VECTORS AND MECHANICS

CONTEXT

The mathematics curriculum provides student teachers with a background in the theory and application of the content needed to understand the underlying structure and nature of mathematics. In addition, it exposes student teachers to the content knowledge needed in preparing them sufficiently to teach mathematics beyond what they will be expected to teach at the basic education level. The demands of rapid change in an information- based society today have influenced mathematics programs in various ways. The skills needed for jobs require thoughtful workers who are oriented to problem solving, irrespective of their gender, cultural and socio-economic backgrounds. By studying mathematics, students are taught to reason, to analyse, to think for themselves, while it imparts confidence in their own reasoning powers, and strengthens their mental faculties. Students need to use rules and thought processes of mathematics along with facts, to develop a reasoning pattern that will translate to their everyday lives, making them better thinkers and problem solvers. It is important for students to view mathematics as a significant part of our culture, not only for its valuable scientific applications but also for its enrichment of our cultural life. This mathematics curriculum is, therefore, intended to equip student teachers with the knowledge, skills and values needed to teach mathematics to basic school pupils in everyday life context. Besides, it provides the requisite resource material for preparing student teachers to teach mathematics sufficiently and effectively in our basic schools.

rse Title	VECTORS AND MECHANICS							
rse Code	EBS 424		Course Level	400	Credit Value:	3	Semester	2
requisite	Algebra II, Geometry							
rse Delivery es	Face - to - face [1] X	Practical Activity [2] X	Work-Based Learning [3] X	Seminars [4]	Independent Study [5] X	e-learning opportunities [6] X	Practicur	
rse cription for ificant ning cate NTS, CF, BSC to be	The course is designed to further expose students to Vectors and Mechanics. This course introdu students to elementary vector algebra and its applications in solving routine problems in geometry mechanics; geometrical and physical interpretation of fundamental vector concepts; multiplication vector by a scalar, dot product, resultant of a system of forces and the vector equation of its lin action, Newton's Laws of motion with emphasis on the use of the second law, motion of a par under uniform acceleration; impulse, momentum and conservation of linear momentum; collisio particles (direct impact). The approaches that would be used in the delivery of this course sh							

essed)	prepare trainees to ensure the learning progress of all students by projecting gender roles and iss relating to equity and inclusivity. NTS 1a, 1b, 2c, NTECF Pillar 1, (p. 21), P. 39, P.45)			
rse ning comes [8]: ding CATORS each ning ome	Outcomes: By the end of the course students will be able to:			Indicators:
	1. demonstrate a sound knowledge of the topics and apply them in real life situations (NTS 2c)			Differentiate between scalar and vector quantities Solve real life problems involving applicatio of vectors and mechanics
	2. pose mathematics tasks in the content studied and solve them using appropriate procedures and tools including ICT (calculators, excel/ spread sheets, and SPSS, etc.).(NTS 2c, NTECF Pillar 1 expectation 3, pages 20 and 21)			Pose and create relevant problems vectors and mechanics and solve perform basic algebraic operations on vec quantities; resolve forces acting at a point its applications to real-life situation
	Units	Topics:	Sub-topics (if any):	Teaching and learning activities to achieve learning outcomes
rse Content	1	Vectors	Scalar and vector quantities, vector representation, Vector in magnitude-bearing form types of vectors, column vector, row vector, and position vector	Provide opportunities for students distingu between scalar and vector quantities. Expose students to the various forms of representing vectors Provide opportunities for students to expre a vector from magnitude bearing form to component form and vice versa
	2	Algebra of vectors	Addition and subtraction of vectors, Scalar multiplication of vectors, Scalar or Dot product, Parallel and perpendicular lines	Provide practical opportunities (e.g. graph sheets or board) for students to perform various operations on vectors Provide suitable opportunities to apply dot product to find angle between two vectors to show if two given vectors are perpendic
	3	Vector Equations and properties of a straight line	Vector equations of a line of the form $\underline{r} = \underline{a} + k\underline{u}$, where '$\underline{a}$' and '$\underline{u}$' are given vectors and	Discuss the steps in finding the vector equation of a line in the form $\underline{r} = \underline{a} + k\underline{u}$ Expose students to the use vector approac

			‘k’ is a parameter. Dividing a line in a given ratio using vector approach.	find a point that divides a line segment in a given ratio
	4	Resultant of a system of forces	Resultant of a system of two or more forces	Provide real situations that leads to finding resultant of two or more forces
	5	Newton’s Laws of motion with emphasis on the use of the second law	Newton’s Laws of motion	Provide opportunities to students to find momentum of particles.
	6	Motion of a particle under uniform acceleration	Motion under uniform acceleration Vertical motion under gravity	Expose students to problems involving mo under uniform acceleration (derivation of equations of motion) Expose students to problems involving ver motion under gravity (vertical projection ar body falling freely form a height)
	7	Impulse, momentum and conservation of linear momentum	Impulse Momentum Conservation of linear momentum	Explain the concepts of impulse, momentu and linear conservation to students Expose students to relevant problems on impulse, momentum and linear conservatio
	8	Collision of particles (direct impact)	Direct impact	Discuss the concept of collision of particle using real life situations e.g., cars colliding headlong (moving opposite direction) and moving in the same directions
rse essment nponents[9] : ucative essment of, and as ning)	Component 1: Formative Assessment (Individual and Group presentations) **Summary of Assessment Method:** Critical Thinking, problem solving skills, creative and innovative skills, life-long learning/ personal skills, collaborative/ social skills, communication skills, literacy ar numeracy skills, leadership skills, digital literacy/ICT skills (NTECF p. 45) • Presentations Weighting (10%) Assesses Learning Outcomes: CLO 1 and 2 (Units 1 and 8)			
	Component 2: Formative Assessment **Summary of Assessment Method:** Critical Thinking, problem solving skills, creative and innovative			

	skills (NTECF p. 45) • Assignments • Class exercises • Quizzes Weighting (30%) Assesses Learning Outcomes: CLO 1 and 2 (Units 1, 2, 3, 5, 6 and 7)
	Component 3: Summative Assessment **Summary of Assessment Method:** End of Semester Examinations Unit 1 – 8 (Core skills to be developed: Critical Thinking, problem solving skills, creative and innovative skills (NTECF p. 45)) Weighting (60%) • Assesses Learning Outcomes: CLO 1 -8 (Units 1-7)
uctional ources	Computer with internet connectivity, Projector, Online resource (YouTube videos)
uired Text e)	Asare-Inkoom, A. (2012). Further/elective Mathematics for Senior Secondary Schools (Vol.1). Cape Coast, Hampton Printing Press.
tional ding List [10]	Anna J. (2001). Comprehensive elective mathematics for senior secondary school students, volum Kumasi: UGCC Publishing House. Backhouse, J. K. & Houldsworth, S.P.T (2005). *Pure Mathematics 1*. London, Longman.Barnett, A. B Ziegler, M. R., & B yleen, K. E. (2000). *College algebra. A graphical approach*. McGraw –Hill Educatio Bittinger, M. L., & Beecher J. A. (1989). *Algebra and geometry*. Reading: Addison-Wesley. Egbe, E., Odili, G.A. & Ugbebor, O.O. (1999) *Further mathematics* (2nd ed.). Onitsa, Nigeria: Africana F Publishers. Humphrey, D., & Topping J. (Ed.). (1971), *Intermediate mechanics Vol. 1 Dynamics.* London: Longm Group Limited.

Unit 1: Vectors

Mathematics begins with numbers, but it is not restricted only to the study of quantities specified only by single number. We may say that 'The time is 9am' or 'The temperature is 37°C' or 'The distance from Cape Coast to Accra is 145 km'. In all these examples, there is the notion of expression of single number. Such quantities that specify the notion of a single number, as the above examples, are called ***scalars.*** In the above examples, we have 9am; 37°C and 145 km as the single quantities. More examples of scalars include height (3.5m); mass (20kg); area ($12cm^2$); volume ($45cm^3$); speed ($4ms^{-1}$); etc.

A scalar is that quantity which possesses only magnitude..

Besides temperature, mass and density, other examples of scalars are energy, speed, length and time.

There are other quantities that convey the notion of two 'numbers'. For example the displacement of a particle might be 10 meters in the northward direction. This displacement is expressed in terms of magnitude (length) of 10 meters in the direction towards north. Such a quantity that is expressed in terms of magnitude and direction is called a *vector.* Thus a vector may be described as any *quantity that has magnitude (length) and direction.* Examples of a vector include displacement, velocity, acceleration, force, etc.

REPRESENTATION OF A VECTOR

A displacement from point *O* to another point *P* is the *vector OP*, [often written as $\overrightarrow{OP}$]. The length of the line segment *OP* $|OP|$ is the *magnitude* of the vector *OP*, and is in the direction of *OP*. This vector may be represented by one of the following ways:

(i)

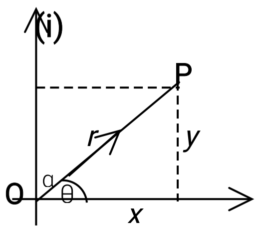


Fig.1

(ii)

$OP = \begin{pmatrix} x \\ y \end{pmatrix}$

(iii)

$OP = (r, \alpha)$

(i) is the diagrammatic representation of the vector *OP*. Its magnitude is $r = |OP|$ and

its direction is along *OP*.

(ii) represents vector *OP* in the *component form.* The vector $\begin{pmatrix} x \\ y \end{pmatrix}$ indicates that the vector

is *x units* along the *x*-axis and *y units* along the *y*-axis*.* The values of *x* and *y* are positive or negative depending on whether the vector is the *first, second*, *third* or *fourth* quadrant.

(iii) represents vector *OP* in the *magnitude – bearing form*. The vector *OP* = (r, α) has magnitude given by $r = \sqrt{x^2 + y^2}$, and its direction is on the bearing of α. At times, the value (r, α) are called the *polar* coordinates of point *P.*

The displacement *OP* has *O* as its initial point and *P* as its terminal point. In other words,

the vector *OP* has its direction from *O* to *P.* Accordingly, the displacement *PO,* which has its direction from *P* to *O* but the same magnitude as *OP* is denoted by *PO* and we write *OP* = - *PO.* Generally, therefore, given a non-zero vector *AB*, then *AB* $\neq$ *BA.*

A vector may also be represented by a single letter; like ***a*** .

VECTORS IN THE MAGNITUDE-BEARING FORMS

At the beginning of this chapter it was explained that a vector can be represented by means of

(i) *diagram,* (ii) *components form* or (iii) *magnitude* (*distance*)-*bearing form.* We have looked at vectors in components form. Let us now consider vectors in distance-bearing forms. The bearing of point *A* from *B* is the *angle measured* (at point *B*) from the *north* in the clock-wise direction in order to get the direction of point *A.* The vector ***a*** = 4i – 3j may be written in magnitude-bearing form as follows:

We first find the magnitude of the vector: $|\boldsymbol{a}| = \sqrt{(4)^2 + (-3)^2} = \sqrt{25} = 5$

We then find the direction of the vector. The vector is shown in Fig.5 below.

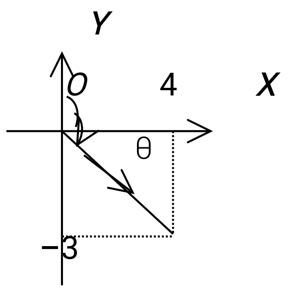


Fig.13

If the angle the vector makes with the *x*-axis is θ

Note that the vector is in the *fourth quadrant.*

In the triangle,

$\tan\theta = ¾ \Rightarrow \theta = \tan^{-1}(¾) = 36.87^\circ \cong 37^\circ$

Hence the bearing of the vector is ($90^\circ + 37^\circ$) 127°.

Therefore, 4i – 3j = (5, 127°).

Exercise.

Write the following vectors in the *distance-bearing* forms.

(i) $3i + 2j$ (ii) $-6i + 8j$ (iii) $-5i - 12j$ (iv) $5i - 8j$

Unit 2:
ALGEBRA OF VECTORS

Addition and Subtraction of Vectors

As we perform basic operations on real numbers, so ca we perform similar operations on vectors. However, since vectors have both magnitudes and directions we cannot find the sum of two vectors in the same way as we do with real numbers. When vectors are written in magnitude-bearing forms then to add two or more vectors, we need to either draw them to scale and do some measurements of distances as well as angles, or resolve or change the vectors to components forms and add them, before changing them back to magnitude-bearing form.

<u>Illustration of equal vectors</u>: Use Fig.1 to answer the following questions.

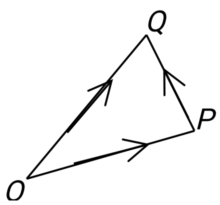
What is the displacement **AB**?
What about displacement **CD**?
What is the displacement **PA**?
What about displacement **RC**?

Fig. 1

The displacement **AB** is a motion of 2 units to the right and 4 units up. This is written as

AB = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Again the displacement **CD** is a motion of 2 units to the right and 4 units up. It is also CD $= \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Are these two displacements *equal*? We can say that they are the same since they are equal in *length* and have the same *direction*. However, it may be argued that they are not equal since they have different starting-points (*initial points*) and ending-points (*terminal points*). Hence it seems that when we are considering a displacement (vector), we should make it clear whether we wish its *location* to be included or not. If we *draw* or *locate* a vector in a particular position, we say that we have *localized* it, otherwise, the vector is called a *free* vector. It is a free vector because it can be located at any position. In other words, a vector can have any position as its initial point. Note that the displacements **PA** and **RC** are also equal. Either **PA** or **RC** is 6 units to the left and 2 units down. This may be written as:

$$\text{PA} = \text{RC} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}.$$



It is very important to write the order of the letters (in a vector) correctly. The length of the vector **OP** equals the length of vector **PO**, but they have *opposite* directions. If a particle moves from *P* to *Q* (Fig.2), the displacement of *Q* to *P* is represented by the vector **PQ**.

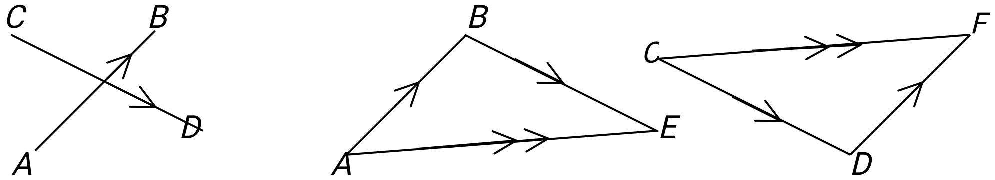
Since the displacement of O to P is followed by displacement of P to Q, geometrically, these two displacements are equivalent to the single displacement OQ.

Hence OQ = OP + PQ

This is known as the *Triangle Law* of *Vector Addition.*

Fig.2 [OQ is called the *resultant* of OP and PQ]

This law states that *'the sum of any two vectors represented by AB and BC is the vector represented by AC where the vectors AB, BC and AC form triangle ABC'.*



(i) (ii) (iii) Fig. 3

In Fig.2, we have OQ = OP + PQ . Therefore PQ = OQ – OP.

