# AI Agents as Universal Task Solvers: It's All About Time

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AWS Agentic AI

September 12, 2025

#### Abstract

AI reasoning agents are already able to solve a variety of tasks by deploying tools, simulating outcomes of multiple hypotheses and reflecting on them. In doing so, they perform computation, although not in the classical sense — there is no *program* being executed. Still, if they perform computation, can AI agents be *universal*? Can chain-of-thought reasoning solve any computable task? How does an AI Agent 'learn to reason'? Is it a matter of model size? Or training dataset size?

In this work, we reinterpret the role of learning in the context of AI Agents, viewing them as compute-capable stochastic dynamical systems, and highlight the role of time in a foundational principle for learning to reason. In doing so, we propose a shift from classical inductive learning to transductive learning — where the objective is not to approximate the distribution of past data, but to capture the algorithmic structure in the data in order to reduce the reasoning time needed to find solutions to new tasks. Transductive learning suggests that, counter to Shannon's theory, a key role of information in learning is about reduction of time rather than (or in addition to) reconstruction error. In particular, we show that the optimal speed-up in finding a solution that a universal solver can achieve using past data is tightly related to the shared algorithmic information in the training data. Using this, we derive a theoretical justification for empirically observed power-law scaling of inference time versus training time in reasoning models. While the compression view of learning, rooted in Occam's Razor, highlights the value of simplicity and regularization, we show that transductive learning yields the most benefits precisely when the data generation mechanism is highly complex.

We then show that scaling *space* (model size and training data) can lead to behaviors that, while technically improving accuracy on benchmarks, fails any reasonable test of intelligence, let alone super-intelligence: In the limit of infinite space and time, large models can behave as 'savants,' able to brute-force through any task without any insight, even without any learning. Instead, we argue that the key quantity to optimize when scaling reasoning models is time, whose critical role in learning has so far only been indirectly considered.

**Keywords:** AI Agents, Algorithmic Information, Computability, Generative AI, Inductive Learning, Large Language Models, Reasoning, Reinforcement Learning, Scaling Laws, Stochastic Dynamical Systems, Transductive Inference, Occam's Razor.

# 1 Introduction

Most of machine learning focuses on *induction*: fitting a function to labeled data and expecting it to generalize to similar inputs. This perspective is valuable, but incomplete: In an agentic setting, we instead want a pre-trained model to be able to tackle a *specific instance of a novel task* and solve *that instance*. We call this process *transduction*: at test time, the model leverages all available data and actively *reasons* to solve the task at hand, rather than apply a solution fit to past data.

Formalizing the general transduction problem is difficult, since it is unclear what constitutes a task, and how it should be specified. Yet, for many practical tasks it is at least clear how to evaluate candidate solutions once they are proposed. Examples include running unit tests on generated code, gathering user feedback in a goal-directed dialogue, applying a formal verifier to a proof, or computing the energy of a candidate protein configuration. In such cases, the *verifier itself* unambiguously defines the task. Motivated by this, we focus on *verifiable* tasks: tasks where each problem instance x is paired with a task-specific function f(x, y) that can be used interactively to verify or score any candidate solution y.

Importantly, on such tasks, finding a correct answer is, in principle, trivial. One could simply enumerate candidates y until one satisfies f(x, y). This brute-force strategy guarantees success – provided we are

willing to wait exponential time in the length of the solution. By contrast, a bespoke solver, carefully designed for the specific task, may find a solution extremely fast. But what if there was a general solver that can tackle any unseen task as efficiently as the best task-specific solver?

That would seem too good to be true, and likely to violate some kind of 'no free lunch theorem.' Yet, Levin [34] and Solomonoff [49] showed that a universal solver U can be constructed that solves any instance x of a task essentially as well as a solver A that is optimal for that task. In particular, the universal solver finds a solution in time  $T_U$  bounded by:

$$T_U(x) \le 2^{\ell(A)} T_A(x),$$

where  $\ell(A)$  is the description length of A and  $T_A(x)$  is the optimal time to find a solution to that instance. That is, the universal solver requires only a constant factor longer than a task-specific solver, where the constant depends on the *complexity of the optimal solver*, not on the particular instance x. The catch is that such constant factor  $2^{\ell(A)}$  can be astronomically large. This is where *learning* comes in.

In [49] Solomonoff observed that, even if a task has never been faced before, prior experience lets us encode effective problem-solving programs more succinctly — e.g., by reusing components of the solution — thereby reducing the factor  $2^{\ell(A)}$ . Thus, in the transductive setting the value of learning is measured not by a reduction in error rate as in induction, but in the reduction of the time it takes to find solutions to unforeseen tasks. This points to a foundational principle for transductive learning: Rather than trying to capture the statistical structure (joint distribution) of past data in hope that future data will respect it, as in induction, in transduction we want to capture the shared algorithmic structure of past data, given which an agent can reason quickly to find solutions to new computable tasks. Accordingly, to be able to solve general unforeseen tasks an agent should inductively learn to perform transductive inference, also known as "learning to reason" or "learning to learn" (meta-learning).

Inductive learning of transductive inference has been mostly an academic concern for decades, because it seems to require (meta-)learning a conditional distributions over *programs* – impractical until recently. But modern large-scale sequence models, such as large language models (LLMs) or reasoning models (LRMs) can encode distributions over programs, and moreover serve as powerful *computation engines* that can plan, search, call tools, and coordinate multi-step reasoning in ways that do not fit the classical inductive mold of machine learning.

In this expository work, we embrace the transductive view of learning and formalize it in the context of modern reasoning models. We study resource-aware objectives, contextualize Levin's guarantees when using LLMs instead of Universal Turing Machines, which requires entirely different proof techniques, and ground these ideas into practical algorithms that trade compute for generality. We aim for this paper to be a step in shifting the focus of learning from imitation to task solving, with theory that explains why efficiency on unseen problems is not only desirable but provably attainable and conducive to reasoning. The rest of this introduction briefly summarizes the main results of this work, with details in subsequent sections.

## 1.1 Can an LLM-powered AI Agent be a universal solver?

Levin and Solomonoff showed that a universal solver exists, but the construction hinges on using (deterministic) Universal Turing Machines (UTMs). LLMs, by contrast, are neither Turing Machines nor deterministic, nor do they execute code in the conventional sense. Their *computational mechanism* is chain-of-thought reasoning (CoT), which does not map easily to any standard computational paradigm.

To study whether an LLM-powered AI agent can be a universal solver, we need more flexible foundations. In Section 2, we extend universal solvers from programs to general stochastic dynamical systems, allowing us to map the theory directly to LLMs with CoT. A key challenge is defining the *time that an LLM-powered agent needs to solve a problem* since naively using expected time would lead to degenerate values. We address this by introducing a new notion of *proper time*  $\tau$ . Using this, we generalize Levin's and Solomonoff's results to general dynamical systems, thus showing, in particular, that LLMs can indeed power universal task solvers, despite being unlike any Turing Machine.

#### 1.2 Intelligence is about time

Having secured the foundations, we turn to learning. Universal solvers of verifiable tasks are peculiar in that no information needs to be learned to achieve perfect accuracy on any task. For instance, to prove a

theorem one could simply iterate over all possible proofs until a correct one appears, without ever having to study math. Indeed, if we were to measure the information that a training dataset provides to such a task using classical notions (Shannon's [14] or Kolmogorov's [36]), we would find it is null.

The role of learning in agentic AI is instead to identify statistical or algorithmic structures that makes future inference more time-efficient. This suggests a notion, orthogonal to Shannon's, that **information** can be about time rather than space. In particular we show that the optimal speed-up in finding a solution that a universal solver can achieve using past data is tightly related to the algorithmic mutual information [36] between the data and the solution:

**Theorem 1.1:** (Information is Speed) The maximum speed-up a task universal solver can achieve in finding an optimal solution h to a task from training on a dataset D, is

$$\log \text{speed-up} = I(h:D)$$

where I(h:D) is the algorithmic mutual information between the data and the solution.

We will give detailed definitions and proofs in the next sections; for now, we call attention to the fact that data can make a solver *exponentially faster*, consistent with our view of learning as transduction, which can be thought of as *amortized inference computation*.

Scaling laws for speed. Having established that past data can speed up universal solvers, we next examine how much speed-up is achievable as a function of the training dataset size. This requires making modeling assumptions about the data generating process or underlying mechanism.

A common assumption, including in Solomonoff's work, is that real data, while complex on the surface, is generated from low-complexity processes (Occam's Razor [9]). This would intuitively suggest that there are 'common reusable components' we can learn from past data to help future reasoning. This intuition is, however, severely misleading: We show that the maximum speed-up obtainable by a solver is bounded by the complexity of the data generating distribution.

**Theorem 1.2:** (Maximum speed-up Bound) The maximum speed-up in an optimal solution h of a task sampled from a data generating process q is

$$\log \operatorname{speed-up} \leq K(q)$$

where the Kolmogorov complexity K(q) is the length of the shortest program for q.

Notably, if the data was generated by the Universal Prior (as in Solomonoff Induction [47, 48]), there would be precisely nothing to learn (zero information). This challenges the fundamental assumption in generalization theory that *simplicity aids learning* [12]. While simplicity benefits *explainability*, it doesn't necessarily improve learning effectiveness. The fact that simpler models generalize better is a consequence of the definition of generalization in the inductive setting, but in reality simpler models are less effective at transduction.

From the result above, we see that the effectiveness of learning – and the validity of scaling laws – hinges on the data generation process having effectively unbounded complexity. If complexity were bounded at K(q), scaling laws would plateau, yet empirically we observe non-saturating power-law scaling [32, 45]. This characteristic power-law trend is captured by Hilberg's conjecture for human-generated data [29, 17]:

**Definition 1.3: (Hilberg Scaling)** Let  $X_n$  be a training dataset of n tokens and  $Y_n$  be a test set of n tokens, then:

$$I(X_n:Y_n) \propto n^{\beta}$$

grows unbounded according to some distribution-specific rate  $\beta \in (0,1)$ .

We introduce a generalization of this conjecture for arbitrarily-sized  $X_n$  and  $Y_m$ , and prove the following scaling law for speed after learning from data:

Theorem 1.4: (Power Law of Inference vs Training) Let h be a chain-of-thought trajectory solving a task, and let  $T_h$  be its length. If the generalized Hilberg's conjecture holds, the log-speed-up from training on n tokens is

$$\log \text{speed-up} = T_h^\beta - \beta \frac{T_h}{n^{1-\beta}}$$

This result provides a *strong theoretical justification* for the empirically observed power-law scaling of inference time versus training time in reasoning LLMs [45], and can also be used to predict the scaling of space-bound LLMs (when the number of weights, rather than data, is the limit), thus providing guidance on how to scale resources when training universal solvers.

**Inversion of scaling laws.** The results so far dealt with the best model that *could* be learned from the data as we scale up. Empirically, models follow predicted power-law trends, suggesting optimal learning. But is this necessarily true? And is bigger always better?

