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Finding an Alternative to the 12-Tone Equal Temperament

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Abstract

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Since the standardization of the 12-tone equal temperament (12-TET) just over 200 years ago, many musicians and mathematicians have criticized the system extensively, emphasizing the impurity of its consonant intervals and the lack of expressive freedom it allows. This study analyzes existing tone systems and describes attempts to design new tone systems in order to find a tone system better than the 12-TET. First, two of the most renowned 12-tone well temperaments, Werckmeister III and Vallotti, were identified as potential candidates. Then, the n-tone equal temperaments with $9 \leq n \leq 23$ were tested for consonant interval accuracies, and the 19-TET and 22-TET displayed potential. The 19-TET was then used as the basis for constructing four non-equal temperaments. The purity of the consonant intervals of the previously mentioned tone systems were compared to those of the 12-TET, and all four 19-tone non-equal temperaments have purer major 6ths and minor 3rds for all keys and three out of those four systems have purer major 3rds and minor 6ths for all keys. The two 12-tone non-equal temperaments have certain purer consonant intervals and certain less pure intervals compared to the 12-TET.

Literature Review

Eric Li

Introduction

Music is an essential part of society. Its impact is apparent in the fine arts; people like Mozart and Beethoven are household names that even non-musicians recognize. Music is, in a sense, everywhere. However, there is a fundamental problem with music that most people do not realize, and it lies in the imperfection of music tuning and tuning systems. Though there are many problems associated with music tuning, especially in ensembles, the majority of them can be fixed through the training and practice of each musician. However, no matter how much time a musician puts into perfecting intonation, there is one problem that cannot be fixed.

This is a problem that originates from a combination of music theory, mathematics, and physics. No tuning system can sound perfect. Because of this issue, instruments like the piano cannot sound as good as they need to. Orchestras and choirs cannot sound as good as they should. Rock bands cannot sound as good, either. Rather, musicians have to resort to a compromise to sound as good as they can, but they cannot mathematically sound perfect. This problem has tormented mathematicians, musicians, and physicists alike for nearly two thousand years, and still, the current tuning system is merely a compromise that, while having many strengths, is lacking in many ways.

Basic Overview of Mathematical Theory in Music

Sound

Music is comprised of sounds. A sound is comprised of sine waves, and their properties determine the various characteristics of a sound. For instance, the frequency of a soundwave denotes the pitch of the sound, the amplitude of a soundwave determines the volume of the sound, the frequency spectrum of the soundwave determines the timbre (type of sound) of the sound, etc.[1] As it applies to the topic at hand, the frequency (the pitch of a sound) is most important. Frequency is measured in hertz, the SI derived unit for frequency. Sound is a complex, and important component of music, but one singular sound is not music; it is the combination of sounds that make up music.

Intervals and Consonance

The distance or space between two sounds is called an interval. An interval that sounds "good" is referred to as consonant while an interval that sounds "bad" is referred to as dissonant[9]. While the degree of pleasantness of an interval may seem completely subjective and perhaps even random, it is very closely related to the ratio of the frequencies of the two sounds: the simpler this ratio is, the more consonant the interval sounds[9]. For ease of explanation, the ratio will be regarded as a fraction. The simplicity of a ratio, for this purpose, is determinant on how small the numerator and denominator of the most simplified version of the fraction are[9]. The reasoning behind this definition is rather simple; it lies behind the fact that once again, sound is a sine wave. Two sine waves whose frequencies are related by simple ratios conflict less when they oscillate[3]. For instance, observe the ratio 2:1. This is the simplest a ratio can get by definition (technically, 1:1 is the simplest, but 1:1 denotes the interval between a pitch and itself). If two sine waves with this frequency ratio were plotted on a graph, one wave would oscillate exactly one more time than the other wave during one period and the two sine functions would overlap every period (refer to figure 1).

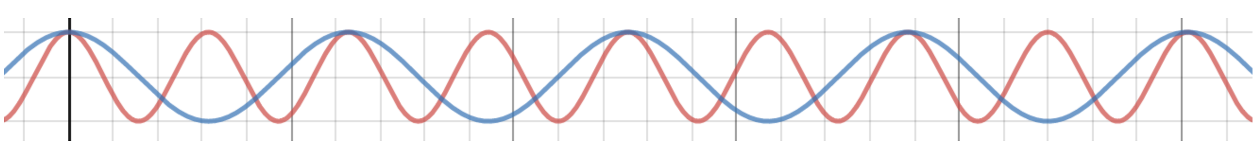

Figure 1. The graph of two sine functions with a frequency ratio of 2:1. The sine wave with the higher frequency (in red) oscillates exactly twice as fast as the sine wave with the lower frequency (in blue). The two functions align in peak every period (two periods if measured from higher frequency wave).

The interval formed by this 2:1 ratio is called an octave, and it is so consonant in music that the pitches sound identical, except one is higher than the other[9]. The consonance of this ratio led to the idea of octave equivalence, where sounds that differ by an octave are referred to by the same pitch name[1]. Now, take the second-most simple ratio: 3:2. The interval formed by this ratio is called a perfect fifth, and is regarded as one of the most important intervals in music due to its consonance. However, a ratio formed by two relatively large prime numbers such as 487/311 between the frequencies would result in very dissonant intervals.

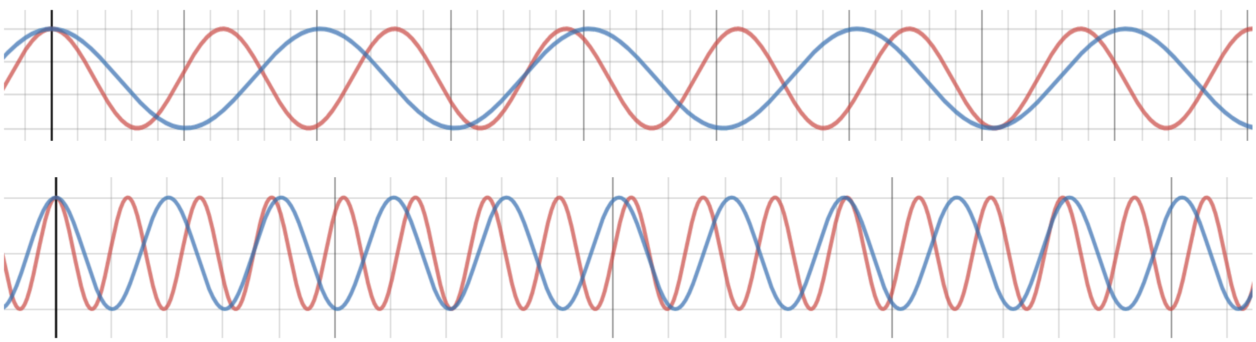

Figure 2. The graphs above each have two sine functions with a frequency ratio of 487:311. The higher frequency sine wave (in red) conflicts often with the lower frequency sine wave (in blue), rarely aligning at any peak (once every 311 periods of the blue function). The top graph is a 'zoomed-in' version of the graph below.

These two soundwaves conflict very often, and as a result, the interval formed by this ratio is very dissonant. The exception to this "rule" is if the ratio is very close to a consonant ratio[9]. For example, the ratio 797/397 is complex by the definition stated previously, but it is so close to 2/1 that to the human ear, there is no difference and the dissonance becomes negligible.

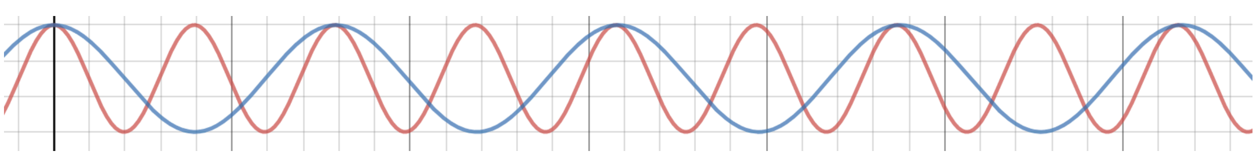

Figure 3. The graph of two sine functions with a frequency ratio of 797:397. Compare this with Figure 1. The difference is barely noticeable even when graphed. This difference is indistinguishable by the human ear.

Tuning, Tone systems, and Temperaments

However, two sounds are still not enough to make music. To make music, a series of sounds over broad ranges is needed. An instrument cannot produce every single frequency accurately simply because there is an infinite number of tones (or frequencies). Moreover, it would be impractical for the musician to know how to play every possible tone, even if the instrument allows it. To make music practical for performance, there must be a standard set of tones that is known and understood by all composers and musicians. This standard set of tones is known as a tone system[3]. With a tone system in place, music can actually be created and performed. As demonstrated previously, music theory and consonance is built upon ratios. But ratios cannot exist without a base pitch, not to mention that the base pitch must be universally agreed upon. Otherwise, every tone system would be based on different pitches, and it would become very difficult for composers, musicians, and instrument-makers to create and share music. To accommodate for all this, mathematicians came up with the frequency 440 hz and agreed that every tone system would include the pitch at 440 Hz[3], a pitch they called A.

Using the concept of octave equivalence, a tone system would be considered completely defined if given a starting tone t_0 where the subscript denotes the octave, the system reaches t_1 (which is an octave above t_0) after some n number of tones[1,2,3,4,9]. This definition ensures that the system loops around every n tones to the same pitch name, which removes the possibility of an infinite string of tones that never reach its starting tone (assuming octave equivalency). The danger of an incomplete or unclosed system, while not apparent at first, is that it would generate an infinite sequence of distinct tones. Like previously mentioned, an infinite sequence of distinct tones would make the system inapplicable to composition and performance because it is not possible for a composer or musician to learn, utilize, and perform an infinite series of tones[6,8].

Therefore, a tone system must loop back to the starting tone on a higher octave after some *n* number of tones so that when it reaches that tone, it would repeat and define the same tones that had already been established, limiting the composer/musician to only those *n* tones. For instance, the current tone system has 12 distinct tones: A-A#-B-C-C#-D-D#-E-F-F#-G-G# and back to A (western music theory designates letters to each pitch, and the # symbol indicates a higher sound than the base letter, but lower than the next base letter). Because the system goes back to A, it will then go through the same series of letters again and again, ensuring that the composer/musician does not have to learn any more than those 12 tones.

The creation of a tone system is difficult because a tone system must incorporate a set of tones that create as many consonant interval combinations as possible. This high quantity of consonant intervals is to allow composers the freedom to use better sounding intervals in their melodies/compositions and in turn allow better sounding performances[1,8]. However, there cannot be too many tones in the system, since that would lead to significant difficulty in performance. Also, the tones should be groupable into sets called keys, which are used as basis for composition. A common method many mathematicians use to create tone systems is to compound a consonant interval ratio onto a starting tone until it reaches the same tone eventually (by octave equivalency of course). This method is an elaborate way to ensure that the tone system closes as well as include at least one consonant interval. Unfortunately, the consonant intervals do not work well mathematically. For instance, take the two most consonant intervals: the octave (2/1) and the perfect fifth (3/2). Theoretically, if the system would close, the following equation should be satisfied by some ordered pair (m, n) where m and n are positive integers greater than 0.

$$\left(\frac{2}{1}\right)^{m} = \left(\frac{3}{2}\right)^{n} \tag{1}$$

However, there is no (m, n) pair that satisfies that equation because the numerator and denominator of both terms will always be only divisible by the original numerator/denominator, thus making it impossible for the two terms to ever be identical, regardless of the m and n values. This is the case for every single consonant interval due to the way a consonant interval is defined: ratios formed by small prime numbers. Therefore, the following equation applies where p_1, p_2, are relatively prime integers, p_3, and p_4 are relatively prime integers, and $\frac{p_1}{p_2}$ is not equal to $\frac{p_3}{p_4}$:

$$\left(\frac{p_1}{p_2}\right)^m = \left(\frac{p_3}{p_4}\right)^n \tag{2}$$

That equation cannot be satisfied with any m or n value. To avoid this problem, some ratios must be altered. The most common approach to every tone system is to keep the octave (2/1) and the perfect fifth (3/2) as pure as possible. When m = 7 and n = 12, $\left(\frac{2}{1}\right)^7$ evaluates to 128 and $\left(\frac{3}{2}\right)^{12}$ is approximately 129.746. The ratio of 129.746 to 128, approximately 1.014, is known as the Pythagorean comma[1,3,9]. A system cannot be left like this because, by definition, the system is not closed. One way to fix this issue is to lower all of the fifths equally by 1/12th of the Pythagorean comma to reach 128. Each of the fifths would then have a ratio of $2^{\frac{7}{12}}: 1$, which happens to be the basis of the current 12-tone equal temperament[1,3,4,9].

There are a multitude of existing temperaments. Many mathematicians have come up with their own tuning systems, but because it is mathematically impossible to create a perfect system, each existing temperament has its own strengths and weaknesses. Therefore, tuning continues to be an issue, as nobody agrees unanimously on which temperament is the best.

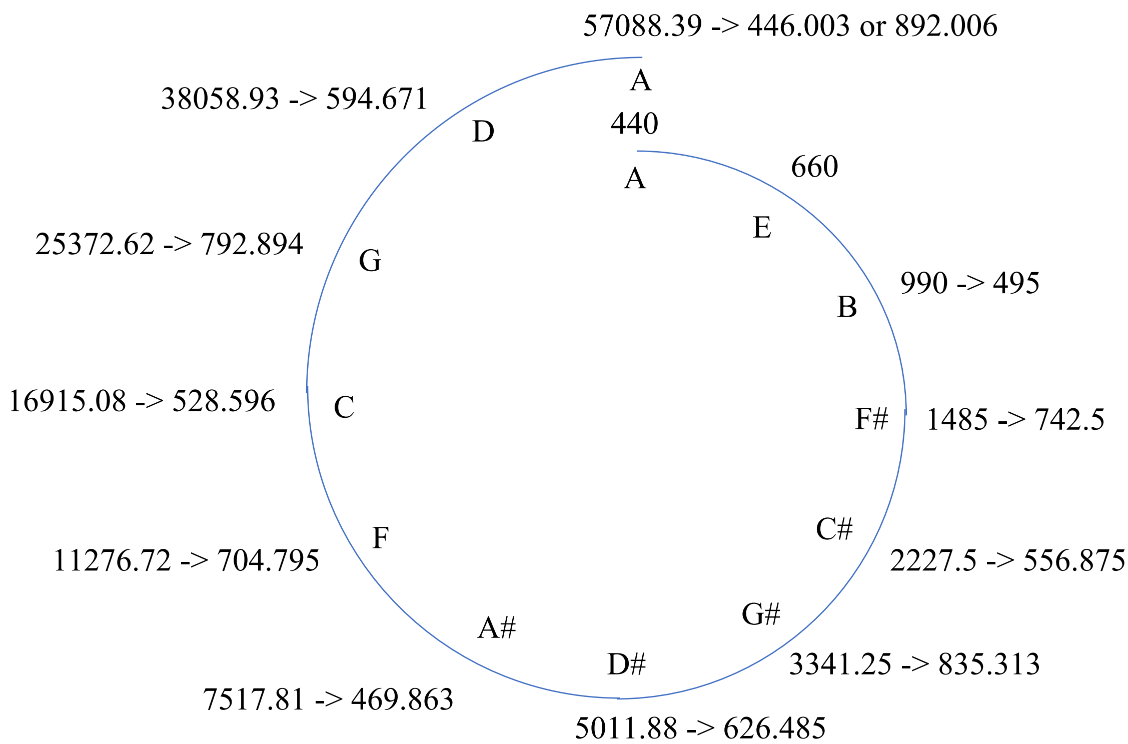

Figure 4. The spiral of pure perfect fifths. Notice how the system does not close visually and when each pitch is translated down to make the pitch frequency lie in the octave between 440 hz and 880 hz, the final A is 446.003 instead of 440.

Pythagoras, Pythagorean Tuning, and Just Intonation

Pythagoras was a mathematician renowned for his work in geometry and algebra. He is famous for the Pythagorean theorem, which is used all around the world today[9]. He was fascinated with numbers. He thought that everything in the universe could be described with numbers. In turn, he thought music could be described with numbers. Pythagoras demonstrated this with a lyre, an ancient instrument that resembles the modern-day harp, by comparing the sounds made by strings of different lengths. After experimenting with various ratios, he discovered that strings of the same lengths made the same pitches. He also realized that strings with different lengths often sounded dissonant when played together, but some sounded less dissonant than others. This led him to believe that the ratios between the lengths of the strings (which is inversely proportional to the frequency of the sounds produced by them) were directly related to how well the two strings sound together.

Pythagoras' interest in music eventually led him to develop the first tone system to ever exist, fittingly known as Pythagorean Tuning. In this system, Pythagoras attempted to build a tone system based on pure perfect fifths[1,3,4,9]. Tone systems that are built off of pure intervals are known as just intonation[1,4,9]. However, this system cannot be closed as shown in equation 1, so Pythagoras altered the last fifth only and attributed the whole Pythagorean comma to that interval[1,3,4,9]. Just intonation has some strengths and some weaknesses. One strength is that it keeps the rest of the fifths completely pure, so 11 out of 12 fifths are pure. However, that one altered fifth, known as the wolf fifth, is practically unusable due to how out-of-tune it is. Another weakness of this tone system is that many of the major and minor thirds (ratios of 5/4 and 6/5 respectively) are significantly out of tune, meaning they are not accurate relative to the pure interval ratios, since the use of pure fifths increases the dissonance of the thirds[1,3,4,9]. Despite its various weaknesses, the Pythagorean temperament is still a largely viable tone system and it laid the foundation for the development of future temperaments.

The Rise of Meantone Temperaments and Circulating Temperaments

The Pythagorean system worked for many centuries, but in the 11th century, the emergence of polyphonic music forced mathematicians and musicians to come up with an alternative[1]. Polyphony is the use of multiple voices or melodies in a piece of music. Prior to this, music was sung by choirs in unison, meaning everyone sang the same pitches in the same order at the same times. Polyphonic music revealed the problem with the Pythagorean tuning because it called for the combination of distinct pitches, and occasionally, the voices form major or minor thirds or even worse, a wolf fifth. This problem had to be solved, and it was remedied by the engineering of another type of tone system called a temperament. A temperament is a tone system based off pure perfect fifths like Pythagorean tuning, except it compromises some of those pure fifths to alleviate certain problems[1,9]. The first type of temperament that came about was the meantone temperament, where little fractions of the Pythagorean comma are split amongst some or occasionally all of the fifths[1,3,4,9]. This adjustment makes the thirds sound better, at the cost of the perfect fifths, and it also makes the wolf fifth sound slightly better. However, the wolf fifth was still an issue because most of the Pythagorean comma was still given to this one wolf fifth[1,4,9]. This issue gave rise to circulating temperaments. In a circulating temperament, the Pythagorean comma is unevenly distributed amongst often all of the fifths. This method completely removes the problem of the wolf fifth, but now, most of the fifths are no longer pure. They are all narrower than the 3:2 ratio. However, both the major and minor thirds sound even better than they did in meantone temperaments[3,9].