If we wish to *add* vectors AB and CD, we first replace them by *equal* vectors placed "*head to tail"* and then use the triangle law of vector addition. [Note: *Two vectors are said to be equal if they have the same magnitude and the same direction.*]

In Fig.3 (ii), vector CD is replaced by BE [CD = BE] which has its initial point at the terminal point of vector AB. Thus AB + CD = AB + BE = AE. In Fig.3 (iii) vector AB is replaced by vector DF [AB = DF] which has its initial point at the terminal point of vector CD.

Hence AB + CD = CD + DF = CF. A special case of the triangle law of vector addition is when the two vectors have a common initial point. Consider two vectors AB and AC with A as the initial common point (Fig.4). If the parallelogram *ABPC,* is completed, then we have:

AC = BP and AB = CP

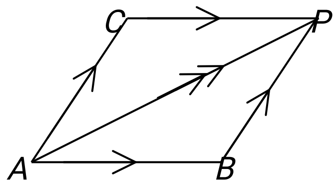


Fig.4

Now AB + AC = AB + BP = AP and

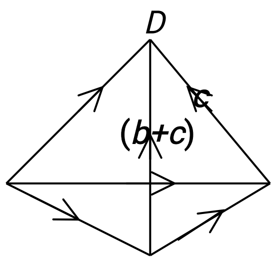
AB + AC = AC + CP = AP

[AP is the *resultant* of AB and AC]

This is known as the *parallelogram law of vector addition.* This law states that "*the sum of two vectors AB and AC is represented by the diagonal AP of the parallelogram ABPC having AB and AC as adjacent sides*".

<u>Remarks:</u> The parallelogram law of vector addition proves that *vector addition is commutative.* From Fig.5; AP = AB + BP = AB + AC [since BP = AC]
AP = AC + CP = AC + AB [since CP = AB].

ASSOCIATIVE LAW (Order of Adding Several Vectors)



Let vectors AB, BC and CD be *a, b* and *c* respectively. Then vector AC = $a + b$ and vector BD = $b + c$.

Now AD = AB + BD = $a + (b + c)$

AD = AC + CD = $(a + b) + c$

A (*a* + *b*) *C*
a
b
B
Fig.5

Therefore $a + (b + c) = (a + b) + c$, indicating that *vector addition is associative.*

Subtraction of Vectors (Negative Vectors)

If *a* is a non-zero vector, the vector with the same magnitude, but in the opposite direction to *a* is called the *negative* of *a.* This vector is denoted by $-a$. Now $-a = (-1)a$. Hence the negative of vector *AB* is the vector $-AB$. Note: $-AB = (-1)AB = BA$. Therefore *subtraction* of vector *AC* from vector *AB* is defined as:

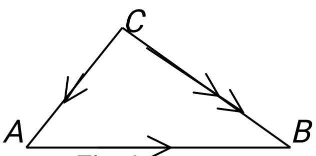


Fig.6

$$\begin{aligned} AB - AC &= AB + (-AC) \\ &= AB + CA \\ &= CA + AB \\ &= CB \end{aligned}$$

Multiplication of a Vector by a Scalar (Parallel Vectors)

Let *k* be a non-zero real number. Then the vector *k*AB is defined as follows:

(i) If $k > 0$, the vector *k*AB has the same direction as the vector AB, but is *k* times as long

as the vector AB.

(ii) If $k < 0$, the vector *k*AB is in the opposite direction to the vector AB and is $(-k)$ times

as long as vector AB. Since the vectors *k*AB and AB have *either* the same direction

or opposite direction to each other, they are parallel. The vector *k*AB is a *scalar multiple* of the vector AB. We say that *if two vectors are parallel, then one is a scalar multiple of the other.*

Example 1

If the vector $c = a + 2b$ and $2c = a - 3b$ show that vectors *a* and *c* have the same direction, but *a* and *b* have opposite direction.

Solution

Write the vectors as:

$a + 2b = c$..........(1)

$a - 3b = 2c$..........(2)

Multiply (1) by 3 and (2) by 2:

$3a + 6b = 3c$.........(3)

$\underline{2a - 6b = 4c}$..........(4)

(3) + (4): $5a = 7c \Rightarrow a = \frac{7}{5}c$ indicating vectors *a* and *c* have the same

direction. Again if we multiply (1) by 2 we get:

$$2\boldsymbol{a} + 4\boldsymbol{b} = 2\boldsymbol{c} \quad \text{.........(5)}$$
$$\boldsymbol{a} - 3\boldsymbol{b} = 2\boldsymbol{c} \quad \text{.........(2)}$$

(5) − (2): $\overline{\boldsymbol{a} + 7\boldsymbol{b} = 0} \Rightarrow \boldsymbol{a} = -7\boldsymbol{b}$ indicating that vectors ***a*** and ***b*** have opposite directions.

Position Vectors

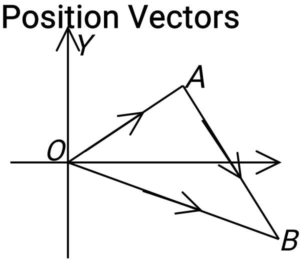


Fig.7

Let *O* be a fixed point (the origin). The vector **OA** is called the *position vector* of point *A.*
Now let position vectors of points *A* and *B* be ***a*** and ***b*** respectively. Vector **AB** may be found as follows: In Fig.7,
AB = **AO** + **OB** = − OA + OB = **OB** − **OA** = ***b*** − ***a***

We can always write any given vector in terms of position vectors. For example

PR = PO + OR = −OP + OR = OR − OP

Example 2

If ***a*** and ***b*** are the position vectors of points *A* and *B* and *M* is the mid-point of *AB*, show that the position vector of *M* is given by ½ (***a + b***).

Solution

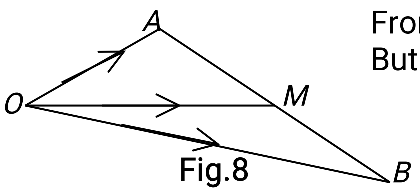


Fig.8

From the diagram; **OM** = **OA** + **AM**
But **AM** = ½**AB**. Thus: **OM** = **OA** + ½ **AB**
= ***OA*** + ½ (***OB*** − ***OA***)
= ***a*** + ½(***b*** − ***a***)
= ½(***a*** + ***b***) as required.

This is known as **vector median property of a triangle.**
From **OM** = ½ (**OA** + **OB**) we have **OA** + **OB** = 2**OM**

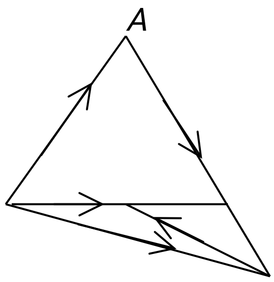
Example 3

(*a*) *OAB* is a triangle, where *O* is the origin. Let ***a, b*** be the position vectors of *A* and *B* respectively. If *X* is a point on *AB* such that *AX* = 2*XB* and *Y* is the mid-point of *OX*, show that $\overrightarrow{BY} = \frac{1}{6}a - \frac{2}{3}b$.

(*b*) What is the resultant of the following five vectors? 3**OA**, 6**BZ**, 2**AO** **AB** and 5**OB**.

Solution

(*a*)



BY = **BO** + **OY**
= − **OB** + ½ **OX**
= − **OB** + ½ (**OA** + **AX**)

a $\qquad = -b + ½\,(a + 2XB)$

But AB = AX + XB

O $\quad Y$ $\quad X$ $\qquad = 2XB + XB$

b $\qquad = 3XB$

B Therefore;

Fig.9 $\qquad XB = \frac{1}{3}AB = \frac{1}{3}(OB - OA)$

Hence BY $= -b + ½[a + 2(\frac{1}{3})(OB - OA)] = -b + ½\,a + \frac{1}{3}b - \frac{1}{3}a$

$$= \frac{-6b + 3a + 2(b - a)}{6} = \frac{-6b + 3a + 2b - 2a}{6}$$

$$= \frac{a - 4b}{6} = \frac{1}{6}a - \frac{2}{3}b$$ as required.

(*c*) $3OA + 6BZ + 2AO + AB + 5OB = 3OA + 6(OZ - OB) + 2(-OA) + OB - OA + 5OB$

$= 3OA + 6OZ - 6OB - 2OA + OB - OA + 5OB$

$= 6OZ$

Unit Vector

A unit vector is defined as a *vector with magnitude one*. In other words, any vector with its length as one, is a unit vector. The letters **i** and **j** are used to define unit vectors in the positive directions of the *x*-and *y*-axes respectively. Any vector can be written in terms of these unit vectors. For example vector **OP** in Fig.11may be written as

OP = ON + NP = 5i + 3j. $\quad$ Note that $i = \begin{pmatrix}1\\0\end{pmatrix}$ and $\quad j = \begin{pmatrix}0\\1\end{pmatrix}$.

Example 4

A, *B* and *C* are the points (1, 2), (4, 3) and (3, –1) respectively. *D* is the mid-point of *BC*.

(i) Write down the position vector of *D*. (ii) Express BA and CA in terms of i and j

(iii) Show that BA + CA = 2DA`

Solution

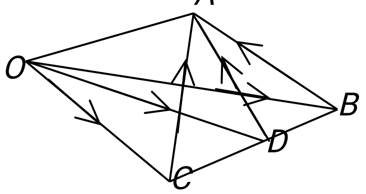


Fig.10

(i) Position vectors of the points are:

OA = i + 2j, $\quad$ OB = 4i + 3j $\quad$ and $\quad$ OC = 3i – j.

Now $\quad$ OD = OB + ½ BC

= OB + ½ (OC – OB)

= (4i + 3j) + ½ [(3i – j) – (4i + 3j)]

= 4i + 3j + ½ (-i – 4j)

$= \frac{7}{2}i + j$

(ii) BA = OA – OB = (i + 2j) – (4i + 3j) = –3i – j

CA = OA − OC = (i + 2j) − (3i − j) = −2i + 3j

(iii) BA + CA = (OA − OB) + (OA − OC)
= 2OA − OB − OC
= 2OA − (OB + OC)
= 2OA − 2OD since OD = ½ (OB + OC)
= 2(OA − OD)
= 2DA as required.

Example 5

If *D, E, F* are mid-points of the sides *BC, CA* and *AB* of triangle *ABC,* show that
(i) AB + BC + CA = 0; (ii) 2AB + 3BC + CA = 2FC

Solution

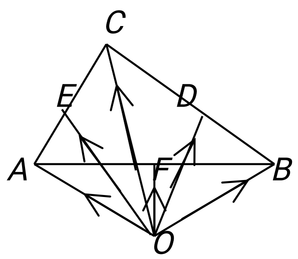


Fig.11

Let the position vectors of points *A, B, C, E, F* and *D* be respectively **OA, OB, OC, OE, OF** and **OD** (See Fig.12)

(i)

AB + BC + CA = (OB − OA) + (OC − OB)
+ (OA − OC)
= OB − OA + OC − OB + OA − OC
= 0 as required

(ii) 2AB + 3BC + CA = 2(OB − OA) + 3(OC − OB) + OA − OC
= 2OB − 2OA + 3OC − 3OB + OA − OC
= − OB − OA + 2OC
= −(OB + OA) + 2OC
= −2OF + 2OC [Since OF = ½ (OA + OB)]
= 2(OC − OF) = 2FC as required.

A unit vector has been defined as any vector with *magnitude* one. Indeed we can always find a unit vector with the *same direction* as any given vector. A unit vector is often denoted by

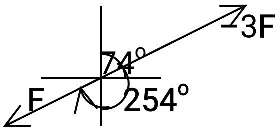
$\hat{a}$ [called *a* cap].

<u>Practice Exercise</u>. Find a unit vector with the same direction as the vector *a* = 3i − 4j.

Scalar Multiple of a Vector (in Magnitude-Bearing form)

Just like vectors in component-forms, we can find a *scalar multiple* of a vector in magnitude-bearing form. For example if F = (15km, 254°), then 3F = 3(15km, 254°) = (45km, 254°). This is because the vectors F and 3F are parallel and have the same direction since 3 is positive. However, if the scalar multiplying the vector is negative, then the two vectors will have *opposite* directions. Therefore, if F = (15km, 254°), then −3F = -3(15km, 254°)

$= (45\text{km}, 254^\circ + 180^\circ) = (45\text{km}, 074^\circ)$



Generally, if the vector F = (k, θ) and p is any scalar (constant), then

pF = (pk, θ) if $p > 0$ [p is positive] and pF = (pk, $\theta + 180^\circ$) if $p < 0$ [p is negative].

Vectors in Three-Dimensional Space

So far we have considered *vectors in two-dimensional space.*

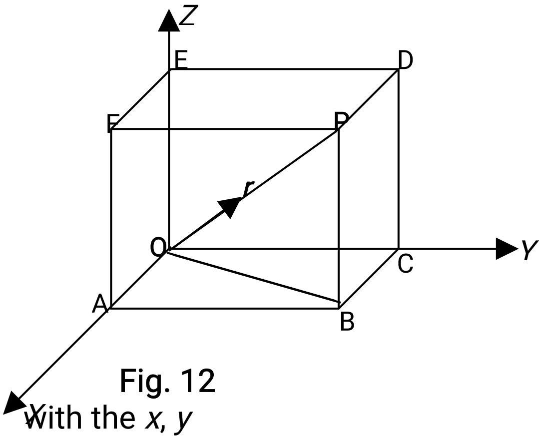


Fig. 12

In three-dimensional space, we have the x-axis, y-axis and the z-axis. Corresponding unit vectors along these axes are **i**, **j** and **k** respectively. See Fig. 14.

If $|OA| = x$, $|OC| = y$ and $|OE| = z$ then

OP = $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Note that the length of OP is

$|OP| = |r| = \sqrt{x^2 + y^2 + z^2}$

Let OP = r make angles α, β and φ

With the x, y

and z axes respectively. Then

$x = rCos\alpha$, $y = rCos\beta$ and $z = Cos\varphi$

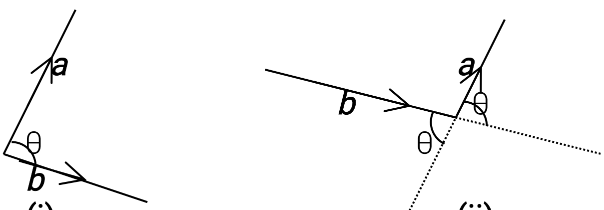
The numbers $Cos\alpha$, $Cos\beta$ and $Cos\varphi$ are called the direction cosine of the vector OP. All that we have considered under two-dimensional vector space are applicable in three-dimensional vector space.

Scalar/Dot Product of Vectors

The *scalar* or *dot* product of two vectors is defined as the *product of their magnitudes and the cosine of the angle between them.* In symbols; the *dot* product of the vectors a and b is:

$a\,.\,b = |a|\,|b|\,\text{Cos}\theta$ where θ is the angle between the two vectors.

It should be noted that the angle θ between the two vectors a and b is the angle between their directions. See Fig.16 below.



