9.1

February 26, 2019

```
In [2]: # Content from Proakis
        # Code © 2019, Alexander Kain
        import numpy as np
        from matplotlib import pyplot as plt
        %matplotlib inline
```

1 9 Implementation of Discrete-Time Systems

1.1 9.1 Structures for the realization of discrete-time systems

A linear time-invariant (LTI) discrete-time systems given by the general constant-coefficient difference equation

$$y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{k=0}^{M-1} b_k x[n-k]$$

with z-transform

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N-1} a_k z^{-k}}$$

can be implemented in various ways algorithmically, depending on these major factors:

- Computational complexity
- Memory requirements
- Finite-word-length effects

1.2 9.2 Structures for FIR systems

Given are

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k]$$

with z-transform

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

Note that the unit sample response of a FIR system is identical to the b_k coefficients:

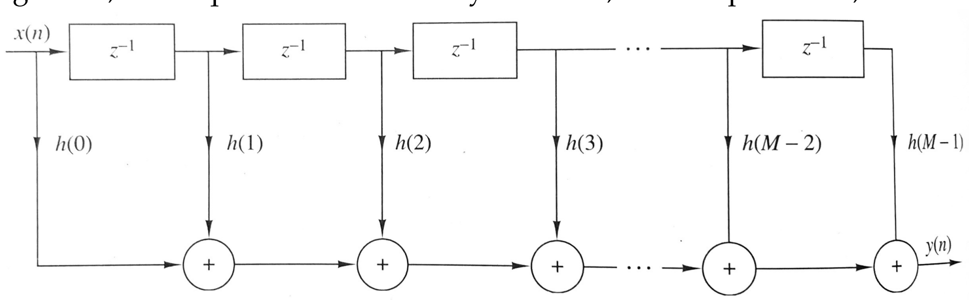
$$h[n] = \begin{cases} b_n, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

One implementation that we have already seen elsewhere is via the FFT.

1.2.1 9.2.1. Direct-Form Structure

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k]$$

In general, this requires $M-1$ memory locations, M multiplications, and $M-1$ additions.




```
In [3]: def fir_filter(b, x):
            Nb = len(b)
            assert Nb > 1
            zb = np.zeros(Nb - 1)
            y = np.empty_like(x)
            for n in range(len(x)):
                y[n] = b[0] * x[n] + (b[1:] * zb).sum()
                zb[1:] = zb[:-1]  # shift stack
                zb[0] = x[n]    # feed stack
            return y

        from scipy.signal import unit_impulse

        x = unit_impulse(7)
        b = np.array([9, 7, 5, 3, -1, 2])
        y = fir_filter(b, x)
        y
```

```
Out[3]: array([ 9.,  7.,  5.,  3., -1.,  2.,  0.])
```

```
In [4]: # using the library (but this implementation is different as we will see later)
        from scipy.signal import lfilter

        y2 = lfilter(b, [1], x)  # setting a_0 = 1
        assert np.allclose(y2, y)
```

```
In [5]: # equivalent
        y2 = np.convolve(b, x)
```

```
assert np.allclose(y2[:len(y)], y)
y2
```

```
Out[5]: array([ 9.,  7.,  5.,  3., -1.,  2.,  0.,  0.,  0.,  0.,  0.,  0.])
```

```
In [6]: # equivalent; for very long b, x, or both
        from scipy.signal import fftconvolve

        y2 = fftconvolve(b, x)  # note numerical precision errors
        assert np.allclose(y2[:len(y)], y)
        y2
```

```
Out[6]: array([ 9.00000000e+00,  7.00000000e+00,  5.00000000e+00,  3.00000000e+00,
              -1.00000000e+00,  2.00000000e+00,  0.00000000e+00, -4.44089210e-16,
               0.00000000e+00,  2.22044605e-16,  5.55111512e-16, -4.44089210e-16])
```

1.2.2 9.2.2. Cascade-Form Structures

We factor $H(z)$ into second-order systems (SOS) so that

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k} = \prod_{k=1}^{K} H_k(z)$$

where

$$H_k(z) = b_{k,0} + b_{k,1} z^{-1} + b_{k,2} z^{-2}$$

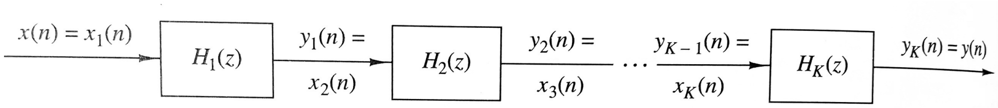
and K is $(M+1)\backslash 2$ (integer division).