Equal Temperament and the Current System

Meantone temperaments and circulating temperaments were used extensively during the golden ages of music, spanning from the medieval period, through the Baroque era, and into the classical period. However, during the latter half of the classical period, a new temperament became the system of choice: the equal temperament. Briefly mentioned earlier, the equal temperament is a tone system where the Pythagorean comma is split evenly amongst all of the fifths[1,3,4,9]. This system removes the wolf fifth completely. Though all of the fifths are compromised, they are all tempered equally. This consistency makes the impurity of the fifths seem less noticeable[1,7,9]. The thirds are also more out of tune than many circulating and meantone temperaments, but the consistency of the equal temperament alleviates the issue considerably[1,7,9].

However, as nice as the equal temperament seems, it also has its weaknesses. The equal distribution of the Pythagorean comma makes every fifth sound the same, and in turn makes every key sound identical. This method restrains one of the most important aspects of music, emotional expression, because the key of a piece of music no longer impacts the emotional effect of that piece, as it sounds identical to the piece composed in any other key[1,4,5,7,9]. In fact, this is one of the primary reasons why equal temperament did not become prominent until the mid-to-late 18th century. Mathematicians and musicians knew that equal temperament would solve many of the problems associated with meantone and circulating temperaments, but the restraint it puts onto key character made them reluctant to switch. Even now, some people criticize equal temperament, asserting that it has ruined harmony[1,2,3,9].

Conclusion

Music tuning is a complex subject. The fundamental problem with tuning and tone systems is one that is without solution. The compromises to this issue, as seen through meantone, circulating, and equal temperaments, each have their strengths and weaknesses, and there is no current consensus on which system is truly ideal. The standard system used today, equal temperament, is disparaged by a number of musicians who believe different temperaments to be ideal[1,5,6]. The need for a solution to this problem is dire because even when all other issues with tuning can be solved through advanced technology and experience/practice, this problem will remain a limiting factor, as there is no perfect system[3]. Thus, a tone system that indisputably exceeds all of the current temperaments must be discovered or created.

Plan

Researchable Question

Is there a useable tone system that surpasses the 12-tone equal temperament in terms of consonant interval purity, key character, or both?

Hypothesis

There is at least one useable tone system that surpasses the 12-tone equal temperament in terms of consonant interval purity, key character, or both.

Methodology

First, before anything else, a method to determine success is needed. For consonant interval purity, the interval in the system of interest closest to the ideal ratio of a known consonant interval is compared to that ideal ratio and if the error was less than the 12-TET, it was considered purer. Key character, the other major metric by which a temperament system is judged, is difficult to quantify objectively; it is typically treated as a subjective topic. However, there are several ways to standardize key character analysis without the need for human testing. One such method is to measure the number of distinct intervals in the tone system. The more distinct intervals there are, the more key character there is. This approach is founded upon the definition of key character, which states that depending on the starting tone, a specific interval built off of it will sound different compared to the same interval built on different starting tone. The more distinct intervals a system has, the more differences there would be between different keys, thus leading to increased key character. This method may, however, run into difficulties when comparing key color between 12-tone systems and 19-tone systems since 19-tone systems inherently has more distinct intervals than 12-tone systems. This is simply due to the fact that the 19-tone system has more tones. To accommodate for this, the number of distinct intervals of an *n*-tone system is compared to the number of distinct intervals in the *n*-tone equal temperament. Table 2 illustrates the number of distinct intervals in each tone system as well as the comparison between each system and its relative equal temperament.

There are many potential directions to take in discovering a tone system that contains purer consonant intervals than the 12-TET or incorporates some level of key character. If a system were to satisfy the second criteria, one thing is clear: the temperament must not be equal-tempered. However, this condition does little to narrow down the list of potential temperaments to explore due to the infinite number of possible non-equal temperaments. Moreover, a system does not have to satisfy both criteria for it to be considered better than the 12-TET. Thus, to further narrow down the list of potential temperaments, a limit on the number of tones in the system is placed so that the number of tones has to lie between 9 tones and 23 tones. The reason for this is that a tone system with any number of tones below 9 would be too simplistic for modern music and would end up limiting musical expression while a tone system with any number of tones above 23 would be too difficult for performance and impractical for real-world application.

To look for promising systems with 9-23 tones, equal temperaments in that tone range were tested for the purity of its consonant intervals. Equal temperaments are not only potential competition for the 12-TET in consonant interval purity, but they also serve as good approximations of other types of temperaments with the same number of tones. In other words, the purity of consonant intervals of an *n*-tone equal temperament is usually representative of the purity of consonant intervals of any *n*-tone temperament. The purity of consonant intervals was tested by comparing the ratio of the consonant intervals in each system to the ideal ratio of that specific consonant interval and using the difference, in cents, to gauge its purity. Through such experimentation, the 19-tone equal temperament and 22-tone equal temperament were the only two systems to display any potential against the 12-tone equal temperament (12-TET), with both systems exceeding the 12-TET in the purity of all of their consonant intervals besides the perfect 5^{th}/perfect 4^{th}.

Due to the lack of equal temperament systems that surpassed the 12-TET, it is logical to consider potential nonequal 12-tone temperaments as well as nonequal temperaments with 19 or 22 tones. There are two main types of nonequal temperaments, namely circulating and meantone temperaments. However, because meantone temperaments have wolf intervals, they are omitted from consideration. From the collection of hundreds of circulating temperaments, two temperaments were chosen based on their popularity before the 12-TET became standard: the Werckmeister III and Vallotti.

As for 19-tone systems and 22-tone systems, an approach must be formulated to design non-equal temperaments that surpass the 12-TET. First, an approach for 19-tone systems is devised. Due to the impurity of the perfect 5ths of the 19-TET, the classic circle of fifths cannot be used effectively as a basis for non-equal 19-tone systems due to the excessively large comma that would result. However, 19-tone systems do have incredibly pure major 6ths, with an error of only 0.1482 cents. Thus, a circle of major 6ths (which will be referred to as simply a circle of 6ths from here forth) can be formed for 19-tone systems, resulting in a comma of only about 2.81547 cents.

A comma of 2.815547 is almost negligible, suggesting that a Pythagorean approach may work out amazingly well. After observing the 19 distinct intervals formed by pure major 6ths, the comma was attributed completely to the 18th interval in the circle, making the minor third of the tonal center pure. The resulting system, a 19-tone just intonation system (Prototype 1, P1), has some key character and preserves the purity of the 19-TET consonant intervals. However, it can be predicted that since only one major sixth was altered out of 19, the number of distinct intervals would not be too much higher than that of the 19-TET, therefore suggesting that P1 would have little key color.

To allow more key character, higher interval variation must be introduced. There are two different methods to approach this. One is to focus only on the consonant intervals with respect to the tonal center while the other is to assume a gradient of importance of keys, with the keys closer to the tonal center being more important than the keys further away from the tonal center (concentric tuning). If the system based solely on the tonal center, every consonant interval of the tonal center can be made completely pure by calculating the comma of the major 6ths with respect to the consonant interval and distributing it equally across the intervals before it. For instance, the perfect 4^{th} is the sixth interval on the circle of 6ths and the first consonant interval of interest on the circle. This means that a perfect 4^{th} is comprised of six major 6ths. Six pure major 6ths make an interval of about 1.3396 instead of 4/3, resulting in a small comma. This comma is divided equally amongst those six intervals so that the perfect 4^{th} is pure. A similar procedure can be applied to the next consonant interval of interest on the circle of 6ths, except that the already altered intervals have to be considered. The resulting system is described as follows (check Appendix B for specific calculations):

$$P2 = \begin{cases} \frac{(1.66537)^1}{2^0} \\ \frac{(1.66537)^2}{2^1} \\ \frac{(1.66537)^3}{2^2} \\ \frac{(1.66537)^4}{2^2} \\ \frac{(1.66537)^5}{2^3} \\ \frac{(1.66537)^6}{2^4} \\ \frac{(1.66537)^6(1.66851)^1}{2^5} \\ \frac{(1.66537)^6(1.66851)^2}{2^5} \\ \frac{(1.66537)^6(1.66851)^3}{2^6} \\ \frac{(1.66537)^6(1.66851)^4}{2^7} \\ \frac{(1.66537)^6(1.66851)^5}{2^8} \\ \frac{(1.66537)^6(1.66851)^6}{2^8} \\ \frac{(1.66537)^6(1.66851)^7}{2^9} \\ \frac{(1.66537)^7(1.66851)^7}{2^{10}} \\ \frac{(1.66537)^8(1.66851)^7}{2^{11}} \\ \frac{(1.66537)^9(1.66851)^7}{2^{11}} \\ \frac{(1.66537)^{10}(1.66851)^7}{2^{12}} \\ \frac{(1.66537)^{11}(1.66851)^7}{2^{13}} \\ \frac{(1.66537)^{12}(1.66851)^7}{2^{14}} \end{cases} \tag{3}$$

If concentric tuning is assumed, a more difficult derivation is needed. To accomplish this, a novel approach, first detailed by Dr. Farup from the Norwegian University of Science and Technology in 2014, is used. Instead of the starting with preset intervals or an objective metric of consonance, the temperament will be designed based on a set of musical requirements. Using these requirements, a parametric temperament can be derived by refining the requirements into equations that will eliminate every degree of freedom until there is only one left. This approach will be covered in detail in Appendix C. The general system described using this approach is in terms of a variable arbitrarily referred to as *a*, which can be any value between -1.35121 and 0.148187 cents.

$$P3\ and\ P4 = \begin{cases} a \\ a \\ a \\ a \\ a \\ a \\ a \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ a \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ a \end{cases} \tag{4}$$

Results and Discussion

Table 1. Purity of Consonant Intervals in Cents

Purity of Consonant Intervals in Cents							
	Perfect 5th	Perfect 4th	Major 3rd	Minor 3rd	Major 6th	Minor 6th	Average Abs. Val.
12-TET	-1.955	+1.955	+13.6863	-15.6413	+15.6413	-13.6863	10.4275
9-TET	-35.2883	+35.2883	+13.6863	-48.9746	+48.9746	-13.6863	32.6497
10-TET	+18.045	-18.045	-26.3137	-75.6413	+75.6413	+26.3137	40.000
11-TET	-47.4095	+47.4095	+50.0499	+11.6314	-11.6314	-50.0499	36.3636
13-TET	+36.5065	-36.5065	-17.0829	-38.7182	+38.7182	+17.0829	30.7692
14-TET	-16.2407	+16.2407	+42.2577	+27.2159	-27.2159	-42.2577	28.5714
15-TET	+18.045	-18.045	+13.6863	+4.35871	-4.35871	-13.6863	12.03
16-TET	-26.955	+26.955	-11.3137	-15.6413	+15.6413	+11.3137	17.97
17-TET	+3.92735	-3.92735	-33.3725	-33.2883	+33.2883	+33.3725	23.5277
18-TET	+31.3783	-31.3783	+13.6863	+17.692	-17.692	-13.6863	20.9198
19-TET	-7.21816	+7.21816	-7.36625	+0.148187	-0.148187	+7.36625	4.9109
20-TET	+18.045	-18.045	-26.3137	-15.6413	+15.6413	+26.3137	20.00
21-TET	-16.2407	+16.2407	+13.6863	+27.2159	-27.2159	-13.6863	19.0476
22-TET	+7.13591	-7.13591	-4.49553	+11.6314	-11.6314	+4.49553	7.7543
23-TET	-23.6941	+23.6941	-21.0963	-2.59781	+2.59781	+21.0963	15.7961

Over the course of this study, a total of seven tone systems, namely the Werckmeister III, Vallotti, 19-TET, and four 19-tone engineered systems, were analyzed and compared to the 12-TET. The 22-TET, though originally promising, was omitted from the final analysis because the 22-TET resembles the 19-TET in almost every way, except that the 22-TET is slightly worse in terms of consonant interval accuracy. Compounded with its high number of tones and nonexistent key character, the 22-TET was assumed to be inferior and as a result, not analyzed. An overview of the consonant interval purities of the seven systems are displayed in figure 5 and the key color of each system is summarized in table 2.

Table 2. A table that displays the total number of distinct intervals in each analyzed tone system and the ratio of that number to the number of distinct intervals of the corresponding equal temperament system, as a way of measuring how much key color each system is displaying. The number of distinct intervals ignore the inverses of intervals.

System	Number of distinct intervals	Ratio of number of distinct intervals to number of base distinct intervals

12-TET (base for 12-tone systems)	6	1.00
Werckmeister III	22	3.67
Vallotti	27	4.50
19-TET (base for 19-tone systems)	9	1.00
Prototype 1 (P1)	18	2.00
Prototype 2 (P2)	51	5.67
Prototype 3 (P3)	49	5.44
Prototype 4 (P4)	49	5.44

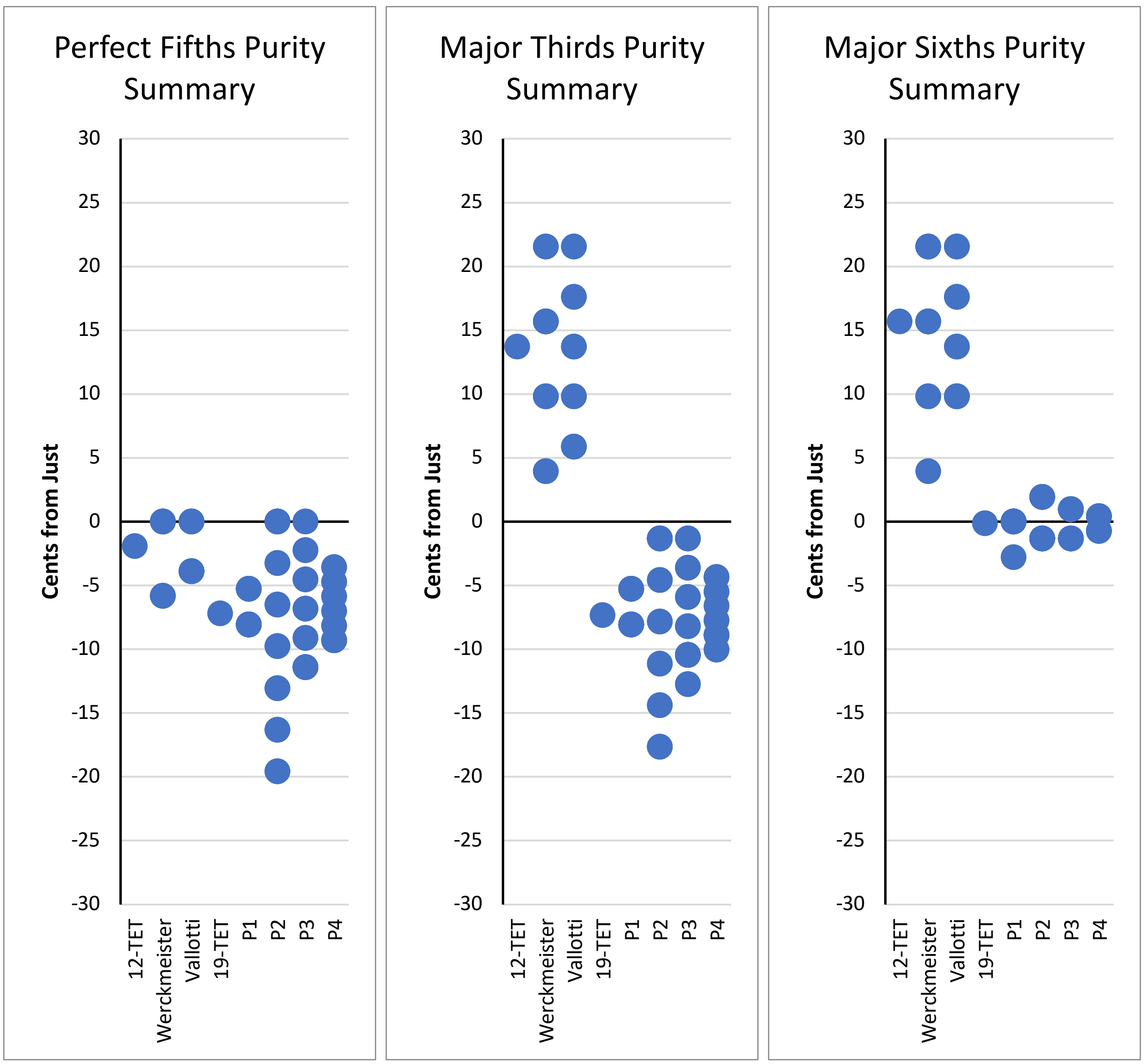

Figure 5. The three graphs above show the purities of every distinct version of the intervals perfect fifth (left), major sixth (middle), and major third (right) in each of the seven systems.

Appendix D contains a more comprehensive collection of all the data. Each tone system explored in this study will now be discussed in detail.

Werckmeister III

The Werckmeister III is a tone system proposed by Andreas Werckmeister in 1691 and is regarded as one of the best well temperaments in history. It is known as the "correct temperament" and is used prominently by musicians ever since the early 1700s. Werckmeister III's strength lies in its 8 pure perfect fifths (and perfect fourths) out of 12, the rest of which are tempered by 1/4th of the Pythagorean comma (only ~5.865 cents). Even though all of the fifths in the 12-TET are tempered only by 1/12th of the Pythagorean comma, the fact remains that none of the fifths are pure. Moreover, the fifths in 12-TET that are far away from an arbitrarily chosen tonal center are just as in-tune as the fifths close to that tonal center, which is often unnecessary since most of the fifths far away from the tonal center will never be used in a piece of music with on that tonal center. Werckmeister actually does not focus on the tonal center in the tempering of the fifths, but the number of pure fifths in the system essentially negates that. A similar situation applies to the major thirds and major sixths, and consequently minor sixths and minor thirds, respectively. Werckmeister applied the concept of concentric tuning, which he came up with himself, to the major thirds and major sixths of Werckmeister III, making the thirds and sixths closest to the tonal center off by only 3.91 cents for both. However, the major thirds and sixths furthest away from the tonal center are off by a maximum of 21.51 cents, which is extremely high. In the 12-TET, though, all of the major thirds and major sixths are off by 13.69 cents and 15.64 cents, respectively. That is already quite impure, and though none of the thirds and sixths are never off by as much as 21.51 cents, they are nevertheless very impure, even near the tonal center. In addition to this, not only does Werckmeister III contest the 12-TET in the purity of its consonant intervals, Werckmeister III also introduces a decent amount of key color. The system has a total of 22 distinct intervals (not counting inverses of intervals and octaves/unisons) compared to the 6 distinct intervals in the 12-TET, which by definition, displays no key character. Overall, Werckmeister III's pure fifths, concentric major thirds and major sixths, and key color makes Werckmeister III a better choice than 12-TET.

The Vallotti

The Vallotti temperament is a tone system devised by Francesco Vallotti in 1779 based on attributing 1/6th of the Pythagorean comma to 6 of the 12 perfect fifths. This system is known for its versatility and is one of the most utilized tone systems for classical and late baroque music. As suggested earlier, the Vallotti temperament has 6 pure fifths out of 12, which is still significantly more than the zero pure fifths in the 12-TET. The remaining fifths are only off by 3.91 cents, which is still somewhat pure. A similar argument to that of Werckmeister III can be made about the Vallotti temperament versus the 12-TET, with the major thirds and major sixths of the Vallotti temperament concentrically designed versus the consistently impure major thirds and major sixths of the 12-TET. For key color, the Vallotti temperament has a total of 27 distinct intervals, which is even more than Werckmeister III, indicating that it demonstrates extensive key character. Therefore, the Vallotti temperament is also a better choice than the 12-TET.

