Notes for Measure theory

Based on *Measure Theory* by Donald.Cohn

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Preface

The cover is from the album ***Orange Moon*** by *Khalil Fong*. The note is based on *Measure Theory* by Donald.Cohn, *Set Theory* by Jech.

— Eric

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Chapter 1

Introduction

“也许是天气，也许是运气，也许是因为有人不放弃。”

—方大同，《因为你》

1.1 Riemann integral – Darboux's Definition

Let $[a, b]$ be a closed bounded interval. A **partition** of $[a, b]$ is a **finite sequence** $\{a_i\}_{i=0}^{k}$ of real numbers such that

$$a = a_0 < a_1 < \cdots < a_k = b \tag{1.1}$$

we call the value a_i the *division points* of the partition. the partition is denoted by a symbol $\mathscr{P}$.

Suppose that f is a bounded real-valued function on $[a, b]$ and that $\mathscr{P}$ is a partition of $[a, b]$, say with division points $\{a_i\}_{i=0}^{k}$. We define m_i and M_i by $m_i = inf\{f(x) : x \in [a_{i-1}, a_i]\}$ and $M_i = sup\{f(x) : x \in [a_{i-1}, a_i]\}$. Then the *lower sum* is define to be $\sum_{i=1}^{k} m_i(a_i - a_{i-1})$ and the *upper sum* is defined to be $\sum_{i=1}^{k} M_i(a_i - a_{i-1})$.

Since f is bounded, so there are m and M such that $m \leq f \leq M$. So the lower sum satisfies

$$\sum_{i=1}^{k} m_i(a_i - a_{i-1}) \leq \sum_{i=1}^{k} M_i(a_i - a_{i-1}) = M(b - a) \tag{1.2}$$

So the set of lower sums of f which denoted by $\underline{\int}_a^b$ has a supremum. Similarly, the upper sums of f which denoted by $\overline{\int}_a^b$ has a infimum. If $\underline{\int}_a^b = \overline{\int}_a^b$, which means that when the number of division points tends to infinity, the lower sum equals the upper sum, then f is said to be *Riemann integrable* on $[a, b]$.

1.2 Riemann Integral – Riemann's Definition

A *tagged partition* of an interval $[a, b]$ is a partition $\{a_i\}_{i=0}k$ of $[a, b]$, together with a sequence $\{x_i\}_{i=1}k$ of *tags* such that $a_{i-1} \leq x_i \leq a_i$ holds for $i = 1, \cdots, k$.

The *mesh* $\|\mathscr{P}\|$ of a partition $\mathscr{P}$ is defined by $\|\mathscr{P}\| = max_i(a_i - a_{i-1})$, THe mesh of a partition is the length of the longest of its subintervals.

The Riemann sum $\mathscr{R}(f, \mathscr{P})$ is defined by

$$\mathscr{R}(f, \mathscr{P}) = \sum_{i=1}^{k} f(x_i)(a_i - a_{i-1}) \tag{1.3}$$

If there is a number L such that

$$\lim_{\mathscr{P}} \mathscr{R}(f, \mathscr{P}) = L \tag{1.4}$$

The limit is taken as the *mesh* of $\mathscr{P}$ approachs 0.

Darboux's and Riemann's definition are equivalent.Every continuous function on $[a, b]$ is Riemann integrable.

Theorem 1.2.1: The Fundamental Theorem of Calculus

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $F : [a, b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f(t)dt$. Then F is differentiable at each x in $[a, b]$ and its derivative is given by $F'(x) = f(x)$.

1.3 From Riemann to Lebesgue

In many situations, it's necessary to reverse the order of taking limits and evaluating integrals, which means that

$$\int_a^b \lim_n f_n(x)dx = \lim_n \int_a^b f_n(x)dx \tag{1.5}$$

We need a theorem about that.

Theorem 1.3.1: When can we reverse the order of limit and integral

Suppose that $\{f_n(x)\}$ is a sequence of integrable functions on the interval $[a, b]$ and that f is a function that $\{f_n\}$ converges to f in a suitable [a] Then f is integrable and

$$\int_a^b f(x)dx = \lim_n \int_a^b f_n(x)dx \tag{1.6}$$

$$f(x) = \lim_n f_n(x) \tag{1.7}$$

which means that $\int_a^b \lim_n f_n(x) = \lim_n \int_a^b f_n(x)dx$

[a] Of course we should figure out what "suitable" means

If the converge is *"converge uniformly"*[1], we would say the theorem is valid for the Riemann integral.

But if it is only pointwise[2] converge. Then the theorem may fail.

