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Q#1. (a) irreflexive, symmetric;

(b) reflexive, symmetric, transitive ;

(c) irreflexive, antisymmetric, transitive;

(d) symmetric, transitive;

(e) reflexive, antisymmetric, transitive;

Q#2. (a) False

R is reflexive $\Rightarrow \forall a \in A,\ aRa$

R is symmetric $\Rightarrow aRb \to bRa$

For arbitary elements s.t. $aRb \wedge bRc$

there is no reason for aRc.

Here provides a counterexample: $A=\{1,2,3\}$

$R=\{(1,1),(2,2),(3,3),(1,2),(2,3),(3,2),(2,1)\}$

Here $(1,2)\wedge(2,3)\to(1,3)$ is false.

Q#2. (b) True.

$\because R_1$ is reflective on set A

$\therefore$ Any arbitary element a in A, we have $(a,a)\in R_1$

and R_1 is a subset of $R_1 \cup R_2$

$\therefore (a,a)\in R_1\cup R_2$

$R_1\cup R_2$ is also on A since R_2 is on set A

$\therefore R_1\cup R_2$ is reflective.

Q2#. (c) It is False. Disprove: Here provides a counterexample.

Consider $R_1 = \{(1,2)\}$ on $A = \{1,2\}$

and $R_2 = \{(2,1)\}$ on $A = \{1,2\}$

R_1 and R_2 are both antisymmetric but

$R_1 \cup R_2 = \{(1,2),(2,1)\}$ is not antisymmetric

since $\begin{cases} (1,2) \in R_1 \cup R_2 \\ (2,1) \in R_1 \cup R_2 \end{cases} \nRightarrow 1 = 2$

Q3#. (a)

$$S_{C_1 \wedge C_2}(R) = \{a \in R \mid (C_1 \wedge C_2)(a)\}$$
$$= \{a \in R \mid C_1(a) \wedge C_2(a)\}$$

$$S_{C_1}(S_{C_2}(R)) = \{a \in S_{C_2}(R) \mid C_1(a) \wedge a \in R\}$$
$$= \{a \in R \wedge C_2(a) \mid C_1(a)\}$$
$$= \{a \in R \mid C_1(a) \wedge C_2(a)\}$$

$\therefore S_{C_1 \wedge C_2}(R) = S_{C_1}(S_{C_2}(R))$

Q3#. (b)

$P_{i_1, i_2 \cdots i_m}(R \cup S)$

$= \{(a_{i_1}, a_{i_2} \cdots a_{i_m}) \mid a = (a_1, a_2 \cdots a_n) \wedge a \in R \cup S\}$

$= \{(a_{i_1}, a_{i_2} \cdots a_{i_m}) \mid a = (a_1, a_2 \cdots a_n) \wedge (a \in R \vee a \in S)\}$

$= \{(a_{i_1}, a_{i_2} \cdots a_{i_m}) \mid [a = (a_1, a_2 \cdots a_n) \wedge a \in R] \vee$
$[a = (a_1, a_2 \cdots a_n) \wedge a \in S]\}$

$P_{i_1, i_2 \cdots i_m}(R) \cup P_{i_1, i_2 \cdots i_m}(S)$

$= \{(a_{i_1}, a_{i_2} \cdots a_{i_m}) \mid a = (a_1, a_2 \cdots a_n) \wedge a \in R\} \cup$
$\{(a_{i_1}, a_{i_2} \cdots a_{i_m}) \mid a = (a_1, a_2 \cdots a_n) \wedge a \in S\}$

$= \{(a_{i_1}, a_{i_2} \cdots a_{i_m}) \mid a = (a_1, a_2 \cdots a_n) \wedge (a \in R \vee a \in S)\}$

$\therefore P_{i_1, i_2 \cdots i_m}(R \cup S) = P_{i_1, i_2 \cdots i_m}(R) \cup P_{i_1, i_2 \cdots i_m}(S)$

Q4#.

(a) Basic step: Show R^2 is symmetric:

For an arbitary element $(a,b) \in R$, we know $(b,a) \in R$

Since R is symmetric.

$\therefore (a,a) \in R^2 \therefore R^2$ is non-empty.

Then for an arbitary element $(a,b) \in R^2$,

suppose there are (a,c) and (c,b) correspondingly in R.