19-Tone Equal Temperament

The 19-TET is the only equal temperament analyzed in this study that could rival the 12-TET. Since the 19-TET is an equal temperament, it has no key color, so that can be disregarded for this tone system. The purity of its consonant intervals, however, is worth discussing. The major sixths of the 19-TET are only off by 0.1482 cents, which makes its major sixths incredibly pure (though not perfectly pure). The 12-TET's major sixths are off by 15.64 cents, which is significantly higher than 0.1482. The 19-TET's major thirds are off by 7.37 cents, which is quite purer than the 12-TET's major thirds, which are off by 13.69 cents. However, the 19-TET's perfect fifths is where it falls to the 12-TET. The fifths of the 19-TET are off by 7.22 cents compared to the fifths of the 12-TET, which are off by only 1.955 cents. The 19-TET's impure fifths present a problem since the perfect fifth is more consonant, and in turn arguably more important, than the major third and major sixth. Furthermore, the increased number of tones makes the 19-TET more difficult to learn and perform for musicians. Therefore, despite its purer major thirds and major sixths, the 19-TET can only be regarded as an equal to the 12-TET, if not worse.

Pythagorean 19-Tone System (Prototype 1 or P1)

Prototype 1 has 18 out of 19 pure major sixths, with the last sixth off by 2.82 cents; these values represent a significant improvement compared to the 12-TET's sixths. Moreover, prototype 1 has two versions of the major third, both of which are purer than those of the 12-TET. The first version is off by 5.29 cents and the other is off by 8.11 cents. However, just like the 19-TET, the two versions of the perfect fifths of prototype 1 are lacking compared to those of the 12-TET. Each version of the perfect fifth is off by the same amount as the major thirds (5.29 cents and 8.11 cents). At this point, prototype 1 seems to compare to the 12-TET just like the 19-TET, but there's one additional factor: key color. Prototype 1 has a total of 18 distinct intervals, which is higher than 19-TET's 9 distinct intervals. But even with this factor, prototype 1 is still not an improvement over the 12-TET because the key character displayed by prototype 1 is not that much and the problems of the 19-TET carry over here. Therefore, prototype 1 is also at most only an equal to the 12-TET.

19-Tone Circulating Temperament #1 (Prototype 2 or P2)

Prototype 2 is vastly different from the 19-TET and prototype 1. The thing that distinguishes prototype 1 from other 19-tone systems is that it has 7 pure perfect fifths out of 19. This alleviates one of the largest factors that held back the 19-TET and prototype 1. However, these 7 pure fifths come at a cost. The two fifths furthest away from the tonal center in prototype 2 are off by 19.59 cents, which is so impure for a perfect fifth to the degree that it is debatably unusable. As for the major sixths, all of them are tempered less than 2 cents, which is significantly better than those of the 12-TET. This is a common trait for all 19-tone systems. Now, for the major thirds, even though the three major thirds furthest from the tonal center are off by 17.678 cents, the 8 major thirds closest to the tonal center are only tempered by 1.351213 cents. Moreover, 14 out of the 19 major thirds of P2 are tempered less than 13.6863 cents, which is the amount each major third in the 12-TET are tempered. For key character, P2 displays a surprisingly large number of 51 distinct intervals compared to the 9 distinct intervals in 19-TET, suggesting it incorporates a lot of key color. Overall, P2's pure consonant intervals around the tonal center and extensive key color makes it better than the 12-TET. However, it is important to note that P2 has two (arguably four) wolf fifths and three wolf thirds, making it much less versatile than the 12-TET and systems like Werckmeister III and Vallotti, which do not have wolf intervals.

19-Tone Circulating Temperament #2 and #3 (Prototype 3 or P3 and Prototype 4 or P4)

Prototype 3 is a compromise between prototype 1 and 2 regarding the purity of its perfect fifths. Prototype 3 has 3 pure fifths, but the most impure fifth is no longer off by 19.59 cents. Rather, in prototype 3, this value is off by 11.43 cents. The error is still high, but useable. Plus, this prototype retains all the other advantages of the 19-TET and most of the advantages of P2, with major sixths off by no more than 1.5 cents. Even P3's most impure major third, which is off by 12.78 cents, is still purer than all of the thirds of the 12-TET. The key color presented in prototype 3 is also impressive, with a total of 49 distinct intervals compared to the 9 distinct intervals in 12-TET. Because of the purity of its consonant intervals compared to the 12-TET and the key color it displays, prototype 3 is better than the 12-TET.

Prototype 4 is a variation of prototype 3 that is more versatile than prototype 3, but at the cost of the pure perfect fifths. P4 has no perfect fifths, but its most impure fifth is only off by 9.323 cents, making it more versatile than P2 and P3. This tradeoff is seen in the major thirds as well. P4's purest major third (tempered by 4.359 cents) is less pure than P3's purest major third (tempered by 1.3512 cents) but P4's most impure major third (tempered by 10.073 cents) is purer than P3's most impure major third (tempered by 12.780 cents). This takes away from the keys close to the tonal center, but it adds to P4's versatility by making the keys far away from the tonal center more useable. Furthermore, all of its major sixths are tempered less than 1 cent. P4, since it is based off of the same parametric system as P3, also has 49 distinct intervals, which is quite high. However, its lack of pure perfect fifths, or any completely pure consonant interval, may render the music composed with this system unpleasant to the ear. Overall, due to its versatility and key character, P4 seems to be slightly better than the 12-TET.

Conclusion

Based on the data collected from seven different tone systems, five of them possess the qualities that allow them to surpass the 12-TET. Those five systems are the Werckmeister III, the Vallotti temperament, P2, P3, and P4. All of these systems naturally exhibit more key color than the 12-TET and they all have purer consonant intervals near the tonal center. Some of these systems even display purer consonant intervals than the 12-TET far away from the tonal center.

Therefore, to conclude, there are multiple potential alternatives to the 12-TET that are arguably superior to the 12-TET.

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Appendix

Appendix A: Limitations and Assumptions

- Time is a great limitation, as the derivation would often require multiple approaches or attempts to reach one reliable system.
- Calculation accuracy is another big limiting factor. The frequency differences between a pure interval and a classic impure interval is often very minute, but these small frequency differences are still audible to the human ear. To work with these numbers, it is crucial to keep the numbers as precise as possible. However, in the process of manual number manipulation, some precision is lost. Despite this, much of this source of error was mitigated by using excel.
- There is no objective way to measure key character. It is a very subjective concept and will be difficult to test for, even with human testing since there are few instruments and musicians adapted to 19-tone or 22-tone systems.

System	Assumptions
19-Tone Pythagorean Temperament (P1)	1. Comma is too small to greatly affect keys other than the tonal center itself. 2. Most important intervals are as followed (ranked as ordered): Perfect 5ths, Perfect 4ths, Major 6ths, Major 3rds, Minor 3rds, Minor 6ths 3. Key character will be noticeable to a listener

19-Tone Circulating Temperament #1 (P2)	1. No other key of importance other than the tonal center 2 (or 1a). Comma is too small to greatly affect keys other than the tonal center itself. 3. Key character will be noticeable to a listener.
19-Tone Circulating Temperament #2 and #3 (P3 + P4)	1. Tonal center is most important key. 2. The keys closer to the tonal center are more important than the keys further from the tonal center 3. Key character will be noticeable to a listener.

Appendix B: P2 Calculations

To make sure that there aren't too many distinct , the major thirds were ignored because it was already much purer than the 12-TET.

Let x be the ratio of the first six major sixths, y be the ratio of the next 7 major sixths, and z be the ratio of the final six major sixths.

$$\frac{4}{3} = x^6 \quad (5)$$
$$x \approx 1.66537$$

$$\left(\frac{4}{3}\right)(2^4)y^6 = (1.5)(2^9) \quad (6)$$
$$y \approx 1.66851$$

$$(1.5)(2^9)z^6 = (1)(2^{14}) \quad (7)$$
$$z \approx 1.66537$$

Appendix C: P3 + P4 Derivation

A 19-tone system is completely defined when the tempering of all 19 major sixths in the circle is known. This can be narrowed down to a definitive parametric solution by constructing equations based on a set of music requirements that restrict the system.

Requirements

Based on trends and patterns in historic temperaments as well as several noticed during the course of the study, a set of music requirements were constructed:

1. No perfect fifth is tempered wide and no major third is tempered wide
2. The perfect fifths will be tuned concentrically with respect to the tonal center
3. The tone system should be as symmetrical as possible with respect to the tonal center
4. The number of distinct intervals should not be too high and must be limited

The first criteria is done to ensure that other intervals do not need to be tempered excessively, including dissonant intervals. The second requirement provides an efficient way to include as many pure (or close to pure) consonant intervals and placing the very impure consonant intervals (sometimes wolf), which arise as a result of the pure consonant intervals, into rarely used or even unused keys. The third requirement is one that is formulated in agreement with many historical temperaments and scholars[1, 2,. Its purpose is to ensure that flat keys aren't favored over sharp keys or vice versa, given that the tonal center is C. The last criteria is placed in order to make sure that the system isn't so different one key to another that it sounds disjointed and not related at all. Even though more distinct intervals mean more key color, there is a point where there's too much. The 4th requirement is there to prevent that from being too much.

Constructing the Temperament

First, a circle of major sixths with 19 tones is constructed. Each tone, starting with an arbitrarily chosen tonal center, is labeled from $i = 0$ to $i = 18$ so that the tonal center is 0 and the tone representing the major sixth above that is 1 and so on. Next, let s(i) represent the tempering of the major sixth above tone i in cents. Let f(i) represent the tempering of the perfect fifth above tone i and let T(i) represent the tempering of the major third above tone i. According to the circle of major sixths, 19 major sixths must make up an octave. However, 19 major sixths make the ratio 1.0016:1 instead of 1, resulting in a comma of about 2.8156 cents. Using the definitions presented prior, this can be expressed as:

$$\sum_{i=0}^{18} s(i) = -2.8156 \tag{8}$$

Similarly, requirement 1 can then be rewritten as:

$$f(i) < 0 \ \forall i \tag{9}$$

$$T(i) < 0 \ \forall i \tag{10}$$

Originally, the focus was on the perfect fifths, due to the obvious importance of the interval. However, every attempt to construct a 19-tone system based on that was unsuccessful because 13 consecutive major sixths make up one perfect fifth, which made all of the requirements too strict and result in the 19-TET instead of a circulating temperament. To avoid this, the focus was shifted onto perfect fourths, which are made of only six consecutive major sixths. Despite the shift in focus, the perfect fifths are still taken into account since the perfect fourth is the inverse of the perfect fifth. Therefore, if the perfect fourths are concentrically tuned the tonal center, then the perfect fifths will be concentrically tuned to a different tonal center, which can then be transposed to be the true tonal center.

It is known that a perfect fourth is made up of six consecutive major sixths. However, six pure consecutive major sixth, forms a ratio of 1.3396, which is about 8.1073 cents higher than the ideal ratio of 4/3. Using this, the tempering of each perfect fourth can be written in terms of the tempering of major sixths:

$$g(i) = \sum_{j=i}^{i+5} s(j) - 8.1073 \tag{11}$$

The second criteria indicates that the perfect fifths must be tempered concentrically about the tonal center. It follows that the perfect fourths are also tempered concentrically, though with respect to a different tonal center. For simplicity, however, it will be assumed that the perfect fourths will be concentrically tuned to the tonal center $i = 0$. Like mentioned previously, the system can be transposed later to make the perfect fifths concentric to the tonal center. Since perfect fourths close to the tonal center are most important in a concentrically tuned temperament, the following equation can be constructed:

$$|g(0)| \leq |g(i)| \tag{12}$$

In order for this temperament to be parametric, the size of the g(0) is specified by a variable a. Due to the way this variable is used, the value of a will determine the purity of the perfect fourths (or fifths, depending on if the temperament was shifted or not) close to the tonal center. Also, to guarantee that none of the major sixths close to the tonal center are over-tempered, the tempering of g(0) is split equally between the first six major sixths. With that, the following equation can be constructed from equation (11) and equation (12):

$$s(0) = s(1) = s(2) = s(3) = s(4) = s(5) = a \tag{13}$$

Criteria 3 can be represented by equation 11:

$$g(i) = g(-i)\ \forall i \tag{14}$$

This can then be rewritten in terms of major sixths based on equation (11) and a symmetry in terms of major sixths can be derived with simple arithmetic:

$$s\left(\frac{5}{2}+\left(i+\frac{1}{2}\right)\right)=s\left(\frac{5}{2}-\left(i+\frac{1}{2}\right)\right) \tag{15}$$

This equation basically means that the major sixths must be symmetric with respect to the i = 5/2 axis in order to make the perfect fourths symmetric with respect to the tonal center (i = 0).

At this point, there are still too many degrees of freedom that there is no definitive solution to the equations already formulated. Therefore, other restrictions must be set. The relationship between major sixths and octaves and the relationship between major sixths and perfect fourths have already been exploited. It is most logical to therefore exploit the relationship between perfect fourths and octaves. There are multiple options for this relation, as shown in the following equations:

$$\sum_{j=0}^{4} g(i+6j) = -90.225 \tag{16}$$

$$\sum_{j=0}^{6} g(i+6j) = 113.685 \tag{17}$$

$$\sum_{j=0}^{11} g(i+6j) = 23.46 \tag{18}$$

An octave can be composed of either 5 major sixths, 7 major sixths, or 12 major sixths, each of them resulting in commas of different sizes. There are pros and cons to using any of these relations over the other two. A small comma is beneficial because an excessive comma would lead to unnecessarily excessive tempering of other intervals. However, equation (18), the relation with the only acceptable comma, states that 12 perfect fourths make up an octave, which would restrict too many intervals. Equation (16) presents the lowest number of intervals, but the comma is too high. Therefore, another relationship must be formed.

So far, the focus has been primarily on perfect fourths. Two other intervals, the major sixth and the major third, have not been factored in greatly. It is true that the major sixth was used to construct the circle that the temperament is based on, but the purity of any major sixth is not really restricted. All of the equations up till now can be satisfied with any size major sixths. Therefore, to place such a restriction, two things are done. First, a relationship can be exploited between perfect fourths and minor thirds, the inverse of major sixths. Three perfect fourths make a minor third with a comma of about 21.51 cents, which can be descried by the equation below:

$$\sum_{j=0}^{2} g(i+6j) = 21.51 \tag{19}$$

The relation presented in equation (19) can be rewritten as the following:

$$g(0) + g(6) + g(12) = 21.51$$

Referring back to criteria 2, the system must be concentrically oriented. Therefore, to minimize the tempering of g(0) and all major sixths in general, most of the comma of 21.51 should be split between g(6) and g(12). This relation can be written as follows:

$$g(0) \leq g(6) = g(12) \tag{20}$$

Now, there are just enough restrictions on the system that there are now only two degrees of freedom. The system, described by the equations derived from the requirements, can be expressed by this:

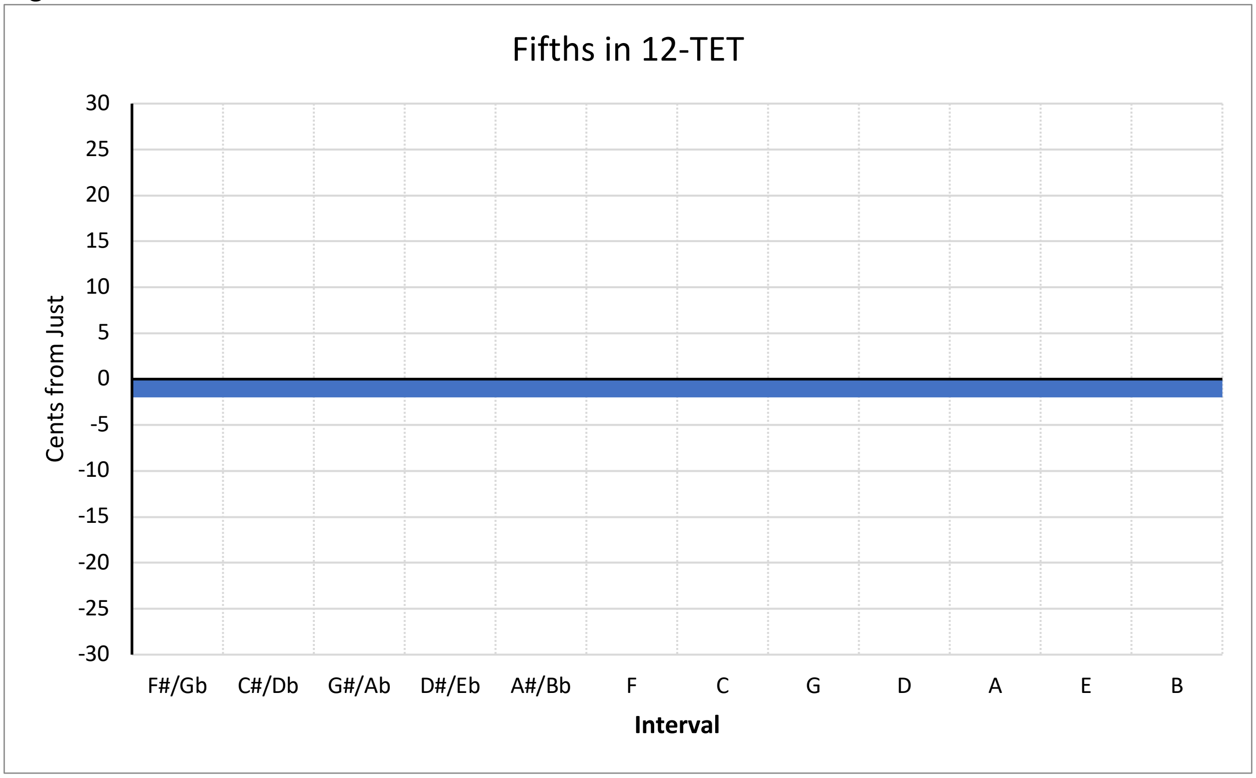
$$
P3\ and\ P4 = \begin{cases} s(0) = a \\ s(1) = a \\ s(2) = a \\ s(3) = a \\ s(4) = a \\ s(5) = a \\ s(6) = c \\ s(7) = b \\ s(8) = b \\ s(9) = b \\ s(10) = b \\ s(11) = b \\ s(12) = c \\ s(13) = b \\ s(14) = b \\ s(15) = b \\ s(16) = b \\ s(17) = b \\ s(18) = c \end{cases} \tag{21}
$$