(i) (ii)

Fig.13

In Fig.13 (i), the two vectors ***a*** and ***b*** have the same initial point and θ is the angle between their directions. In Fig.13(ii), vectors ***a*** and ***b*** have different initial points and θ is the angle between their directions.

<u>Remarks:</u>

1. If the vectors ***a*** and ***b*** are parallel, then $\theta = 0^\circ$ and ***a . b*** = $|a|\,|b|$, since $\text{Cos}0^\circ = 1$

2. If the vectors ***a*** and ***b*** are perpendicular, then $\theta = 90^\circ$ and ***a . b*** = 0, since $\text{Cos}90^\circ = 0$

   [We use this condition to find whether two vectors are perpendicular.]

3. The angle θ, between the two vectors is given by $\theta = Cos^{-1}\left(\dfrac{a.b}{|a||b|}\right)$.

Note also that since the unit vectors **i** and **j** are perpendicular, we have the following:

i . **i** = **j** . **j** = 1 and **i** . **j** = **j** . **i** = 0.

Scalar/Dot Product in terms of Component

Let the vector ***a*** = a_1**i** + a_2**j** and the vector ***b*** = b_1**i** + b_2**j**. Then the dot product is

$$a\,.\,b = (a_1 i + a_2 j)\,.\,(b_1 i + b_2 j)$$
$$= a_1 i\,.\,(b_1 i + b_2 j) + a_2 j\,.\,(b_1 i + b_2 j)$$
$$= a_1 b_1 i.i + a_1 b_2 i.j + a_2 b_1 j.i + a_2 b_2 j.j = a_1 b_1 + a_2 b_2.$$

Note that the value $a_1b_1 + a_2b_2$ is a scalar, hence the name *scalar product.* In other words, dot product of two vectors gives a *scalar* <u>not</u> a *vector.*

Exercise

(i) Find the scalar product of the following pairs of vectors:

(*a*) 2i + 6j and 3i – j (*b*) 2i + 3j and 3i – 4j (*c*) 5i + 2j and 4i + 6j

(*d*) 4i – 2j and 3i + 6j

(ii) Which pairs are perpendicular?

(iii) For those that are not perpendicular, find the angle between them.

In three dimensional vector space, the scalar product of the vectors ***a*** = x_1**i** + y_1**j** + z_1**k** and

b = x_2**i** + y_2**j** + z_2**k** is defined as ***a .b*** = $x_1x_2 + y_1y_2 + z_1z_2$.

It is just like the definition in the two-dimensional vector space.

Angle between two vectors in three-dimensional space

Suppose we require the angle between the free vectors **PQ** and **RS** (Fig. 17). Consider the two corresponding localized vectors **OA** and **OB** passing through the origin O. Then the angle between **OA** and **OB** is the required angle. If the angle between the vectors **OA** and **OB** is θ then $\cos\theta = ll_1 + mm_1 + nn_1$

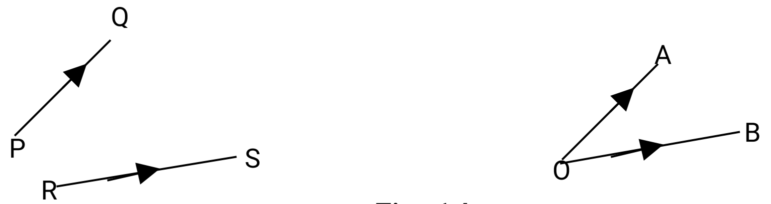


Fig. 14

where l, m, n and l_1, m_1, n_1 are the direction cosines of OA and OB.

Example 7
If the position vectors of points A and B are **4i + 4j – 7k** and **5i – 2j + 6k** respectively, find the
angle between the vectors OA and AB.

Solution

OA = 4i + 4j – 7k $\Rightarrow |OA| = \sqrt{4^2 + 4^2 + 7^2} = 9$

Hence directed cosines of OA are $\frac{4}{9}, \frac{4}{9}$ *and* $-\frac{7}{9}$

Again AB = (5i – 2j + 6k) – (4i + 4j – 7k) = i – 6j + 13k

Thus $|AB| = \sqrt{1^2 + 6^2 + 13^2} = \sqrt{206}$

So the directed cosines of AB are $\frac{1}{\sqrt{206}}, -\frac{6}{\sqrt{206}}$ *and* $\frac{13}{\sqrt{206}}$

If θ is the angle between OA and AB, then

$$\cos\theta = \left(\frac{4}{9}\right)\left(\frac{1}{\sqrt{206}}\right) + \left(\frac{4}{9}\right)\left(-\frac{6}{\sqrt{206}}\right) + \left(-\frac{7}{9}\right)\left(\frac{13}{\sqrt{206}}\right) = \frac{4 - 24 - 91}{9\sqrt{206}} = \frac{-111}{9\sqrt{206}} = \frac{-37}{3\sqrt{206}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-37}{3\sqrt{206}}\right) = 149.24^{\circ}$$

Note that we can also use *dot product* to find the angle between the two vectors.

Exercise.
Show that vectors ***a*** = 3i + 5j + 2k and ***b*** = 2i + 2j – 8k are perpendicular.

The Cosine Rule

The *Cosine* and *Sine* rules are used to “solve triangles”.

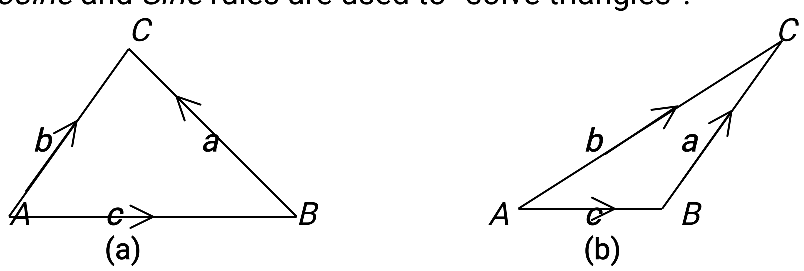


Fig.15

In Fig.15 [in either triangle], let *AB = c, BC = a* and *AC = b.*

In either triangle, *BC = a = b – c.*

Therefore $\boldsymbol{a.a} = (\boldsymbol{b} - \boldsymbol{c}).(\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{b.b} - 2\boldsymbol{b.c} + \boldsymbol{c.c} = \boldsymbol{b.b} + \boldsymbol{c.c} - 2\boldsymbol{b.c}$

[If we let $|\boldsymbol{a}| = a$, $|\boldsymbol{b}| = b$ and $|\boldsymbol{c}| = c$, then $\boldsymbol{a.a} = |\boldsymbol{a}|\,|\boldsymbol{a}|\,Cos0^{\circ} = a^2$,

$\boldsymbol{b.b} = |\boldsymbol{b}|\,|\boldsymbol{b}|\,Cos0^{\circ} = b^2$ and $\boldsymbol{c.c} = |\boldsymbol{c}|\,|\boldsymbol{c}|\,Cos0^{\circ} = c^2$.]

Note that angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ is A. Thus the above *dot* product becomes:

$$a^2 = b^2 + c^2 - 2\boldsymbol{b.c} = b^2 + c^2 - 2|\boldsymbol{b}|\,|\boldsymbol{c}|\,CosA$$

That is

$$\boxed{a^2 = b^2 + c^2 - 2bcCosA}$$

This is known as the *Cosine Rule.* If we re-label triangle *ABC* and we go through similar process we shall also arrive at:

$$\boxed{b^2 = a^2 + c^2 - 2acCosB} \quad \text{OR} \quad \boxed{c^2 = a^2 + b^2 - 2abCosC}$$

Note that if angle *A*, *B* or *C* is 90°, then triangle *ABC* becomes a right-angled triangle and the above formulae become: $a^2 = b^2 + c^2$; $b^2 = a^2 + c^2$ or $c^2 = a^2 + b^2$

[since $Cos90^{\circ} = 0$].

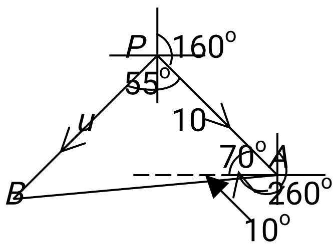
Hence the *Cosine Rule* is simply an extension of *Pythagoras Theorem*.

Example 7

Two ships A and B leave a port simultaneously. A steams at 10 kmh^{-1} on a bearing of 160° and B steams on a bearing of 215°. Just after one hour the bearing of B from A is 260°. Find the speed of B, correct to two significant figures.

Solution

Let *P* be the port, *A* and *B* the positions of the ships A and B just after one hour. Let also the speed of ship B be *u* kmh^{-1}.



Note that since the bearing of point *A* from *P* is 160° and bearing of *B* from *P* is 215°,

thus angle *APQ* = 55° [i.e. 215° – 160° = 55°]

Again angle *BAP* = 80° [70° + 10°]

Hence angle *ABP* = 180° – (55° + 80°) = 45°.

By *Sine rule*

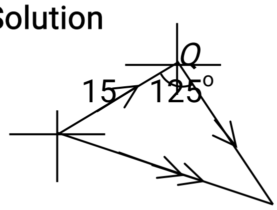
$$\frac{u}{Sin\,80^{\circ}} = \frac{10}{Sin\,45^{\circ}} \Rightarrow u = \frac{10\,Sin\,80^{\circ}}{Sin\,45^{\circ}} = 13.93 \cong 14$$

So the speed of ship B is 14 kmh^{-1}.

Example 8

A ship steams from port *P* for a distance of 15 *km* on a bearing of 070° to port *Q*. It then steams from port *Q* to port *R* a distance of 20 *km* on a bearing of 125°. Find the distance and bearing of port *R* from port *P*.

Solution



Note that angle *PQR* is 125°

In triangle *PQR*, by *Cosine rule*,

P θ 20

R

$|PR|^2 = 15^2 + 20^2 - 2(15)(20)Cos125^{\circ}$
$= 225 + 400 + 344.1458618$
$= 969.1458618$

Therefore, $|PQ| = 31.13\ km$

Let angle *QPR* be θ. In triangle *PQR*, by *Sine rule*;

$$\frac{20}{Sin\ \theta} = \frac{31.13}{Sin125^{\circ}} \Rightarrow Sin\theta = \frac{20\ Sin125^{\circ}}{31.13}.\ \text{Hence}\ \theta = Sin^{-1}\left(\frac{20\ Sin125^{\circ}}{31.13}\right) = 31.75^{\circ}.$$

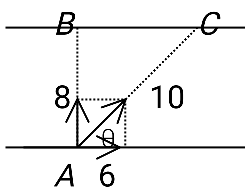
Thus the distance and bearing of *R* from *P* are 31.13 *km* and 102°.

Example 9

A boat is rowed with speed 8 kmh^{-1} straight across a river, which is flowing at 6 kmh^{-1}.

Find the resultant velocity of the boat. If the breadth of the river is 100*m*, find how far down the river will the boat reach the opposite bank.

Solution



Let $v\ kmh^{-1}$ be the speed of the resultant velocity of the boat.

From the diagram,

$v = \sqrt{(8^2 + 6^2)} = \sqrt{100} = 10$

If θ is the angle the resultant makes with the bank of the river,

then $Cos\theta = 6/10 = 0.6 \Rightarrow \theta = Cos^{-1}(0.6) = 53.13^{\circ} \cong 53^{\circ}$.

Let *A* be the starting point of the boat and *B* is the point directly opposite on the other bank. *C* is a point down river where the boat reaches. By similar triangles;

$\frac{|BC|}{|AB|} = \frac{6}{8}$. [$|AB| = 100m$] Therefore $|BC| = \frac{6}{8} \times 100m = 75m$.

Hence the boat will be carried down -stream a distance of 75*m*.

Unit 3: Vector Equation and Properties of a Straight Line

Vector equation of a Line

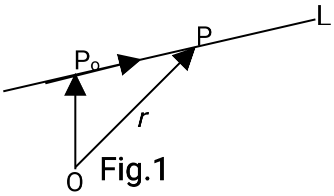


Fig.1

Let *r* be the position vector of any point P on the line *L*.

If P_o is any initial point on *L*, then

$r = \overrightarrow{OP_o} + \overrightarrow{P_oP}$

This implies that P is a specific point on the line.

If P is a variable point on the line, then the vector equation of the line can be written as:

$r = \overrightarrow{OP_o} + t\overrightarrow{P_oP}$ where $t \in R$ (Real Number)

Note that the vector $\overrightarrow{P_oP}$ is the direction vector of the line *L*.

Position Vector of a Point that divides a line segment AB into a given Ratio

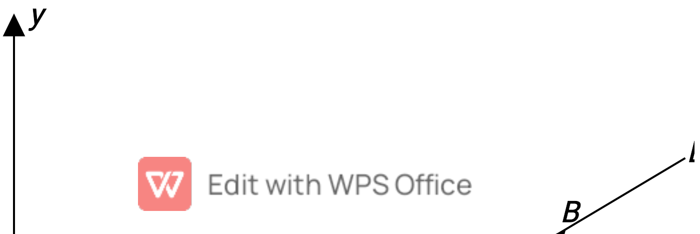


Fig. 2

Let *L* be any line in the coordinate plane points *A* and *B* any points on the line *L.* Let also *P* be a point on *L* dividing *AB* in the ratio *m:n.*
Let also the position vectors of points *A, B* and *P* be respectively ***a b*** and ***r***.

We have $\left|\vec{AP}\right| : \left|\vec{PB}\right| = m:n$ (see Fig. 2)

This relation can be rewritten as : $\left|\vec{AP}\right| = \frac{m}{m+n}\vec{AB}$

Writing $\vec{AP}$, and $\vec{AB}$ in terms of their position vectors, with *O* as the origin gives

$$,\vec{OP} = \vec{OA} + \frac{m}{m+n}\left(\vec{OB}-\vec{OA}\right)$$

$$.= a + \frac{m}{m+n}(b-a)$$

$$,= \frac{na+mb}{m+n}$$

The scalars *m* and *n* can be positive or negative according to whether the division of the line segment is internal or external. In the worked examples below efforts will be made to distinguish between these two types of divisions.

