MAT 442: HOMEWORK 6
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1. Problem 16:

   (a)

$$x_{t+1} = r + x_t + x_t^2$$

To find the equilibrium points, we set

$$f(x) = r + x + x^2 = x$$
$$\Rightarrow x^2 + r = 0$$

Using the quadratic formula we obtain

$$x = \frac{0 \pm 2\sqrt{-r}}{2} = \pm\sqrt{-r} = \boxed{\sqrt{-r} \quad \text{or} \quad -\sqrt{-r}.}$$

Since the derivative of

$$f(x) = r + x + x^2$$

is

$$\boxed{f'(x) = 2x + 1},$$

it follows that when $x = \sqrt{-r}$, $f'(\sqrt{-r}) = 1 + 2\sqrt{-r} > 1$ so that $x = \sqrt{-r}$ is unstable.But for $x = -\sqrt{-r}$, $f'(-\sqrt{-r}) = 1 - 2 - \sqrt{-r}$ so that for $-1 < r < 0$, we have $\left|f'(-\sqrt{-r})\right| < 1$ thus equilibrium $x = -\sqrt{-r}$ is LAS.Therefore the bifurcation diagram has the form given below.

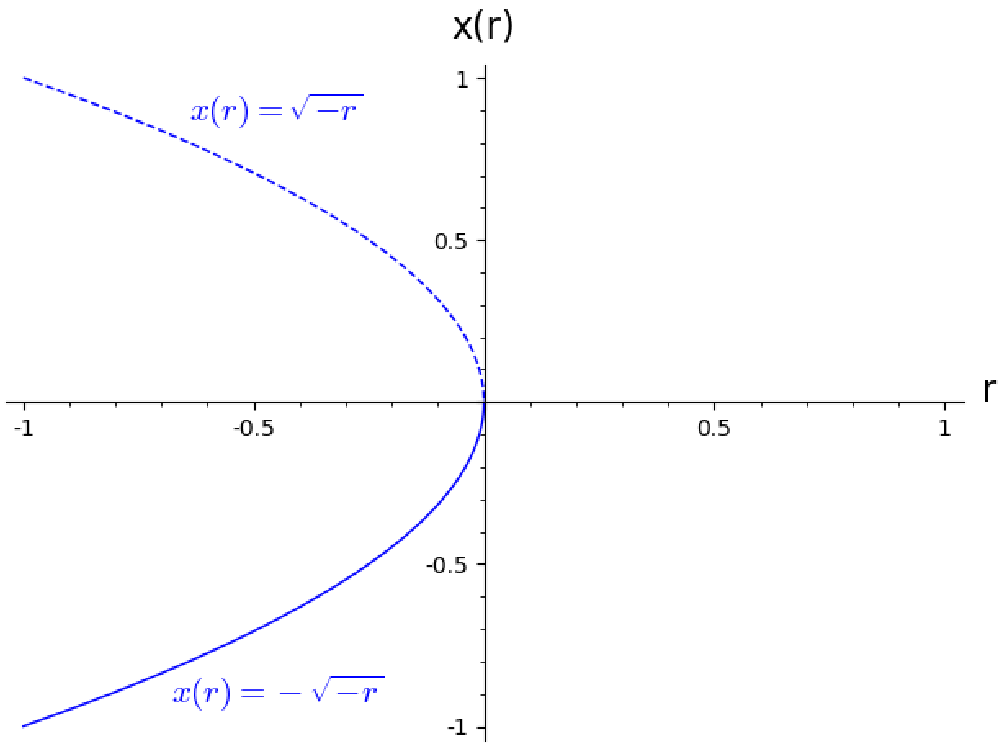

Figure 1: The dashed curve in the graph represents unstability and the solid curve represents stability.

(b)

$$x_{t+1} = (r+1)x_t - x_t^3$$

To find the equilibrium points, we set

$$\begin{aligned} &f(x) = (r+1)x - x^3 = x \\ &\Rightarrow -x + x(r+1) - x^3 = 0 \\ &\Rightarrow x(r - x^3) = 0 \end{aligned}$$

Using the quadratic formula we obtain

$$x = \frac{0 \pm 2\sqrt{r}}{2} = \pm\sqrt{r} = \boxed{x = 0 \text{ or } \sqrt{r} \text{ or } -\sqrt{r}}.$$

Since the derivative of

$$f(x) = (r+1)x - x^3$$

is

$$\boxed{f'(x) = r + 1 - 3x^2},$$

it follows that when $x = 0$, $f'(0) = r + 1$ and set $-1 < r + 1 < 1$ as usual so that $-2 < r < 0$ is unstable.