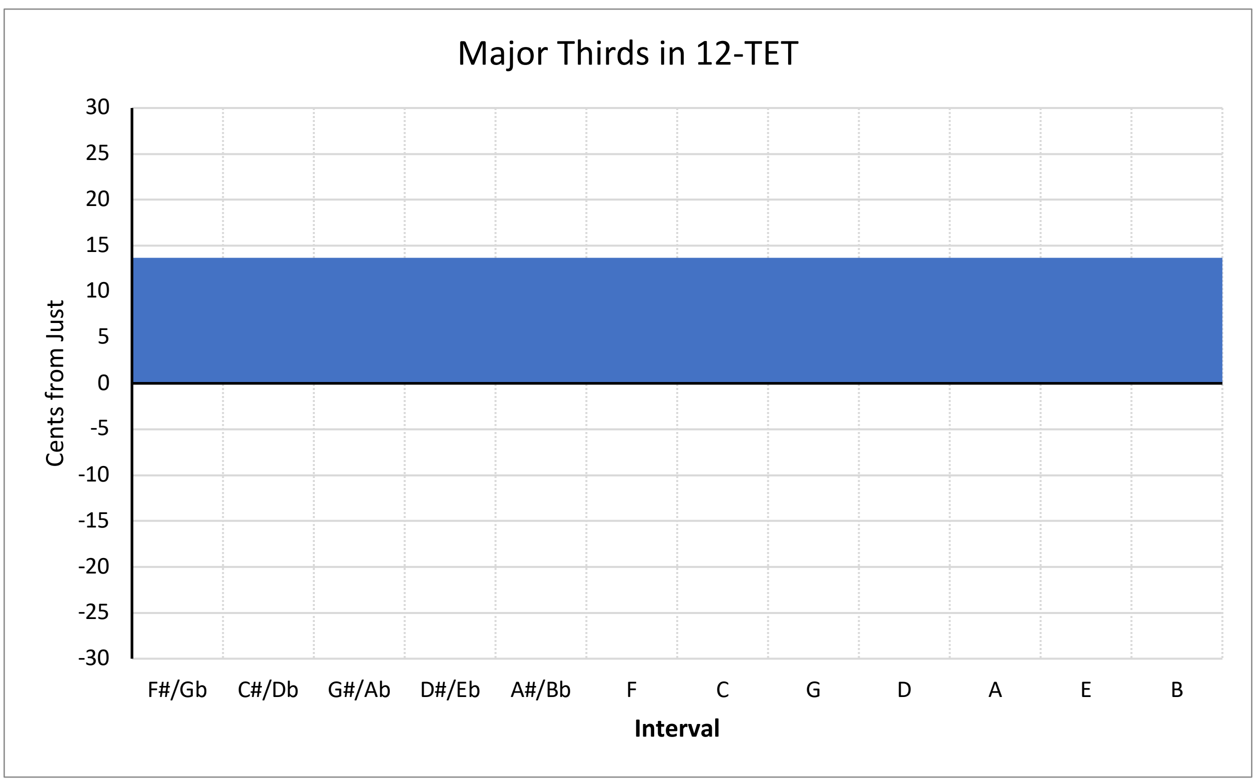
To remove that last degree of freedom, the fourth criteria can be used. The number of distinct intervals should not be too high, and to do this, either *b* or *c* needs to be equal to *a*. If *b* is set equal to *a*, this would mean that only 3 intervals are tempered differently than the 16 others, which would result in very low key color. To maintain the amount of key character in the system, *c* is set equal to *a*. To close the system, *b* would have to equal 0.9*a* -0.28155. Thus, the system can be expressed as this:

$$
P3\ and\ P4 = \begin{cases} a \\ a \\ a \\ a \\ a \\ a \\ a \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ a \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ -0.9a - 0.281555 \\ a \end{cases} \tag{22}
$$

Appendix D: Data

Figure 6-8. 12-TET Consonant Interval Purities





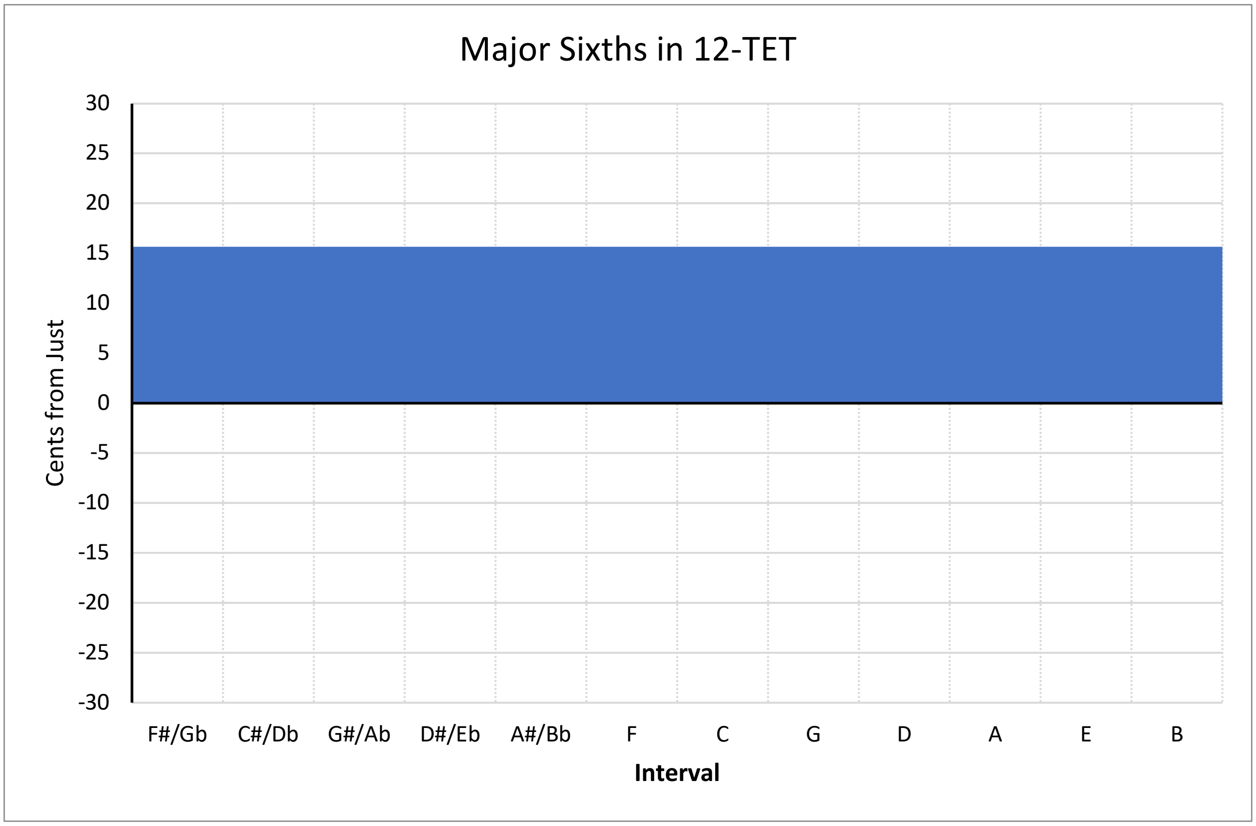
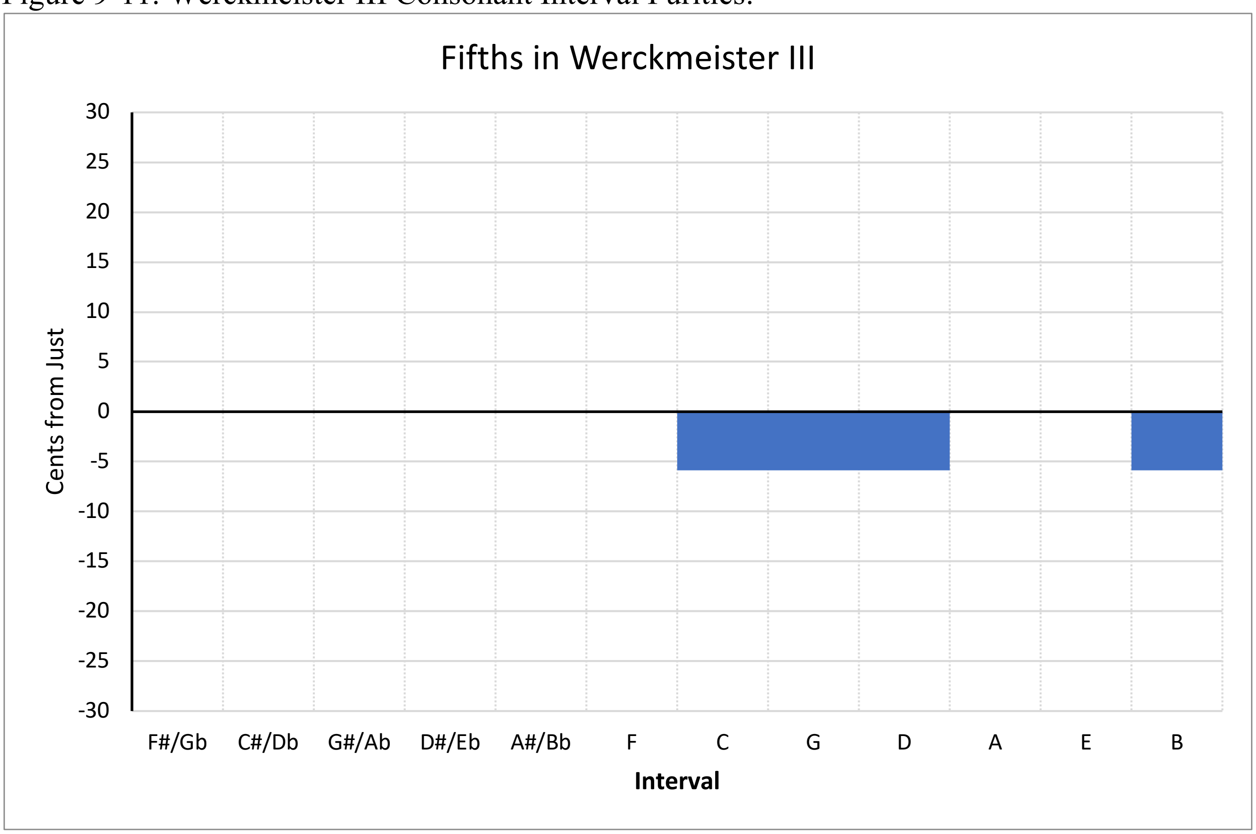
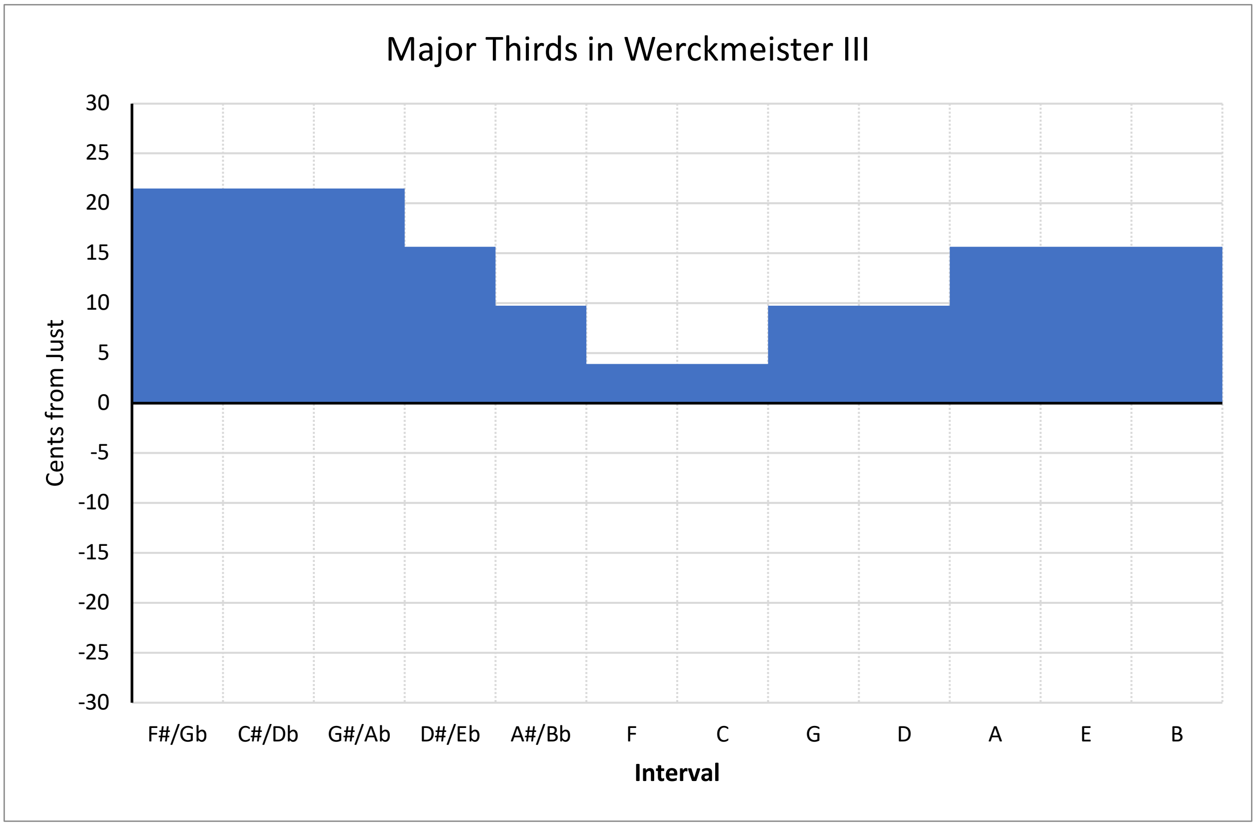


Figure 9-11. Werckmeister III Consonant Interval Purities:




Major Thirds in Werckmeister III
Cents from Just
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
F#/Gb
C#/Db
G#/Ab
D#/Eb
A#/Bb
F
C
G
D
A
E
B
Interval

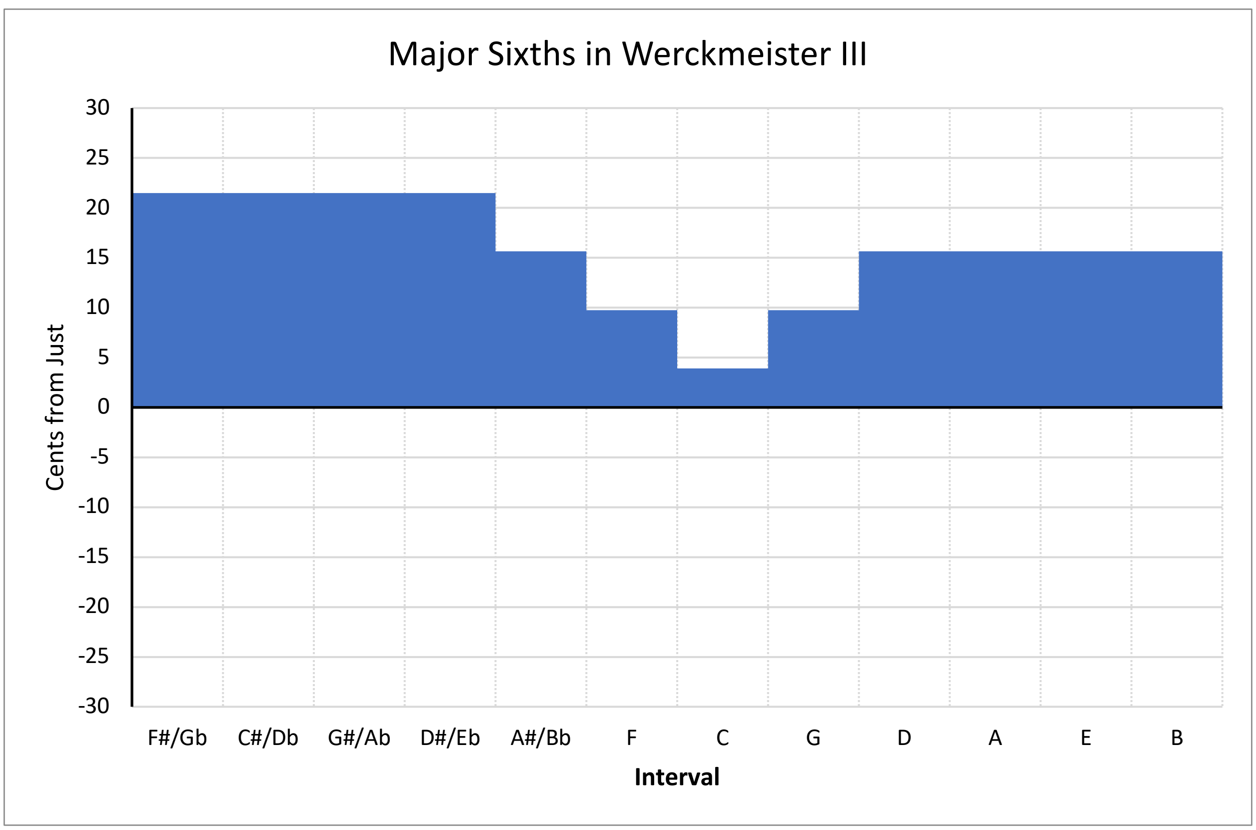
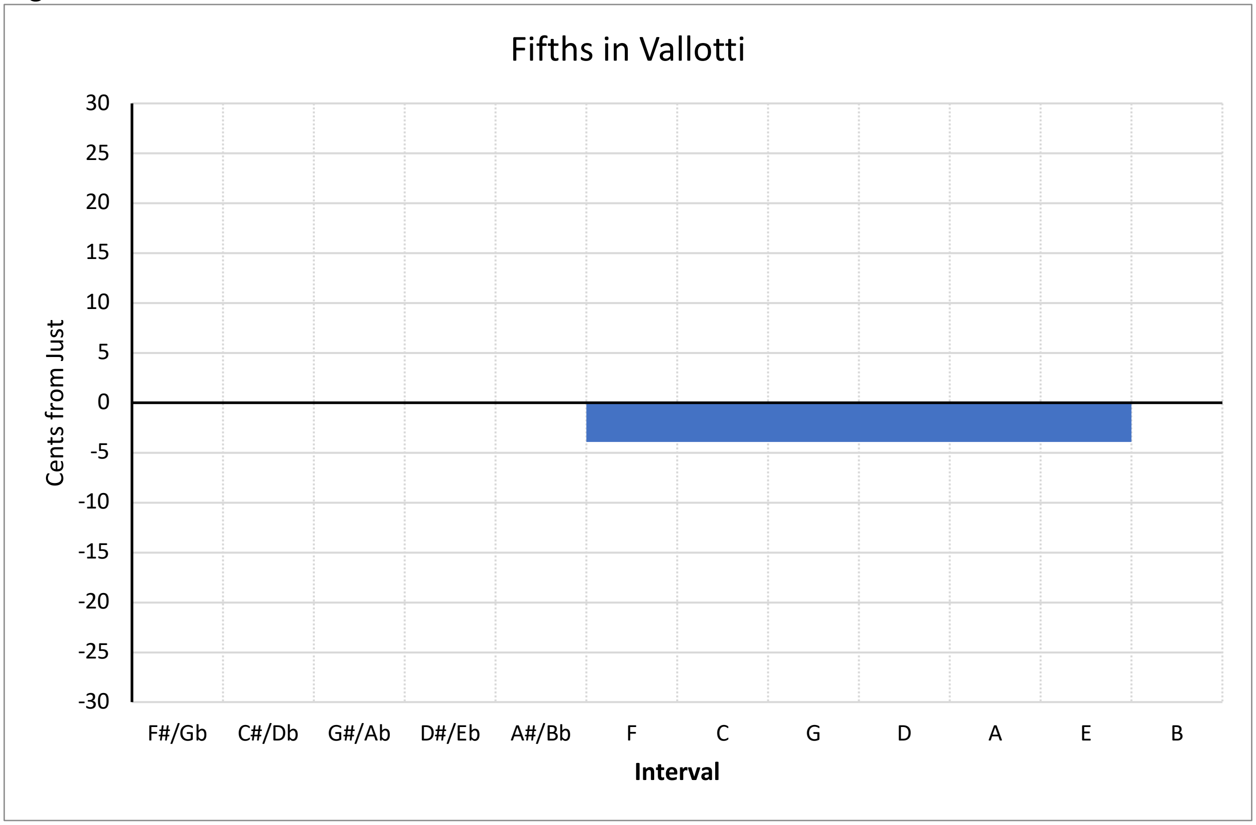
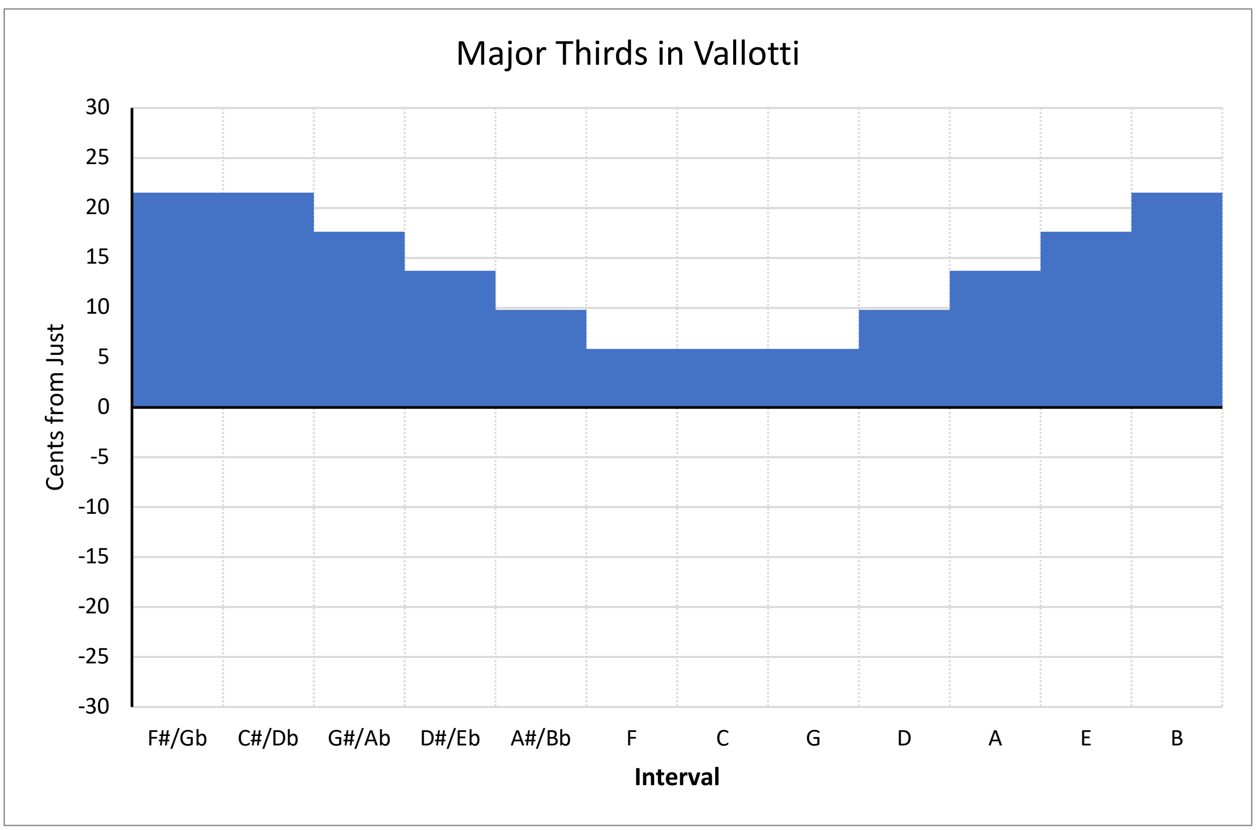

