CS201: Discrete Mathematics (Fall 2023)
Written Assignment #6 - Solutions
(100 points maximum but 110 points in total)
Deadline: 11:59pm on Jan 7 (please submit via Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**10p**) Let G be a *simple* graph with n vertices.

(a) (**3p**) What is the *maximum* and *minimum* numbers of edges G can have? Explain.

(b) (**3p**) What is the *maximum* and *minimum* degrees each vertex in G can have? Explain.

(c) (**4p**) Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.

Solution:

(a) Since G is simple, it can have at most one edges between each pair of distinct vertices, hence the maximum number of edges is $\binom{n}{2} = n(n-1)/2$. The minimum number of edges G can have is clearly 0, i.e., G has no edge.

(b) The maximum degree is $n-1$ because each vertex can be adjacent to at most all the other $n-1$ vertices, and the minimum degree is 0, in which case all vertices are isolated.

(c) We prove this by contradiction. Suppose that the minimum degree is $(n-1)/2$ and G is not connected. Then G has at least two connected components. In each of these components, the minimum vertex degree is still $(n-1)/2$, and this means that each connected component must have at least $(n-1)/2+1$ vertices. Since there are at least two components, this means that the graph has at least $2(\frac{n-1}{2}+1) = n+1$ vertices, which is a contradiction.

□

Q.2 (**8p**) The *complementary graph* $\overline{G}$ of a simple graph G is a simple graph that has the same vertices as G, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of the following graphs: $\overline{K_n}$, $\overline{K_{m,n}}$, $\overline{C_n}$, $\overline{Q_n}$.

Solution: $\overline{K_n}$ is the graph with n vertices and no edges. $\overline{K_{m,n}}$ is the disjoint union of K_m and K_n. $\overline{C_n}$ is the graph with n vertices $\{v_1, \dots, v_n\}$ and an edge between any two distinct vertices v_i and v_j with $i \neq j \pm 1 \pmod{n}$. $\overline{Q_n}$ is the graph whose vertices are represented by bit strings of length n and with an edge between two distinct vertices if the associated bit strings differ in *more than one* bit.

□

Q.3 (**8p**) Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

(a) (**3p**) Use a bipartite graph to model the four employees and their qualifications.

(b) (**5p**) Use Hall's marriage theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support. If such an assignment exists, find one.

Solution:

(a) Let $V_1 = \{P, Q, R, S\}$ denote the set of four employees and $V_2 = \{h, s, n, w\}$ denote the set of four areas. It is easy to draw a bipartite graph G with bipartition (V_1, V_2) to model the four employees and their qualifications. The graph is omitted here.

(b) There is a complete matching from V_1 to V_2 in G. By Hall's marriage theorem, we only need to show that $|N(A)| \geq |A|$ holds for any $A \subseteq V_1$. Since each vertex in V_1 has degree ≥ 2, $|N(A)| \geq |A|$ holds if $|A| \leq 2$. It is also easy to see that $|N(A)| = |A|$ when $|A| = 4$. We are only left to show that $|N(A)| \geq |A|$ when $|A| = 3$. Since $\deg(P) = 3$, any A consisting of P satisfies $|N(A)| \geq |A|$. We only need to check the case where $A = \{S, Q, R\}$, in which case $|N(A)| = 4 \geq 3 = |A|$. This concludes the proof.

There are several complete matchings in G. One of them is $\{\{P, h\}, \{Q, n\}, \{R, w\}, \{S, s\}\}$.

□

Q.4 (**5p**) A simple graph G is called *self-complementary* if G and $\overline{G}$ are isomorphic. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

Solution: If G is self-complementary, then the number of edges of G must equal to the number of edges of $\overline{G}$. Since the union of G and $\overline{G}$ is K_v, the sum of their edges is $v(v-1)/2$. Therefore, the number of edges in G and $\overline{G}$ must both be $v(v-1)/4$. In order for this number to be an integer, it is easy to check that $v \equiv 0$ or $1 \pmod 4$ and $v \not\equiv 2$ or $3 \pmod 4$.

□

Q.5 (**9p**) Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimum* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(a) (**3p**) $dist(u, v) \geq 0$ and $dist(u, v) = 0$ if and only if $u = v$.

(b) (**3p**) $dist(u, v) = dist(v, u)$.

(c) (**3p**) $dist(u, v) \leq dist(u, w) + dist(w, v)$.