Example 1.3.1: a example that makes Theorem 1.3.1 fail

For each positive integer n let f_n be a piecewise linear function 分段线性函数 on $[0, 1]$ whose graph is made up of three line segments, connecting the points $(0, 0), (\frac{1}{2n}, 2n), (\frac{1}{n}, 0)$ and $(1, 0)$. Then for each n the triangle formed by the graph of f_n and the x-axis has area 1. So for every f_n we have $\int_0^1 f_n(x) = 1$. but $\int_0^1 \lim_n f_n(x)dx$ doesn't exist in Riemann integrals, which means that $\int_a^b \lim_n f(x)dx \neq \lim_n \int_a^b f_n(x)$ So Theorem 1.3.1 fails in this example.

Remark 1.3.1: Explaination

But why did it fail? It's because $f_n(x)$ is not bounded uniformly, which means that there exist a M and $|f_n(x)| \leq M$ holds for all n and x.

Next we look at an example in which the f_n are uniformly bounded but the theorem still fails.

Example 1.3.2: A bounded function but not Riemann integrable

Since rational numbers are *countable*.

for each n we define a function $f_n(x) : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases} \tag{1.8}$$

Since the rationals are **dense** in $[0, 1]$. So the $\underline{\int}_0^1 = 0$ and $\overline{\int}_0^1 = 1$. So f is not Riemann integrable.

[1] We will figure it out later

[2] $\{f_n\}$ converges pointwise to f on $[a, b]$ if $\lim_n f_n(x) = f(x)$ for each x in $[a, b]$

Example 1.3.3: The difficulty to make Theorem 1.3.1 valid

The difficulty in the previous example comes from the fact that the f_n fail to be continuous.But you can also produce a sequence $\{f_n(x)\}$ such that

1. each $f_n(x)$ is continuous,
2. $0 \leq f_n(x) \leq 1$ holds for each n and x
3. $\{f_n(x)\}$ converges pointwise to a function that is not Riemann integrable.

Lebesgue showed that Theorem 1.3.1, when formulated in terms of his new integral, holds for pointwise convergence of the sequence f_n, subject only to some rather natural boundedness conditions on that sequence.

The definition of Riemann Integral deals with partition of the interval $[a, b]$, which is the domain of f. But the Lebesgue Integral deals with the partition of the range of f.

Suppose we have c to be a positive number and $0 \leq f(x) \leq c$ holds for $x \in [a, b]$. Suppose that is a partition of $[0, c]$, and we have

$$A_i = x \in [a, b] : f(x) \in [a_{i-1}, a_i) \tag{1.9}$$

the set A_i can be empty, unions of finite collection of subintervals, or even more complicated sets. the sum $s(f, \mathscr{P})$ is given by

$$s(f, \mathscr{P}) = \sum_{i=1}^{k} a_{i-1} meas(A_i) \tag{1.10}$$

meas(A_i)is the **size** of the **set** A_i. How to define the size of set is the problem we need to solve.

Chapter 2

Measures

"人间的青草地需要浇水，内心的花园就不会枯萎。"

—方大同，《每个人都会》

2.1 Algebras and Sigma-Algebras

Definition 2.1.1: Algebra

Let X is an arbitrary set.A collection $\mathscr{A}$ of subsets of X is an *algebra* on X if

1. $X \in \mathscr{A}$,
2. for each set A that belongs to $\mathscr{A}$, the set A^c belongs to $\mathscr{A}$.
3. for each finite sequence $A_1 \cdots, A_n$ of sets that belongs to $\mathscr{A}$, the set $\cup_{i=1}^n A_i$ belongs to $\mathscr{A}$, and
4. for each finite sequence $A_1 \cdots, A_n$ of sets that belongs to $\mathscr{A}$, the set $\cap_{i=1}^n A_i$ belongs to $\mathscr{A}$.

From the definition we can find out that $\mathscr{A}$ is closed under complementation, finite unions 有限并集 and finite intersections 有限交集 and condition(3) is equal to condition(4) under condition(2), because $\cap_{i=1}^n A_i = (\cup_{i=1}^n A_i^c)^c$

Definition 2.1.2: σ-Algebra

Let X is an arbitrary set.A collection $\mathscr{A}$ of subsets of X is an σ*-algebra* on X if

1. $X \in \mathscr{A}$,
2. for each set A that belongs to $\mathscr{A}$, the set A^c belongs to $\mathscr{A}$.

3. for each countable infinite sequence $\{A_i\}$ of sets that belongs to $\mathscr{A}$, the set $\cup_{i=1}^{\infty} A_i$ belongs to $\mathscr{A}$, and
4. for each countable infinite sequence $\{A_i\}$ of sets that belongs to $\mathscr{A}$, the set $\cap_{i=1}^{\infty} A_i$ belongs to $\mathscr{A}$.

From the definition we can find out that $\mathscr{A}$ is closed under complementation, countable unions 可数并集 and countable intersections 可数交集 Note that, as in the case of algebras, we could have used only conditions (1), (2), and (3), or only conditions (1), (2), and (4), in our definition.

