Literature Review

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Introduction

Music is an essential part of society. Its impact is apparent in the fine arts; people like Mozart and Beethoven are household names that even non-musicians recognize. Music is, in a sense, everywhere. However, there is a fundamental problem with music that most people do not realize, and it lies in the imperfection of music tuning and tuning systems. Though there are many problems associated with music tuning, especially in ensembles, the majority of them can be fixed through the training and practice of each musician. However, no matter how much time a musician puts into perfecting intonation, there is one problem that cannot be fixed.

This is a problem that originates from a combination of music theory, mathematics, and physics. No tuning system can sound perfect. Because of this issue, instruments like the piano cannot sound as good as they need to. Orchestras and choirs cannot sound as good as they should. Rock bands cannot sound as good, either. Rather, musicians have to resort to a compromise to sound as good as they can, but they cannot mathematically sound perfect. This problem has tormented mathematicians, musicians, and physicists alike for nearly two thousand years, and still, the current tuning system is merely a compromise that, while having many strengths, is lacking in many ways.

Basic Overview of Mathematical Theory in Music

Sound

Music is comprised of sounds. A sound comprised of sine waves, and their properties the various characteristics of a sound. For instance, the frequency of a soundwave denotes the pitch of the sound, the amplitude of a soundwave determines the volume of the sound, the frequency spectrum of the soundwave determines the timbre (type of sound) of the sound, etc.[1] As it applies to the topic at hand, the frequency (the pitch of a sound) is most important. Frequency is measured in hertz, the SI derived unit for frequency. Sound is a complex, and important component of music, but one singular sound is not music; it is the combination of sounds that make up music.

Intervals and Consonance

The distance or space between two sounds is called an interval. An interval that sounds "good" is referred to as consonant while an interval that sounds "bad" is referred to as dissonant[9]. While the degree of pleasantness of an interval may seem completely subjective and perhaps even random, it is very closely related to the ratio of the frequencies of the two sounds: the simpler this ratio is, the more consonant the interval sounds[9]. For ease of explanation, the ratio will be regarded as a fraction. The simplicity of a ratio, for this purpose, is determinant on how small the numerator and denominator of the most simplified version of the fraction are[9]. The reasoning behind this definition is rather simple; it lies behind the fact that once again, sound is a sine wave. Two sine waves whose frequencies are related by simple ratios conflict less when they oscillate[3]. For instance, observe the ratio 2:1. This is the simplest a ratio can get by definition (technically, 1:1 is the simplest, but 1:1 denotes the interval between a pitch and itself). If two sine waves with this frequency ratio were plotted on a graph, one wave would oscillate exactly one more time than the other wave during one period and the two sine functions would overlap every period (refer to figure 1).

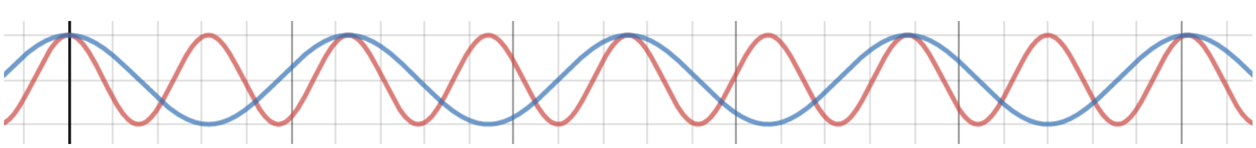

Figure 1. The graph of two sine functions with a frequency ratio of 2:1. The sine wave with the higher frequency (in red) oscillates exactly twice as fast as the sine wave with the lower frequency (in blue). The two functions align in peak every period (two periods if measured from higher frequency wave).

The interval formed by this 2:1 ratio is called an octave, and it is so consonant in music that the pitches sound identical, except one is higher than the other[9]. The consonance of this ratio led to the idea of octave equivalence, where sounds that differ by an octave are referred to by the same pitch name[1]. Now, take the second-most simple ratio: 3:2. The interval formed by this ratio is called a perfect fifth, and is regarded as one of the most important intervals in music due to its consonance. However, a ratio formed by two relatively large prime numbers such as 487/311 between the frequencies would result in very dissonant intervals.