Solution:

(a) By definition, the $dist(u, v)$ is the minimum length of a path from u to v, and the length is the number of edges in the path. Thus, $dist(u, v)$ cannot be negative. Furthermore, $dist(u, v) = 0$ if and only if there is a path of length 0 from u to v, which means that $u = v$.

(b) Suppose that P is a path from u to v of the minimum length. We reverse all the edges in the path P, and will get a path P' from v to u. Note that P' must be the minimum path from v to u. Otherwise, there exists another path P'' with shorter path from v to u and we can reverse P'' to get a path from u to v shorter than P, which is a contradiction. Thus, $dist(u, v) = dist(v, u)$.

(c) By definition, $dist(u,v)$ equals the number of edges in a path P from u to v of minimum length. Suppose that P_1 and P_2 are paths of minimum length from u to w and from w to v, respectively. Then connecting P_1 and P_2 yields a path P' from u to v. By the minimality of the length of P, we have $dist(u,v) \leq dist(u,w) + dist(w,v)$.

□

Q.6 (**10p**) In an n-player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner — there are no ties. The results of such a tournament can then be represented with a *tournament directed graph* where the vertices correspond to players and there is an edge $x \to y$ if and only if x beats y in their game.

(a) (**3p**) Explain why a tournament directed graph cannot have cycles of length 1 or 2.

(b) (**3p**) Is the "beats" relation for a tournament graph always/sometimes/never: antisymmetric? reflexive? irreflexive? transitive?

(c) (**4p**) A relation is called a *strict total ordering* if it is irreflexive, antisymmetric, transitive, and every two distinct elements in the set are comparable. Show that a tournament directed graph represents a strict total ordering if and only if there are no cycles of length 3.

Solution:

(a) Since no player plays with himself, there are no loops in a tournament graph, i.e., no length-1 cycles. Also, because every pair of players competes exactly once without ties, it cannot be the case that x beats y and y beats x for $x \neq y$, i.e., there are no length-2 cycles.

(b) No loops implies that the "beats" relation is always irreflexive and never reflexive. It is also always antisymmetric since for every pair of distinct players, exactly one game is played and exactly one of the players wins. Finally, it is not hard to see that some tournament directed graphs represent transitive relations and some do not.

(c) As observed in (b), the tournament directed graph is antisymmetric and irreflexive. Since every pair of distinct players is comparable in this graph, it represents a strict total ordering if and only if the relation is transitive. By definition, a relation is transitive if and only if for any distinct players x, y and z, $x \to y$ and $y \to z$ imply that $x \to z$. Therefore, if a relation is transitive, then there are no cycles of length 3. The other direction is also true. Suppose there are no length-3 cycles and the relation is not transitive, then there must exist three distinct players x, y and z such that $x \to y$ and $y \to z$ hold but $x \to z$ does not hold. Then, $z \to x$ must hold because x, z must complete and have one winner. However, this results in a length-3 cycle among x, y, z, a contradiction. Therefore, no length-3 cycles implies that the relation is transitive. The proof is concluded.

□

Q.7 (**5p**) Let G be a connected simple graph. Show that if an edge in a connected graph is not contained in any simple circuit, then this edge is a *cut edge*.

Solution: We prove it by contraposition. Suppose that the considered edge $\{u,v\}$ is not a cut edge, then we show that $\{u,v\}$ must be contained in a simple circuit. Since the edge is not a cut-edge, the graph obtained by removing the edge is connected. So there exists a path from u to v which does not traverse the edge $\{u,v\}$. We proved in class that if there exists a path from u to v, then there exists a simple path from u to v. Then, this simple path together with the edge $\{u,v\}$ forms a simple circuit that contains $\{u,v\}$. □

Q.8 (**5p**) Given a graph G, its *line graph* $L(G)$ is defined as follows: every edge of G corresponds to a unique vertex of $L(G)$; any two vertices of $L(G)$ are adjacent if and only if their corresponding distinct edges of G share a common endpoint. Prove that if a connected simple graph G is regular (i.e., all vertices have the same degree), then $L(G)$ has an Euler circuit.

Solution: If the vertex degree of a regular graph G is d, then each endpoint of any edge in G has $d-1$ edges incident to it. That is, every vertex in $L(G)$ has degree equal to $2(d-1)$. Since all vertices have even degree, $L(G)$ has an Euler circuit. □

Q.9 (**5p**) Prove that every n-cube Q_n $(n > 1)$ has a Hamilton circuit. (Hint: Q_{n+1} can be constructed from two copies of Q_n.)