Current scaling laws use prediction error (or perplexity) as a proxy for intelligence, arguing that more data, bigger models and more compute will lead to "super-intelligence." But, counterintuitively, as the models become more powerful, learning becomes unnecessary since the model can rely more on exhaustive computations rather than insights coming from learned structure in the data. As ordinary scaling happens, better and better performance in the benchmarks comes with less and less insight in the data, all the way to the limit where infinite resources allow solving any task by brute force without any learning. More precisely, emergence of "intelligence" (in the etymological sense *inter legere* or "to pick out from the data") goes hand-in-hand with optimizing a solution under time constraints:

**Theorem 1.5:** (Learning and Time Optimization) Without time penalties, optimal inference can be achieved brute-force without learning. Conversely, any system that optimizes time must learn at least  $I(h:D) = \log$  speed-up bits from past data.

The results above reveal that plots of accuracy-versus-size, routinely used to predict progress towards 'super-intelligence,' can be misleading. By ignoring the cost of time they encourage savantry over intelligence. Intelligent behavior should instead be measured by error reduction per unit time/compute. Properly accounting for the cost of time using the net reward, we see that an optimal agent is one that balances time and accuracy, rather than blindly maximizing reward through brute-force reasoning. This optimal operating point depends on the task, environment and user needs. Hence, it cannot be predetermined during training, but rather has to be calibrated and controlled by the user at deployment to fit the situation at hand.

The theorems above highlight a core tension: in inductive learning, we seek to *minimize* the amount of information we acquire during training, in hope of achieving *generalization*, while in universal solvers we seek to *maximize* memorization in case some "facts" may turn out to be handy when reasoning to solve future tasks.

# 1.3 Maximizing a Continuous Reward

Thus far we focused on systems that need to reach a verifiable goal. More in general, the system may be trying to optimize a continuous reward function R(x, y). This introduces two challenges: we do not know when we found the maximum, and we do not know if the increase in reward from further exploration will compensate the cost (which also depends on the user).

If we have no way to estimate the predicted improvement in reward as a function of the additional effort, tree search algorithms are worst-case optimal [41]. If, on the other hand, we can forecast future rewards we show that the problem can be formalized as a Pandora's box problem [55, 10], which provides both an optimal search strategy and an optimal stopping rule.

# 2 Dynamical Systems as Computation

Let us consider an example of an LLM solving a task using Chain-of-Thought (CoT):

# Example 2.1 What is the largest zero of $x^3 + 5x^2 - 74x + 120$ ?

I need to find the zeros of the polynomial  $f(x) = x^3 + 5x^2 - 74x + 120$ . Let me try to factor this polynomial. First, I'll check if there are any rational roots using the rational root theorem. The possible rational roots are the factors of 120 divided by the factors of 1, which means the possible rational roots are:  $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 15, \pm 20, \pm 24, \pm 30, \pm 40, \pm 60, \pm 120$ .

Let me test some of these: For  $x=1:1+5-74+120=52\neq 0$  For  $x=-1:-1+5+74+120=198\neq 0$  For x=2:8+20-148+120=0

So x = 2 is a root! This means (x - 2) is a factor.

Now I'll perform polynomial division to find the other factor:  $x^3 + 5x^2 - 74x + 120 = (x-2)(x^2 + ax + b)$ 

Expanding the right side:  $(x-2)(x^2+ax+b) = x^3+ax^2+bx-2x^2-2ax-2b = x^3+(a-2)x^2+(b-2a)x-2b$ 

Comparing coefficients:

- Coefficient of  $x^2$ : a-2=5, so a=7
- Coefficient of x: b 2a = -74, so b 14 = -74, thus b = -60

I'll continue factoring  $x^2 + 7x - 60$  by finding two numbers that multiply to -60 and add to 7. The numbers 12 and -5 work perfectly, since  $12 \times (-5) = -60$  and 12 + (-5) = 7. This means I can split the quadratic into (x + 12)(x - 5).

Combining all factors gives  $x^3 + 5x^2 - 74x + 120 = (x-2)(x+12)(x-5)$ , revealing the zeros at x = 2, x = -12, and x = 5. The largest zero is 5.

Clearly, this chain of thought is acting as some kind of computational device. However, it cannot be anything resembling a program since the results of the computation are intermixed with the 'code', e.g.,

For 
$$x = 1, \underbrace{1 + 5 - 74 + 120}_{\text{code}} = \underbrace{52}_{\text{result}}$$

It also cannot be the trace of execution of an underlying algorithm, since the code is generated dynamically based on the output of previous operations. And, unlike standard programs, the particular CoT tokens are not necessarily meaningful: just outputting dots "..." may still lead to the correct result.

A more sound view of CoT is that it is a generic trajectory of a stochastic dynamical system that is *performing computation*: starting from an initial state, it evolves until it becomes a *terminating* state (one where the network is confident it can answer), at which point it outputs a final answer.

Of course, it is well understood that a (deterministic or stochastic) dynamical system can solve computational tasks (Deterministic Finite Automata, Turing Machines, Game of Life, etc.). But LLM systems are quite peculiar: they were not designed to solve a specific task, but rather aim to be *universal solvers*: given a description of a task, the system should be able to find a solution. Moreover, rather than brute-forcing a solution, we expect it to find the fastest way to solve the problem, as well as accessing past information (e.g., the "rational root theorem" in the example above) to significantly speed up the solution.

Our objective is to develop a theory of AI Agents as stochastic dynamical systems that are universal task solvers, and study how learning and information relate to *time*. Our first step is to give a definition of the "time that a stochastic system needs to solve a task," which is surprisingly non-trivial to define.

#### 2.1 Notation

Let  $s \in \mathcal{S}$  be a state in a potentially infinite state space  $\mathcal{S}$ , and let  $t \in \mathbb{N}$  be a time index. A sequence of states  $h = (s_1, \ldots, s_n)$  is called a trajectory or path. Its length is the time T(h) = n. A stochastic dynamical system is defined by the transition probability  $\nu(s_{t+1}|s_t)$ . We say h is a trajectory between two states  $u, v \in \mathcal{S}$  if  $s_1 = u$  and  $s_n = v$ . The probability  $\nu(h) = \prod_{t=1}^{n-1} \nu(s_{t+1}|s_t)$  of a trajectory is the product of the transition probabilities along the path.

The system should be able to read inputs and output answers. Let  $\Sigma$  be the input/output alphabet. We assume that the system has a set  $\mathcal{F} \subset \mathcal{S}$  of terminating states, and a function dec :  $\mathcal{F} \to \Sigma^*$  that, given a

terminal state generates an output. We also assume that there is a function enc:  $\Sigma^* \to \mathcal{S}$  that encodes the input into a state of the dynamical system, where  $\Sigma^*$  is the set of all possible finite trajectories. We say that a trajectory  $h = (h_1, \ldots, h_t)$  terminates with output a— which we write  $h \downarrow a$ — if  $h_t \in \mathcal{F}$  and  $dec(h_t) = a$ .

Let x be an input. For simplicity, we assume that all trajectories starting from  $\operatorname{enc}(x)$  and ending in a terminating state terminate with the same output, or with a special  $\operatorname{\mathsf{<error}}$  token. This allows us to write  $\nu(x \downarrow a)$  meaning the dynamical system  $\nu$  starting from  $\operatorname{enc}(x)$  terminates with a. While this is generally restrictive, we mainly study settings where the answer is verifiable, in which case we can trivially return error if the output is not correct.

#### **Example 2.2** Two key systems we are interested in are:

**LLMs.** The state is the state of the LLM after reading some tokens. This is the KV cache for an autoregressive Transformer, or more generally the hidden state for a State Space Model. The transition function  $\nu(s_{t+1}|s_t)$  generates the next token given the state  $s_t$  and uses it as input to generate state  $s_{t+1}$ . The final states are the states at which the LLM outputs an  $\end_{of\_thought}$  token. The decoding function consists of letting the network generate the answer after  $\end_{of\_thought}$ . The encoding function simply lets the LLM read the input tokens to update its initial state.

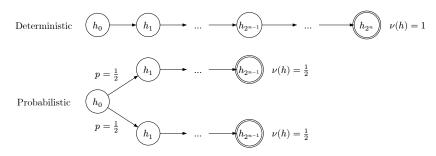
**Turing Machines.** The state of a Turing machine is the content of its tape at a given time, plus its internal state. The transition function  $\nu(s_{t+1}|s_t)$  updates the tape and its internal state as usual, either deterministically in a standard Turing machine, or randomly in a probabilistic machine.

We also make use of several notions from algorithmic information theory [36]. Let x be a string, its Kolmogorov Complexity K(x) is defined as the length (in bits) of the shortest program that terminates outputting x. Given two strings x and y, their algorithmic mutual information is I(x:y):=K(x)+K(y)-K(x,y)=K(x)-K(x|y) (up to logarithmic additive terms). This can be interpreted as how much more x can be compressed if we have already observed y. Recall that, by the coding theorem, given any probability distribution  $\nu(x)$  over binary strings, there is a corresponding encoding algorithm that encodes a string x in  $\ell_{\nu}(x) := -\log_2 \nu(x)$  bits.

# 2.2 Proper Time

As we have anticipated, transductive learning is about solving generic tasks quickly. But how do we define the *time* that a *stochastic* system needs to solve the task? The question is subtle, since if we look at the length of a particular sampled trajectory, randomness can make an algorithm look arbitrarily faster or slower, without changing what it effectively computes.

Let us first consider a motivating example. Let f(x) be a function that is easy to evaluate, but can be inverted only through brute-force search (i.e., a 'one-way function'). Given y, the task is to find a binary string x of length |x| = n such that y = f(x). A deterministic Turing machine must try all  $2^n$  candidates for x, so the total time is  $T = 2^n$ . On the other hand, a stochastic machine can guess the first k of x, and brute-force the remaining n - k, so every terminating trajectory has length  $T = 2^{n-k}$ , but occurs with probability only  $\nu(h) = 2^{-k}$ .



<sup>&</sup>lt;sup>1</sup>A more standard and less restrictive definition is to ask that the answer is correct at better than chance level, that is we would say  $\nu(x\downarrow a)$  if  $\mathbb{P}[h\downarrow a|h_1=\mathrm{enc}(x)]>2/3$ .

The probabilistic machine can then be arbitrarily faster than the deterministic machine as measured by the trajectory length T(h) even if, effectively, both are doing the same brute force search. If however we consider the ratio  $\tau(h) = T(h)/\nu(h) = 2^n$ , we see that it remains constant, no matter how we branch the computation path: randomness shortens paths but also makes them rarer. This invariance suggests the following definition of "proper" time for single-trajectory targets.

