

CS201: Discrete Mathematics (Fall 2023)

Written Assignment #5 - Solutions

(100 points maximum but 110 points in total)

Deadline: 11:59pm on Dec 25 (please submit via Blackboard)

PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (15p) Let S be the set of all strings of English letters. Determine whether the following relations are *reflexive*, *irreflexive*, *symmetric*, *antisymmetric*, and/or *transitive*. No proof is required.

- (a) (3p) $R_1 = \{(a, b) | a \text{ and } b \text{ have no letters in common}\}$
- (b) (3p) $R_2 = \{(a, b) | a \text{ and } b \text{ are of the same length}\}$
- (c) (3p) $R_3 = \{(a, b) | a \text{ is shorter than } b\}$
- (d) (3p) $R_4 = \{(a, b) | a \text{ and } b \text{ have exactly one letter in common}\}$
- (e) (3p) $R_5 = \{(a, b) | a \text{ contains } b \text{ as a substring}\}$

Solution:

- (a) symmetric. (not irreflexive due to empty string)
- (b) reflexive, symmetric, transitive.
- (c) irreflexive, antisymmetric, transitive.
- (d) symmetric.
- (e) reflexive, antisymmetric, transitive.

□

Q.2 (15p) Consider relations on a set A . **Prove or disprove** the following statements:

- (a) (5p) If R is reflexive and symmetric, then R is transitive.
- (b) (5p) If R_1, R_2 are reflexive, then $R_1 \cup R_2$ is reflexive.
- (c) (5p) If R_1, R_2 are antisymmetric, then $R_1 \cup R_2$ is antisymmetric.

Solution:

- (a) False. Counterexample: Consider $A = \{1, 2, 3\}$ and

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}.$$

Then R is reflexive and symmetric, but not transitive, e.g., $(1, 2), (2, 3) \in R$ but $(1, 3) \notin R$.

- (b) True. By definition, for any $a \in A$, we have $(a, a) \in R_1$ and $(a, a) \in R_2$, so $(a, a) \in R_1 \cup R_2$.
- (c) False. Counterexample: Consider $A = \{1, 2\}$ and $R_1 = \{(1, 2)\}$, $R_2 = \{(2, 1)\}$. Then R_1, R_2 are antisymmetric, but it is obvious that $R_1 \cup R_2 = \{(1, 2), (2, 1)\}$ is not antisymmetric.

□

Q.3 (10p) Prove the following statements about n -ary relations:

- (a) (5p) If C_1 and C_2 are conditions that elements of the n -ary relation $R : A_1, \dots, A_n$ may satisfy, then $s_{C_1 \wedge C_2}(R) = s_{C_1}(s_{C_2}(R))$.
- (b) (5p) If R and S are n -ary relations, then $P_{i_1, i_2, \dots, i_m}(R \cup S) = P_{i_1, i_2, \dots, i_m}(R) \cup P_{i_1, i_2, \dots, i_m}(S)$.

Solution:

- (a) By definition, we have $s_{C_1 \wedge C_2}(R) = \{a \in R \mid (C_1 \wedge C_2)(a) = T\}$. Furthermore, we have $s_{C_1}(s_{C_2}(R)) = s_{C_2}(R) \cap \{a \in R \mid C_1(a) = T\} = \{a \in R \mid C_1(a) = T\} \cap \{a \in R \mid C_2(a) = T\} = \{a \in R \mid C_2(a) = T \wedge C_1(a) = T\} = \{a \in R \mid C_1(a) \wedge C_2(a) = T\} = s_{C_1 \wedge C_2}(R)$.
- (b) By definition of the projection operator, both sides of this equation pick out the m -tuples consisting of the i_1 -th, i_2 -th, \dots , i_m -th components of n -tuples in R or S .

□

Q.4 (10p) Suppose that a relation R on a set A is symmetric.

- (a) (7p) Prove that, for any positive integer $n \geq 1$, R^n is symmetric.
- (b) (3p) Prove that R^* is symmetric.

Solution:

- (a) Let $P(n)$ denote the statement “ R^n is symmetric”. We prove it by strong induction.

Basis step: It is known that $P(1)$ is true, i.e., R is symmetric. We show that $P(2)$ is also true. By definition, if $(a, b) \in R^2$, then there exists $c \in A$ such that $(a, c) \in R$ and $(c, b) \in R$. Since R is symmetric, we have $(c, a) \in R$ and $(b, c) \in R$. By the definition of relation composition, this implies $(b, a) \in R^2$ and hence $P(2)$ is true.

Inductive step: Assume that $P(j)$ is true for all $1 \leq j \leq k$ where $k \geq 2$, i.e., R^j is symmetric. We show that $R^{k+1} = R^k \circ R$ is also symmetric. If $(a, b) \in R^{k+1}$, by definition there exists $c \in A$ such that $(a, c) \in R$ and $(c, b) \in R^k$. By inductive hypothesis, we know $(c, a) \in R$ and $(b, c) \in R^k$. Furthermore, since $R^k = R^{k-1} \circ R$, there exists $d \in A$ such that $(b, d) \in R$ and $(d, c) \in R^{k-1}$. Therefore, we have $(d, a) \in R^k$. Note that $(b, d) \in R$, we get $(b, a) \in R^{k+1}$.

By strong induction, we have that R^n is symmetric for all $n \geq 1$.

- (b) Recall that $R^* = \bigcup_{n=1}^{\infty} R^n$. If $(a, b) \in R^*$, then there exists some positive integer n such that $(a, b) \in R^n$. In (a) we already proved that R^n is symmetric, so $(b, a) \in R^n$, and hence $(b, a) \in R^*$. By definition, R^* is symmetric.

