CS201: Discrete Mathematics (Fall 2023)
Written Assignment #4 - Solutions
(100 points maximum but 110 points in total)
Deadline: 11:59pm on Dec 7 (please submit via Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**10p**) In this assignment, we show that the principle of mathematical induction (weak induction), the second principle of mathematical induction (strong induction), and the well-ordering principle are all equivalent; that is, each can be shown to be valid from the other.

(a) (**5p**) Prove that weak induction and strong induction are equivalent.

(b) (**5p**) In class, we already proved that weak induction can be derived from the well-ordering principle. Now, prove that weak induction implies the well-ordering principle. (Hint: proof by contradiction, i.e., a non-empty set with no least element must be empty by induction.)

Solution:

(a) The strong induction principle clearly implies weak induction, because if one has shown that $P(k) \to P(k+1)$, then it automatically follows that $P(1) \wedge \cdots \wedge P(k) \to P(k+1)$; in other words, strong induction can always be invoked whenever weak induction is used.

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Clearly $\forall n P(n)$ is logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using weak induction. That is, given $Q(1)$ is true and $Q(k) \to Q(k+1)$ is true, we need to prove $\forall n Q(n)$ is true by applying strong induction on $P(n)$. First, we note that $P(1) = Q(1)$, so $P(1)$ is true. Then, by definition we know $Q(k) \to Q(k+1)$ can be written as $Q(k) \to Q(k) \wedge P(k+1)$, which is equivalent to $Q(k) \to P(k+1)$. By definition, $Q(k) \to P(k+1)$ is equivalent to $P(1) \wedge \cdots \wedge P(k) \to P(k+1)$, which is exactly the inductive step of strong induction. Therefore, by strong induction, we can conclude that $\forall n P(n)$ is true, and hence $\forall n Q(n)$ is true.

(b) Suppose that the well-ordering principle were false. Let S be a nonempty set of nonnegative integers that has no least element. Let $P(n)$ be the statement "$i \notin S$ for $i = 0, 1, \ldots, n$". Then, $P(0)$ is true, because if $0 \in S$ then S has 0 as its least element. Now, suppose that $P(k)$ is true, i.e., $i \notin S$ for all $i = 0, 1, \ldots, k$. Then, $k+1 \notin S$, because otherwise $n+1$ would be its least element. Therefore, $P(k+1)$ is true. By weak induction, we have $n \notin S$ for all nonnegative integers n. Therefore, $S = \varnothing$, a contradiction.

(Note that proof by induction on the size of S does not work because S could be infinite.)

□

Q.2 (**5p**) Prove by induction that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \quad = \quad (A_1 \cap A_2 \cap \cdots \cap A_n) - B.$$

(Note that similarly one can use mathematical induction to prove that the De Morgan's law and distributive law can also be generalized to the n-set case.)

Solution: If $n = 1$, there is nothing to prove, and then $n = 2$, this says that $(A_1 \cap \bar{B}) \cap (A_2 \cap \bar{B}) = (A_1 \cap A_2) \cap \bar{B})$, which is the distributive law. For the inductive step, assume that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B;$$

we must show that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) = (A_1 \cap A_2 \cap \cdots A_n \cap A_{n+1}) - B.$$

We have

$$\begin{aligned}
& (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) \\
= \quad & ((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B)) \cap (A_{n+1} - B) \\
= \quad & ((A_1 \cap A_2 \cap \cdots \cap A_n) - B) \cap (A_{n+1} - B) \quad \text{inductive hypothesis} \\
= \quad & (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B. \quad \text{following from the } n = 2 \text{ case}
\end{aligned}$$

□

Q.3 (**5p**) Use mathematical induction to prove that "if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where each a_i is an integer, then $p \mid a_i$ for some integer $i \in \{1, 2, \ldots, n\}$".

Solution: Let $P(n)$ be the above statement "if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where each a_i is an integer, then $p \mid a_i$ for some integer $i \in \{1, 2, \ldots, n\}$".

Basis step: $P(1)$ is true, because if $p \mid a_1$ then obviously $p \mid a_1$ holds.