Example 1

Find the position vector of a point which divides the join A and B whose position vectors are

$a = 3i - 2j$ and $b = 5i + 3j$ respectively in the ratio:

(i) $3 : 2$

(ii) $4 : -1$

(iii) $-2:5$

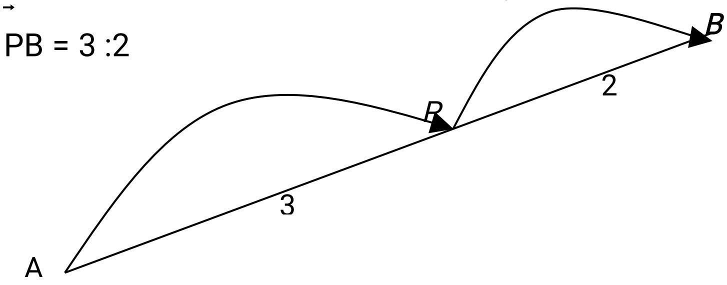
Solution

(i)Since the numbers 3: 2 are positive then the division of the segment is internal

We want to find $\vec{OP}$ if $\vec{AP} : \vec{PB} = 3:2$

Let $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$

$\Rightarrow \vec{AP} = \frac{3}{5}\left(\vec{AB}\right)$



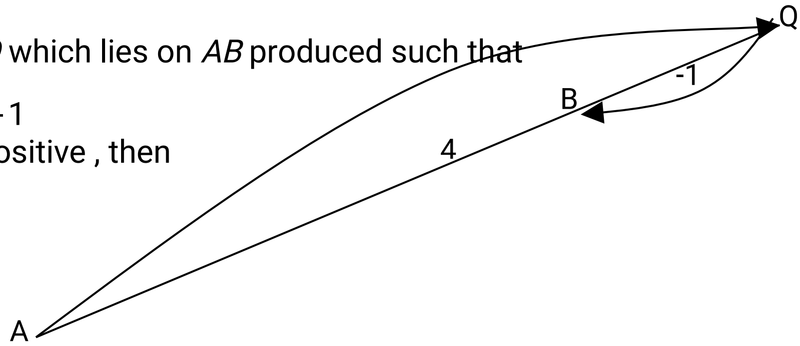
$$\text{Or } \vec{OP} = \vec{OA} + \frac{3}{5}\left(\vec{OB} - \vec{OA}\right)$$
$$= \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$$
$$= \frac{2}{5}(3i-2j) + \frac{3}{5}(5i+3j)$$
$$= \frac{2}{5}i + j$$

(ii) We find a point Q which lies on AB produced such that

$\left|\vec{AP}\right| : \left[\vec{PB}\right] = 4 : -1$

Taking A to B as positive , then

$$\vec{AQ} = \frac{4}{4+(-1)}\vec{AB}$$

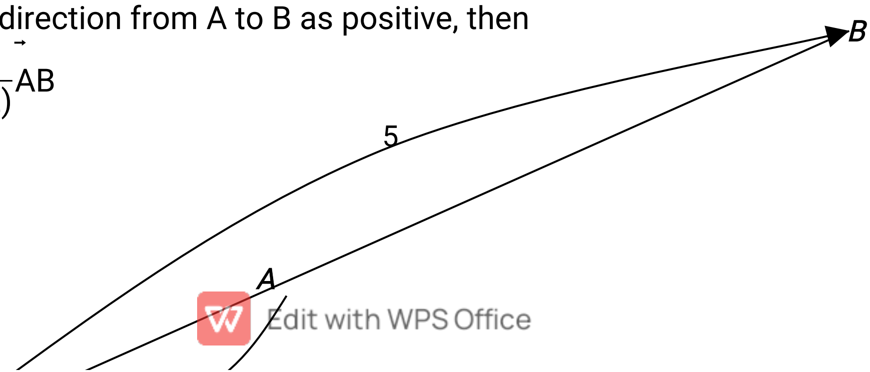


$$\text{Or } \vec{OQ} = \mathbf{a} + \frac{4}{3}(\mathbf{b}-\mathbf{a}) = -\frac{1}{3}\mathbf{a} + \frac{4}{3}\mathbf{b}$$
$$= -\frac{1}{3}(3i-2j) + \frac{4}{3}(5i + 3j)$$
$$= \frac{17}{3}i + \frac{14}{3}j$$

(iii) We require to find point R which lies on BA produced such that $\vec{AR} : \vec{RB} = -2:5$

Taking the direction from A to B as positive, then

$$\vec{AR} = \frac{-2}{5+(-2)}\vec{AB}$$



Or $\vec{OR} = \vec{OA} - \frac{2}{3}\left(\vec{OB} - \vec{OA}\right)$

$= \frac{5}{3}a - \frac{2}{3}b$

$= \frac{5}{3}(3i-2j) - \frac{2}{3}(5i + 3j)$

$\frac{5}{3}i - \frac{16}{3}j$

Example 2

Find the vector equation of a line that passes through points A(2, 5) and B(-4, 1).

Solution

Position vector of point A is 2i + 5j. Now direction vector of **AB** is :

AB = **OB** − **OA** = (−4i + j) − (2i + 5j) = −6i − 4j

Hence the vector equation of the line is $r = (2i + 5j) + t(-6i - 4j)$

t is a scalar or any real number. Now when $t = 0$, we have point A(2, 5). When $t = 1$, we have point B(− 4 , 1).

This equation can also be written as $x = 2 - 6t$ and $y = 5 - 4t$.

This is the *parametric* form of the equation. Again if we solve for t we shall arrive at

$\frac{x-2}{-6} = \frac{y-5}{-4}$. This is called the *symmetric* form of the equation.

Exercise.

1.Find the *vector*, *parametric* and *symmetric* equations of each of the following lines:

(i) passes through the points A(1, 7) and B(3, 4)

(ii) passes through the points A(3, 2, 4) and B(−1, 7, 2).

2. If the position vectors of points A and B are 4i + 4j − 7k and 5i − 2j + 6k respectively,

find the direction cosines of the vectors **OA**, **OB** and **AB**.

Properties of a Straight Line

In the equation, $\vec{OP} = \vec{OA} + \frac{m}{m+n}\left(\vec{OB} - \vec{OA}\right)$

$$= a + \frac{m}{m+n}(b-a)$$

$$= \frac{na+mb}{m+n}$$

If *P* is the mid-point of *AB,* then we have $m = n = 1$ and the equation becomes

$$\vec{OP} = a + \frac{1}{1+1}(b-a)$$

$$= \frac{1}{2}(a+b)$$

Example

Find the mid-point of *AB* if *A*(2, 3) and *B*(4, 7).

Solution

The position vectors of A and B are respectively 2i + 3j and 4i + 7j

Let the position vector of the mid-point be $\vec{OP}$

$$\vec{OP} = \frac{1}{2}(6i + 10j)$$

$$= 3i + 5j$$

So the mid-point is (3, 5)

Collinear Points

Three points *A*, *B* and *P* are said to be collinear if they all lie on the same straight line,

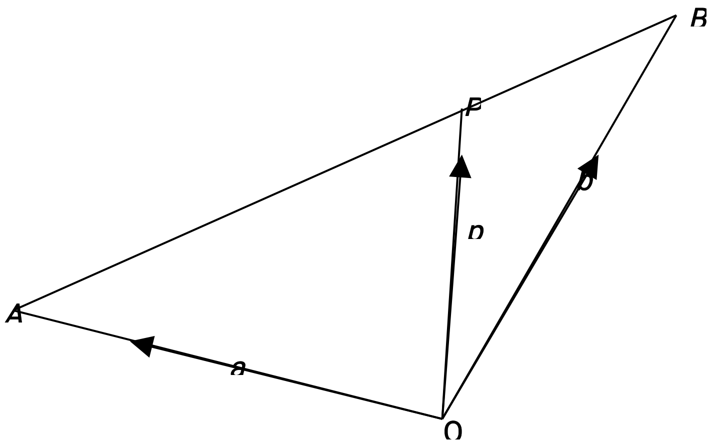


Fig. 3

In Figure 3, let O be the origin *AB* a line, *P* a point on the line. Let also *a*, *b* and *p* be the position vectors of the points *A*, *B* and *P* respectively.

Then the position of *P* relative to both *A* and *B* must have a simple ratio of the type

$$|\vec{AP}| : |\vec{PB}| = m : n$$

Where *m* and *n* are scalars.

In other words, *P* has the position vector

$$p = \frac{na+mb}{m+n}$$

This equation is the same as

$$na + mb - (m+n)p = 0.$$ This equation can be rewritten as
$\gamma a + \mu b + \beta p = 0$ with $\gamma + \mu + \beta = 0$.
This implies that points *A, B* and *P* will be linear with the properties that

$$\gamma a + \mu b + \beta p = 0$$

and
$$\gamma + \mu + \beta = 0$$

Example

The position vectors of points *A*, *P* and *B* are $11i + j$, $5i + \frac{13}{3}j$ and $2i + 6j$ respectively.

a. Show that *A*, *P* and *B* lie on the same straight line.

Solution

Let *a, p* and *b* be the position vectors of *A, P* and *B* respectively. If *A, P* and *B* are n collinear, then there are scalars *l, m* and *n* not all zero with the properties that

$$la + mp + nb = 0$$

and
$$l + m + n = 0$$

Using the first equation, we have : $l(11i + j) + m(5i + \frac{13}{3}j) + n(2i + 6j) = 0$

Equating the components of the vector to zero gives:

i: $\quad 11l + 5m + 2n = 0$

j: $\quad l + \frac{13}{3}m + 6n = 0$

Dividing each equation through by *n* we have:

$$11\frac{l}{n} + 5\frac{m}{n} + 2 = 0$$

$$\frac{l}{n} + \frac{13}{3}\frac{m}{n} + 6 = 0$$

Letting $\frac{l}{n} = \gamma$ and $\frac{m}{n} = \mu$ the equations become:

$$11\gamma + 5\mu + 2 = 0\ldots\ldots\ldots\ldots\ldots(1)$$

$$\gamma + \frac{13}{3}\mu + 6 = 0\ldots\ldots\ldots\ldots\ldots(2)$$

We then solve these equations simultaneously for γ and μ.

$$11\gamma + 5\mu + 2 = 0\ldots\ldots\ldots\ldots\ldots(1)$$

Multiply (2) by 11

$$11\gamma + \frac{143}{3}\mu + 66 = 0\ldots\ldots\ldots(3)$$

(3) – (1): $\quad \frac{128}{3}\mu + 64 = 0$

Hence $\mu = -\frac{3}{2}$

But

Putting this value in (1) gives $\mu = \frac{1}{2}$

But $\frac{l}{n} = \gamma$ and $\frac{m}{n} = \mu$,

Hence $\frac{l}{n} = \frac{-3}{2}$ and $\frac{m}{n} = \frac{1}{2}$

Therefore the ratio l:m:n = 1: -3:2

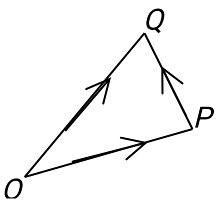
Since $l + m + n = 1 + (-3) + 2 = 0$, it follows that points *A*, *P* and *B* are collinear.

Exercise

Show that the points *A*(2, 7), *B*(8, 11) and *P*(5, 8) are collinear.

Unit 4 RESULTANT OF A SYSTEM OF VECTORS /FORCES

We introduced the resultants of vectors in Unit 2. Re visit Unit 2, we said that It is very important to write the order of the letters (in a vector) correctly. The length of



the vector **OP** equals the length of vector **PO**, but they have *opposite* directions. If a particle moves from *P* to *Q* (Fig.2), the displacement of *Q* to *P* is represented by the vector **PQ**.

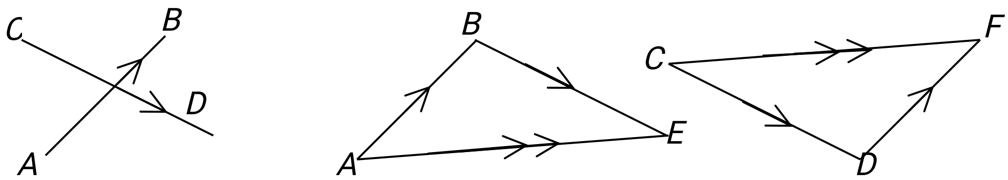
Since the displacement of *O* to *P* is followed by displacement of *P* to *Q*, geometrically, these two displacements are equivalent to the single displacement *OQ*.

Hence **OQ** = **OP** + **PQ**

This is known as the *Triangle Law* of *Vector Addition.*

[OQ is called the ***resultant*** of OP and PQ]

This law states that *'the sum of any two vectors represented by* ***AB*** *and* ***BC*** *is the vector represented by* ***AC*** *where the vectors* ***AB, BC*** *and* ***AC*** *form triangle ABC'.*



(i) (ii) (iii)

Fig. 3

In Fig.2, we have **OQ** = **OP** + **PQ** . Therefore, **PQ** = **OQ** – **OP**.

If we wish to *add* vectors **AB** and **CD**, we first replace them by *equal* vectors placed "*head to tail"* and then use the triangle law of vector addition. [**Note**: *Two vectors are said to be equal if they have the same magnitude and the same direction.*]

In Fig.3 (ii), vector **CD** is replaced by **BE** [**CD** = **BE**] which has its initial point at the terminal point of vector AB. Thus AB + CD = AB + BE = AE. In Fig.3 (iii) vector AB is replaced by vector DF [AB = DF] which has its initial point at the terminal point of vector CD.

Hence AB + CD = CD + DF = CF. A special case of the triangle law of vector addition is when the two vectors have a common initial point. Consider two vectors AB and AC with A as the initial common point (Fig.4). If the parallelogram *ABPC,* is completed, then we have:

AC = BP and AB = CP

Now AB + AC = AB + BP = AP and

AB + AC = AC + CP = AP

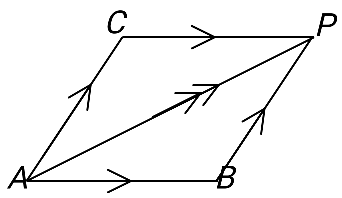


Fig.4

[AP is the *resultant* of AB and AC]

This is known as the *parallelogram law of vector addition.* This law states that "*the sum of two vectors AB and AC is represented by the diagonal AP of the parallelogram ABPC having AB and AC as adjacent sides*".

Remarks: The parallelogram law of vector addition proves that *vector addition is commutative.* From Fig.5; AP = AB + BP = AB + AC [since BP = AC]

AP = AC + CP = AC + AB [since CP = AB].

ASSOCIATIVE LAW (Order of Adding Several Vectors)

Let vectors AB, BC and CD be *a, b* and *c* respectively. Then vector AC $= a + b$ and vector BD $= b + c$.

Now AD = AB + BD $= a + (b + c)$

AD = AC + CD $= (a + b) + c$

Therefore $a + (b + c) = (a + b) + c$, indicating that *vector addition is associative.*

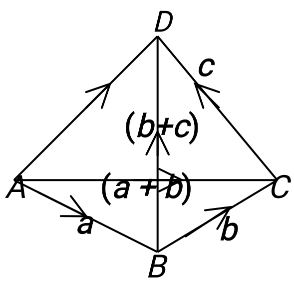


Fig. 5

If the vectors are written in components form, the resultant of the vectors is simply the vector sum.

For example, the resultant of a = 2i + 3j, b = 5i – 6j and c = 4i + 5j

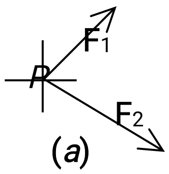
= $a + b + c$ = (2i + 3j) + (5i – 6j) + (4i + 5j)

= 11i + 2j

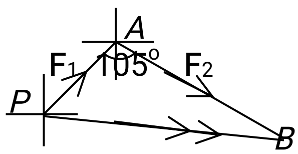
If the vectors are written in magnitude-bearing form and there are only two vectors involved, then the resultant may be found by sketching them, apply the Cosine and Sine Rules to find the magnitude and bearing of the resultant of the vectors. However, if the vectors are more that two, the resultant may be found by first resolving them to components forms, add them before changing it to magnitude-bearing form. We work an example of each.

