Assignment 4 12212726 Yang Yanzhuo

Sorry. I correted some answers and resubmitted it 57 minutes after the deadline. But the first submission shoulb be on time. I'd be very grateful if you score the latest submission. Thank you!

Q1. (a) Let A be the set of all propositional functions, which can be proved by weak induction, and B can be proved by strong induction.

I. Show that weak induction implies strong induction

For an arbitrary element in B, says P, we know that:

P is true if $\begin{cases} P(1) \text{ is true} \\ P(1) \wedge P(2) \cdots P(k) \to P(k+1) \text{ is true.} \end{cases}$

Let $q(n) = P_2(1) \wedge P_2(2) \cdots P_2(n)$

then P_2 is true if $\begin{cases} q(1) \text{ is true} \\ q(k) \to P_2(k+1) \text{ is true} \end{cases}$

~~$P_2(k+1) =$~~ And if $q(k) \to P_2(k+1)$ is true, then $q(k+1) = q(k) \wedge P_2(k+1)$ is true,

then P is true if $\begin{cases} q(1) \text{ is true} \\ q(k) \to q(k+1) \text{ is true} \end{cases}$

We prove P by weak induction, which means $P_2 \in A$.

$\therefore B \subseteq A$

II. Show that strong induction implies weak induction.

For an arbitrary element P in A, we know that:

P is true if $\begin{cases} P(1) \text{ is true} \\ P(k) \to P(k+1) \text{ is true} \end{cases}$

Obviously, $(P(1) \wedge P(2) \cdots P(k) \to P(k+1) \text{ is true})$

$\to (P(k) \to P(k+1) \text{ is true})$

because $P(1) \wedge P(2) \cdots P(k) \to P(k)$

$\therefore$ P is true if $\begin{cases} P(1) \text{ is true} \\ P(1) \wedge P(2) \cdots P(k) \to P(k+1) \text{ is true.} \end{cases}$

Q1. (b) Let $p(n)$ be the proposition function that.

for any non-empty set. which is a subset of positive integers. if it contains an element i s.t. $i \leq n$. then it has least element.

Obviously, $P(1)$ is true since 1 is the least positive integers.
Assume $p(k)$ is true. That means. for an arbitrary set $A \subseteq Z^+$. if there $\exists\, i \leq k$, then it has least element.
Then assume $p(k) \to p(k+1)$ is not true. which means ~~p(k+1) is false~~.
$\neg p(k) \vee p(k+1)$ is false $\Rightarrow$ $\neg(\neg p(k) \vee p(k+1))$ is true.
$\Rightarrow p(k) \wedge \neg p(k+1)$ is true $\wedge$ $p(k)$ is true $\Rightarrow$ $p(k+1)$ is false
That means, $A \subseteq Z^+$. if $\exists\, i \leq k+1$, then it dosen't has least.

I. $i \leq k$.
Since $p(k)$ is true. so A has least, which forms contradiction.

II. $i > k \wedge i \leq k+1 \Rightarrow i = k+1 \Rightarrow k+1 \in A$
If $\exists\, j < k+1 \Rightarrow j < k \wedge p(k)$ is true $\Rightarrow$ A has least $\Rightarrow$ contradiction
If there doesn't $\exists\, j < k+1$. then $k+1$ is the least in A. which forms contradiction.

$\therefore$ "$p(k+1)$ is false" is contradiction
$\therefore$ $p(k+1)$ is true.
$\therefore$ $p(k) \to p(k+1)$ is true.

Thus. we prove Well-Ordering principle by induction.

Q2. Basic step: When ~~∗~~ $n=1$:

Obviously, $A_1-B = A_1-B$

Inductive step: Assume that $(A_1-B)\cap(A_2-B)\cdots(A_n-B)$

$=(A_1\cap A_2\cdots A_n)-B$. then $(A_1-B)\cap(A_2-B)\cdots(A_n-B)\cap(A_{n+1}-B)$

$=[(A_1\cap A_2\cdots A_n)-B]\cap(A_{n+1}-B)$

$=(A_1\cap A_2\cdots A_n\cap\overline{B})\cap(A_{n+1}\cap\overline{B})$ (By definition, $A-B=\{a\in A\wedge a\notin B\}$)