The gain factor can be distributed:

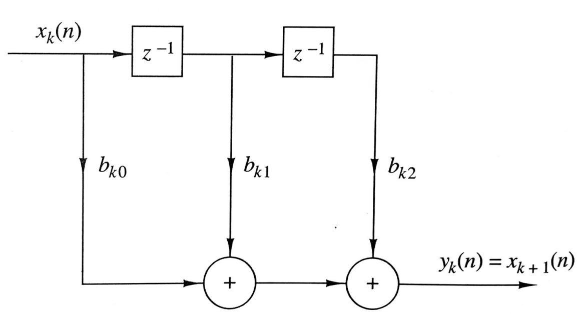
$$b_0 = b_{1,0} b_{2,0} \cdots b_{K,0}$$

It is desireable to form pairs of complex-conjugate roots so that the coefficients remain real-valued; real-valued roots can be paired in an arbitrary manner.

The advantage of this structure is that it can provide better numerical precision.



(a)



(b)

In [7]:
```
# example
x, b
```

Out[7]: (array([1., 0., 0., 0., 0., 0., 0.]), array([9, 7, 5, 3, -1, 2]))

In [8]:
```
k = b[0]  # gain
z = np.roots(b / k)
z
```

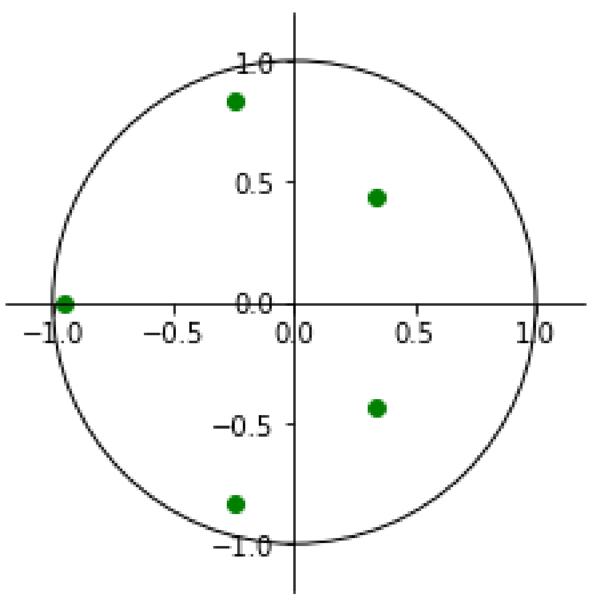
Out[8]:
```
array([-0.95547214+0.j        , -0.25008834+0.83349746j,
       -0.25008834-0.83349746j,  0.33893552+0.43846708j,
        0.33893552-0.43846708j])
```

In [9]:
```
# visualize
from matplotlib.patches import Circle

def zplane(z=np.array([]), p=np.array([])):
    """draw z-plane with poles and zeros"""
    plt.plot(p.real, p.imag, 'rx')  # poles
    plt.plot(z.real, z.imag, 'go')  # zeros

    ax = plt.gca()
    ax.add_patch(Circle((0,0), radius=1, fill=False))
    ax.spines['left'].set_position('center')
    ax.spines['bottom'].set_position('center')
    ax.spines['right'].set_visible(False)
    ax.spines['top'].set_visible(False)
    ax.set_aspect('equal')
    ax.axis([-1.2, 1.2, -1.2, 1.2])

zplane(z)
```



```
In [10]: b0 = np.poly(z[0])
         b1 = np.poly(z[1:3]) * k  # applying gain here, to match later built-in solution
         b2 = np.poly(z[3:5])
         b0, b1, b2
```

```
Out[10]: (array([1.        , 0.95547214]),
          array([9.        , 4.50159008, 6.8153597 ]),
          array([ 1.        , -0.67787104,  0.30713067]))
```

```
In [11]: # check correctness
         b_hat = np.convolve(np.convolve(b0, b1), b2)
         assert np.allclose(b_hat, b)
```

```
In [12]: y0 = fir_filter(b0, x)
         y1 = fir_filter(b1, y0)
         y2 = fir_filter(b2, y1)
         assert np.allclose(y2,  y)
```

```
In [13]: # using the library
         from scipy.signal import tf2sos, sosfilt

         sos = tf2sos(b, [1])  # setting a_0 = 1
         sos
         # Each row corresponds to a second-order section,
         # with the first three columns providing the numerator coefficients,
         # and the last three providing the denominator coefficients.
```

```
Out[13]: array([[ 9.        ,  4.50159008,  6.8153597 ,  1.        ,  0.        ,
                  0.        ],
                [ 1.        , -0.67787104,  0.30713067,  1.        ,  0.        ,
                  0.        ],
                [ 1.        ,  0.95547214,  0.        ,  1.        ,  0.        ,
                  0.        ]])
```

```
In [15]: y2 = sosfilt(sos, x)
         assert np.allclose(y2,  y)
```

```
In [ ]: # example from
        from scipy.signal import ellip  # elliptical filter design

        b, a = ellip(13, 0.009, 80, 0.05, output='ba')
        sos = ellip(13, 0.009, 80, 0.05, output='sos')
        x = unit_impulse(700)
        y_tf = lfilter(b, a, x)
        y_sos = sosfilt(sos, x)
        plt.plot(y_tf, 'r', label='TF')
        plt.plot(y_sos, 'k', label='SOS')
```