Major Sixths in Werckmeister III
Cents from Just
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
F#/Gb
C#/Db
G#/Ab
D#/Eb
A#/Bb
F
C
G
D
A
E
B
Interval

Figure 12-14. Vallotti Consonant Interval Purities:





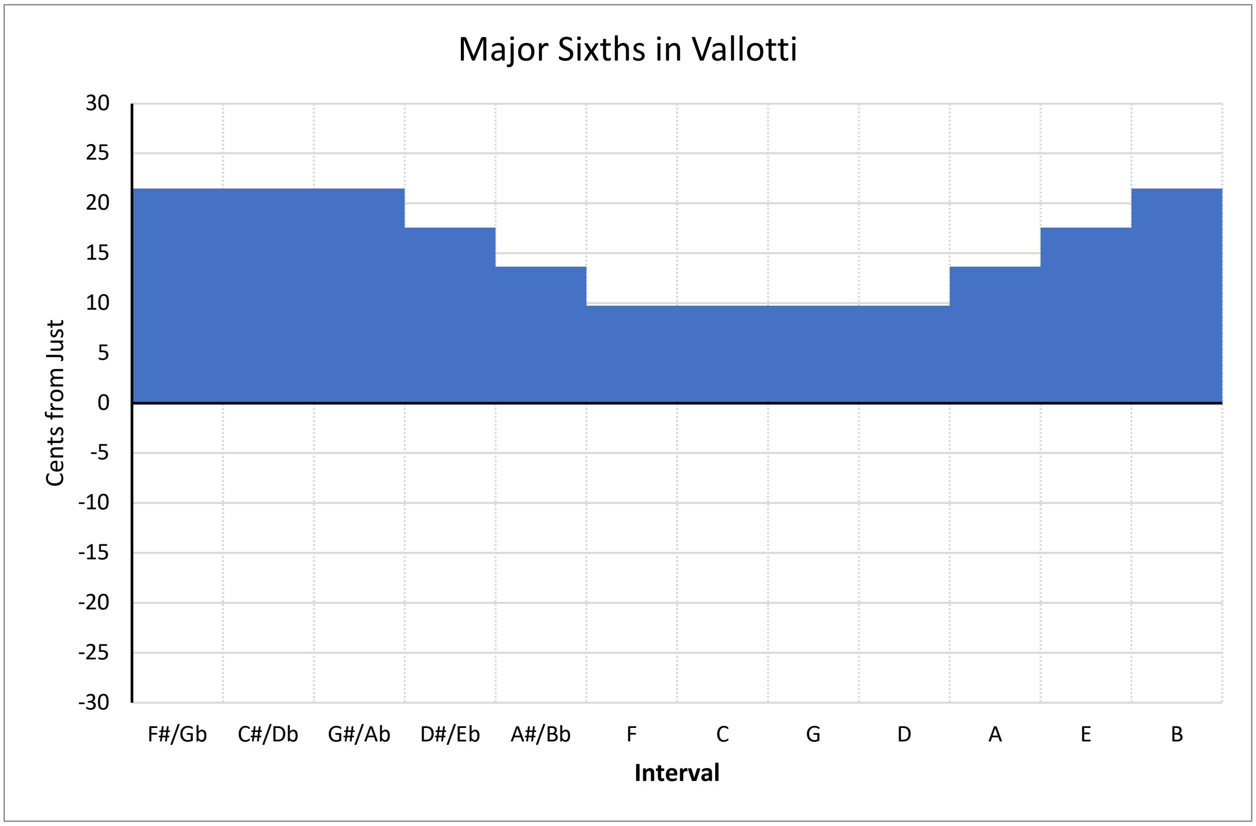
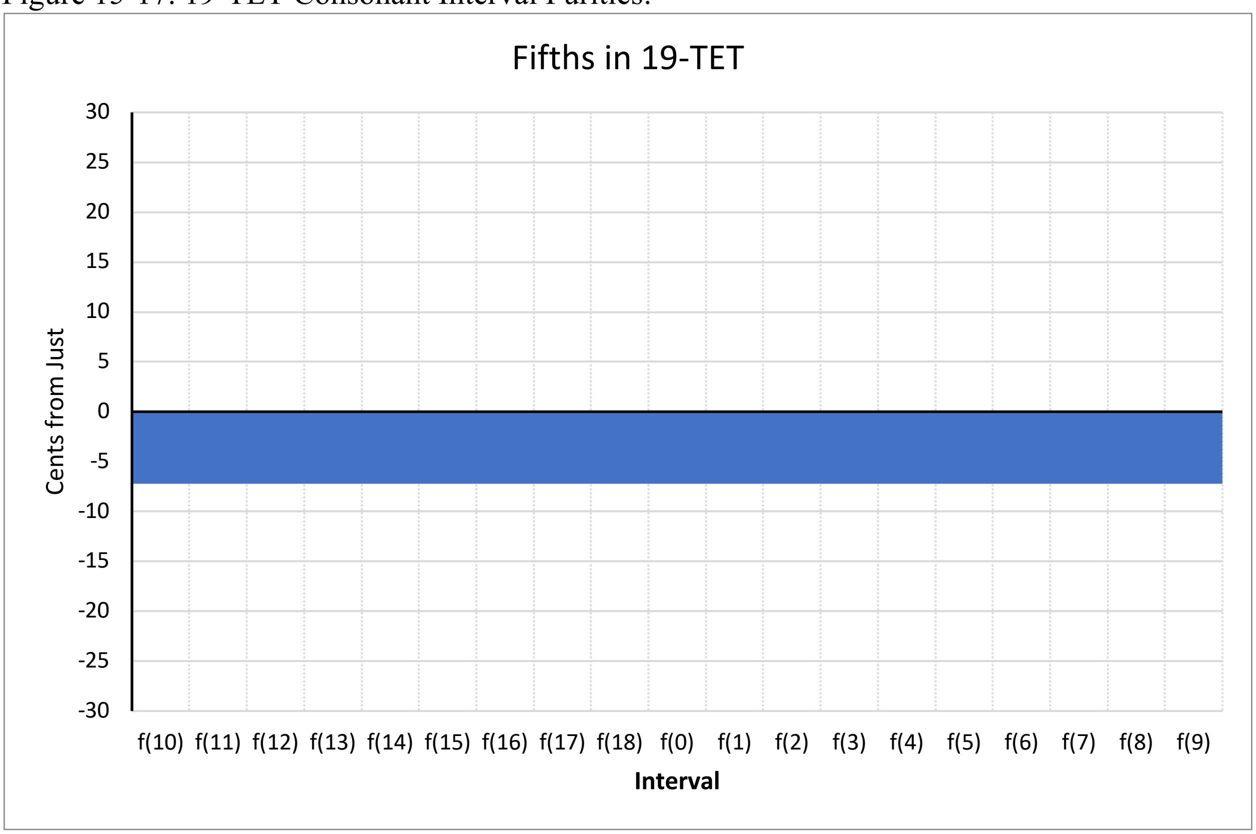
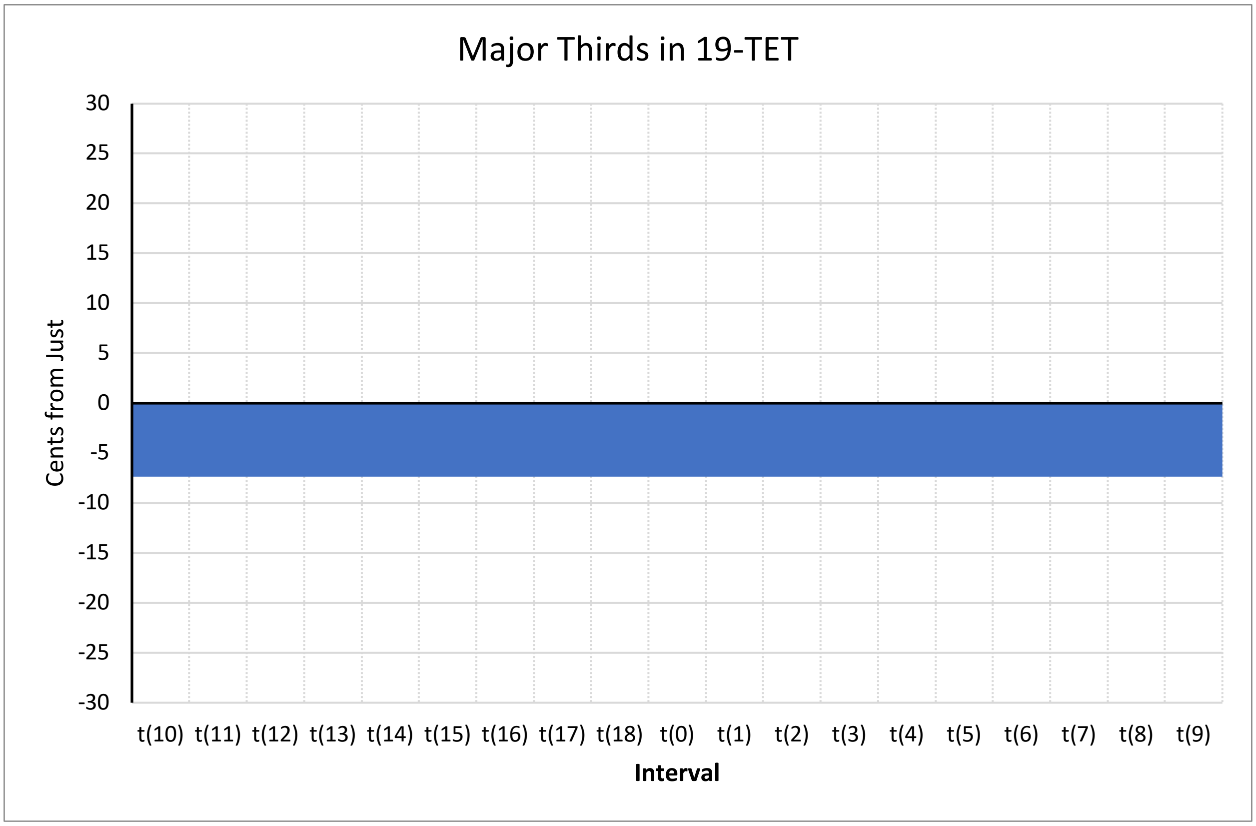


Figure 15-17. 19-TET Consonant Interval Purities:




Major Thirds in 19-TET
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
Cents from Just
t(10) t(11) t(12) t(13) t(14) t(15) t(16) t(17) t(18) t(0) t(1) t(2) t(3) t(4) t(5) t(6) t(7) t(8) t(9)
Interval

