CS201: Discrete Mathematics (Fall 2023)
Written Assignment #1 - Solutions
(100 points maximum but 105 points in total)
Deadline: 11:59pm on Oct 12 (please submit via Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**5p**) Consider the following propositions:

p : You get an A on the final.

q : You do all the assignments.

r : You get an A in this course.

Translate the following statements to formulas using p, q, r and logical connectives.

(a) (**1p**) You get an A either in this course or on the final.

(b) (**1p**) To get an A in this course, it is necessary for you to do all the assignments.

(c) (**1p**) You do all the assignments, but you don't get an A on the final; nevertheless, you get an A in this course.

(d) (**1p**) If you don't get an A in this course, then you don't get an A on the final or don't do all the assignments.

(e) (**1p**) You get an A in this course if and only if you do all the assignments and get an A on the final.

Solution:

(a) $r \oplus p$

(b) $r \rightarrow q$

(c) $q \wedge \neg p \wedge r$

(d) $\neg r \rightarrow (\neg p \vee \neg q)$

(e) $r \leftrightarrow (q \wedge p)$

$\square$

Q.2 (**10p**) Construct a truth table for each of the following compound propositions:

(a) (**1p**) $p \oplus \neg p$

(b) (**2p**) $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

(c) (**2p**) $(p \oplus q) \rightarrow (p \vee \neg q)$

(d) (**5p**) $(p \rightarrow \neg q) \leftrightarrow (r \rightarrow (p \vee \neg q))$

Solution:

(a)

p	$\neg p$	$(p \oplus \neg p)$
T	F	T
F	T	T

(b)

p	q	$(p \to q) \wedge (\neg p \leftrightarrow q)$
T	T	F
T	F	F
F	T	T
F	F	F

(c)

p	q	$(p \oplus q) \to (p \vee \neg q)$
T	T	T
T	F	T
F	T	F
F	F	T

(d)

p	q	r	$p \to \neg q$	$(p \vee \neg q)$	$r \to (p \vee \neg q)$	$(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	T	T
F	T	T	T	F	F	F
T	F	F	T	T	T	T
T	F	T	T	T	T	T
T	T	F	F	T	T	F
T	T	T	F	T	T	F

□

Q.3 (**15p**) Use logical equivalences to prove the following statements. Please write out the names of laws used at each step (see the lecture slides for examples).

(a) (**4p**) $\neg(p \to q) \to p$ is a tautology.

(b) (**3p**) $(p \wedge \neg q) \to r$ and $p \to (q \vee r)$ are equivalent.

(c) (**8p**) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

Solution:

(a) We have

$$
\begin{aligned}
&\neg(p \to q) \to p \\
\equiv\ & \neg\neg(p \to q) \vee p \quad \text{Useful} \\
\equiv\ & (p \to q) \vee p \quad \text{Double negation} \\
\equiv\ & (\neg p \vee q) \vee p \quad \text{Useful} \\
\equiv\ & (\neg p \vee p) \vee q \quad \text{Commutative} \\
\equiv\ & T \qquad \text{Domination}
\end{aligned}
$$

Therefore, it is a tautology.

(b) We have

$$\begin{aligned}
&(p \wedge \neg q) \to r \\
&\equiv \neg(p \wedge \neg q) \vee r \quad \text{Useful} \\
&\equiv (\neg p \vee q) \vee r \quad \text{De Morgan's} \\
&\equiv \neg p \vee (q \vee r) \quad \text{Associative} \\
&\equiv p \to (q \vee r) \quad \text{Useful}
\end{aligned}$$

Therefore, they are equivalent.

(c) We have

$$\begin{aligned}
&(p \to q) \to ((r \to p) \to (r \to q)) \\
&\equiv \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) \quad \text{Useful} \\
&\equiv \neg(\neg p \vee q) \vee ((r \wedge \neg p) \vee (\neg r \vee q)) \quad \text{De Morgan} \\
&\equiv \neg(\neg p \vee q) \vee ((r \vee (\neg r \vee q)) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Distributive} \\
&\equiv \neg(\neg p \vee q) \vee (((r \vee \neg r) \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Associative} \\
&\equiv \neg(\neg p \vee q) \vee ((T \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Complement} \\
&\equiv \neg(\neg p \vee q) \vee (T \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee (\neg r \vee q)) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee ((\neg p \vee q) \vee \neg r) \quad \text{Associative} \\
&\equiv (\neg(\neg p \vee q) \vee (\neg p \vee q)) \vee \neg r \quad \text{Associative} \\
&\equiv T \vee \neg r \quad \text{Complement} \\
&\equiv T \quad \text{Identity.}
\end{aligned}$$

Thus, it is a tautology.

□

Q.4 (**10p**) Determine whether or not the following pairs of statements are logically equivalent, and explain your answer. (Truth tables are not necessary if your explanation is clear.)