**Definition 2.3:** (Proper Time) Let  $\nu(h_{t+1}|h_t)$  be a stochastic dynamical system. Define the proper time to reach v from u as:

$$\tau_{\nu}(u \to v) = \min_{h_{u \to v}} \frac{T(h)}{\nu(h|x)}$$

where the minimum is over all trajectories h from u to v, or  $\tau_{\nu} = \infty$  if no trajectories exist. For an input-output specification, the proper time to terminate from input x with output a is:

$$\tau_{\nu}(x \downarrow a) = \min_{h \downarrow a \text{ and } h_1 = \operatorname{enc}(x)} \frac{T(h)}{\nu(h|x)},$$

where the minimum is over trajectories starting from enc(x) and terminating with output a.

This definition is closely related with Levin Complexity [35], and its extension to tree search [41]. If the system is deterministic, then  $\nu(h|x)=1$  for any path, and  $\tau_{\nu}$  reduces to standard running time. Conversely, we now show that  $\tau_{\nu}$  indeed captures the actual computational effort required by a stochastic dynamical system when simulated deterministically.

Theorem 2.4: (Dynamical Systems  $\Rightarrow$  Turing Machines with same  $\tau$ ) Let  $\nu(s_{t+1}|s_t)$  be a dynamical system. There is a deterministic Turing machine  $M_{\nu}$ , with access to an oracle to compute  $\nu(\cdot|\cdot)$ , such that:

$$T_{M_{\nu}}(x \downarrow a) \leq 2 \tau_{\nu}(x \downarrow a)$$

The theorem follows directly by taking  $M_{\nu}$  to be the Turing Machine that implements the algorithm in the following key lemma:

**Lemma 2.5:** (Levin Tree Search [41]) Let u, v be two states. There is a deterministic algorithm A that discovers a path between them (if it exists) while visiting at most T nodes where

$$T = 2 \tau_{\nu}(u \to v)$$

Proof sketch. Let  $h^*$  be a path realizing the minimum  $\tau_{\nu}(h^*) = \tau_{\nu}(u \to v)$ . We show that a simple greedy priority search will find  $h^*$  in at most  $\tau_{\nu}(h^*)$  steps. Maintain a frontier  $S_t$  of partial paths under consideration, initialized with the trivial path at u. At each step t, extend the  $h \in S_t$  whose one-step extension h' minimizes  $\tau_{\nu}(h')$ . An induction on the search tree shows that within at most  $T = 2\tau_{\nu}(h^*)$  steps  $h^*$  enters the frontier.

Since all computation today is executed on deterministic logic hardware, Theorem 2.4 validates  $\tau_{\nu}$  as the "proper" way to measure time for a stochastic dynamical system.<sup>2</sup> It also frames  $\tau$  as a fundamental property of the algorithm we are executing, rather than a function of the stochasticity of its implementation.

In a deterministic system, the time (path length) between states is a distance. Similarly, the following theorem establishes that for a path  $x \to y \to z$ , the proper time to go from  $x \to z$  cannot be greater than the time it takes to first go to y and then to z. It will play an important role in multiple proofs.

Lemma 2.6: (Proper Time is submultiplicative) Let x, y, z be three states. Then:

$$\tau_{\nu}(x \to z) \le \tau_{\nu}(x \to y) \cdot \tau_{\nu}(y \to z)$$

<sup>&</sup>lt;sup>2</sup>The name has a loose analogy with relativistic proper time: like  $\tau = t^2 - x^2$ , our  $\log \tau_v = T - \log v(h)$  mixes temporal and the probabilistic 'distances,' providing a representation-invariant clock.

*Proof.* Let  $h_{x\to y}$  and  $h_{y\to z}$  be paths that realize the minimum in the definition of  $\tau$ . We can construct the path  $h_{x\to z} = h_{x\to y} \circ h_{y\to z}$  composing the two paths. By Definition 2.3 we have:

$$\tau_{\nu}(x \to z) \le \frac{T(h_{x \to z})}{\nu(h_{x \to z})} = \frac{T(h_{x \to y}) + T(h_{y \to z})}{\nu(h_{x \to y})\nu(h_{y \to z})}$$
(1)

$$\leq \frac{T(h_{x\to y})T(h_{y\to z})}{\nu(h_{x\to y})\nu(h_{y\to z})} \tag{2}$$

$$= \tau_{\nu}(x \to y) \, \tau_{\nu}(y \to z) \tag{3}$$

Where in the second line (2) we used the fact that  $T_1 + T_2 \le T_1 T_2$  whenever  $T_1 \ge 2$  and  $T_2 \ge 2$  which is automatically satisfied when the states are distinct. If two or more states are the same, the property can be easily checked by hand.

This also implies that  $\log \tau(x \to z) \le \log \tau(x \to y) + \log \tau(y \to z)$ , which makes  $\log \tau$  an asymmetric distance between states. Note that  $\tau$  is sub-multiplicative, while deterministic time is sub-additive. This is because in a stochastic system, time may be dominated by the time to find a suitable combination of paths, and probability of the composition of two paths is a product.

These two lemmas suffice to prove the key theorems in the rest of this work, including providing a straightforward construction for a generalization of Solomonoff-Levin Universal Search Algorithm [34, 49].

### 2.2.1 Side note: Multiple successful paths

The quantity in Definition 2.3 measures the cost to uncover a particular trajectory. Many tasks, however, accept any trajectory leading to one of the final states  $s \in \mathcal{F}$ . In that setting, multiple distinct paths can succeed, and the right notion aggregates their probabilities. Let

$$F_{\nu}(t; \mathcal{F}) \stackrel{\text{def}}{=} \Pr(\text{reach } \mathcal{F} \text{ within } t \text{ steps})$$

be the *success-by-time* curve. If we run independent trials of length t (restarting after t steps), we need in expectation  $1/F_{\nu}(t;\mathcal{F})$  trials for one success, for total expected work  $t/F_{\nu}(t;\mathcal{F})$ . Optimizing over the cutoff gives a canonical baseline:

$$\tau_{\nu}^{*}(\mathcal{F}) \stackrel{\text{def}}{=} \inf_{t>1} \frac{t}{F_{\nu}(t;\mathcal{F})}.$$

This general notion (i) strictly improves over any single-path bound when many solutions exist, and (ii) collapses to the proper time when there is effectively one successful path. In principle, to simulate the stochastic system in total time  $\tau_{\nu}^{*}(\mathcal{F})$  we would need the unknown optimal cutoff t. However, universal Luby-type restart schedules [39, 41] achieve expected work within a logarithmic factor of the optimum fixed-cutoff policy:

Expected work = 
$$O(\tau_{\nu}^*(\mathcal{F}) \log \tau_{\nu}^*(\mathcal{F}))$$
.

Thus  $\tau_{\nu}^{*}(\mathcal{F})$  characterizes intrinsic difficulty 'up to logs.' For clarity of exposition, in the rest of the paper we focus on  $\tau_{\nu}(h)$ , but all results extend naturally to  $\tau_{\nu}^{*}(\mathcal{F})$ .

# 3 Universal Solvers

We are now finally ready to introduce Universal Solvers, which are our main focus. A universal solver is a dynamical system that can efficiently find a solution to an arbitrary problem, if one exists. We formalize it as follows.

Let  $f(x,y): X \times Y \to \{0,1\}$  be a computable function. We say that y is a witness of x if f(x,y) = 1. A universal search program is any program S that, provided with an oracle for f and an input x, terminates with outputting y such that f(x,y) = 1 (we generalize this to continuous rewards in Section 6):

$$S(x \downarrow y) \iff f(x,y) = 1$$

If y does not exist, the program is allowed to terminate with an error or not terminate at all. Generally, together with the input x we may also pass a description of the objective f(x, y) so the search program is not blind. To keep the notation uncluttered, we do not denote this additional input.

It is always possible to find a witness to any problem by just enumerating all possible y in a dovetail fashion and checking for f(x,y) = 1 using the oracle for f. However, we are interested in search programs that are as efficient as possible.

**Definition 3.1:** (Universal Solver) A dynamical system U is a universal solver system if, for any objective f(x,y) and any other system A that solves the problem — i.e., for all x,  $A(x \downarrow y)$  with f(x,y) = 1 — we have:

$$\tau_U(x\downarrow y) \le C_A \tau_A(x\downarrow y).$$

That is, for any task, a universal solver is at most a constant factor slower than the best possible system A that solves that particular task. The existence of a universal solver is non-trivial. Levin introduced the notion of universal search, as well as a sketch of the existence of such a system in the same paper that introduced the notion of NP-Complete problems [34]. Solomonoff later realized its importance for machine learning, and provided a detailed proof [49]. With the formalism we introduced, the existence proof is straightforward and can be generalized to any stochastic system, with Turing Machine as a special case.

**Theorem 3.2:** (Existence of Universal Solver) Let m be any distribution encoding programs from which we can sample. Then, there is a dynamical system  $U_m$  such that for any solver A:

$$\tau_{U_m}(x\downarrow y) \le C_A' 2^{-\log m(A)} \tau_A(x\downarrow y).$$

In particular,  $U_m$  is a universal solver with constant  $C_A = C'_A 2^{-\log m(A)}$ .

*Proof.* Let U be a universal solver. Construct a composite dynamical system  $U_m$  as follows. First, given x use m to sample a program encoding [A], and append it to the input to get x[A]. Then run the universal system U to execute [A] on x.

Let A be any algorithm that solves the search problem. Then  $U(x[A] \downarrow y)$ , giving us a path  $x \to x[A] \to y$  from the input x to the solution y. Applying the submultiplicativity of  $\tau$  to this path from Lemma 2.6, and using the definition of universal solver (Definition 3.1) we have:

$$\tau_{U_m}(x \downarrow y) \le \tau_m(x \to x[A])\tau_U(x[A] \downarrow y) \tag{4}$$

$$\leq C'_A \tau_m(x \to x[A]) \tau_A(x \downarrow y)$$
 (5)

Note that by definition of  $U_m$  we have  $\tau_{U_m}(x \to x[A]) = T/m(A) = 1/m(A)$  (where we assume the entire program [A] is sampled in one step). Replacing this identity in the above we get:

$$\tau_{U_m}(x\downarrow y) \le C_A' 2^{-\log m[A]} \tau_A(x\downarrow y)$$

Which gives the desired time bound.

The construction above instantiates a particular universal solver which first 'guesses' a program that may solve the task and then executes it. Of course, in general, universal solvers need not be a one-shot guess: human problem-solvers will not blindly guess an algorithm and execute it, but will rather interleave partial computations, observations, backtracking and shortcuts. Our general stochastic dynamical-system view already subsumes such interactive behavior. Nonetheless, the search algorithm presented is universal (as in, no other can be significantly faster) and already highlights some interesting and general observations:

**Speed of a universal solver.** For any solver A that succeeds on x, the universal solver above solves the problem in time:

$$\tau_{U_m}(x \downarrow y) \leq C_A 2^{-\log m(A|x)} \tau_A(x \downarrow y),$$

that is, the slowdown with respect to an arbitraty solver is the simulation constant  $C_A$  times the inverse prior weight of the right program. This bound highlights two levers for learning. The term  $2^{-\log m(A|x)} = 2^{\ell_m(A|x)}$  depends on the code length  $\ell_m$ : if tasks reuse a small set of subroutines, reshaping m to give these short codes yields exponential gains (we return to this in Section 4). The factor  $C_A$  reflects how many steps our base dynamics spend simulating a single step of A; when particular transition patterns recur, we can 'macro-step' them—effectively adding shortcuts in the dynamics—to shrink  $C_A$ . These ideas extend beyond the guess–execute prototype to any universal search system and we will study them in details in later sections.