Inductive step: Assume $P(k)$ is true, i.e., the above statement is true when considering k a_is. Then, we show $P(k+1)$ is true by considering two cases.

- $p \mid a_1 a_2 \cdots a_k$: In this case, we know $p \mid a_i$ for some integer $i \in \{1, 2, \ldots, k\}$ by inductive hypothesis, so $P(k+1)$ is true.
- $p \nmid a_1 a_2 \cdots a_k$: In this case, we have $\gcd(p, a_1 a_2 \cdots a_k) = 1$ because p is prime and p is not a common divisor. Then, since $p \mid a_1 a_2 \cdots a_{k+1}$, we have $p \mid a_{k+1}$ and hence $P(k+1)$ is true.

By mathematical induction, we have $P(n)$ is true for all nonnegative integers n.

□

Q.4 (**10p**) Let $P(n)$ be the statement that postage of n cents can be formed using just 3-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 12$.

(a) (**2p**) Show statements $P(12)$, $P(13)$, $P(14)$ are true, completing the basis step of the proof.

(b) (**2p**) What is the inductive hypothesis of the proof?

(c) (**2p**) What do you need to prove in the inductive step?

(d) (**2p**) Complete the inductive step for $k + 1 \geq 15$.

(e) (**2p**) Explain why these steps show that this statement is true whenever $n \geq 12$.

Solution:

(a) $P(12)$ is true, because we can form postage of 12 cents with four 3-cent stamps. $P(13)$ is true, because we can form postage of 13 cents with two 3-cent stamps and one 7-cent stamp. $P(14)$ is true, because we can form postage of 14 cents with two 7-cent stamps.

(b) The inductive hypothesis is the statement that using just 3-cent and 7-cent stamps we can form postage of j cents for all $12 \leq j \leq k$, where we assume that $k \geq 14$.

(c) In the inductive step we must show, assuming the inductive hypothesis, that we can form $(k+1)$-cent postage using just 4-cent and 7-cent stamps.

(d) We want to form postage of $k+1$ cents. Since $k \geq 14$, we know that $P(k-2)$ is true, that is, we can form postage of $k-2$ cents. We can form $(k+1)$-cent postage by putting one more 3-cent stamp on the envelope, which concludes the inductive step.

(e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement $P(n)$ is true for every integer n greater than or equal to 12.

□

Q.5 (**6p**) Design a recursive algorithm for binary search (as described in Assignment 2, Q.14). Write out the pseudocode.

Solution:

Algorithm 1 BinarySearch (x: target integer, i, j: the smallest and largest indices searched in the n increasingly ordered numbers $a_1, a_2, \ldots, a_n$)

---

if $i > j$ **then return** 0
$m := \lfloor (i+j)/2 \rfloor$
if $x = a_m$ **then return** m
if $x < a_m$ **then**
  return BinarySearch$(x, i, m-1)$
else
  return BinarySearch$(x, m+1, j)$
{the returned value is the subscript of a_i such that $a_i = x$, or 0 if no such a_i is found}

□

Q.6 (**6p**) Prove that the number of divisions required by the Euclidean algorithm to find $\gcd(a, b)$, where $a \geq b > 0$, is $O(\log b)$. (Hint: prove that the remainders r_i satisfy $r_{i+2} < r_i/2$.)

Solution: Recall that in the Euclidean algorithm, we have $r_0 = a$, $r_1 = b$ and $r_i = r_{i+1}q_{i+1} + r_{i+2}$ for $i \geq 0$. We prove that $r_{i+2} < r_i/2$ by considering two cases: $r_{i+1} \leq r_i/2$ and $r_{i+1} > r_i/2$.

First, note that $0 < r_{i+1} < r_i$ holds for all $i \geq 0$ such that $r_{i+1} > 0$. If $r_{i+1} \leq r_i/2$, it is easy to see that $r_{i+2} < r_{i+1} \leq r_i/2$ holds. If $r_{i+1} > r_i/2$, we have $r_{i+2} = r_i - r_{i+1}q_{i+1} \leq r_i - r_{i+1} < r_i - r_i/2 = r_i/2$.

