Exercise 2.2

Task a)

In[2169]:=

```
(*Task a)*)
ClearAll["Global`*"]
A = {{σ + 1, 3}, {-2, σ - 1}};
Eigenvalues[A]
```

Out[2171]=

$$\left\{-i\sqrt{5}+\sigma,\ i\sqrt{5}+\sigma\right\}$$

Task b)

In[2172]:=

```
(*Task b)*)
ClearAll["Global`*"]
A = {{σ + 1, 3}, {-2, σ - 1}};

x0 = {u, v};

x[t_] = {x1[t], x2[t]};

solution[t_, u_, v_] = DSolve[{x'[t] == A.x[t], x[0] == x0}, x[t], t];

xSolution[t_, u_, v_] = x[t] /. solution[t, u, v][[1]];

{x1Solution[t_, u_, v_], x2Solution[t_, u_, v_]} = Simplify[xSolution[t, u, v]]
```

Out[2178]=

$$\left\{\frac{1}{5} e^{t\sigma}\left(5\,u\,\mathrm{Cos}\left[\sqrt{5}\,t\right]+\sqrt{5}\,(u+3\,v)\,\mathrm{Sin}\left[\sqrt{5}\,t\right]\right),\right.$$
$$\left.\frac{1}{5} e^{t\sigma}\left(5\,v\,\mathrm{Cos}\left[\sqrt{5}\,t\right]-\sqrt{5}\,(2\,u+v)\,\mathrm{Sin}\left[\sqrt{5}\,t\right]\right)\right\}$$

Task c)

In[2179]:=

```
(*Task c)*)
Clear[σ, u, v, values]
σ = -0.1;

(*u and v changes starting position and amplitude of oscillations.*)

xSol[t_, u_, v_] = xSolution[t, u, v]〚1〛;
ySol[t_, u_, v_] = xSolution[t, u, v]〚2〛;

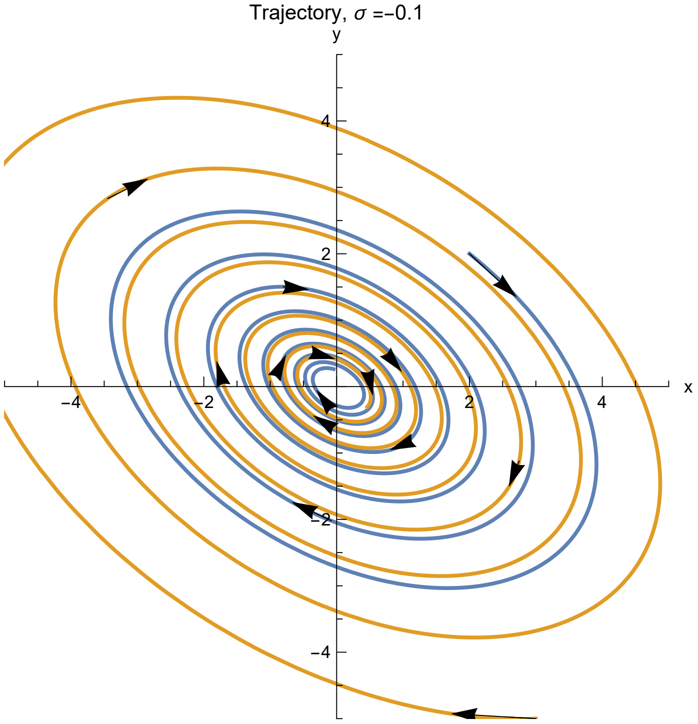
(*Initial conditions*)
values = {{2, 2}, {3, -5}};

plots = Table[Module[{u, v}, {u, v} = values〚i〛;
    ParametricPlot[Evaluate[{xSol[t, u, v], ySol[t, u, v]}], {t, 0, 25},
     PlotStyle → Directive[ColorData[97, i], Thick],
     PlotRange → {{-5, 5}, {-5, 5}},
     AxesLabel → {"x", "y"},
     PlotLabel → "Trajectory, σ =" <> ToString[σ]]],
   {i, 1, Length[values]}];

arrows = Graphics[Table[Module[{u, v}, {u, v} = values〚i〛;
     Table[Arrow[{Evaluate[{xSol[t, u, v], ySol[t, u, v]}],
        Evaluate[{xSol[t + 0.1, u, v], ySol[t + 0.1, u, v]}] }],
      {t, 0, 24, 4}]], {i, 1, Length[values]}]];

Show[plots, arrows, ImageSize → Medium]
```