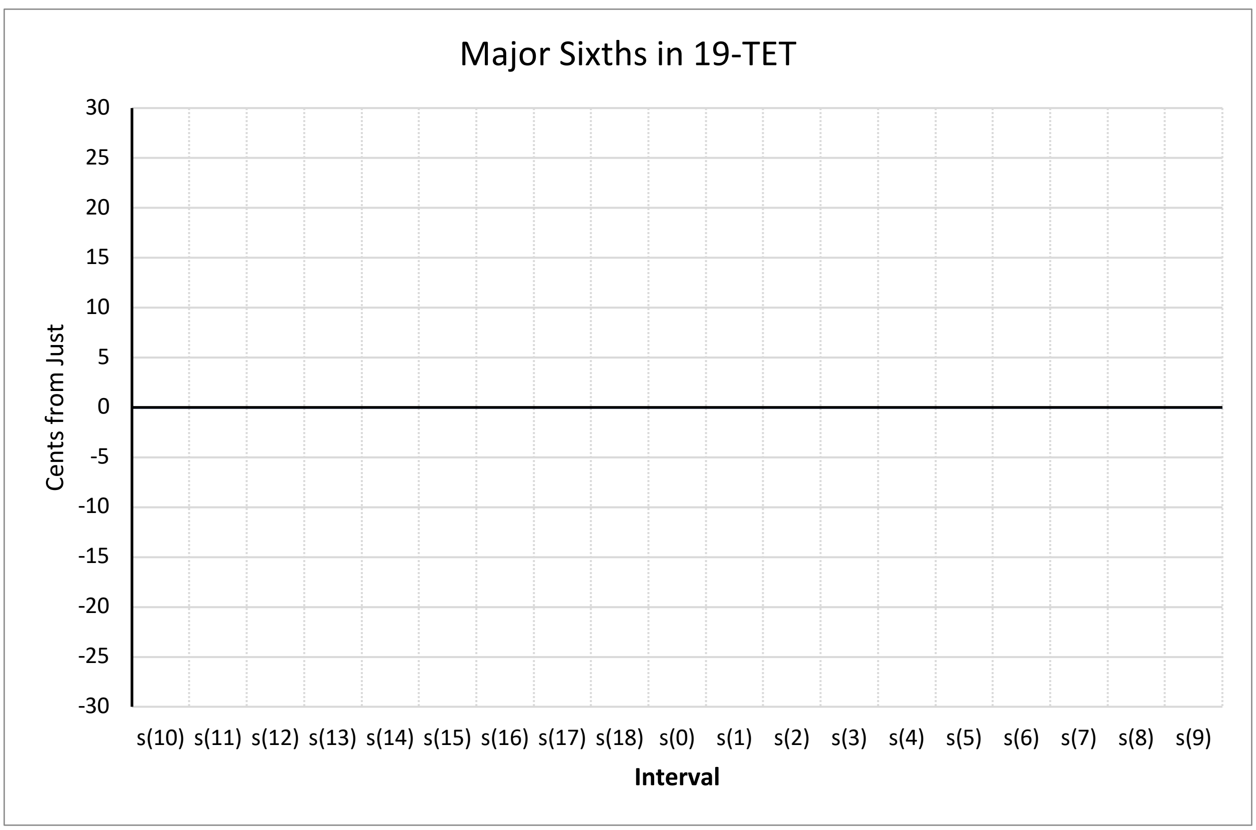
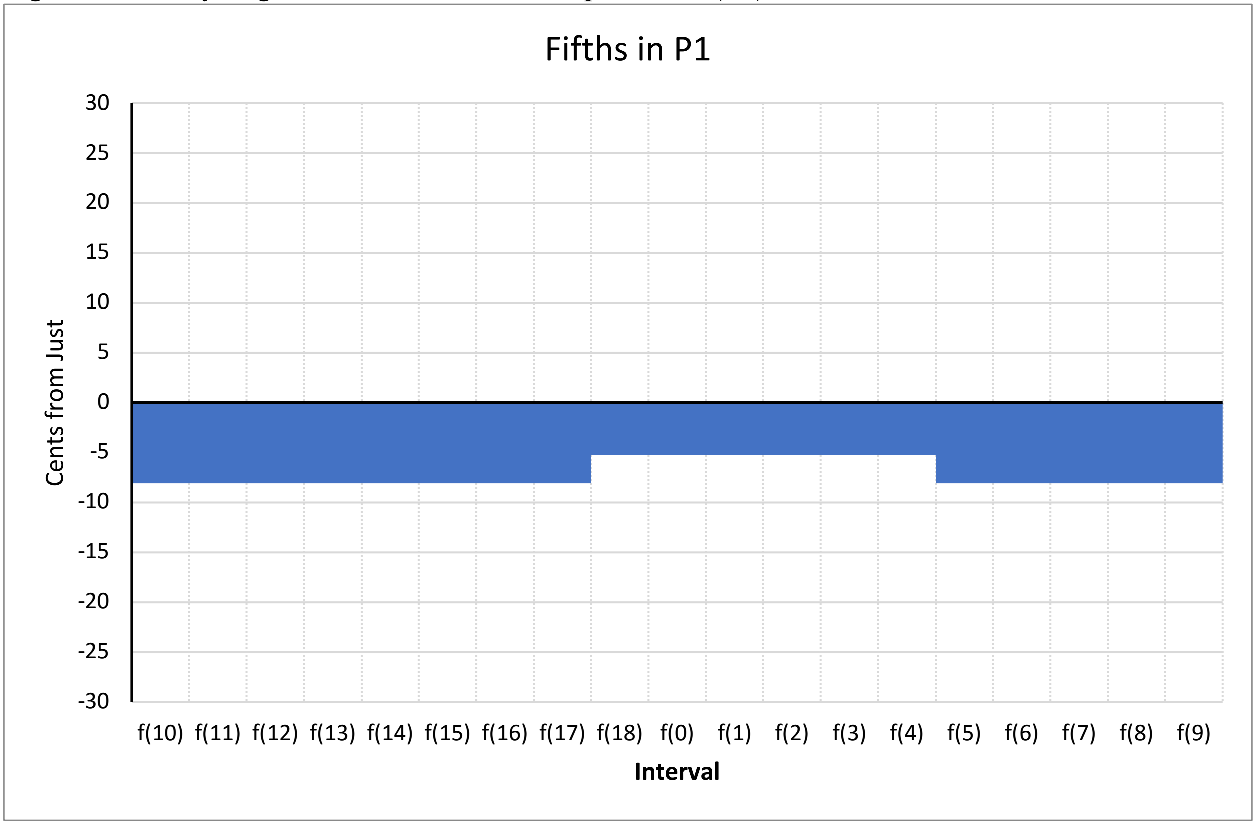
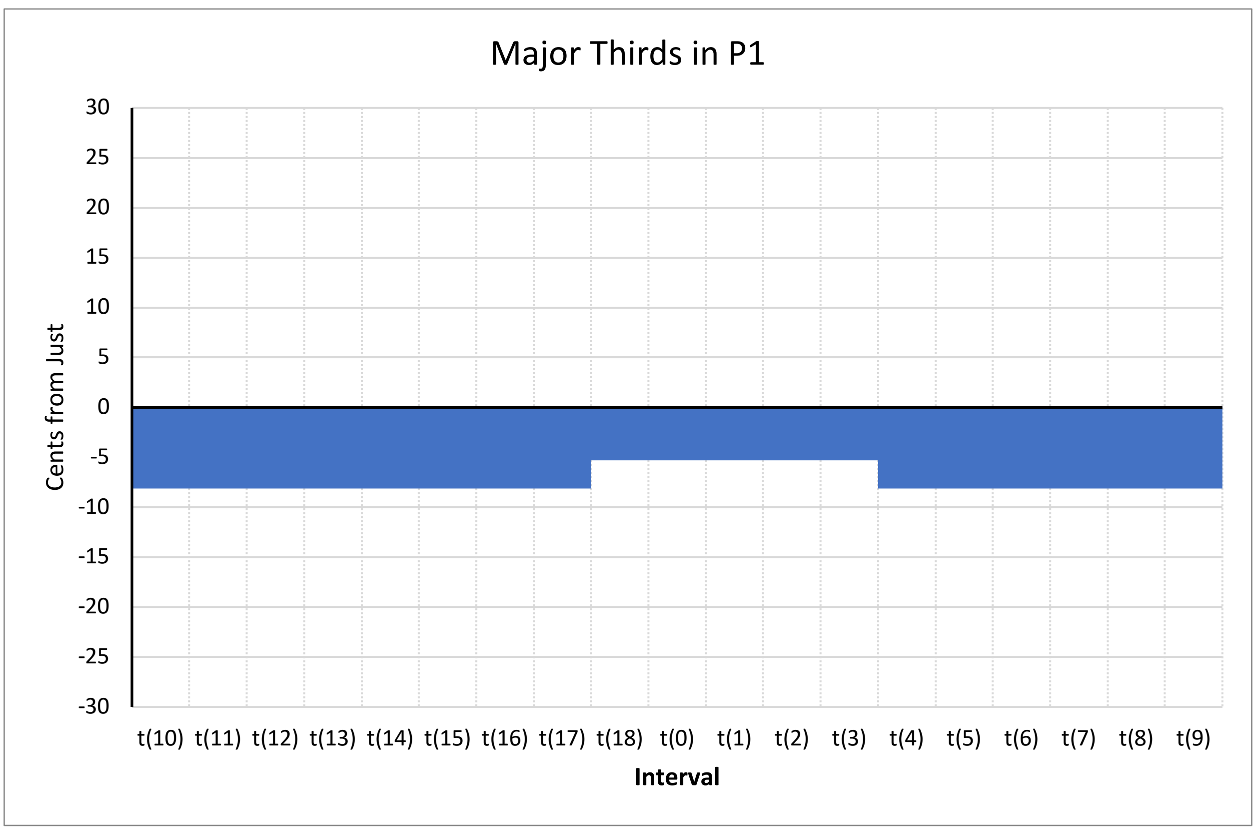

Major Sixths in 19-TET
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
Cents from Just
s(10) s(11) s(12) s(13) s(14) s(15) s(16) s(17) s(18) s(0) s(1) s(2) s(3) s(4) s(5) s(6) s(7) s(8) s(9)
Interval

Figure 18-20. Pythagorean Just 19-Tone Temperament (P1)





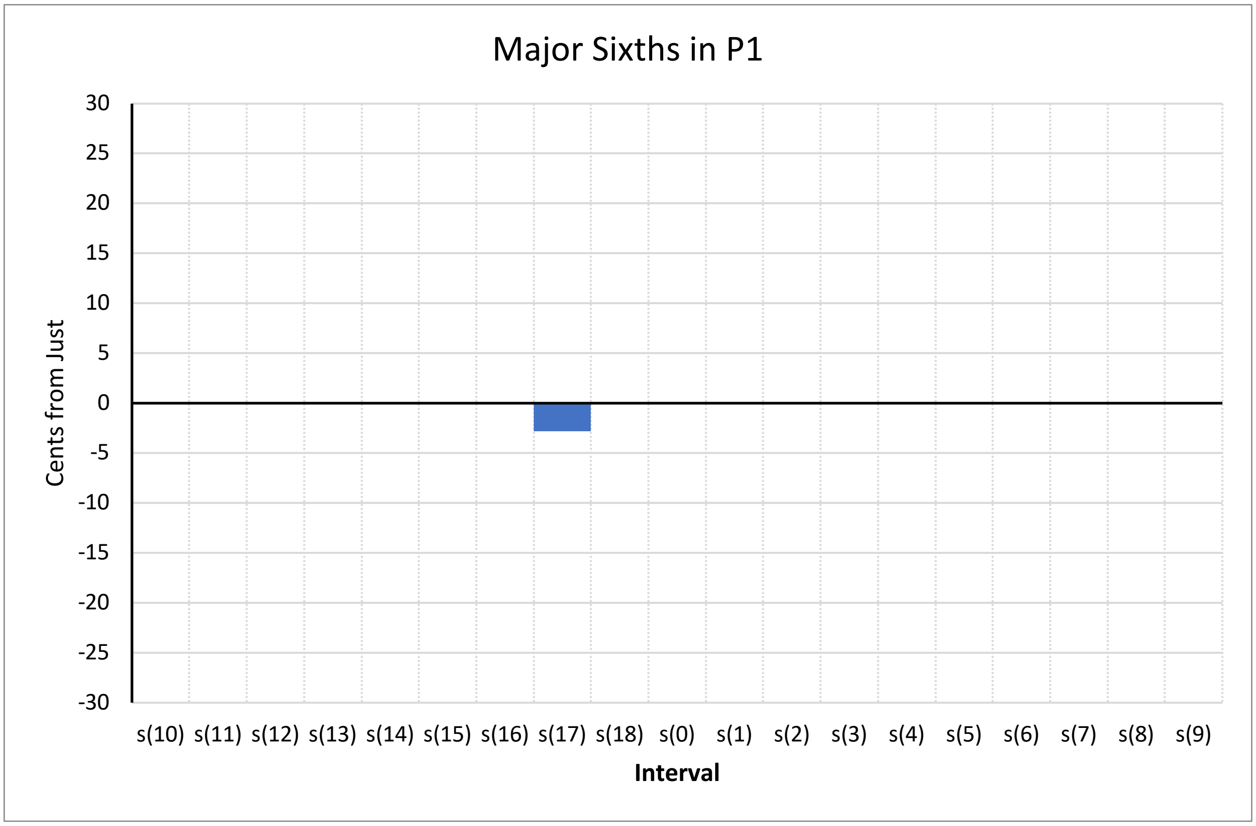
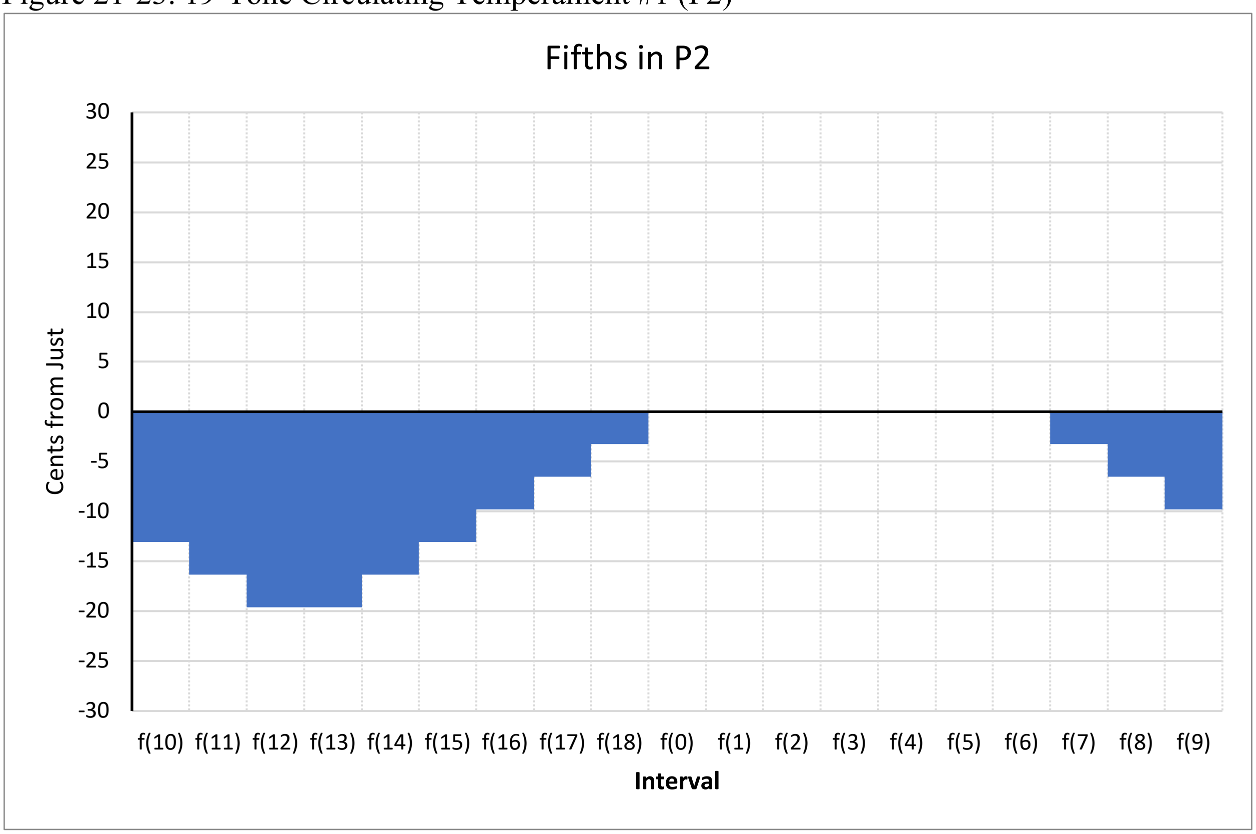
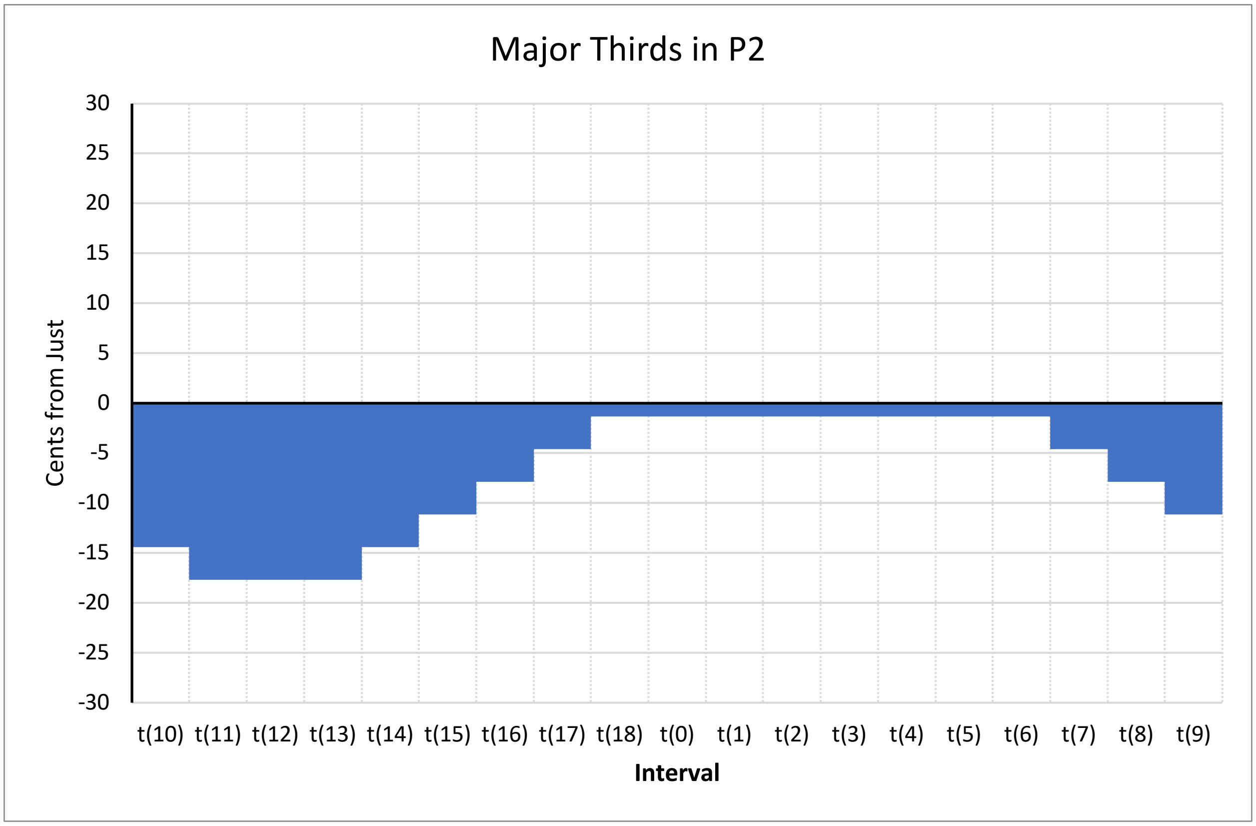


Figure 21-23. 19-Tone Circulating Temperament #1 (P2)




Major Thirds in P2
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
Cents from Just
t(10) t(11) t(12) t(13) t(14) t(15) t(16) t(17) t(18) t(0) t(1) t(2) t(3) t(4) t(5) t(6) t(7) t(8) t(9)
Interval

