Notes for Linear Algebra

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Preface

The cover is from the album ***Dark side of the Moon*** by *Pink Floyd*.

— Eric

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Chapter 1

Introduction

> "But the greatest thing by far is to have a command of metaphor. This alone cannot be imparted by another; it is the mark of genius, for to make good metaphors implies an eye for resemblances."
>
> — *Aristotle, Poetics, 1459a7-10, tr. by S.H. Butcher.*

1.1 Why is Linear Algebra important?

Example 1.1.1. matrix and information

class \ student	1	2	3	4	5
1	1	0	0	1	1
2	0	1	1	0	1
3	1	0	1	0	1
4	1	0	0	1	1

we have 4 students and 5 classes, 1 implies this student is in the class while 0 means not.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

$A_{ij} = 1$ means the ith student is in the jth class, sum the row we can get the number of classes a student attends, sum the column we can get the number of students a class have. Our question is how can we calculate the number of students that take both class 1 and 3? With a simple matrix we can easily get the answer. But if there are thousands of students or class what way we should use? Look at this matrix multiplication.

$$A^T A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 2 \\ 3 & 1 & 2 & 2 & 4 \end{bmatrix}$$

This matrix is symmetric, Next, look at the diagonal entries: 3, 1, 2, 2, 4 - these give the total enrollments in class 1, 2, 3, 4, and 5. Next, let's look at the entry in row 1, column

4. This is a 2. It's also how many students are enrolled in both classes 1 and 4. In fact, you can check that the ijth entry of A^TA is the number of students enrolled in both class i and class j.

Proof. if the ijth entry of $p \times q$ matrix B is B_{ij} and the ijth entry of $q \times r$ matrix C is C_{ij} then the ijth entry of $p \times r$ matrix BC is $\sum_{\ell=1}^{q} B_{i\ell}C_{\ell j}$. If A is a $m \times n$ matrix then A^T is a $n \times m$ matrix, so $(A^TA)_{ij} = \sum_{\ell=1}^{m} A^T_{i\ell}A_{\ell j} = \sum_{\ell=1}^{m} A_{\ell i}A_{\ell j}$. $A_{\ell i}A_{\ell j} = 1$ means that $A_{\ell i} = A_{\ell j} = 1$. So it means that the ℓth student is in the i class and the ℓth student is in the jth class. So we can check the $(A^TA)_{ij}$ entry to see the number of students who attends both the ith and jth class. □

1.2 Graphs and Matrices

Encoding a graph into a matrix.

Definition 1.2.1. A **Graph** is a pair (V, E), where V is a set of textbfvertices and E is a subset of $V \times V$. That is: E is a subset of ordered pairs(u, v) where u and v are vertices. Elements of E are called **edges or links** $(u, v) \in E$ means that u and v are connected. We require that $(u, u) \notin E$ $(u, v) \in E$ if and only if $(v, u) \in E$. That means we consider undirected graphs.

A **Path** is a finite sequence of vertices$(i_1, i_2, i_3, \ldots, i_k)$ that (i_j, i_{j+1}) is an textbfedge. In other words, a path is a sequence of vertices in the graph such that each vertex is connected to the following vertex. The textbflengthof a path is the number of edges that we travel along the path.

The **Degree** of a vertex i is the number of vertices that are connected to i, we write deg$i$ for the degree of i.

Definition 1.2.2. The adjacency matrix A of a graph(V, E) has entries given by:

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

Notice that the entries in the diagonal are all 0 because we do not consider loops. And the adjacency matrix is symmetric because we consider undirected graph.

Proof. Suppose there are two zero vectors $\mathbf{0}$ and $\mathbf{0}'$. Then:

$$\mathbf{0} + \mathbf{0}' = \mathbf{0}', \quad \text{by definition of } \mathbf{0}$$

But also:

$$\mathbf{0} + \mathbf{0}' = \mathbf{0}, \quad \text{by definition of } \mathbf{0}'$$

Hence, $\mathbf{0} = \mathbf{0}'$, proving uniqueness. □

Remark 1.2.3. The zero vector is a specific element in the space, not an absence of element.

Conclusion 1.2.4. Every vector space has a unique zero vector satisfying $v + 0 = v$ for all v.