$=(A_1\cap A_2\cdots A_n\cap A_{n+1})\cap\overline{B}$ (Association law)

$(A_1\cap A_2\cdots A_n\cap A_{n+1})-B$

$=A_1\cap A_2\cdots A_n\cap A_{n+1}\cap\overline{B}$

$\therefore$ $(A_1-B)\cap(A_2-B)\cdots(A_{n+1}-B)=(A_1\cap A_2\cdots A_n\cap A_{n+1})-B$

$\therefore$ $(A_1-B)\cap(A_2-B)\cdots(A_{n+1}-B)=(A_1\cap A_2\cdots A_n\cap A_{n+1})-B$)

is true for all non-negative integers n.

Q3. Basic step: Obviously: "If p is prime and $p|a_1$, then $\exists i=1$ s.t. $p|a_i$" is true.

Inductive step: Assume that if p is prime and $p|a_1a_2\cdots a_n$, then $\exists a_i$: s.t. $p|a_i$, $a_i\in\{a_1,a_2\cdots a_n\}$

If p is prime and $p|a_1a_2\cdots a_na_{n+1}$, then:

I. $p\nmid a_{n+1}\Rightarrow \gcd(p,a_{n+1})=1$ since p is prime

$\begin{cases} p|a_1a_2\cdots a_na_{n+1} \\ \gcd(p,a_{n+1}) \end{cases}\Rightarrow p|a_1a_2\cdots a_n\Rightarrow\exists a_i$ s.t. $p|a_i$, $a_i\in\{a_1a_2\cdots a_n\}$

$\{a_1,a_2\cdots a_n\}\subseteq\{a_1,a_2\cdots a_{n+1}\}\Rightarrow a_i\in\{a_1,a_2\cdots a_{n+1}\}$

Also means $\exists a_i$ s.t. $p|a_i$, $a_i\in\{a_1,a_2\cdots a_{n+1}\}$

II. $p|a_{n+1}\Rightarrow\exists a_i$ s.t. $p|a_i$, $a_i=a_{n+1}$.

Thus, we get $\exists a_i$ s.t. $p|a_i$, $a_i\in\{a_1,a_2\cdots a_{i+1}\}$ from the assumption.

Then, we know the original proposition is tautology.

Q4. (a). 12 cents can be formed with four 3-cent stamps.
13 cents can be formed with two 3-cents and one 7-cent
14 cents can be formed with two ~~2~~ 7-cent stamps.

(b). Assume $p(n)$, $p(n+1)$, $p(n+2)$ are true.

(c). Prove $p(n+3)$ is true with the assumption.

(d). 1°. $k+1=15 \Rightarrow k=14$. we know $p(14)$ is true from (a)
2°. $k+1>15 \Rightarrow k>14$.
If $p(k) \wedge p(k+1) \wedge p(k+2)$ is true,
then we know $p(k)$ is true, i.e.
$\exists$ integers a and b s.t. $k = 3a+7b$
And $k+3 = 3(a+1)+7b$
$\therefore$ $p(k+3)$ is true since $a+1$ is integer.

From (a), we know $p(12) \wedge p(13) \wedge p(14)$ is true,
then $p(k)$ is true for $k>14$.

Thus, we prove the inductive step for $k+1 \geq 15$.

(e). We define a new proposition $Q(n) = p(n) \wedge p(n+1) \wedge p(n+2)$
Basic step shows $p(12) \wedge p(13) \wedge p(14) = Q(12)$ is true.
With the IH that $Q(n)$ is true, we prove ~~p(k+3) p(n+1)~~ $p(n+3)$ is true.
$Q(n+1) = p(n+1) \wedge p(n+2) \wedge p(n+3)$ is true.

$\begin{cases} Q(12) \text{ is true} \\ Q(n) \rightarrow Q(n+1) \text{ is true} \end{cases} \Rightarrow Q(n)$ is true for $n \geq 12$.
(By the weak induction)

$\Rightarrow p(n)$ is true for $n \geq 12$

Q5. Algorithm Binary Search by Recursive: (short as BSR)

BSR (integer X, integer left, integer right, integers $a_1, a_2 \cdots a_n$ with ordered) (here is increasingly or non-decreasingly)

~~if $i \geq j$~~

we have X for target integer and ~~a_i~~ as for sought sequence.