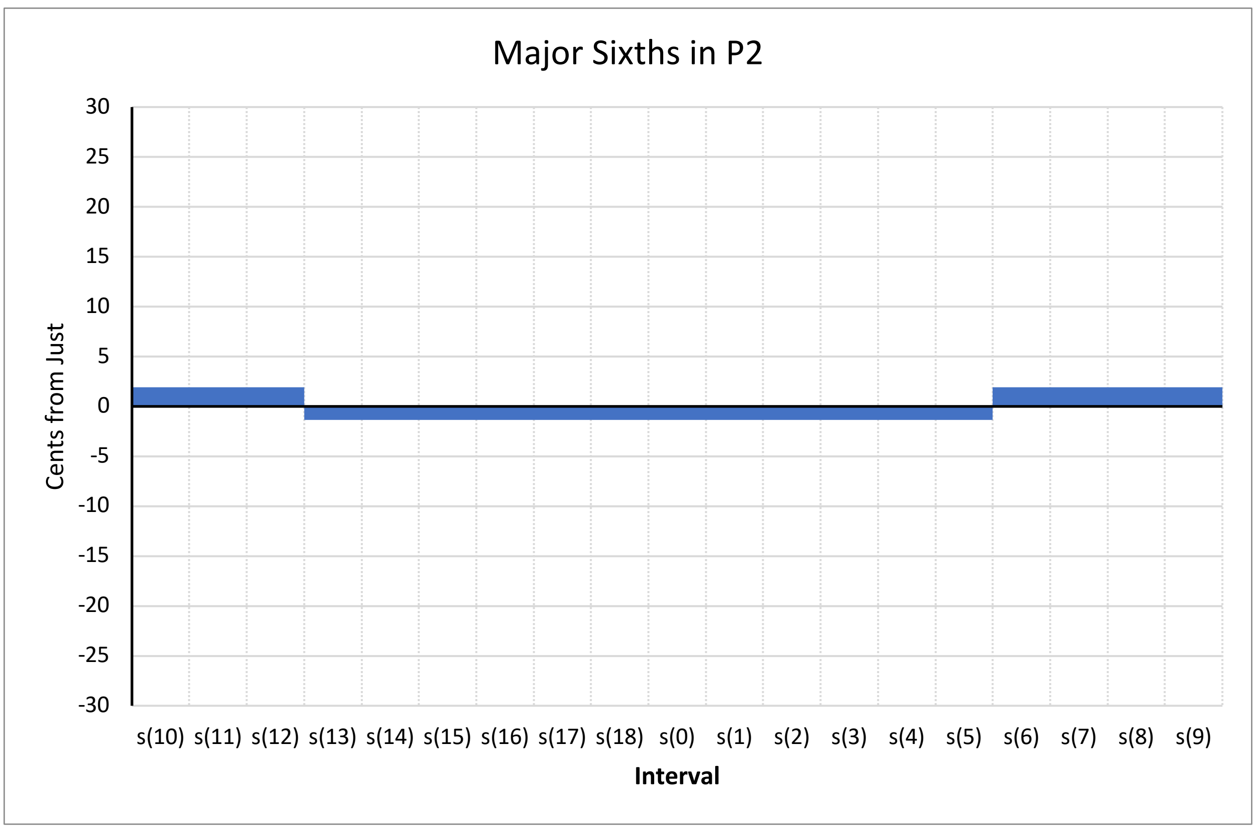
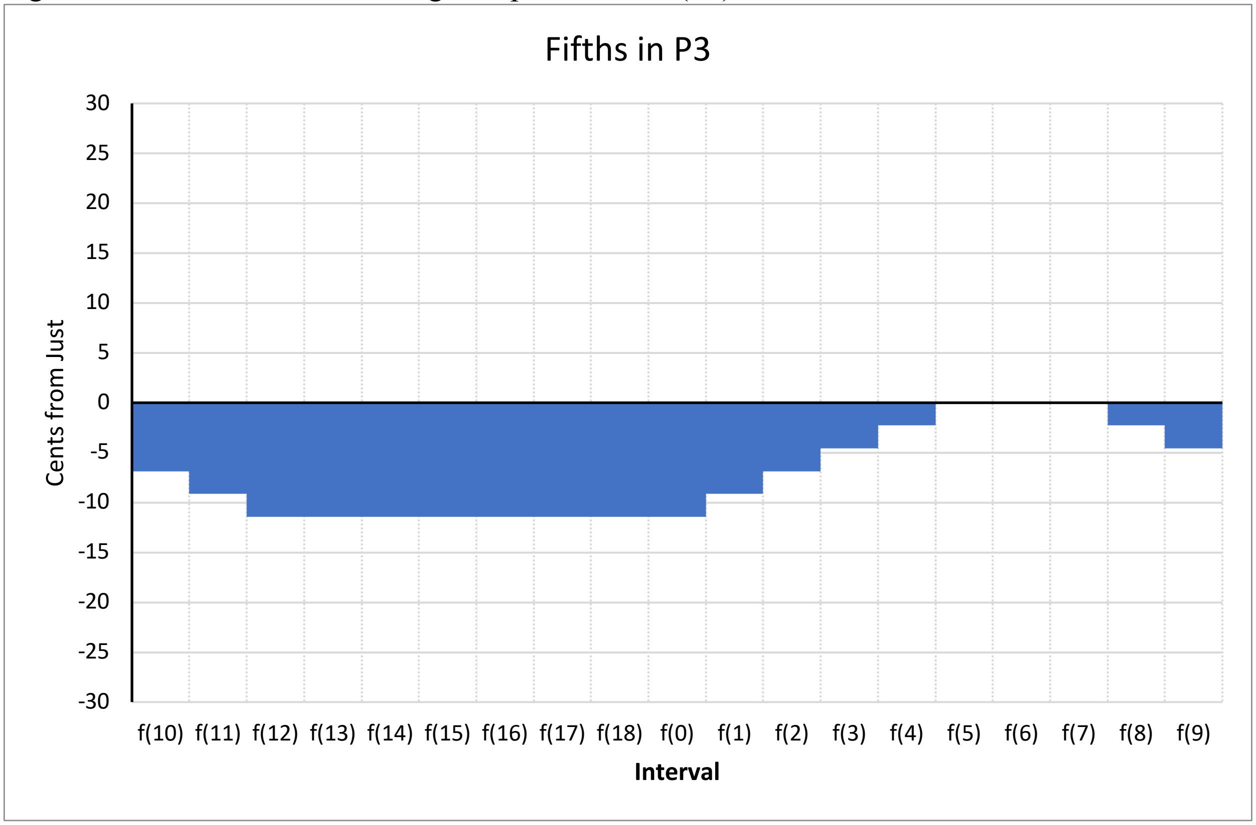
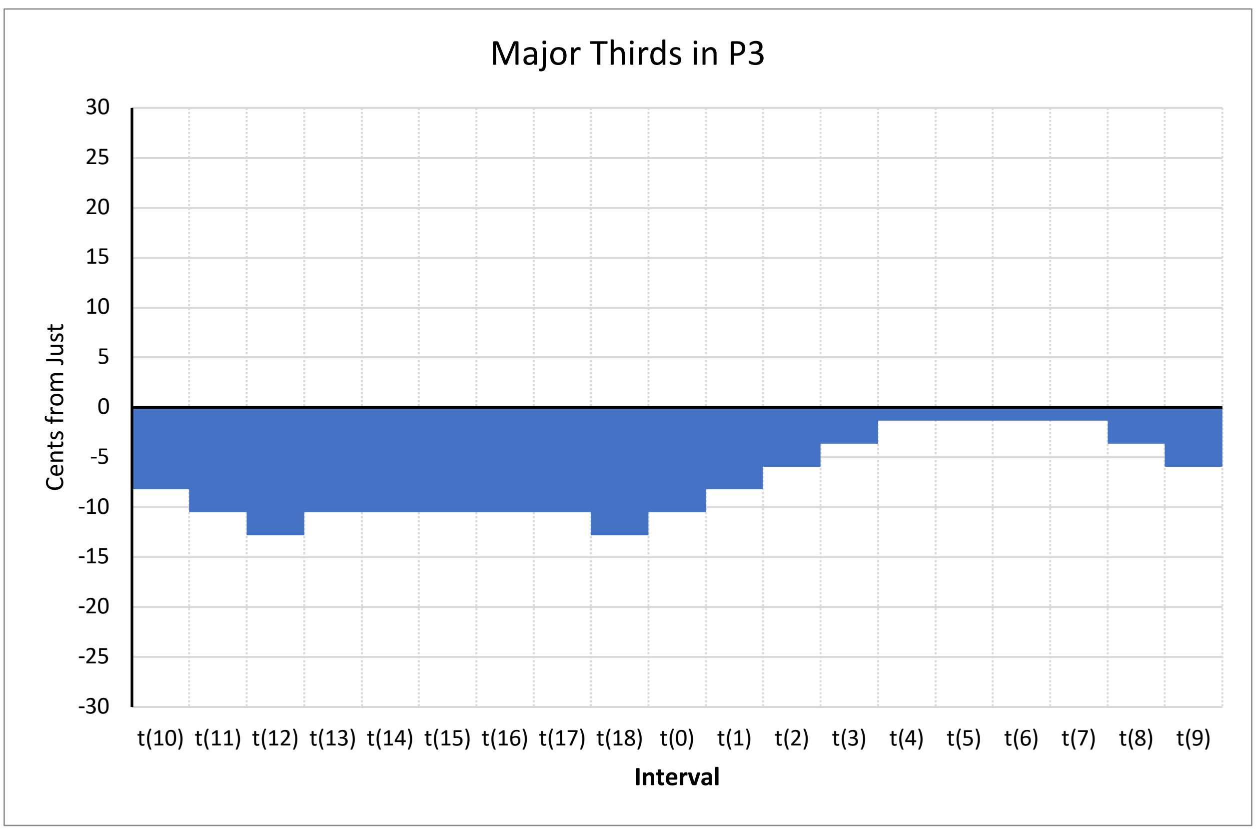

Major Sixths in P2
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
Cents from Just
s(10) s(11) s(12) s(13) s(14) s(15) s(16) s(17) s(18) s(0) s(1) s(2) s(3) s(4) s(5) s(6) s(7) s(8) s(9)
Interval

Figure 24-26. 19-Tone Circulating Temperament #2 (P3)





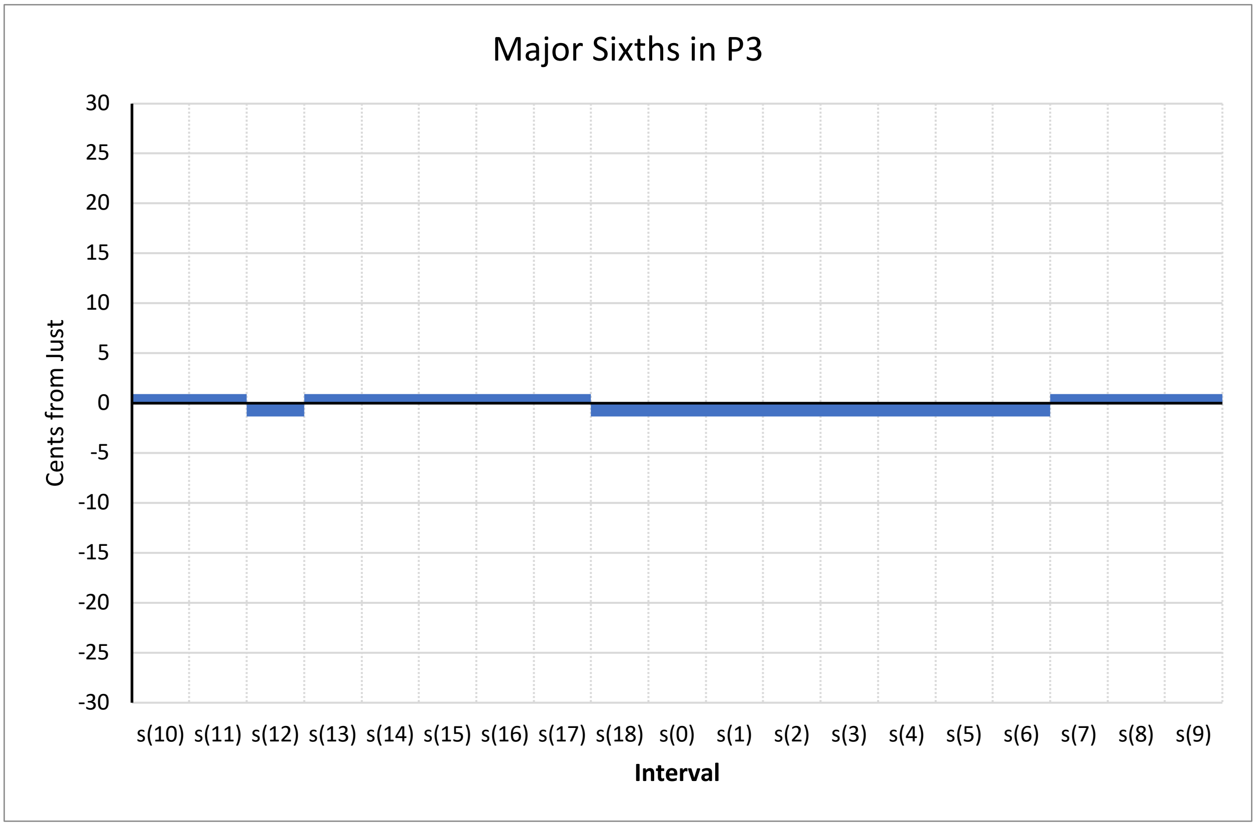
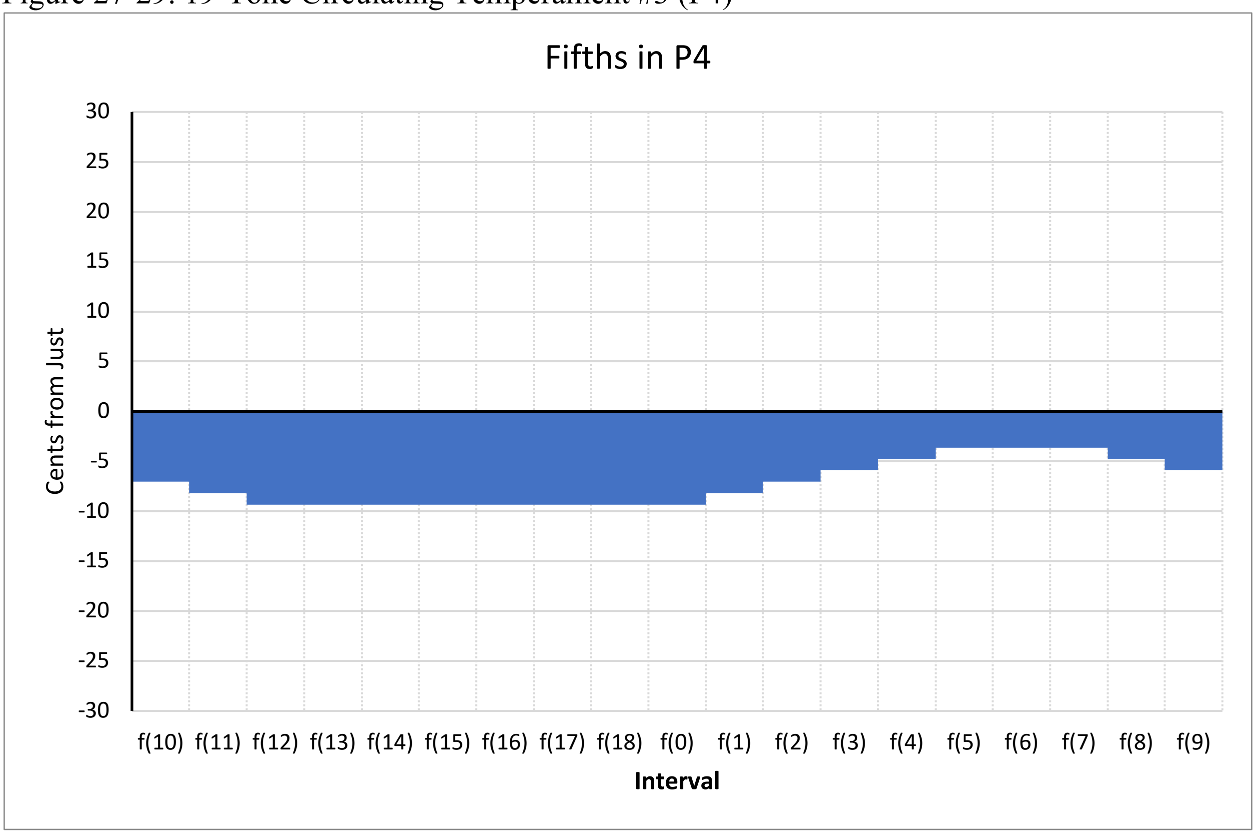
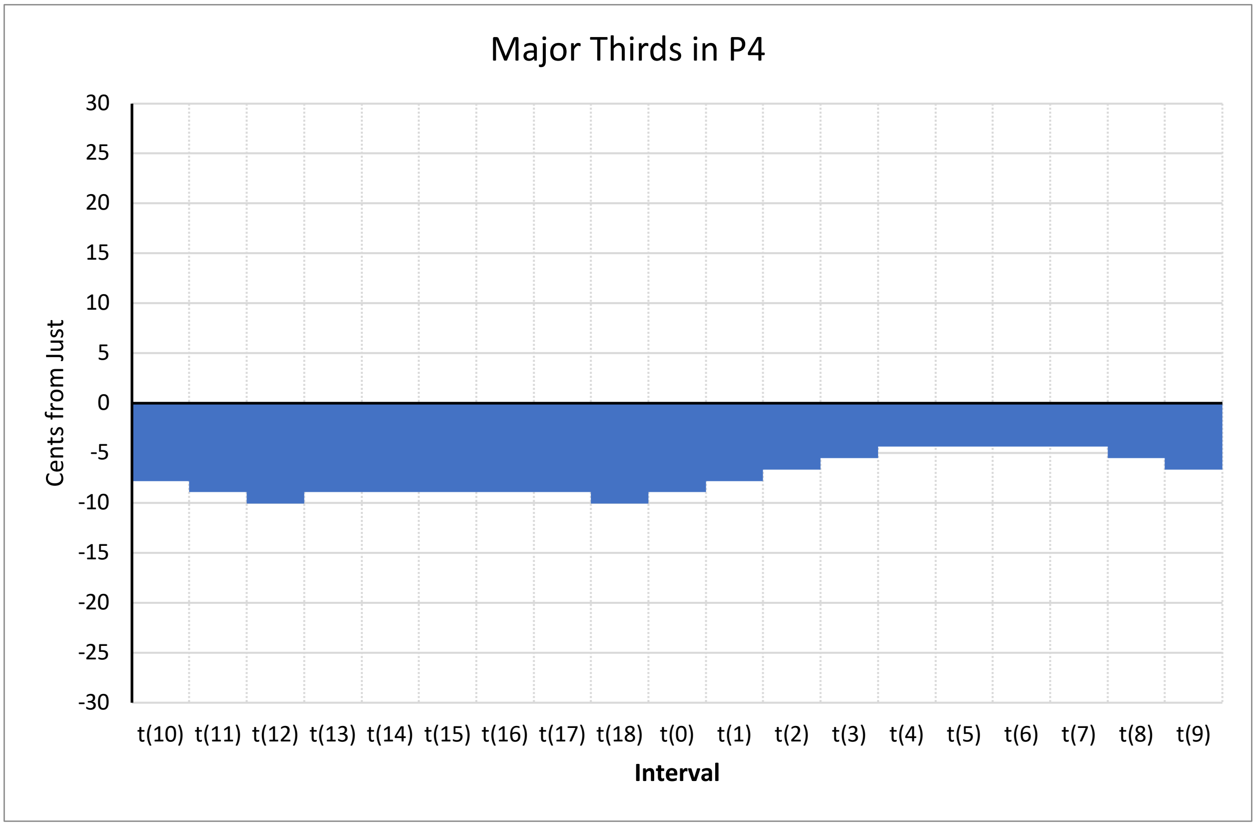
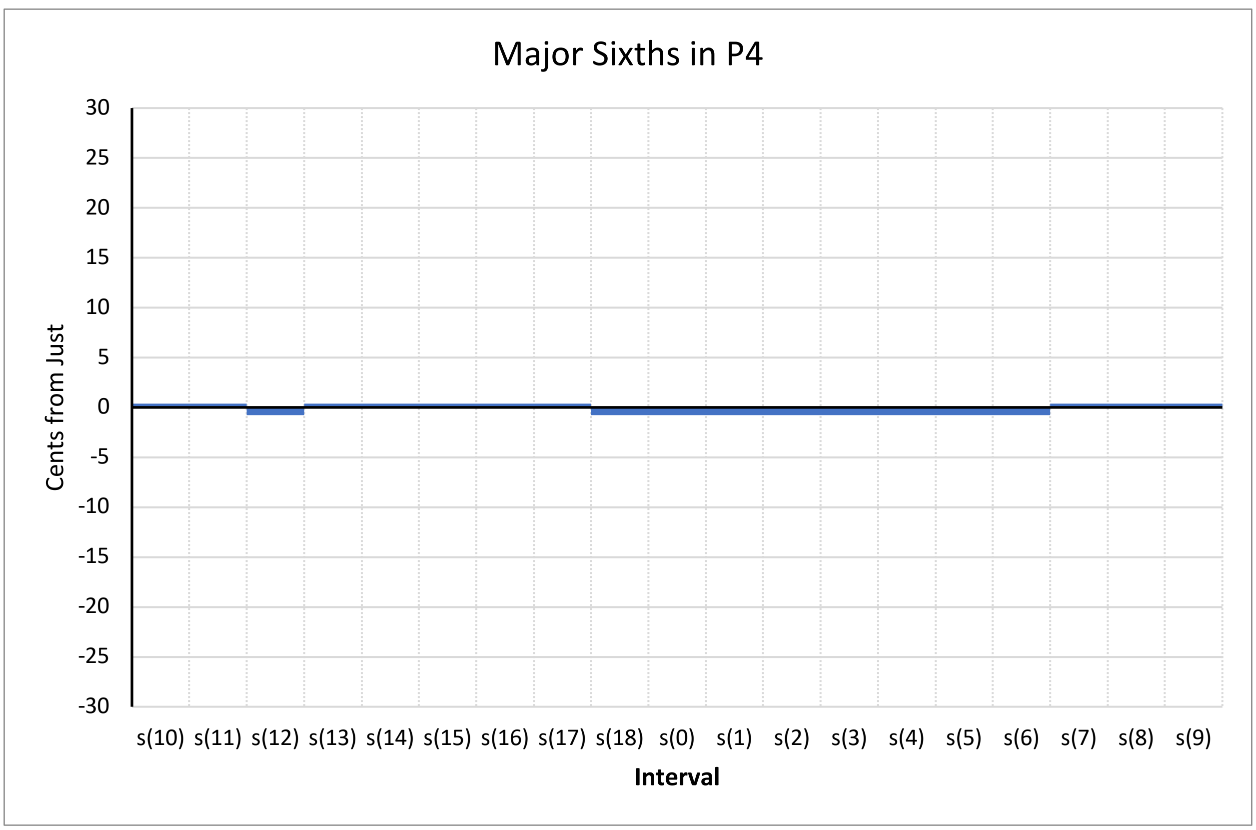