Example 1

Forces F_1 = (4N, 045°) and F_2 = (8N, 120°) act on a particle p at a point. Find the resultant force of F_1 and F_2.



(a)



(b)

Solution

Diagram (a) shows the forces acting on the particle p. Diagram (b) shows the forces drawn such that initial point (A) of F_2 coincides with the terminal point of F_1. Note that angle PAB = 105°. By *Cosine rule;*

$PB^2 = 4^2 + 8^2 - 2(4)(8)\cos 105^\circ = 16 + 64 + 16.56441889 = 96.56441889$

Thus $PB = \sqrt{96.56441889} = 9.8267964 \cong 9.8$

Let angle $APB = \theta$, then by *Sine rule*;

$$\frac{8}{\sin\theta} = \frac{9.827}{\sin 105^\circ} \Rightarrow \sin\theta = \frac{8\sin 105^\circ}{9.827} \Rightarrow \theta = \sin^{-1}\left(\frac{8\sin 105^\circ}{9.827}\right) = 51.845^\circ \cong 51.8^\circ .= 52^\circ$$

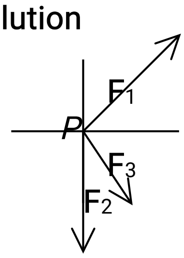
(The angle of the bearing is 45° + 52° = 97°)

So the resultant of the forces is (9.8N, 097°).

Example 2

A body is acted on by the forces F_1 = (50N, 040°), F_2 = (63N, 180°) and F_3 = (28N, 150°). Find the resultant force of these three forces.

Solution



Let P be the body being acted on the forces F_1, F_2 and F_3. Note that we cannot apply the *Cosine rule* since there are more than two forces. [The *Cosine rule* is applicable when two forces are only involved]. Hence we need to *resolve* the forces into components forms before we find the resultant. Note:

$$F_1 = (50N, 040^\circ) = \begin{pmatrix} 50\cos 50^\circ \\ 50\sin 50^\circ \end{pmatrix} = \begin{pmatrix} 32.1394 \\ 38.3022 \end{pmatrix}$$

$$F_2 = (63N, 180^o) = \begin{pmatrix} 0 \\ -63 \end{pmatrix}$$

$$F_3 = (28N, 150^o) = \begin{pmatrix} 28\,Cos\,60^{\circ} \\ -28\,Sin\,60^{\circ} \end{pmatrix} = \begin{pmatrix} 14 \\ -24.2487 \end{pmatrix}$$

Thus $F_1 + F_2 + F_3 = \begin{pmatrix} 32.1394 \\ 38.3022 \end{pmatrix} + \begin{pmatrix} 0 \\ -63 \end{pmatrix} + \begin{pmatrix} 14 \\ -24.2487 \end{pmatrix}$

$= \begin{pmatrix} 46.1394 \\ -48.9465 \end{pmatrix}$

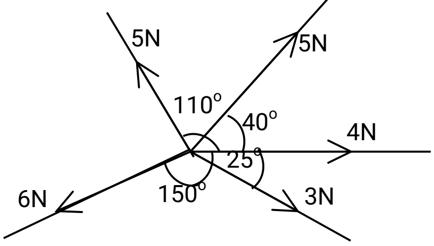
Magnitude = $\sqrt{(46.1394)^2 + (-48.9465)^2} = 67.265 \cong 67.3N$

To find the direction, we find $\tan^{-1}\left(\frac{48.9465}{46.1394}\right) = 46.69^{\circ} = 47^o$

Note that the resultant force lies in the *fourth quadrant*.
Thus the bearing of the resultant force is 137^o (i.e. $90^o + 47^o$).
Therefore, the resultant of the three forces is R = (67.3N. 137^o).

Example 3

Find the magnitude and direction of the resultant of the forces shown in the diagram below.



Solution

We write the forces in magnitude-bearing forms:, starting from the 4N on the east in the

anti-clockwise direction, we have:

(4N. 090^0), (5N, 050^0), (5N, 340^0), (6N, 240^0) and (3N, 115^0)

Resolving these forces to components, we have:

(4N. 090^0) = $\begin{pmatrix} 4Sin90^0 \\ 4Cos90^0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

(5N, 050^0) = $\begin{pmatrix} 5Sin50^0 \\ 5Cos50^0 \end{pmatrix} = \begin{pmatrix} 3.8302 \\ 3.2139 \end{pmatrix}$

(5N, 340^0) = $\begin{pmatrix} 5Sin340^0 \\ 5Cos340^0 \end{pmatrix} = \begin{pmatrix} -1.7101 \\ 4.6985 \end{pmatrix}$

$(6N, 240^0) = \begin{pmatrix} 6Sin240^0 \\ 6Cos240^0 \end{pmatrix} = \begin{pmatrix} -3\sqrt{3} \\ -3.000 \end{pmatrix}$

$(3N, 115^0) = \begin{pmatrix} 3Sin115^0 \\ 3Cois115^0 \end{pmatrix} = \begin{pmatrix} 2.7189 \\ -1.2679 \end{pmatrix}$

Hence the resultant = $\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 3.8302 \\ 3.2139 \end{pmatrix} + \begin{pmatrix} -1.7101 \\ 4.6985 \end{pmatrix} + \begin{pmatrix} -3\sqrt{3} \\ -3.0000 \end{pmatrix} + \begin{pmatrix} 2.7189 \\ -1.2679 \end{pmatrix}$

$= \begin{pmatrix} 3.6428 \\ 3.6445 \end{pmatrix}$

This vector lies in the first quadrant and we need to change it to magnitude-bearing form/

Magnitude = $\sqrt{(3.6428)^2+(3.6445)^2}$ = 5.15289 ≅ 5.15 N

To find the direction, note that $\tan\theta = \dfrac{3.6445}{3.6428}$ where θ is the angle the vector makes with the *x-axis.* So $\theta = \tan^{-1}\left(\dfrac{3.6445}{3.6428}\right) = 45.01^0 = 45^0$

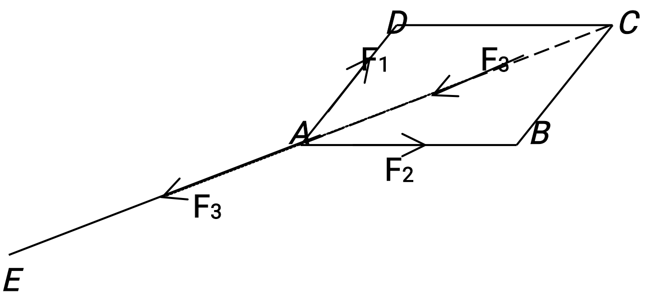
Hence the resultant of the forces is $(5.15N, 045^0)$

Forces in Equilibrium

In a parallWe saw in the first diagram in the introduction, that $\overrightarrow{AC} = R$ represents the resultant of forces

F_1 and F_2. Now if we consider the opposite resultant $\overrightarrow{CA} = F_3$, then the forces F_1, F_2 and F_3 are

said to be in *equilibrium* [see diagram below]. Note also that $\overrightarrow{AD} = \overrightarrow{BC}$.



When we find the *vector sum* of these forces, we get zero. In the diagram, we have $F_1 + F_2 + F_3 = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$. Thus the forces F_1, F_2 and F_3 are in equilibrium.

If the vector sum of given vectors is zero, we say such vectors are in equilibrium.

Lami's Theorem

In the diagram, the forces F_1, F_2 and F_3 act at point *A*. By the *Sine rule*,

$\frac{|F_1|}{SinBAC} = \frac{|F_2|}{SinACB} = \frac{|F_3|}{SinABC}$. Again in the diagram; $\angle BAC = 180^o - \angle EAB$

$\angle ACB = \angle DAC$ [Alternate angles] $= 180^o - \angle EAD$; $\angle ABC = 180^o - \angle DAB$

But from trigonometry, $Sin(180^o - \varphi) = Sin\varphi$. Therefore the above relation becomes:

$$\frac{|F_1|}{SinEAB} = \frac{|F_2|}{SinEAD} = \frac{|F_3|}{SinDAB}$$

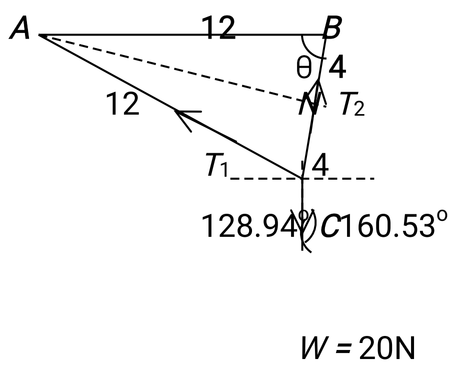
This is known as the *Lami's Theorem*, which states that:

"*If three forces acting at a point are in equilibrium, then each is proportional to the sine of the angle between the other two forces.*"

Example 5

A mass *C* of weight 20 N is suspended by two light strings *AC* and *BC* from points *A* and *B* on
the same horizontal level above *C*. *AB* = *AC* = 12 cm and *CB* = 8 cm. Find the tensions in *AC*
and *BC*.

Solution



Let T_1 and T_2 be the tensions in *CA* and *CB* respectively. Also let the foot of the perpendicular from *A* to *BC* be *N* and $\angle ABC = \theta$. Then in triangle *ABN*, $Cos\theta = \frac{4}{12} = \frac{1}{3}$

$\Rightarrow \theta = Cos^{-1}(\frac{1}{3}) = 70.53^o$

Since $\triangle ABC$ is isosceles triangle, $\angle ACB = 70.53^o$. Thus angle between forces T_2 and the weight ($W = 20N$) is 160.53^o. Again angle between forces T_1 and T_2 is 70.53^o, and angle between forces T_1 and the weight ($W = 20N$) is

$[360^o - (160.53^o + 70.53^o)]$ 128.94^o. Since the forces T_1, T_2 and W are in equilibrium, then by *Lami's Theorem*, we have; $\frac{T_1}{Sin160.53^\circ} = \frac{T_2}{Sin128.94^\circ} = \frac{20\ N}{Sin\ 70.53^\circ}$

Thus $\frac{T_1}{Sin160.53^\circ} = \frac{20\ N}{Sin\ 70.53^\circ} \Rightarrow T_1 = \frac{20\ NSin\ 160.53^\circ}{Sin\ 70.53^\circ}$ = 7.07N

Again $\frac{T_2}{Sin128.94^\circ} = \frac{20\ N}{Sin\ 70.53^\circ} \Rightarrow T_2 = \frac{20\ NSin\ 128.94^\circ}{Sin\ 70.53^\circ}$ = 16.50N

Alternatively: Since the system is in equilibrium, we could resolve the forces into components and add them since their sum is zero.

Note: $T_1 + T_2 + W = 0$. Resolving these forces into components, we get

$$T_1 = \begin{pmatrix} -T_1 Cos\ 38.94^\circ \\ T_1 Sin\ 38.94^\circ \end{pmatrix} = \begin{pmatrix} -0.7778\ T_1 \\ 0.6285\ T_1 \end{pmatrix} \quad T_2 = \begin{pmatrix} T_2 Cos\ 70.53^\circ \\ T_2 Sin\ 70.53^\circ \end{pmatrix} = \begin{pmatrix} 0.3333\ T_2 \\ 0.9428\ T_2 \end{pmatrix}$$

and $W = \begin{pmatrix} 0 \\ -20\ N \end{pmatrix}$. Hence,

$$\begin{pmatrix} -0.7778\ T_1 \\ 0.6285\ T_1 \end{pmatrix} + \begin{pmatrix} 0.3333\ T_2 \\ 0.9428\ T_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -20\ N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, we have $-0.7778T_1 + 0.3333T_2 = 0$..............(1)

$0.6285T_1 + 0.9428T_2 = 20N$.........(2)

From (1) $0.7778T_1 = 0.3333T_2 \Rightarrow T_1 = 0.4285T_2$. Substituting for T_1 in (2):

$0.6285(0.4285T_2) + 0.9428T_2 = 20N \Leftrightarrow 1.2121\,T_2 = 20N \Rightarrow \underline{T_2 = 16.5N}$

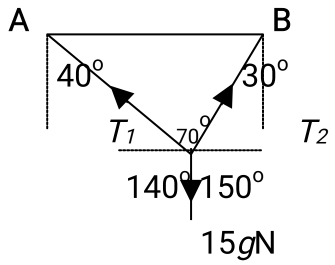
substituting for $T_2 = 16.5N$ gives $T_1 = 0.4285(16.5N) \Rightarrow \underline{T_1 = 7.07N}$

Example 6

A body P of mass 15 *kg* is suspended by two light inextensible strings AP and BP which are

inclined at 40° and 30° respectively to the downward vertical when P is at rest. Find the magnitudes of the tensions in the strings.

Solution



Let the magnitudes in AP and BP be T_1 and T_2 respectively. Note the angles between the forces are shown in the diagram above. By *Lami's* Theorem,

$$\frac{T_1}{Sin\,150^\circ} = \frac{T_2}{Sin\,140^\circ} = \frac{15\,gN}{Sin\,70^\circ} \Rightarrow T_1 = \frac{15\,gSin\,150^\circ}{Sin\,70^\circ} = 7.98g\text{N and}$$

$$T_2 = \frac{15\,gSin\,140^\circ}{Sin\,70^\circ} = 10.26g\text{N}$$

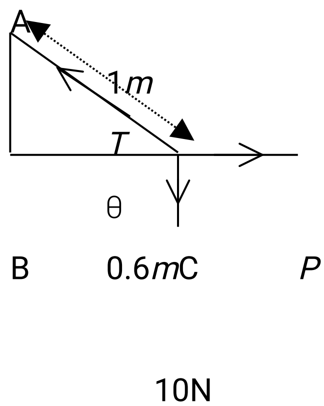
[That is T_1 = 78.2N, T_2 = 100.5N if g = 9.8 and T_1 = 79.8N, T_2 = 102.6N if g = 10].

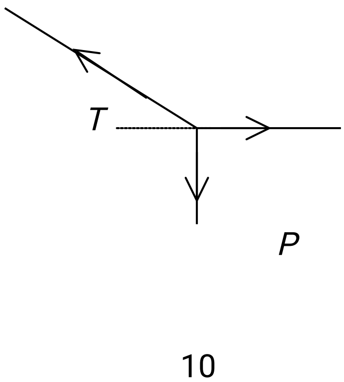
Example 7

One end of a string 1*m* long is fixed at a point A and the other end is fastened to a small weight

10N. The object is pulled aside by a horizontal force until it is 0.6*m* from the vertical through A. Find the magnitudes of the tension in the string and the horizontal force.