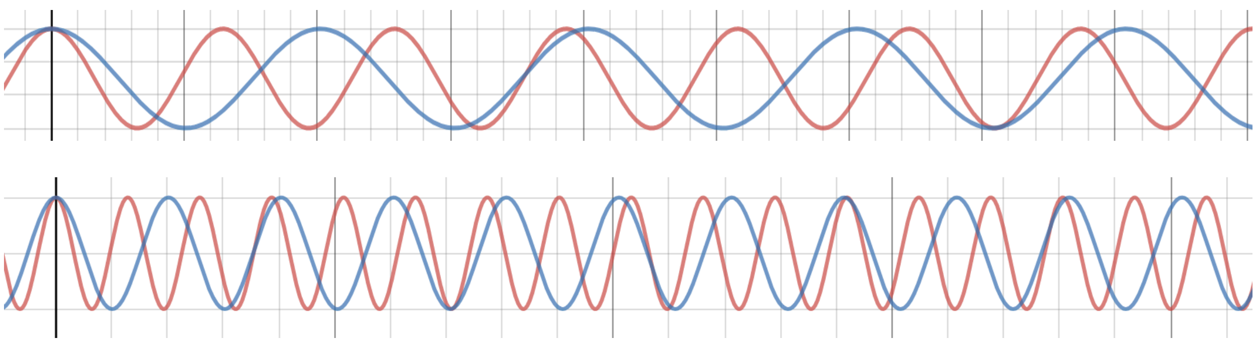

Figure 2. The graphs above each have two sine functions with a frequency ratio of 487:311. The higher frequency sine wave (in red) conflicts often with the lower frequency sine wave (in blue), rarely aligning at any peak (once every 311 periods of the blue function). The top graph is a 'zoomed-in' version of the graph below.

These two soundwaves conflict very often, and as a result, the interval formed by this ratio is very dissonant. The exception to this "rule" is if the ratio is very close to a consonant ratio[9]. For example, the ratio 797/397 is complex by the definition stated previously, but it is so close to 2/1 that to the human ear, there is no difference and the dissonance becomes negligible.

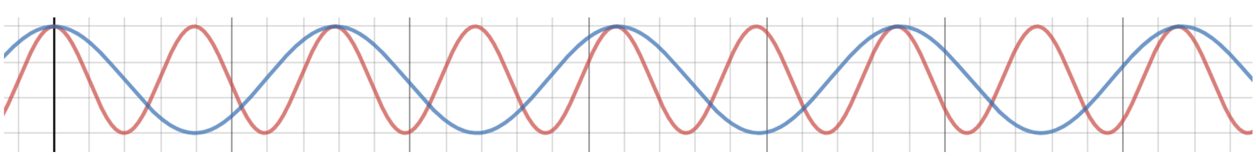

Figure 3. The graph of two sine functions with a frequency ratio of 797:397. Compare this with Figure 1. The difference is barely noticeable even when graphed. This difference is indistinguishable by the human ear.

Tuning, Tone systems, and Temperaments

However, two sounds are still not enough to make music. To make music, a series of sounds over broad ranges is needed. An instrument cannot produce every single frequency accurately simply because there is an infinite number of tones (or frequencies). Moreover, it would be impractical for the musician to know how to play every possible tone, even if the instrument allows it. To make music practical for performance, there must be a standard set of tones that is known and understood by all composers and musicians. This standard set of tones is known as a tone system[3]. With a tone system in place, music can actually be created and performed. As demonstrated previously, music theory and consonance is built upon ratios. But ratios cannot exist without a base pitch, not to mention that the base pitch must be universally agreed upon. Otherwise, every tone system would be based on different pitches, and it would become very difficult for composers, musicians, and instrument-makers to create and share music. To accommodate for all this, mathematicians came up with the frequency 440 hz and agreed that every tone system would include the pitch at 440 Hz[3], a pitch they called A.