Howerver, for $x = \sqrt{r}$, $f'(\sqrt{r}) = r + 1 - 3r$ we have

$$-1 < r + 1 - 3r < 1 \Leftrightarrow -2 < -2r < 0 \Leftrightarrow \boxed{0 < r < 1}$$

which is stable and for $x = -\sqrt{r}$, $f'(-\sqrt{r}) = r + 1 + 3r$ we get

$$-1 < r + 1 + 3r < 1 \Leftrightarrow -2 < 4r < 0 \Leftrightarrow \boxed{-1/2 < r < 0}$$

which is also stable.

Therefore the bifurcation diagram has the form given below.

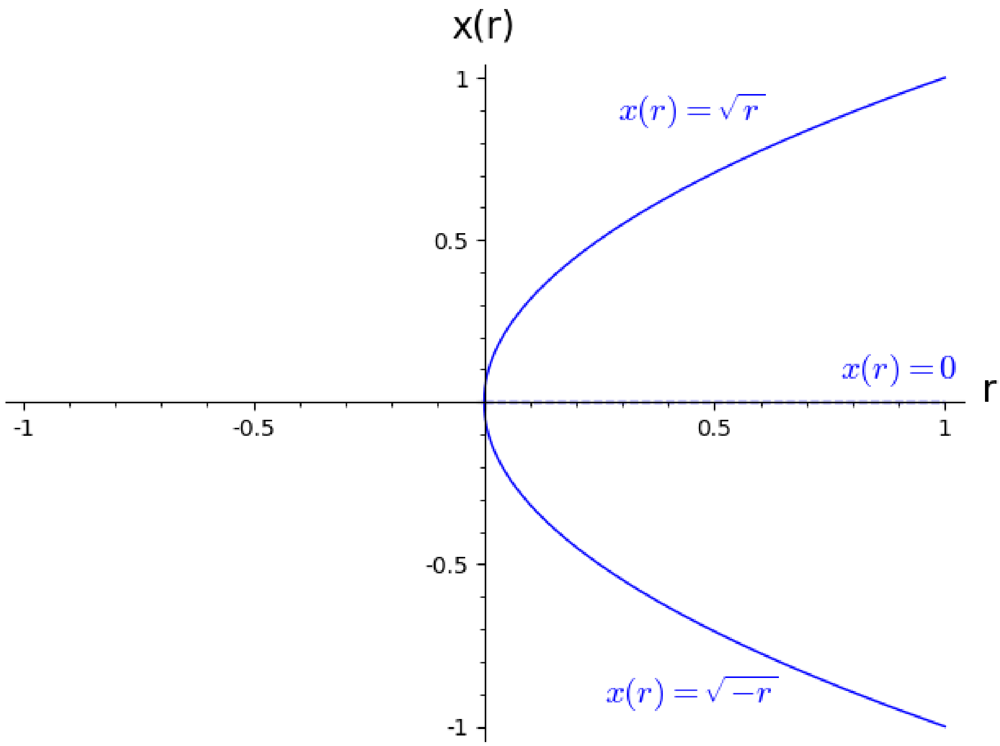

Figure 2: The dashed line in the graph represents unstability and the solid curve represents stability.

(c)

$$x_{t+1} = (r+1)x_t + x_t^2$$

To find the equilibrium points, we set

$$\begin{aligned}
&f(x) = (r+1)x + x^2 = x \\
&\Rightarrow -x + x(r+1) + x^2 = 0 \\
&\Rightarrow rx + x^2 = 0
\end{aligned}$$

Using the factor method we obtain

$$\begin{aligned}
&\Rightarrow x(r+x) = 0 \\
&\Rightarrow \boxed{x = 0 \text{ or } x = -r}.
\end{aligned}$$

Since the derivative of

$$f(x) = (r+1)x + x^2$$

is

$$\boxed{f'(x) = r + 1 + 2x},$$

it follows that when $x = 0$, $f'(0) = r + 1$ and set $-1 < f'(0) < 1$ as usual to get $\boxed{-2 < r < 0}$ so that $x = 0$ is unstable.But for $x = -r$, $f'(-r) = r + 1 + 2(-r)$ we set $-1 < f'(0) < 1$ as usual to get

$$-1 < -r + 1 < 1 \Leftrightarrow -2 < -r < 0 \Leftrightarrow \boxed{0 < r < 2}$$

thus r is stable.Therefore the bifurcation diagram has the form given below.