(a) (**2p**) $p \oplus q$ and $\neg p \vee \neg q$

(b) (**2p**) $\neg q \wedge (p \leftrightarrow q)$ and $\neg p$

(c) (**3p**) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

(d) (**3p**) $(p \to q) \to r$ and $p \to (q \to r)$

Solution:

(a) The combined truth table is:

p	q	$p \oplus q$	$\neg p \vee \neg q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	F	F

By comparing the last two columns, we have that they are not equivalent.

(b) The combined truth table is:

p	q	$\neg q$	$p \leftrightarrow q$	$\neg q \wedge (p \leftrightarrow q)$	$\neg p$
F	F	T	T	T	T
F	T	F	F	F	T
T	F	T	F	F	F
T	T	F	T	F	F

By comparing the last two columns, we have that they are not equivalent.

(c) The second statement is false only when p is true and $q \vee r$ is false, which means both q and r are false.

The first statement if false only when both $p \to q$ and $p \to r$ are false. This only happens when p is true, and both q and r are false.

Thus, these two statements are logically equivalent.

(d) These two statements are not logically equivalent. It suffices to give a counterexample. When p, q and r are all false, $(p \to q) \to r$ is false, but $p \to (q \to r)$ is true.

□

Q.5 (**5p**) Determine for which values of p, q, r the statement $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true and for which values it is false, and explain your reasoning. Do not use a truth table.

Solution: The statement is false when p, q, r have the same truth value and is true otherwise. The explanation is as follows. The first clause is true if and only if at least one of p, q and r is true. The second clause is true if and only if at least one of the three variables is false. Therefore the entire statement is true if and only if there is at least one T and one F among the truth values of the variables, in other words, that they don't all have the same truth value.

□

Q.6 (**5p**) Prove that if $p \to q$, $\neg p \to \neg r$, $s \vee r$, then $q \vee s$. Please write out the names of inference rules used at each step (see the lecture slides for examples).

Solution:

Step	**Reason**
(1) $p \to q$	Premise
(2) $\neg q \to \neg p$	Contrapositive of (1)
(3) $\neg p \to \neg r$	Premise
(4) $\neg q \to \neg r$	Hypothetical syllogism (2) (3)
(5) $q \vee \neg r$	Useful
(6) $s \vee r$	Premise
(7) $q \vee s$	Resolution

□

Q.7 (**5p**) Prove that if $p \wedge q$, $q \to \neg(p \wedge r)$, $s \to r$, then $\neg s$. Please write out the names of inference rules used at each step (see the lecture slides for examples).

Solution:

Step	Reason
(1) $p \wedge q$	Premise
(2) q	Simplication of (1)
(3) $q \to \neg(p \wedge r)$	Premise
(4) $\neg(p \wedge r)$	Modens ponens (2) (3)
(5) $\neg p \vee \neg r$	De Morgan's
(6) p	Simplication of (1)
(7) $\neg r$	Disjunctive syllogism (5) (6)
(8) $s \to r$	Premise
(9) $\neg s$	Modus tollens (7) (8)

□

Q.8 (**5p**) Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++". Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at SUSTech.

(a) (**1p**) There is a student who either can speak Russian or knows C++.

(b) (**1p**) There is a student who can speak Russian but who doesn't know C++.

(c) (**1p**) Every student can speak Russian and knows C++.

(d) (**1p**) No student can speak Russian or knows C++.

(e) (**1p**) If a student can speak Russian then he/she does not know C++.

Solution:

(a) $\exists x(P(x) \oplus Q(x))$

(b) $\exists x(P(x) \wedge \neg Q(x))$

(c) $\forall x(P(x) \wedge Q(x))$

(d) $\forall x \neg(P(x) \vee Q(x))$

(e) $\forall x(P(x) \to \neg Q(x))$

□

Q.9 (**8p**) Let $L(x, y)$ be the statement "x loves y", where the domain for both x and y consists of all people in the world. Translate the following statements to quantified formulas. (Hint: you can use $=, \neq$ to connect variables.)

(a) (**1p**) Everybody loves somebody.

(b) (**2p**) There is someone who loves only himself or herself but no other person.

(c) (**5p**) There are exactly two people whom Lynn loves.

Solution:

(a) $\forall x \exists y\ L(x, y)$

(b) $\exists x \forall y (L(x,y) \leftrightarrow x = y)$

(c) $\exists x \exists y (x \neq y \wedge L(Lynn, x) \wedge L(Lynn, y) \wedge (\forall z (L(Lynn, z) \rightarrow (z = x \vee z = y)))$

$\square$

Q.10 (**5p**) Express the negations of each of the following statements such that all negation symbols *immediately precede* predicates.