Universal Search and Universal Computation. From the proof of Theorem 3.2, we see that starting from a universal system we can easily construct a time-optimal universal search system. The following straightforward theorem shows that the converse also holds. For a system to be a universal solver it needs to be a universal computation system.

Theorem 3.3: (Universal Search  $\Rightarrow$  Linear-Time Universal System) Let U be an optimal universal search program. Then it is also a universal dynamical system.

*Proof.* Let M be a Turing machine. Construct the function f(x,y)=1 if M(x)=y and 0 otherwise. By definition of optimal universal search we have  $U(x[f])\downarrow y\iff f(x,y)=1$  which implies M(x)=y by construction. Moreover

$$\tau_U(x) \le C_M \tau_M(x \to x[M]) T_M(x) \tag{6}$$

$$=C_M'T_M(x) (7)$$

therefore it is linear-time universal.

The claim is straightforward but it has an important implication: if we train a model to solve a sufficiently general set of tasks, the model will necessarily learn to simulate a Universal Turing Machine.

**Optimality through meta-learning.** An important property of a universal search system is that it is necessarily optimal, meaning that no other universal search system can be arbitrarily faster:

**Lemma 3.4** Let U and U' be two universal search systems. Then, for any task f we have:

$$\tau_{U}(x\downarrow y) \leq C_{U'}\tau_{U'}(x\downarrow y).$$

where the constant  $C_{U'}$  does not depend on the task f or the input x.

*Proof.* Since by universality U' finds the solution to the task f, we can take A = U' in the definition of universal search, giving us

$$\tau_U(x \downarrow y) \le C_{U'}\tau_{U'}(x \downarrow y)$$

Hence, U is at most  $C_{U'}$  times slower than U', where  $C_{U'}$  does not depend on the task f.

The proof is a trivial manipulation of the definitions, but it underlies a key concept, which in modern terminology would be called meta-learning. Let's use the particular universal system in Theorem 3.2 to make the point explicit. For it, the time required to solve a task depends on  $-\log m(A)$ , the encoding length of its optimal solution. It is a priori possible that a system U' may achieve a better time on some tasks by learning a better encoding m' specific for them. However, U can just search (meta-learn) the solver U', and use it to solve the task leading to a slow down of at most  $2^{-\log m(U')}$ . In practice, the constant  $2^{-\log m(U')}$  is too large, and we need to amortize it through learning, which is our focus for most of this work.

#### 3.1 Universal Solvers and Sampling

By Theorem 2.4 we can convert a stochastic system  $\nu$  to a deterministic program that finds a solution in time  $T = \tau_{\nu}(x \mapsto y)$ . However, this program is *not* obtained by naively sampling a random trajectory up to completion, as one may be tempted to do. In fact, doing that would have an infinite expected runtime:

Note 3.5 Let  $\nu$  be any computable prior that gives mass to all programs, the Universal Prior), then  $\mathbb{E}_{A \sim \nu}[T_A] = \sum \nu(A)T_A = \infty$  even assuming we have an oracle preventing us from running algorithms that do not terminate. To see it, consider the program A that computes  $\nu(A)$  and runs for  $\nu(A)^{-1}$  steps before terminating. Then  $\nu(A)T(A) = 1$  and there are infinite such programs in the expectation.

This highlights an important principle: if we have a way to guess a possible solution, in general it is not time-optimal to keep generating guesses and testing them. For an LLM, this means that sampling CoT traces until one succeeds is not a good idea. Rather, we need to keep open multiple possibilities and smartly allocate time budget between all of them. To add some color, imagine trying to prove a theorem. You will likely start with the most likely guess, but if it starts to take too long with no solution in sight, you will try spending some time on another approach and perhaps come back to the original approach later.

The construction in Theorem 2.4 which achieves  $\tau$  on a deterministic system can be made into a stochastic algorithm. The algorithm above hinges on keeping multiple hypotheses at the same time and continuing to explore them with increasingly more budget. What prevents us from having a system that achieves the same expected time by sampling individual trajectories?

We have seen before that such a system cannot sample programs directly from  $\nu(A)$  as the expected time could easily be infinite. A good guess is that we need to sample from the distribution<sup>3</sup>

$$\nu_t(A) = \frac{1}{Z} \frac{\nu(A)}{T(A)},\tag{8}$$

which prioritizes programs that have a short running time. This is indeed the case:

Theorem 3.6: (Time-Weighted Sampling) Let  $\nu$  be a universal search system. If we sample trajectories from:

$$h \sim \nu_t(h|x) \propto \frac{\nu(h|x)}{T(h)}$$
 (9)

and run them to completion, the total amount of operations we need to perform before finding a solution is:

$$\mathbb{E}[T_{\text{total}}] = \tau_{\nu}(x \mapsto y). \tag{10}$$

*Proof.* Let  $h_*$  be a trajectory solving the task, and let  $n_*$  denote the number of iterations before  $h^*$  is sampled. In expectation, we have  $\mathbb{E}[n_*] = \frac{1}{\nu_t(A)} = Z\frac{T(h)}{\nu(h)}$ . We now need to compute how much time is spent validating each of the  $n_*$  samples. The expected time that we need to spend validating a single sample from  $\nu_t$  is:

$$\mathbb{E}_{A \sim \nu_t}[T_A] = \frac{1}{Z} \sum_i \frac{\nu(A_i)}{T_{A_i}} T_{A_i} = \frac{1}{Z}$$
 (11)

so the total time we need to spend validating the  $n_*$  is:

$$T_{\text{total}} = \mathbb{E}[T_1 + T_2 + \dots + T_{n^*}] = \mathbb{E}[n_*]\mathbb{E}[T_i] = Z\frac{T(A)}{\nu(A)}\frac{1}{Z} = \frac{T(A)}{\nu(A)}$$
(12)

which gives the desired result.

Hence, a universal search algorithm that only wants to consider one guess at the time has to learn how to sample from  $\nu_t(h)$ , which means that in addition to estimating the probability  $\nu(h)$  that a solution is correct, it should also be able to predict the time T(h) that it will take to run it.

The distribution  $\nu_t$  is actually computable (the time T(h) may be undecidable, but to upper bound  $\nu_t(h)$  within  $\epsilon$  we just need to show that  $\nu(h)/T(h) < \epsilon$  and hence run for  $T = \nu(h)/\epsilon$  steps). However, in [24] it is shown that  $\nu_t(h)$  takes double exponential time in  $1/\epsilon$  to approximate, and doing so essentially requires running multiple programs, which we want to avoid to begin with.

 $<sup>^3</sup>$ This distribution is closely related to Schmidhuber's Speed Prior [43] and Filan et al.'s  $S_{\mathrm{Kt}}$  prior [24].

Hence, the only option left if we want to avoid searching over trajectories is to *train* a system to approximate both the likelihood of solution and the cost of time. While this will not be our focus, the following importance weighted training scheme gives a way to train:

**Theorem 3.7:** (Importance-Weighted Training) Let  $\nu$  be a dynamical system. Let  $\mathbf{h} = (h^1, \dots, h^n)$  be a batch of trajectories sampled from  $\nu$ . Then the distribution  $\mu^*$  minimizing:

$$\mu^* = \arg\min_{\mu} \mathbb{E}_{\mathbf{h}} \left[ \sum_{i} w_i \mu(h^i) \right] \quad \text{with} \quad w_i = \frac{T(h^i)}{\sum_{j} T(h^j)}$$
 (13)

is exactly  $\mu^* = \nu_t$ .

# 4 Scaling Laws for Speed

By definition of universal solver, given a function f(x,y) and an input x, there is a trajectory h finding a witness x if such a witness exists. Let h be the shortest such trajectory, i.e., the one with minimal T(h). The total time required by the universal search system to find it is:

$$\tau_U = 2^{-\log\nu(h)} T(h) = 2^{\ell_\nu(h)} T(h) \tag{14}$$

where we defined  $\ell_{\nu}(h) = -\log \nu(h)$  to be the compression length of the trajectory using  $\nu$ . How do we reduce the search time  $\tau_U$ ? We could reduce the thinking time T(h) by learning to skip some steps to get directly to the solution (see Section 7). But the largest improvement will come from reducing the exponential factor  $2^{\ell_{\nu}(h)}$ . This is the time needed to guess the correct solution to the problem. Due to the coding theorem, we can improve the probability of guessing the solution, thus speeding up the search, by instead finding a way to reduce the compression length of h. We can do this by learning from a dataset D.

For example, suppose we have a list of programs that have worked well in the past. If we notice that some pieces of code tend to appear frequently (say, the code to compute an FFT), we could change the encoding to replace those pieces of code with a unique name. This reduces the length of those programs making them more likely to be sampled. Not only that, but any program reusing those components is more likely to be guessed in the future.

Another example to add color: suppose that while proving theorems we often use the same sequence of steps. We probably will want to turn it into a named theorem — e.g., "Cauchy–Schwarz inequality" — which will also make us more likely to try to use it in future problems. In this spirit, let's crystallize this in the following:

**Theorem 4.1:** (Better compression  $\Leftrightarrow$  Faster search) For a universal search system with model  $\nu$ , improving the compression of a trajectory h by  $\Delta$  bits accelerates its discovery by a factor of  $2^{\Delta}$ .

Let's now formalize what learning from data means. Given some data D, we denote by:

$$\ell_{\nu}(h|D) = -\log \nu(h|D) \tag{15}$$

the negative log-likelihood given by the model to a trajectory after observing the dataset D. One possibility is that we train on the data D. In this case, we assume that  $\nu_{\theta}$  is a parametrized family of distributions. Let  $\theta_D$  be the parameters obtained after training on D. Then we define  $\nu(h|D) := \nu_{\theta_D}(h)$  as the likelihood given to h by the trained model. Alternatively, we can do in-context learning (ICL) or prompting where we feed the data D to the model to obtain a state  $s_D$ , and then we set  $\nu(h|D) := \nu(h|s_D)$  the likelihood of the trajectory after having seen the data. It could also be that the model  $\nu(h)$  has a way to retrieve information from D, a process known as retrieval-augmented generation (RAG). And any mix of these methods may be used (some data is used to train, other to prompt, other is used for retrieval). While different in implementation, from a theoretical perspective there is no fundamental difference between these ways of using past data, and which is why we can write generically  $\nu(h|D)$ .