Remark 2.1.1: Each σ-Algebra on X is an algebra on X

the union of the finite sequence $A_1, A_2, \cdots, A_n$ is the same as the union of the infinite sequence $A_1, A_2, \cdots, A_n, A_n, A_n, \cdots$

Thus in the definitions of algebras and σ-algebras given above, we can replace condition (1) with the requirement that ϕ be a member of $\mathscr{A}$. Furthermore, if $\mathscr{A}$ is a family of subsets of X that is nonempty, closed under complementation, and closed under the formation of finite or countable unions, then $\mathscr{A}$ must contain X: if the set A belongs to $\mathscr{A}$, then X, since it is the union of A and A^c, must also belong to $\mathscr{A}$. Thus in our definitions of algebras and σ-algebras, we can replace condition (1) with the requirement that $\mathscr{A}$ be nonempty.

If $\mathscr{A}$ is a σ-algebra on X, we call a subset of X $\mathscr{A} - \mathit{Measurable}$ if it belongs to $\mathscr{A}$.

Example 2.1.1: Some example of Families of sets

1. Let X be a set, and let $\mathscr{A}$ be the collection of all subsets of X. Then $\mathscr{A}$ is a σ-algebra of X.
2. Let X be a infinite set, and let $\mathscr{A}$ be the collection of all subsets A of X such that either A or A^c is finite. Then $\mathscr{A}$ is an algebra on X but is not closed under countable unions; hence it's not a σ-algebra.
3. Let X be a set, and let $\mathscr{A}$ be the collection of all subsets A of X such that X such that either A or A^c is countable. Then $\mathscr{A}$ is a σ-algebra.
4. Let $\mathscr{A}$ be the collection of all subsets of $\mathbb{R}$ that are unions of finitely many intervals of the form $(a, b]$, $(a, +\infty)$ or $(-\infty, b]$. The $\mathscr{A}$ is an algebra but is not an σ-algebra. Suppose we have $A_n = (0, 2 - \frac{1}{n}]$, for every certain n, $A_n \in \mathscr{A}$, but $\cup_{i=1}^{\infty} A_i = (0, 2) \notin \mathscr{A}$. So it's not an σ-algebra.

Now let's consider way of constructing σ-algebra.

Proposition 2.1.1: constructing σ-algebra

Let X be a set. Then the intersection of an arbitrary nonempty collection of σ-algebras on X is a σ-algebra on X.

Proof

This proof is easy so I just ignore it. $\square$

We should notice that the union of a family of σ-algebra may not be a σ-algebra. From the proof we can see that the key is that the property of intersection makes the transition of sets possible, but union may not succeed in doing this.

Corollary 2.1.1: σ-algebra on X includes a family subsets of X

Let X be a set, and let $\mathscr{F}$ be a family of subsets of X. Then there is a smallest σ-algebra on X that includes $\mathscr{F}$.

Remark 2.1.2: smallest σ-algebra

When we say the *smallest* σ-algebra(denoted by $\mathscr{A}$) that includes $\mathscr{F}$, we are saying that any σ-algebra that includes $\mathscr{F}$ also includes $\mathscr{A}$. The smallest σ-algebra is called the σ-algebra *generated by* $\mathscr{F}$ and denoted by $\sigma(\mathscr{F})$.

Proof

Let $\mathscr{L}$ be the collection of all the σ-algebra that include $\mathscr{F}$. Then $\mathscr{L}$ is nonempty, since it contains the σ-algebra that consists of all subsets of X. Based on the preceding proposition. The intersection of these σ-algebras which belongs to $\mathscr{L}$ is also a σ-algebra. It includes $\mathscr{F}$ and is included in every σ-algebra in $\mathscr{L}$. $\square$

Definition 2.1.3: Borel σ-algebra on $\mathbb{R}^d$

The *Borel* σ-algebra on $\mathbb{R}^d$ is the σ-algebra on $\mathbb{R}^d$ generated by the collection of open subsets of $\mathbb{R}^d$. It is denoted by $\mathscr{B}(\mathbb{R}^d)$. The *Borel subsets* of $\mathbb{R}^d$ are those that belong to $\mathscr{B}(\mathbb{R}^d)$.

Proposition 2.1.2: generating Borel σ-algebra

The σ-algebra $\mathscr{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ is generated by each of the following collections of sets:

1. the collection of all closed subsets of $\mathbb{R}$.

2. the collection of all subintervals of $\mathbb{R}$ of the form $(-\infty, b]$.
3. the collection of all subintervals of $\mathbb{R}$ of the form $(a, b]$.