Solution





Let the horizontal force be *P* and the tension in the string *T* and angle ACB = θ. In triangle ABC; θ = $Cos^{-1}(0.6) = 53.13^\circ$. Thus by *Lami's* Theorem,

$$\frac{T}{Sin\,90^\circ} = \frac{P}{Sin\,143.13^\circ} = \frac{10\,N}{Sin\,126.87^\circ}$$

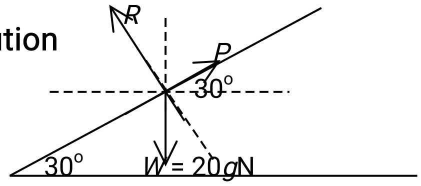
$$\Rightarrow T = \frac{10\,Sin\,90^\circ}{Sin\,126.87^\circ} = 12.5N \text{ and } P = \frac{10\,NSin\,143.13^\circ}{Sin\,126.87^\circ} = 7.5N$$

Example 8

A body of mass 20*kg* is kept at rest on a smooth plane inclined at an angle of 30^o to the horizontal by a force *P* up the plane. Find: (i) the magnitude of *P*;

(ii) the magnitude of the normal reaction of the plane on the body.

Solution



Let *R* be the normal reaction of the plane on the body, *W* (20*g*N) is the weight of the body and *P* is the force up the plane that keeps the body at rest. The only forces acting on the body are *P*, *W* and *R*. Since the body is at rest, we have

$P + R + W = 0$.

Note: Angle between *P* and *W* is 120^o, angle between *R* and *W* is 150^o and that between *R* and *P* is 90^o. Hence by *Lami's* theorem;

$$\frac{P}{Sin 150^\circ} = \frac{R}{Sin 120^\circ} = \frac{20\,gN}{Sin 90^\circ} \Rightarrow P = \frac{20\,g Sin 150^\circ}{Sin 90^\circ} = 10gN$$

[98N, if $g = 9.8$ and 100N, if $g = 10$]

and $R = \dfrac{20\,g Sin 120^\circ}{Sin 90^\circ}$ =17.32*g*N [169.74N if $g = 9.8$ and 173.2N if $g = 10$].

UNIT 5: NEWTON'S LAWS OF MOTION WITH EMPHASIS ON THE USE OF THE SECOND LAW

DYNAMICS

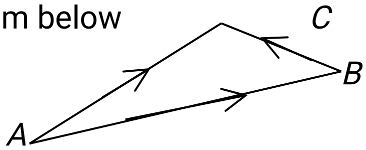
Dynamics is that part of mechanics that deals with motions of bodies or particles as a result of actions of forces on them. Let a moving particle be at *P* at time *t* and move from *P* to P^1 in the time interval dt . Then the vector PP^1 is the displacement of the particle and $\frac{PP^1}{dt}$ is the average velocity during the interval time of dt . If this average velocity approaches a *limit* as dt 0, the limiting value is called the velocity of the particle at the point *P* or at time *t*. Now if *s* is the displacement, then $s = f(t)$, velocity (v) = $\frac{ds}{dt}$ = $f^1(t)$ and acceleration (a) = $\frac{d^2s}{dt^2}$ = $f^{11}(t)$. All these are vector quantities since each has a magnitude and direction. Examples of these quantities may include: s = (10m, 057°); v = (5ms^{-1}, 128°) and a = (9.8ms^{-2}, 180°). Velocity (v) may be constant or not. If it is constant, it is called *uniform.* A body moving with uniform velocity will travel; (1) the same distance in any time interval, (2) in a fixed direction (in a straight line). A body may have several different velocities at the same time. For example, a man walking on the deck of a moving ship may have (i) his own velocity, (ii) velocity of the ship and (iii) velocity of the ocean current. The single velocity that is equivalent to several velocities is called their *resultant.* Note: While *velocity* expresses the rate of motion of a particle in terms of magnitude and direction, *speed* expresses the rate of motion of the particle *without specifying the direction of motion.* Thus while *speed* is a *scalar* quantity, *velocity* is a *vector* quantity.

Relative Velocity

Velocity, like any other vector, is related to a *frame of reference.* When we say a car has a velocity of 120 kmh^{-1} in a certain direction, we mean 120 kmh^{-1} relative to the earth. The earth itself is moving relative to the sun, so the velocity of the car relative to the sun has a value that may be deduced from the velocity of the car relative to the earth and the velocity of the earth relative to the sun. In general ; velocity of *A* relative to *B* + velocity of *B* relative to *C* = velocity of *A* relative to *C.* In symbols this may be written as

${}_AV_B + {}_BV_C = {}_AV_C$. This may be deduced from *triangular law of vector addition.*

From diagram below $\vec{AC} = \vec{AB} + \vec{BC}$



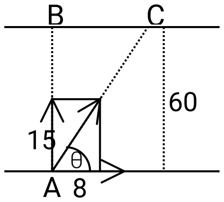
From ${}_AV_B + {}_BV_C = {}_AV_C$, we have ${}_AV_B = {}_AV_C - {}_BV_C$. That is

Velocity of *A* relative to *B* = velocity of *A* relative to *C* + REVERSED velocity of *B* relative to *C.*

Example 1

A boat is rowed with a velocity of 15 kmh^{-1} straight across a river that is flowing at 8 kmh^{-1}. Find the magnitude and direction of the resultant velocity of the boat. If the breadth of the river is 60m, find how far downstream the boat will reach the other bank.

Solution



Let θ be the angle of direction of the resultant velocity of the boat. The resultant is along AC. From the diagram,

the resultant $v = \sqrt{15^2 + 8^2} = \sqrt{289} = 17\ kmh^{-1}$.

Direction = θ = $Cos^{-1}(8/17)$. = 61.93° with the current of the river.

A is the starting point, B, directly opposite and C is the point down

stream where the boat reaches. Note △ ABC is similar to $\triangle AB^1C^1$,

Using similar triangles:

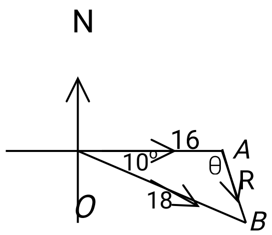
$\frac{15}{60} = \frac{8}{BC}$, Thus BC = 32 m

So the boat will reach 32 meters downstream at the other bank.

Example 2

A boat which would be traveling due east at $16kmh^{-1}$ if the water were still is actually moving at $18kmh^{-1}$ on a bearing of 100°. Find the speed and direction of the current.

Solution



Let the current of the river be *R*. From the diagram;

By *Cosine Rule:* $R^2 = 16^2 + 18^2 - 2(16)(18)Cos10^o$

$= 580 - 576Cos10^o = 12.75073426$

$\Rightarrow R = \sqrt{12.75073426} = 3.57081703 \cong 3.6$

Let angle *OAB* = θ. By Cosine Rule

$18^2 = 16^2 + (3.57081703)^2 - 2(16)(3.57081703)Cos\theta$

$\Rightarrow \theta = Cos^{-1}\left(\frac{18^2 - 16^2 - 3.57081703^2}{-2(16)(3.57081703)}\right) = 118.91515^o$

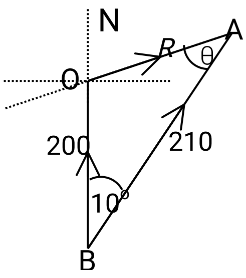
So the current is ($3.6kmh^{-1}$, 151°).

Example 3

An aircraft whose speed in still air is 200 knots is heading north. If its apparent speed is

210 knots on the bearing 010°, find the velocity of the wind.

Solution



Let the velocity of the wind be *R* (vector *OA*). Also let angle OAB = θ.

In △OAB, by *Cosine rule*

$R^2 = 200^2 + 210^2 - 2(200)(210)\text{Cos}10^\circ = 1376.148747$

Thus $R = 37.096 \cong 37$ knots.

Again by *Sine rule*

$$\frac{\sin\theta}{200} = \frac{\sin 10^\circ}{37.0964789} \Rightarrow \sin\theta = 0.9361976274$$

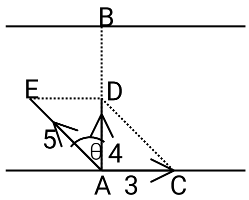
Hence θ = 69.42°. Therefore, angle BOA = 100.58°. Thus angle NOA = 79.42°.

The velocity of the wind is *R* = (37 knots, 079°).

Example 4

A river is flowing at 3 km^{-1} and its breadth is 150m. If a man can paddle a boat at 5 kmh^{-1}, find the direction in which he must paddle in order to go straight across the river, and the time it takes him to cross.

Solution



Let $\overrightarrow{AC}$ be the velocity of the current of the river, $\overrightarrow{AE}$, the velocity of the man and $\overrightarrow{AD}$ the resultant of these two velocities. Let θ be the angle of direction the man should paddle the boat in order to go straight across the river.

Note that ED = AC. In △ ADE; $5^2 = 3^2 + AD^2$.

Thus $AD^2 = 25 - 9 = 16$. Hence AD = 4.

Also $\theta = \tan^{-1}(3/4) = 36.87^\circ$.

So the man should paddle the boat 36.87° upstream from point A so that he will go

straight across the river. Now 4kmh^{-1} = $\frac{4000\ m}{3600\ s}$ = $\frac{10}{9}\,ms^{-1}$. If $\frac{10}{9}\,m \longrightarrow 1s$

Therefore $150m \longrightarrow \frac{150}{10} \times 9s$ = 135*s* (2 minutes, 15 seconds).

Newton's Laws of Motion

Newton's laws of motion may be stated as follows:

First Law: *A body continues in its state of rest or of uniform motion in a straight line, unless an external force acts on it.*

This law implies that if a body is moving with a uniform velocity, there is no external force acting on it; conversely, if there is an external force acting on the body, its velocity changes.

Second Law: *Change of momentum per unit time is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

Momentum : The *mass* of a body is the quantity of matter in the body, and this is often measured in *kilograms.* The *momentum* of a body is defined as the product of its *mass* and its *velocity.* The momentum *p* of a body of mass *m* moving with velocity vector *v* is given by

p = mv. This is a vector quantity and moves in the same direction as the velocity vector. Unit of measurement of momentum is $kg\text{-}ms^{-1}$. Now if *m* is constant [note that this is not always so, for a rocket burns fuel quickly and becomes lighter with time],

then $\frac{d}{dt}(mv) = m\frac{dv}{dt}$ = *ma* where *a* = $\frac{dv}{dt}$ is the acceleration of the body.

Hence the 2nd law is at times stated as *ma* = F.

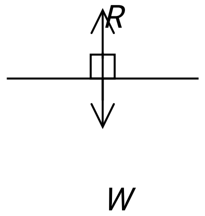
That is Force = *mass x acceleration.*

Third Law: *To every action, there is an equal and opposite reaction.*

Thus if a body *P* exerts a force on another body *Q*, then *Q* exerts an equal and opposite force on *P*. Examples of this phenomenon are *normal reaction .*

Normal Reaction

Consider a body at rest on a horizontal table (see diagram).

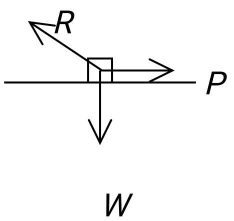


The weight *W* of the body acts vertically downwards on the table. Since the body is at rest, the table also exerts an equal and opposite force *R* on the body to balance *W.* Thus we have *W = R* and *R* acts vertically upwards. This is called the *Normal Reaction* of the table on the body.

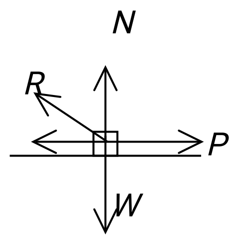
Friction

In many problems it is assumed that the surface on which the body is resting is *smooth* (i.e. there is no force between the surface and the body tending to prevent motion along the surface). Let the body in the diagram below be pulled by an *increasing* horizontal force *P.*

The forces acting on the body are as shown below.



(i)



(ii)

In (i) the forces acting on he body are the pull *P*; the weight *W* and the reaction *R.* The force *R* (the reaction of the table on the body) may be resolved into two forces *N* (normal to the plane) and *F* (along the plane). *N* is the *normal reaction* of the plane on the body and *F* is the *frictional force* that tends to prevent motion [see (ii)]. If *P* (the pull) is not large enough to move the body, the *P = F* and *N = W* since the body is in equilibrium. Suppose *P* is increased until the body is just about to move (i.e. *P* is just equal to maximum value of *F* – the limiting value of *F*), the body is said to be in limiting equilibrium. **Note** the following:

1. *Friction* is a passive force, i.e. up to a certain point [*F = P*].

2. Its direction is always opposite to the direction in which the body tends to move.

3. Only a certain amount of *friction* can be called into play. The maximum amount is

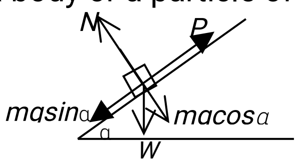
called the *limiting friction.*

4. The *limiting friction F* divided by the *normal reaction N* (i.e. *F/N*) is a constant

usually denoted by μ.

The ratio $\mu = F/N$, which depends on the nature of the surfaces in contact, is called *the coefficient of (limiting) friction.* Thus when the friction is limiting, we have $F = \mu N$.

If a body or a particle of mass m is at rest on a rough inclined plane of slope α



(see diagram), if the coefficient of *friction* is μ,

then the normal reaction $N = mg\cos\alpha$, the *frictional force* $F = mg\sin\alpha$ and the pull up the inclined plane $P = \mu N$ (since $F = \mu N = P$).

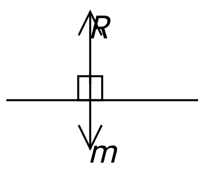
Again the frictional force $F = \mu N = mg\sin\alpha = \mu N$, and the normal reaction $N = mg\cos\alpha$.

Thus $F = \mu N \Rightarrow mg\sin\alpha = \mu mg\cos\alpha$ giving $\mu = \tan\alpha$. The angle α is usually denoted by λ and is called the *angle of friction.* [$\tan\lambda = \mu = \frac{F}{N}$.]

Example 5

A body of mass m *kg* on a horizontal plane is being pulled vertically upward with acceleration $a\ ms^{-2}$. Find the normal reaction of the plane on the body.

Solution



Let the reaction of the plane on the body be R (see diagram).

Since the body is moving vertically upwards, we have

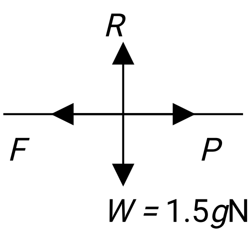
$R - mg = ma \Rightarrow R = ma + mg = m(a + g)$N

where g *is acceleration* due to gravity.

Example 6

A body of mass 1.5 *kg* is being pulled horizontally along a rough plane by a force of magnitude 9N. Find the coefficient of friction. If the horizontal force is increased to 12N so that the body begins to move, find its acceleration. [Take $g = 10ms^{-2}$].

Solution



Let the reaction of the plane on the particle be R, the pull P and the frictional force F. Now $P = 9$N, $R = 1.5g$N and $F = P$. But $F = \mu R \Rightarrow \mu = F/R = 9/1.5g = 9/15 = 3/5$.

