x|y)\,dx\; p(y)\,dy$$

Since $p(y)$ is constant with respect to x we can move it inside the integral with respect to x.

$$E_y(E_{x|y}(x)) = \int_y \int_x F(x)\,p(x|y)\,p(y)\,dx\,dy$$

$$= \int_x \int_y f(x)\,p(x|y)\,p(y)\,dy\,dx$$

We now see that that the integration over y <u>marginalizes</u> the conditional probability.

$$E_y(E_{x|y}(x)) = \int_x f(x)\,P(x)\,dx$$

Therefore, by definition

$$\boxed{E_y(E_{x|y}(x)) = E(x)}$$

4

For a pdf to be proper, its integral over it's defined range must be equal to one.

$$\int p(x)\,dx = 1$$

a) Lets see if this is true:

$$p(x) = 3x^2 \Big|_0^1$$

$$\int_0^1 p(x)\,dx = \int_0^1 3x^2\,dx$$

$$= \left[x^3\right]_0^1$$

$$\boxed{\int p(x)\,dx = 1}$$ So this is a proper pdf.

if $Y = X^2$ then $p(y) = p^2(x)$

The difference is where the bands on y will be

b)

$$\int_a^b 9x^4\,dx = 1$$

$$\left[\frac{9}{5}x^5\right]_a^b = 1$$

$$\boxed{b^5 - a^5 = 5/9,\ 0 \leq a < b}$$

Anywhere this holds true but for a range from $0 \rightarrow$ we let $a = 0$.

$$b^5 = \frac{5}{9},\quad b = \sqrt[5]{5/9}$$

So: $\boxed{P(y) = p(x^2) = 9x^4,\ \left[0, \sqrt[5]{5/9}\right]}$