1.2.3 9.2.4 Lattice Structure

Consider a *sequence* of FIR filters with system functions

$$H_m(z) = A_m(z) \quad m = 0, 1, \ldots, M-1$$

where

$$A_m(z) = 1 + \sum_{k=1}^{m} \alpha_m[k] z^{-k} = \sum_{k=0}^{m} \alpha_m[k] z^{-k}$$

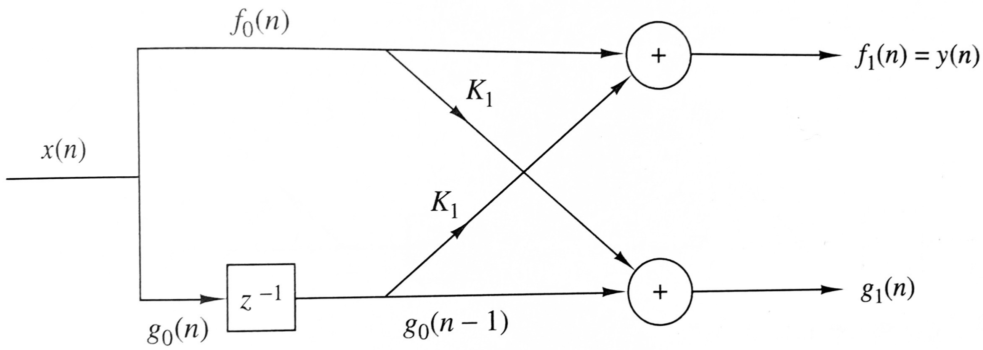
and $\alpha_0 \equiv 1$ and thus $A_0(z) = 1$. The subscript m denotes the degree of the polynomials. The unit sample response of the m^{th} filter is $h_m[k] = \alpha_m[k]$, for $k = 1, 2, \ldots, m$.
Given input $x[n]$ and output $y[n]$ we obtain

$$y[n] = x[n] + \sum_{k=1}^{m} \alpha_m[k] x[n-k]$$

Suppose $m = 1$, then

$$y[n] = x[n] + \alpha_1[1] x[n-1]$$

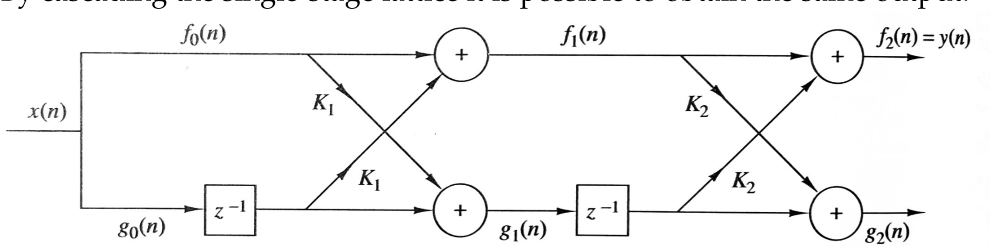
This output can be obtained using a first-order or single-stage *lattice* filter:



where we let the *reflection coefficient* $K_1 = \alpha_1[1]$.
Now let $m = 2$. The output is

$$y[n] = x[n] + \alpha_2[1] x[n-1] + \alpha_2[2] x[n-2]$$

By cascading the single-stage lattice it is possible to obtain the same output:



because

$$f_0[n] = g_0[n] = x[n]$$

the output of the first stage is

$$f_1[n] = f_0[n] + K_1 g_0[n-1] = x[n] + K_1 x[n-1]$$
$$g_1[n] = K_1 f_0[n] + g_0[n-1] = K_1 x[n] + x[n-1]$$

the top output of the second stage is

$$\begin{aligned} y[n] = f_2[n] &= f_1[n] + K_2 g_1[n-1] \\ &= (x[n] + K_1 x[n-1]) + K_2 (K_1 x[n-1] + x[n-2]) \\ &= x[n] + K_1(1+K_2)x[n-1] + K_2 x[n-2] \end{aligned}$$

