CS201: Discrete Mathematics (Fall 2023)
Written Assignment #4
(100 points maximum but 110 points in total)
Deadline: 11:59pm on Dec 7 (please submit via Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**10p**) In this assignment, we show that the principle of mathematical induction (weak induction), the second principle of mathematical induction (strong induction), and the well-ordering principle are all equivalent; that is, each can be shown to be valid from the other.

(a) (**5p**) Prove that weak induction and strong induction are equivalent.

(b) (**5p**) In class, we already proved that weak induction can be derived from the well-ordering principle. Now, prove that weak induction implies the well-ordering principle. (Hint: proof by contradiction, i.e., a non-empty set with no least element must be empty by induction.)

Q.2 (**5p**) Prove by induction that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \quad = \quad (A_1 \cap A_2 \cap \cdots \cap A_n) - B.$$

(Note that similarly one can use mathematical induction to prove that the De Morgan's law and distributive law can also be generalized to the n-set case.)

Q.3 (**5p**) Use mathematical induction to prove that "if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where each a_i is an integer, then $p \mid a_i$ for some integer $i \in \{1, 2, \ldots, n\}$".

Q.4 (**10p**) Let $P(n)$ be the statement that postage of n cents can be formed using just 3-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 12$.

(a) (**2p**) Show statements $P(12)$, $P(13)$, $P(14)$ are true, completing the basis step of the proof.

(b) (**2p**) What is the inductive hypothesis of the proof?

(c) (**2p**) What do you need to prove in the inductive step?

(d) (**2p**) Complete the inductive step for $k + 1 \geq 15$.

(e) (**2p**) Explain why these steps show that this statement is true whenever $n \geq 12$.

Q.5 (**6p**) Describe a recursive algorithm for binary search (as described in Assignment 2, Q.14). Write out the pseudocode.

Q.6 (**6p**) Prove that the number of divisions required by the Euclidean algorithm to find $\gcd(a, b)$, where $a \geq b > 0$, is $O(\log b)$. (Hint: prove that the remainders r_i satisfy $r_{i+2} < r_i/2$.)

Q.7 (**8p**) Iterating the recurrence $T(n) = aT(n/2) + n$ to show that, for $1 \leq a < 2$ and $T(1) \geq 0$ we have $T(n) = \Theta(n)$. Please show your iteration steps.

Q.8 (**12p**) Consider a deck of 52 cards that consists of 4 suits each with one card of each of the 13 ranks. Answer the following questions using combination notations only, e.g., $\binom{12}{2}\binom{3}{1}\binom{42}{3}$.

(a) (**2p**) How many full houses? That is three cards of one rank and two of another rank.

(b) (**2p**) How many two pairs? That is two cards of one rank, two of another rank, and one of a third rank.

(c) (**2p**) How many flushes? That is five cards of the same suit.

(d) (**4p**) How many straights? That is five cards of sequential ranks. Note that a straight with an ace in it can only be "10JQKA" or "A2345" but not other cases like "JQKA2".

(e) (**2p**) How many quads? That is four cards of one rank and one of another rank.

Q.9 (**5p**) How many bit strings of length 8 contain either 4 consecutive 0s or 4 consecutive 1s?

Q.10 (**8p**) Prove that the following binomial coefficient is divisible by 2022.

$$\binom{2020}{1010}$$

(Hint: first note that $2022 = 2 \cdot 1011$ and recall what we learned from number theory to decompose the problem into two subproblems, then use the fact that for all $0 \leq k \leq n$ the combinations $\binom{n}{k}$ are integers.)

Q.11 (**5p**) Prove the hockey-stick identity.

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r} \text{ where } n, r \text{ are positive integers}$$

Use a combinatorial argument and do not use Pascal's identity.

Q.12 (**10p**) Solve the recurrence relation $a_n = 3a_{n-2} + 2a_{n-3}$, $n \geq 3$, with initial conditions $a_0 = 1$, $a_1 = -5$ and $a_2 = 0$.

Q.13 (**10p**) Solve nonhomogenenous recurrence relations.

(a) (**8p**) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + n^2$.

(b) (**2p**) Find the solution of the recurrence relation in part (a) with the initial condition $a_1 = 2$.

Q.14 (**10p**) Use generating functions to solve the recurrence relation $a_n = 4a_{n-1} + 8^{n-1}$ with the initial condition $a_0 = 0$.