In our setting, the benefit of learning is not measured by better accuracy — since we have a verifier, sooner or later, we will find a correct solution — but rather by the reduction in search time. The speed-up

factor achieved after training on the data is given by the ratio:

$$\frac{\tau_{\nu}(h)}{\tau_{\nu_D}(h)} = 2^{\ell_{\nu}(h) - \ell_{\nu}(h|D)} = 2^{I_{\nu}(h:D)}$$
(16)

where we defined the  $\nu$ -algorithmic mutual information:

$$I_{\nu}(h:D) = \ell_{\nu}(h) - \ell_{\nu}(h|D) \tag{17}$$

So, the speed up of universal search is given by the algorithmic mutual information between inference time trajectories and past data. Let's highlight this:

**Theorem 4.2: (Information is speed)** The log-speedup of a search algorithm after seeing some data is:

$$\log \frac{\tau_{\nu}(h)}{\tau_{\nu}(h|D)} = I_{\nu}(h:D) \tag{18}$$

We are interested in  $\nu$  that are very good compressors (since we want to minimize  $\ell_{\nu}(h)$ ). Asymptotically, the best compressor is the universal prior  $m(h) \propto 2^{-K(h)}$  for which  $I_m(X:Y)$  becomes the algorithmic mutual information [36]:

$$I(h:D) = K(h) - K(h|D) = K(h) + K(D) - K(Dh).$$
(19)

While we are interested in  $I_{\nu}(h:D)$ , we can use I(h:D) as a proxy of what is the best we could achieve asymptotically. The advantage is that I(h:D) has a number of theoretical properties that make it easier to work with.

The key question now is: what is the maximum possible log-speedup I(h:D) we can get from learning? As it turns out, the answer is not straightforward and depends on some key assumptions about how real data works. Let's get there step-by-step.

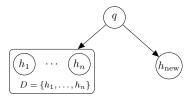
The trajectory h is the trajectory of an optimal solution to a task (e.g., the optimal CoT to get to a solution, or the shortest execution trace of a program that solves the problem). Meanwhile  $D = \{h_1, \ldots, h_n\}$  is presumably created by collecting examples of trajectories that optimally solved tasks in the past.

A first guess (often done in the Minimum Description Length literature) is that solutions to real world problem tend to have low complexity. It therefore may make sense to hypothesize that  $h_i \sim m(h) = 2^{-K(h)}$  is sampled from the universal prior itself, which favors low-complexity solutions. What would be I(h:D) in this setting? Disappointingly, we can show that:

$$\mathbb{P}[I(h:D) > k] \le nc2^{-k} \tag{20}$$

so the probability that past data D share substantial information with the solution to the present task h, and therefore can lead to substantial speedup through learning, is vanishingly small. This does not bode well for the possibility to learn a fast universal solver.

To see what happened, it is useful to abstract a bit. Suppose we have a mechanism q(h) generating trajectories. Let  $D = \{h_1, \ldots, h_n\} \sim q$  be data seen in the past (our training set), and let  $h_{\text{new}} \sim q$  be a new data we are trying to find at inference time. This forms a graphical model:



where q acts as a separator between past and future data. By the Data Processing Inequality [14] this implies

$$I(D:h_{\text{new}}) \le I(D:q) \le K(q). \tag{21}$$

That is, since  $h_{\text{new}}$  is sampled i.i.d. from q, the only information that the past data D can provide about  $h_{\text{new}}$  is information about q itself, and this cannot be larger than its description length K(q).

Theorem 4.3: (Maximum speedup is bound by world complexity) The maximum speed-up we can obtain using data generated by a process q is:

$$\log \frac{\tau_{\nu}(h)}{\tau_{\nu}(h|D)} = K(q) \tag{22}$$

Since the universal prior m(h) has low Kolmogorov complexity K(m) = O(1), there is nothing we can learn from it. More generally, whenever the data is generated by a low-complexity distribution, no matter how much data we observe, we will never be able to obtain more than a constant time speed up.

This gets to a key question about what is the scaling law of information for real world data. To study it further, it is useful to reframe the question a bit. We have been thinking of q as a mechanism that generates i.i.d. sample of trajectories. This may be restrictive. More generally, let q be a dynamical process generating a sequence  $x_1, x_2, \ldots$  of tokens. Let  $X_n = x_{1:n}$  and  $Y_m = x_{n:n+m}$  be an initial sequence of length n and its continuation of length n. We can think of  $X_n$  as our training set (past data) and  $Y_m$  as our test set (future data we are trying to predict). It may be useful to think of  $X_n$  and  $Y_m$  to be natural language text, or code.

We want to know how  $I(X_n; Y_m)$  scales when  $n, m \to \infty$ . Let's suppose q is a finite-dimensional Markov process with a discrete D-dimensional hidden state  $s \in S^D$  over some alphabet S. What information can  $X_n$  provide about  $Y_m$ ? By the Markov hypothesis, the only information that  $X_n$  can provide to help predicting  $Y_m$  are the parameters  $\theta$  of the underlying process, and the final state  $s_n$ , so we have:

$$I(X_n:Y_m) \le c|\theta| + D\log|S| \tag{23}$$

where  $|\theta|$  is the number of parameters of the process, and c is how many bits we need to encode the parameters. Again, we find that for a very large class of processes  $I(X_n : Y_m)$  is bounded by a constant, and asymptotically there is nothing to learn as long as (i) the parameters of the process are finite-dimensional and (ii) the size of the state is bounded (or, equivalently, the process has finite or fading memory).

# 4.1 Hilberg's conjecture for scaling

Is this what happens on real data? A particularly well studied case is when the process q generating the data is a human writing natural language text. In the special case that n = m, the **Hilberg's conjecture** [29, 15, 16, 18, 19], posits that:

$$I(X_n:Y_n) \propto n^{\beta} \tag{24}$$

for some  $0 < \beta < 1$ . This is in sharp contrast with the results above. If Hilberg's conjecture holds (which, empirically, it does [51]), then the process generating real data is very unlike any standard dynamical process.<sup>5</sup>

Since we care about real data, let's introduce the following generalized Hilberg's conjecture (GHC) scaling to arbitrary n and m, and take it as our assumption of how physically-generated data, including human-generated ones, behave.

**Definition 4.4:** (Generalized Hilberg's conjecture) Let  $X_n = x_{1:n}$  and  $Y_m = x_{n:n+m}$  be data generated by a stochastic process. We say that it has GHC scaling if:

$$I(X_n:Y_m) \propto n^{\beta} + m^{\beta} - (m+n)^{\beta} \tag{25}$$

This expression reduces to the standard Hilberg's conjecture when n = m. It is symmetric, and is always positive.<sup>6</sup> To get an intuition of how a process may satisfy the GHC, in Section 4.4 we will show one

 $<sup>^{4}</sup>$ This may be slightly confusing, m can generate programs of arbitrary complexity, but its own complexity is low. In fact, we just need a few lines to define it.

<sup>&</sup>lt;sup>5</sup>An unrelated consequence is that a pure LLM implemented by a *state space model* or an attention model with finite context cannot possibly be a perfect model for natural language. Since its state is bounded, it satisfies eq. (23) and cannot asymptotically scale like natural text. However, RAG RAG side-steps the issue, so an agent with external memory can be a model of language, or a model of the world, in ways in which an ordinary Transformer cannot no matter how many parameters it has and how much data it is trained on.

<sup>&</sup>lt;sup>6</sup>Define s = n/(m+n) and t = m/(n+m). The function  $f(x) = x^{\beta}$  is convex, so  $s^{\beta} + t^{\beta} \ge (s+t)^{\beta} = 1$ .

explicitly based on the Santa Fe process [19]. The key intuition will be that the GHC is satisfied whenever the "world" (whatever is generating the data) contains an unbounded amount of unchanging facts that are referenced in the data with a heavy tail distribution. For now, let's assume our process satisfies the conjecture and derive the scaling laws for speed up of a universal search agent.

## 4.2 Scaling Laws for Time

Assume the training data  $X_n$  and the inference data  $Y_m$  are generated by a process satisfying the GHC:

$$I(X_n; Y_m) = m^{\beta} + n^{\beta} - (n+m)^{\beta}. \tag{26}$$

We are interested in the case where  $X_n$  is the training set, so  $n \gg m$ , in which case we can approximate:

$$I(X_n; Y_m) \approx m^{\beta} - \beta \frac{m}{n^{1-\beta}}$$
 (27)

From Theorem 4.2, the log-speed-up we get from training is exactly  $I(X_n; Y_m)$  and m = T(h) is the length of the inference-time trajectory. Therefore we conclude:

**Theorem 4.5:** (Time Scaling Law) The log-speed up we obtain training on a large enough dataset D of n tokens is:

$$\log \frac{\tau_{\nu}(h)}{\tau_{\nu}(h|D)} = T(h)^{\beta} - \beta T(h)/n^{1-\beta}$$
(28)

This tells us a few interesting things. First, the speed up is upper-bounded not by a constant (like we previously obtained for simple models) but by  $T(h)^{\beta}$ . That is, the longer the trajectory is, the more it is sped up. This makes intuitive sense: if finding a solution required just a few steps, even without any learning we could have brute-forced it quickly. Complex problems are the ones that benefit from learning. We also get  $O(T(h)/n^{1-\beta})$  convergence to the optimal speed up, so we want the number of training tokens n to be:

$$n \propto L^{1/(1-\beta)}. (29)$$

where L is the maximum length of a trajectory we expect to need to solve a problem. That is, we need more training tokens if we expect to solve challenging problems.

The parameter  $0 < \beta < 1$  relates to the *complexity* of the task distribution; in particular, it controls how long-tailed the distribution of 'useful facts' is, with  $\beta \to 1$  implying that the distribution is very heavy tailed. When  $\beta$  is high, we need significantly more training tokens to achieve the optimal rate, since there are many more facts that are commonly used. But we also get a better payback, since the speed up  $T(h)^{\beta}$  is also going to be larger.

**Note 4.6** For natural language,  $\beta \approx 0.5$ , which gives  $n \propto L^2$ . So if we are going to generate trajectories of 10K tokens, we need  $\approx 100M$  training tokens.

This ratio of test to train data is realistic when fine-tuning a model for reasoning. But when training from scratch it is a clear underestimate. There are a few factors to consider. First, the initial training data is needed to put the weights in a proper configuration, which depends more on the amount of weights than on the information in the training data. Indeed, it is common to pretrain on lower-information content with size proportional to the number of weights. Second, we are assuming that the mechanism generating the test data is the same as the training data, which is not the case. Facts that are useful at test time may appear very rarely in the training set (e.g., if we ask PhD-level questions to a model trained on generic data). Third, the scaling laws are derived under the assumption that we can identify useful facts and memorize them the first time we see them. But realistically, we need to see a fact multiple times to identify it as useful, which inflates the number of required tokens.

