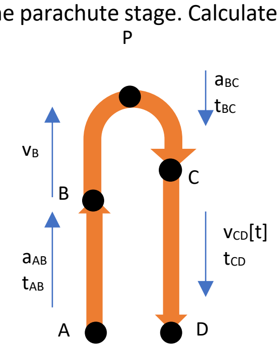
Eric Li September 29, 2018 Section H

Uber Problem – Hamster Huey and Calculus Cam

Problem Description

One calm afternoon Calculus Cam decides to launch Hamster Huey into the air using a model rocket. The rocket is launched straight up off the ground, from rest. The rocket engine is designed to burn for specified time while producing non-constant net acceleration given by the equations below. After the engine stops the rocket continues upward in free-fall. A parachute opens after the rocket falls a specified vertical distance from its maximum height. When the parachute opens, assume the rocket instantly stops, and then increases speed to a terminal velocity given by the equation below. Assume the air resistance affects the rocket only during the parachute stage. Calculate the total time the rocket is in the air



The problem is divided into three parts, described with points A, B, C, D, and P:

A: Point at which rocket starts

B: Point at which engine cuts out

C: Point at which parachute deploys

D: Point at which rocket hits the ground

P: Point at which rocket reaches max height

Givens:

Acceleration of rocket while engine burns	$a_{AB}[t] = -t^2 + 23$	m/s²
Engine burn time	$t_{AB} = 5.2$	sec
Vertical distance rocket falls from max height before parachute opens	$h = 140$	m
Speed of rocket with parachute	$v_{CD}[t] = -16(1 - e^{-t/11})$	m/s
Initial launch velocity of rocket	$v_A = 0$	m/s
Initial position of rocket	$y_A = 0$	m

Method and Strategy

Part AB:

Step 1: Find v_B

Given the acceleration as a function of time: $a_{AB}[t] = -t^2 + 23$, $v_{AB}[t]$ can be solved for by taking the indefinite integral of $a_{AB}[t]$.

$$v_{AB}[t] = \int (-t^2 + 23)dt$$

$$v_{AB}[t] = -\tfrac{1}{3}t^3 + 23t + C$$

Since the rocket is launched from rest ($v_A = 0$ m/s), $v_{AB}[0] = 0$. If 0 is plugged in for t, C evaluates to 0. Therefore,

$$v_{AB}[t] = -\tfrac{1}{3}t^3 + 23t$$

With that equation and with the given engine burn time of 5.2 seconds, v_B can be solved for by plugging in 5.2 in for time into the equation above.

$$v_B = -\tfrac{1}{3}(5.2)^3 + 23(5.2)$$

$$\underline{v_B = 72.731\ m/s}$$

Step 2: Find y_B

To find y_B, the $y_{AB}[t]$ equation is needed. That can be done by taking the indefinite integral of the $v_{AB}[t]$ equation derived earlier.

$$y_{AB}[t] = \int \left(-\tfrac{1}{3}t^3 + 23t\right) dt$$

$$y_{AB}[t] = -\tfrac{1}{12}t^4 + \tfrac{23}{2}t^2 + C$$

Similarly, because the rocket starts from the ground, $y_{AB}[0] = 0$. If 0 is plugged in for t, C evaluates to 0 once again. Thus,

$$y_{AB}[t] = -\tfrac{1}{12}t^4 + \tfrac{23}{2}t^2$$

Again, 5.2 is plugged in for t to find y_B.

$$y_B = -\tfrac{1}{12}(5.2)^4 + \tfrac{23}{2}(5.2)^2$$

$$\underline{y_B = 250.03\ m}$$

Part BC:

Step 1: Find t_{BP}

To do this, the first thing that needs to be solved for is the time it takes for the rocket, now a projectile, to reach the maximum height, which can be done using the kinematic equation below.

$$v_f = v_0 + at$$

$$v_P = v_B + a_{BP}t_{BP}$$

The only acceleration in this portion is caused by gravity ($a_{BC} = a_{BP} = -9.8\ m/s^2$) and it is known that $v_B = 72.731\ m/s$. At max height, the velocity is 0 m/s. Plugging those values into that kinematic equation,

$$0 = 72.731 - 9.8t_{BP}$$

$$\underline{t_{BP} = 7.4215\ s}$$

Step 2: Find y_P and y_C

With t_{BP}, y_P can be found by using the following kinematic equation.