Then $(c,a) \in R$ and $(b,c) \in R$

$\begin{cases} (b,c) \in R \\ (c,a) \in R \end{cases} \Rightarrow (b,a) \in R^2 \Rightarrow R^2$ is symmetric.

Inductive step: Suppose that R^n is symmetric.

Similarly, we know R^{n+1} is non-empty because R^n is symmetric.

For an arbitary element $(a,b) \in R^{n+1}$,

suppose there are $(a,c) \in R^n$ and $(c,b) \in R$

$$\left.\begin{array}{l} \begin{cases} (a,c) \in R^n \\ R^n \text{ is symmetric} \end{cases} \Rightarrow (c,a) \in R^n \\ \begin{cases} (c,b) \in R \\ R \text{ is symmetric} \end{cases} \Rightarrow (b,c) \in R \end{array}\right\} \Rightarrow (b,a) \in R^{n+1} \Rightarrow R^{n+1} \text{ is symmetric}$$

Thus. R^n is symmetric for any integers n, where $n \geq 1$ if R is symmetric.

(b) $R^* = \bigsqcup_{k=1}^{\infty} R^k$

For an arbitary element $(a,b) \in R^*$,

where $(a,b) \in R^i$, we know $(b,a) \in R^i$ since R^i is symmetric

$\therefore (b,a) \in R^*$

$\therefore R^*$ is symmetric.

Q5#. The symmetric closure of a relation R is:

$S = R \cup \{(b,a) \mid (a,b) \in R\}$

The transitive of S is:

$T_S = S \cup \{(a,c) \mid (a,b) \in S, (b,c) \in S\}$

$= R \cup \{(b,a) \mid (a,b) \in R\} \cup \{(a,c) \mid (a,b) \in S, (b,c) \in S\}$

The transitive closure of R is

$T = R \cup \{(a,c) \mid (a,b) \in R, (b,c) \in R\}$

The symmetric closure of T is

$S_T = T \cup \{(b,a) \mid (a,b) \in T\}$

$= R \cup \{(a,c) \mid (a,b) \in R, (b,c) \in R\} \cup \{(b,a) \mid (a,b) \in T\}$

Now, we need to prove that $S_T \subseteq T_S$.

For an arbitary element $(a_i, b_i) \in S_T$:

1°. $(a_i, b_i) \in R$

$(a_i, b_i) \in R \subseteq T_S \Rightarrow (a_i, b_i) \in T_S$

2°. $(a_i, b_i) \in \{(a,c) \mid (a,b) \in R, (b,c) \in R\}$

There must $\exists c_i$ s.t. $(a_i, c_i) \in R \wedge (c_i, b_i) \in R$

$\begin{cases} (a_i, c_i) \in R \subseteq S \\ (c_i, b_i) \in R \subseteq S \end{cases} \Rightarrow \begin{cases} (a_i, c_i) \in S \\ (c_i, b_i) \in S \end{cases}$

$\Rightarrow (a_i, b_i) \in \{(a,c) \mid (a,b) \in S, (b,c) \in S\}$

$\Rightarrow (a_i, b_i) \in T_S$ since $\{(a,c) \mid (a,b) \in S, (b,c) \in S\} \subseteq T_S$

3°. $(a_i, b_i) \in \{(b,a) \mid (a,b) \in T\}$

There must exist $(b_i, a_i) \in T$

Then there must exist c_i s.t. $(b_i, c_i) \in R \wedge (c_i, a_i) \in R$

$(b_i, c_i) \in R \Rightarrow (c_i, b_i) \in S$ since $\{(b,a) \mid (a,b) \in R\} \subseteq S$

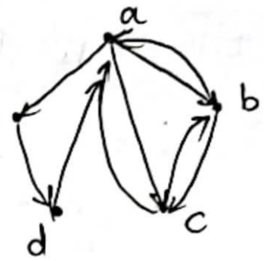
Similarly, $(c_i, a_i) \in R \Rightarrow (a_i, c_i) \in S$

$\begin{cases} (a_i, c_i) \in S \\ (c_i, b_i) \in S \end{cases}$ similarly we know $(a_i, b_i) \in T_S$

Thus, for $\forall (a_i, b_i) \in S_T$, $(a_i, b_i) \in T_S$. Then we know $S_T \subseteq T_S$

Q#6.

