NM-HW1-EricVidal

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2.9. Homework: Damped vibrating string
This document is dedicated to explaining the results demanded in **2.9.5 Homework implementation**.

1 Input data and discretization parameters

Most of the input data and discretization parameters are already set, and it is just a matter of initializing the variables. However, the t_{max} variable has to be derived previously. It is defined as the time needed to observe the initial amplitude decaying by a factor 10^{-2}. So, we start by reminding the notes Eq. 2.73 of the square modulus of the amplitude of a virtual mode,

$$|v_n(x,t)|^2 = |\phi_n(x)|^2 \, e^{-2\gamma t}, \tag{1}$$

where by applying the t_{max} definition, we find that the amplitude at t_{max} is,

$$|v_n(x,t_{max})| = |\phi_n(x)| \, e^{-\gamma t_{max}} = |\phi_n(x)| \, 10^{-2}, \tag{2}$$

then we can find the t_{max} expression,

$$t_{max} = \frac{2}{\gamma} \ln 10. \tag{3}$$

As we set that the gamma factor is $\gamma = 100$ Hz, then $t_{max} = 0.04605$ s.

2 Types of pinches

Once, the array of discretized values of x and the initial conditions are computed we can plot them. In Figures 1 and 2, the pinches of 2.44 and 2.45 are represented, respectively. The $f(x)$ is the initial position function, and $g(x)$ is the initial velocities function which is 0 along the whole string for both cases.

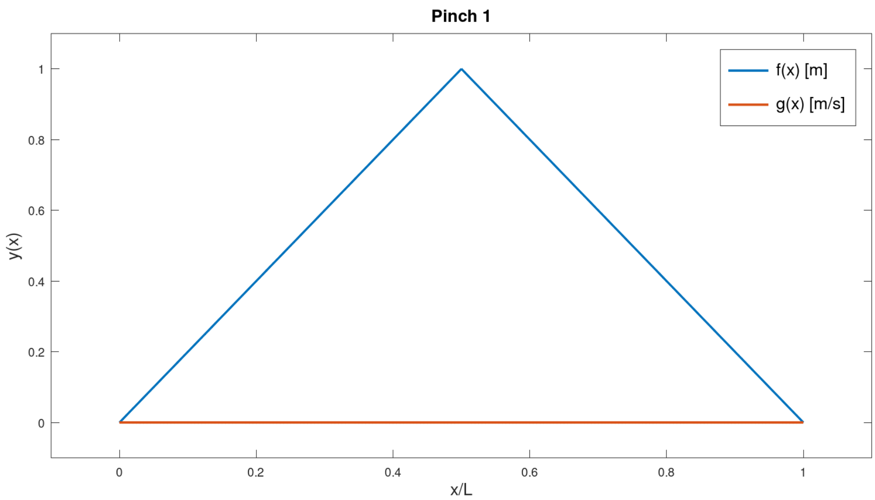

Fig. 1: This figure represents the initial position, $f(x)$, in meters and the initial velocities, $g(x)$, in meters per second, according to Eq. 2.44 of the notes. The abscissas are the discretized positions normalized with the string longitude, L, so they do not have dimensions. It is worth to note that it is arbitrarily pinched at $p = L/2$, however, it could have been $p \in (0, L)$.