BSR $(X, i, j,$ ~~$a(s)$~~ $as)$

if $(i \geq j$ or $a_i \underset{>}{\leq} X$ or $a_j < X)$ return 0;

$m := \lfloor (i+j)/2 \rfloor$;

if $(X > a_m)$ then return BSR$(X, m+1, j, as)$;

if $(X < a_m)$ then return BSR$(X, i, m-1, as)$;

if $(X = a_m)$ then return m;

Note: 1. "return 0" means not found;

2. Initialize the BSR with ~~$i:=0$~~ $i:=1$ and $j:=n$.

Q7. $T(n)=\begin{cases} T(1) & , n=1 \\ aT(\frac{n}{2})+n, & n>1 \end{cases}$

Then $T(n) = a[aT(\frac{n}{4})+\frac{n}{2}]+n$

$= a^2T(\frac{n}{4}) \quad +n+\frac{1}{2}an$

$= a^2[aT(\frac{n}{8})+\frac{n}{4}]+n+\frac{1}{2}an$

$= a^3T(\frac{n}{8})+n+\frac{1}{2}an+\frac{1}{4}a^2n$

$= a^3[aT(\frac{n}{16})+\frac{n}{8}]+\frac{1}{4}a^2n+\cancel{\frac{1}{4}}\frac{1}{2}an+n$

$(\cdots)$

I. $a\neq 1$. $= a^{\log_2 n}T(\frac{n}{2^{\log_2 n}})+\frac{1}{2^{\log_2 n}}a^{\log_2 n}\cdot n+\frac{1}{2^{\log_2 n-1}}a^{\log_2 n-1}\cdot n+\cdots n$

$= a^{\log_2 n}T(1)+\frac{1}{n}\cdot a^{\log_2 n}\cdot n+\frac{2}{n}\cdot a^{\log_2 n-1}n+\cdots n$

$\underline{\underline{m=a^{\log_2 n}}}\ mT(1)+\frac{m}{n}\cdot n+\cancel{\frac{2m}{A}\frac{m}{2}\cdot n+\frac{4}{n}\frac{m}{4}}$

$\frac{2\cancel{m}}{n}\cdot\frac{m}{a}\cdot n+\frac{4}{n}\cdot\frac{m}{a^2}\cdot n+\cdots n$

$= mT(1)+m+\frac{2m}{a}+\frac{4m}{a^2}+\cdots\frac{2^{\log_2 n}m}{a^{\log_2 n}}$

$= mT(1)+m+\frac{2}{a}m+\frac{4}{a^2}m+\cdots\frac{n}{a^{\log_2 n}}m$

$= mT(1)+m+\frac{2}{a}m+\frac{4}{a^2}m+\cdots\frac{n}{m}m$

$= mT(1)+(1+\frac{2}{a}+\frac{4}{a^2}+\cdots\frac{n}{m})m$

$= mT(1)+\frac{1-(\frac{2}{a})^m}{1-\frac{2}{a}}\cdot m$

$= m\left[T(1)+\frac{1-(\frac{2}{a})^m}{1-\frac{2}{a}}\right]$

$\cancel{=[T(1)+\frac{1}{1-\frac{2}{a}}]m-\frac{(\frac{2}{a})^m}{1-\frac{2}{a}}\cdot m}$

~~$= Am-\frac{m\cdot C^m}{B}$, where A, B, C are constant~~

$\cancel{= A\sqrt[\log_a 2]{n}}$

$= \sqrt[\log_a 2]{n}\left(T(1)+\frac{1}{1-\frac{2}{a}}\right)-\frac{(\frac{2}{a})^m\cdot m}{1-\frac{2}{a}}$

Obviously $\theta(n)=\theta(\sqrt[\log_a 2]{n})$ since $a\in(1,2)$

and $\frac{(\frac{2}{a})^m\cdot m}{1-\frac{2}{a}}=O(n) \Rightarrow T(n)=\theta(n)$

II. $a=1$

then we have $T(n) = T(\frac{n}{2^{\log_2 n}}) + n + n + \cdots n$

$$= T(1) + n \cdot \log_2 n$$

~~Ot~~ $T(n) = \theta(n\log_2 n) = \theta(n)$

Q8. (a) $\binom{13}{2}\binom{4}{3}\binom{4}{2}\binom{2}{1}$

(b) $\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}$

(c) $\binom{13}{5}\binom{4}{1}$

(d) $\cancel{9\binom{4}{1}^9}$ $9\binom{4}{1}^5$

(e) $\binom{13}{2}\binom{2}{1}\binom{4}{4}\binom{4}{1}$

Q9. step1: _ 0 0 0 0 _ _ _

there are 4 positions for these 4 0s.

step2: each other positions have two possible value.

step3: Substitude all 0 with 1, all 1 with 0.

step4: Cross the duplication and out of requirements.

