Exercise 4.2

task a)

In[1139]:=

```
rdot[r_, μ_] := μ * r - r^3
θdot[ω_, r_, ν_] := ω + ν * r^2

sol = Solve[rdot[r, μ] == 0, r]
```

Out[1141]=

$$\{\{r \to 0\}, \{r \to -\sqrt{\mu}\}, \{r \to \sqrt{\mu}\}\}$$

r = 0 => no limit cycle, r < 0 does not make sense since r is a radius => r0 = sqrt(mu)

In[1142]:=

```
(*T = 2π / angular velocity = 2π/θdot*)
T = 2 π / θdot[ω, r /. sol[[3]], ν]
```

Out[1142]=

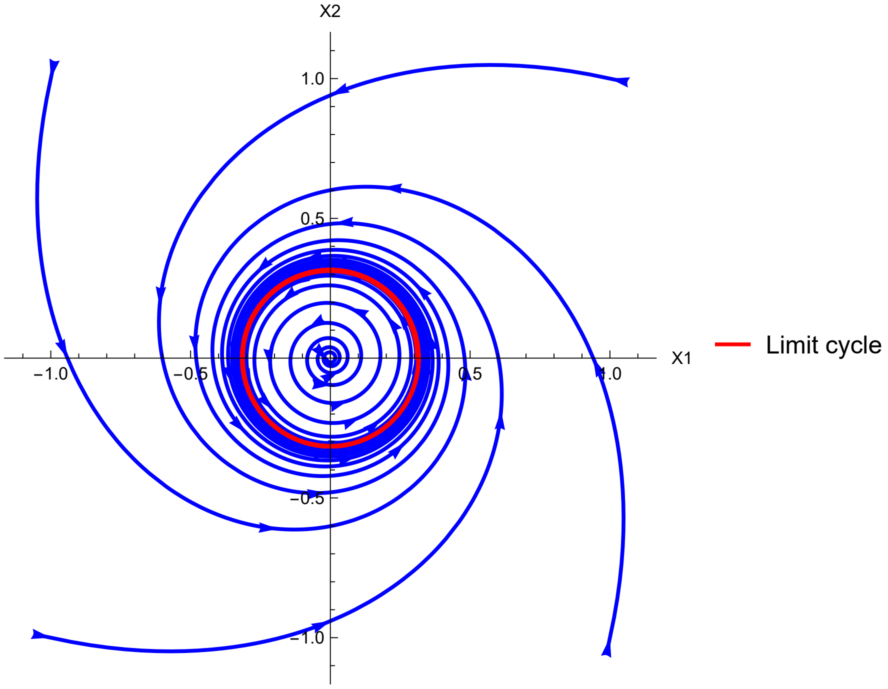
$$\frac{2 \pi}{\mu \nu + \omega}$$

task b)

In[1143]:=

```
ClearAll["Global`*"]

X1dot[X1_, X2_] := (1/10) * X1 - X2^3 - X1 * X2^2 - X1^2 * X2 - X2 - X1^3
X2dot[X1_, X2_] := X1 + (1/10) * X2 + X1 * X2^2 + X1^3 - X2^3 - X1^2 * X2

initialConditions = Flatten[Table[{x, y}, {x, -1, 1, 0.2}, {y, -1, 1, 0.2}], 1];

initialConditions = Join[
   {{-0.01, -0.01}}, {{1, 1}}, {{-1, 1}}, {{1, -1}}, {{-1, -1}}
  ];


solutions = Table[NDSolve[{
      X1'[t] == X1dot[X1[t], X2[t]],
      X2'[t] == X2dot[X1[t], X2[t]],
      X1[0] == init〚1〛,
      X2[0] == init〚2〛},
     {X1, X2}, {t, 0, 50}],
    {init, initialConditions}];

limitCycleSolution = NDSolve[{
     X1'[t] == X1dot[X1[t], X2[t]],
     X2'[t] == X2dot[X1[t], X2[t]],
     X1[0] == 0.31,
     X2[0] == 0 },
    {X1, X2}, {t, 0, 1000} ];

arrows = Table[{0.03, t}, {t, 0, 1, 0.05}];

p1 = ParametricPlot[
     Evaluate[Table[{X1[t], X2[t]} /. sol, {sol, solutions}]], {t, 0, 50},
     PlotRange → All,
     AxesLabel → {"X1", "X2"},
     PlotStyle → Blue] /. Line[x_] ⧴ {Arrowheads[arrows], Arrow[x]};

p2 = ParametricPlot[Evaluate[{X1[t], X2[t]} /. limitCycleSolution], {t, 0, 50},
    PlotStyle → Red, PlotLegends → {"Limit cycle"}
  ];

Show[p1, p2]
```