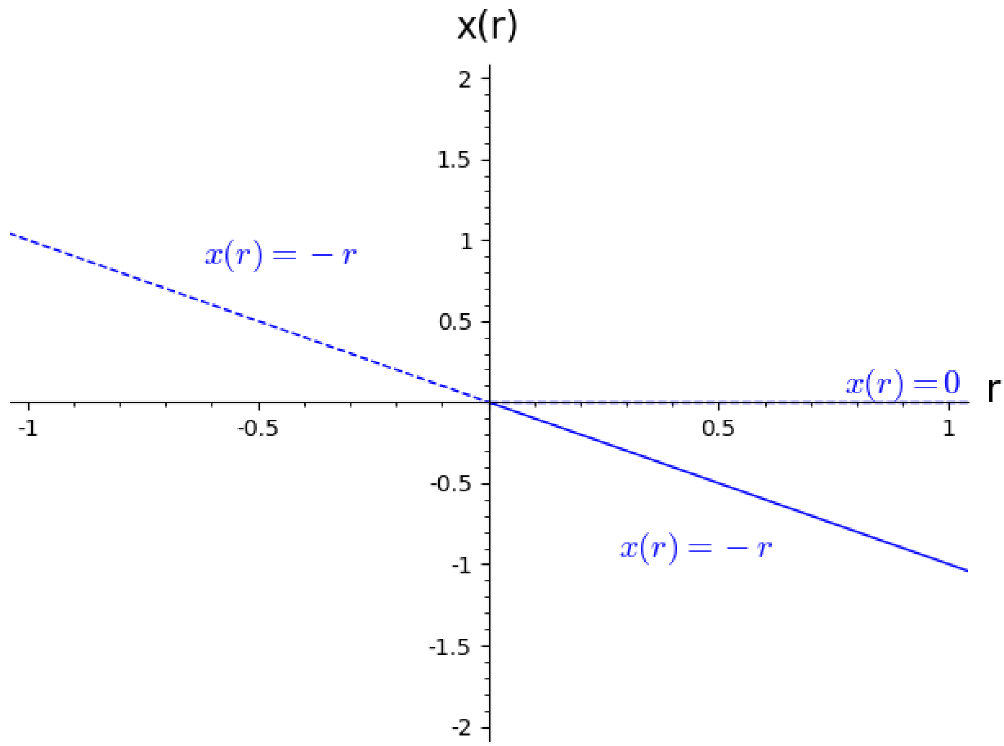

Figure 3: The dashed line in the graph represents unstability and the solid lines represents stability.

2. Problem 17:

$$x_{t+1} = r - x_t - x_t^2$$

To find the equilibrium points, we set

$$f(x) = r - x - x^2 = x$$
$$\Rightarrow -r + 2x + x^2 = 0$$

Using the quadratic formula we obtain

$$x = \frac{-2 \pm \sqrt{4r+4}}{2} = -1 \pm \sqrt{r+1} = \boxed{\sqrt{r+1} - 1 \quad \text{or} \quad -\sqrt{r+1} - 1.}$$

Since the derivative of

$$f(x) = r - x - x^2$$

is

$$\boxed{f'(x) = -1 - 2x}.$$

Now to check for the stability we have $1 + r > 0 \Leftrightarrow r > -1$.Then

$$f'(-1 + \sqrt{r+1}) = -1 - 2(-1 + \sqrt{1+r}) = 1 - 2\sqrt{1+r},$$

and as always we set

$$-1 < 1 - \sqrt{1+r} < 1 \Leftrightarrow 1 > \sqrt{1+r} > 0 \Leftrightarrow 0 < 1 + r < 1 \Leftrightarrow \boxed{-1 < r < 0}$$

Hence r is stable. For

$$f'(-1 - \sqrt{1+r}) = -1 - 2(-1 - \sqrt{1+r}) = 1 + 2\sqrt{1+r} > 1$$

is unstable.

