CS201: Discrete Mathematics (Fall 2023)
Written Assignment #2
(100 points maximum but 110 points in total)

Deadline: 11:59pm on Oct 23 (please submit via Blackboard)

PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**5p**) Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) (**1p**) $(A \cap B \neq \varnothing) \rightarrow ((A - B) \subset A)$

(b) (**1p**) $(A \subseteq B) \rightarrow (|A \cup B| \geq 2|A|)$

(c) (**3p**) $\overline{(A - B)} \cap \overline{(B - A)} = \overline{A \cup B}$

Q.2 (**10p**) Let A, B and C be sets. Prove the following using set identities. Please write out the names of identities used at each step (see the lecture slides for examples).

(a) (**4p**) $\overline{A \cap (B \cup C)} = (\overline{C} \cap \overline{B}) \cup \overline{A}$

(c) (**6p**) $(A - B) \cap (B - A) = \varnothing$

Q.3 (**10p**) Prove the following statements:

(a) (**5p**) Any subset of a countable set A is still countable. (Note that the contrapositive statement is also useful: If A is uncountable, then any set having A as a subset is also uncountable.)

(b) (**5p**) If A is a countable and there is an onto function from A to B, then B is also countable.

Q.4 (**15p**) The *symmetric difference* of A and B, denoted by $A \Delta B$, is the set containing those elements in either A or B (not in both A and B).

(a) (**5p**) Determine whether the symmetric difference is associative; that is, if A, B and C are sets, does it follow that $A \Delta (B \Delta C) = (A \Delta B) \Delta C$? Explain your answer.

(b) (**5p**) Suppose that A, B, C are sets such that $A \Delta C = B \Delta C$. Prove or disprove $A = B$.

(c) (**5p**) Give an example of two uncountable sets A and B such that $A \Delta B$ is infinite and countable. (Hint: we learned from class that $[0, 1]$ is uncountable.)

Q.5 (**5p**) For finite sets A, B and C, explain why the following inclusion-exclusion formula is true:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Q.6 (**5p**) Show that if A, B, C and D are (probably infinite) sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

Q.7 (**10p**) Consider two functions $g : A \rightarrow B$ and $f : B \rightarrow C$ and its composition function $f \circ g$. Answer the following questions and explain.

(a) (**2p**) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one?

(b) (**2p**) If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one?

(c) (**2p**) If $f \circ g$ is one-to-one, must g be one-to-one?

(d) (**2p**) If $f \circ g$ is onto, must f be onto?

(e) (**2p**) If $f \circ g$ is onto, must g be onto?

Q.8 (**5p**) Let x be a real number. Prove that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Q.9 (**10p**) Derive the *closed formula* for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, where m is a positive integer. (Hint: express the summation as a function $f(m, n)$, where $n = \lfloor \sqrt{m} \rfloor - 1$.)

Q.10 (**5p**) Apply the Schröder-Bernstein theorem to prove $(0, 1)$ and $[0, 2]$ have the same cardinality.

Q.11 (**5p**) Show that when the Hilbert's Grand Hotel (see lecture slides "03 Sets and Functions") is fully occupied one can still accommodate countably infinite new guests in it.

Q.12 (**10p**) In order to show that there exist uncomputable functions, it suffices to prove the following two parts:

(a) (**5p**) The set of all computer programs in all existing programming languages is countable.

(b) (**5p**) The set of all functions from $\mathbf{Z}^+$ to the set of digits $\{0, 1, \ldots, 9\}$ is uncountable.

Then, one can conclude that there exists a function $f^* : \mathbf{Z}^+ \to \{0, 1, \ldots, 9\}$ that is uncomputable, i.e., no computer program in any programming language can find the values of this function f^*. Prove the above two statements from scratch, i.e., do not use proved theorems taught in class. (Hint: refer to the proof of the theorem "the set of all Java programs are uncountable" for (a) and refer to the proof of the theorem "the set of real numbers is uncountable" for (b).)

Q.13 (**5p**) Prove that for any $a > 1$, $\Theta(\log_a n) = \Theta(\log_2 n)$. (This means the base of logarithm does not matter for measuring the complexity so we can simply write $\Theta(\log n)$.)

Q.14 (**10p**) Consider Algorithm 1 for the *binary search* algorithm, which searches an integer x in any increasingly ordered sequence of n distinct integers $a_1, a_2, \ldots, a_n$.

Algorithm 1 Binary Search (x: target integer, $a_1, a_2, \ldots, a_n$: increasingly ordered integers)

$i := 1$
$j := n$
while $i < j$ **do**
  $m := \lfloor (i + j)/2 \rfloor$
  if $x > a_m$ **then** $i := m + 1$
  else $j := m$
if $x = a_i$ **then** *location* $:= i$
else *location* $:= 0$
return *location* {*location* is the subscript of a_i such that $a_i = x$, or 0 if no such a_i is found}

Answer the following questions:

(a) (**3p**) What is the time complexity of Algorithm 1 in terms of n? Explain your answer. (Just need to count the number of comparison operations and express it with the $\Theta(\cdot)$ notation.)

(b) (**3p**) Improve Algorithm 1 such that its best-case time complexity is $\Theta(1)$ and its worst-case time complexity is unchanged (when measured with $\Theta(\cdot)$). Explain your answer.

(c) (**2p**) What is the space complexity of Algorithm 1? Explain. (Use the $\Theta(\cdot)$ notation.)

(d) (**2p**) Considering binary representation of integers on computers, what is the input size of the above binary search problem using fixed-length encoding?