Proof

Let $\mathscr{B}_1, \mathscr{B}_2$,and $\mathscr{B}_3$ be the σ-algebras generated by the collections of sets in parts (1),(2), and (3) of the proposition. We will show that $\mathscr{B}(\mathbb{R}) \supseteq \mathscr{B}_1 \supseteq \mathscr{B}_2 \supseteq \mathscr{B}_3$ and then that $\mathscr{B}_3 \supseteq \mathscr{B}(\mathbb{R})$; this will establish the proposition. Since $\mathscr{B}(\mathbb{R})$ includes the family of open subsets of $\mathbb{R}$ and is closed under complementation, it includes the family of closed subsets of $\mathbb{R}$; thus it includes the σ-algebra generated by the closed subsets of $\mathbb{R}$, namely $\mathscr{B}_1$. The sets of the form $(-\infty, b]$ are closed and so belong to $\mathscr{B}_1$; consequently $\mathscr{B}_1 \supseteq \mathscr{B}_2$. Since $(a, b] = (-\infty, b] \cap (-\infty, a]^c$,each set of the form $(a, b]$ belongs to $\mathscr{B}_2$; thus $\mathscr{B}_2 \supseteq \mathscr{B}_3$. Finally, note that each open subinterval of $\mathbb{R}$ is the union of a sequence of sets of the form $(a, b]$ and that each open subset of $\mathbb{R}$ is the union of a sequence of open intervals (this proof needs the dense property of rational numbers). Thus each open subset of $\mathbb{R}$ belongs to $\mathscr{B}_3$, and so $\mathscr{B}_3 \supseteq \mathscr{B}(\mathbb{R})$.

□

The Borel σ-algebra $\mathscr{B}(\mathbb{R})$ contains virtually every subset of $\mathbb{R}$ that is interest of analysis, and it's small enough that it can be dealt with in a fairly constructive manner.

Let $\mathscr{G}$ be the family of all open subsets of $\mathbb{R}^d$, and let $\mathscr{F}$ be the family of all closed subsets of $\mathbb{R}^d$. Let $\mathscr{G}_\delta$ be the collection of all intersections of sequence of sets in $\mathscr{G}$. Let $\mathscr{F}_\sigma$ be the collection of all intersections of sequence of sets in $\mathscr{F}$. Sets in $\mathscr{G}_\delta$ are often called $G'_\delta s$, sets in $\mathscr{F}\sigma$ are called $F'_\sigma s$.

Proposition 2.1.3: Each closed subset of $\mathbb{R}^d$ is a G_δ

Proof

suppose F is a closed set in $\mathbb{R}^d$, construct open a sequence $\{U_n\}$ of open subsets of $\mathbb{R}^d$ such that $F = \cap_n U_n$. We define U_n by:

$$U_n = \{x \in \mathbb{R}^d : \|x - y\| < 1/n \text{ for some } y \text{ in } F\}$$

□

Proposition 2.1.4: Each open subset of $\mathbb{R}^d$ is an $\mathscr{F}_\sigma$

Proof

If U is an open set, then U^c is a closed set, so U^c is a G_δ. Therefore there is a sequence $\{U_n\}$ of open sets such that $U^c = \cap_n U_n$. Based on De Morgan's Law, U is an F_σ. $\square$

A sequence $\{A_i\}$ of sets is called *increasing* if $A_i \subseteq A_{i+1}$ holds for each i and *decreasing* if $A_{i+1} \subseteq A_i$ holds for each i.

Proposition 2.1.5: When can algebra be σ-algebra?

Let X be a set, and let $\mathscr{A}$ be an algebra on X. Then $\mathscr{A}$ is a σ-algebra if either:

1. $\mathscr{A}$ is closed under the formation of unions of increasing sequences of sets, or
2. $\mathscr{A}$ is closed under the formation of intersections of decreasing sequences of sets.

Proof

Since $\mathscr{A}$ is an algebra, Suppose $\{A_i\}$ is a sequence of sets that belong to $\mathscr{A}$. For each n let $B_n = \cup_{i=1}^n A_i$. The sequence $\{B_n\}$ is increasing, since $\mathscr{A}$ is an algebra. each B_n belongs to $\mathscr{A}$. The assumption(1) implies that $\cup_n B_n$ belongs to $\mathscr{A}$. However $\cup_i A_i$ is equal to $\cup_n B_n$ and so belongs to $\mathscr{A}$. So $\mathscr{A}$ is an σ-algebra.
If $\{A_i\}$ is an increasing sequence of sets that belong to $\mathscr{A}$, then $\{A_i^c\}$ is a decreasing sequence of sets that belong to $\mathscr{A}$, so condition(2) implies $\cap_i A_i^c$ belongs to $\mathscr{A}$. Since $\cup_i A_i = (\cap_i A_i^c)^c$, so $\cup_i A_i$ belongs to $\mathscr{A}$. So we have $\mathscr{A}$ is a σ-algebra. $\square$

2.2 Measures

References