Out[2186]=



In[2187]:=

```
Clear[σ, u, v]
σ = 0;

xSol[t_, u_, v_] = xSolution[t, u, v][[1]];
ySol[t_, u_, v_] = xSolution[t, u, v][[2]];

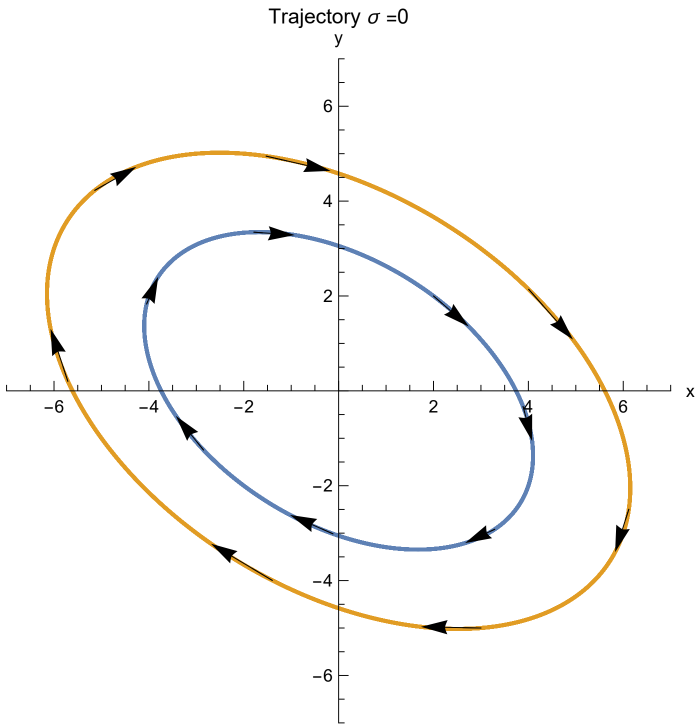
(*Initial conditions*)
values = {{2, 2}, {3, -5}};

plots = Table[Module[{u, v}, {u, v} = values[[i]];
    ParametricPlot[Evaluate[{xSol[t, u, v], ySol[t, u, v]}], {t, 0, 25},
     PlotStyle → Directive[ColorData[97, i], Thick],
     PlotRange → {{-7, 7}, {-7, 7}},
     AxesLabel → {"x", "y"},
     PlotLabel → "Trajectory σ =" <> ToString[σ]]],
   {i, 1, Length[values]}];

arrows = Graphics[Table[Module[{u, v}, {u, v} = values[[i]];
     Table[Arrow[{Evaluate[{xSol[t, u, v], ySol[t, u, v]}],
        Evaluate[{xSol[t + 0.1, u, v], ySol[t + 0.1, u, v]} ]}],
      {t, 0, 24, 4}]], {i, 1, Length[values]}]];

Show[plots, arrows, ImageSize → Medium]
```