## 4.3 Memory-Time Trade-Off

So far we have assumed we can use all information in X, but in practice the available memory M may be a bottleneck. On a dataset of length n there are  $k = n^{\beta}$  facts to memorize, requiring M = ck bits of memory. Replacing n with M using this relationship in Theorem 4.5 we get the scaling if memory (rather than n) is the bottleneck.

Corollary 4.7: (Time-Memory Scaling Law) Assuming memory is used optimally, the speed-up as a function of the used memory is given by:

$$\log \frac{\tau_{\nu}(h)}{\tau_{\nu}(h|D)} = T(h)^{\beta} - \frac{T(h)}{M^{1/\beta - 1}}$$
(30)

However, this assumes that we are somehow able to extract from the training data the most useful facts and store them (and only them) in memory. Since we are using an online learning algorithm, the memory also needs to store information about the facts in the training data that we have not yet deemed useful, since we need to wait to see them again to confirm if they are useful.

Proposition 4.8: (Online Memory Overhead) An online agent needs a constant factor

$$M_{\text{online}} = C_{\beta} M_{\text{offline}}$$
 (31)

of additional memory compared to an offline to achieve the same performance.

This reflects a realistic issue: it is easier (i.e., faster) to learn from a textbook that gives us directly the useful facts (offline learning) rather than having to 'connect the dots' and try to guess the useful facts from online experience.

#### 4.3.1 Prompting and RAG

So far we have focused on the speed-up provided by training on a dataset D of past data. What is instead the effect of adding a prompt p to the request? First, note that the key result:

$$speed-up = 2^{I_{\nu}(h:p)} \tag{32}$$

where the dataset D is replaced by the prompt p, remains valid, so the speed up is still determined by the  $\nu$ -algorithmic mutual information between the prompt and the trajectory.

If the prompt is an *in-context learning* prompt, which provides examples of the task, then the theory is identical to the case of a dataset (effectively the prompt is a dataset). However, we expect it to provide much more algorithmic information per-sample than the pre-training dataset D, since presumably it will contain only examples directly relevant to the task.

The prompt could also contain information directly relevant to the trajectory, which does not follow a GHC scaling law. For example, if the prompt is a *plan* describing exactly what to do, then:

$$I_{\nu}(h:p) = \ell_{\nu}(h) - \overbrace{\ell_{\nu}(h|p)}^{0} = \ell_{\nu}(h)$$
 (33)

and we get the maximum possible speed-up, meaning that the time to execute the search becomes merely  $\tau_U = T(h^*)$ , the minimum possible time required by a trajectory to solve the task.

Alternatively, a well crafted prompt p may not specify the whole trajectory, but all the information that it has may be relevant to the trajectory, that is  $I(h:p) = \ell_{\nu}(p)$ . In this case, we get a significant speed-up  $2^{\ell_{\nu}(p)}$ . For example, just 10 good bits of prompt (a few tokens) can reduce the time to find a solution by  $\sim 1024$  times. We can think of this as a useful *hint* that brings down the time to solve a problem from hours to minutes.

### 4.4 Example of GHC scaling: Santa Fe process

So far we have assumed that our data generating process satisfies the Generalized Hilberg's Conjecture scaling  $I(X_n:Y_m)=n^{\beta}+m^{\beta}-(n+m)^{\beta}$ , and we anticipated that this relates to having an infinite distribution of facts that appear in the data following a long-tail distribution. Following [15], we now explicitly construct such a process, showing that the GHC scaling definition makes sense and how exactly 'facts' relate to scaling.

Let  $\{Z_k\}_{k=1}^{\infty} \sim \text{Bern}(1/2)$  be an infinite set of binary properties which are sampled before any text is generated. We can think of them as facts about the world, which may be referenced in the text.

Importantly, since the  $Z_k$  are sampled only once at the very beginning and do not change over time, once a fact is first encountered, we know its value in any future text.<sup>7</sup> Some facts are referenced very often, others very rarely. Empirically, natural frequencies are well captured by a Zipf power law:

$$p(k) = ck^{-1/\beta} \tag{34}$$

for some normalization factor c.

To generate a sequence X, we concatenate the index of a random fact and its value:

$$X = ((k_1, Z_{k_1}), (k_2, Z_{k_2}), \dots, (k_n, Z_{k_n}))$$

where  $k_i \sim p(k)$ . We generate Y similarly.

**Theorem 4.9: (Santa Fe Process GHC Scaling)** The Santa Fe process described above follows GHC scaling:

 $I(X_n:Y_m) = n^{\beta} + m^{\beta} - (n+m)^{\beta}$ 

*Proof.* We now want to show that for this process  $I(X_n; Y_m)$  follows GHC scaling. Let's first rewrite:

$$I(X : Y) = H(X) + H(Y) - H(XY).$$

To compute the compression cost H(X) of X, the key observation is that we only need to encode the value of a fact the first time we see it (since it remains constant). How many unique facts appear in X? Asymptotically, this is given by:

$$U(n) = \sum_{k=1}^{\infty} [1 - (1 - p_k)^n] \approx C_{\beta} n^{\beta}.$$

Using this, the compression cost H(X) is the cost of encoding the n random indices (using H(p) bits per index), plus the cost of encoding the U(X) unique properties:

$$H(X) = nH(p) + C_{\beta}n^{\beta}.$$

The cost of H(Y) and H(XY) are computed similarly. Putting all together we get the desired result:

$$I(X;Y) = H(X) + H(Y) - H(XY)$$

$$= [nH(p) + C_{\beta}n^{\beta}] + [mH(p) + C_{\beta}m^{\beta}] - [(n+m)H(p) + C_{\beta}(n+m)^{\beta}]$$

$$= C_{\beta}[n^{\beta} + m^{\beta} - (n+m)^{\beta}]$$

This construction is quite artificial – real world data are not a stream of random facts, indices, and immutable binary values. But it does highlight in a simple way some key, and more fundamental, issues: First, the process has infinite complexity since it needs to store the value of all the  $Z_k$ . Remember that infinite complexity is exactly what we want for learning, lest the benefit of learning would be limited per Theorem 1.2. It makes sense to postulate that the world has infinite complexity (or at least effectively infinite as we can't fully observe it in the lifetime of the agent), and natural text describing the world would naturally reflect this truism. Second, the distribution of facts has to follow a power law. Some facts ("the color of the sky") are more referenced than others ("the house number of John Doe") and power laws are abundant in real data and have several theoretical justifications (rich-get-richer effect, least-effort/max-entropy trade-off, etc.).

In terms of reasoning traces, one may think of 'facts' as functions or theorems that one may invoke by calling their name (index). Since functions/theorems are reused and always remain constant, after memorizing them we can significantly compress future reasoning. It may seem wrong to think of a theorem as a randomly generated fact. But that particular theorem being deemed useful is indeed a random artifact of the history of mathematics. Without learning it in its context, it would be just a random string of symbols that happens to be true (like infinite many other unnamed strings do).

<sup>&</sup>lt;sup>7</sup>An alternative view is that the process is extremely long memory: after the first time it generates a value for  $Z_k$  it remembers it and reuses at all later times.

# 5 Inversion of Scaling Laws

So far we have established the fact that learning from data leads to a speed up in finding a solution to an unforeseen task. However, the equivalence

$$\log(\text{speed-up}) = I_{\nu}(h:D) \tag{35}$$

tells us something stronger: we learn to be faster if and only if we learn from data. This suggests that we can learn something from data if and only if we train with a time optimization objective.

Let's work through an example. Suppose we want to train a universal solver, and (as is natural) we use as reward function the expected number of correct solutions, determined by a function R, averaged over some distribution  $f \sim q$  of tasks:

$$L = \mathbb{E}_{f \sim a} \mathbb{E}_{h \sim \nu} [R(h)]. \tag{36}$$

Further suppose that our agent has unlimited compute power available, so that we have no need to optimize resources over their usage. What will such a system learn?

If the distributions of task is generic enough, we know that the system has to learn to perform universal computation (Section 3). But that is *the only* thing that it needs to learn. Having universal computation, it can implement the basic Solomonoff-Levin Universal search algorithm, which will always find the solution to the task, thus achieving maximum reward. It will take eons to find the solution, but since compute is free for this agent, that is not a problem.

To further clarify, suppose we want to teach the model to play chess. Training is not necessary to achieve a better reward, since a standard tree-search over all the possible moves will eventually find the best move to make. Training is required only to reduce the time that it takes to find the best move.

Claim 5.1: (Only time bound systems learn) If a system is not penalized for the time it takes to find a solution to the task, it is optimal to always brute-force a solution without learning anything. Vice versa, any system that optimizes time has to learn at least  $I_{\nu}(h:D) = \log(\text{speed-up})$  bits of information from the data.

Going more in depth, we can look at how we expect  $I_{\nu}(h;D)$  to behave as we scale the model. First, note that as we scale the maximum time allowed for a trajectory, we also usually want to jointly scale the amount of weights of the model. So if T is the maximum time for the trajectory, the number of weights will be some monotone function  $|\theta| = f(M)$ . Note that the number of weights puts a constraint on the maximum amount of information  $I(\nu:D)$  about the data that we can store in the model parameters. But since  $I_{\nu}(h:D) \leq I(\nu:D)$ , this also puts an upper-bound on the per-trajectory information  $I_{\nu}(h:D)$ . Next, let's look at how much information the model is forced to capture if it wants to have perfect performance on the task. We need  $\tau_{\nu}(h|D) \leq T$  so we need to store enough information to speed up the search until it takes less than T total time. This means:

$$\tau_{\nu}(h|D) = \frac{\tau_{\nu}(h)}{2^{I(h:D)}} < T$$
(37)

$$\Rightarrow I(h:D) < \log \tau_{\nu}(h) - \log T \tag{38}$$

so as expected from the discussion before, the amount of information we need decreases as T increases. Putting the two bounds together we obtain the curve for  $I_{\nu}(h:D)$  shown in Figure 1.

As we scale the model, when the inference time budget and the number of weights are small, the model learns as expected, acquiring information from the data, storing it into the weights. The expected reward it obtains steadily increases as more problems become solvable within the allotted time budget. At some point, the amount of information the weights can store is large enough that, thanks to the speed up, all trajectories are solvable within the time budget. At that point the reward is always optimal and stabilizes. But, paradoxically, if we further increase the time budget the model can use brute force search more, and the information it needs to acquire starts decreasing until it reaches zero. This is the savant regime, where the model can default to expensive brute force search with no learning whatsoever, yet still achieving optimal reward. In this regime, optimal performance (orange curve) comes from excess capacity rather than "insight," as measured by learned algorithmic information (blue curve).

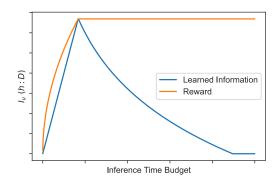


Figure 1: Scaling law inversion curve showing information vs. model scale

To avoid entering the savant regime, one option is to penalize time using a reward like:

$$\mathcal{R} = R(x, y) - \lambda \log \tau_{\nu}(x \mapsto y) \tag{39}$$

$$= R(x,y) + \lambda I_{\nu}(h:D) \tag{40}$$

Which forces the model to actually learn algorithmic information.