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Form the representative matrix above, where the rows are the first element on the set.

$$M_{R_0} = M_R$$

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R_3} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad M_{R_4} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{R_5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Q#7. (a) I. Reflexive

$$\left\{\begin{array}{l} x-x=0 \\ 0\in\mathbb{Z} \end{array}\right. \Rightarrow (x,x)\in R$$

Every real numbers satify this condition and R is on $\mathbb{R}$ } R is reflexive

II. Symmetric

For $\forall (a,b)\in R$. we know that $a-b\in\mathbb{Z}$, which means a and b have the same fractional part

$\therefore b-a\in\mathbb{Z}$ $\therefore (b,a)\in R$ $\therefore$ R is symmetric.

III. Transitive

For $\forall (a,b)\in R \wedge (b,c)\in R$, we know that a and b ha the same fraction and so do b and c.
Thus, a and c have the same fraction.
$\therefore a-c\in\mathbb{Z}$ $\therefore (a,c)\in R$ $\therefore$ R is transitive.

(b) $[1]_R=\{b\in\mathbb{R}: (1,b)\in R\}=\{b\in\mathbb{R} \mid 1-b\in\mathbb{Z}\}$
The fraction of 1 is zero, thus, the fraction of b is also 0.
That means b can be any integers.

$\therefore [1]_R=\mathbb{Z}$

Similarly: $[\frac{1}{2}]_R=\{\frac{1}{2}+k \mid k\in\mathbb{Z}\}$
$[\pi]_R=\{\pi+k \mid k\in\mathbb{Z}\}$

Q#8. (a) Reflexive: Obviously, for $\forall x \in \mathbb{R}$, we know $f(x) \leq f(x)$

Antisymmetric: for $\forall x \in \mathbb{R}$, if we know that:

$\begin{cases} f(x) \leq g(x) \\ g(x) \leq f(x) \end{cases}$, then $f(x) = g(x)$ is always true.

Thus, $\preceq$ is antisymmetric.

Transitive: for $\forall x \in \mathbb{R}$, if we know $f(x) \leq g(x)$ and $g(x) \leq h(x)$, then $f(x) \leq h(x)$

Thus, $\preceq$ is transitive.

Above all, f is dominated by f;

it is impossible that $f \preceq g$ and $g \preceq f$ and $f \neq g$.

if $f \preceq g$ and $g \preceq h$, then $f \preceq h$.

Thus, $\preceq$ is a partial ordering.

(b) The relation is not a total ordering.

There must exist two functions s.t. $\forall x \in (-\infty, C_0)$, $f(x) \leq g(x)$

and $\forall x \in (C_0, +\infty)$, $g(x) \leq f(x)$.

Here provides a counterexample:

$f(x) = x$, $g(x) = 1$. Then $C_0 = 1$.

Q#9.

(a) l, m

(b) a, b, c

(c) No

(d) No

(e) l, k, m

(f) k

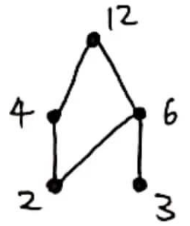
(g) a, d, b

(h) d

Q#10. Suppose the relation R describe this relation.

Then $R = \{(2,2), (2,4), (2,6), (2,12), (3,3), (3,6), (3,12), (4,12), (4,4), (6,6), (6,12), (12,12)\}$

Construct the Hasse diagram of R:



Find the compatible total ordering by topological sorting.

1°. $a_1 = 2$.

1.1°. $a_2 = 3$. then the orders are $\{2,3,4,6,12\}$ or $\{2,3,6,4,12\}$

1.2°. $a_2 = 4$. then the order is $\{2,4,3,6,12\}$

$a_5 = 12$

2°. $a_1 = 3$. then $a_2 = 2$

2.1°. $a_3 = 4$. then $a_4 = 6$.

2.2°. $a_3 = 6$. then $a_4 = 4$.

$a_5 = 12$.

Thus, these are all desired:

$\{2, 3, 4, 6, 12\}$

$\{2, 3, 6, 4, 12\}$

$\{2, 4, 3, 6, 12\}$

$\{3, 2, 4, 6, 12\}$

$\{3, 2, 6, 4, 12\}$