The above shows that it takes at most $2\lceil \log_2(b+1) \rceil$ divisions to complete the Euclidean algorithm to find $\gcd(a, b)$, i.e., to reach some $r_n = 0$. That is, the number of divisions is $O(\log b)$.

□

Q.7 (**8p**) Iterating the recurrence $T(n) = aT(n/2) + n$ to show that, for $1 \leq a < 2$ and $T(1) \geq 0$ we have $T(n) = \Theta(n)$. Please show your iteration steps.

Solution: By iterating the recurrence, we have:

$$\begin{aligned}
T(n) &= aT\left(\frac{n}{2}\right) + n \\
&= a\left[aT\left(\frac{n}{2^2}\right) + \frac{n}{2}\right] + n \\
&= a^2 T\left(\frac{n}{2^2}\right) + \left(\frac{a}{2}\right) n + n \\
&= a^2\left[aT\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right] + \left(\frac{a}{2}\right) n + n \\
&= a^3 T\left(\frac{n}{2^3}\right) + \left(\frac{a}{2}\right)^2 n + \frac{a}{2} n + n \\
&= \cdots \\
&= a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i \\
&= n^{\log_2 a} T(1) + n \frac{1 - \left(\frac{a}{2}\right)^{\log_2 n}}{1 - \frac{a}{2}} \\
&= \left(T(1) + \frac{2}{a-2}\right) n^{\log_2 a} + \frac{2}{2-a} n \\
&= \Theta(n),
\end{aligned}$$

where we are using the fact that for $1 \le a < 2$,

$$a^{\log_2 n} = (2^{\log_2 a})^{\log_2 n} = (2^{\log_2 n})^{\log_2 a} = n^{\log_2 a} = O(n).$$

Note that actually the above shows that $T(n) = \Theta(n)$ for any constant $T(1)$ and $0 < a < 2$. $\square$

Q.8 (**12p**) Consider a deck of 52 cards that consists of 4 suits each with one card of each of the 13 ranks. Answer the following questions using combination notations only, e.g., $\binom{12}{2}\binom{3}{1}\binom{42}{3}$.

(a) (**2p**) How many full houses? That is three cards of one rank and two of another rank.

(b) (**2p**) How many two pairs? That is two cards of one rank, two of another rank, and one of a third rank.

(c) (**2p**) How many flushes? That is five cards of the same suit.

(d) (**4p**) How many straights? That is five cards of sequential ranks. Note that a straight with an ace in it can only be "10JQKA" or "A2345" but not other cases like "JQKA2".

(e) (**2p**) How many quads? That is four cards of one rank and one of another rank.

Solution:

(a) $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$

(b) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$

(c) $\binom{4}{1}\binom{13}{5}$

(d) $\binom{10}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}$

(e) $\binom{13}{1}\binom{48}{1}$

$\square$

Q.9 (**5p**) How many bit strings of length 8 contain either 4 consecutive 0s or 4 consecutive 1s?

Solution: First, we count the number of bit strings of length 8 that contain four consecutive 0s. We will count it based on where the string of four or more consecutive 0s starts. If it starts from the first bit, then the first four bits are all 0s, but the last four bits are arbitrary, therefore there are $2^4 = 16$ such strings. If it starts from the second bit, then the first bit must be a 1, the next four bits are all 0s, and the last three bits are arbitrary; therefore there are $2^3 = 8$ such strings. If it starts in the third bit, then the second bit must be a 1 but the first bit and the last two bits are arbitrary, therefore there are $2^3 = 8$ such strings. Similarly, there are 8 such strings that have consecutive 0s starting from each of fourth and fifth bits. This gives us a total of $16 + 4 \cdot 8 = 48$ strings that contain four consecutive 0s. Symmetrically, there are 48 strings that contain four consecutive 1s.