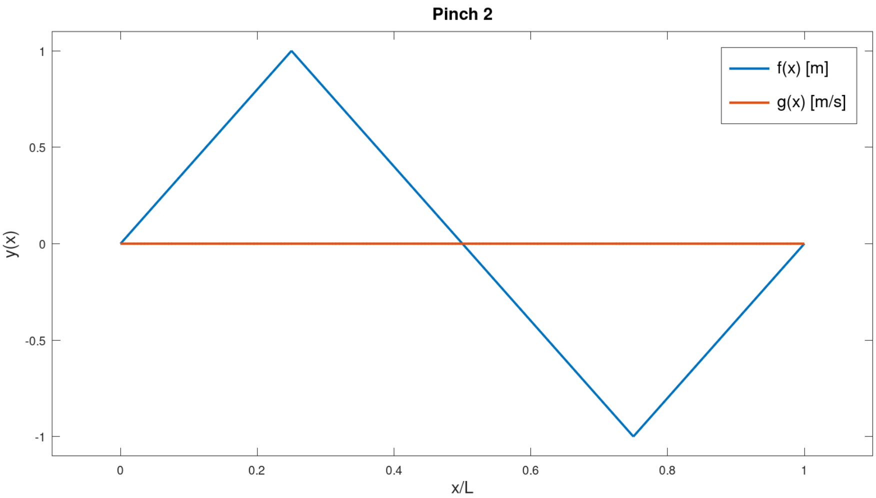

Fig. 2: This figure represents the initial position, $f(x)$, in meters and the initial velocities, $g(x)$, in meters per second, according to Eq. 2.45 of the notes. The abscissas are the discretized positions normalized with the string longitude, L, so they do not have dimensions. It is worth to note that it is arbitrarily pinched first at $p_1 = L/4$, and secondly at $p_2 = 3L/4$, however it could have been whatever two pinch values fulfilling $0 < p_1 < p_2 < L$.

3 Eigenfunctions

The spatial part of the solution remains the same as the ideal vibrating string, so we still have eigenfunctions, $\Phi(x)$. The eigenfunctions used for the development of the task are plotted at Fig. 3.

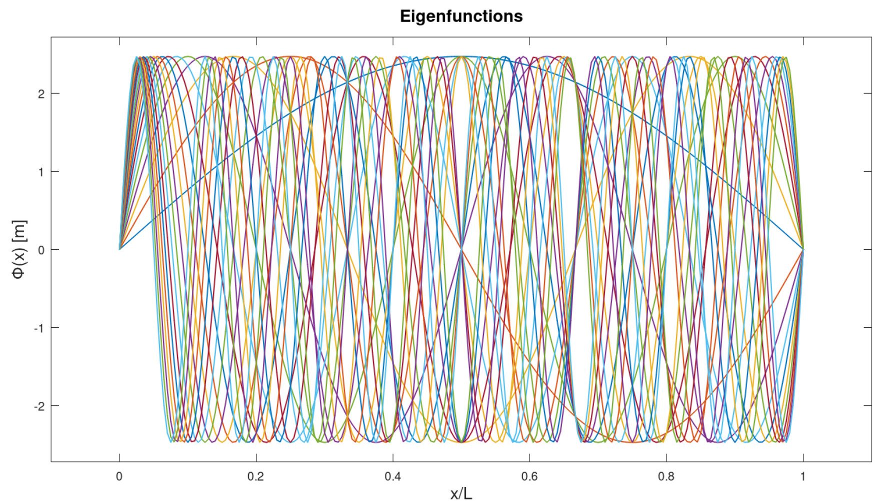

Fig. 3: This figure represents the eigenfunctions, $\phi(x)$, in meters, according to Eq. 2.75 of the notes. The abscissas are the discretized positions normalized with the string longitude, L, so they do not have dimensions. The number of modes chosen to characterize the string is 20, so there are 20 eigenfunctions plotted.

4 Parameters

In this section important parameters for characterizing the string are shown in Table 1.

Moreover, with the **Appendix** Table 4, we can compared the difference between the damped angular frequency $\Omega_n = \sqrt{\omega_n^2 - \gamma^2}$ and the angular frequency ω_n (with $gamma = 0$) for the different n modes. It is clear that both increase with n, and the difference between them decreases with n. When we study the role that plays $\gamma \in [10, \omega_1/2]$ Hz, we see that the greater it is, the more difference between them with

