Homework Assignment 2

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1) **Alternating direction method of multipliers for finding hidden partition of a graph:** In class, we discussed the problem of finding the hidden partitions of a graph by solving the following optimization problem:

$$\begin{aligned} \min_{\mathbf{X}} \quad & \langle \mathbf{X}, \widetilde{\mathbf{A}} \rangle \\ \text{s.t.} \quad & \mathbf{X} \succeq \mathbf{0}, \\ & X_{ii} = 1, \ \ \forall i = 1, \ldots, n, \end{aligned} \tag{1}$$

where $\widetilde{\mathbf{A}} = \lambda \mathbf{1}\mathbf{1}^T - \mathbf{A}$ and $\mathbf{A}$ is the adjacency matrix of the graph. Notice that here we relaxed the constraint $\mathrm{rank}(\mathbf{X}) \leq 1$ in the slides due to its nonconvexity. In this homework assignment, we use ADMM algorithm for solving (1) on a small dataset for recovering the hidden partition of a given graph.

- Download the file *A.txt* which contains the adjacency matrix of a graph with $n = 100$ nodes.
- Use *imagesc($\cdot$)* in Matlab or *imshow($\cdot$)* in Python (or some other command) to visualize the adjacency matrix $\mathbf{A}$. This plot corresponds to the left image discussed in slide 9 of lecture 10. We are going to find a permutation of the nodes leading to the right image and reveal the hidden partition in the graph.
- Explain in few sentences that (1) can be reformulated as the following problem:

$$\begin{aligned} \min_{\mathbf{X}, \mathbf{Z}} \quad & \langle \mathbf{X}, \widetilde{\mathbf{A}} \rangle \\ \text{s.t.} \quad & \mathbf{X} \succeq \mathbf{0} \\ & Z_{ii} = 1, \ \ \forall i = 1, \ldots, n \\ & \mathbf{Z} = \mathbf{X} \end{aligned} \tag{2}$$

- Set $\lambda = 5$ and compute $\widetilde{\mathbf{A}}$.
- Implement ADMM algorithm to find the optimal solution of (2).

- Ideally, we want the optimal solution $\mathbf{X}^*$ to be rank one as we discussed in the lectures. However, since the rank constraint is relaxed initially due to its nonconvexity, the optimal solution will not necessarily be rank one. Since $\mathbf{X}^*$ is PSD, the best rank one approximation of $\mathbf{X}^*$ is given by $\widehat{\mathbf{X}} = \alpha \mathbf{x}\mathbf{x}^T$, where α the largest eigenvalue of $\mathbf{X}^*$ and $\mathbf{x}$ is the corresponding eigenvector.
- Use the sign of the vector $\mathbf{x}$ to partition the nodes into two groups. Use the resulting partition to plot the permuted adjacency matrix.
- Explain in a few sentences why the sign vector should reveal the underlying partition.
- Report the indices of the nodes in each partition.

In the steps of the ADMM algorithm, you may need the following fact: Let $\mathcal{S}_+$ be the set of symmetric positive semidefinite matrices and $\mathbf{B}$ be an arbitrary symmetric matrix. Assume $\mathbf{B}$ has the eigenvalue decomposition $\mathbf{B} = \mathbf{U}\mathbf{D}\mathbf{U}^T$. Then, the projection of the matrix $\mathbf{B}$ onto the set $\mathcal{S}_+$ is given by $\mathbf{B}_+ = \mathbf{U}\widetilde{\mathbf{D}}\mathbf{U}^T$ where $\widetilde{\mathbf{D}}$ is obtained by setting the negative entries of $\mathbf{D}$ to zero.