Note the sharp contrast with information theoretic regularizers. In a classic machine learning setting, we want to maximize [2]:

$$\mathcal{R}_{\text{reg}} = R(x, y) - \alpha I(w : D) \tag{41}$$

That is, we want to **minimize** the amount of information in the weights [1]. This ensures generalization, and is the basis of the Minimum Description Length (MDL) principle. For reasoning, however, we want to **maximize** the information in the weights in order to minimize time. In transduction there is no issue of generalization, since we have access to all relevant data and a verifier can that tells us whether the task has been solved. More generally the trade-off is not trivial.

Consider a real biological agent that has to model a physical environment in order to act. The agent may learn the correct, generalizable, underlying physical rules which would be optimal from an MDL perspective. But if those rules involve a large amount of computations, the agent may not be able to use them in a reasonable time for survival. It may instead be optimal for the agent to memorize several case-by-case rules (e.g., a feather falls with a certain speed) even if they do not generalize (a feather doesn't fall with the same speed in a vacuum), if they allow it to quickly come up with approximate, but timely, estimates that are better for survival than the best estimate rendered belatedly.

It is common to refer to transduction as *System-2* behavior and induction (restricted to a single forward pass) as *System-1* behavior [30]. Encoding reasoning traces into the weights can then be thought as *automatization*: the common thinking pattern are made faster by moving them to System-1. Whether this is advantageous depends on environmental stability. In stationary environments, automatization allows faster reaction and better energy usage. However, in time-varying environments, the ability to reason at inference time cannot be fully replaced by a set of fixed learned behaviors.

# 6 Maximizing a Continuous Reward

So far we have studied universal solvers under the assumption that the reward function R(x,y) has a binary "success/failure" value. In the more general case, the universal solver is required to maximize a continuous reward function. If we know the maximum achievable reward  $R_{\rm max}$ , and we know that the reward can be achieved, we can define a new binary reward function "has achieved the maximum," thus reducing to the binary case. However, generally we do not know what is the achievable maximum and, more importantly, we do not know if the search cost to find a better solution will be worth the value. That is, a practical universal solver has to decide when to stop.

The objective underlying this decision is optimization of the user's net gain. Consider a universal solver U that over time outputs various candidate solutions  $y_t$ , and let  $R^* = \max R(x, y_t)$  be the maximum

reward achieved by any solution up to now. The value obtained by the user stopping at this point is:

$$J = R^* - \lambda T_{\text{total}}$$

where  $R^*$  represents the best reward achieved across all explored traces,  $\lambda$  is the per-token cost, and  $T_{\text{total}}$  is the total compute time consumed until now. The question is whether investing additional compute to search for a solution is likely to improve J.

Deciding whether a reward function can be improved, let alone whether it is convenient to do so, is generally undecidable. Hence, we need to assume we have learned a forecasting model  $\psi_{\theta}$  to estimate the distribution of rewards for each potential continuation h, enabling principled decisions about which reasoning paths to pursue. Under this assumption, we can formulate the search for optimal solutions as a Pandora's box problem with order constraints [55, 10].

Universal search as Pandora's box problem. We can visualize all possible computations to solve a task as a rooted tree  $\mathcal{T}$ . A node n represents a partial reasoning trace; exploring an extension of a reasoning trace (i.e., a child of n) for t tokens incurs a cost  $c = \lambda t$  and, if we terminate after the extension, yields a terminal reward  $R \sim \psi_{\theta}(\cdot \mid x, z)$ . Nodes naturally obey order constraints: a child may be explored only after its parent.

Given the current best obtained reward  $R^*$ , the incremental value of exploring a new child node n is

$$\Delta(n) = \mathbb{E}[(R - R^*)_+] - \lambda t. \tag{42}$$

While we could greedily pick the next computation to perform by maximizing  $\Delta$ , this generally leads to suboptimal solutions. Searching for the best possible strategy is, in principle, exponentially complex. However, Weitzman [55] showed that a simple greedy strategy does exist to optimally solve this problem. In particular, Weitzman defines a *conditional reservation value* for each candidate extension as the unique  $z_n$  solving

$$\mathbb{E}[(R-z_n)_+] = \lambda t. \tag{43}$$

The resulting value of  $z_n$  is called the *Gittins index* of the node. The provably optimal policy is then to visit at each step the unexplored node which has the currently highest Gittins index. It is instead optimal to *terminate* the search when:

$$R^{\star} \ge \max_{n \in \text{unexplored}} z_n \tag{44}$$

at which point there is no node that we can visit that is expected to improve the final objective J. That is, the expected improvement in reward does not compensate the cost of exploring the node.

This framework can be extended [10] to the case when there are constraints on the order of opening the boxes (e.g., testing a solution obtained after 1,024 thinking tokens requires first reaching 512 tokens), in which case the value of a box also needs to take into account the value of the boxes it allows access to.

Making decisions with Gittins Indexes. Once we use the Gittins index  $z_i$  using our forecasting model, we have a simple criterion to make several key decisions during our search. For example,  $z_i$  allows us to decide: (i) when to continue extending a reasoning trace (if the Gittins index of a child is the highest); (ii) when to branch a trace (if the child of a parent node achieves the highest index), in particular when it is optimal to restart from scratch by sampling a new reasoning trace), and (iii) when it is optimal to stop attempting to improve the solution of a problem and return the current best (when no node has a better index than the current reward).

Need for a Forecasting Model. Exploring greedily based on Gittins indexes remain optimal as long as: (i) the reward distribution is known in advance, and (ii) the distribution of different boxes are independent (the reward observed for one box does not affect the predicted reward on others). When this is not the case, using Gittins indices may not be optimal, but generally remains a strong policy [44].

In general, however, we do not know a priori the distribution of possible rewards we may obtain. For example, we can't know in advance that thinking for 1,000 or 10,000 tokens will lead to finding a correct solution. In order to efficiently optimize the net reward J an agent also has to learn to forecast both the cost of an exploration attempt, and the proability of it improving over the current best solution.

# 7 Maximalistic Models of Computation

The scaling law for speed discussed in Section 4 describes the gains obtainable by assigning higher probability to common patterns learned from past reasoning traces. However, this is not yet the maximum speed-up we can achieve. Consider for example a universal task solver that is frequently asked to solve tasks requiring the computation of a Fast-Fourier Transform (FFT). Once it learns from past data the correct algorithm to compute an FFT, it can instantiate that procedure each time, significantly reducing exploration cost. However, it still must execute hundreds of algorithmic steps – thousands of "thinking tokens" for an LLM – to reach a solution. This is far from optimal: a better solution would be to compute the FFT in a single step, either by modifying the weights to implement the relevant circuitry or by invoking a specialized tool.

Two forces drive such further speed gains: automatization and abstraction. In automatization, as we have discussed, skills migrate from costly System-2 processes to fast System-1 routines. Instead of re-deriving physical laws, the model learns stable approximations of relevant dynamics; instead of re-searching for an FFT, it executes a compact surrogate. In abstraction, long chains of reasoning are compressed into atomic operations. For example, rather than re-proving a property whenever needed, the model introduces a lemma — an invocable unit that replaces a lengthy sub-derivation.

Both phenomena can be understood as expansions of the model's effective instruction set. In a CPU, each executed instruction modifies the internal state, and a carefully constructed minimal instruction set (RISC)<sup>8</sup> suffices for universality. However, some long patterns of operations may appear frequently. This motivates the introduction of complex instructions (CISC) that, even if technically redundant, can perform the same long state transition in a single operation, trading additional real estate (e.g., area) for reduced latency. That is, space redundancy is used to buy time minimality. Similarly, each generated token in a reasoning LLM updates its internal state. The system may be universal even with simple states and transitions. Yet, instead of emitting hundreds of primitive tokens, we can introduce specialized tokens/dynamics – or dedicated callable tools – that let the model 'jump' directly to the same final state without having to run through the individual steps.

This introduces a familiar trade-off. Adding tokens, tools, or weight-level circuitry increases parameter count and training complexity, and raises orchestration and verification burden. At the same time, it can markedly reduce wall-clock latency, exploration cost, and the memory pressure of long reasoning traces. For modern LLMs, the balance differs from classical CPUs: scaling parameter count is comparatively straightforward and efficient, while long sequential chains of thought are expensive, hard to parallelize, and brittle with respect to context-length and recall.

These constraints suggest a natural design pressure toward maximalistic models of computation: computational engines equipped with rich tool libraries, learned subcircuits for frequent subproblems, and token types representing complex operations. Concretely, this includes (i) a library of skills—APIs, theorem banks, solvers—with learnable dispatch; (ii) token or type extensions that encode compound operators; and (iii) fine-tuned subnets that implement high-value routines (e.g., parsing, algebraic transforms, approximate simulation). Training signals should reward short-horizon solutions and penalize unnecessary long-form reasoning when a reliable jump exists.

Maximalistic models of computation stand in contrast with *minimalistic* ones, such as Turing Machines, which are designed to operate with the smallest possible instruction set on the smallest possible dictionary through a computer with the smallest number of components. Such simplicity makes sense if the goal is analysis by humans. However, beyond interpretability, this pressure towards simplicity does not reflect the structure of the world that an AI Agents must interact with. Such world instead engenders pressure towards speed, which can be optimized by increased complexity.

Maximalism is not mere memorization. Automatized routines and abstracted lemmas are structured compressions of procedures, not rote patterns. They improve reliability and latency on recurring structures (FFT, parsing, algebraic simplifications), while remaining fallible on rare, non-stationary, or adversarial cases where general reasoning must reassert itself. Selecting which operations deserve "instruction status" — balancing parameter growth against step reduction — and validating the safety of state jumps is a key problem for reasoning LLMs.

As minimizing reasoning-trace length becomes the dominant objective, we should expect models to shift

<sup>&</sup>lt;sup>8</sup>Reduced Instruction-Set Computer

toward maximalistic computation, mirroring the biological transition from System-2 deliberation to System-1 fluency. The path to greater speed is not only better search, but better *steps*.

# 8 Discussion and related prior work

By its expository nature, this paper relates to a vast body of work. While we have pointed to related work throughout the paper using the earliest instantiation of the relevant ideas as a reference, each such seed spawned a vast body of work which cannot be realistically surveyed here. We welcome suggestion on relevant work we may have missed, as well as earlier attributions of the relevant concepts.