n	k (m^{-1})	Ω_n (Hz)	ν (Hz)	T_n (s)
1	9.57803	2763.13	439.766	0.00227394
2	19.1561	5528.97	879.963	0.00113641
3	28.7341	8294.21	1320.06	0.000757539
4	38.3121	11059.3	1760.14	0.000568136
5	47.8901	13824.3	2200.21	0.000454502
6	57.4682	16589.3	2640.27	0.000378749
7	67.0462	19354.3	3080.33	0.000324640
8	76.6242	22119.3	3520.39	0.000284059
9	86.2022	24884.2	3960.45	0.000252497
10	95.7803	27649.2	4400.51	0.000227247
11	105.358	30414.2	4840.56	0.000206588
12	114.936	33179.1	5280.62	0.000189372
13	124.514	35944.1	5720.67	0.000174805
14	134.092	38709.0	6160.73	0.000162318
15	143.670	41474.0	6600.78	0.000151497
16	153.248	44238.9	7040.84	0.000142029
17	162.826	47003.8	7480.89	0.000133674
18	172.404	49768.8	7920.95	0.000126248
19	181.983	52533.7	8361.00	0.000119603
20	191.561	55298.7	8801.06	0.000113623

Table 1: The table shows for each mode n, the quantized k values in inverse meters, the damped angular frequency, Ω_n, in Hz, the frequency, ν, in Hz, and the period, T_n, in seconds. It is worth to note that the frequencies are increasing with the normal modes n, so the k and period are decreasing with n.

the smaller ns is. Plus, the greater γ, the more difficult they converge. The γs considered for the study are 100 Hz, 400 Hz, 700 Hz and 1000 Hz. Just in case, it is clear that $\Omega_n < \omega_n$, but $\Omega_{n\to\infty} \to \omega_{n\to\infty}$.

5 Orthonormalisation

Using the trapezoidal rule for computing the integral in the scalar product of $\langle \Phi_n | f \rangle$, we can proof that the eigenmodes are orthonormal, and the scalar product of two same normal modes is 1, as we can see in Table 2.

n	$\langle \Phi_n \vert \Phi_n \rangle$
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	1

n	$\langle \Phi_n \vert \Phi_n \rangle$
11	1
12	1
13	1
14	1
15	1
16	1
17	1
18	1
19	1
20	1

Table 2: The overlap integrals between the same eigenmode, $\langle \Phi_n | \Phi_n \rangle$ for the 20 considered. For all of them is 1, so it proves that the eigenfunctions are well defined.

6 Overlap integrals

Making use of the orthonormal property we use the overlap integrals $\langle \Phi_n | f \rangle$ and $\langle \Phi_n | g \rangle$ for finding the solution of the string amplitude in position and time $\psi(x, t)$. For the second pinch we compute the overlap integrals between the initial conditions and the eigenfunctions shown in Table 3.

n	$\langle \Phi_n \| f \rangle$	$\langle \Phi_n \| g \rangle$
1	-9.57603E-18	0
2	0.328283	0
3	6.34881E-17	0
4	-2.92307E-17	0
5	3.83210E-17	0
6	-0.0364999	0
7	-8.66006E-18	0
8	2.16908E-17	0
9	4.99512E-17	0
10	0.0131573	0

n	$\langle \Phi_n \| f \rangle$	$\langle \Phi_n \| g \rangle$
11	3.82308E-17	0
12	4.08287E-17	0
13	7.49387E-17	0
14	-0.00672616	0
15	-1.39235E-17	0
16	5.01105E-17	0
17	-2.08099E-17	0
18	0.00407964	0
19	-7.58196E-17	0
20	-1.99646E-17	0

Table 3: The overlap integrals between the initial position and the eigenfunctions, $\langle \Phi_n | f \rangle$, and between the initial velocities and the eigenfunctions, $\langle \Phi_n | g \rangle$, are shown. It is important to see that the scalar product $\langle \Phi_n | g \rangle$ is 0 as we could expect because $g(x)$ is set to 0. Moreover, due to the symmetry and pinch position of $f(x)$, its scalar product with the odd n modes is practically 0, thus it is overlaping mainly with the $n = 2$ mode and a little bit with some other even n modes.