Comparing to the direct-form equation above we see that $\alpha_2[1] = K_1(1+K_2)$ and $\alpha_2[2] = K_2$, or equivalently $K_2 = \alpha_2[2]$ and $K_1 = \alpha_2[1]/(1+\alpha_2[2])$.

Using a proof by induction, we can demonstrate the equivalence between an m^{th}-order direct-form FIR filter and an m^{th}-order/stage lattice filter, generally described by the following set of order-recursive equations. To start, let $m = 0$:

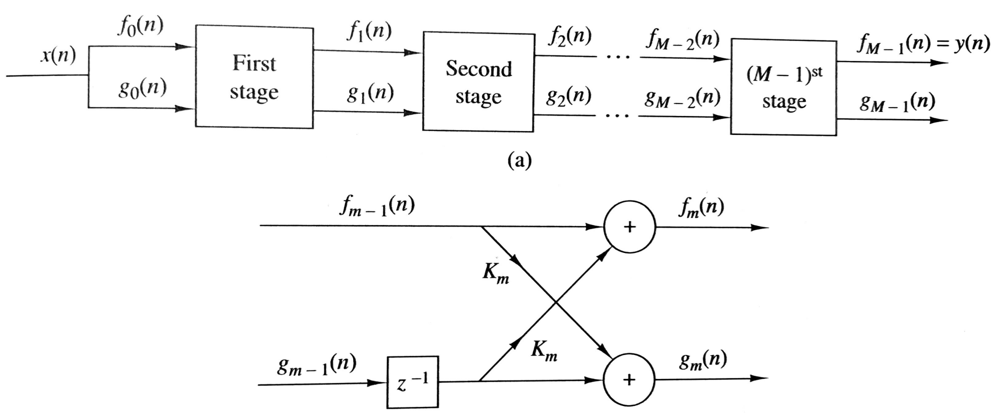
$$f_0[n] = g_0[n] = x[n]$$

Then, for $m = 1, 2, \ldots, M-1$

$$f_m[n] = f_{m-1}[n] + K_m g_{m-1}[n-1]$$
$$g_m[n] = K_m f_{m-1}[n] + g_{m-1}[n-1]$$

Finally,

$$y[n] = f_{M-1}[n]$$



To convert direct-form filter coefficients to lattice coefficients, let $m = M-1, M-2, \ldots, 1$ and calculate

$$K_m = \alpha_m[m]$$
$$\alpha_{m-1}[0] = 1$$

$$\alpha_{m-1}[k] = \frac{\alpha_m[k] - \alpha_m[m]\alpha_m[m-k]}{1 - \alpha_m^2[m]} \qquad k = 1, 2, \ldots, m-1$$

Note that this requires $|K_m| < 1$, otherwise $A_{m-1}(z)$ has a root on the unit circle, which needs to be factored out first.

```
In [14]: # Example 9.2.3

         def tf2latc(b):
             M = len(b)
             K = np.zeros(M)
             K[0] = 1  # implicit
             b = b / b[0]
             b1 = np.empty_like(b)  # allocate memory for the new time-step
             for m in range(M - 1, 0, -1):
                 K[m] = b[m]
                 b1[0] = 1
                 for k in range(1, m):
                     b1[k] = (b[k] - b[m] * b[m-k]) / (1 - b[m] * b[m])
                 b[:] = b1  # switch
             return K

         b = np.array([1, 13/24, 5/8, 1/3])
         k = tf2latc(b)
         k

Out[14]: array([1.        , 0.25      , 0.5       , 0.33333333])
```

To convert from lattice coefficients to direct-form filter coefficients, let $m = 1, 2, \ldots, M-1$ and calculate

$$\alpha_m[0] = 1$$

$$\alpha_m[m] = K_m$$

$$\alpha_m[k] = \alpha_{m-1}[k] + \alpha_m[m]\alpha_{m-1}[m-k] \quad k = 1, 2, \ldots, m-1$$