Out[1153]=



task c)

In[1154]:=

```
ClearAll["Global`*"]
X1[r_, θ_] = r * Cos[θ];
X2[r_, θ_] = r * Sin[θ];

X1computed[r_, θ_] = D[X1[r[t], θ[t]], t] // FullSimplify
X2computed[r_, θ_] = D[X2[r[t], θ[t]], t] // FullSimplify
```

Out[1157]=

Cos[θ[t]] r′[t] − r[t] Sin[θ[t]] θ′[t]

Out[1158]=

Sin[θ[t]] r′[t] + Cos[θ[t]] r[t] θ′[t]

In[1159]:=

```
X1dot[X1_, X2_] = (1 / 10) * X1[r, θ] - X2[r, θ] ^3 - X1[r, θ] * X2[r, θ] ^2 -
    X1[r, θ] ^2 * X2[r, θ] - X2[r, θ] - X1[r, θ] ^3 // FullSimplify
X2dot[X1_, X2_] = X1[r, θ] + (1 / 10) * X2[r, θ] + X1[r, θ] * X2[r, θ] ^2 +
    X1[r, θ] ^3 - X2[r, θ] ^3 - X1[r, θ] ^2 * X2[r, θ] // FullSimplify
```

Out[1159]=

$$\left(\frac{r}{10} - r^3\right) \text{Cos}[\theta] - r \left(1 + r^2\right) \text{Sin}[\theta]$$

Out[1160]=

$$\left(r + r^3\right) \text{Cos}[\theta] + \left(\frac{r}{10} - r^3\right) \text{Sin}[\theta]$$

In[1161]:=

```
rdot[r_, μ_] = μ * r - r^3
θdot[ω_, r_, ν_] = ω + ν * r^2
```

Out[1161]=

$-r^3 + r\mu$

Out[1162]=

$r^2 \nu + \omega$

By comparing constants of the computed xdot and the given xdot (eq 2) we find that $\mu = 1/10$, $\nu = 1$, $\omega = 1$.

task d)

In[1163]:=

```
ClearAll["Global`*"]
r0 = Sqrt[1 / 10];
T = 2 π / ((1 / 10) + 1);
M0 = IdentityMatrix[2];

X1dot[X1_, X2_] := (1 / 10) * X1 - X2^3 - X1 * X2^2 - X1^2 * X2 - X2 - X1^3;
X2dot[X1_, X2_] := X1 + (1 / 10) * X2 + X1 * X2^2 + X1^3 - X2^3 - X1^2 * X2;

jacobi = D[{X1dot[X1, X2], X2dot[X1, X2]}, {{X1, X2}}];

MPrime[M_, X_] := (jacobi /. {X1 → X[[1]], X2 → X[[2]]}).M;

solution = NDSolve[{
    X1'[t] == X1dot[X1[t], X2[t]],
    X2'[t] == X2dot[X1[t], X2[t]],
    M'[t] == MPrime[M[t], {X1[t], X2[t]}],
    X1[0] == r0,
    X2[0] == 0,
    M[0] == M0},
   {X1, X2, M}, {t, 0, T}];

solM[t_] = M[t] /. solution[[1]];

solX[t_] = {X1[t], X2[t]} /. solution[[1]];

X1[t_] := solX[t][[1]];
X2[t_] := solX[t][[2]];
M11[t_] := solM[t][[1, 1]];
M12[t_] := solM[t][[1, 2]];
M21[t_] := solM[t][[2, 1]];
M22[t_] := solM[t][[2, 2]];


Plot[Evaluate[{X1[t], X2[t], M11[t], M12[t], M21[t], M22[t]}],
 {t, 0, T}, PlotLegends → {"X1", "X2", "M11", "M12", "M21", "M22"},
 PlotRange → Full,
 PlotStyle → {Thick, Red, Blue, Green, Orange, Purple, Magenta},
 AxesLabel → {"t", "Values"}]
```

Part: Part specification

InterpolatingFunction[{{0., 5.71199}}, {5, 3, 1, {69}, {4}, 0, 0, 0, 0, Automatic, {}, {}, False}, {{0., 0.000140231, ≪48≫, ≪19≫}}, {{{{1., 0.}, {0., 1.}}, {{−0.2, −1.1}, {1.3, 0.}}}, {{{0.999972, −0.000154254}, {0.000182291, 1.}}, {{−0.200226, −1.1}, {1.29993, −0.000169677}}}, ≪48≫, ≪19≫}, {Automatic}][t][[1, 1]] is longer than depth of object.