7 Initial conditions and its approximation

Now we can compare how good it is the approximation for the second pinch with the 20 selected eigenmodes in Fig. 4.

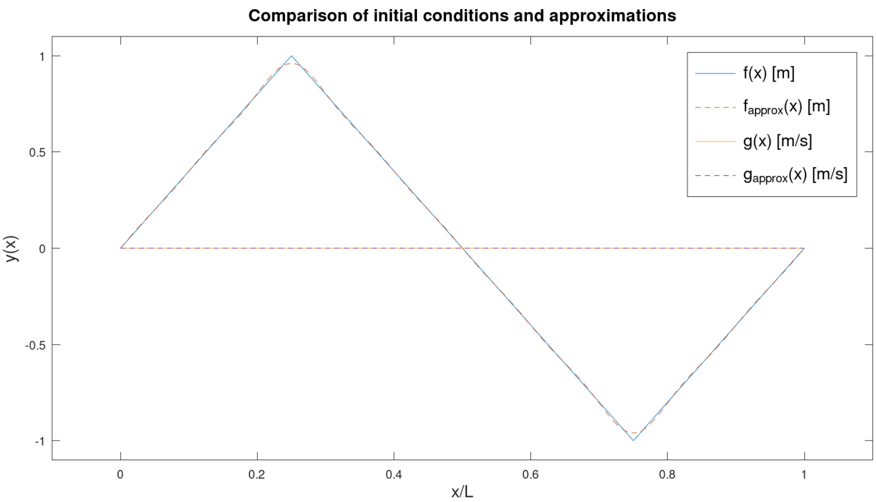

Fig. 4: In this figure, it is plotted the second pinch initial conditions for position, $f(x)$, in meters, the initial velocities, $g(x)$, in meters per second, and its approximations with the same units, $f_{approx}(x)$ and $g_{approx}(x)$, respectively. The abscissas are the discretized positions normalized with the string longitude, L, so they do not have dimensions. We can observe that the velocities are 0 for all cases, and in terms of the position the it can be seen that the approximation is good, so resembles very much the actual one, despite the pinches points where struggles a bit.

8 Wave evolution

Finally, we can plot amplitudes, $\psi(x, t)$, for different time moments in Fig. 5 and realize how the wave is decaying until the t_{max} defined in the Eq. (3).

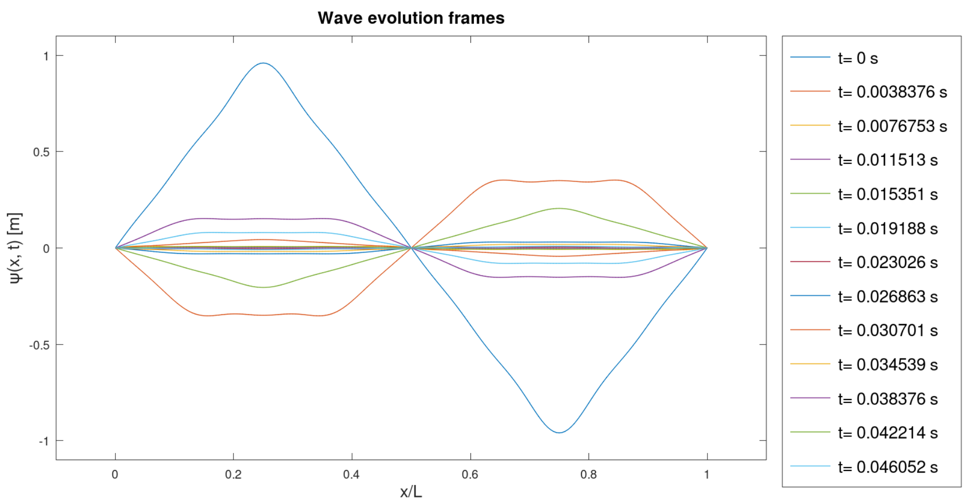

Fig. 5: This figure is a plot of the amplitude $\psi(x,t)$ for 13 equidistant times, $t \in [0, t_{max}]$, where $t_{max} = 0.046052$s. The abscissas are the discretized positions normalized with the string longitude, L, so they do not have dimensions. It is easy to see how the wave decays as expected