Thus. $4 \cdot \binom{2}{1}^4 \cdot \binom{2}{1} - 2 = 126$ is the answer.

Q10. That is, prove $2022 \mid \frac{2020!}{1010!\ 1010!}$

$$\frac{2020!}{1010!\ 1010!} = \frac{1011\times1012\times\cdots2020}{1010\times1009\times\cdots1} = 2\times1010\times\frac{2019\times2018\cdots1011}{1010\times1009\times\cdots1}$$

$$=2\times\frac{2019\times2018\cdots1011}{1009!}$$

$$=2\times1011\times\frac{2019\times2018\cdots1012\cdot1011!}{2020\ 1009!\cdot1011!}\cdot\frac{2020}{2020}$$

$$=2022\times\binom{\cancel{2019}\ 2020}{1009}\times\frac{1}{2020}$$

$\binom{2020}{1010}$ is an integer. then $2022 \mid \left(\frac{2020}{1010}\right)$

Q11.

Q12. The characteristic equation is $r^3-3r-2=0$

solve it and get $r_1=r_2=-1$ and $r_3=2$

If $a_n=\alpha_1(-1)^n+\alpha_2(-1)^n n+\alpha_3(2)^n$

then
$$\begin{cases} a_0=1=\alpha_1+\alpha_3+\alpha_2\cdot 0 \\ a_1=-5=-\alpha_1\mp\alpha_2+2\mp2\alpha_3 \\ a_2=0=\alpha_1+2\alpha_2+4\alpha_3 \end{cases} \Rightarrow \begin{cases} \alpha_1=2 \\ \alpha_2=1 \\ \alpha_3=-1 \end{cases}$$

Thus. we get $a_n=2(-1)^n+n(-1)^n-2^n$

Q13. The C.E. is $r-2=0 \Rightarrow r=2$

Then $a_n=\alpha\cdot2^n+p(n)$. Try $p(n)=an^2+bn+c$

Then $a_n=2a_{n-1}+n^2 \Rightarrow \alpha 2^n+an^2+bn+c=\left(\alpha\cdot2^{n-1}+a(n-1)^2+b(n-1)+c\right)\cdot2+n^2$

$=\alpha\cdot2^n+2a(n^2-2n+1)+2bn+2c-2b+n^2$

$\Rightarrow (a-2a-1)n^2+(b+4a-2b)n+c-2c+2b-2a$

$$\Rightarrow \begin{cases} 1+a=0 \\ b+4a-2b=0 \\ -c+2b-2a=0 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=-4 \\ c=-6 \end{cases} \Rightarrow a_n=\alpha 2^n-n^2-4n-6$$

$a_1=2\alpha-1-4-6=2 \Rightarrow \alpha=\frac{13}{2}$ $\Rightarrow a_n=\frac{13}{2}\cdot2^n-n^2-4n-6$

Q14. $a_n = 4a_{n-1} + 8^{n-1}, \ a_0 = 0$

$$G(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 4a_{n-1} x^n + 8^{n-1} x^n$$

$$= 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 8^{n-1} x^{n-1}$$

$$= 4x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 8^n x^n$$

$$= 4x\, G(x) + \frac{x}{1-8x}$$

$$\Rightarrow G(x) = \frac{x}{(1-8x)(1-4x)} = \frac{1}{4}\left[\frac{1}{1-8x} - \frac{1}{1-4x}\right] = \frac{1}{4} \sum_{n=0}^{\infty} \left(8^n x^n - 4^n x^n\right)$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} (8^n - 4^n) x^n = \sum_{n=1}^{\infty} a_n x^n$$

$$\Rightarrow a_n = \frac{1}{4}(8^n - 4^n)$$