Solution: Note: I changed this problem description, so do not deduct points if the students proved the basis step starting from $n = 1$ (but this is wrong).

We prove it by induction. Recall that vertices in Q_n represent n-bit strings and there is an edge between two vertices if their strings differ in exactly one bit.

Basis step: If $n = 2$, $Q_2 = C_4$, which itself is a Hamilton circuit.

Inductive step: Assume Q_n has a Hamilton circuit, we can construct a Hamilton circuit for Q_{n+1} as follows. First, note that Q_{n+1} can be constructed from two copies of Q_n by adding an edge between all 2^n pairs of vertices that represent the *same* n-bit string in these two copies. This is because we can get all vertices of Q_{n+1} by adding an extra leading 0 to all n-bit strings in the first copy of Q_n and adding an extra leading 1 to all n-bit strings in the second copy of Q_n, then all edges in the two copies of Q_n remain unchanged and we need 2^n extra edges between the same n-bit strings of the two copies. Now, take a Hamilton circuit H in one Q_n and reverse it on the other Q_n denoted by Q'_n to get H'; then, delete an arbitrary edge $\{u, v\}$ from H and delete the corresponding edge $\{u', v'\}$ from H'; finally, add edges $\{u, u'\}$ and $\{v, v'\}$. This is a Hamilton circuit in Q_{n+1}. □

Q.10 (**5p**) Show that if G is a simple graph with at least 11 vertices, then G or $\overline{G}$ is nonplanar.

Solution: Suppose G has v vertices and e edges. By definition, $\overline{G}$ also has v vertices, and if $\overline{G}$ has e' edges then $e + e'$ is equal to the number of edges in the complete graph K_v. Since there are $v(v-1)/2$ edges in K_v, one of G or $\overline{G}$ must have no more than $v(v-1)/4$ edges. By Corollary 1 of Euler's formula, we only need to show $v(v-1)/4 > 3v - 6$ holds for $v \geq 11$. The inequality is equivalent to $v^2 - 13v + 24 > 0$, i.e., $(v - 13/2)^2 - 73/4 > 0$. It is easy to check that this inequality indeed holds for $v \geq 11$. □

Q.11 (**5p**) Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - 10/3$ if $v \geq 4$.

Solution: Each region has degree no less than 5, because $v \geq 4$ and there exists no simple circuits of length ≤ 4. Then, since all region degrees sum to $2e$, we have $2e \geq 5r$. Plug this into Euler's formula $r = e - v + 2$. We have $2e/5 \geq e - v + 2$, which implies that $e \leq (5/3)v - 10/3$. □

Q.12 (**10p**) There are 17 students who communicate with each other discussing problems in discrete mathematics. There are only 3 possible problems, and each pair of students discuss one of these 3 problems. Prove that there are at least 3 students who all pairwise discuss the same problem.

Solution: We use vertices $A, B, C, \ldots$ to denote the 17 students and edges to denote the communications among these students. In addition, we use 3 different colors (e.g., red, yellow, blue) to color the edges to denote the 3 problems they discuss. Then, the original problem becomes finding a triangle with the same edge color from K_{17}.

For any fixed student A, he or she communicates with the other 16 students. By the pigeonhole principle, at least 6 edges are of the same color. Without loss of generality, we assume that the edges AB, AC, AD, AE, AF, AG are all of color red. There are two cases.

If among the six students B, C, D, E, F, G there is one edge (e.g., BC) whose color is also red, then we find a red-colored triangle (e.g., ABC).

If among the six students B, C, D, E, F, G there is no red edge, we consider the edges BC, BD, BE, BF, BG. There are only two colors for these 5 edges, so at least 3 of them share the same color. Without loss of generality, assume BC, BD, BE are of the same color, yellow. We consider the triangle CDE. If it has at least one yellow edge (e.g., CD), then we find a yellow-colored triangle (e.g., BCD). If CDE does not have yellow edge, then all edges of CDE must be blue, and we find a blue-colored triangle. □

Q.13 (**8p**) The *rooted Fibonacci trees* T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n \geq 3$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree. Answer the following questions: (you can write your answers in terms of the Fibonacci numbers f_n)

(a) (**6p**) How many vertices, leaves, and internal vertices does T_n ($n \geq 1$) have? Explain.

(b) (**2p**) What is the height of T_n? Explain.