9 Video

The video cannot be displayed here, it is worth to note that the number of frames corresponds to,

$$n_{frames} = \frac{t_{max}}{\min T_n} \cdot 10, \tag{4}$$

and the time evolution represented is $t \in [0, t_{max}]$

If the number of frames is reduced, the resolution is worse, but the video is faster as less frames have to be displayed.

For technical reasons, instead of 20 eigenfunctions it is set to 10 (which means less frames and worse approximation) in order to have a more fluid video without having to save all the results previously, thus it can be run in all sort of computers. It can easily be changed if a good hardware is used to run the code.

Appendix

In the appendix we can find the extra tables computed to conduct and justify the study of the damped angular frequency and the angular frequency in section 4.

$\gamma = 100$ Hz		
n	ω_n	Ω_n
1	2764.94	2763.13
2	5529.88	5528.97
3	8294.81	8294.21
4	11059.8	11059.3
5	13824.7	13824.3
6	16589.6	16589.3
7	19354.6	19354.3
8	22119.5	22119.3
9	24884.4	24884.2
10	27649.4	27649.2
11	30414.3	30414.2
12	33179.3	33179.1
13	35944.2	35944.1
14	38709.1	38709.0
15	41474.1	41474.0
16	44239.0	44238.9
17	47003.9	47003.8
18	49768.9	49768.8
19	52533.8	52533.7
20	55298.8	55298.7

$\gamma = 400$ Hz		
n	ω_n	Ω_n
1	2764.94	2735.85
2	5529.88	5515.39
3	8294.81	8285.16
4	11059.8	11052.5
5	13824.7	13818.9
6	16589.6	16584.8
7	19354.6	19350.4
8	22119.5	22115.9
9	24884.4	24881.2
10	27649.4	27646.5
11	30414.3	30411.7
12	33179.3	33176.8
13	35944.2	35942.0
14	38709.1	38707.1
15	41474.1	41472.1
16	44239.0	44237.2
17	47003.9	47002.2
18	49768.9	49767.3
19	52533.8	52532.3
20	55298.8	55297.3

$\gamma = 700$ Hz		
n	ω_n	Ω_n
1	2764.94	2577.77
2	5529.88	5438.71
3	8294.81	8234.31
4	11059.8	11014.5
5	13824.7	13788.5
6	16589.6	16559.5
7	19354.6	19328.7
8	22119.5	22096.9
9	24884.4	24864.3
10	27649.4	27631.3
11	30414.3	30397.9
12	33179.3	33164.2
13	35944.2	35930.3
14	38709.1	38696.2
15	41474.1	41462.0
16	44239.0	44227.7
17	47003.9	46993.3
18	49768.9	49758.8
19	52533.8	52524.3
20	55298.8	55289.7

$\gamma = 1000$ Hz		
n	ω_n	Ω_n
1	2764.94	2674.86
2	5529.88	5485.39
3	8294.81	8265.22
4	11059.8	11037.6
5	13824.7	13807.0
6	16589.6	16574.9
7	19354.6	19341.9
8	22119.5	22108.4
9	24884.4	24874.6
10	27649.4	27640.5
11	30414.3	30406.3
12	33179.3	33171.9
13	35944.2	35937.4
14	38709.1	38702.8
15	41474.1	41468.2
16	44239.0	44233.5
17	47003.9	46998.7
18	49768.9	49764.0
19	52533.8	52529.2
20	55298.8	55294.3

Table 4: These tables display the angular frequency, ω_n, damped angular frequency Ω_n for the n modes ranging from 1 to 20, and *gamma*s 100 Hz, 400 Hz, 700 Hz and 1000 Hz.