Ultimately, this paper falls within the scope of statistical machine learning, where one starts with instantiated data which implicitly define the task, and arrives at a model that performs the same inference computation on all future instances of the same task. However, our exposition is prompted by the fact that trained models, once used as generative distributions, exhibit behaviors that were not explicitly encoded in the training data nor the loss. A key such behavior is the ability to solve previously unseen tasks. In classical (inductive) machine learning, there is no feedback mechanism at inference time, so one can only evaluate the quality of a model post-hoc, typically on data other than the one at hand. Agents, on the other hand, interact with the environment, which provides feedback, and/or can call tools to solicit feedback. Inference computation can therefore adapt depending on the resulting feedback. This mode of interaction calls for a different approach to learning, which aims to empower transductive inference. The power of LLMs stems from the fact that, despite being trained inductively with supervised classification (next-token prediction is a standard multi-class supervised classification problem, albeit improperly referred to as "unsupervised pre-training"), they operate transductively by performing variable-length inference computation. In this paper, we explore the foundational principles of such transductive learning, and its limits, including bounds and power laws.

Transduction, In-context Learning, and Solomonoff Inference. Transduction in the form of learning jointly from labeled training examples and unlabeled test samples was championed by Vapnik [13, 53, 25]. An early observation was that language models can learn multiple tasks implicitly through the unsupervised language modelling objective [42], and exhibit diverse behaviors when adequately prompted. In-context learning [21], which is a form of transductive inference, introduces demonstration examples into the model context to elicit desired behavior. It has been shown that LLMs can perform optimization algorithms such as gradient descent and and ridge regression transductively at inference time from in-context examples [5]. [26] demonstrates that sparse linear functions, decision trees, and two-layer networks can be learned in-context. [56] investigates what minimal pretraining is necessary to induce in-context learning, showing a small pretrained model can achieve close to the Bayes optimal algorithm. [37] introduce a theory of in-context learning where a hypothesis is formed at inference time, obtaining generalization bounds. [59] demonstrates that a single self-attention layer trained by gradient flow to perform in-context learning converges to a global minimum that is competitive with the best predictor on the test distribution. The connection between in-context learning and Solomonoff inference was identified in [27], where the authors attempt to learn the Solomonoff semimeasure directly by sampling programs and training on their outputs. In [20], motivated by the inductive theory of Solomonoff [47, 48, 46], the authors demonstrate that LLMs can outperform general purpose compressors, even for audio and visual data. The connection between Solomonoff induction and neural network optimization as a form of program search was mentioned earlier in [31].

As we have noted in the introduction, time plays no role in Solomonoff Inference, nor in in-context learning. Neither involve actual "learning" in the classical inductive sense: The weights are fixed and the same task, presented in-context multiple times, requires repeating the same effort to no different outcome each time. However, time plays a key role in *learning* transduction, which is the core motivation of this work.

LRMs, SLMs, VLMs, World Models, etc. (nomenclature) The term LLM originally referred to large-scale Transformer-based models (pre-)trained as next-token predictors using large-scale corpora of natural language, then co-opted as probability distribution to sample new natural language text. Optionally, these models could be fine-tuned by scoring such generated expressions using human preference, an external reward mechanism, or by the model itself through self-assessment. It is also common to call the same exact artifact a 'World Model' (WM) if trained on sensory data such as video or audio instead of natural language, or 'vision-language model' (VLM) if trained on both, or 'vision-language-action' model

if the output expression is used to issue commands to an actuator, or 'large reasoning models' (LRMs) if they are used to generate variable-length trajectories prior to rendering the decision or action. In our nomenclature, any large-scale predictor trained on sequential data with latent logical/linguistic structure (with objects, relations, functions, etc.) develops an inner state space or an internal "Neuralese language" [52]. Sensory data are replete with latent discrete entities [6], their relations (topological, geometric, dynamic, semantic) and (de)composition into parts (meronomies) or abstract concepts (taxonomies). In our definition of LLM, therefore, where 'language' is not restricted to natural language, VLMs, WMs, LRMs and other variants are also LLMs. We also include in the term LLMs models that use different architectures, so long as they have a 'state' (memory), whether explicit (as in state-space models) or implicit by co-opting a sliding window of data, as in autoregressive Transformers [58]. Since the largest LLMs at this point comprise trillions of parameters, some now refer to models with merely billions of parameters as 'small language models' or SLM. Obviously, 'small' is subjective, and these models have no architecture, structural, functional, or conceptual difference from their 'large' counterpart, so they too are just LLMs. Empirically some emergent phenomena are only observed at scale, but this does not mean that there is a clear divider between 'large' and 'small', even phenomenologically since smaller models can still be distilled from larger ones and maintain their behavior even if it would not have emerged from cold-start using the same training prototocol [28].

Embodied AI The results described in this paper pertain to both software agents that exist within the world of bits, as well as embodied agents that interact with the physical environment. While this may seem counter to the forced dichotomy between LLMs and so-called World Models, once the sensory data is tokenized the two become the same mathematically. Regardless of how a model is trained inductively, once it acquires the ability to perform transductive inference, it can act as an agent. This could be in the world of bits, where interaction with the surrounding environment is through APIs and function calls, or in the world of atoms, where sensors provide measurements that the agent turns into a representation of the environment (which is an abstract concept finitely encoded in the state of the agent [3]) and operate on it (i.e., reason), to produce actuator commands that affect the relation between the agent and the environment. Such environment then provides a feedback signal, ultimately in the form of "verification" (e.g., survival or rewards). The reasoning agents exists in a finitely-encoded world, and interfaces with the physical world through encoders and decoders. The core of all these agents is the ability to perform transductive inference within the discrete/discretized representation, which requires computation as described in this document. While evolution proves that processing sensory data is sufficient to foster the emergence of reasoning, language is already conveniently distilled (symbolized and compressed) making the traversal of the evolutionary path unnecessary for the emergence of reasoning. In this sense, agentic AI subsumes embodied AI, where the latter focuses on the source of the data (sampled physical sensory measurements) and the outcome of actuators command (physical motion).

Universal Computation, Universal Search, Algorithmic Information Theoretical Computer Science has devoted decades to the development of universal algorithms; indeed, Levin's paper that introduced his universal search algorithm seeded a large portion of the subsequent literature on computational complexity theory. Since we only use the concepts driving conceptual search, we do not review this sizeable body of work here and refer the reader to any textbook on complexity theory. One exception we make is to comment on the literature of Kolmogorov Complexity and Algorithmic Information Theory. While there are also textbooks covering this material [36] we feel that they do not adequately cover the fundamental limitations of this approach. Specifically, Kolmogorov's theory is portrayed as fundamental and fundamentally right, just impractical. Therefore, minimum-description length and other principles are developed, which do not possess the universality and objectivity which are the raison d'être of the theory in the first place.

In reality, it is the very aim of the theory to attain universality that causes its demise. As shown in [4, Appendix], any attempt to define a canonical notion of "useful" information in instantiated data, or equivalently separate *structure* from *randomness* or *signal* from *noise*, leads to a vacuous or tautological theory, where all data is noise. This fact is well know. However, rather than clearly stating that celebrated algorithmic information devices (such as Kolmogorov's Minimal Sufficient Statistics) are fundamentally dengenerate and only reflects the relation between the data and the Halting problem, the methods are described as "fundamentally right."

While Kolmogorov's theory is useful for certain asymptotic analysis, and we make heavy use of it in this manuscript, we believe it is important for the reader to be aware of its fundamental limitations.

Reinforcement Learning Similarly, our viewing AI Agents as a stochastic dynamical system operating in a partially unknown and dynamic environment makes our work naturally fit in the body framework of reinforcement learning (RL), or more generally stochastic optimal control [8]. Since we are not focusing on any particular RL algorithm, we do not review this already vast and still growing literature. The basic concepts of RL have long been established and are the subject of textbooks. Here we merely remark on the fact that, in our setting, we are not seeking for a fixed policy to solve a particular task in closed-loop, nor to learn a policy from multiple rollouts or direct experience. Instead, we are looking to how to foster the emergence of general task-solving tools that allow an agent to craft a new policy for tasks that no agent may have ever experienced. Rather than exploring/exploiting in data space during training to arrive at a fixed policy, this leads to explore/exploit in policy space during inference.

System-1, System-2. As inductively-trained transductive inference engines, LLMs perform transduction at inference time. Transductive inference is akin to what cognitive psychologists call deliberative or 'System-2,' thinking, whereas inference from inductive learning is akin to automatic or 'System-1' thinking [30]. As discussed in Section 7, part of inductive learning of transductive inference is akin to the process of automatization, also related to meta learning. We note that transductive inference is known in the LLM jargon as "in-context learning" even if, properly speaking, there not learning: repeating the same task with the same context over and over will always yield the same outcome in the same amount of time, rather than resulting in the dramatic speed we expect from an agent that learns from experiences.

**Solomonoff Induction.** Similarly, Solomonoff Induction is often taken as a prototype for asymptotically optimal learning. However, the inference algorithm is fixed once and for all, and the same task presented repeatedly requires the same effort every time, with no improvement from experience. While this ensures optimal loss minimization, it is hardly 'intelligence' in any reasonable sense of the term. It does not look into the structure of the data, but simply performs a brute force search and averaging to find a good fit.

Markov Chains We use general dynamical systems as a model of computation, but arguably the most important piece is the transition probability  $\nu(s_{t+1}|s_t)$ , which defines a Markov Chain. One may wonder why we need to introduce such general machinery, including 'proper time,' when standard concepts from Markov Chains, such as the expected hitting time, would suffice. As noted, however, hitting time could be made arbitrarily small or large without fundamentally changing the computations performed. We also note that since AI Agents interact with the unknown environment, they are not closed systems describable with a Markov chain, but can still be described as a dynamical system. We also note that our use of dynamical systems or Markov structure is restricted to modeling the computations of the agent solving the task, not on the data itself. In fact, we stay clear of ever making any assumption on the data generating distribution aside from it satisfying the Hilberg's conjecture.

Memorization and Generalization. Information complexity based generalization theory formalizes the notion that generalization occurs whenever the information the learned hypothesis contains about the training data is minimal (low memorization). [7] demonstrated that a single information exponential inequality is sufficient to derive PAC Bayes bounds [12], the mutual information bound [57], and a version of the conditional mutual information bound [50]. Even the classical finite-hypothesis and VC dimension bound [54] can be viewed as primitive versions of such bounds. All the aforementioned results assume the training and test data are drawn as i.i.d. samples from a common distribution. In contrast with the aforementioned theory, it has been demonstrated that there are learning tasks where memorization is provably necessary to achieve high accuracy [11], and that mitigating memorization can cause the model to fail on long-tailed tasks [23, 22]. There is evidence that natural language is akin to such long-tailed tasks that require memorization. Recent work demonstrate that there are models for language which explicitly memorize the training data that perform well. [33] introduce nearest-neighbor language models (kNN-LM) which predict the next token according to the k nearest neighbors of the context embedding, which requires explicitly encoding all context embeddings and their subsequent token. [40] demonstrates that augmenting parameteric LLMs with a kNN-LM can significantly boost performance. Even stronger, [38] demonstrates that a generalization of an n-gram model (dubbed  $\infty$ -gram) outperforms kNN-LM, while losslessly encoding the training data into a suffix array data structure.

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