Solution:

(a) The number of vertices in the tree T_n satisfies the recurrence relation $v_n = v_{n-1} + v_{n-2} + 1$ (the "+1" is for the root), with $v_1 = v_2 = 1$. Thus the sequence begins $1, 1, 3, 5, 9, 15, 25, \ldots$. One can prove by induction (omitted here) that $v_n = 2f_n - 1$. The number of leaves satisfies the recurrence relation $l_n = l_{n-1} + l_{n-2}$, with $l_1 = l_2 = 1$, so $l_n = f_n$. Since the number of internal vertices i_n satisfies $i_n = v_n - l_n$, we have $i_n = f_n - 1$.

(b) It is clear that the height of the tree T_n ($n \geq 3$) equals the height of the tree T_{n-1} plus 1, and the heights of T_1 and T_2 are both 0. Therefore, the height of T_n is $n - 2$ for all $n \geq 2$, and the height of T_1 is 0.

□

Q.14 (**5p**) Calculate the value of the postfix expression "$3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -$" and show your computation steps.

Solution:

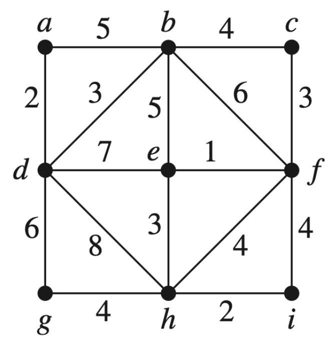
$$
\begin{aligned}
&3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ - \\
&= (3\ 2\ *)\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ - \\
&= (6\ 2\uparrow)\ 5\ 3\ -\ 8\ 4\ /\ *\ - \\
&= 36\ (5\ 3\ -)\ 8\ 4\ /\ *\ - \\
&= 36\ 2\ (8\ 4\ /)\ *\ - \\
&= 36\ (2\ 2\ *)\ - \\
&= (36\ 4\ -) \\
&= 32
\end{aligned}
$$

□

Q.15 (**12p**) Consider the graph shown below and answer the following questions:

(a) (**6p**) Use Dijkstra's algorithm to find a shortest path and its length from a to f. Show your iteration steps.

(b) (**6p**) Use Prim's algorithm to find the minimum spanning tree of the following weighted connected undirected graph. For every spanning step, draw the corresponding subtree.



Solution: Note: I changed this problem, so it is fine if the students used both Kruskal's and Prim's algorithm to find the minimum spanning tree. Just grade their solutions accordingly.

(a) Run Dijkstra's algorithm and keep track of the shortest path from a to all other vertices and their length $L(\cdot)$, with $L(v)$ denoting the shortest length from a to v. The iteration steps are shown as follows:

S	a	b	c	d	e	f	g	h	i
$\varnothing$	0	∞	∞	∞	∞	∞	∞	∞	∞
$\{a\}$	0	5 (a)	∞	2 (a)	∞	∞	∞	∞	∞
$\{a,d\}$	0	5 (a)	∞	2 (a)	9 (a,d)	∞	8 (a,d)	10 (a,d)	∞
$\{a,d,b\}$	0	5 (a)	9 (a,b)	2 (a)	9 (a,d)	11 (a,b)	8 (a,d)	10 (a,d)	∞
$\{a,d,b,g\}$	0	5 (a)	9 (a,b)	2 (a)	9 (a,d)	11 (a,b)	8 (a,d)	10 (a,d)	∞
$\{a,d,b,g,c\}$	0	5 (a)	9 (a,b)	2 (a)	9 (a,d)	11 (a,b)	8 (a,d)	10 (a,d)	∞
$\{a,d,b,g,c,e\}$	0	5 (a)	9 (a,b)	2 (a)	9 (a,d)	10 (a,d,e)	8 (a,d)	10 (a,d)	∞
$\{a,d,b,g,c,e,f\}$	0	5 (a)	9 (a,b)	2 (a)	9 (a,d)	10 (a,d,e)	8 (a,d)	10 (a,d)	14

From the above algorithm run, we find a shortest path from a to f: a, d, e, f, of length 10.

(b) There could be more than one ways to get the minimum spanning tree using Prim's algorithm. Here we list one of such possibilities. The edges added to the minimum spanning tree in each spanning step are as follows: (1) $\{e,f\}$ (2) $\{c,f\}$ (3) $\{e,h\}$ (4) $\{h,i\}$ (5) $\{b,c\}$ (6) $\{b,d\}$ (7) $\{a,d\}$ (8) $\{g,h\}$. The drawing of the subtrees are omitted here.

□