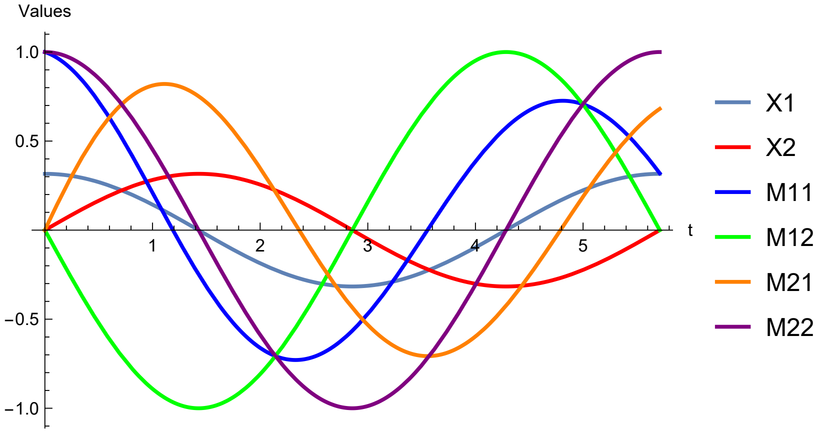
Part: Part specification

InterpolatingFunction[{{0., 5.71199}}, {5, 3, 1, {69}, {4}, 0, 0, 0, 0, Automatic, {}, {}, False}, {{0., 0.000140231, ≪48≫, ≪19≫} }, {{{{1., 0.}, {0., 1.}}, {{−0.2, −1.1}, {1.3, 0.}}}, {{{0.999972, −0.000154254}, {0.000182291, 1.}}, {{−0.200226, −1.1}, { 1.29993, −0.000169677}}}, ≪48≫, ≪19≫}, {Automatic}][t][[1, 2]] is longer than depth of object.

Part: Part 2 of InterpolatingFunction[{{0., 5.71199}}, {5, 3, 1, {69}, {4}, 0, 0, 0, 0, Automatic, {}, {}, False}, {{0., 0.000140231, ≪48≫, ≪19≫} }, {{{{1., 0.}, {0., 1.}}, {{−0.2, −1.1}, {1.3, 0.}}}, {{{0.999972, −0.000154254}, {0.000182291, 1.}}, {{−0.200226, −1.1}, { 1.29993, −0.000169677}}}, ≪48≫, ≪19≫}, {Automatic}][t] does not exist.

Part: Part 2 of InterpolatingFunction[{{0., 5.71199}}, {5, 3, 1, {69}, {4}, 0, 0, 0, 0, Automatic, {}, {}, False}, {{0., 0.000140231, ≪48≫, ≪19≫} }, {{{{1., 0.}, {0., 1.}}, {{−0.2, −1.1}, {1.3, 0.}}}, {{{0.999972, −0.000154254}, {0.000182291, 1.}}, {{−0.200226, −1.1}, { 1.29993, −0.000169677}}}, ≪48≫, ≪19≫}, {Automatic}][t] does not exist.

Out[1180]=



task e)

In[1181]:=

```
solM[T]
```

Out[1181]=

$\{\{0.319053, 2.12317\times10^{-8}\}, \{0.680947, 1.\}\}$

task f)

In[1182]:=

```
eig = Eigenvalues[solM[T]];
σ = (1 / T) * Log[eig]
```

Set: Tag Times in σ is Protected.

Out[1183]=

$\{5.78753\times10^{-9}, -0.2\}$

task g)

In[1184]:=

```
ClearAll["Global`*"]
T = 2 π / ((1 / 10) + 1);

JgI[r_, θ_] = {{Cos[θ], -r Sin[θ]}, {Sin[θ], r Cos[θ]}};
Jg[r_, θ_] = Inverse[JgI[r, θ]];

rdot[r_] := (1 / 10) * r - r^3;
θdot[r_] := 1 + r^2;

r0 = Sqrt[1 / 10];
θ0 = 0;

Mcart0 = IdentityMatrix[2];
Mpol0 = JgI[r0, θ0].Mcart0.Jg[r0, θ0];

Jpolar = D[{rdot[r], θdot[r]}, {{r, θ}}];

Mpol = Mpol0.MatrixExp[Jpolar * T];

Mcart[r_, θ_] = JgI[r, θ].Mpol.Jg[r, θ];

sol = Mcart[r0, θ0]
```

Out[1197]=

$$\left\{\left\{e^{-4\pi/11}, 0\right\}, \left\{1 - e^{-4\pi/11}, 1\right\}\right\}$$

task h)

In[1198]:=

```
eig = Eigenvalues[Mcart[r0, θ0]];
σ = (1 / T) * Log[eig]
```

Out[1199]=

$$\left\{0, -\frac{1}{5}\right\}$$