Using the concept of octave equivalence, a tone system would be considered completely defined if given a starting tone t_0 where the subscript denotes the octave, the system reaches t_1 (which is an octave above t_0) after some n number of tones[1,2,3,4,9]. This definition ensures that the system loops around every n tones to the same pitch name, which removes the possibility of an infinite string of tones that never reach its starting tone (assuming octave equivalency). The danger of an incomplete or unclosed system, while not apparent at first, is that it would generate an infinite sequence of distinct tones. Like previously mentioned, an infinite sequence of distinct tones would make the system inapplicable to composition and performance because it is not possible for a composer or musician to learn, utilize, and perform an infinite series of tones[6,8].

Therefore, a tone system must loop back to the starting tone on a higher octave after some *n* number of tones so that when it reaches that tone, it would repeat and define the same tones that had already been established, limiting the composer/musician to only those *n* tones. For instance, the current tone system has 12 distinct tones: A-A#-B-C-C#-D-D#-E-F-F#-G-G# and back to A (western music theory designates letters to each pitch, and the # symbol indicates a higher sound than the base letter, but lower than the next base letter). Because the system goes back to A, it will then go through the same series of letters again and again, ensuring that the composer/musician does not have to learn any more than those 12 tones.

The creation of a tone system is difficult because a tone system must incorporate a set of tones that create as many consonant interval combinations as possible. This high quantity of consonant intervals is to allow composers the freedom to use better sounding intervals in their melodies/compositions and in turn allow better sounding performances[1,8]. However, there cannot be too many tones in the system, since that would lead to significant difficulty in performance. Also, the tones should be groupable into sets called keys, which are used as basis for composition. A common method many mathematicians use to create tone systems is to compound a consonant interval ratio onto a starting tone until it reaches the same tone eventually (by octave equivalency of course). This method is an elaborate way to ensure that the tone system closes as well as include at least one consonant interval. Unfortunately, the consonant intervals do not work well mathematically. For instance, take the two most consonant intervals: the octave (2/1) and the perfect fifth (3/2). Theoretically, if the system would close, the following equation should be satisfied by some ordered pair (m, n) where m and n are positive integers greater than 0.

$$\left(\frac{2}{1}\right)^{m} = \left(\frac{3}{2}\right)^{n} \tag{1}$$

However, there is no (m, n) pair that satisfies that equation because the numerator and denominator of both terms will always be only divisible by the original numerator/denominator, thus making it impossible for the two terms to ever be identical, regardless of the m and n values. This is the case for every single consonant interval due to the way a consonant interval is defined: ratios formed by small prime numbers. Therefore, the following equation applies where p_1, p_2, are relatively prime integers, p_3, and p_4 are relatively prime integers, and $\frac{p_1}{p_2}$ is not equal to $\frac{p_3}{p_4}$:

$$\left(\frac{p_1}{p_2}\right)^m = \left(\frac{p_3}{p_4}\right)^n \tag{2}$$

That equation cannot be satisfied with any m or n value. To avoid this problem, some ratios must be altered. The most common approach to every tone system is to keep the octave (2/1) and the perfect fifth (3/2) as pure as possible. When m = 7 and n = 12, $\left(\frac{2}{1}\right)^7$ evaluates to 128 and $\left(\frac{3}{2}\right)^{12}$ is approximately 129.746. The ratio of 129.746 to 128, approximately 1.014, is known as the Pythagorean comma[1,3,9]. A system cannot be left like this because, by definition, the system is not closed. One way to fix this issue is to lower all of the fifths equally by 1/12th of the Pythagorean comma to reach 128. Each of the fifths would then have a ratio of $2^{\frac{7}{12}}: 1$, which happens to be the basis of the current 12-tone equal temperament[1,3,4,9].

There are a multitude of existing temperaments. Many mathematicians have come up with their own tuning systems, but because it is mathematically impossible to create a perfect system, each existing temperament has its own strengths and weaknesses. Therefore, tuning continues to be an issue, as nobody agrees unanimously on which temperament is the best.