For 2-cycle,I used *mathematica* command

$$\text{solve}[f[f[x]] == x, x]$$

to give us the solutions

$$-\sqrt{r}, \sqrt{r}, -1 - \sqrt{1+r}, -1 + \sqrt{1+r}.$$

Since $f'(y) = -1 - 2y$ we compute

$$f'(-\sqrt{r}) = -1 + 2(\sqrt{r}) \text{ and } f'(\sqrt{r}) = -1 - 2\sqrt{r}.$$

Now to check the stability of the 2-cycles, we have

$$f'(-\sqrt{r})f'(\sqrt{r}) = (-1 + 2\sqrt{r})(-1 - 2\sqrt{r}) = 1 - 4\sqrt{r}$$

thus

$$-1 < 1 - 4r < 1 \Leftrightarrow -2 < -4r < 0 \Leftrightarrow 0 < 4r < 2 \Leftrightarrow \boxed{0 < r < 1/2}$$

Therefore 2-cycles is LAS.The bifurcation diagram has the form given below.

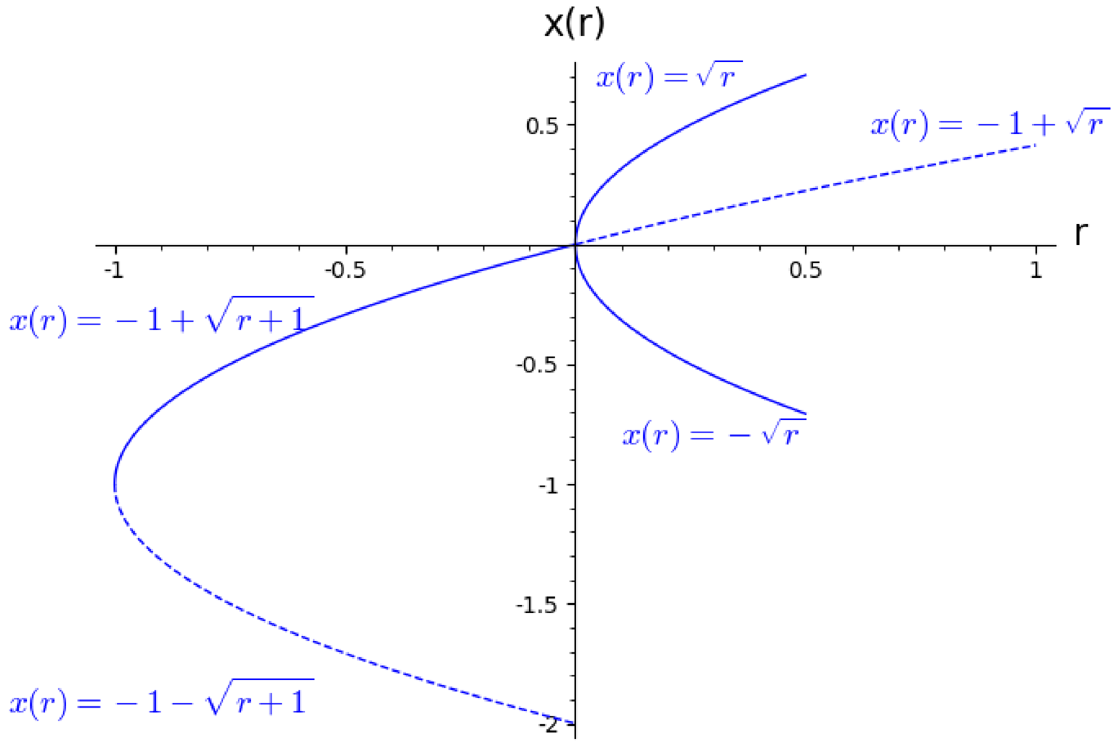

Figure 4: The dashed curve in the graph represents unstability and the solid curve represents stability

References

[1] Linda Allen's book, An Introduction to Mathematical Biology, Pearson, 2007