Clearly there are exactly two strings that contain both (00001111 and 11110000) and hence are counted twice. By the inclusion-exclusion principle, there are $48 + 48 - 2 = 94$ strings that contain 4 consecutive 0s or 4 consecutive 1s. Since "either... or..." excludes the above two strings, we have the answer is $94 - 2 = 92$.

$\square$

Q.10 (**8p**) Prove that the following binomial coefficient is divisible by 2022.

$$\binom{2020}{1010}$$

(Hint: first note that $2022 = 2 \cdot 1011$ and recall what we learned from number theory to decompose the problem into two subproblems, then use the fact that for all $0 \leq k \leq n$ the combinations $\binom{n}{k}$ are integers.)

Solution: First, note that $\gcd(2, 1011) = 1$, so it suffices to prove that $2 \mid \binom{2020}{1010}$ and $1011 \mid \binom{2020}{1010}$. That is, for two coprime integers a, b, if $a \mid c$ and $b \mid c$, then $ab \mid c$. The proof is not hard. By definition, we know $c = ak$ for some integer k. Then, from $b \mid ak$ and $\gcd(b, a) = 1$, one can conclude that $b \mid k$ and therefore $ab \mid c$.

Next, we prove $2 \mid \binom{2020}{1010}$ and $1011 \mid \binom{2020}{1010}$ in the general form, i.e.,

$$2 \Big| \binom{2n}{n} \text{ and } (n+1) \Big| \binom{2n}{n}.$$

Since

$$\begin{aligned}
\frac{1}{2} \cdot \binom{2n}{n} &= \frac{1}{2} \cdot \frac{(2n)!}{n!n!} \\
&= \frac{1}{2} \cdot \frac{2n \cdot (2n-1)!}{n!n!} \\
&= \frac{(2n-1)!}{(n-1)!n!} \\
&= \binom{2n-1}{n}
\end{aligned}$$

is an integer, we have 2 divides $\binom{2n}{n}$.

Since

$$\begin{aligned}\frac{1}{n+1}\binom{2n}{n} &= \frac{(2n)!}{(n+1)!n!} \\ &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\ &= \binom{2n}{n} - \binom{2n}{n-1}\end{aligned}$$

is an integer, we have $n+1$ divides $\binom{2n}{n}$. $\square$

Q.11 (**5p**) Prove the hockey-stick identity.

$$\sum_{k=0}^{r}\binom{n+k}{k} = \binom{n+r+1}{r} \text{ where } n, r \text{ are positive integers}$$

Use a combinatorial argument and do not use Pascal's identity.

Solution: There could be many ways to form a combinatorial proof. For instance, consider $\binom{n+r+1}{r}$ as counting the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternatively, suppose that the j-th bit is the rightmost 1 bit, so we have $n+1 \le j \le n+r+1$. Once we have determined where the last 1 is, we know the last $n+r+1-j$ bits must be all 0s. We are only left to determine where to place the remaining 0s in the $j-1$ spaces before the rightmost 1 bit (at the j-th position). Note that there are n 1s and $(j-1-n)$ 0s in the first $j-1$ bits. By the sum rule, there are $\sum_{j=n+1}^{n+r+1}\binom{j-1}{j-1-n} = \sum_{k=0}^{r}\binom{n+k}{k}$ ways. $\square$

Q.12 (**10p**) Solve the recurrence relation $a_n = 3a_{n-2} + 2a_{n-3}$, $n \ge 3$, with initial conditions $a_0 = 1$, $a_1 = -5$ and $a_2 = 0$.

Solution: The characteristic equation is

$$r^3 - 3r - 2 = (r+1)^2(r-2).$$

The roots are $r = -1$ with multiplicity 2 and $r = 2$ with multiplicity 1. Hence, the solutions to this recurrence are of the form

$$a_n = (\alpha_1 n + \alpha_2)(-1)^n + \alpha_3 2^n.$$

To find the constants α_1, α_2 and α_3, we use the initial conditions. Plugging in $n = 0, 1, 2$, we have

$$\begin{aligned} a_0 = 1 &= \alpha_2 + \alpha_3, \\ a_1 = -5 &= -\alpha_1 - \alpha_2 + 2\alpha_3, \\ a_2 = 0 &= 2\alpha_1 + \alpha_2 + 4\alpha_3. \end{aligned}$$

We then have $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_3 = -1$. Hence,

$$a_n = (n+2)(-1)^n - 2^n.$$

$\square$

Q.13 (**10p**) Solve nonhomogenenous recurrence relations.