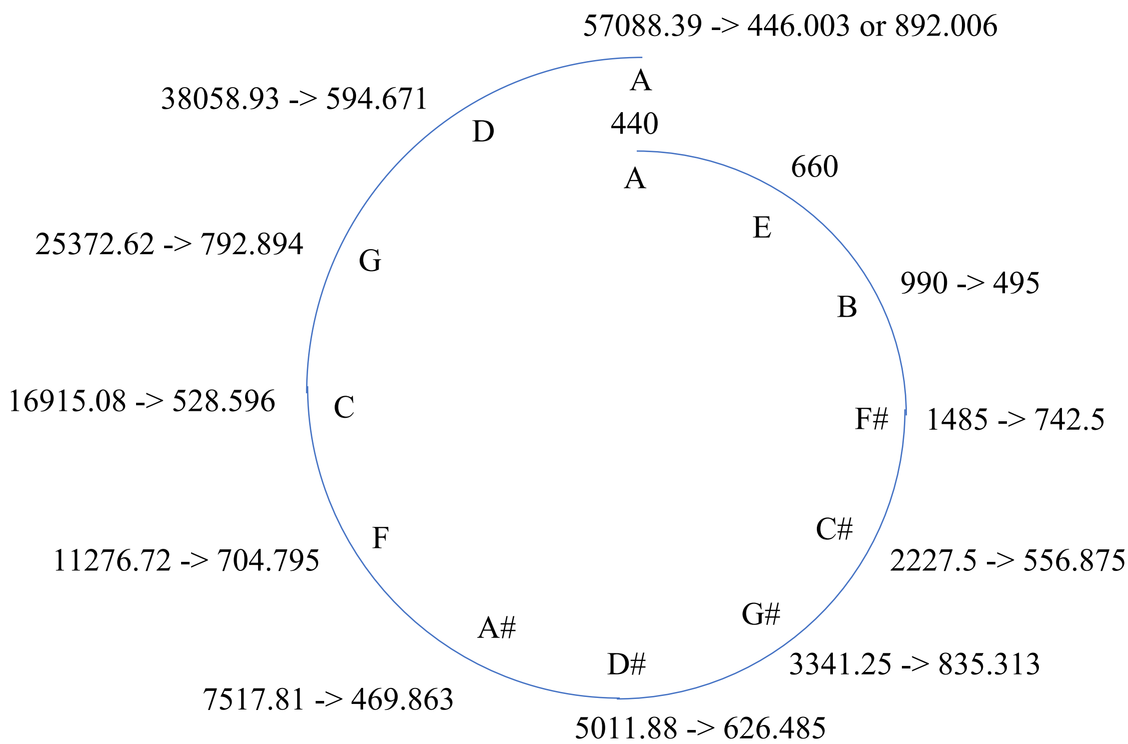

Figure 4. The spiral of pure perfect fifths. Notice how the system does not close visually and when each pitch is translated down to make the pitch frequency lie in the octave between 440 hz and 880 hz, the final A is 446.003 instead of 440.

Pythagoras, Pythagorean Tuning, and Just Intonation

Pythagoras was a mathematician renowned for his work in geometry and algebra. He is famous for the Pythagorean theorem, which is used all around the world today[9]. He was fascinated with numbers. He thought that everything in the universe could be described with numbers. In turn, he thought music could be described with numbers. Pythagoras demonstrated this with a lyre, an ancient instrument that resembles the modern-day harp, by comparing the sounds made by strings of different lengths. After experimenting with various ratios, he discovered that strings of the same lengths made the same pitches. He also realized that strings with different lengths often sounded dissonant when played together, but some sounded less dissonant than others. This led him to believe that the ratios between the lengths of the strings (which is inversely proportional to the frequency of the sounds produced by them) were directly related to how well the two strings sound together.