Figure 27-29. 19-Tone Circulating Temperament #3 (P4)




Major Thirds in P4
Cents from Just
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
t(10) t(11) t(12) t(13) t(14) t(15) t(16) t(17) t(18) t(0) t(1) t(2) t(3) t(4) t(5) t(6) t(7) t(8) t(9)
Interval


Major Sixths in P4
Cents from Just
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
s(10) s(11) s(12) s(13) s(14) s(15) s(16) s(17) s(18) s(0) s(1) s(2) s(3) s(4) s(5) s(6) s(7) s(8) s(9)
Interval

Appendix E: Preliminary Research and Notes

Knowledge Gaps

Knowledge Gap	Resolved by	Information is located
The math behind music intervals and scales	Reading	

Literature Search Parameters

Database/search engine	Keywords
MathSciNet	Music, temperament, intonation, frequency, tone

Sources

Format:

Source Title	
Source Citation	
Source Type	
Keywords	
Summary	
Reason for Interest	
Notes	
Questions	

1. Alternatives to Semitones and Quartertones: Music-Theoretical Suggestions

Source Title	Alternatives to Semitones and Quartertones: Music-Theoretical Suggestions
Source Citation	Smethurst, R. (2018). Alternatives to semitones and quartertones: Music-theoretical suggestions. *The Mathematical Intelligencer, 40*(3), 37-42. doi:10.1007/s00283-018-9800-z
Source Type	Journal Article
Keywords	
Summary	
Reason for Interest	Ideas for different temperaments and tuning systems

Notes	Status: Preliminary Abstract/Introduction: • 24-tone equal temperament critique • Whole tone scale with new symmetrical nodes, source from Olivier Messaien • Heinz Bolen's non-octave scale • Article uses approximate ratios, not absolute Article : - 24-tone equal temperament is harder to play and is no better than the 12-tone equal temperament - 18-tone equal temperament is described by cents. Each step is about 66.67 cents - Scale with six equal steps of 200 cents (whole tone scale) can be played with 18-tone equal temperament - Scale with nine equal steps of 133.33 cents - Ratios of primes form consonances. Septimal minor seventh (7/4) arguably more consonant than minor seventh in 12-tone equal temperament (9/5) - SYMMETRICAL MODES (research that) - A symmetrical mode in 12-tone equal temperament is a sequence of distinct notes that has a non-trivial group of symmetries by the actions of translations of mod 12. A symmetrical mode in 18-tone equal temperament is the same, except of mod 18. - Bohlen's peculiar 11-tone scale that doesn't repeat at 2:1 frequency ratio, but instead at 3:1 frequency ratio (perfect twelfth). Something to consider?
Questions	

2. A sequential pattern mining approach to design taxonomies for hierarchical music genre recognition

Source Title	A sequential pattern mining approach to design taxonomies for hierarchical music genre reognition
Source Citation	Sylvain Iloga, Olivier Romain, & Maurice Tchuenté. (2018). A sequential pattern mining approach to design taxonomies for hierarchical music genre recognition. *PAA : Pattern Analysis and Applications, 21*(2), 363-380. doi:10.1007/s10044-016-0582-7
Source Type	Journal Article
Keywords	Audio Processing, Automatic Taxonomy Generation, Music Genre Recognition, Hierarchical Classification, Sequential Pattern Mining, Histogram Mining
Summary	

Reason for Interest	
Notes	Status: Reviewed Abstract: • New method to detect the genre of music based on hierarchical classifiers vs direct/flat classifiers • Method uses sequential pattern mining to identify characteristics of the music and produce a vector representation of the music, which is then put through a hierarchical-based algorithm that identifies the genre Introduction: • Radio transmits information through FM waves • Audio is indexed to facilitate radio on demand applications like criterion-based programming • Music genre classification remains a problem • Hierarchical system is more efficient than flat system because “it applies a classifier to an inner node of the taxonomy” (research what this means_ • Advantages of the hierarchical system proposed: 1. Taxonomies are more reliable 2. Hierarchical systems better analyze the links between different genres of music 3. If the system is wrong, it will be relatively close to the correct genre • Human-generated manual taxonomies are fairly inaccurate • New approach uses sequential pattern mining to generate a taxonomy Article: - Manually generated taxonomies are based on human feelings and descriptions (semanics) and are therefore usable by humans. Automatically generated taxonomies are better suited for computational methods. - Brecheisen used 5 music descriptors to create a hierarchical classifier for 11 different genres: o Mel Frequency Cepstral Coefficients (MFCCs) o Spectral flux o Spectra rolloff o Beat histogram o Pitch histogram - Brecheisen’s hierarchical system reached 70.03% accuracy, which was worse than his own flat system (with 72.01% accuracy)

	- Many other attempts at a hierarchical system have been made, but none of them exceeded 79% accuracy and were statistically indifferent from their flat equivalents
	- Sequential Pattern Mining have been used before for both flat and hierarchical systems before, but the accuracy was not high.
	- Background math concepts:
	    o Given sequential pattern database δ
	    o Let X = $\{x_1, x_2, \ldots x_k\}$, where X is a set of all the items that appear in δ
	    o A subset of X is an itemset
	    o Let s = $\{s_1, s_2, \ldots s_m\}$, where s is an ordered list of itemsets of X. s represents a sequential pattern.
	    o I DO NOT UNDERSTAND, REREAD LATER
	- Overview of Proposed Method:
	    o An automatic approach to taxonomy generation
	    o Sequential pattern mining techniques to music genre descriptors in the form of vectors
	    o The vectors are inputted into a hierarchical clustering algorithm
	    o Cross-validation applied during classification procedure
	- Specific logistics do not relate to project idea
	- Method reaches 91.6% accuracy

Questions	

3. Music Temperament (eBook/PDF)

Source Title	Music Temperament
Source Citation	Daniel A. Steck, *Musical Temperament*, available online at http://steck.us/teaching (revision 0.2.5, 10 September 2017)
Source Type	eBook
Keywords	
Summary	
Reason for Interest	
Notes	Chapter 1.2: Consonance and Harmonics - Consonance is a group of pitches that sound pleasant or good to people - Dissonance is a group of pitches that sound sad or tense - Given a periodic frequency f, there is a fundamental frequency component f_1 and all of its multiples, called harmonics. Together, f_1 and all of its harmonics are called a harmonic series. - Consonance comes from two pitches that are of the same harmonic series. Two harmonics of the same fundamental are

	consonant. (200Hz and 300Hz are consonant because they are harmonics of 100Hz) - Because of that, with simple algebraic manipulation, two sounds are consonant if the ratio of their frequencies is a rational number. Exception to this rule: If the ratio is too complex (say 301:187), the sounds may not form a consonance. Therefore, this rule applies only to simple ratios. Also, if the ratio is really close to a simple, rational number, then it will still be consonant to human ears. - An octave is defined as two pitches that whose frequency ratio is 2:1. This is the only interval that is defined definitively in terms of mathematics.
Questions	

3. A System for Tuning Instruments Using Recorded Music Instead of Theory-Based Frequency Presets

Source Title	A System for Tuning Instruments Using Recorded Music Instead of Theory-Based Frequency Presets
Source Citation	
Source Type	Journal Article
Keywords	
Summary	
Reason for Interest	Considered idea of making a tuner that utilizes a different, more obscure tuning system.
Notes	Abstract : - Tuner based on a user-uploaded recording - Gives audio and visual feedback Article: - This is useful for cultures who intentially want to sound different intonation-wise as a way to expresse individuality - This is also useful for people who want to tune to or learn to tune like professional musicians - Also useful for people who play certain instruments that are hard to tune, such as an instrument with many strings - Skips frequency estimation step - Consists of analysis system and application system - Recording is provided in MP3 or WAV format - Analysis system extracts list of frames as reference signals - Application system uses the reference signals and provides audio and visual feedback on the user’s intonation - Recording has to be monophonic - Dynamic user interface

	- Manual Pruning Project Idea altered, article is irrelevant
Questions	

4. Unified Music Theories for General Equal Temperament Systems

Source Title	Unifed Music Theories for General Equal Temperament Systems
Source Citation	
Source Type	Paper
Keywords	
Summary	
Reason for Interest	Relevancy to temperament
Notes	Introduction - Begins with questioning on piano key arrangements for any n-tone equal temperament system - The total number of combinations, restricted by the first key being a white key and that no two black keys may be adjacent, ends up being a Fibonacci sequence, starting with 1 at n=1 and 2 at n=2. o As n increases, the number of combinations increases so drastically that there needs to be other restrictions applied Key Signatures and Abstract Algebra - Interval notation: usage of integers to represent each number of the scale (C = 0, C# = 1, etc.). This is apparently useful for equal temperament systems because the intervals in an equal temperament system can be described with integer subtraction due to the equality of the system. - To apply integers like this to equal temperament systems, a cyclical-quotient group must be used. This basically loops the integers around at a certain number, making the set of all integers restricted to a range, with the largest number looping back to the smallest. For instance, for 12-TET, 11 loops back to 0. A lot of complex math later… Part 3: Key Configurations - From the complex math described in previous parts of the paper, several axioms about positioning of white and black keys are determined

	○ Axiom 1: Three consecutive white keys not separated by a black key cannot exist. There must also be at least one white key between any two black keys. ○ Axiom 2: The key signature of any scale must either have sharps or flats, but never both. ○ Axiom 3: For n values that don't include the traditional dominant, tonic, subdominant, leading tone, etc. , injective mapping must apply (I don't really understand this one quite well) Part 4: Key Configuration for Some Cases - Several tables regarding possible white-black key combinations, refer to - Using the three axioms above, the number of possible white/black key combinations for 12-TET has been reduced from 233 (Fibonacci sequence) to 3. More complex math later… Part 6: The Universal Perfect Fifth (Fourth) Constant - A lot of temperament systems are not explored because of how inaccurate they are in regards to important intervals like perfect fifth at first glance. (Reference table 11) - Personal Note: Try out 16-TET - Joseph Yasser made insights into 19-TET with evolving tonality and scales, along with an adequate way to determine white/black key combinations - Yasser's evolving tonality scales follow the Fibonacci sequence discovered earlier by the bicycle website. ○ A whole lot of complex math…
Questions	

5. Temperament: A Beginner's Guide

Source Title	Temperament: A Beginner's Guide
Source Citation	https://www.albany.edu/piporg-l/tmprment.html
Source Type	Website
Keywords	
Summary	
Reason for Interest	This is unlike the other source here for several reasons. This source is primarily to provide more background knowledge and aid with the understanding of jargon.
Notes	- A comma is the mathematical phenomenon where the starting note in a circle of fifth is not identical to the ending note.

	o Equal temperaments shift all notes by an equal amount to solve this issue. o Other temperaments shift notes by different amounts to account for this issue - Pythagorean Temperament o Nearly all fifths and fourths are in tune and the whole comma is concentrated in one interval, which is unusable as a result. o This temperament was ideal for early music, where musical concepts like modulation and complex polyphony were not included - Meantone Temperament o Major thirds are perfectly in tune o Fifths and fourths are slightly tempered except one fifth, which is extremely tempered. It is known as the wolf fifth. o Despite the wolf fifth, the temperament was still very good and allowed for composers to modulate in the middle of a piece. o Modified meantone temperament came about in an attempt to alleviate the ugliness of the wolf fifth, tempering the thirds a little bit. - Circulating Temperament o The wolf fifth in meantone temperament limited the freedom of composers, so circulating temperaments came about to fix it. o These systems allowed for different colors to different scales/keys. o A performer can play in any key in circulating temperament, but there is one key that is very in tune, and the further you get from it, the less in tune it is. - Equal Temperament o Modulation in all keys can be done without fear that one key will sound worse than another. o All keys are the same (both good and bad at the same time) o None of the intervals is perfectly in tune, but all of them are close enough to sound acceptable. o
Questions	

6. The temperament police

Source Title	
Source Citation	

Source Type	
Keywords	
Summary	
Reason for Interest	Relevancy to extracting temperament from recordings, potential idea for tuner

Notes	- Explores harpsichord temperaments through recordings
	- Steps in identifying temperament from recordings
	    o Automatic Transcription
	        ▪ Detecting existence and timing of notes
	        ▪ Discusses harpsichord-specific transcription system
	    o Precise Frequency Estimation
	        ▪ Obtaining precise estimates of the fundamental frequencies of each note in the recording
	    o Temperament Estimation
	        ▪ Determining temperament from those frequencies
	- Unfortunately, the results and analyses of harpsichord recordings is unrelated. There is no in-depth description of the temperament systems they discovered.

Questions	

7. Constructing an optimal circulating temperament based on a set of musical requirements

Source Title	Constructing an optimal circulating temperament based on a set of musical requirements
Source Citation	
Source Type	
Keywords	
Summary	
Reason for Interest	

Notes	Introduction
	- In this paper, they constructed a temperament system based on the requirements prioritized as follows:
	    o 1. No wide fifth nor narrow major thirds because it results in strong tempering of other intervals
	    o 2. There must be a tonal center corresponding to the least tempered major third, and the major thirds closest to this tonal center on the circle of fifth should have priority over the more distant ones. (not sure what this means yet)

	o 3. Tonal center should be as symmetric as possible in terms of major thirds. (basically flat keys shouldn't be favored over sharp keys) o 4. Resulting number of new, unique intervals should be kept minimal o 5. No interval must be tempered more than "absolutely necessary" to obey other criteria Development of the Temperament - A temperament is described when the sizes of all 12 fifths are known - Beginning procedure/assumptions o Number the 12 notes in the circle of fifths, starting with C = 1, G = 2, etc. o Let f(i) denote the tempering of the upper note in a fifth starting on i, where if f(i) = 0, the fifth is perfectly 3/2 ratio. If f(i) < 0, the fifth is too narrow and if f(i) > 0, the fifth is too wide. o Total tempering of all of the fifth must add up to the Pythagorean comma o Pythagorean comma is the size of the interval between two enharmonic equivalent o Let t(i) denote the tempering of the upper note in a major third starting on i. A major third can be constructed with four consecutive fifths, and if all four fifths are pure, the resulting interval is one syntonic comma wide. -
Questions	

8. On the construction, comparison, and exchangeability of tuning systems

Source Title	On the construction, comparison, and exchangeability of tuning systems
Source Citation	
Source Type	
Keywords	
Summary	
Reason for Interest	
Notes	Introduction - Paper provides short mathematical reviews on several of the most popular existing temperaments

<table>
<tr><td></td><td>Pythagorean tuning and just intonation

Definition for 12-tone systems:

Given two notes a and b, where b is a whole tone above a, b is one octave below two fifths above a.
Given two notes a and b, where b is a chromatic semitone above a, b is four octaves below seven fifths above a
Given two notes a and b, where b is a diatonic semitone above a, b is three octaves above four fifths below a

Many tuning systems tune fifths and octaves and then use them to determine the other tones in the system.
In Pythagorean tuning system, the set of tuned notes f based on a base note f_0 are defined as $(3/2)^n(2)^m(f_0) = f$

$3^2/2^3 = 9/8 \rightarrow$ Whole Tone
$3^7/2^{11} \rightarrow$ Chromatic Semitone
$2^8/3^5 \rightarrow$ Diatonic Semitone

All just intonations add the 5/4 (major third) interval to Pythagorean tuning
One way this can be done is altering some of the fifths slightly. Fifths (ratio of 3/2) is often approximated with 40/27.

Fifths with ratio of 3/2 are called Pythagorean fifths. Fifths with ratio of 40/27 are called syntonic fifths.
The interval between a Pythagorean fifth and a syntonic fifth is called a syntonic comma

Most notable just intonation system is the Zarlinean system
The biggest problem with Pythagorean and Zarlinean systems are that the circle of fifths are not closed.

Temperaments

Circulating Temperaments

They close the circle of fifths
Common examples:

12-note equal temperament
Marpurg’s temperament I

Meantone Temperaments

They add an intermediate tone between the major tone (9/8) and minor tone (10/9)

</td></tr>
<tr><td>Questions</td><td></td></tr>
</table>

9. The Mathematical Theory of Tone Systems

Source Title	The Mathematical Theory of Tone Systems
Source Citation	
Source Type	Book

Keywords	
Summary	
Reason for Interest	
Notes	12-tone equal temperament - Fifths off by -2 cents - Fourths off by +2 cents - Minor 3rds off by -16 cents - Major 3rds off by +14 cents - Minor 6ths off by -14 cents - Major 6ths off by +16 cents
Questions	

10. Music: A Mathematical Offering

Source Title	Music: A Mathematical Offering
Source Citation	
Source Type	Book/eBook
Keywords	
Summary	
Reason for Interest	

Notes	Temperaments
	- Irregular temperaments and meantone temperaments tend to alter each interval by some fraction of a relative comma
	    o Notation often reflects this, when each interval has a superscript number that represents the multiplier of the relative comma that the interval is altered by
	-
	
	Cents:
	- Cents is a way to linearize the ratios of intervals
	- It is based on the 12-tone equal temperament, where each tone in the 12-TET is a multiple of 100.
	    o 100 -> half step
	    o 700 -> Perfect fifth
	    o 500 -> Perfect fourth
	    o 400 -> Major Third
	    o etc.
	- This can be converted back and forth to ratios by the equation cents = $1200\log_2(\text{ratio})$

Questions	