(a) (**1p**) $\exists z \forall y \forall x T(x,y,z)$

(b) (**2p**) $\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)$

(c) (**2p**) $\forall x \exists y (P(x,y) \rightarrow Q(x,y))$

Solution:

(a)

$$\begin{aligned} \neg \exists z \forall y \forall x T(x,y,z) &\equiv \forall z \neg \forall y \forall x T(x,y,z) \\ &\equiv \forall z \exists y \neg \forall x T(x,y,z) \\ &\equiv \forall z \exists y \exists x \neg T(x,y,z) \end{aligned}$$

(b)

$$\begin{aligned} \neg(\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)) &\equiv \neg \exists x \exists y P(x,y) \vee \neg \forall x \forall y Q(x,y) \\ &\equiv \forall x \neg \exists y P(x,y) \vee \exists x \neg \forall y Q(x,y) \\ &\equiv \forall x \forall y \neg P(x,y) \vee \exists x \exists y \neg Q(x,y) \end{aligned}$$

(c)

$$\begin{aligned} \neg \forall x \exists y (P(x,y) \rightarrow Q(x,y)) &\equiv \exists x \neg \exists y (P(x,y) \rightarrow Q(x,y)) \\ &\equiv \exists x \forall y \neg(\neg P(x,y) \vee Q(x,y)) \\ &\equiv \exists x \forall y (P(x,y) \wedge \neg Q(x,y)). \end{aligned}$$

$\square$

Q.11 (**10p**) Consider this argument: "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners." Answer the following questions.

(a) (**3p**) Define the predicates and translate each sentence to a quantified formula.

(b) (**7p**) Show the formal proof steps and explain which rule of inference is used for each step.

Solution:

(a) Let $s(x)$ be "x is a movie produced by Sayles," let $c(x)$ be "x is a movie about coal miners", and let $w(x)$ be "movie x is wonderful", where the universe for x is all movies. We are given premises $\forall x(s(x) \rightarrow w(x))$ and $\exists x(s(x) \wedge c(x))$, and we want to conclude $\exists x(c(x) \wedge w(x))$.

(b) The proof steps and rules of inferences are shown as follows:

Step	**Reason**
(1) $\exists x(s(x) \land c(x))$	Premise
(2) $s(y) \land c(y)$	Existential instantiation using (1)
(3) $s(y)$	Simplification of (2)
(4) $\forall x(s(x) \to w(x))$	Premise
(5) $s(y) \to w(y)$	Universal instantiation of (4)
(6) $w(y)$	Modus ponens using (3) and (5)
(7) $c(y)$	Simplification of (2)
(8) $w(y) \land c(y)$	Conjunction using (6) and (7)
(9) $\exists x(c(x) \land w(x))$	Existential generalization using (8)

$\square$

Q.12 (**5p**) Prove that $\sqrt[3]{2}$ is irrational.

Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

$\square$

Q.13 (**5p**) Prove that there is an irrational number between every two distinct rational numbers. (Hint: you can use the in-class learned fact that $\sqrt{2}$ is irrational.)

Solution: By finding a common denominator, we can assume the given rational numbers are a/b and c/b, where b is a positive integer and a and c are integers with $a < c$. In particular, $(a+1)/b \leq c/b$. Thus, $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the two given rational numbers, because $0 < \sqrt{2} < 2$. Furthermore, x is irrational, because if x were rational, then $2(bx - a) = \sqrt{2}$ would be as well, which is wrong.

$\square$

Q.14 (**12p**) Prove that all integral solutions to the equation

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2}$$

such that $m, n \geq 3$ and $e > 0$ are in this table:

m	n	e
3	3	6
3	4	12
3	5	30
4	3	12
5	3	30

(Hint: use proof by cases.)

Solution: We use case analysis. Since $m \geq 3$, one of the following four cases must hold:

- $m = 3$. There are now four subcases:

  - $n = 3$. Substituting $m = n = 3$ into the equation implies that $e = 6$, which is the first solution.
  - $n = 4$. Similar to the above subcase, we have $e = 12$, which is the second solution.
  - $n = 5$. Similar to the above two subcases, we have $e = 30$, which is the third solution.
  - $n \geq 6$. This implies:
$$\frac{1}{m} + \frac{1}{n} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$
    Thus, the left side of the equation is strictly less than the right for all $e > 0$, so there are no solutions in this case.

- $m = 4$. There are two subcases:
  - $n = 3$. Substituting $m = 4$ and $n = 3$ into the equation implies that $e = 12$, which is the fourth solution.
  - $n \geq 4$. This implies:
$$\frac{1}{m} + \frac{1}{n} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
    Again, the left side of the equation is strictly less than the right for all $e > 0$, so there are no solutions in this case.

- $m = 5$. There are two subcases:
  - $n = 3$. Substituting $m = 5$ and $n = 3$ into the equation implies that $e = 30$, which is the fifth solution.
  - $n \geq 4$. This implies:
$$\frac{1}{m} + \frac{1}{n} \leq \frac{1}{5} + \frac{1}{4} < \frac{1}{2}$$
    Again, the equation cannot hold for all $e > 0$, so there are no solutions in this case.

- $m \geq 6$. This implies:
$$\frac{1}{m} + \frac{1}{n} \leq \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$
  Once more, the equation can not hold, so there are no solutions in this case.

$\square$