Pythagoras' interest in music eventually led him to develop the first tone system to ever exist, fittingly known as Pythagorean Tuning. In this system, Pythagoras attempted to build a tone system based on pure perfect fifths[1,3,4,9]. Tone systems that are built off of pure intervals are known as just intonation[1,4,9]. However, this system cannot be closed as shown in equation 1, so Pythagoras altered the last fifth only and attributed the whole Pythagorean comma to that interval[1,3,4,9]. Just intonation has some strengths and some weaknesses. One strength is that it keeps the rest of the fifths completely pure, so 11 out of 12 fifths are pure. However, that one altered fifth, known as the wolf fifth, is practically unusable due to how out-of-tune it is. Another weakness of this tone system is that many of the major and minor thirds (ratios of 5/4 and 6/5 respectively) are significantly out of tune, meaning they are not accurate relative to the pure interval ratios, since the use of pure fifths increases the dissonance of the thirds[1,3,4,9]. Despite its various weaknesses, the Pythagorean temperament is still a largely viable tone system and it laid the foundation for the development of future temperaments.

The Rise of Meantone Temperaments and Circulating Temperaments

The Pythagorean system worked for many centuries, but in the 11th century, the emergence of polyphonic music forced mathematicians and musicians to come up with an alternative[1]. Polyphony is the use of multiple voices or melodies in a piece of music. Prior to this, music was sung by choirs in unison, meaning everyone sang the same pitches in the same order at the same times. Polyphonic music revealed the problem with the Pythagorean tuning because it called for the combination of distinct pitches, and occasionally, the voices form major or minor thirds or even worse, a wolf fifth. This problem had to be solved, and it was remedied by the engineering of another type of tone system called a temperament. A temperament is a tone system based off pure perfect fifths like Pythagorean tuning, except it compromises some of those pure fifths to alleviate certain problems[1,9]. The first type of temperament that came about was the meantone temperament, where little fractions of the Pythagorean comma are split amongst some or occasionally all of the fifths[1,3,4,9]. This adjustment makes the thirds sound better, at the cost of the perfect fifths, and it also makes the wolf fifth sound slightly better. However, the wolf fifth was still an issue because most of the Pythagorean comma was still given to this one wolf fifth[1,4,9]. This issue gave rise to circulating temperaments. In a circulating temperament, the Pythagorean comma is unevenly distributed amongst often all of the fifths. This method completely removes the problem of the wolf fifth, but now, most of the fifths are no longer pure. They are all narrower than the 3:2 ratio. However, both the major and minor thirds sound even better than they did in meantone temperaments[3,9].

Equal Temperament and the Current System

Meantone temperaments and circulating temperaments were used extensively during the golden ages of music, spanning from the medieval period, through the Baroque era, and into the classical period. However, during the latter half of the classical period, a new temperament became the system of choice: the equal temperament. Briefly mentioned earlier, the equal temperament is a tone system where the Pythagorean comma is split evenly amongst all of the fifths[1,3,4,9]. This system removes the wolf fifth completely. Though all of the fifths are compromised, they are all tempered equally. This consistency makes the impurity of the fifths seem less noticeable[1,7,9]. The thirds are also more out of tune than many circulating and meantone temperaments, but the consistency of the equal temperament alleviates the issue considerably[1,7,9].

However, as nice as the equal temperament seems, it also has its weaknesses. The equal distribution of the Pythagorean comma makes every fifth sound the same, and in turn makes every key sound identical. This method restrains one of the most important aspects of music, emotional expression, because the key of a piece of music no longer impacts the emotional effect of that piece, as it sounds identical to the piece composed in any other key[1,4,5,7,9]. In fact, this is one of the primary reasons why equal temperament did not become prominent until the mid-to-late 18th century. Mathematicians and musicians knew that equal temperament would solve many of the problems associated with meantone and circulating temperaments, but the restraint it puts onto key character made them reluctant to switch. Even now, some people criticize equal temperament, asserting that it has ruined harmony[1,2,3,9].

Conclusion

Music tuning is a complex subject. The fundamental problem with tuning and tone systems is one that is without solution. The compromises to this issue, as seen through meantone, circulating, and equal temperaments, each have their strengths and weaknesses, and there is no current consensus on which system is truly ideal. The standard system used today, equal temperament, is disparaged by a number of musicians who believe different temperaments to be ideal[1,5,6]. The need for a solution to this problem is dire because even when all other issues with tuning can be solved through advanced technology and experience/practice, this problem will remain a limiting factor, as there is no perfect system[3]. Thus, a tone system that indisputably exceeds all of the current temperaments must be discovered or created.

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