(a) (**8p**) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + n^2$.

(b) (**2p**) Find the solution of the recurrence relation in part (a) with the initial condition $a_1 = 2$.

Solution:

(a) The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next, we need a particular solution to the given recurrence relation. For this we guess a function of the form $a_n = p_2 n^2 + p_1 n + p_0$. We plug this into the recurrence relation and obtain $p_2 n^2 + p_1 n + p_0 = 2(p_2(n-1)^2 + p_1(n-1) + p_0) + n^2$. We rewrite this by grouping terms with equal powers of n, obtaining $(-p_2 - 1)n^2 + (4p_2 - p_1)n + (-2p_2 + 2p_1 - p_0) = 0$. In order for this equation to be true for all n, we must have $p_2 = -1$, $4p_2 = p_1$, and $-2p_2 + 2p_1 - p_0 = 0$. This tells us that $p_1 = -4$ and $p_0 = -6$. Therefore the particular solution we seek is $a_n^{(p)} = -n^2 - 4n - 6$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n - n^2 - 4n - 6$.

(b) We plug the initial condition into our solution from part (a) to obtain $a_1 = 2 = 2\alpha - 1 - 4 - 6$. This tells us that $\alpha = 13/2 = 6.5$. So the solution is $a_n = 6.5 \cdot 2^n - n^2 - 4n - 6$.

$\square$

Q.14 (**10p**) Use generating functions to solve the recurrence relation $a_n = 4a_{n-1} + 8^{n-1}$ with the initial condition $a_0 = 0$.

Solution: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $\{a_n\}$. By multiplying x^n to both sides of the recurrence relation, we have

$$a_n x^n = 4a_{n-1}x^n + 8^{n-1}x^n.$$

We sum both sides starting from $n = 1$ and get

$$\begin{aligned} G(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} 4a_{n-1}x^n + \sum_{n=1}^{\infty} 8^{n-1}x^n \\ &= 4x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x \sum_{n=1}^{\infty} 8^{n-1}x^{n-1} \\ &= 4xG(x) + x \cdot \frac{1}{1-8x}. \end{aligned}$$

Solving for $G(x)$ and plugging in $a_0 = 0$, we have

$$G(x) = \frac{1}{1-4x}(a_0 + \frac{x}{1-8x}) = \frac{x}{(1-4x)(1-8x)}.$$

Next, we aim to expand the right-hand side of the above equation into two partial fractions, i.e., finding α_1 and α_2 such that

$$G(x) = \alpha_1 \cdot \frac{1}{1-4x} + \alpha_2 \cdot \frac{1}{1-8x} = \frac{\alpha_1 + \alpha_2 - (8\alpha_1 + 4\alpha_2)x}{(1-4x)(1-8x)}.$$

This can be translated into solving the equations: $\alpha_1 + \alpha_2 = 0$ and $8\alpha_1 + 4\alpha_2 = -1$, which yields $\alpha_1 = -1/4$ and $\alpha_2 = 1/4$.

Therefore, we have

$$\begin{aligned} G(x) &= -\frac{1}{4}\cdot\frac{1}{1-4x}+\frac{1}{4}\cdot\frac{1}{1-8x} \\ &= -\frac{1}{4}\sum_{n=0}^{\infty}4^n x^n+\frac{1}{4}\sum_{n=0}^{\infty}8^n x^n \\ &= \sum_{n=0}^{\infty}(-\frac{1}{4}\cdot 4^n+\frac{1}{4}\cdot 8^n)x^n, \end{aligned}$$

and hence $a_n = -\frac{1}{4}\cdot 4^n + \frac{1}{4}\cdot 8^n$.

$\square$