□

Q.5 (5p) Prove that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Solution: Suppose that (a, b) is in the symmetric closure of the transitive closure of an arbitrary relation R . It is sufficient to show that (a, b) is also in the transitive closure of the symmetric closure of R . We know that at least one of (a, b) and (b, a) is in the transitive closure of R . Hence, there is either a path from a to b in R or a path from b to a in R (or both). In the former case,

there is a path from a to b in the symmetric closure of R . In the latter case, we can form a path from a to b in the symmetric closure of R by reversing the directions of all the edges in a path from b to a , going backward. Therefore, (a, b) is in the transitive closure of the symmetric closure of R . □

Q.6 (10p) Use the Floyd-Warshall algorithm to find the transitive closures of the relation $R = \{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$ on set $\{a, b, c, d, e\}$.

Solution: Since R is defined on a set consisting of 5 elements, we only need to compute W_5 .

$$\begin{aligned} \mathbf{W}_0 &= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} & \mathbf{W}_1 &= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} & \mathbf{W}_2 &= \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \mathbf{W}_3 &= \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} & \mathbf{W}_4 &= \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} & \mathbf{W}_5 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

□

Q.7 (10p) Consider the relation $R = \{(x, y) \mid x - y \in \mathbf{Z}\}$.

- (a) **(7p)** Prove that R is an equivalence relation on the set of real numbers \mathbf{R} .
- (b) **(3p)** Describe what elements the following equivalence classes consist of: $[1]$, $[\frac{1}{2}]$, and $[\pi]$.

Solution:

- (a) We prove that the given relation R is an equivalence relation:

- *Reflexive:* For all $a \in \mathbf{Z}$, $a - a \in \mathbf{Z}$, so $(a, a) \in R$.
- *Symmetric:* If $(a, b) \in R$, then by definition $a - b \in \mathbf{Z}$. It is clear that $b - a = -(a - b) \in \mathbf{Z}$ and hence $(b, a) \in R$.
- *Transitive:* If $(a, b) \in R$ and $(b, c) \in R$, then we have $a - b \in \mathbf{Z}$ and $b - c \in \mathbf{Z}$. Therefore, we have $a - c = a - b + b - c \in \mathbf{Z}$ and hence $(a, c) \in R$.

- (b) By definition, we have $[1] = \mathbf{Z}$, $[\frac{1}{2}] = \{\frac{1}{2} + n \mid n \in \mathbf{Z}\}$, and $[\pi] = \{\pi + n \mid n \in \mathbf{Z}\}$. □

Q.8 (10p) For any functions $f : \mathbf{R} \rightarrow \mathbf{R}$ and $g : \mathbf{R} \rightarrow \mathbf{R}$. We say f is *dominated* by g , denoted by $f \preceq g$, if and only if $\forall x \in \mathbf{R}, f(x) \leq g(x)$ holds. Prove or disprove the following:

- (a) **(7p)** The relation \preceq is a partial ordering.
- (b) **(3p)** The relation \preceq is a total ordering.

Solution:

(a) True. We can prove it as follows:

Reflexive: For all $x \in \mathbf{R}$, $f(x) \leq f(x)$, so $f \preceq f$.

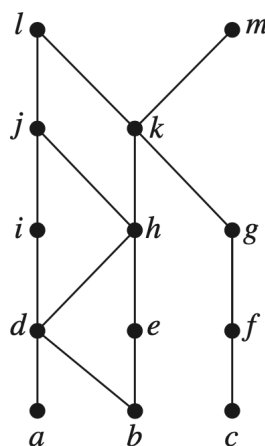
Antisymmetric: If $f \preceq g$ and $g \preceq f$, then for all $x \in \mathbf{R}$ we have $f(x) \leq g(x) \leq f(x)$ and hence $f(x) = g(x)$, i.e., $f = g$.

Transitive: If $f \preceq g$ and $g \preceq h$, then for all $x \in \mathbf{R}$ we have $f(x) \leq g(x) \leq h(x)$ and hence $f(x) \leq h(x)$, i.e., $f \preceq h$.

(b) False. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\leq -1 = g(1)$ and $g(-1) = 1 \not\leq -1 = f(-1)$. So it is not the case that for all x , $f(x) \leq g(x)$, and it is not the case that for all x , $g(x) \leq f(x)$. That is, these two functions are not comparable.

□

Q.9 (20p) Answer questions about the partial order represented by the Hasse diagram below:



(a) (3p) Find the maximal elements.

(b) (3p) Find the minimal elements.

(c) (2p) Is there a greatest element?

(d) (2p) Is there a least element?

(e) (3p) Find all upper bounds of $\{a, e, f\}$.

(f) (2p) Find the least upper bound of $\{a, e, f\}$, if it exists.

(g) (3p) Find all lower bounds of $\{h, i, j\}$.

(h) (2p) Find the greatest lower bound of $\{h, i, j\}$, if it exists.

Solution:

(a) l, m

(b) a, b, c

(c) No

(d) No

(e) k, l, m

(f) k

(g) d, a, b

(h) d

□

Q.10 (5p) Topological sorting. Find **all** compatible total orderings for the poset $(\{2, 3, 4, 6, 12\}, |)$.

Solution: It is not hard to draw a Hasse diagram for the given poset (omitted here). Then, one can follow the topological algorithm learned in class and list all the compatible total orderings as follows: $\{2, 3, 4, 6, 12\}$, $\{2, 3, 6, 4, 12\}$, $\{2, 4, 3, 6, 12\}$, $\{3, 2, 4, 6, 12\}$, $\{3, 2, 6, 4, 12\}$.

□