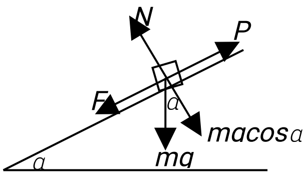
Let the acceleration be ***a***. If P now is 12N, then

$P - F = ma$ i.e. $12 - 9 = 1.5a \Rightarrow a = 3/1.5 = 2ms^{-2}$.

Example 7

A body of mass m *kg* slides down an inclined plane of inclination α. If μ is the coefficient of friction, find the acceleration that it begins to slide with. Evaluate this acceleration, correct to 2 significant figures, if the mass of the body is 10 *kg*, coefficient of friction is ¼ and the angle of inclination is 30°. [Take $g = 9.8\ ms^{-2}$]

Solution



Let N be the normal reaction of the plane on the particle, F, the frictional force and P the force up the inclined pane. Now $N = mgcos\alpha$, $F = mgsin\alpha$ and $F = \mu N = P$.

Since the particle slides down the plane, we have $F - P = ma$ where a is acceleration. That is:

$mgsin\alpha - \mu mgcos\alpha = ma \Rightarrow a = (mgsin\alpha - \mu mgcos\alpha)/m$

$= gsin\alpha - \mu gcos\alpha = g(sin\alpha - \mu cos\alpha)$.

Now if $g = 9.8$, $\mu = ¼$ and $\alpha = 30°$, then we have

$a = 9.8(\sin 30° - ¼\cos 30°) = 9.8\left(\frac{1}{2} - \left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}\right)\right) = 2.778 \cong 2.8\ ms^{-2}$.

UNIT 6: MOTION OF A PARTICLE UNDER UNIFORM ACCELERATION

Equations of Motion

If a body moves with a uniform acceleration ***a*** ms^{-2},

then $\frac{dv}{dt} = a \Rightarrow dv = adt$ and integrating gives $v = at + c$.

If the body starts with speed $u\ ms^{-1}$ (i.e. when $t = 0$, $v = u$) then $c = u$ and we obtain

$\underline{v = u + at}$ ………………….(1)

Note: If the body starts from rest then $u = 0$, giving $v = at$.

Now $v = ds/dt \Rightarrow ds/dt = u + at$ or $ds = (u + at)dt$ and integrating gives:

$s = ut + ½ at^2 + k$. But when $t = 0$ $s = 0$ giving $k = 0$. Thus

$\underline{s = ut + ½at^2}$ ………………..(2)

From (1) $t = \frac{v-u}{a}$, and substituting for t in (2) gives

$\underline{v^2 = u^2 + 2as}$…………………..(3)

Again from (1), $at = v - u$ and from (3) $v^2 - u^2 = 2as. \Rightarrow (v + u)(v - u) = 2as$ and substituting for $at = v - u$ we have $2as = at(v + u)$ i.e.

$\underline{s = ½ (v + u)t}$…………………(4)

These four relations are known as ***equations of motion*** under uniform acceleration. They are of fundamental importance and must be remembered. In working problems we select the relations that contain the quantities we are given and the one we want to find.

Example 1

A body starts with speed $5ms^{-1}$ and $5s$ later its speed becomes $35ms^{-1}$. Find its acceleration and the distance traveled within this time interval.

Solution

Note: $u = 5$, $v = 35$ and $t = 5$ and we want to find a and s (distance). Using equation (1)

$v = u + at$, we have $35 = 5 + 5a \Rightarrow a = 6$. Again using equation (4) $s = ½(v + u)t$ we

obtain $s = ½(35 + 5)5 = 100$.

Therefore the acceleration is $6ms^{-2}$ and the particle travels $100m$ within the time.

Example 2

A body of mass 12 kg increases its speed from 10.8 kmh^{-1} to 64.8 kmh^{-1} in $5s$. Find the force that was acting on the body.

Solution

We first change the speeds to ms^{-1}. $10.8\ kmh^{-1} = \frac{10.8(1000)}{60 \times 60} ms^{-1} = 3\ ms^{-1}$

and $64.8\ kmh^{-1} = \frac{64.8(1000)}{60 \times 60} ms^{-1} = 18 ms^{-1}$. Using the relation $v = u + at$

we have $18 = 3 + 5a \Rightarrow a = 3\ ms^{-2}$. But $F = ma$

Therefore the force is $F = 12(3) = 36N$

Example 3

If an express train reduces its speed from 96 kmh^{-1} to 24 kmh^{-1} in 0.8 km, for how long is the brakes applied and how much longer would it come to rest?

Solution

Assuming the retardation $a\ kmh^{-2}$ is uniform and using the relation $v^2 = u^2 + 2as$

we have $24^2 = 96^2 + 2(0.8)a \Rightarrow a = -5400\ kmh^{-2}$. Again using the relation $v = u + at$

we have: $24 = 96 - 5400t \Rightarrow t = \frac{1}{75} h = 48$ seconds.

To find the additional time $t^1 h$ taken to come to rest, we have:

$0 = 24 - 5400t^1 \Rightarrow t^1 = \frac{1}{225} h = 16$ seconds.

Example 4

A bullet of mass 0.5g is fired horizontally with speed 350 ms^{-1} into a plank of wood. The wood offers a constant resistance of 50 N. Find how deep the bullet will go.

Solution

We first change the mass from g to kg. i.e. $0.5g = \frac{0.5}{1000} kg = \frac{5}{10000} kg = \frac{1}{2000} kg$

Since the wood offers resistance, we have $F = ma = -50N$.

That is $\frac{a}{2000} = -50 \Rightarrow a = -100000\ ms^{-2}$. Using the relation $v^2 = u^2 + 2as$,

$$0 = 350^2 + 2(-100000)s \Rightarrow s = \frac{-122500}{-200000} = 0.6125\ m = 61.25\ cm.$$

Thus, the bullet can go a distance of 61.25 *cm* into the plank of wood before it will stop.

Vertical Motion Under Gravity

When a particle is projected vertically upwards and if air resistance is neglected, then acceleration of the particle is only affected by acceleration due to gravity g. Since g offers resistance to the motion, we have

$$a = -g; \qquad v = u - gt; \qquad h = ut - \tfrac{1}{2}gt^2; \text{ and } \qquad v^2 = u^2 - 2gh.$$

where g is acceleration due to gravity, v is final speed, u initial speed, t time h is the height attained.

Example 5

A body is projected vertically upwards at 24.5 ms^{-1}. Find:

(*a*) when it will attain maximum height? (*b*) maximum height attained,

(*c*) when it passes its starting point and with what speed?

(*d*) when will it be 29.4 *m* below its starting point? [Take $g = 9.8\ ms^{-2}$]

Solution

(*a*) Using $v = u - gt$ with $u = 24.5$, $g = 9.8$ and $v = 0$ since at the maximum height, $v = 0$. Thus we have $0 = 24.5 - 9.8t \Rightarrow t = 2.5$ seconds.

(*b*) Using the relation $h = ut - \frac{1}{2}gt^2$ with $t = 2.5$, we have

$$h = 24.5(2.5) - \tfrac{1}{2}(9.8)(2.5)^2 = 30.625\ m$$

(*c*) The body passes its starting point when $h = 0$.

That is $24.5t - \frac{1}{2}(9.8)t^2 = 0 \Rightarrow 245t - 49t^2 = 0 \Rightarrow t(245 - 49t) = 0$

given $t = 0$ or $t = 5$. $t = 0$ is the starting time, so the body passes its starting point

5 seconds after projection.

(*d*) When, $h = -29.4\ m$, we have $-29.4 = 24.5t - \frac{1}{2}(9.8)t^2 \Rightarrow 4.9t^2 - 24.5t - 29.4 = 0$

$\Rightarrow t^2 - 5t - 6 = 0 \Rightarrow (t - 6)(t + 1) = 0 \Rightarrow t = 6$ or $t = -1$.

Since time cannot be negative, the body will be 29.4 *m* below its starting point 6 seconds after projection.

Example 6

A stone is dropped from the top of a tower 39.2m high, and at the same instant another stone is projected vertically upwards from the bottom of the tower at 19.6ms^{-1}. Find when and where the stones pass each other. Is it true to say that the second stone could reach half way the height of the tower? [Take g = 9.8ms^{-2}]

Solution

Let h_1 be the height from the top of the tower that the first stone falls at time t. Let h_2 be the height attained by the second stone at time t. Using the relation $h = ut - ½gt^2$,

for the first stone, since it was dropped ,we have $h_1 = 4.9t^2$.

For the second stone, since it was projected from the bottom of the tower, we have

$h_2 = 19.6t - 4.9t^2$. Now since both stone start to move at the same time

$h_1 = h_2$ that is $4.9t^2 = 19.6t - 4.9t^2 \Rightarrow 9.8t^2 - 19.6t = 0 \Rightarrow t(9.8t - 19.6) = 0$

Thus $t = 0$ or $t = 2s$.

Now when $t = 2$, we have $h_1 = 4.9(4)m = 19.6m$.

For the second, it reaches its maximum height when its speed is zero.

That is, $19.6 - 9.8t = 0 \Rightarrow t = 2$.

When $t = 2$, $h_2 = 19.6(2) - 4.9(4) = 19.6m$ which is half of 39.2m.

So it true to say that the second stone could just reach half way the height of the tower.

LECTURE 7: Impulse, Momentum and Conservation of Linear Momentum

We have already been using the terms acceleration to depict the variation in velocity per unit time. In this unit, we wish to investigate the relationship existing between acceleration and forces. In doing so we shall be guided by the *laws* formatted by Isaac Newton (17th Centry)

Force*s*

A force is a vector quantity that produces or turns to produce a change in the existing state of a body.

Its effects can be felt for the body at rest or one that is in motion. While investigating the effect of a force on a body, Isaac Newton (1642 – 1727) instituted the laws of motion.

NEWTON'S FIRST LAW OF MOTION

Every body remains at rest or of uniform motion unless compelled by an external force to act otherwise.

This law is at times referred to as the law of *inertia* Closely related to force concept of mass.

MASS

In a loose sense *mass* means *matter* What then is matter? Matter is that we perceive by our senses – especially the sense of touch. We can perceive matter by noting any sudden change in its state of rest or state of motion. This property in which the state of matter experiences a change or is about to experience a change is called *inertia*. We can therefore define *mass* as *the measure a inertia in a body*. Mass is a scalar quantity We measure mass by the use of a common balance. The popular uiit used in measuring mass are the kilogram and gram.

Momentum of a particle

The linear momentum of a particle is the product of its mass and its velocity. If the mass of the particle is *m* and is moving with velocity *v*, then its linear momentum is *mv*. Momentum is a vector quantity

Unit of Momentum

The unit of momentum is Newton-second (Ns)

Newton's 2nd Law of Motion – Measure of Force

The rate of change of momentum is proportional to the applied force and takes place in the direction of the force.

This can be stated mathematically as follows:

$$
\begin{aligned}
p &= \frac{d}{dt}(mv) \\
&= m\frac{dv}{dt} \\
&= ma
\end{aligned}
$$

Where *a* is the acceleration of the particle.

The equation **F** = m***a*** is the definition of a force.

Unit of Force

The equation **F** = m***a*** holds if F is expressed in Newtons (N) while *m* and *a* are expressed in kg and ms^{-2} respectively.

Types of Forces

1. **Weight**

   The *weight* of a body is the amount of gravitational attraction on it. This should not be confused with *mass* of a body.

   For example if a body of mass *m* is lying on a horizontal plane (see Fig.1), its weight will be directed downwards

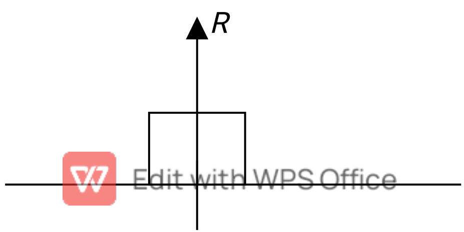


Fig. 1

Usually, the weight of a body is concentrated at the centre of gravity (*CG*) of the body.

Note the centre gravity varies from one object to another. For example for a regular block which has the shape of a cuboid, its *CG* is at the intersection of the diagonals as shown in Fig. 2.

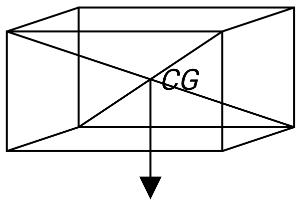


Fig. 2

For a triangular lamina, its *CG* is at the intersection of the medians (see Fig. 3): and so on.

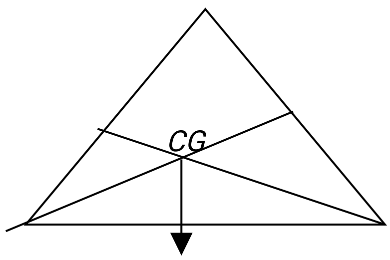


Fig. 3

If 20 kg mass is resting on a table, then it has a weight of 20 ×9.8N(or 20 ×10N) = 196 N(or 200N) depending on the value of *g*. Remember weight is a force.

Newton’s 3rd Law of Motion

Action and rection are equal and opposite.

2. Reaction

In Fig. 1, the block exerts a force *Wg* on the surface directly down, the surface also exerts a force on the block. This is shown as *R* the reaction. The block is in *equilibrium* (that is its not moving, it is completely at rest). The weight of the block must balance the reaction *R* on the block.

3. Tension

If an inelastic string is tied to a ceiling and the free end is tied to a load of weight *W* N. Then the string has a force on it. The string itself is light and has no weight of its own. The force which acts on the string is called ***tension*** (See Fig. 3)

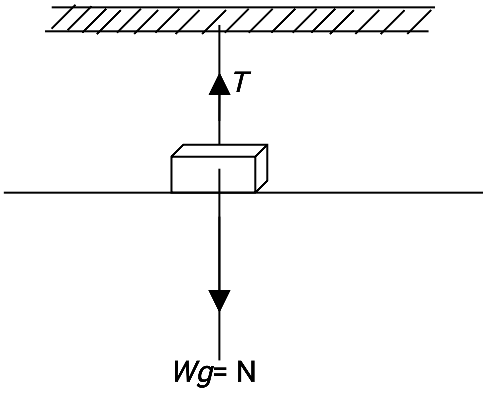


Fig. 3

4. Thrust

If a load is placed on a spring, the spring is depressed If more load is placed on top of the spring, the spring is depressed further. If the loads are removed , the spring shoots back upwards The restoration of the spring to its original position is called ***thrust***.

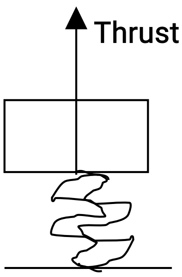
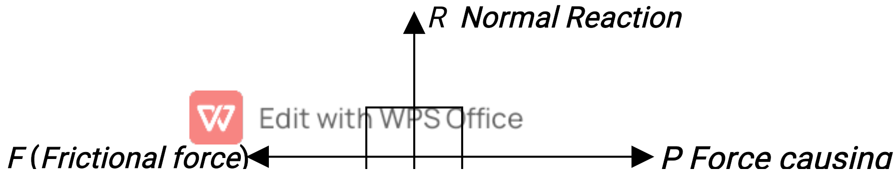


Fig. 4

Thrust is a force.