Out[2194]=



In[2195]:=

```
Clear[σ, u, v]
σ = 0.1;
xSol[t_, u_, v_] = xSolution[t, u, v][[1]];
ySol[t_, u_, v_] = xSolution[t, u, v][[2]];

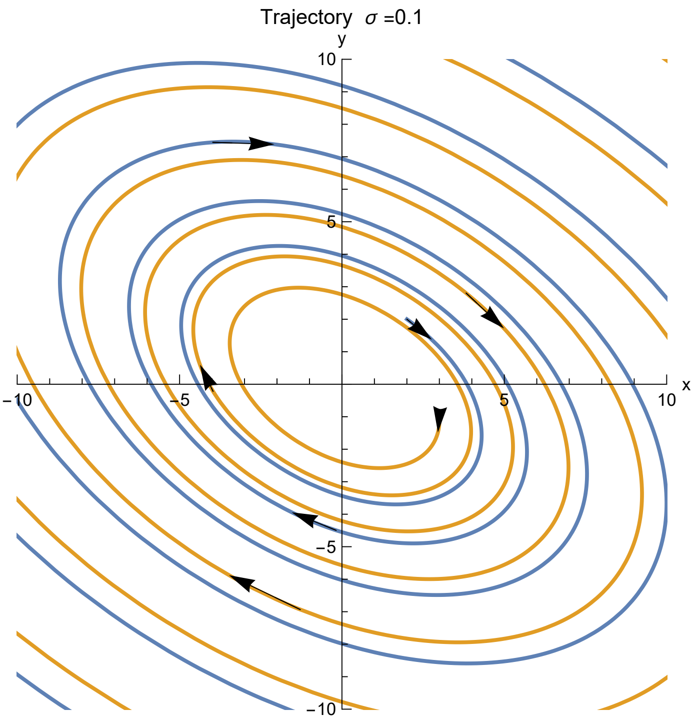
(*Initial conditions*)
values = {{2, 2}, {3, -1}};

plots = Table[Module[{u, v}, {u, v} = values[[i]];
    ParametricPlot[Evaluate[{xSol[t, u, v], ySol[t, u, v]}], {t, 0, 25},
     PlotStyle → Directive[ColorData[97, i], Thick],
     PlotRange → {{-10, 10}, {-10, 10}},
     AxesLabel → {"x", "y"},
     PlotLabel → "Trajectory  σ =" <> ToString[σ]]],
   {i, 1, Length[values]}];

arrows = Graphics[Table[Module[{u, v}, {u, v} = values[[i]];
     Table[Arrow[{Evaluate[{xSol[t, u, v], ySol[t, u, v]}],
        Evaluate[{xSol[t + 0.1, u, v], ySol[t + 0.1, u, v]}] }],
      {t, 0, 24, 4}]], {i, 1, Length[values]}]];

Show[plots, arrows, ImageSize → Medium]
```

Out[2202]=



Task d)

In[2203]:=

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(*Task d)*)

(*Angular frequency from studying the answers [x(t),y(t)]]*)
ω = Sqrt[5];

periodTime = 2 * π / ω
```

Out[2204]=

$$\frac{2\pi}{\sqrt{5}}$$

Task e)

In[2205]:=

```
(*Task e)*)
(*Choose initial conditions (u,v) to (1,1)*)
(*Have tried for multiple initial conditions
 and they all give the same ratio which is reasonable.*)
u = 1;
v = 1;
σ = 0;

(*Copied from task b), use initial conditions u=v=1*)
```

$$x[t\_, u\_, v\_] := \frac{1}{5} e^{t\sigma} \left(5\, u \operatorname{Cos}\left[\sqrt{5}\, t\right] + \sqrt{5}\, (u + 3\, v) \operatorname{Sin}\left[\sqrt{5}\, t\right]\right)$$

$$y[t\_, u\_, v\_] := \frac{1}{5} e^{t\sigma} \left(5\, v \operatorname{Cos}\left[\sqrt{5}\, t\right] - \sqrt{5}\, (2\, u + v) \operatorname{Sin}\left[\sqrt{5}\, t\right]\right)$$

```
(*The distance of any point on the ellipse from the origin*)
l[t_] := Sqrt[x[t, u, v]^2 + y[t, u, v]^2] // FullSimplify

major = Maximize[l[t], t] // FullSimplify;
minor = Minimize[l[t], t] // FullSimplify;
ratio = major[[1]] / minor[[1]] // FullSimplify
```

Out[2213]=

$$\frac{1}{2}\left(1 + \sqrt{5}\right)$$

Task f)

In[2214]:=

```
(*Initial conditions:*)
u = 1;
v = 1;

(*Insert the time at which we reach the major*)
majorVec = {x[t, u, v] /. major[[2, 1]], y[t, u, v] /. major[[2, 1]]} // FullSimplify;

normalization = Norm[majorVec] // FullSimplify;

normalizedVec = majorVec / normalization  // Simplify
```

Out[2218]=

$$\left\{\sqrt{\frac{1}{10}\left(5+\sqrt{5}\right)},\ -\sqrt{\frac{2}{5+\sqrt{5}}}\right\}$$