$$y_f = \tfrac{1}{2}at^2 + v_0t + y_0$$

$$y_P = \tfrac{1}{2}a_{BP}{t_{BP}}^2 + v_Bt_{BP} + y_B$$

Plug in the values already known.

$$y_P = \tfrac{1}{2}(-9.8)(7.4215)^2 + (72.731)(7.4215) + (250.03)$$

$$\underline{y_P = 519.92\ m}$$

Given that the rocket falls 140 m from max height before deploying the parachute,

$$y_C = y_P - 140$$

$$y_C = 519.92 - 140$$

$$\underline{y_C = 379.92\ m}$$

Step 3: Find t_{BC}

Now that y_C is known, the kinematic equation used in Step 2 can be applied to find t_{BC}:

$$y_C = \tfrac{1}{2}a_{BC}{t_{BC}}^2 + v_Bt_{BC} + y_B$$

$$379.92 = \tfrac{1}{2}(-9.8){t_{BC}}^2 + (72.731)t_{BC} + 250.03$$

$$0 = -4.9{t_{BC}}^2 + 72.731t_{BC} - 129.89$$

$$t_{BC} = 2.0764\ s \text{ or } t_{BC} = 12.767\ s$$

Since it took 7.4215 seconds for the rocket to reach max height, t_{BC} must be greater than 7.4215. Therefore,

$$\underline{y_{BC} = 12.767\ s}$$

Part CD:

Step 1: Find y_{CD} as a function of time ($y_{CD}[t]$)

For this part, the equation for speed as a function of time is given as:

$$v_{CD}[t] = -16(1 - e^{-\frac{t}{11}})$$

Taking the indefinite integral of $v_{CD}[t]$ will get y_{CD} as a function of time.

$$y_{CD}[t] = \int \left(-16(1 - e^{-\frac{t}{11}})\right) dt$$

$$y_{CD}[t] = -16 \int \left(1 - e^{-\frac{t}{11}}\right) dt$$

$$y_{CD}[t] = -16\left(\int (1)dt - \int \left(e^{-t/11}\right)dt\right)$$
$$y_{CD}[t] = -16\left(t - \left(-11e^{-t/11}\right)\right) + C$$
$$y_{CD}[t] = -16t - 176e^{-\frac{t}{11}} + C$$

The parachute is deployed at $y_C = 379.92\ m$, so:

$$y_{CD}[0] = 379.92$$
$$379.92 = -16(0) - 176e^{-\frac{(0)}{11}} + C$$
$$379.92 = C - 176$$
$$C = 555.92$$

Therefore, the official equation for y_{CD} as a function of time is:

$$y_{CD}[t] = -16t - 176e^{-\frac{t}{11}} + 555.92$$

Step 2: Find t_{CD}

Since the rocket lands where it was launched ($y_D = 0\ m$),

$$0 = -16t_{CD} - 176e^{-\frac{t_{CD}}{11}} + 555.92$$
$$176e^{-\frac{t_{CD}}{11}} = 16t_{CD} + 555.92$$
$$e^{-\frac{t_{CD}}{11}} = \frac{16t_{CD} + 555.92}{176}$$
$$\ln\left(\frac{16t_{CD} + 555.92}{176}\right) = -\frac{t_{CD}}{11}$$
$$\ln(16t_{CD} + 555.92) - \ln(176) = -\frac{t_{CD}}{11}$$
$$\underline{t_{CD} = 34.257\ s}$$

Final Part:

At this point, t_{AB}, t_{BC}, and t_{CD} are all known. All that's left to do is to add them up to find the total time the rocket is in the air.

$$t_{total} = t_{AB} + t_{BC} + t_{CD}$$
$$t_{total} = 5.2 + 12.767 + 34.257$$
$$\boxed{t_{total} = 52.22\ s}$$