5. Friction

Another force is *friction.* Friction is a force which opposes motion. If a body is



about to move, there are forces that must operate

These forces are illustrated in Fig. 5. These are the weight ($W = mg\ N$) of the body, the normal reaction (R) , the force producing motion (P) and the frictional force (F) opposing motion.

Frictional force is proportional to the normal reaction and the constant of proportionality is called coefficient of friction.

That is **F = μ** R, where F is the frictional force, R the normal reaction and μ is dimensionless constant. From the above equation, we have $\mu = \frac{F}{R}$.

Worked Example 1

A man that weighs 60 kg stands on a lift which moves with acceleration of $8ms^{-2}$. Calculate

the reaction of the floor when,

(i) the lift is moving up

(ii) the lift is moving down

(Take $g = 10ms^{-2}$)

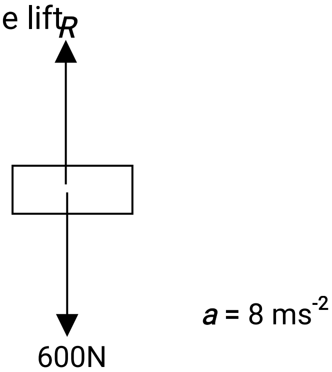
Solution

(i) Let $R\ N$ be the reaction of the floor of the lift

When the lift is moving up:

$R - 600 = 60 \times 8$

$R = 600 + 480$



$= 1080$ N

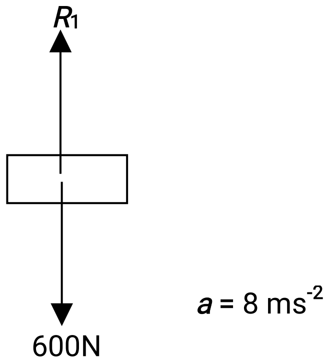
(ii) Let the reaction of the floor of the lift be R_1 when the lift is moving down.

When the lift is moving down, we have:

$600 - R_1 = 60 \times 8$

$\Rightarrow R_1 = 600 - 480$

$= 120$ N



Example 2

A body of mass 5 kg is acted upon by a force which changes the body from rest to have a velocity of 6 ms^{-1} for 5 seconds. Find :

(iii) The force that acted on the body;

(iv) Change in momentum of the body.

Solution

(i) Not that $t = 5$s, $m = 5$kg, $u = 0$, $v = 6$ms^{-1}

Using the equation v = u + at, we have

$6 = 0 + 5a$

Hence $a = 1.2$ ms^{-2}

Force $F = ma = 5 \times 1.2 = 6$ N

(ii) We saw in Unit 5 that *Impulse* which is the time effect of a force on the body is the same as the change in Momentum and its given by

$$Ft = mv - mu$$

Hence change in momentum = 6 × 5 = 30 N*s*

If we use Ft = 6 × 5 = 30 N*s*

Conservation of Linear Momentum

As explained earlier, the idea of impulse was introduced in Unit 5. Let us re call what we said.

We said that, From *Newton's* 2nd *Law of motion* momentum p of a body is defined as the product of the mass of the body and its velocity. Symbolically, $p = mv$. This is a vector quantity and has the same direction as the velocity vector, the unit being $kg\text{-}ms^{-1}$.

Again from *Newton's* 1st *Law of motion*, a force is required to change the velocity of a moving body. Thus, a force must act on a body to change its momentum.

When two bodies moving in the same straight line collide, the **collision** is said to be *direct*. A body of mass m, moving with velocity u and acted on by a force F changes its momentum.

The equation $Ft = mv - mu$ shows that the change in momentum is in the *direction* of the force F since, t is a scalar. Suppose body P_1 of mass m_1 moving with speed u_1 collides with anther body P_2 also of mass m_2 and moving with speed u_2. Suppose, after collision, P_1 and P_2 move in the same straight line with speeds v_1 and v_2 respectively. Then

Change in momentum of $P_1 = m_1v_1 - m_1u_1$.

Change in momentum of $P_2 = m_2v_2 - m_2u_2$.

During collision, body P_1 is acted upon by a force F_1 for a time t and body P_2 is also acted upon by a force F_2 for the same time t. If no other forces act on either of the two particles, the ***impulses*** are equal to changes in momentum. Symbolically,

$$tF_1 = m_1v_1 - m_1u_1 \quad \text{and} \quad tF_2 = m_2v_2 - m_2u_2.$$

Now by the 3rd *law of Newton*, $tF_1 = -tF_2$, or $tF_1 + tF_2 = 0$.

In terms of changes in momentum,

we have: $m_1v_1 - m_1u_1 = -(m_2v_2 - m_2u_2) \Rightarrow m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$.

That is: *Total momentum of the bodies before collision equals total momentum after collision.* This result illustrates the *Principle of Conservation of Linear Momentum.*

Now if the particles *coalesce* (they get stuck to each other) after impact, both particles will move with a common speed v. In this case, the *principle of conservation of linear momentum* becomes

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v.$$

Example 3

Two bodies of masses 8kg and 5kg move towards one another in the same straight line with speeds 20ms^{-1} and 10ms^{-1} respectively. If the bodies coalesce on impact, find their common speed after collision.

Solution

Let the direction of the mass of 8kg be positive. Let also the common speed be Vms^{-1}.

Total momentum before collision = (8)(20) + (5)(-10) = 110

Total momentum after collision = $(8 + 5)V = 13V$.

But by the principle of conservation of linear momentum, we have:

$$13V = 110 \Rightarrow V = 8.461538.$$

So the common speed after impact is 8.5ms^{-1}.

Example 4

A body E of mass 60kg moving with velocity 5ms^{-1} due West, collides with a body F of mass 50kg moving with velocity 16ms^{-1} due East. After collision E moves with velocity 4ms^{-1} due East. Find:

(a) the velocity of F immediately after collision and state its direction;

(b) the change in momentum of E and F respectively;

(c) the time it takes E to come to stop after the collision, if it moved with a constant retardation of 0.25ms^{-2}.

Solution

(a) Let the velocity of F immediately after collision be Vms^{-1}

Momentum of E before collision = (60)(−5) = −300$kgms^{-1}$ (due West)

Momentum of E after collision = (60)(4) = 240$kgms^{-1}$ (due East)

Momentum of F before collision = (50)(16) = 800$kgms^{-1}$ (due East)

Momentum of F after collision = 50$Vkgms^{-1}$.

By the principle of conservation of linear momentum, we have

$800 - 300 = 240 + 50V \Rightarrow V = 5.2ms^{-1}$

Thus, the velocity of F immediately after collision is $5.2ms^{-1}$ due East.

(*b*) Change in momentum of $E = (60)(4) - (60)(-5) = 240 + 300 = 540kgms^{-1}$.

Change in momentum of $F = (50)(5.2) - (50)(16) = 260 - 800 = -540kgms^{-1}$.

(*c*) For E its velocity is $4ms^{-1}$ due East after collision. To find the time taken for E to stop, we use the relation $v = u + at$ with $v = 0$, $u = 4$ and $a = -0.25$.

Thus $0 = 4 - 0.25t \Rightarrow t = 16$.

So it takes E 16 *seconds* to stop after collision.

UNIT 8: COLLISION OF PARTICLES (DIRECT IMPACT)

Direct Impact

We saw in *Newton's* 2nd *Law of motion* that momentum p of a body is defined as the product of the mass of the body and its velocity. Symbolically, $p = mv$. This is a vector quantity and has the same direction as the velocity vector, the unit being $kg\text{-}ms^{-1}$. Again from *Newton's* 1st *Law of motion*, a force is required to change the velocity of a moving body. Thus a force must act on a body to change its momentum.

Now $F = ma$……….(1) and $v = u + at$…………..(2).

From (1) $a = \frac{F}{m}$ and from (2) $at = v - u$. Thus $\left(\frac{F}{m}\right)t = v - u \Rightarrow Ft = mv - mu.$

The quantity Ft (the product of a force and time) is the *time effect* of the force on the body and it is called the *Impulse* of the force. Impulse of a force equals the *change in momentum* it produces.

Example 1

A body of mass $6kg$ is moving in a fixed direction with speed $4ms^{-1}$. The body is hit by an object that causes it to change its speed. Find the change in momentum if after the impact, the body

(i) moves with a speed of $6ms^{-1}$ in the same direction;

(ii) moves with speed of $6ms^{-1}$ in the opposite direction.

Solution

(i) Let the direction be eastward:

$\xrightarrow[u = 4]{}$ $\xrightarrow[v = 6]{}$ and $m = 6$

Momentum before impact = $6 \times 4 = 24\ kgms^{-1}$.

Momentum after impact = $6 \times 6 = 36\ kgms^{-1}$.

Thus change in momentum = $36 - 24 = \underline{12\ kgms^{-1}}$

(ii) $\xrightarrow[u = 4]{}$ $\xleftarrow[v = -6]{}$

Momentum before impact = $6 \times 4 = 24\ kgms^{-1}$.

Momentum after impact = $6 \times (-6) = -36\ kgms^{-1}$.

Thus change in momentum = $-36 - 24 = -60\ kgms^{-1}$.

Collision

When two bodies moving in the same straight line collide, the collision is said to be *direct*. A body of mass m, moving with velocity u and acted on by a force F changes its momentum.

The equation $Ft = mv - mu$ shows that the change in momentum is in the *direction* of the force F since t is a scalar. Suppose body P_1 of mass m_1 moving with speed u_1 collides with anther body P_2 also of mass m_2 and moving with speed u_2. Suppose after collision, P_1 and P_2 move in the same straight line with speeds v_1 and v_2 respectively. Then

Change in momentum of $P_1 = m_1 v_1 - m_1 u_1$.

Change in momentum of $P_2 = m_2 v_2 - m_2 u_2$.

During collision, body P_1 is acted upon by a force F_1 for a time t and body P_2 is also acted upon by a force F_2 for the same time t. If no other forces act on either of the two particles, the *impulses* are equal to changes in momentum. Symbolically,

$tF_1 = m_1 v_1 - m_1 u_1$ and $tF_2 = m_2 v_2 - m_2 u_2$.

Now by the 3rd *law of Newton*, $tF_1 = - tF_2$, or $tF_1 + tF_2 = 0$.

In terms of changes in momentum,

we have: $m_1 v_1 - m_1 u_1 = -(m_2 v_2 - m_2 u_2) \Rightarrow m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$.

That is: *Total momentum of the bodies before collision equals total momentum after collision.* This result illustrates the *Principle of Conservation of Linear Momentum.*

Now if the particles *coalesce* (they get stuck to each other) after impact, both particles will move with a common speed v. In this case, the *principle of conservation of linear momentum* becomes

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v.$$

Example 2

Two bodies of masses $8kg$ and $5kg$ move towards one another in the same straight line with speeds $20ms^{-1}$ and $10ms^{-1}$ respectively. If the bodies coalesce on impact, find their common speed after collision.

Solution

Let the direction of the mass of $8kg$ be positive. Let also the common speed be Vms^{-1}.

Total momentum before collision = (8)(20) + (5)(-10) = 110

Total momentum after collision = $(8 + 5)V = 13V$.

But by the principle of conservation of linear momentum, we have:

$13V = 110 \Rightarrow V = 8.461538$.

So the common speed after impact is $8.5ms^{-1}$.

Example 3

A body E of mass $60kg$ moving with velocity $5ms^{-1}$ due West, collides with a body F of mass $50kg$ moving with velocity $16ms^{-1}$ due East. After collision E moves with velocity $4ms^{-1}$ due East. Find:

(*a*) the velocity of F immediately after collision and state its direction;

(*b*) the change in momentum of E and F respectively;

(*c*) the time it takes E to come to stop after the collision, if it moved with a constant retardation of $0.25ms^{-2}$.

Solution

(*a*) Let the velocity of F immediately after collision be Vms^{-1}

Momentum of E before collision = $(60)(-5) = -300kgms^{-1}$ (due West)

Momentum of E after collision = $(60)(4) = 240kgms^{-1}$ (due East)

Momentum of F before collision = $(50)(16) = 800kgms^{-1}$ (due East)

Momentum of F after collision = $50Vkgms^{-1}$.

By the principle of conservation of linear momentum, we have

$800 - 300 = 240 + 50V \Rightarrow V = 5.2ms^{-1}$

Thus the velocity of F immediately after collision is $5.2ms^{-1}$ due East.

(*b*) Change in momentum of $E = (60)(4) - (60)(-5) = 240 + 300 = 540kgms^{-1}$.

Change in momentum of $F = (50)(5.2) - (50)(16) = 260 - 800 = -540 kgms^{-1}$.

(c)

For E its velocity is $4ms^{-1}$ due East after collision. To find the time taken for E to

stop, we use the relation $v = u + at$ with $v = 0$, $u = 4$ and $a = -0.25$.
Thus $0 = 4 - 0.25t \Rightarrow t = 16$.
So it takes E 16 *seconds* to stop after collision.

Example 4

A ball of mass 1*kg* strikes the cushion of a sofa perpendicularly at a speed of $5ms^{-1}$ and rebounds at $2ms^{-1}$. What is the change in momentum? If the total time of impact is 0.02*s* and the force of the cushion is constant, find the value of the force.

Solution

Let the direction of the ball towards the cushion be positive.

Momentum before impact = $(1)(5) = 5kg\text{-}ms^{-1}$.

Momentum after impact = $(1)(-2) = -2kg\text{-}ms^{-1}$

Therefore change in momentum = $-2 - 5 = -7kg\text{-}ms^{-1}$.

Now time effect of impact equals change in momentum. That is $tF = mv - mu$.

Hence $0.02F = -7 \Rightarrow F = -350$. Hence the value of the force is 350*N* in the opposite

direction to the direction of the strike of the ball on the cushion.

Example 5

A lead ball A of mass 2*kg* moving with speed $8ms^{-1}$ collides with another ball B of mass 1*kg* which is at rest. If ball A now moves with speed $6ms^{-1}$, find the speed with which ball B moves.

Solution

Let the velocity of B be V after collision with A.

By the principle of conservation of linear momentum, we have

$$2 \times 8 + 1 \times 0 = 2 \times 6 + 1 \times V \qquad \Rightarrow V = 16 - 12 = 4$$

Therefore B begins to move with speed $4ms^{-1}$.

Example 6

A bullet of mass 0.02*kg* traveling at $200ms^{-1}$ hits a stationary block of wood of mass 5*kg*, passes through it and emerges horizontally with speed $50ms^{-1}$. If the block is free to move on a smooth horizontal plane, find the speed with which it is moving after the bullet passes through it.

Solution

Let the speed of the block of wood be Vms^{-1}. By the principle of conservation of momentum, we have (0.02)(200) + (5)(0) = (0.02)(50) + 5V

$$\Rightarrow V = 0.6ms^{-1}.$$